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New Directions in Mean Field Games: MFG Subpopulation Behaviours and Graphon-MFG Systems

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14. ABSTRACT
A new form of Mean Field Games (MFGs) called Graphon Mean Field Games (GMFGs) is developed and analysed in this program of research together with closely associated theories and their methodologies. GMFGs are non-cooperative games defined for large sub-populations of agents concentrated on the nodes of large, heterogeneous, networks and their asymptotic (graphon) limits. In this setting, game theoretic Nash equilibria are expressed in terms of the newly de- fined GMFG PDE equations, which are a sweeping generalization of the classical MFG PDEs. A key step in the project is the completion of the rigorous deriva- tion of the central theoretical results of GMFG theory. In a closely related development, the theory of graphons is used as a basis for Graphon Linear Control (GLC) theory in which large populations of centrally controlled dynamical systems which interact over large, heterogeneous networks are analyzed and optimized in a tractable way. Specifically, the project includes the generation of a Graphon Linear Control (GLC) theory for the control of large populations of dynamical linear system agents with quadratic performance functions on large networks. In particular, an approximation theory of associ- ated numerical methods is established based upon finite dimensional invariant subspaces of the system graphon operator. This program is currently being ex- tended for GLC systems subject to network-wide Q-noise. Finally, a feature of the research program has been the development of the theory of GMFG systems embedded in Euclidean space; this underlies the variational analysis of agent performance in networks as a function of network position, where this theory includes sparse as well as dense networks.

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New Directions in Mean Field Games: MFG Subpopulation Behaviours and Graphon-MFG Systems

Peter E. Caines

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1 Overview of Research Objectives

1.1 Background and Overview

Very large networks linking dynamical agents are now ubiquitous and the need to analyse, design and control them is evident. The emergence of the graphon theory of large networks and their infinite limits has enabled the formulation of a theory of the centralized control of dynamical systems distributed on asymptotically infinite networks.

Large-scale systems of competing agents are frequently formulated as dynamic games, this however confronts the problem that for even small populations non-cooperative game theory is intractable. The 2006 breakthrough of Minyi Huang, Roland Malhamé and the PI of FA9550-19-1-0138, Peter Caines, was the introduction of Mean Field Game (MFG) theory (originally termed Nash Certainty Equivalence theory) which was also simultaneously discovered by J.M. Lasry and P.L. Lions.

Mean Field Game (MFG) theory analyzes the existence of Nash equilibria for dynamical games in terms of infinite populations of negligible agents forming the limits of the original finite population systems (Caines, 2020). The core of MFG theory is a pair of partial differential equations consisting of (i) the Hamilton-Jacobi-Bellman equation of optimal control for a generic agent, and (ii) the Fokker-Planck-Kolmogorov equation describing the controlled evolution of the state of a generic agent. These are linked by the generic agent's state

distribution, known as the system's mean field.

A fundamental aspect of MFG theory justifying its application in a vast range of cases is the ϵ-Nash property by which controls derived from the MFG equations yield near-equilibrium performance for finite systems for sufficiently large populations.

The MFG field has expanded tremendously since its inception. Google search results now run to 1.10 billion, citations to papers are in the thousands, conferences and workshops on MFG have proliferated and at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively). In addition to the explosion of mathematical and theoretical investigations over last 16 years, applications of MFG theory have spread to a vast range of domains including: optimization of decentralized EV charging (Ma et al., 2011; Zhu et al., 2016; Foguen Tchuendom et al., 2019); nonlinear system state estimation (Yang et al., 2013); systemic risk in banking (Carmona et al., 2013); cell phone energy management (Aziz and Caines, 2016); economics of energy production (Ludkovski and Sircar, 2015); macro-economics of growth (Gomes et al., 2016); cloud-based networks in information technology (Hanif et al., 2015); algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Nourian et al., 2011; Grover et al., 2018), AI reinforcement learning for MFG systems (Subramanian and Mahajan, 2019; Agarwal et al., 2022), deep learning applications to the solution of MFGs (Ruthotto et al., 2020) and polymer science (Welch et al., 2019).

In this context it is important to note that the research on MFG by the Principal Investigator has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, most recently, for the period 2022-2025.

1.2 Major Research Goals under Grant FA9550-19-1-0138

The program of research under FA9550-19-1-0138 is focused on the development of a new form of Mean Field Games called Graphon Mean Field Games (GMFGs); these are large population games defined on large, heterogeneous, networks and their asymptotic (graphon) limits.

In this setting, game theoretic equilibria are expressed in terms of the newly defined GMFG PDE equations, which are a sweeping generalization of the classical MFG PDEs.

In a closely related development, the theory of graphons is used as a basis for Graphon Linear Control (GLC) theory in which large populations of centrally controlled dynamical systems interacting over large, heterogeneous networks

may be analyzed and optimized in a tractable way.

Canonical Design Procedure for GMFG and GLC Systems

A major goal has been to generate large scale network game strategies and, respectively, control design procedures with the following standard pattern:

The first step is to consider the graphon limit of the network over which the individual systems are distributed and which constitutes the global system.

In the second step, the resulting graphon based GMFG equations, or, respectively, the graphon based infinite dimensional linear system optimization problems are specified. The GMFG equations are then solved to generate the Nash best response strategies or, respectively, the linear feedback controls are synthesized using the well established infinite dimensional linear systems control theory.

Finally, the best response strategies, or, respectively, the infinite dimensional linear feedback controls are approximated in such a way that they may be applied to the original complex finite systems.

The key feature of this strategy is the following: In both the network mean field game and the network control settings it is demonstrated that as the size of the network tends to infinity, the performance of the finite systems when they are subject to the infinite limit system best response game strategies, or, respectively, centralized control laws, yield pay-offs which converge to their Nash equilibrium, or, respectively, optimal, infinite system limit values.

1.3 Organization of the Sequence of Research Objectives

The work over the full reporting period has been organized with the following sequence of objectives:

(i) The establishment of a rigorous theory of the central results of Graphon Mean Field Game (GMFG) theory,

(ii) The generation of a specific Graphon Linear Control (GLC) theory for the control of large populations of dynamical linear system agents with quadratic performance functions on large, complex networks.

(iii) The development of an approximation theory of associated numerical methods, especially those based upon finite dimensional invariant subspaces of the graphon operator in the GLC case.

(iv) The pioneering development of the theory of GMFG systems embedded in Euclidean space; this underlies our currently developing variational analysis of agent performance in networks as a function of position, where this theory also includes sparse as well as dense networks.

(v) The development of an SIR epidemic model software which permits the simulation of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity, these models are constructed on a data base of real world US surface and air-traffic travel data.

2 Details of Accomplishments

2.1 Graphon Mean Field Game Theory: Foundations

The work in (Caines and Huang, 2021) (Caines and Huang, 2018), (Caines and Huang, 2019), established the foundations and central results for graphon mean field game theory for nonlinear systems, that is to say games of infinite populations on infinite heterogeneous networks as limits of the corresponding finite systems.

Specifically, in the conference papers (Caines and Huang, 2018), (Caines and Huang, 2019) Graphon Mean Field Games (GMFG) and the GMFG equations were introduced for the analysis of non-cooperative dynamical games on unbounded networks. In this formulation, asymptotically infinite sub-populations are located at the nodes of large heterogeneous networks, the asymptotic network limits then constitute a part of the analysis. In (Caines and Huang, 2021), the basic existence and uniqueness results were established for the GMFG equations, and a full derivation of the fundamental results was provided, including the key epsilon-Nash theory for GMFG systems which relates the infinite population equilibria on infinite networks to finite population equilibria on finite networks.

2.2 LQG GMFG Decomposition Methods

The theory of invariant spaces of linear operators developed in the papers (Gao and Caines, 2021) is applied in (Gao et al., 2021b), (Gao et al., 2021a) and (Gao et al., 2022) to yield a tractable methodology for generating decentralized game strategies for systems distributed over complex networks. The fundamental properties of the existence and uniqueness of solutions are established in (Gao et al., 2021b,a, 2022) in two principal ways: (i) Riccati equation decomposition and (ii) fixed point contraction theorem methods.

2.3 LQG GMFG Master Equation Analysis

First, using a probabilistic approach, solutions to the Linear Quadratic Graphon Mean Field Games are expressed as solutions to coupled Forward Backward Stochastic Differential Equations (FBSDEs) of McKean-Vlasov type. Next, the

existence of the so-called Master field, which allows for the decoupling of these FBSDEs is established. Finally, the infinite dimensional Partial Differential Equation (PDE), so-called Master Equation, is established for which the Master Field is a solution (see (Foguen Tchuendom et al., 2020)).

2.4 GMFG with Affine Dynamics

GMFG theory for nonlinear problems with individual affine dynamics has been established in Caines et al. (2022) and the Nash value properties for control affine systems has been analyzed in Caines et al. (2023).

2.5 Graphon Linear Systems Control with Approximation Theory

As explained above, the theory of graphons is further used as a basis upon which centrally controlled linear dynamical systems interacting over large, heterogeneous networks are analyzed in a tractable way.

The graphon control methodology is developed in (Gao and Caines, 2020, 2019a,b) to study control problems for arbitrary-size networks of linear dynamical systems. The complete treatment of problems involving well-posedness, controllability, approximation errors for state-to-state control and regulations is established in (Gao and Caines, 2020).

Scalable and tractable solution methods are established for a class of graphon control problem in (Gao and Caines, 2019a,b). Next, the graphon linear system control framework is extended to more general cases where the state of each individual agent on the network is multi-dimensional (Gao and Caines, 2021); moreover, this work provides a generalized solution method with an application to large-scale networks of coupled harmonic oscillators.

2.6 LQ Graphon Dynamical Systems and Control with Q-space Noise

The work in (Dunyak and Caines, 2022) extends the formulation and analysis of (Gao and Caines, 2021) to include stochastic processes in infinite dimensions as a random process source driving the network's controlled dynamical systems; this generalization involves the formulation of a form of graphon Q-space noise.

2.7 Critical Nodes for GMFG and Embedded GMFG

The works (Caines, 2022),(Foguen Tchuendom et al., 2021), (Foguen Tchuendom et al., 2022a), (Foguen Tchuendom et al., 2022b) and (Caines et al., 2023)

pioneer the theory of GMFG systems embedded in Euclidean space by constructing limit graphs and limit graphons for sequences of graphs sitting in Euclidean space. The crucial notion here is that for embedded graphs and graphons we may define the differentiation of functions on the embedded graph limits. In particular we formulate an extended version of mean field game systems, termed Embedded Graphon Mean Field Games (EGMFG), or Graphexon Mean Field Games (GXMFG), which are amenable to the application of the calculus to the optimization of node location in such systems, this being one of the main objectives of the new development. Moreover this formulation includes the case of sparse graph limits (see (Caines, 2022)) which is not covered by the standard graphon formulation.

2.8 MFG Theory and Related Stochastic Adaptive and Hybrid Control Theory

The paper (Pakniyat and Caines, 2021) constitutes the climax of several decades of work by my own group and many other researchers aimed at producing a rigorous, general, hybrid optimal systems control theory (HOCT). These and related results have great applicability in many domains of engineering and scientific modelling, and furthermore in economics and finance. In fact the paper (Firoozi et al., 2022) presents an extension of HOCT to stochastic MFG systems motivated by trading dynamics in finance.

In (Saude and Caines, 2022), we study the Markowitz dynamic portfolio problem, where an agent seeks to maximize its expected return while minimizing the variance of the return (risk). For the first time in this topic, the impact of the aggregate of the agents' trades is incorporated into the dynamics and costs using an extended mean-field games framework.

2.9 Data Based Protocols for Modelling Large Populations on Large Networks with Applications to SIR Problems

The initial investigation of applying graphon theory to the control of epidemic spread is proposed in (Gao and Caines, 2019a) where scalable solutions based on spectral approximation and linearization are established for controlling virus spreads on networks following a type of controlled network SIS model.
The work (Dunyak and Caines, 2021; Salii, 2021) includes the development of SIR epidemic models which permits the study of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity; the latter models were constructed on a data base of real world US surface and air-traffic travel data.

2.10 Centrality Analysis for Finite and Infinite Networks

This work advances the theory of centralities for dynamical systems on networks and its associated applications. In (Gao, 2022) a family of centralities characterized by fixed-points of networks is identified which includes the standard centralities (e.g. PageRank, Katz-Bonacich, eigen centralities) and equilbrirum cost for games on networks (e.g. Nash values of LQG GMFG). Variation bounds of centralities with respect to the network variations are established. In (Gao, 2021) network centralities have been utilized in the modelling of opinion dynamics on social networks and in identifying significant network partitions based upon opinion disagreement states.

3 Dissemination to Communities of Interest

Research findings and results have been published in the leading systems and control and networks journals including the SIAM Journal on Control and Optimization, IEEE Transactions on Automatic Control, Automatica, IEEE Transactions on Control of Network Systems, and in conferences proceedings including the IEEE Conference on Decision and Control.

3.1 Publications: Journals

S. Gao, P. E. Caines, and M. Huang, "LQG graphon mean field games: analysis via graphon invariant subspaces," Conditionally accepted by IEEE Transactions on Automatic Control, 2021. (Gao et al., 2022)

D. Firoozi, A. Pakniyat, and P. E. Caines, "A class of hybrid LQG mean field games with state-invariant switching and stopping strategies," Automatica, vol. 141, p. 110244, 2022. (Firoozi et al., 2022)

P. E. Caines, D. Ho, M. Huang, J. Jian, and Q. Song. On the graphon mean field game equations: Individual agent affine dynamics and mean field dependent performance functions. ESAIM: Control, Optimisation and Calculus of Variations, 28:24, 2022 (Caines et al., 2022)

P. E. Caines and M. Huang, "Graphon mean field games and their equations," SIAM Journal on Control and Optimization, vol. 59, no. 6, pp. 4373–4399, 2021. (Caines and Huang, 2021)

A. Pakniyat and P. E. Caines, "On the hybrid minimum principle: The Hamiltonian and adjoint boundary conditions," IEEE Transactions on Automatic Control, vol. 66, no. 3, pp. 1246–1253, 2021. (Pakniyat and Caines, 2021)

S. Gao and P. E. Caines, "Subspace decomposition for graphon LQR: Applications to VLSNs of harmonic oscillators," IEEE Transactions on Control of Network Systems, vol. 8, no. 2, pp. 576–586, 2021. (Gao and Caines, 2021)

S. Gao, R. Foguen Tchuendom, and P. E. Caines, "Linear quadratic graphon field games," Communications in Information and Systems, vol. 21, no. 3, pp. 341–369, 2021. (Gao et al., 2021b)

D. Firoozi, and P. E. Caines, "ϵ-Nash Equilibria for Major-Minor LQG Mean Field Games with Partial Observations of All Agents", *IEEE Trans.on Automatic Control*, July, 2020, Vol 66, No. 6, pp 2778 - 2786. On-line: DOI: 10.1109/TAC.2020.3010129 (Firoozi and Caines, 2020)

P. E. Caines. Mean field games. In Encyclopedia of Systems and Control, pages 1197–1202. Springer, 2021. (Caines, 2021)

P. E. Caines. Mean field game theory: A tractable methodology for large population problems. SIAM News, 53(3):5–6, 2020. (Caines, 2020)

D. Firoozi, S. Jaimungal, and P. E. Caines. Convex analysis for lqg systems with applications to major–minor LQG mean–field game systems. Systems & Control Letters, 142:104734, 2020. (Firoozi et al., 2020)

S. Gao and P. E. Caines. Graphon control of large-scale networks of linear systems. IEEE Transactions on Automatic Control, 65(10):4090–4105, 2020 (Gao and Caines, 2020)

N. Sen and P. E. Caines. Mean field games with partial observation. SIAM Journal on Control and Optimization, 57(3):2064–2091, 2019 (Sen and Caines, 2019)

P. E. Caines, D. Ho, and Q. Song. The density evolution of the killed mckean–vlasov process. Stochastics, 92(4):642–657, 2020. (Caines et al., 2020)

3.2 Publications: Conference Proceedings

A. Dunyak and P. E. Caines, "Linear stochastic graphon systems with q- space noise," in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 3926–3932. (Dunyak and Caines, 2022)

P. E. Caines, "Embedded vertexon-graphons and embedded GMFG systems," Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022,

pp. 5550–5557. (Caines, 2022)

S. Gao, “Fixed-point centrality for networks,” in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 1628–1635. (Gao, 2022).

R. Foguen Tchuendom, S. Gao, P. E. Caines, and M. Huang, “Optimal network location in infinite horizon LQG graphon mean field games,” in Proceedings of the 61th IEEE Conference on Decision and Control, Cancun, Mexico, December 2022, pp. 5558–5565. (Foguen Tchuendom et al., 2022b)

R. Foguen Tchuendom, S. Gao, and P. E. Caines “Stationary cost nodes in infinite horizon LQG GMFGs,” IFAC-PapersOnLine for the 25th International Symposium on Mathematical Theory of Networks and Systems (MTNS), vol. 55, no. 30, pp. 284–289, 2022. (Foguen Tchuendom et al., 2022a)

J. Saude and P. E. Caines, “Markowitz portfolio optimization extended quadratic mean- field games approach,” Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 5523–5528. (Saude and Caines, 2022)

Y. V. Salii, “Benchmarking optimal control for network dynamic systems with plausible epidemic models,” in International Conference on Complex Networks and Their Applications. Springer, 2021, pp. 194–206. (Salii, 2021)

S. Gao, P. E. Caines, and M. Huang, “LQG graphon mean field games: Graphon invariant subspaces,” Proceedings of the 60th IEEE Conference on Decision and Control, Austin, Texas, USA, December 2021, pp. 5253–5260. (Gao et al., 2021a)

R. Foguen Tchuendom, P. E. Caines, and M. Huang, “Critical nodes in graphon mean field games,” Proceedings of the 60th IEEE Conference on Decision and Control (CDC). IEEE, 2021, pp. 166–170. (Foguen Tchuendom et al., 2021)

A. Dunyak and P. E. Caines, “Large scale systems and SIR models: A featured graphon approach,” Proceedings of the 60th IEEE Conference on Decision and Control (CDC), 2021, pp. 6928–6933. (Dunyak and Caines, 2021)

S. Gao, “Centrality-weighted opinion dynamics: Disagreement and social network partition,” Proceedings of the 60th IEEE Conference on Decision and Control (CDC), Austin, Texas, USA, December 2021, pp. 5496–5501. (Gao, 2021)

R. Foguen Tchuendom, P. E. Caines, and M. Huang. “On the master equation for linear quadratic graphon mean field games”, Proceedings of the 59th

IEEE Conference on Decision and Control (CDC), pages 1026–1031, 2020. (Foguen Tchuendom et al., 2020)

S. Gao and P. E. Caines, Optimal and approximate solutions to linear quadratic regulation of a class of graphon dynamical systems, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 8359–8365, Nice, France, December 2019b. (Gao and Caines, 2019b)

S. Gao and P. E. Caines, Spectral representations of graphons in very large network systems control, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 5068–5075, Nice, France, December 2019a.(Gao and Caines, 2019a)

P. E. Caines and M. Huang. Graphon mean field games and the GMFG equations: ϵ-Nash equilibria, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 286–292, December 2019. (Caines and Huang, 2019)

3.3 Talks: April 1st, 2019 to December 31st, 2022

1. P. E. Caines,"Vertexons and Embedded Graphon Mean Field Games", the Simons Institute for the Theory of Computing, UC Berkeley, CA, Sept.29, 2022 (Invited talk) in the "Graph Limits, Nonparametric Models, and Estimation Workshop" 26 - 30 September, 2022.

2. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", (Invited talk) 29th August, 2022, in the Conference on Many Player Games and Applications, Humbolt University Berlin, 29 - 31 August 2022.

3. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Large Populations on Complex Networks", Canadian Applied and Industrial Mathematics Society (CAIMS), June 23rd, 2021 (Plenary talk).

4. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", the Financial Mathematics/Engineering Seminar Series, Hong Kong Polytechnic University, June 16th 2021 (Invited talk).

5. P. E. Caines, "Static and Time Varying Graphons in Economic and Pandemic GMFG Models" SIAM Financial Mathematics and Engineering (FM21), June 1, 2021 (Invited talk).

6. S. Gao, "Subspace Decompositions in Graphon Control and Graphon Mean Field Games", Informal Systems Seminar, Centre for Intelligent Machines,

McGill University, Groupe d'Etudes et de Recherche en Analyse des Decisions, Montreal, CA, Dec. 11, 2020 (Invited talk).

7. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Networked Systems", Invited Speaker at the "Conference on Mean Field Games and Related Topics - 5". 9th - 13th September, 2019

8. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for a Networked World", University of Saint Petersburg, Russia, 6th September, 2019 (Distinguished Speaker).

9. P. E. Caines, "Graphon Control and Graphon Mean Field Games", Department of Applied Mathemaics, Hong Kong Polythenic Universty, Hong Kong), 28th June, 2019 (Distinguished Lecturer).

3.4 Workshop Organization: April 1st, 2019 to December 31st, 2022

Name: **Mean Field Games on Networks Workshop**

Location: Vancouver, BC, Canada (Virtual Meeting)

Dates: October 26 - 29, 2021

Organizers: Peter E. Caines, Shuang Gao, Minyi Huang, and Rinel Foguen Tchuendom

This workshop (PIMS et al., 2021) brought together researchers in applied mathematics, mean field games, network science, network games, and systems and control theory to exchange ideas and to work on the extensions of mean field game theory to dynamic game problems on heterogeneous large-scale networks. The workshop offered a platform to current results and stimulate discussions on the open challenges in this emerging field, on mathematical theory, computational algorithms, and the applications methodologies of MFG on general networks.

Sponsors: Pacific Institute for the Mathematical Sciences (PIMS), Groupe d'études et de recherche en analyse des décisions (GERAD), and the Fields Institute for Research in Mathematical Sciences.

Participants: Peter E. Caines (PI) and PI's Postdoctoral Researchers: Rinel Foguen Tchuendom, Shuang Gao, Joao Saude, and Yaroslav Salii, plus 50 other international participants.

4 Impacts

As stated earlier, the research on MFG and related topics by the Principal Investigator of the grant FA9550-19-1-0138 has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, presently 2022 - 2025. Consequently, the impacts of research under the 2019 - 2022 grant, FA9550-19-1-0138, may be viewed as a continuation and development of those of the entire program.

Indeed, as explained in Section 1, the MFG field has expanded enormously since its inception. Google links now run to 1.10 billion, citations to papers are in the thousands and conferences and workshops on MFG have proliferated. A sample of the most recent workshops and conference sessions is the following, where we note that the PI and group members have been invited to participate in all of them: Workshop "Mean Field Games on Networks", Pacific Institute for the Mathematical Sciences, Vancouver, BC, Canada, 26 - 29, October, 2021; "Conference on Mathematics and Financial Economics, "Many Player Games and Applications", Humbolt University, Berlin, 29 - 31 August 2022; Workshop on Mean Field Games, Montreal: Centre de Recherches Mathématiques (CRM), 11-15 April, 2022; "Graph Limits, Nonparametric Models, and Estimation Workshop", Simons Institute for the Theory of Computation, UCB, Berkeley, CA, 26-30 September, 2022; IEEE, Control Systems Society Conference on Decision and Control, Cancun, December, 2022, Mean Field Games Session; Mean Field Games Invited Session, IFAC World Congress, Yokahama, Japan, July 2023.

Furthermore, at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively), the last appearing in the period covered by this report. In addition to the very rapid expansion of mathematical and theoretical investigations, applications of MFG theory have spread to a vast range of domains, a sample of the most recent being the optimization of decentralized electric vehicle charging (Foguen Tchuendom et al., 2019), algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Grover et al., 2018), AI reinforcement learning for MFG systems (Agarwal et al., 2022) and polymer science (Welch et al., 2019).

As is shown by this indicative list of publications, meetings and presentations, it is reasonable to assert that the influence of the AFOSR supported research carried out on Mean Field Games and, in particular, on Mean Field Games on networks, has had, and is having, an impact on fields ranging from the core area of systems and control science to economics, finance, energy generation and distribution, operations research, the life sciences and the physical sciences.

5 Changes

There were no changes in the project or its direction.

6 Technical Updates

There are no technical updates to report beyond the continuing research progress and dissemination of results described in report.

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New Directions in Mean Field Games: MFG Subpopulation Behaviours and Graphon-MFG Systems

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1 Overview of Research Objectives

1.1 Background and Overview

Very large networks linking dynamical agents are now ubiquitous and the need to analyse, design and control them is evident. The emergence of the graphon theory of large networks and their infinite limits has enabled the formulation of a theory of the centralized control of dynamical systems distributed on asymptotically infinite networks.

Large-scale systems of competing agents are frequently formulated as dynamic games, this however confronts the problem that for even small populations non-cooperative game theory is intractable. The 2006 breakthrough of Minyi Huang, Roland Malhamé and the PI of FA9550-19-1-0138, Peter Caines, was the introduction of Mean Field Game (MFG) theory (originally termed Nash Certainty Equivalence theory) which was also simultaneously discovered by J.M. Lasry and P.L. Lions.

Mean Field Game (MFG) theory analyzes the existence of Nash equilibria for dynamical games in terms of infinite populations of negligible agents forming the limits of the original finite population systems (Caines, 2020). The core of MFG theory is a pair of partial differential equations consisting of (i) the Hamilton-Jacobi-Bellman equation of optimal control for a generic agent, and (ii) the Fokker-Planck-Kolmogorov equation describing the controlled evolution of the state of a generic agent. These are linked by the generic agent's state

distribution, known as the system's mean field.

A fundamental aspect of MFG theory justifying its application in a vast range of cases is the ϵ-Nash property by which controls derived from the MFG equations yield near-equilibrium performance for finite systems for sufficiently large populations.

The MFG field has expanded tremendously since its inception. Google search results now run to 1.10 billion, citations to papers are in the thousands, conferences and workshops on MFG have proliferated and at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively). In addition to the explosion of mathematical and theoretical investigations over last 16 years, applications of MFG theory have spread to a vast range of domains including: optimization of decentralized EV charging (Ma et al., 2011; Zhu et al., 2016; Foguen Tchuendom et al., 2019); nonlinear system state estimation (Yang et al., 2013); systemic risk in banking (Carmona et al., 2013); cell phone energy management (Aziz and Caines, 2016); economics of energy production (Ludkovski and Sircar, 2015); macro-economics of growth (Gomes et al., 2016); cloud-based networks in information technology (Hanif et al., 2015); algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Nourian et al., 2011; Grover et al., 2018), AI reinforcement learning for MFG systems (Subramanian and Mahajan, 2019; Agarwal et al., 2022), deep learning applications to the solution of MFGs (Ruthotto et al., 2020) and polymer science (Welch et al., 2019).

In this context it is important to note that the research on MFG by the Principal Investigator has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, most recently, for the period 2022-2025.

1.2 Major Research Goals under Grant FA9550-19-1-0138

The program of research under FA9550-19-1-0138 is focused on the development of a new form of Mean Field Games called Graphon Mean Field Games (GMFGs); these are large population games defined on large, heterogeneous, networks and their asymptotic (graphon) limits.

In this setting, game theoretic equilibria are expressed in terms of the newly defined GMFG PDE equations, which are a sweeping generalization of the classical MFG PDEs.

In a closely related development, the theory of graphons is used as a basis for Graphon Linear Control (GLC) theory in which large populations of centrally controlled dynamical systems interacting over large, heterogeneous networks

may be analyzed and optimized in a tractable way.

Canonical Design Procedure for GMFG and GLC Systems

A major goal has been to generate large scale network game strategies and, respectively, control design procedures with the following standard pattern:

The first step is to consider the graphon limit of the network over which the individual systems are distributed and which constitutes the global system.

In the second step, the resulting graphon based GMFG equations, or, respectively, the graphon based infinite dimensional linear system optimization problems are specified. The GMFG equations are then solved to generate the Nash best response strategies or, respectively, the linear feedback controls are synthesized using the well established infinite dimensional linear systems control theory.

Finally, the best response strategies, or, respectively, the infinite dimensional linear feedback controls are approximated in such a way that they may be applied to the original complex finite systems.

The key feature of this strategy is the following: In both the network mean field game and the network control settings it is demonstrated that as the size of the network tends to infinity, the performance of the finite systems when they are subject to the infinite limit system best response game strategies, or, respectively, centralized control laws, yield pay-offs which converge to their Nash equilibrium, or, respectively, optimal, infinite system limit values.

1.3 Organization of the Sequence of Research Objectives

The work over the full reporting period has been organized with the following sequence of objectives:

(i) The establishment of a rigorous theory of the central results of Graphon Mean Field Game (GMFG) theory,

(ii) The generation of a specific Graphon Linear Control (GLC) theory for the control of large populations of dynamical linear system agents with quadratic performance functions on large, complex networks.

(iii) The development of an approximation theory of associated numerical methods, especially those based upon finite dimensional invariant subspaces of the graphon operator in the GLC case.

(iv) The pioneering development of the theory of GMFG systems embedded in Euclidean space; this underlies our currently developing variational analysis of agent performance in networks as a function of position, where this theory also includes sparse as well as dense networks.

(v) The development of an SIR epidemic model software which permits the simulation of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity, these models are constructed on a data base of real world US surface and air-traffic travel data.

2 Details of Accomplishments

2.1 Graphon Mean Field Game Theory: Foundations

The work in (Caines and Huang, 2021) (Caines and Huang, 2018), (Caines and Huang, 2019), established the foundations and central results for graphon mean field game theory for nonlinear systems, that is to say games of infinite populations on infinite heterogeneous networks as limits of the corresponding finite systems.

Specifically, in the conference papers (Caines and Huang, 2018), (Caines and Huang, 2019) Graphon Mean Field Games (GMFG) and the GMFG equations were introduced for the analysis of non-cooperative dynamical games on unbounded networks. In this formulation, asymptotically infinite sub-populations are located at the nodes of large heterogeneous networks, the asymptotic network limits then constitute a part of the analysis. In (Caines and Huang, 2021), the basic existence and uniqueness results were established for the GMFG equations, and a full derivation of the fundamental results was provided, including the key epsilon-Nash theory for GMFG systems which relates the infinite population equilibria on infinite networks to finite population equilibria on finite networks.

2.2 LQG GMFG Decomposition Methods

The theory of invariant spaces of linear operators developed in the papers (Gao and Caines, 2021) is applied in (Gao et al., 2021b), (Gao et al., 2021a) and (Gao et al., 2022) to yield a tractable methodology for generating decentralized game strategies for systems distributed over complex networks. The fundamental properties of the existence and uniqueness of solutions are established in (Gao et al., 2021b,a, 2022) in two principal ways: (i) Riccati equation decomposition and (ii) fixed point contraction theorem methods.

2.3 LQG GMFG Master Equation Analysis

First, using a probabilistic approach, solutions to the Linear Quadratic Graphon Mean Field Games are expressed as solutions to coupled Forward Backward Stochastic Differential Equations (FBSDEs) of McKean-Vlasov type. Next, the

existence of the so-called Master field, which allows for the decoupling of these FBSDEs is established. Finally, the infinite dimensional Partial Differential Equation (PDE), so-called Master Equation, is established for which the Master Field is a solution (see (Foguen Tchuendom et al., 2020)).

2.4 GMFG with Affine Dynamics

GMFG theory for nonlinear problems with individual affine dynamics has been established in Caines et al. (2022) and the Nash value properties for control affine systems has been analyzed in Caines et al. (2023).

2.5 Graphon Linear Systems Control with Approximation Theory

As explained above, the theory of graphons is further used as a basis upon which centrally controlled linear dynamical systems interacting over large, heterogeneous networks are analyzed in a tractable way.

The graphon control methodology is developed in (Gao and Caines, 2020, 2019a,b) to study control problems for arbitrary-size networks of linear dynamical systems. The complete treatment of problems involving well-posedness, controllability, approximation errors for state-to-state control and regulations is established in (Gao and Caines, 2020).

Scalable and tractable solution methods are established for a class of graphon control problem in (Gao and Caines, 2019a,b). Next, the graphon linear system control framework is extended to more general cases where the state of each individual agent on the network is multi-dimensional (Gao and Caines, 2021); moreover, this work provides a generalized solution method with an application to large-scale networks of coupled harmonic oscillators.

2.6 LQ Graphon Dynamical Systems and Control with Q-space Noise

The work in (Dunyak and Caines, 2022) extends the formulation and analysis of (Gao and Caines, 2021) to include stochastic processes in infinite dimensions as a random process source driving the network's controlled dynamical systems; this generalization involves the formulation of a form of graphon Q-space noise.

2.7 Critical Nodes for GMFG and Embedded GMFG

The works (Caines, 2022),(Foguen Tchuendom et al., 2021), (Foguen Tchuendom et al., 2022a), (Foguen Tchuendom et al., 2022b) and (Caines et al., 2023)

pioneer the theory of GMFG systems embedded in Euclidean space by constructing limit graphs and limit graphons for sequences of graphs sitting in Euclidean space. The crucial notion here is that for embedded graphs and graphons we may define the differentiation of functions on the embedded graph limits. In particular we formulate an extended version of mean field game systems, termed Embedded Graphon Mean Field Games (EGMFG), or Graphexon Mean Field Games (GXMFG), which are amenable to the application of the calculus to the optimization of node location in such systems, this being one of the main objectives of the new development. Moreover this formulation includes the case of sparse graph limits (see (Caines, 2022)) which is not covered by the standard graphon formulation.

2.8 MFG Theory and Related Stochastic Adaptive and Hybrid Control Theory

The paper (Pakniyat and Caines, 2021) constitutes the climax of several decades of work by my own group and many other researchers aimed at producing a rigorous, general, hybrid optimal systems control theory (HOCT). These and related results have great applicability in many domains of engineering and scientific modelling, and furthermore in economics and finance. In fact the paper (Firoozi et al., 2022) presents an extension of HOCT to stochastic MFG systems motivated by trading dynamics in finance.

In (Saude and Caines, 2022), we study the Markowitz dynamic portfolio problem, where an agent seeks to maximize its expected return while minimizing the variance of the return (risk). For the first time in this topic, the impact of the aggregate of the agents' trades is incorporated into the dynamics and costs using an extended mean-field games framework.

2.9 Data Based Protocols for Modelling Large Populations on Large Networks with Applications to SIR Problems

The initial investigation of applying graphon theory to the control of epidemic spread is proposed in (Gao and Caines, 2019a) where scalable solutions based on spectral approximation and linearization are established for controlling virus spreads on networks following a type of controlled network SIS model.
The work (Dunyak and Caines, 2021; Salii, 2021) includes the development of SIR epidemic models which permits the study of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity; the latter models were constructed on a data base of real world US surface and air-traffic travel data.

2.10 Centrality Analysis for Finite and Infinite Networks

This work advances the theory of centralities for dynamical systems on networks and its associated applications. In (Gao, 2022) a family of centralities characterized by fixed-points of networks is identified which includes the standard centralities (e.g. PageRank, Katz-Bonacich, eigen centralities) and equilbrirum cost for games on networks (e.g. Nash values of LQG GMFG). Variation bounds of centralities with respect to the network variations are established. In (Gao, 2021) network centralities have been utilized in the modelling of opinion dynamics on social networks and in identifying significant network partitions based upon opinion disagreement states.

3 Dissemination to Communities of Interest

Research findings and results have been published in the leading systems and control and networks journals including the SIAM Journal on Control and Optimization, IEEE Transactions on Automatic Control, Automatica, IEEE Transactions on Control of Network Systems, and in conferences proceedings including the IEEE Conference on Decision and Control.

3.1 Publications: Journals

S. Gao, P. E. Caines, and M. Huang, "LQG graphon mean field games: analysis via graphon invariant subspaces," Conditionally accepted by IEEE Transactions on Automatic Control, 2021. (Gao et al., 2022)

D. Firoozi, A. Pakniyat, and P. E. Caines, "A class of hybrid LQG mean field games with state-invariant switching and stopping strategies," Automatica, vol. 141, p. 110244, 2022. (Firoozi et al., 2022)

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P. E. Caines and M. Huang, "Graphon mean field games and their equations," SIAM Journal on Control and Optimization, vol. 59, no. 6, pp. 4373–4399, 2021. (Caines and Huang, 2021)

A. Pakniyat and P. E. Caines, "On the hybrid minimum principle: The Hamiltonian and adjoint boundary conditions," IEEE Transactions on Automatic Control, vol. 66, no. 3, pp. 1246–1253, 2021. (Pakniyat and Caines, 2021)

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N. Sen and P. E. Caines. Mean field games with partial observation. SIAM Journal on Control and Optimization, 57(3):2064–2091, 2019 (Sen and Caines, 2019)

P. E. Caines, D. Ho, and Q. Song. The density evolution of the killed mckean–vlasov process. Stochastics, 92(4):642–657, 2020. (Caines et al., 2020)

3.2 Publications: Conference Proceedings

A. Dunyak and P. E. Caines, "Linear stochastic graphon systems with q- space noise," in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 3926–3932. (Dunyak and Caines, 2022)

P. E. Caines, "Embedded vertexon-graphons and embedded GMFG systems," Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022,

pp. 5550–5557. (Caines, 2022)

S. Gao, "Fixed-point centrality for networks," in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 1628–1635. (Gao, 2022).

R. Foguen Tchuendom, S. Gao, P. E. Caines, and M. Huang, "Optimal network location in infinite horizon LQG graphon mean field games," in Proceedings of the 61th IEEE Conference on Decision and Control, Cancun, Mexico, December 2022, pp. 5558–5565. (Foguen Tchuendom et al., 2022b)

R. Foguen Tchuendom, S. Gao, and P. E. Caines "Stationary cost nodes in infinite horizon LQG GMFGs," IFAC-PapersOnLine for the 25th International Symposium on Mathematical Theory of Networks and Systems (MTNS), vol. 55, no. 30, pp. 284–289, 2022. (Foguen Tchuendom et al., 2022a)

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Y. V. Salii, "Benchmarking optimal control for network dynamic systems with plausible epidemic models," in International Conference on Complex Networks and Their Applications. Springer, 2021, pp. 194–206. (Salii, 2021)

S. Gao, P. E. Caines, and M. Huang, "LQG graphon mean field games: Graphon invariant subspaces," Proceedings of the 60th IEEE Conference on Decision and Control, Austin, Texas, USA, December 2021, pp. 5253–5260. (Gao et al., 2021a)

R. Foguen Tchuendom, P. E. Caines, and M. Huang, "Critical nodes in graphon mean field games," Proceedings of the 60th IEEE Conference on Decision and Control (CDC). IEEE, 2021, pp. 166–170. (Foguen Tchuendom et al., 2021)

A. Dunyak and P. E. Caines, "Large scale systems and SIR models: A featured graphon approach," Proceedings of the 60th IEEE Conference on Decision and Control (CDC), 2021, pp. 6928–6933. (Dunyak and Caines, 2021)

S. Gao, "Centrality-weighted opinion dynamics: Disagreement and social network partition," Proceedings of the 60th IEEE Conference on Decision and Control (CDC), Austin, Texas, USA, December 2021, pp. 5496–5501. (Gao, 2021)

R. Foguen Tchuendom, P. E. Caines, and M. Huang. "On the master equation for linear quadratic graphon mean field games", Proceedings of the 59th

IEEE Conference on Decision and Control (CDC), pages 1026–1031, 2020. (Foguen Tchuendom et al., 2020)

S. Gao and P. E. Caines, Optimal and approximate solutions to linear quadratic regulation of a class of graphon dynamical systems, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 8359–8365, Nice, France, December 2019b. (Gao and Caines, 2019b)

S. Gao and P. E. Caines, Spectral representations of graphons in very large network systems control, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 5068–5075, Nice, France, December 2019a.(Gao and Caines, 2019a)

P. E. Caines and M. Huang. Graphon mean field games and the GMFG equations: ϵ-Nash equilibria, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 286–292, December 2019. (Caines and Huang, 2019)

3.3 Talks: April 1st, 2019 to December 31st, 2022

1. P. E. Caines,"Vertexons and Embedded Graphon Mean Field Games", the Simons Institute for the Theory of Computing, UC Berkeley, CA, Sept.29, 2022 (Invited talk) in the "Graph Limits, Nonparametric Models, and Estimation Workshop" 26 - 30 September, 2022.

2. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", (Invited talk) 29th August, 2022, in the Conference on Many Player Games and Applications, Humbolt University Berlin, 29 - 31 August 2022.

3. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Large Populations on Complex Networks", Canadian Applied and Industrial Mathematics Society (CAIMS), June 23rd, 2021 (Plenary talk).

4. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", the Financial Mathematics/Engineering Seminar Series, Hong Kong Polytechnic University, June 16th 2021 (Invited talk).

5. P. E. Caines, "Static and Time Varying Graphons in Economic and Pandemic GMFG Models" SIAM Financial Mathematics and Engineering (FM21), June 1, 2021 (Invited talk).

6. S. Gao, "Subspace Decompositions in Graphon Control and Graphon Mean Field Games", Informal Systems Seminar, Centre for Intelligent Machines,

McGill University, Groupe d'Etudes et de Recherche en Analyse des Decisions, Montreal, CA, Dec. 11, 2020 (Invited talk).

7. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Networked Systems", Invited Speaker at the "Conference on Mean Field Games and Related Topics - 5". 9th - 13th September, 2019

8. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for a Networked World", University of Saint Petersburg, Russia, 6th September, 2019 (Distinguished Speaker).

9. P. E. Caines, "Graphon Control and Graphon Mean Field Games", Department of Applied Mathemaics, Hong Kong Polythenic Universty, Hong Kong), 28th June, 2019 (Distinguished Lecturer).

3.4 Workshop Organization: April 1st, 2019 to December 31st, 2022

Name: **Mean Field Games on Networks Workshop**

Location: Vancouver, BC, Canada (Virtual Meeting)

Dates: October 26 - 29, 2021

Organizers: Peter E. Caines, Shuang Gao, Minyi Huang, and Rinel Foguen Tchuendom

This workshop (PIMS et al., 2021) brought together researchers in applied mathematics, mean field games, network science, network games, and systems and control theory to exchange ideas and to work on the extensions of mean field game theory to dynamic game problems on heterogeneous large-scale networks. The workshop offered a platform to current results and stimulate discussions on the open challenges in this emerging field, on mathematical theory, computational algorithms, and the applications methodologies of MFG on general networks.

Sponsors: Pacific Institute for the Mathematical Sciences (PIMS), Groupe d'études et de recherche en analyse des décisions (GERAD), and the Fields Institute for Research in Mathematical Sciences.

Participants: Peter E. Caines (PI) and PI's Postdoctoral Researchers: Rinel Foguen Tchuendom, Shuang Gao, Joao Saude, and Yaroslav Salii, plus 50 other international participants.

4 Impacts

As stated earlier, the research on MFG and related topics by the Principal Investigator of the grant FA9550-19-1-0138 has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, presently 2022 - 2025. Consequently, the impacts of research under the 2019 - 2022 grant, FA9550-19-1-0138, may be viewed as a continuation and development of those of the entire program.

Indeed, as explained in Section 1, the MFG field has expanded enormously since its inception. Google links now run to 1.10 billion, citations to papers are in the thousands and conferences and workshops on MFG have proliferated. A sample of the most recent workshops and conference sessions is the following, where we note that the PI and group members have been invited to participate in all of them: Workshop "Mean Field Games on Networks", Pacific Institute for the Mathematical Sciences, Vancouver, BC, Canada, 26 - 29, October, 2021; "Conference on Mathematics and Financial Economics, "Many Player Games and Applications", Humbolt University, Berlin, 29 - 31 August 2022; Workshop on Mean Field Games, Montreal: Centre de Recherches Mathématiques (CRM), 11-15 April, 2022; "Graph Limits, Nonparametric Models, and Estimation Workshop", Simons Institute for the Theory of Computation, UCB, Berkeley, CA, 26-30 September, 2022; IEEE, Control Systems Society Conference on Decision and Control, Cancun, December, 2022, Mean Field Games Session; Mean Field Games Invited Session, IFAC World Congress, Yokahama, Japan, July 2023.

Furthermore, at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively), the last appearing in the period covered by this report. In addition to the very rapid expansion of mathematical and theoretical investigations, applications of MFG theory have spread to a vast range of domains, a sample of the most recent being the optimization of decentralized electric vehicle charging (Foguen Tchuendom et al., 2019), algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Grover et al., 2018), AI reinforcement learning for MFG systems (Agarwal et al., 2022) and polymer science (Welch et al., 2019).

As is shown by this indicative list of publications, meetings and presentations, it is reasonable to assert that the influence of the AFOSR supported research carried out on Mean Field Games and, in particular, on Mean Field Games on networks, has had, and is having, an impact on fields ranging from the core area of systems and control science to economics, finance, energy generation and distribution, operations research, the life sciences and the physical sciences.

5 Changes

There were no changes in the project or its direction.

6 Technical Updates

There are no technical updates to report beyond the continuing research progress and dissemination of results described in report.

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AFOSR Final Report 2023-03-30
Grant FA9550-19-1-0138
New Directions in Mean Field Games: MFG Subpopulation Behaviours and Graphon-MFG Systems

Peter E. Caines

Reporting period April 1st, 2019 to December 31st, 2022

March 24, 2023

1 Overview of Research Objectives

1.1 Background and Overview

Very large networks linking dynamical agents are now ubiquitous and the need to analyse, design and control them is evident. The emergence of the graphon theory of large networks and their infinite limits has enabled the formulation of a theory of the centralized control of dynamical systems distributed on asymptotically infinite networks.

Large-scale systems of competing agents are frequently formulated as dynamic games, this however confronts the problem that for even small populations non-cooperative game theory is intractable. The 2006 breakthrough of Minyi Huang, Roland Malhamé and the PI of FA9550-19-1-0138, Peter Caines, was the introduction of Mean Field Game (MFG) theory (originally termed Nash Certainty Equivalence theory) which was also simultaneously discovered by J.M. Lasry and P.L. Lions.

Mean Field Game (MFG) theory analyzes the existence of Nash equilibria for dynamical games in terms of infinite populations of negligible agents forming the limits of the original finite population systems (Caines, 2020). The core of MFG theory is a pair of partial differential equations consisting of (i) the Hamilton-Jacobi-Bellman equation of optimal control for a generic agent, and (ii) the Fokker-Planck-Kolmogorov equation describing the controlled evolution of the state of a generic agent. These are linked by the generic agent's state

distribution, known as the system's mean field.

A fundamental aspect of MFG theory justifying its application in a vast range of cases is the ϵ-Nash property by which controls derived from the MFG equations yield near-equilibrium performance for finite systems for sufficiently large populations.

The MFG field has expanded tremendously since its inception. Google search results now run to 1.10 billion, citations to papers are in the thousands, conferences and workshops on MFG have proliferated and at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively). In addition to the explosion of mathematical and theoretical investigations over last 16 years, applications of MFG theory have spread to a vast range of domains including: optimization of decentralized EV charging (Ma et al., 2011; Zhu et al., 2016; Foguen Tchuendom et al., 2019); nonlinear system state estimation (Yang et al., 2013); systemic risk in banking (Carmona et al., 2013); cell phone energy management (Aziz and Caines, 2016); economics of energy production (Ludkovski and Sircar, 2015); macro-economics of growth (Gomes et al., 2016); cloud-based networks in information technology (Hanif et al., 2015); algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Nourian et al., 2011; Grover et al., 2018), AI reinforcement learning for MFG systems (Subramanian and Mahajan, 2019; Agarwal et al., 2022), deep learning applications to the solution of MFGs (Ruthotto et al., 2020) and polymer science (Welch et al., 2019).

In this context it is important to note that the research on MFG by the Principal Investigator has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, most recently, for the period 2022-2025.

1.2 Major Research Goals under Grant FA9550-19-1-0138

The program of research under FA9550-19-1-0138 is focused on the development of a new form of Mean Field Games called Graphon Mean Field Games (GMFGs); these are large population games defined on large, heterogeneous, networks and their asymptotic (graphon) limits.

In this setting, game theoretic equilibria are expressed in terms of the newly defined GMFG PDE equations, which are a sweeping generalization of the classical MFG PDEs.

In a closely related development, the theory of graphons is used as a basis for Graphon Linear Control (GLC) theory in which large populations of centrally controlled dynamical systems interacting over large, heterogeneous networks

may be analyzed and optimized in a tractable way.

Canonical Design Procedure for GMFG and GLC Systems

A major goal has been to generate large scale network game strategies and, respectively, control design procedures with the following standard pattern:

The first step is to consider the graphon limit of the network over which the individual systems are distributed and which constitutes the global system.

In the second step, the resulting graphon based GMFG equations, or, respectively, the graphon based infinite dimensional linear system optimization problems are specified. The GMFG equations are then solved to generate the Nash best response strategies or, respectively, the linear feedback controls are synthesized using the well established infinite dimensional linear systems control theory.

Finally, the best response strategies, or, respectively, the infinite dimensional linear feedback controls are approximated in such a way that they may be applied to the original complex finite systems.

The key feature of this strategy is the following: In both the network mean field game and the network control settings it is demonstrated that as the size of the network tends to infinity, the performance of the finite systems when they are subject to the infinite limit system best response game strategies, or, respectively, centralized control laws, yield pay-offs which converge to their Nash equilibrium, or, respectively, optimal, infinite system limit values.

1.3 Organization of the Sequence of Research Objectives

The work over the full reporting period has been organized with the following sequence of objectives:

(i) The establishment of a rigorous theory of the central results of Graphon Mean Field Game (GMFG) theory,

(ii) The generation of a specific Graphon Linear Control (GLC) theory for the control of large populations of dynamical linear system agents with quadratic performance functions on large, complex networks.

(iii) The development of an approximation theory of associated numerical methods, especially those based upon finite dimensional invariant subspaces of the graphon operator in the GLC case.

(iv) The pioneering development of the theory of GMFG systems embedded in Euclidean space; this underlies our currently developing variational analysis of agent performance in networks as a function of position, where this theory also includes sparse as well as dense networks.

(v) The development of an SIR epidemic model software which permits the simulation of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity, these models are constructed on a data base of real world US surface and air-traffic travel data.

2 Details of Accomplishments

2.1 Graphon Mean Field Game Theory: Foundations

The work in (Caines and Huang, 2021) (Caines and Huang, 2018), (Caines and Huang, 2019), established the foundations and central results for graphon mean field game theory for nonlinear systems, that is to say games of infinite populations on infinite heterogeneous networks as limits of the corresponding finite systems.

Specifically, in the conference papers (Caines and Huang, 2018), (Caines and Huang, 2019) Graphon Mean Field Games (GMFG) and the GMFG equations were introduced for the analysis of non-cooperative dynamical games on unbounded networks. In this formulation, asymptotically infinite sub-populations are located at the nodes of large heterogeneous networks, the asymptotic network limits then constitute a part of the analysis. In (Caines and Huang, 2021), the basic existence and uniqueness results were established for the GMFG equations, and a full derivation of the fundamental results was provided, including the key epsilon-Nash theory for GMFG systems which relates the infinite population equilibria on infinite networks to finite population equilibria on finite networks.

2.2 LQG GMFG Decomposition Methods

The theory of invariant spaces of linear operators developed in the papers (Gao and Caines, 2021) is applied in (Gao et al., 2021b), (Gao et al., 2021a) and (Gao et al., 2022) to yield a tractable methodology for generating decentralized game strategies for systems distributed over complex networks. The fundamental properties of the existence and uniqueness of solutions are established in (Gao et al., 2021b,a, 2022) in two principal ways: (i) Riccati equation decomposition and (ii) fixed point contraction theorem methods.

2.3 LQG GMFG Master Equation Analysis

First, using a probabilistic approach, solutions to the Linear Quadratic Graphon Mean Field Games are expressed as solutions to coupled Forward Backward Stochastic Differential Equations (FBSDEs) of McKean-Vlasov type. Next, the

existence of the so-called Master field, which allows for the decoupling of these FBSDEs is established. Finally, the infinite dimensional Partial Differential Equation (PDE), so-called Master Equation, is established for which the Master Field is a solution (see (Foguen Tchuendom et al., 2020)).

2.4 GMFG with Affine Dynamics

GMFG theory for nonlinear problems with individual affine dynamics has been established in Caines et al. (2022) and the Nash value properties for control affine systems has been analyzed in Caines et al. (2023).

2.5 Graphon Linear Systems Control with Approximation Theory

As explained above, the theory of graphons is further used as a basis upon which centrally controlled linear dynamical systems interacting over large, heterogeneous networks are analyzed in a tractable way.

The graphon control methodology is developed in (Gao and Caines, 2020, 2019a,b) to study control problems for arbitrary-size networks of linear dynamical systems. The complete treatment of problems involving well-posedness, controllability, approximation errors for state-to-state control and regulations is established in (Gao and Caines, 2020).

Scalable and tractable solution methods are established for a class of graphon control problem in (Gao and Caines, 2019a,b). Next, the graphon linear system control framework is extended to more general cases where the state of each individual agent on the network is multi-dimensional (Gao and Caines, 2021); moreover, this work provides a generalized solution method with an application to large-scale networks of coupled harmonic oscillators.

2.6 LQ Graphon Dynamical Systems and Control with Q-space Noise

The work in (Dunyak and Caines, 2022) extends the formulation and analysis of (Gao and Caines, 2021) to include stochastic processes in infinite dimensions as a random process source driving the network's controlled dynamical systems; this generalization involves the formulation of a form of graphon Q-space noise.

2.7 Critical Nodes for GMFG and Embedded GMFG

The works (Caines, 2022),(Foguen Tchuendom et al., 2021), (Foguen Tchuendom et al., 2022a), (Foguen Tchuendom et al., 2022b) and (Caines et al., 2023)

pioneer the theory of GMFG systems embedded in Euclidean space by constructing limit graphs and limit graphons for sequences of graphs sitting in Euclidean space. The crucial notion here is that for embedded graphs and graphons we may define the differentiation of functions on the embedded graph limits. In particular we formulate an extended version of mean field game systems, termed Embedded Graphon Mean Field Games (EGMFG), or Graphexon Mean Field Games (GXMFG), which are amenable to the application of the calculus to the optimization of node location in such systems, this being one of the main objectives of the new development. Moreover this formulation includes the case of sparse graph limits (see (Caines, 2022)) which is not covered by the standard graphon formulation.

2.8 MFG Theory and Related Stochastic Adaptive and Hybrid Control Theory

The paper (Pakniyat and Caines, 2021) constitutes the climax of several decades of work by my own group and many other researchers aimed at producing a rigorous, general, hybrid optimal systems control theory (HOCT). These and related results have great applicability in many domains of engineering and scientific modelling, and furthermore in economics and finance. In fact the paper (Firoozi et al., 2022) presents an extension of HOCT to stochastic MFG systems motivated by trading dynamics in finance.

In (Saude and Caines, 2022), we study the Markowitz dynamic portfolio problem, where an agent seeks to maximize its expected return while minimizing the variance of the return (risk). For the first time in this topic, the impact of the aggregate of the agents' trades is incorporated into the dynamics and costs using an extended mean-field games framework.

2.9 Data Based Protocols for Modelling Large Populations on Large Networks with Applications to SIR Problems

The initial investigation of applying graphon theory to the control of epidemic spread is proposed in (Gao and Caines, 2019a) where scalable solutions based on spectral approximation and linearization are established for controlling virus spreads on networks following a type of controlled network SIS model.
The work (Dunyak and Caines, 2021; Salii, 2021) includes the development of SIR epidemic models which permits the study of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity; the latter models were constructed on a data base of real world US surface and air-traffic travel data.

2.10 Centrality Analysis for Finite and Infinite Networks

This work advances the theory of centralities for dynamical systems on networks and its associated applications. In (Gao, 2022) a family of centralities characterized by fixed-points of networks is identified which includes the standard centralities (e.g. PageRank, Katz-Bonacich, eigen centralities) and equilbrirum cost for games on networks (e.g. Nash values of LQG GMFG). Variation bounds of centralities with respect to the network variations are established. In (Gao, 2021) network centralities have been utilized in the modelling of opinion dynamics on social networks and in identifying significant network partitions based upon opinion disagreement states.

3 Dissemination to Communities of Interest

Research findings and results have been published in the leading systems and control and networks journals including the SIAM Journal on Control and Optimization, IEEE Transactions on Automatic Control, Automatica, IEEE Transactions on Control of Network Systems, and in conferences proceedings including the IEEE Conference on Decision and Control.

3.1 Publications: Journals

S. Gao, P. E. Caines, and M. Huang, "LQG graphon mean field games: analysis via graphon invariant subspaces," Conditionally accepted by IEEE Transactions on Automatic Control, 2021. (Gao et al., 2022)

D. Firoozi, A. Pakniyat, and P. E. Caines, "A class of hybrid LQG mean field games with state-invariant switching and stopping strategies," Automatica, vol. 141, p. 110244, 2022. (Firoozi et al., 2022)

P. E. Caines, D. Ho, M. Huang, J. Jian, and Q. Song. On the graphon mean field game equations: Individual agent affine dynamics and mean field dependent performance functions. ESAIM: Control, Optimisation and Calculus of Variations, 28:24, 2022 (Caines et al., 2022)

P. E. Caines and M. Huang, "Graphon mean field games and their equations," SIAM Journal on Control and Optimization, vol. 59, no. 6, pp. 4373–4399, 2021. (Caines and Huang, 2021)

A. Pakniyat and P. E. Caines, "On the hybrid minimum principle: The Hamiltonian and adjoint boundary conditions," IEEE Transactions on Automatic Control, vol. 66, no. 3, pp. 1246–1253, 2021. (Pakniyat and Caines, 2021)

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S. Gao and P. E. Caines. Graphon control of large-scale networks of linear systems. IEEE Transactions on Automatic Control, 65(10):4090–4105, 2020 (Gao and Caines, 2020)

N. Sen and P. E. Caines. Mean field games with partial observation. SIAM Journal on Control and Optimization, 57(3):2064–2091, 2019 (Sen and Caines, 2019)

P. E. Caines, D. Ho, and Q. Song. The density evolution of the killed mckean–vlasov process. Stochastics, 92(4):642–657, 2020. (Caines et al., 2020)

3.2 Publications: Conference Proceedings

A. Dunyak and P. E. Caines, "Linear stochastic graphon systems with q- space noise," in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 3926–3932. (Dunyak and Caines, 2022)

P. E. Caines, "Embedded vertexon-graphons and embedded GMFG systems," Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022,

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3.3 Talks: April 1st, 2019 to December 31st, 2022

1. P. E. Caines,"Vertexons and Embedded Graphon Mean Field Games", the Simons Institute for the Theory of Computing, UC Berkeley, CA, Sept.29, 2022 (Invited talk) in the "Graph Limits, Nonparametric Models, and Estimation Workshop" 26 - 30 September, 2022.

2. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", (Invited talk) 29th August, 2022, in the Conference on Many Player Games and Applications, Humbolt University Berlin, 29 - 31 August 2022.

3. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Large Populations on Complex Networks", Canadian Applied and Industrial Mathematics Society (CAIMS), June 23rd, 2021 (Plenary talk).

4. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", the Financial Mathematics/Engineering Seminar Series, Hong Kong Polytechnic University, June 16th 2021 (Invited talk).

5. P. E. Caines, "Static and Time Varying Graphons in Economic and Pandemic GMFG Models" SIAM Financial Mathematics and Engineering (FM21), June 1, 2021 (Invited talk).

6. S. Gao, "Subspace Decompositions in Graphon Control and Graphon Mean Field Games", Informal Systems Seminar, Centre for Intelligent Machines,

McGill University, Groupe d'Etudes et de Recherche en Analyse des Decisions, Montreal, CA, Dec. 11, 2020 (Invited talk).

7. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Networked Systems", Invited Speaker at the "Conference on Mean Field Games and Related Topics - 5". 9th - 13th September, 2019

8. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for a Networked World", University of Saint Petersburg, Russia, 6th September, 2019 (Distinguished Speaker).

9. P. E. Caines, "Graphon Control and Graphon Mean Field Games", Department of Applied Mathemaics, Hong Kong Polythenic Universty, Hong Kong), 28th June, 2019 (Distinguished Lecturer).

3.4 Workshop Organization: April 1st, 2019 to December 31st, 2022

Name: **Mean Field Games on Networks Workshop**

Location: Vancouver, BC, Canada (Virtual Meeting)

Dates: October 26 - 29, 2021

Organizers: Peter E. Caines, Shuang Gao, Minyi Huang, and Rinel Foguen Tchuendom

This workshop (PIMS et al., 2021) brought together researchers in applied mathematics, mean field games, network science, network games, and systems and control theory to exchange ideas and to work on the extensions of mean field game theory to dynamic game problems on heterogeneous large-scale networks. The workshop offered a platform to current results and stimulate discussions on the open challenges in this emerging field, on mathematical theory, computational algorithms, and the applications methodologies of MFG on general networks.

Sponsors: Pacific Institute for the Mathematical Sciences (PIMS), Groupe d'études et de recherche en analyse des décisions (GERAD), and the Fields Institute for Research in Mathematical Sciences.

Participants: Peter E. Caines (PI) and PI's Postdoctoral Researchers: Rinel Foguen Tchuendom, Shuang Gao, Joao Saude, and Yaroslav Salii, plus 50 other international participants.

4 Impacts

As stated earlier, the research on MFG and related topics by the Principal Investigator of the grant FA9550-19-1-0138 has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, presently 2022 - 2025. Consequently, the impacts of research under the 2019 - 2022 grant, FA9550-19-1-0138, may be viewed as a continuation and development of those of the entire program.

Indeed, as explained in Section 1, the MFG field has expanded enormously since its inception. Google links now run to 1.10 billion, citations to papers are in the thousands and conferences and workshops on MFG have proliferated. A sample of the most recent workshops and conference sessions is the following, where we note that the PI and group members have been invited to participate in all of them: Workshop "Mean Field Games on Networks", Pacific Institute for the Mathematical Sciences, Vancouver, BC, Canada, 26 - 29, October, 2021; "Conference on Mathematics and Financial Economics, "Many Player Games and Applications", Humbolt University, Berlin, 29 - 31 August 2022; Workshop on Mean Field Games, Montreal: Centre de Recherches Mathématiques (CRM), 11-15 April, 2022; "Graph Limits, Nonparametric Models, and Estimation Workshop", Simons Institute for the Theory of Computation, UCB, Berkeley, CA, 26-30 September, 2022; IEEE, Control Systems Society Conference on Decision and Control, Cancun, December, 2022, Mean Field Games Session; Mean Field Games Invited Session, IFAC World Congress, Yokahama, Japan, July 2023.

Furthermore, at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively), the last appearing in the period covered by this report. In addition to the very rapid expansion of mathematical and theoretical investigations, applications of MFG theory have spread to a vast range of domains, a sample of the most recent being the optimization of decentralized electric vehicle charging (Foguen Tchuendom et al., 2019), algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Grover et al., 2018), AI reinforcement learning for MFG systems (Agarwal et al., 2022) and polymer science (Welch et al., 2019).

As is shown by this indicative list of publications, meetings and presentations, it is reasonable to assert that the influence of the AFOSR supported research carried out on Mean Field Games and, in particular, on Mean Field Games on networks, has had, and is having, an impact on fields ranging from the core area of systems and control science to economics, finance, energy generation and distribution, operations research, the life sciences and the physical sciences.

5 Changes

There were no changes in the project or its direction.

6 Technical Updates

There are no technical updates to report beyond the continuing research progress and dissemination of results described in report.

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AFOSR Final Report 2023-03-30
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New Directions in Mean Field Games: MFG Subpopulation Behaviours and Graphon-MFG Systems

Peter E. Caines

Reporting period April 1st, 2019 to December 31st, 2022

March 24, 2023

1 Overview of Research Objectives

1.1 Background and Overview

Very large networks linking dynamical agents are now ubiquitous and the need to analyse, design and control them is evident. The emergence of the graphon theory of large networks and their infinite limits has enabled the formulation of a theory of the centralized control of dynamical systems distributed on asymptotically infinite networks.

Large-scale systems of competing agents are frequently formulated as dynamic games, this however confronts the problem that for even small populations non-cooperative game theory is intractable. The 2006 breakthrough of Minyi Huang, Roland Malhamé and the PI of FA9550-19-1-0138, Peter Caines, was the introduction of Mean Field Game (MFG) theory (originally termed Nash Certainty Equivalence theory) which was also simultaneously discovered by J.M. Lasry and P.L. Lions.

Mean Field Game (MFG) theory analyzes the existence of Nash equilibria for dynamical games in terms of infinite populations of negligible agents forming the limits of the original finite population systems (Caines, 2020). The core of MFG theory is a pair of partial differential equations consisting of (i) the Hamilton-Jacobi-Bellman equation of optimal control for a generic agent, and (ii) the Fokker-Planck-Kolmogorov equation describing the controlled evolution of the state of a generic agent. These are linked by the generic agent's state

distribution, known as the system's mean field.

A fundamental aspect of MFG theory justifying its application in a vast range of cases is the ϵ-Nash property by which controls derived from the MFG equations yield near-equilibrium performance for finite systems for sufficiently large populations.

The MFG field has expanded tremendously since its inception. Google search results now run to 1.10 billion, citations to papers are in the thousands, conferences and workshops on MFG have proliferated and at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively). In addition to the explosion of mathematical and theoretical investigations over last 16 years, applications of MFG theory have spread to a vast range of domains including: optimization of decentralized EV charging (Ma et al., 2011; Zhu et al., 2016; Foguen Tchuendom et al., 2019); nonlinear system state estimation (Yang et al., 2013); systemic risk in banking (Carmona et al., 2013); cell phone energy management (Aziz and Caines, 2016); economics of energy production (Ludkovski and Sircar, 2015); macro-economics of growth (Gomes et al., 2016); cloud-based networks in information technology (Hanif et al., 2015); algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Nourian et al., 2011; Grover et al., 2018), AI reinforcement learning for MFG systems (Subramanian and Mahajan, 2019; Agarwal et al., 2022), deep learning applications to the solution of MFGs (Ruthotto et al., 2020) and polymer science (Welch et al., 2019).

In this context it is important to note that the research on MFG by the Principal Investigator has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, most recently, for the period 2022-2025.

1.2 Major Research Goals under Grant FA9550-19-1-0138

The program of research under FA9550-19-1-0138 is focused on the development of a new form of Mean Field Games called Graphon Mean Field Games (GMFGs); these are large population games defined on large, heterogeneous, networks and their asymptotic (graphon) limits.

In this setting, game theoretic equilibria are expressed in terms of the newly defined GMFG PDE equations, which are a sweeping generalization of the classical MFG PDEs.

In a closely related development, the theory of graphons is used as a basis for Graphon Linear Control (GLC) theory in which large populations of centrally controlled dynamical systems interacting over large, heterogeneous networks

may be analyzed and optimized in a tractable way.

Canonical Design Procedure for GMFG and GLC Systems

A major goal has been to generate large scale network game strategies and, respectively, control design procedures with the following standard pattern:

The first step is to consider the graphon limit of the network over which the individual systems are distributed and which constitutes the global system.

In the second step, the resulting graphon based GMFG equations, or, respectively, the graphon based infinite dimensional linear system optimization problems are specified. The GMFG equations are then solved to generate the Nash best response strategies or, respectively, the linear feedback controls are synthesized using the well established infinite dimensional linear systems control theory.

Finally, the best response strategies, or, respectively, the infinite dimensional linear feedback controls are approximated in such a way that they may be applied to the original complex finite systems.

The key feature of this strategy is the following: In both the network mean field game and the network control settings it is demonstrated that as the size of the network tends to infinity, the performance of the finite systems when they are subject to the infinite limit system best response game strategies, or, respectively, centralized control laws, yield pay-offs which converge to their Nash equilibrium, or, respectively, optimal, infinite system limit values.

1.3 Organization of the Sequence of Research Objectives

The work over the full reporting period has been organized with the following sequence of objectives:

(i) The establishment of a rigorous theory of the central results of Graphon Mean Field Game (GMFG) theory,

(ii) The generation of a specific Graphon Linear Control (GLC) theory for the control of large populations of dynamical linear system agents with quadratic performance functions on large, complex networks.

(iii) The development of an approximation theory of associated numerical methods, especially those based upon finite dimensional invariant subspaces of the graphon operator in the GLC case.

(iv) The pioneering development of the theory of GMFG systems embedded in Euclidean space; this underlies our currently developing variational analysis of agent performance in networks as a function of position, where this theory also includes sparse as well as dense networks.

(v) The development of an SIR epidemic model software which permits the simulation of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity, these models are constructed on a data base of real world US surface and air-traffic travel data.

2 Details of Accomplishments

2.1 Graphon Mean Field Game Theory: Foundations

The work in (Caines and Huang, 2021) (Caines and Huang, 2018), (Caines and Huang, 2019), established the foundations and central results for graphon mean field game theory for nonlinear systems, that is to say games of infinite populations on infinite heterogeneous networks as limits of the corresponding finite systems.

Specifically, in the conference papers (Caines and Huang, 2018), (Caines and Huang, 2019) Graphon Mean Field Games (GMFG) and the GMFG equations were introduced for the analysis of non-cooperative dynamical games on unbounded networks. In this formulation, asymptotically infinite sub-populations are located at the nodes of large heterogeneous networks, the asymptotic network limits then constitute a part of the analysis. In (Caines and Huang, 2021), the basic existence and uniqueness results were established for the GMFG equations, and a full derivation of the fundamental results was provided, including the key epsilon-Nash theory for GMFG systems which relates the infinite population equilibria on infinite networks to finite population equilibria on finite networks.

2.2 LQG GMFG Decomposition Methods

The theory of invariant spaces of linear operators developed in the papers (Gao and Caines, 2021) is applied in (Gao et al., 2021b), (Gao et al., 2021a) and (Gao et al., 2022) to yield a tractable methodology for generating decentralized game strategies for systems distributed over complex networks. The fundamental properties of the existence and uniqueness of solutions are established in (Gao et al., 2021b,a, 2022) in two principal ways: (i) Riccati equation decomposition and (ii) fixed point contraction theorem methods.

2.3 LQG GMFG Master Equation Analysis

First, using a probabilistic approach, solutions to the Linear Quadratic Graphon Mean Field Games are expressed as solutions to coupled Forward Backward Stochastic Differential Equations (FBSDEs) of McKean-Vlasov type. Next, the

existence of the so-called Master field, which allows for the decoupling of these FBSDEs is established. Finally, the infinite dimensional Partial Differential Equation (PDE), so-called Master Equation, is established for which the Master Field is a solution (see (Foguen Tchuendom et al., 2020)).

2.4 GMFG with Affine Dynamics

GMFG theory for nonlinear problems with individual affine dynamics has been established in Caines et al. (2022) and the Nash value properties for control affine systems has been analyzed in Caines et al. (2023).

2.5 Graphon Linear Systems Control with Approximation Theory

As explained above, the theory of graphons is further used as a basis upon which centrally controlled linear dynamical systems interacting over large, heterogeneous networks are analyzed in a tractable way.

The graphon control methodology is developed in (Gao and Caines, 2020, 2019a,b) to study control problems for arbitrary-size networks of linear dynamical systems. The complete treatment of problems involving well-posedness, controllability, approximation errors for state-to-state control and regulations is established in (Gao and Caines, 2020).

Scalable and tractable solution methods are established for a class of graphon control problem in (Gao and Caines, 2019a,b). Next, the graphon linear system control framework is extended to more general cases where the state of each individual agent on the network is multi-dimensional (Gao and Caines, 2021); moreover, this work provides a generalized solution method with an application to large-scale networks of coupled harmonic oscillators.

2.6 LQ Graphon Dynamical Systems and Control with Q-space Noise

The work in (Dunyak and Caines, 2022) extends the formulation and analysis of (Gao and Caines, 2021) to include stochastic processes in infinite dimensions as a random process source driving the network's controlled dynamical systems; this generalization involves the formulation of a form of graphon Q-space noise.

2.7 Critical Nodes for GMFG and Embedded GMFG

The works (Caines, 2022),(Foguen Tchuendom et al., 2021), (Foguen Tchuendom et al., 2022a), (Foguen Tchuendom et al., 2022b) and (Caines et al., 2023)

pioneer the theory of GMFG systems embedded in Euclidean space by constructing limit graphs and limit graphons for sequences of graphs sitting in Euclidean space. The crucial notion here is that for embedded graphs and graphons we may define the differentiation of functions on the embedded graph limits. In particular we formulate an extended version of mean field game systems, termed Embedded Graphon Mean Field Games (EGMFG), or Graphexon Mean Field Games (GXMFG), which are amenable to the application of the calculus to the optimization of node location in such systems, this being one of the main objectives of the new development. Moreover this formulation includes the case of sparse graph limits (see (Caines, 2022)) which is not covered by the standard graphon formulation.

2.8 MFG Theory and Related Stochastic Adaptive and Hybrid Control Theory

The paper (Pakniyat and Caines, 2021) constitutes the climax of several decades of work by my own group and many other researchers aimed at producing a rigorous, general, hybrid optimal systems control theory (HOCT). These and related results have great applicability in many domains of engineering and scientific modelling, and furthermore in economics and finance. In fact the paper (Firoozi et al., 2022) presents an extension of HOCT to stochastic MFG systems motivated by trading dynamics in finance.

In (Saude and Caines, 2022), we study the Markowitz dynamic portfolio problem, where an agent seeks to maximize its expected return while minimizing the variance of the return (risk). For the first time in this topic, the impact of the aggregate of the agents' trades is incorporated into the dynamics and costs using an extended mean-field games framework.

2.9 Data Based Protocols for Modelling Large Populations on Large Networks with Applications to SIR Problems

The initial investigation of applying graphon theory to the control of epidemic spread is proposed in (Gao and Caines, 2019a) where scalable solutions based on spectral approximation and linearization are established for controlling virus spreads on networks following a type of controlled network SIS model.
The work (Dunyak and Caines, 2021; Salii, 2021) includes the development of SIR epidemic models which permits the study of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity; the latter models were constructed on a data base of real world US surface and air-traffic travel data.

2.10 Centrality Analysis for Finite and Infinite Networks

This work advances the theory of centralities for dynamical systems on networks and its associated applications. In (Gao, 2022) a family of centralities characterized by fixed-points of networks is identified which includes the standard centralities (e.g. PageRank, Katz-Bonacich, eigen centralities) and equilbrirum cost for games on networks (e.g. Nash values of LQG GMFG). Variation bounds of centralities with respect to the network variations are established. In (Gao, 2021) network centralities have been utilized in the modelling of opinion dynamics on social networks and in identifying significant network partitions based upon opinion disagreement states.

3 Dissemination to Communities of Interest

Research findings and results have been published in the leading systems and control and networks journals including the SIAM Journal on Control and Optimization, IEEE Transactions on Automatic Control, Automatica, IEEE Transactions on Control of Network Systems, and in conferences proceedings including the IEEE Conference on Decision and Control.

3.1 Publications: Journals

S. Gao, P. E. Caines, and M. Huang, "LQG graphon mean field games: analysis via graphon invariant subspaces," Conditionally accepted by IEEE Transactions on Automatic Control, 2021. (Gao et al., 2022)

D. Firoozi, A. Pakniyat, and P. E. Caines, "A class of hybrid LQG mean field games with state-invariant switching and stopping strategies," Automatica, vol. 141, p. 110244, 2022. (Firoozi et al., 2022)

P. E. Caines, D. Ho, M. Huang, J. Jian, and Q. Song. On the graphon mean field game equations: Individual agent affine dynamics and mean field dependent performance functions. ESAIM: Control, Optimisation and Calculus of Variations, 28:24, 2022 (Caines et al., 2022)

P. E. Caines and M. Huang, "Graphon mean field games and their equations," SIAM Journal on Control and Optimization, vol. 59, no. 6, pp. 4373–4399, 2021. (Caines and Huang, 2021)

A. Pakniyat and P. E. Caines, "On the hybrid minimum principle: The Hamiltonian and adjoint boundary conditions," IEEE Transactions on Automatic Control, vol. 66, no. 3, pp. 1246–1253, 2021. (Pakniyat and Caines, 2021)

S. Gao and P. E. Caines, "Subspace decomposition for graphon LQR: Applications to VLSNs of harmonic oscillators," IEEE Transactions on Control of Network Systems, vol. 8, no. 2, pp. 576–586, 2021. (Gao and Caines, 2021)

S. Gao, R. Foguen Tchuendom, and P. E. Caines, "Linear quadratic graphon field games," Communications in Information and Systems, vol. 21, no. 3, pp. 341–369, 2021. (Gao et al., 2021b)

D. Firoozi, and P. E. Caines, "ϵ-Nash Equilibria for Major-Minor LQG Mean Field Games with Partial Observations of All Agents", *IEEE Trans.on Automatic Control*, July, 2020, Vol 66, No. 6, pp 2778 - 2786. On-line: DOI: 10.1109/TAC.2020.3010129 (Firoozi and Caines, 2020)

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P. E. Caines. Mean field game theory: A tractable methodology for large population problems. SIAM News, 53(3):5–6, 2020. (Caines, 2020)

D. Firoozi, S. Jaimungal, and P. E. Caines. Convex analysis for lqg systems with applications to major–minor LQG mean–field game systems. Systems & Control Letters, 142:104734, 2020. (Firoozi et al., 2020)

S. Gao and P. E. Caines. Graphon control of large-scale networks of linear systems. IEEE Transactions on Automatic Control, 65(10):4090–4105, 2020 (Gao and Caines, 2020)

N. Sen and P. E. Caines. Mean field games with partial observation. SIAM Journal on Control and Optimization, 57(3):2064–2091, 2019 (Sen and Caines, 2019)

P. E. Caines, D. Ho, and Q. Song. The density evolution of the killed mckean–vlasov process. Stochastics, 92(4):642–657, 2020. (Caines et al., 2020)

3.2 Publications: Conference Proceedings

A. Dunyak and P. E. Caines, "Linear stochastic graphon systems with q- space noise," in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 3926–3932. (Dunyak and Caines, 2022)

P. E. Caines, "Embedded vertexon-graphons and embedded GMFG systems," Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022,

pp. 5550–5557. (Caines, 2022)

S. Gao, "Fixed-point centrality for networks," in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 1628–1635. (Gao, 2022).

R. Foguen Tchuendom, S. Gao, P. E. Caines, and M. Huang, "Optimal network location in infinite horizon LQG graphon mean field games," in Proceedings of the 61th IEEE Conference on Decision and Control, Cancun, Mexico, December 2022, pp. 5558–5565. (Foguen Tchuendom et al., 2022b)

R. Foguen Tchuendom, S. Gao, and P. E. Caines "Stationary cost nodes in infinite horizon LQG GMFGs," IFAC-PapersOnLine for the 25th International Symposium on Mathematical Theory of Networks and Systems (MTNS), vol. 55, no. 30, pp. 284–289, 2022. (Foguen Tchuendom et al., 2022a)

J. Saude and P. E. Caines, "Markowitz portfolio optimization extended quadratic mean- field games approach," Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 5523–5528. (Saude and Caines, 2022)

Y. V. Salii, "Benchmarking optimal control for network dynamic systems with plausible epidemic models," in International Conference on Complex Networks and Their Applications. Springer, 2021, pp. 194–206. (Salii, 2021)

S. Gao, P. E. Caines, and M. Huang, "LQG graphon mean field games: Graphon invariant subspaces," Proceedings of the 60th IEEE Conference on Decision and Control, Austin, Texas, USA, December 2021, pp. 5253–5260. (Gao et al., 2021a)

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S. Gao, "Centrality-weighted opinion dynamics: Disagreement and social network partition," Proceedings of the 60th IEEE Conference on Decision and Control (CDC), Austin, Texas, USA, December 2021, pp. 5496–5501. (Gao, 2021)

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S. Gao and P. E. Caines, Optimal and approximate solutions to linear quadratic regulation of a class of graphon dynamical systems, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 8359–8365, Nice, France, December 2019b. (Gao and Caines, 2019b)

S. Gao and P. E. Caines, Spectral representations of graphons in very large network systems control, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 5068–5075, Nice, France, December 2019a.(Gao and Caines, 2019a)

P. E. Caines and M. Huang. Graphon mean field games and the GMFG equations: ϵ-Nash equilibria, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 286–292, December 2019. (Caines and Huang, 2019)

3.3 Talks: April 1st, 2019 to December 31st, 2022

1. P. E. Caines,"Vertexons and Embedded Graphon Mean Field Games", the Simons Institute for the Theory of Computing, UC Berkeley, CA, Sept.29, 2022 (Invited talk) in the "Graph Limits, Nonparametric Models, and Estimation Workshop" 26 - 30 September, 2022.

2. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", (Invited talk) 29th August, 2022, in the Conference on Many Player Games and Applications, Humbolt University Berlin, 29 - 31 August 2022.

3. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Large Populations on Complex Networks", Canadian Applied and Industrial Mathematics Society (CAIMS), June 23rd, 2021 (Plenary talk).

4. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", the Financial Mathematics/Engineering Seminar Series, Hong Kong Polytechnic University, June 16th 2021 (Invited talk).

5. P. E. Caines, "Static and Time Varying Graphons in Economic and Pandemic GMFG Models" SIAM Financial Mathematics and Engineering (FM21), June 1, 2021 (Invited talk).

6. S. Gao, "Subspace Decompositions in Graphon Control and Graphon Mean Field Games", Informal Systems Seminar, Centre for Intelligent Machines,

McGill University, Groupe d'Etudes et de Recherche en Analyse des Decisions, Montreal, CA, Dec. 11, 2020 (Invited talk).

7. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Networked Systems", Invited Speaker at the "Conference on Mean Field Games and Related Topics - 5". 9th - 13th September, 2019

8. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for a Networked World", University of Saint Petersburg, Russia, 6th September, 2019 (Distinguished Speaker).

9. P. E. Caines, "Graphon Control and Graphon Mean Field Games", Department of Applied Mathemaics, Hong Kong Polythenic Universty, Hong Kong), 28th June, 2019 (Distinguished Lecturer).

3.4 Workshop Organization: April 1st, 2019 to December 31st, 2022

Name: **Mean Field Games on Networks Workshop**

Location: Vancouver, BC, Canada (Virtual Meeting)

Dates: October 26 - 29, 2021

Organizers: Peter E. Caines, Shuang Gao, Minyi Huang, and Rinel Foguen Tchuendom

This workshop (PIMS et al., 2021) brought together researchers in applied mathematics, mean field games, network science, network games, and systems and control theory to exchange ideas and to work on the extensions of mean field game theory to dynamic game problems on heterogeneous large-scale networks. The workshop offered a platform to current results and stimulate discussions on the open challenges in this emerging field, on mathematical theory, computational algorithms, and the applications methodologies of MFG on general networks.

Sponsors: Pacific Institute for the Mathematical Sciences (PIMS), Groupe d'études et de recherche en analyse des décisions (GERAD), and the Fields Institute for Research in Mathematical Sciences.

Participants: Peter E. Caines (PI) and PI's Postdoctoral Researchers: Rinel Foguen Tchuendom, Shuang Gao, Joao Saude, and Yaroslav Salii, plus 50 other international participants.

4 Impacts

As stated earlier, the research on MFG and related topics by the Principal Investigator of the grant FA9550-19-1-0138 has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, presently 2022 - 2025. Consequently, the impacts of research under the 2019 - 2022 grant, FA9550-19-1-0138, may be viewed as a continuation and development of those of the entire program.

Indeed, as explained in Section 1, the MFG field has expanded enormously since its inception. Google links now run to 1.10 billion, citations to papers are in the thousands and conferences and workshops on MFG have proliferated. A sample of the most recent workshops and conference sessions is the following, where we note that the PI and group members have been invited to participate in all of them: Workshop “Mean Field Games on Networks”, Pacific Institute for the Mathematical Sciences, Vancouver, BC, Canada, 26 - 29, October, 2021; “Conference on Mathematics and Financial Economics, “Many Player Games and Applications”, Humbolt University, Berlin, 29 - 31 August 2022; Workshop on Mean Field Games, Montreal: Centre de Recherches Mathématiques (CRM), 11-15 April, 2022; “Graph Limits, Nonparametric Models, and Estimation Workshop”, Simons Institute for the Theory of Computation, UCB, Berkeley, CA, 26-30 September, 2022; IEEE, Control Systems Society Conference on Decision and Control, Cancun, December, 2022, Mean Field Games Session; Mean Field Games Invited Session, IFAC World Congress, Yokahama, Japan, July 2023.

Furthermore, at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively), the last appearing in the period covered by this report. In addition to the very rapid expansion of mathematical and theoretical investigations, applications of MFG theory have spread to a vast range of domains, a sample of the most recent being the optimization of decentralized electric vehicle charging (Foguen Tchuendom et al., 2019), algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Grover et al., 2018), AI reinforcement learning for MFG systems (Agarwal et al., 2022) and polymer science (Welch et al., 2019).

As is shown by this indicative list of publications, meetings and presentations, it is reasonable to assert that the influence of the AFOSR supported research carried out on Mean Field Games and, in particular, on Mean Field Games on networks, has had, and is having, an impact on fields ranging from the core area of systems and control science to economics, finance, energy generation and distribution, operations research, the life sciences and the physical sciences.

5 Changes

There were no changes in the project or its direction.

6 Technical Updates

There are no technical updates to report beyond the continuing research progress and dissemination of results described in report.

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AFOSR Final Report 2023-03-30
Grant FA9550-19-1-0138
New Directions in Mean Field Games: MFG Subpopulation Behaviours and Graphon-MFG Systems

Peter E. Caines

Reporting period April 1st, 2019 to December 31st, 2022

March 24, 2023

1 Overview of Research Objectives

1.1 Background and Overview

Very large networks linking dynamical agents are now ubiquitous and the need to analyse, design and control them is evident. The emergence of the graphon theory of large networks and their infinite limits has enabled the formulation of a theory of the centralized control of dynamical systems distributed on asymptotically infinite networks.

Large-scale systems of competing agents are frequently formulated as dynamic games, this however confronts the problem that for even small populations non-cooperative game theory is intractable. The 2006 breakthrough of Minyi Huang, Roland Malhamé and the PI of FA9550-19-1-0138, Peter Caines, was the introduction of Mean Field Game (MFG) theory (originally termed Nash Certainty Equivalence theory) which was also simultaneously discovered by J.M. Lasry and P.L. Lions.

Mean Field Game (MFG) theory analyzes the existence of Nash equilibria for dynamical games in terms of infinite populations of negligible agents forming the limits of the original finite population systems (Caines, 2020). The core of MFG theory is a pair of partial differential equations consisting of (i) the Hamilton-Jacobi-Bellman equation of optimal control for a generic agent, and (ii) the Fokker-Planck-Kolmogorov equation describing the controlled evolution of the state of a generic agent. These are linked by the generic agent's state

distribution, known as the system's mean field.

A fundamental aspect of MFG theory justifying its application in a vast range of cases is the ϵ-Nash property by which controls derived from the MFG equations yield near-equilibrium performance for finite systems for sufficiently large populations.

The MFG field has expanded tremendously since its inception. Google search results now run to 1.10 billion, citations to papers are in the thousands, conferences and workshops on MFG have proliferated and at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively). In addition to the explosion of mathematical and theoretical investigations over last 16 years, applications of MFG theory have spread to a vast range of domains including: optimization of decentralized EV charging (Ma et al., 2011; Zhu et al., 2016; Foguen Tchuendom et al., 2019); nonlinear system state estimation (Yang et al., 2013); systemic risk in banking (Carmona et al., 2013); cell phone energy management (Aziz and Caines, 2016); economics of energy production (Ludkovski and Sircar, 2015); macro-economics of growth (Gomes et al., 2016); cloud-based networks in information technology (Hanif et al., 2015); algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Nourian et al., 2011; Grover et al., 2018), AI reinforcement learning for MFG systems (Subramanian and Mahajan, 2019; Agarwal et al., 2022), deep learning applications to the solution of MFGs (Ruthotto et al., 2020) and polymer science (Welch et al., 2019).

In this context it is important to note that the research on MFG by the Principal Investigator has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, most recently, for the period 2022-2025.

1.2 Major Research Goals under Grant FA9550-19-1-0138

The program of research under FA9550-19-1-0138 is focused on the development of a new form of Mean Field Games called Graphon Mean Field Games (GMFGs); these are large population games defined on large, heterogeneous, networks and their asymptotic (graphon) limits.

In this setting, game theoretic equilibria are expressed in terms of the newly defined GMFG PDE equations, which are a sweeping generalization of the classical MFG PDEs.

In a closely related development, the theory of graphons is used as a basis for Graphon Linear Control (GLC) theory in which large populations of centrally controlled dynamical systems interacting over large, heterogeneous networks

may be analyzed and optimized in a tractable way.

Canonical Design Procedure for GMFG and GLC Systems

A major goal has been to generate large scale network game strategies and, respectively, control design procedures with the following standard pattern:

The first step is to consider the graphon limit of the network over which the individual systems are distributed and which constitutes the global system.

In the second step, the resulting graphon based GMFG equations, or, respectively, the graphon based infinite dimensional linear system optimization problems are specified. The GMFG equations are then solved to generate the Nash best response strategies or, respectively, the linear feedback controls are synthesized using the well established infinite dimensional linear systems control theory.

Finally, the best response strategies, or, respectively, the infinite dimensional linear feedback controls are approximated in such a way that they may be applied to the original complex finite systems.

The key feature of this strategy is the following: In both the network mean field game and the network control settings it is demonstrated that as the size of the network tends to infinity, the performance of the finite systems when they are subject to the infinite limit system best response game strategies, or, respectively, centralized control laws, yield pay-offs which converge to their Nash equilibrium, or, respectively, optimal, infinite system limit values.

1.3 Organization of the Sequence of Research Objectives

The work over the full reporting period has been organized with the following sequence of objectives:

(i) The establishment of a rigorous theory of the central results of Graphon Mean Field Game (GMFG) theory,

(ii) The generation of a specific Graphon Linear Control (GLC) theory for the control of large populations of dynamical linear system agents with quadratic performance functions on large, complex networks.

(iii) The development of an approximation theory of associated numerical methods, especially those based upon finite dimensional invariant subspaces of the graphon operator in the GLC case.

(iv) The pioneering development of the theory of GMFG systems embedded in Euclidean space; this underlies our currently developing variational analysis of agent performance in networks as a function of position, where this theory also includes sparse as well as dense networks.

(v) The development of an SIR epidemic model software which permits the simulation of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity, these models are constructed on a data base of real world US surface and air-traffic travel data.

2 Details of Accomplishments

2.1 Graphon Mean Field Game Theory: Foundations

The work in (Caines and Huang, 2021) (Caines and Huang, 2018), (Caines and Huang, 2019), established the foundations and central results for graphon mean field game theory for nonlinear systems, that is to say games of infinite populations on infinite heterogeneous networks as limits of the corresponding finite systems.

Specifically, in the conference papers (Caines and Huang, 2018), (Caines and Huang, 2019) Graphon Mean Field Games (GMFG) and the GMFG equations were introduced for the analysis of non-cooperative dynamical games on unbounded networks. In this formulation, asymptotically infinite sub-populations are located at the nodes of large heterogeneous networks, the asymptotic network limits then constitute a part of the analysis. In (Caines and Huang, 2021), the basic existence and uniqueness results were established for the GMFG equations, and a full derivation of the fundamental results was provided, including the key epsilon-Nash theory for GMFG systems which relates the infinite population equilibria on infinite networks to finite population equilibria on finite networks.

2.2 LQG GMFG Decomposition Methods

The theory of invariant spaces of linear operators developed in the papers (Gao and Caines, 2021) is applied in (Gao et al., 2021b), (Gao et al., 2021a) and (Gao et al., 2022) to yield a tractable methodology for generating decentralized game strategies for systems distributed over complex networks. The fundamental properties of the existence and uniqueness of solutions are established in (Gao et al., 2021b,a, 2022) in two principal ways: (i) Riccati equation decomposition and (ii) fixed point contraction theorem methods.

2.3 LQG GMFG Master Equation Analysis

First, using a probabilistic approach, solutions to the Linear Quadratic Graphon Mean Field Games are expressed as solutions to coupled Forward Backward Stochastic Differential Equations (FBSDEs) of McKean-Vlasov type. Next, the

existence of the so-called Master field, which allows for the decoupling of these FBSDEs is established. Finally, the infinite dimensional Partial Differential Equation (PDE), so-called Master Equation, is established for which the Master Field is a solution (see (Foguen Tchuendom et al., 2020)).

2.4 GMFG with Affine Dynamics

GMFG theory for nonlinear problems with individual affine dynamics has been established in Caines et al. (2022) and the Nash value properties for control affine systems has been analyzed in Caines et al. (2023).

2.5 Graphon Linear Systems Control with Approximation Theory

As explained above, the theory of graphons is further used as a basis upon which centrally controlled linear dynamical systems interacting over large, heterogeneous networks are analyzed in a tractable way.

The graphon control methodology is developed in (Gao and Caines, 2020, 2019a,b) to study control problems for arbitrary-size networks of linear dynamical systems. The complete treatment of problems involving well-posedness, controllability, approximation errors for state-to-state control and regulations is established in (Gao and Caines, 2020).

Scalable and tractable solution methods are established for a class of graphon control problem in (Gao and Caines, 2019a,b). Next, the graphon linear system control framework is extended to more general cases where the state of each individual agent on the network is multi-dimensional (Gao and Caines, 2021); moreover, this work provides a generalized solution method with an application to large-scale networks of coupled harmonic oscillators.

2.6 LQ Graphon Dynamical Systems and Control with Q-space Noise

The work in (Dunyak and Caines, 2022) extends the formulation and analysis of (Gao and Caines, 2021) to include stochastic processes in infinite dimensions as a random process source driving the network's controlled dynamical systems; this generalization involves the formulation of a form of graphon Q-space noise.

2.7 Critical Nodes for GMFG and Embedded GMFG

The works (Caines, 2022),(Foguen Tchuendom et al., 2021), (Foguen Tchuendom et al., 2022a), (Foguen Tchuendom et al., 2022b) and (Caines et al., 2023)

pioneer the theory of GMFG systems embedded in Euclidean space by constructing limit graphs and limit graphons for sequences of graphs sitting in Euclidean space. The crucial notion here is that for embedded graphs and graphons we may define the differentiation of functions on the embedded graph limits. In particular we formulate an extended version of mean field game systems, termed Embedded Graphon Mean Field Games (EGMFG), or Graphexon Mean Field Games (GXMFG), which are amenable to the application of the calculus to the optimization of node location in such systems, this being one of the main objectives of the new development. Moreover this formulation includes the case of sparse graph limits (see (Caines, 2022)) which is not covered by the standard graphon formulation.

2.8 MFG Theory and Related Stochastic Adaptive and Hybrid Control Theory

The paper (Pakniyat and Caines, 2021) constitutes the climax of several decades of work by my own group and many other researchers aimed at producing a rigorous, general, hybrid optimal systems control theory (HOCT). These and related results have great applicability in many domains of engineering and scientific modelling, and furthermore in economics and finance. In fact the paper (Firoozi et al., 2022) presents an extension of HOCT to stochastic MFG systems motivated by trading dynamics in finance.

In (Saude and Caines, 2022), we study the Markowitz dynamic portfolio problem, where an agent seeks to maximize its expected return while minimizing the variance of the return (risk). For the first time in this topic, the impact of the aggregate of the agents' trades is incorporated into the dynamics and costs using an extended mean-field games framework.

2.9 Data Based Protocols for Modelling Large Populations on Large Networks with Applications to SIR Problems

The initial investigation of applying graphon theory to the control of epidemic spread is proposed in (Gao and Caines, 2019a) where scalable solutions based on spectral approximation and linearization are established for controlling virus spreads on networks following a type of controlled network SIS model.
The work (Dunyak and Caines, 2021; Salii, 2021) includes the development of SIR epidemic models which permits the study of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity; the latter models were constructed on a data base of real world US surface and air-traffic travel data.

2.10 Centrality Analysis for Finite and Infinite Networks

This work advances the theory of centralities for dynamical systems on networks and its associated applications. In (Gao, 2022) a family of centralities characterized by fixed-points of networks is identified which includes the standard centralities (e.g. PageRank, Katz-Bonacich, eigen centralities) and equilibrirum cost for games on networks (e.g. Nash values of LQG GMFG). Variation bounds of centralities with respect to the network variations are established. In (Gao, 2021) network centralities have been utilized in the modelling of opinion dynamics on social networks and in identifying significant network partitions based upon opinion disagreement states.

3 Dissemination to Communities of Interest

Research findings and results have been published in the leading systems and control and networks journals including the SIAM Journal on Control and Optimization, IEEE Transactions on Automatic Control, Automatica, IEEE Transactions on Control of Network Systems, and in conferences proceedings including the IEEE Conference on Decision and Control.

3.1 Publications: Journals

S. Gao, P. E. Caines, and M. Huang, "LQG graphon mean field games: analysis via graphon invariant subspaces," Conditionally accepted by IEEE Transactions on Automatic Control, 2021. (Gao et al., 2022)

D. Firoozi, A. Pakniyat, and P. E. Caines, "A class of hybrid LQG mean field games with state-invariant switching and stopping strategies," Automatica, vol. 141, p. 110244, 2022. (Firoozi et al., 2022)

P. E. Caines, D. Ho, M. Huang, J. Jian, and Q. Song. On the graphon mean field game equations: Individual agent affine dynamics and mean field dependent performance functions. ESAIM: Control, Optimisation and Calculus of Variations, 28:24, 2022 (Caines et al., 2022)

P. E. Caines and M. Huang, "Graphon mean field games and their equations," SIAM Journal on Control and Optimization, vol. 59, no. 6, pp. 4373–4399, 2021. (Caines and Huang, 2021)

A. Pakniyat and P. E. Caines, "On the hybrid minimum principle: The Hamiltonian and adjoint boundary conditions," IEEE Transactions on Automatic Control, vol. 66, no. 3, pp. 1246–1253, 2021. (Pakniyat and Caines, 2021)

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N. Sen and P. E. Caines. Mean field games with partial observation. SIAM Journal on Control and Optimization, 57(3):2064–2091, 2019 (Sen and Caines, 2019)

P. E. Caines, D. Ho, and Q. Song. The density evolution of the killed mckean–vlasov process. Stochastics, 92(4):642–657, 2020. (Caines et al., 2020)

3.2 Publications: Conference Proceedings

A. Dunyak and P. E. Caines, "Linear stochastic graphon systems with q- space noise," in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 3926–3932. (Dunyak and Caines, 2022)

P. E. Caines, "Embedded vertexon-graphons and embedded GMFG systems," Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022,

pp. 5550–5557. (Caines, 2022)

S. Gao, “Fixed-point centrality for networks,” in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 1628–1635. (Gao, 2022).

R. Foguen Tchuendom, S. Gao, P. E. Caines, and M. Huang, “Optimal network location in infinite horizon LQG graphon mean field games,” in Proceedings of the 61th IEEE Conference on Decision and Control, Cancun, Mexico, December 2022, pp. 5558–5565. (Foguen Tchuendom et al., 2022b)

R. Foguen Tchuendom, S. Gao, and P. E. Caines “Stationary cost nodes in infinite horizon LQG GMFGs,” IFAC-PapersOnLine for the 25th International Symposium on Mathematical Theory of Networks and Systems (MTNS), vol. 55, no. 30, pp. 284–289, 2022. (Foguen Tchuendom et al., 2022a)

J. Saude and P. E. Caines, “Markowitz portfolio optimization extended quadratic mean- field games approach,” Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 5523–5528. (Saude and Caines, 2022)

Y. V. Salii, “Benchmarking optimal control for network dynamic systems with plausible epidemic models,” in International Conference on Complex Networks and Their Applications. Springer, 2021, pp. 194–206. (Salii, 2021)

S. Gao, P. E. Caines, and M. Huang, “LQG graphon mean field games: Graphon invariant subspaces,” Proceedings of the 60th IEEE Conference on Decision and Control, Austin, Texas, USA, December 2021, pp. 5253–5260. (Gao et al., 2021a)

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S. Gao, “Centrality-weighted opinion dynamics: Disagreement and social network partition,” Proceedings of the 60th IEEE Conference on Decision and Control (CDC), Austin, Texas, USA, December 2021, pp. 5496–5501. (Gao, 2021)

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S. Gao and P. E. Caines, Optimal and approximate solutions to linear quadratic regulation of a class of graphon dynamical systems, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 8359–8365, Nice, France, December 2019b. (Gao and Caines, 2019b)

S. Gao and P. E. Caines, Spectral representations of graphons in very large network systems control, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 5068–5075, Nice, France, December 2019a.(Gao and Caines, 2019a)

P. E. Caines and M. Huang. Graphon mean field games and the GMFG equations: ϵ-Nash equilibria, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 286–292, December 2019. (Caines and Huang, 2019)

3.3 Talks: April 1st, 2019 to December 31st, 2022

1. P. E. Caines,"Vertexons and Embedded Graphon Mean Field Games", the Simons Institute for the Theory of Computing, UC Berkeley, CA, Sept.29, 2022 (Invited talk) in the "Graph Limits, Nonparametric Models, and Estimation Workshop" 26 - 30 September, 2022.

2. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", (Invited talk) 29th August, 2022, in the Conference on Many Player Games and Applications, Humbolt University Berlin, 29 - 31 August 2022.

3. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Large Populations on Complex Networks", Canadian Applied and Industrial Mathematics Society (CAIMS), June 23rd, 2021 (Plenary talk).

4. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", the Financial Mathematics/Engineering Seminar Series, Hong Kong Polytechnic University, June 16th 2021 (Invited talk).

5. P. E. Caines, "Static and Time Varying Graphons in Economic and Pandemic GMFG Models" SIAM Financial Mathematics and Engineering (FM21), June 1, 2021 (Invited talk).

6. S. Gao, "Subspace Decompositions in Graphon Control and Graphon Mean Field Games", Informal Systems Seminar, Centre for Intelligent Machines,

McGill University, Groupe d'Etudes et de Recherche en Analyse des Decisions, Montreal, CA, Dec. 11, 2020 (Invited talk).

7. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Networked Systems", Invited Speaker at the "Conference on Mean Field Games and Related Topics - 5". 9th - 13th September, 2019

8. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for a Networked World", University of Saint Petersburg, Russia, 6th September, 2019 (Distinguished Speaker).

9. P. E. Caines, "Graphon Control and Graphon Mean Field Games", Department of Applied Mathemaics, Hong Kong Polythenic Universty, Hong Kong), 28th June, 2019 (Distinguished Lecturer).

3.4 Workshop Organization: April 1st, 2019 to December 31st, 2022

Name: **Mean Field Games on Networks Workshop**

Location: Vancouver, BC, Canada (Virtual Meeting)

Dates: October 26 - 29, 2021

Organizers: Peter E. Caines, Shuang Gao, Minyi Huang, and Rinel Foguen Tchuendom

This workshop (PIMS et al., 2021) brought together researchers in applied mathematics, mean field games, network science, network games, and systems and control theory to exchange ideas and to work on the extensions of mean field game theory to dynamic game problems on heterogeneous large-scale networks. The workshop offered a platform to current results and stimulate discussions on the open challenges in this emerging field, on mathematical theory, computational algorithms, and the applications methodologies of MFG on general networks.

Sponsors: Pacific Institute for the Mathematical Sciences (PIMS), Groupe d'études et de recherche en analyse des décisions (GERAD), and the Fields Institute for Research in Mathematical Sciences.

Participants: Peter E. Caines (PI) and PI's Postdoctoral Researchers: Rinel Foguen Tchuendom, Shuang Gao, Joao Saude, and Yaroslav Salii, plus 50 other international participants.

4 Impacts

As stated earlier, the research on MFG and related topics by the Principal Investigator of the grant FA9550-19-1-0138 has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, presently 2022 - 2025. Consequently, the impacts of research under the 2019 - 2022 grant, FA9550-19-1-0138, may be viewed as a continuation and development of those of the entire program.

Indeed, as explained in Section 1, the MFG field has expanded enormously since its inception. Google links now run to 1.10 billion, citations to papers are in the thousands and conferences and workshops on MFG have proliferated. A sample of the most recent workshops and conference sessions is the following, where we note that the PI and group members have been invited to participate in all of them: Workshop “Mean Field Games on Networks”, Pacific Institute for the Mathematical Sciences, Vancouver, BC, Canada, 26 - 29, October, 2021; “Conference on Mathematics and Financial Economics, “Many Player Games and Applications”, Humbolt University, Berlin, 29 - 31 August 2022; Workshop on Mean Field Games, Montreal: Centre de Recherches Mathématiques (CRM), 11-15 April, 2022; “Graph Limits, Nonparametric Models, and Estimation Workshop”, Simons Institute for the Theory of Computation, UCB, Berkeley, CA, 26-30 September, 2022; IEEE, Control Systems Society Conference on Decision and Control, Cancun, December, 2022, Mean Field Games Session; Mean Field Games Invited Session, IFAC World Congress, Yokahama, Japan, July 2023.

Furthermore, at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively), the last appearing in the period covered by this report. In addition to the very rapid expansion of mathematical and theoretical investigations, applications of MFG theory have spread to a vast range of domains, a sample of the most recent being the optimization of decentralized electric vehicle charging (Foguen Tchuendom et al., 2019), algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Grover et al., 2018), AI reinforcement learning for MFG systems (Agarwal et al., 2022) and polymer science (Welch et al., 2019).

As is shown by this indicative list of publications, meetings and presentations, it is reasonable to assert that the influence of the AFOSR supported research carried out on Mean Field Games and, in particular, on Mean Field Games on networks, has had, and is having, an impact on fields ranging from the core area of systems and control science to economics, finance, energy generation and distribution, operations research, the life sciences and the physical sciences.

5 Changes

There were no changes in the project or its direction.

6 Technical Updates

There are no technical updates to report beyond the continuing research progress and dissemination of results described in report.

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Grant FA9550-19-1-0138
New Directions in Mean Field Games: MFG Subpopulation Behaviours and Graphon-MFG Systems

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1 Overview of Research Objectives

1.1 Background and Overview

Very large networks linking dynamical agents are now ubiquitous and the need to analyse, design and control them is evident. The emergence of the graphon theory of large networks and their infinite limits has enabled the formulation of a theory of the centralized control of dynamical systems distributed on asymptotically infinite networks.

Large-scale systems of competing agents are frequently formulated as dynamic games, this however confronts the problem that for even small populations non-cooperative game theory is intractable. The 2006 breakthrough of Minyi Huang, Roland Malhamé and the PI of FA9550-19-1-0138, Peter Caines, was the introduction of Mean Field Game (MFG) theory (originally termed Nash Certainty Equivalence theory) which was also simultaneously discovered by J.M. Lasry and P.L. Lions.

Mean Field Game (MFG) theory analyzes the existence of Nash equilibria for dynamical games in terms of infinite populations of negligible agents forming the limits of the original finite population systems (Caines, 2020). The core of MFG theory is a pair of partial differential equations consisting of (i) the Hamilton-Jacobi-Bellman equation of optimal control for a generic agent, and (ii) the Fokker-Planck-Kolmogorov equation describing the controlled evolution of the state of a generic agent. These are linked by the generic agent's state

distribution, known as the system's mean field.

A fundamental aspect of MFG theory justifying its application in a vast range of cases is the ϵ-Nash property by which controls derived from the MFG equations yield near-equilibrium performance for finite systems for sufficiently large populations.

The MFG field has expanded tremendously since its inception. Google search results now run to 1.10 billion, citations to papers are in the thousands, conferences and workshops on MFG have proliferated and at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively). In addition to the explosion of mathematical and theoretical investigations over last 16 years, applications of MFG theory have spread to a vast range of domains including: optimization of decentralized EV charging (Ma et al., 2011; Zhu et al., 2016; Foguen Tchuendom et al., 2019); nonlinear system state estimation (Yang et al., 2013); systemic risk in banking (Carmona et al., 2013); cell phone energy management (Aziz and Caines, 2016); economics of energy production (Ludkovski and Sircar, 2015); macro-economics of growth (Gomes et al., 2016); cloud-based networks in information technology (Hanif et al., 2015); algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Nourian et al., 2011; Grover et al., 2018), AI reinforcement learning for MFG systems (Subramanian and Mahajan, 2019; Agarwal et al., 2022), deep learning applications to the solution of MFGs (Ruthotto et al., 2020) and polymer science (Welch et al., 2019).

In this context it is important to note that the research on MFG by the Principal Investigator has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, most recently, for the period 2022-2025.

1.2 Major Research Goals under Grant FA9550-19-1-0138

The program of research under FA9550-19-1-0138 is focused on the development of a new form of Mean Field Games called Graphon Mean Field Games (GMFGs); these are large population games defined on large, heterogeneous, networks and their asymptotic (graphon) limits.

In this setting, game theoretic equilibria are expressed in terms of the newly defined GMFG PDE equations, which are a sweeping generalization of the classical MFG PDEs.

In a closely related development, the theory of graphons is used as a basis for Graphon Linear Control (GLC) theory in which large populations of centrally controlled dynamical systems interacting over large, heterogeneous networks

may be analyzed and optimized in a tractable way.

Canonical Design Procedure for GMFG and GLC Systems

A major goal has been to generate large scale network game strategies and, respectively, control design procedures with the following standard pattern:

The first step is to consider the graphon limit of the network over which the individual systems are distributed and which constitutes the global system.

In the second step, the resulting graphon based GMFG equations, or, respectively, the graphon based infinite dimensional linear system optimization problems are specified. The GMFG equations are then solved to generate the Nash best response strategies or, respectively, the linear feedback controls are synthesized using the well established infinite dimensional linear systems control theory.

Finally, the best response strategies, or, respectively, the infinite dimensional linear feedback controls are approximated in such a way that they may be applied to the original complex finite systems.

The key feature of this strategy is the following: In both the network mean field game and the network control settings it is demonstrated that as the size of the network tends to infinity, the performance of the finite systems when they are subject to the infinite limit system best response game strategies, or, respectively, centralized control laws, yield pay-offs which converge to their Nash equilibrium, or, respectively, optimal, infinite system limit values.

1.3 Organization of the Sequence of Research Objectives

The work over the full reporting period has been organized with the following sequence of objectives:

(i) The establishment of a rigorous theory of the central results of Graphon Mean Field Game (GMFG) theory,

(ii) The generation of a specific Graphon Linear Control (GLC) theory for the control of large populations of dynamical linear system agents with quadratic performance functions on large, complex networks.

(iii) The development of an approximation theory of associated numerical methods, especially those based upon finite dimensional invariant subspaces of the graphon operator in the GLC case.

(iv) The pioneering development of the theory of GMFG systems embedded in Euclidean space; this underlies our currently developing variational analysis of agent performance in networks as a function of position, where this theory also includes sparse as well as dense networks.

(v) The development of an SIR epidemic model software which permits the simulation of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity, these models are constructed on a data base of real world US surface and air-traffic travel data.

2 Details of Accomplishments

2.1 Graphon Mean Field Game Theory: Foundations

The work in (Caines and Huang, 2021) (Caines and Huang, 2018), (Caines and Huang, 2019), established the foundations and central results for graphon mean field game theory for nonlinear systems, that is to say games of infinite populations on infinite heterogeneous networks as limits of the corresponding finite systems.

Specifically, in the conference papers (Caines and Huang, 2018), (Caines and Huang, 2019) Graphon Mean Field Games (GMFG) and the GMFG equations were introduced for the analysis of non-cooperative dynamical games on unbounded networks. In this formulation, asymptotically infinite sub-populations are located at the nodes of large heterogeneous networks, the asymptotic network limits then constitute a part of the analysis. In (Caines and Huang, 2021), the basic existence and uniqueness results were established for the GMFG equations, and a full derivation of the fundamental results was provided, including the key epsilon-Nash theory for GMFG systems which relates the infinite population equilibria on infinite networks to finite population equilibria on finite networks.

2.2 LQG GMFG Decomposition Methods

The theory of invariant spaces of linear operators developed in the papers (Gao and Caines, 2021) is applied in (Gao et al., 2021b), (Gao et al., 2021a) and (Gao et al., 2022) to yield a tractable methodology for generating decentralized game strategies for systems distributed over complex networks. The fundamental properties of the existence and uniqueness of solutions are established in (Gao et al., 2021b,a, 2022) in two principal ways: (i) Riccati equation decomposition and (ii) fixed point contraction theorem methods.

2.3 LQG GMFG Master Equation Analysis

First, using a probabilistic approach, solutions to the Linear Quadratic Graphon Mean Field Games are expressed as solutions to coupled Forward Backward Stochastic Differential Equations (FBSDEs) of McKean-Vlasov type. Next, the

existence of the so-called Master field, which allows for the decoupling of these FBSDEs is established. Finally, the infinite dimensional Partial Differential Equation (PDE), so-called Master Equation, is established for which the Master Field is a solution (see (Foguen Tchuendom et al., 2020)).

2.4 GMFG with Affine Dynamics

GMFG theory for nonlinear problems with individual affine dynamics has been established in Caines et al. (2022) and the Nash value properties for control affine systems has been analyzed in Caines et al. (2023).

2.5 Graphon Linear Systems Control with Approximation Theory

As explained above, the theory of graphons is further used as a basis upon which centrally controlled linear dynamical systems interacting over large, heterogeneous networks are analyzed in a tractable way.

The graphon control methodology is developed in (Gao and Caines, 2020, 2019a,b) to study control problems for arbitrary-size networks of linear dynamical systems. The complete treatment of problems involving well-posedness, controllability, approximation errors for state-to-state control and regulations is established in (Gao and Caines, 2020).

Scalable and tractable solution methods are established for a class of graphon control problem in (Gao and Caines, 2019a,b). Next, the graphon linear system control framework is extended to more general cases where the state of each individual agent on the network is multi-dimensional (Gao and Caines, 2021); moreover, this work provides a generalized solution method with an application to large-scale networks of coupled harmonic oscillators.

2.6 LQ Graphon Dynamical Systems and Control with Q-space Noise

The work in (Dunyak and Caines, 2022) extends the formulation and analysis of (Gao and Caines, 2021) to include stochastic processes in infinite dimensions as a random process source driving the network's controlled dynamical systems; this generalization involves the formulation of a form of graphon Q-space noise.

2.7 Critical Nodes for GMFG and Embedded GMFG

The works (Caines, 2022),(Foguen Tchuendom et al., 2021), (Foguen Tchuendom et al., 2022a), (Foguen Tchuendom et al., 2022b) and (Caines et al., 2023)

pioneer the theory of GMFG systems embedded in Euclidean space by constructing limit graphs and limit graphons for sequences of graphs sitting in Euclidean space. The crucial notion here is that for embedded graphs and graphons we may define the differentiation of functions on the embedded graph limits. In particular we formulate an extended version of mean field game systems, termed Embedded Graphon Mean Field Games (EGMFG), or Graphexon Mean Field Games (GXMFG), which are amenable to the application of the calculus to the optimization of node location in such systems, this being one of the main objectives of the new development. Moreover this formulation includes the case of sparse graph limits (see (Caines, 2022)) which is not covered by the standard graphon formulation.

2.8 MFG Theory and Related Stochastic Adaptive and Hybrid Control Theory

The paper (Pakniyat and Caines, 2021) constitutes the climax of several decades of work by my own group and many other researchers aimed at producing a rigorous, general, hybrid optimal systems control theory (HOCT). These and related results have great applicability in many domains of engineering and scientific modelling, and furthermore in economics and finance. In fact the paper (Firoozi et al., 2022) presents an extension of HOCT to stochastic MFG systems motivated by trading dynamics in finance.

In (Saude and Caines, 2022), we study the Markowitz dynamic portfolio problem, where an agent seeks to maximize its expected return while minimizing the variance of the return (risk). For the first time in this topic, the impact of the aggregate of the agents' trades is incorporated into the dynamics and costs using an extended mean-field games framework.

2.9 Data Based Protocols for Modelling Large Populations on Large Networks with Applications to SIR Problems

The initial investigation of applying graphon theory to the control of epidemic spread is proposed in (Gao and Caines, 2019a) where scalable solutions based on spectral approximation and linearization are established for controlling virus spreads on networks following a type of controlled network SIS model.
The work (Dunyak and Caines, 2021; Salii, 2021) includes the development of SIR epidemic models which permits the study of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity; the latter models were constructed on a data base of real world US surface and air-traffic travel data.

2.10 Centrality Analysis for Finite and Infinite Networks

This work advances the theory of centralities for dynamical systems on networks and its associated applications. In (Gao, 2022) a family of centralities characterized by fixed-points of networks is identified which includes the standard centralities (e.g. PageRank, Katz-Bonacich, eigen centralities) and equilbrirum cost for games on networks (e.g. Nash values of LQG GMFG). Variation bounds of centralities with respect to the network variations are established. In (Gao, 2021) network centralities have been utilized in the modelling of opinion dynamics on social networks and in identifying significant network partitions based upon opinion disagreement states.

3 Dissemination to Communities of Interest

Research findings and results have been published in the leading systems and control and networks journals including the SIAM Journal on Control and Optimization, IEEE Transactions on Automatic Control, Automatica, IEEE Transactions on Control of Network Systems, and in conferences proceedings including the IEEE Conference on Decision and Control.

3.1 Publications: Journals

S. Gao, P. E. Caines, and M. Huang, “LQG graphon mean field games: analysis via graphon invariant subspaces,” Conditionally accepted by IEEE Transactions on Automatic Control, 2021. (Gao et al., 2022)

D. Firoozi, A. Pakniyat, and P. E. Caines, “A class of hybrid LQG mean field games with state-invariant switching and stopping strategies,” Automatica, vol. 141, p. 110244, 2022. (Firoozi et al., 2022)

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P. E. Caines and M. Huang, “Graphon mean field games and their equations,” SIAM Journal on Control and Optimization, vol. 59, no. 6, pp. 4373–4399, 2021. (Caines and Huang, 2021)

A. Pakniyat and P. E. Caines, “On the hybrid minimum principle: The Hamiltonian and adjoint boundary conditions,” IEEE Transactions on Automatic Control, vol. 66, no. 3, pp. 1246–1253, 2021. (Pakniyat and Caines, 2021)

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S. Gao, R. Foguen Tchuendom, and P. E. Caines, “Linear quadratic graphon field games,” Communications in Information and Systems, vol. 21, no. 3, pp. 341–369, 2021. (Gao et al., 2021b)

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S. Gao and P. E. Caines. Graphon control of large-scale networks of linear systems. IEEE Transactions on Automatic Control, 65(10):4090–4105, 2020 (Gao and Caines, 2020)

N. Sen and P. E. Caines. Mean field games with partial observation. SIAM Journal on Control and Optimization, 57(3):2064–2091, 2019 (Sen and Caines, 2019)

P. E. Caines, D. Ho, and Q. Song. The density evolution of the killed mckean–vlasov process. Stochastics, 92(4):642–657, 2020. (Caines et al., 2020)

3.2 Publications: Conference Proceedings

A. Dunyak and P. E. Caines, “Linear stochastic graphon systems with q- space noise,” in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 3926–3932. (Dunyak and Caines, 2022)

P. E. Caines, “Embedded vertexon-graphons and embedded GMFG systems,” Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022,

pp. 5550–5557. (Caines, 2022)

S. Gao, “Fixed-point centrality for networks,” in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 1628–1635. (Gao, 2022).

R. Foguen Tchuendom, S. Gao, P. E. Caines, and M. Huang, “Optimal network location in infinite horizon LQG graphon mean field games,” in Proceedings of the 61th IEEE Conference on Decision and Control, Cancun, Mexico, December 2022, pp. 5558–5565. (Foguen Tchuendom et al., 2022b)

R. Foguen Tchuendom, S. Gao, and P. E. Caines “Stationary cost nodes in infinite horizon LQG GMFGs,” IFAC-PapersOnLine for the 25th International Symposium on Mathematical Theory of Networks and Systems (MTNS), vol. 55, no. 30, pp. 284–289, 2022. (Foguen Tchuendom et al., 2022a)

J. Saude and P. E. Caines, “Markowitz portfolio optimization extended quadratic mean- field games approach,” Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 5523–5528. (Saude and Caines, 2022)

Y. V. Salii, “Benchmarking optimal control for network dynamic systems with plausible epidemic models,” in International Conference on Complex Networks and Their Applications. Springer, 2021, pp. 194–206. (Salii, 2021)

S. Gao, P. E. Caines, and M. Huang, “LQG graphon mean field games: Graphon invariant subspaces,” Proceedings of the 60th IEEE Conference on Decision and Control, Austin, Texas, USA, December 2021, pp. 5253–5260. (Gao et al., 2021a)

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IEEE Conference on Decision and Control (CDC), pages 1026–1031, 2020. (Foguen Tchuendom et al., 2020)

S. Gao and P. E. Caines, Optimal and approximate solutions to linear quadratic regulation of a class of graphon dynamical systems, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 8359–8365, Nice, France, December 2019b. (Gao and Caines, 2019b)

S. Gao and P. E. Caines, Spectral representations of graphons in very large network systems control, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 5068–5075, Nice, France, December 2019a.(Gao and Caines, 2019a)

P. E. Caines and M. Huang. Graphon mean field games and the GMFG equations: ϵ-Nash equilibria, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 286–292, December 2019. (Caines and Huang, 2019)

3.3 Talks: April 1st, 2019 to December 31st, 2022

1. P. E. Caines,"Vertexons and Embedded Graphon Mean Field Games", the Simons Institute for the Theory of Computing, UC Berkeley, CA, Sept.29, 2022 (Invited talk) in the "Graph Limits, Nonparametric Models, and Estimation Workshop" 26 - 30 September, 2022.

2. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", (Invited talk) 29th August, 2022, in the Conference on Many Player Games and Applications, Humbolt University Berlin, 29 - 31 August 2022.

3. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Large Populations on Complex Networks", Canadian Applied and Industrial Mathematics Society (CAIMS), June 23rd, 2021 (Plenary talk).

4. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", the Financial Mathematics/Engineering Seminar Series, Hong Kong Polytechnic University, June 16th 2021 (Invited talk).

5. P. E. Caines, "Static and Time Varying Graphons in Economic and Pandemic GMFG Models" SIAM Financial Mathematics and Engineering (FM21), June 1, 2021 (Invited talk).

6. S. Gao, "Subspace Decompositions in Graphon Control and Graphon Mean Field Games", Informal Systems Seminar, Centre for Intelligent Machines,

McGill University, Groupe d'Etudes et de Recherche en Analyse des Decisions, Montreal, CA, Dec. 11, 2020 (Invited talk).

7. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Networked Systems", Invited Speaker at the "Conference on Mean Field Games and Related Topics - 5". 9th - 13th September, 2019

8. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for a Networked World", University of Saint Petersburg, Russia, 6th September, 2019 (Distinguished Speaker).

9. P. E. Caines, "Graphon Control and Graphon Mean Field Games", Department of Applied Mathemaics, Hong Kong Polythenic Universty, Hong Kong), 28th June, 2019 (Distinguished Lecturer).

3.4 Workshop Organization: April 1st, 2019 to December 31st, 2022

Name: **Mean Field Games on Networks Workshop**

Location: Vancouver, BC, Canada (Virtual Meeting)

Dates: October 26 - 29, 2021

Organizers: Peter E. Caines, Shuang Gao, Minyi Huang, and Rinel Foguen Tchuendom

This workshop (PIMS et al., 2021) brought together researchers in applied mathematics, mean field games, network science, network games, and systems and control theory to exchange ideas and to work on the extensions of mean field game theory to dynamic game problems on heterogeneous large-scale networks. The workshop offered a platform to current results and stimulate discussions on the open challenges in this emerging field, on mathematical theory, computational algorithms, and the applications methodologies of MFG on general networks.

Sponsors: Pacific Institute for the Mathematical Sciences (PIMS), Groupe d'études et de recherche en analyse des décisions (GERAD), and the Fields Institute for Research in Mathematical Sciences.

Participants: Peter E. Caines (PI) and PI's Postdoctoral Researchers: Rinel Foguen Tchuendom, Shuang Gao, Joao Saude, and Yaroslav Salii, plus 50 other international participants.

4 Impacts

As stated earlier, the research on MFG and related topics by the Principal Investigator of the grant FA9550-19-1-0138 has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, presently 2022 - 2025. Consequently, the impacts of research under the 2019 - 2022 grant, FA9550-19-1-0138, may be viewed as a continuation and development of those of the entire program.

Indeed, as explained in Section 1, the MFG field has expanded enormously since its inception. Google links now run to 1.10 billion, citations to papers are in the thousands and conferences and workshops on MFG have proliferated. A sample of the most recent workshops and conference sessions is the following, where we note that the PI and group members have been invited to participate in all of them: Workshop "Mean Field Games on Networks", Pacific Institute for the Mathematical Sciences, Vancouver, BC, Canada, 26 - 29, October, 2021; "Conference on Mathematics and Financial Economics, "Many Player Games and Applications", Humbolt University, Berlin, 29 - 31 August 2022; Workshop on Mean Field Games, Montreal: Centre de Recherches Mathématiques (CRM), 11-15 April, 2022; "Graph Limits, Nonparametric Models, and Estimation Workshop", Simons Institute for the Theory of Computation, UCB, Berkeley, CA, 26-30 September, 2022; IEEE, Control Systems Society Conference on Decision and Control, Cancun, December, 2022, Mean Field Games Session; Mean Field Games Invited Session, IFAC World Congress, Yokahama, Japan, July 2023.

Furthermore, at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively), the last appearing in the period covered by this report. In addition to the very rapid expansion of mathematical and theoretical investigations, applications of MFG theory have spread to a vast range of domains, a sample of the most recent being the optimization of decentralized electric vehicle charging (Foguen Tchuendom et al., 2019), algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Grover et al., 2018), AI reinforcement learning for MFG systems (Agarwal et al., 2022) and polymer science (Welch et al., 2019).

As is shown by this indicative list of publications, meetings and presentations, it is reasonable to assert that the influence of the AFOSR supported research carried out on Mean Field Games and, in particular, on Mean Field Games on networks, has had, and is having, an impact on fields ranging from the core area of systems and control science to economics, finance, energy generation and distribution, operations research, the life sciences and the physical sciences.

5 Changes

There were no changes in the project or its direction.

6 Technical Updates

There are no technical updates to report beyond the continuing research progress and dissemination of results described in report.

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AFOSR Final Report 2023-03-30
Grant FA9550-19-1-0138
New Directions in Mean Field Games: MFG Subpopulation Behaviours and Graphon-MFG Systems

Peter E. Caines

Reporting period April 1st, 2019 to December 31st, 2022

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1 Overview of Research Objectives

1.1 Background and Overview

Very large networks linking dynamical agents are now ubiquitous and the need to analyse, design and control them is evident. The emergence of the graphon theory of large networks and their infinite limits has enabled the formulation of a theory of the centralized control of dynamical systems distributed on asymptotically infinite networks.

Large-scale systems of competing agents are frequently formulated as dynamic games, this however confronts the problem that for even small populations non-cooperative game theory is intractable. The 2006 breakthrough of Minyi Huang, Roland Malhamé and the PI of FA9550-19-1-0138, Peter Caines, was the introduction of Mean Field Game (MFG) theory (originally termed Nash Certainty Equivalence theory) which was also simultaneously discovered by J.M. Lasry and P.L. Lions.

Mean Field Game (MFG) theory analyzes the existence of Nash equilibria for dynamical games in terms of infinite populations of negligible agents forming the limits of the original finite population systems (Caines, 2020). The core of MFG theory is a pair of partial differential equations consisting of (i) the Hamilton-Jacobi-Bellman equation of optimal control for a generic agent, and (ii) the Fokker-Planck-Kolmogorov equation describing the controlled evolution of the state of a generic agent. These are linked by the generic agent's state

distribution, known as the system's mean field.

A fundamental aspect of MFG theory justifying its application in a vast range of cases is the ϵ-Nash property by which controls derived from the MFG equations yield near-equilibrium performance for finite systems for sufficiently large populations.

The MFG field has expanded tremendously since its inception. Google search results now run to 1.10 billion, citations to papers are in the thousands, conferences and workshops on MFG have proliferated and at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively). In addition to the explosion of mathematical and theoretical investigations over last 16 years, applications of MFG theory have spread to a vast range of domains including: optimization of decentralized EV charging (Ma et al., 2011; Zhu et al., 2016; Foguen Tchuendom et al., 2019); nonlinear system state estimation (Yang et al., 2013); systemic risk in banking (Carmona et al., 2013); cell phone energy management (Aziz and Caines, 2016); economics of energy production (Ludkovski and Sircar, 2015); macro-economics of growth (Gomes et al., 2016); cloud-based networks in information technology (Hanif et al., 2015); algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Nourian et al., 2011; Grover et al., 2018), AI reinforcement learning for MFG systems (Subramanian and Mahajan, 2019; Agarwal et al., 2022), deep learning applications to the solution of MFGs (Ruthotto et al., 2020) and polymer science (Welch et al., 2019).

In this context it is important to note that the research on MFG by the Principal Investigator has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, most recently, for the period 2022-2025.

1.2 Major Research Goals under Grant FA9550-19-1-0138

The program of research under FA9550-19-1-0138 is focused on the development of a new form of Mean Field Games called Graphon Mean Field Games (GMFGs); these are large population games defined on large, heterogeneous, networks and their asymptotic (graphon) limits.

In this setting, game theoretic equilibria are expressed in terms of the newly defined GMFG PDE equations, which are a sweeping generalization of the classical MFG PDEs.

In a closely related development, the theory of graphons is used as a basis for Graphon Linear Control (GLC) theory in which large populations of centrally controlled dynamical systems interacting over large, heterogeneous networks

may be analyzed and optimized in a tractable way.

Canonical Design Procedure for GMFG and GLC Systems

A major goal has been to generate large scale network game strategies and, respectively, control design procedures with the following standard pattern:

The first step is to consider the graphon limit of the network over which the individual systems are distributed and which constitutes the global system.

In the second step, the resulting graphon based GMFG equations, or, respectively, the graphon based infinite dimensional linear system optimization problems are specified. The GMFG equations are then solved to generate the Nash best response strategies or, respectively, the linear feedback controls are synthesized using the well established infinite dimensional linear systems control theory.

Finally, the best response strategies, or, respectively, the infinite dimensional linear feedback controls are approximated in such a way that they may be applied to the original complex finite systems.

The key feature of this strategy is the following: In both the network mean field game and the network control settings it is demonstrated that as the size of the network tends to infinity, the performance of the finite systems when they are subject to the infinite limit system best response game strategies, or, respectively, centralized control laws, yield pay-offs which converge to their Nash equilibrium, or, respectively, optimal, infinite system limit values.

1.3 Organization of the Sequence of Research Objectives

The work over the full reporting period has been organized with the following sequence of objectives:

(i) The establishment of a rigorous theory of the central results of Graphon Mean Field Game (GMFG) theory,

(ii) The generation of a specific Graphon Linear Control (GLC) theory for the control of large populations of dynamical linear system agents with quadratic performance functions on large, complex networks.

(iii) The development of an approximation theory of associated numerical methods, especially those based upon finite dimensional invariant subspaces of the graphon operator in the GLC case.

(iv) The pioneering development of the theory of GMFG systems embedded in Euclidean space; this underlies our currently developing variational analysis of agent performance in networks as a function of position, where this theory also includes sparse as well as dense networks.

(v) The development of an SIR epidemic model software which permits the simulation of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity, these models are constructed on a data base of real world US surface and air-traffic travel data.

2 Details of Accomplishments

2.1 Graphon Mean Field Game Theory: Foundations

The work in (Caines and Huang, 2021) (Caines and Huang, 2018), (Caines and Huang, 2019), established the foundations and central results for graphon mean field game theory for nonlinear systems, that is to say games of infinite populations on infinite heterogeneous networks as limits of the corresponding finite systems.

Specifically, in the conference papers (Caines and Huang, 2018), (Caines and Huang, 2019) Graphon Mean Field Games (GMFG) and the GMFG equations were introduced for the analysis of non-cooperative dynamical games on unbounded networks. In this formulation, asymptotically infinite sub-populations are located at the nodes of large heterogeneous networks, the asymptotic network limits then constitute a part of the analysis. In (Caines and Huang, 2021), the basic existence and uniqueness results were established for the GMFG equations, and a full derivation of the fundamental results was provided, including the key epsilon-Nash theory for GMFG systems which relates the infinite population equilibria on infinite networks to finite population equilibria on finite networks.

2.2 LQG GMFG Decomposition Methods

The theory of invariant spaces of linear operators developed in the papers (Gao and Caines, 2021) is applied in (Gao et al., 2021b), (Gao et al., 2021a) and (Gao et al., 2022) to yield a tractable methodology for generating decentralized game strategies for systems distributed over complex networks. The fundamental properties of the existence and uniqueness of solutions are established in (Gao et al., 2021b,a, 2022) in two principal ways: (i) Riccati equation decomposition and (ii) fixed point contraction theorem methods.

2.3 LQG GMFG Master Equation Analysis

First, using a probabilistic approach, solutions to the Linear Quadratic Graphon Mean Field Games are expressed as solutions to coupled Forward Backward Stochastic Differential Equations (FBSDEs) of McKean-Vlasov type. Next, the

existence of the so-called Master field, which allows for the decoupling of these FBSDEs is established. Finally, the infinite dimensional Partial Differential Equation (PDE), so-called Master Equation, is established for which the Master Field is a solution (see (Foguen Tchuendom et al., 2020)).

2.4 GMFG with Affine Dynamics

GMFG theory for nonlinear problems with individual affine dynamics has been established in Caines et al. (2022) and the Nash value properties for control affine systems has been analyzed in Caines et al. (2023).

2.5 Graphon Linear Systems Control with Approximation Theory

As explained above, the theory of graphons is further used as a basis upon which centrally controlled linear dynamical systems interacting over large, heterogeneous networks are analyzed in a tractable way.

The graphon control methodology is developed in (Gao and Caines, 2020, 2019a,b) to study control problems for arbitrary-size networks of linear dynamical systems. The complete treatment of problems involving well-posedness, controllability, approximation errors for state-to-state control and regulations is established in (Gao and Caines, 2020).

Scalable and tractable solution methods are established for a class of graphon control problem in (Gao and Caines, 2019a,b). Next, the graphon linear system control framework is extended to more general cases where the state of each individual agent on the network is multi-dimensional (Gao and Caines, 2021); moreover, this work provides a generalized solution method with an application to large-scale networks of coupled harmonic oscillators.

2.6 LQ Graphon Dynamical Systems and Control with Q-space Noise

The work in (Dunyak and Caines, 2022) extends the formulation and analysis of (Gao and Caines, 2021) to include stochastic processes in infinite dimensions as a random process source driving the network's controlled dynamical systems; this generalization involves the formulation of a form of graphon Q-space noise.

2.7 Critical Nodes for GMFG and Embedded GMFG

The works (Caines, 2022),(Foguen Tchuendom et al., 2021), (Foguen Tchuendom et al., 2022a), (Foguen Tchuendom et al., 2022b) and (Caines et al., 2023)

pioneer the theory of GMFG systems embedded in Euclidean space by constructing limit graphs and limit graphons for sequences of graphs sitting in Euclidean space. The crucial notion here is that for embedded graphs and graphons we may define the differentiation of functions on the embedded graph limits. In particular we formulate an extended version of mean field game systems, termed Embedded Graphon Mean Field Games (EGMFG), or Graphexon Mean Field Games (GXMFG), which are amenable to the application of the calculus to the optimization of node location in such systems, this being one of the main objectives of the new development. Moreover this formulation includes the case of sparse graph limits (see (Caines, 2022)) which is not covered by the standard graphon formulation.

2.8 MFG Theory and Related Stochastic Adaptive and Hybrid Control Theory

The paper (Pakniyat and Caines, 2021) constitutes the climax of several decades of work by my own group and many other researchers aimed at producing a rigorous, general, hybrid optimal systems control theory (HOCT). These and related results have great applicability in many domains of engineering and scientific modelling, and furthermore in economics and finance. In fact the paper (Firoozi et al., 2022) presents an extension of HOCT to stochastic MFG systems motivated by trading dynamics in finance.

In (Saude and Caines, 2022), we study the Markowitz dynamic portfolio problem, where an agent seeks to maximize its expected return while minimizing the variance of the return (risk). For the first time in this topic, the impact of the aggregate of the agents' trades is incorporated into the dynamics and costs using an extended mean-field games framework.

2.9 Data Based Protocols for Modelling Large Populations on Large Networks with Applications to SIR Problems

The initial investigation of applying graphon theory to the control of epidemic spread is proposed in (Gao and Caines, 2019a) where scalable solutions based on spectral approximation and linearization are established for controlling virus spreads on networks following a type of controlled network SIS model.
The work (Dunyak and Caines, 2021; Salii, 2021) includes the development of SIR epidemic models which permits the study of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity; the latter models were constructed on a data base of real world US surface and air-traffic travel data.

2.10 Centrality Analysis for Finite and Infinite Networks

This work advances the theory of centralities for dynamical systems on networks and its associated applications. In (Gao, 2022) a family of centralities characterized by fixed-points of networks is identified which includes the standard centralities (e.g. PageRank, Katz-Bonacich, eigen centralities) and equilbrirum cost for games on networks (e.g. Nash values of LQG GMFG). Variation bounds of centralities with respect to the network variations are established. In (Gao, 2021) network centralities have been utilized in the modelling of opinion dynamics on social networks and in identifying significant network partitions based upon opinion disagreement states.

3 Dissemination to Communities of Interest

Research findings and results have been published in the leading systems and control and networks journals including the SIAM Journal on Control and Optimization, IEEE Transactions on Automatic Control, Automatica, IEEE Transactions on Control of Network Systems, and in conferences proceedings including the IEEE Conference on Decision and Control.

3.1 Publications: Journals

S. Gao, P. E. Caines, and M. Huang, "LQG graphon mean field games: analysis via graphon invariant subspaces," Conditionally accepted by IEEE Transactions on Automatic Control, 2021. (Gao et al., 2022)

D. Firoozi, A. Pakniyat, and P. E. Caines, "A class of hybrid LQG mean field games with state-invariant switching and stopping strategies," Automatica, vol. 141, p. 110244, 2022. (Firoozi et al., 2022)

P. E. Caines, D. Ho, M. Huang, J. Jian, and Q. Song. On the graphon mean field game equations: Individual agent affine dynamics and mean field dependent performance functions. ESAIM: Control, Optimisation and Calculus of Variations, 28:24, 2022 (Caines et al., 2022)

P. E. Caines and M. Huang, "Graphon mean field games and their equations," SIAM Journal on Control and Optimization, vol. 59, no. 6, pp. 4373–4399, 2021. (Caines and Huang, 2021)

A. Pakniyat and P. E. Caines, "On the hybrid minimum principle: The Hamiltonian and adjoint boundary conditions," IEEE Transactions on Automatic Control, vol. 66, no. 3, pp. 1246–1253, 2021. (Pakniyat and Caines, 2021)

S. Gao and P. E. Caines, “Subspace decomposition for graphon LQR: Applications to VLSNs of harmonic oscillators,” IEEE Transactions on Control of Network Systems, vol. 8, no. 2, pp. 576–586, 2021. (Gao and Caines, 2021)

S. Gao, R. Foguen Tchuendom, and P. E. Caines, “Linear quadratic graphon field games,” Communications in Information and Systems, vol. 21, no. 3, pp. 341–369, 2021. (Gao et al., 2021b)

D. Firoozi, and P. E. Caines, “ϵ-Nash Equilibria for Major-Minor LQG Mean Field Games with Partial Observations of All Agents”, *IEEE Trans.on Automatic Control*, July, 2020, Vol 66, No. 6, pp 2778 - 2786. On-line: DOI: 10.1109/TAC.2020.3010129 (Firoozi and Caines, 2020)

P. E. Caines. Mean field games. In Encyclopedia of Systems and Control, pages 1197–1202. Springer, 2021. (Caines, 2021)

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D. Firoozi, S. Jaimungal, and P. E. Caines. Convex analysis for lqg systems with applications to major–minor LQG mean–field game systems. Systems & Control Letters, 142:104734, 2020. (Firoozi et al., 2020)

S. Gao and P. E. Caines. Graphon control of large-scale networks of linear systems. IEEE Transactions on Automatic Control, 65(10):4090–4105, 2020 (Gao and Caines, 2020)

N. Sen and P. E. Caines. Mean field games with partial observation. SIAM Journal on Control and Optimization, 57(3):2064–2091, 2019 (Sen and Caines, 2019)

P. E. Caines, D. Ho, and Q. Song. The density evolution of the killed mckean–vlasov process. Stochastics, 92(4):642–657, 2020. (Caines et al., 2020)

3.2 Publications: Conference Proceedings

A. Dunyak and P. E. Caines, “Linear stochastic graphon systems with q- space noise,” in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 3926–3932. (Dunyak and Caines, 2022)

P. E. Caines, “Embedded vertexon-graphons and embedded GMFG systems,” Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022,

pp. 5550–5557. (Caines, 2022)

S. Gao, "Fixed-point centrality for networks," in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 1628–1635. (Gao, 2022).

R. Foguen Tchuendom, S. Gao, P. E. Caines, and M. Huang, "Optimal network location in infinite horizon LQG graphon mean field games," in Proceedings of the 61th IEEE Conference on Decision and Control, Cancun, Mexico, December 2022, pp. 5558–5565. (Foguen Tchuendom et al., 2022b)

R. Foguen Tchuendom, S. Gao, and P. E. Caines "Stationary cost nodes in infinite horizon LQG GMFGs," IFAC-PapersOnLine for the 25th International Symposium on Mathematical Theory of Networks and Systems (MTNS), vol. 55, no. 30, pp. 284–289, 2022. (Foguen Tchuendom et al., 2022a)

J. Saude and P. E. Caines, "Markowitz portfolio optimization extended quadratic mean- field games approach," Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 5523–5528. (Saude and Caines, 2022)

Y. V. Salii, "Benchmarking optimal control for network dynamic systems with plausible epidemic models," in International Conference on Complex Networks and Their Applications. Springer, 2021, pp. 194–206. (Salii, 2021)

S. Gao, P. E. Caines, and M. Huang, "LQG graphon mean field games: Graphon invariant subspaces," Proceedings of the 60th IEEE Conference on Decision and Control, Austin, Texas, USA, December 2021, pp. 5253–5260. (Gao et al., 2021a)

R. Foguen Tchuendom, P. E. Caines, and M. Huang, "Critical nodes in graphon mean field games," Proceedings of the 60th IEEE Conference on Decision and Control (CDC). IEEE, 2021, pp. 166–170. (Foguen Tchuendom et al., 2021)

A. Dunyak and P. E. Caines, "Large scale systems and SIR models: A featured graphon approach," Proceedings of the 60th IEEE Conference on Decision and Control (CDC), 2021, pp. 6928–6933. (Dunyak and Caines, 2021)

S. Gao, "Centrality-weighted opinion dynamics: Disagreement and social network partition," Proceedings of the 60th IEEE Conference on Decision and Control (CDC), Austin, Texas, USA, December 2021, pp. 5496–5501. (Gao, 2021)

R. Foguen Tchuendom, P. E. Caines, and M. Huang. "On the master equation for linear quadratic graphon mean field games", Proceedings of the 59th

IEEE Conference on Decision and Control (CDC), pages 1026–1031, 2020. (Foguen Tchuendom et al., 2020)

S. Gao and P. E. Caines, Optimal and approximate solutions to linear quadratic regulation of a class of graphon dynamical systems, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 8359–8365, Nice, France, December 2019b. (Gao and Caines, 2019b)

S. Gao and P. E. Caines, Spectral representations of graphons in very large network systems control, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 5068–5075, Nice, France, December 2019a.(Gao and Caines, 2019a)

P. E. Caines and M. Huang. Graphon mean field games and the GMFG equations: ϵ-Nash equilibria, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 286–292, December 2019. (Caines and Huang, 2019)

3.3 Talks: April 1st, 2019 to December 31st, 2022

1. P. E. Caines,"Vertexons and Embedded Graphon Mean Field Games", the Simons Institute for the Theory of Computing, UC Berkeley, CA, Sept.29, 2022 (Invited talk) in the "Graph Limits, Nonparametric Models, and Estimation Workshop" 26 - 30 September, 2022.

2. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", (Invited talk) 29th August, 2022, in the Conference on Many Player Games and Applications, Humbolt University Berlin, 29 - 31 August 2022.

3. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Large Populations on Complex Networks", Canadian Applied and Industrial Mathematics Society (CAIMS), June 23rd, 2021 (Plenary talk).

4. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", the Financial Mathematics/Engineering Seminar Series, Hong Kong Polytechnic University, June 16th 2021 (Invited talk).

5. P. E. Caines, "Static and Time Varying Graphons in Economic and Pandemic GMFG Models" SIAM Financial Mathematics and Engineering (FM21), June 1, 2021 (Invited talk).

6. S. Gao, "Subspace Decompositions in Graphon Control and Graphon Mean Field Games", Informal Systems Seminar, Centre for Intelligent Machines,

McGill University, Groupe d'Etudes et de Recherche en Analyse des Decisions, Montreal, CA, Dec. 11, 2020 (Invited talk).

7. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Networked Systems", Invited Speaker at the "Conference on Mean Field Games and Related Topics - 5". 9th - 13th September, 2019

8. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for a Networked World", University of Saint Petersburg, Russia, 6th September, 2019 (Distinguished Speaker).

9. P. E. Caines, "Graphon Control and Graphon Mean Field Games", Department of Applied Mathemaics, Hong Kong Polythenic Universty, Hong Kong), 28th June, 2019 (Distinguished Lecturer).

3.4 Workshop Organization: April 1st, 2019 to December 31st, 2022

Name: **Mean Field Games on Networks Workshop**

Location: Vancouver, BC, Canada (Virtual Meeting)

Dates: October 26 - 29, 2021

Organizers: Peter E. Caines, Shuang Gao, Minyi Huang, and Rinel Foguen Tchuendom

This workshop (PIMS et al., 2021) brought together researchers in applied mathematics, mean field games, network science, network games, and systems and control theory to exchange ideas and to work on the extensions of mean field game theory to dynamic game problems on heterogeneous large-scale networks. The workshop offered a platform to current results and stimulate discussions on the open challenges in this emerging field, on mathematical theory, computational algorithms, and the applications methodologies of MFG on general networks.

Sponsors: Pacific Institute for the Mathematical Sciences (PIMS), Groupe d'études et de recherche en analyse des décisions (GERAD), and the Fields Institute for Research in Mathematical Sciences.

Participants: Peter E. Caines (PI) and PI's Postdoctoral Researchers: Rinel Foguen Tchuendom, Shuang Gao, Joao Saude, and Yaroslav Salii, plus 50 other international participants.

4 Impacts

As stated earlier, the research on MFG and related topics by the Principal Investigator of the grant FA9550-19-1-0138 has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, presently 2022 - 2025. Consequently, the impacts of research under the 2019 - 2022 grant, FA9550-19-1-0138, may be viewed as a continuation and development of those of the entire program.

Indeed, as explained in Section 1, the MFG field has expanded enormously since its inception. Google links now run to 1.10 billion, citations to papers are in the thousands and conferences and workshops on MFG have proliferated. A sample of the most recent workshops and conference sessions is the following, where we note that the PI and group members have been invited to participate in all of them: Workshop “Mean Field Games on Networks”, Pacific Institute for the Mathematical Sciences, Vancouver, BC, Canada, 26 - 29, October, 2021; “Conference on Mathematics and Financial Economics, “Many Player Games and Applications”, Humbolt University, Berlin, 29 - 31 August 2022; Workshop on Mean Field Games, Montreal: Centre de Recherches Mathématiques (CRM), 11-15 April, 2022; “Graph Limits, Nonparametric Models, and Estimation Workshop”, Simons Institute for the Theory of Computation, UCB, Berkeley, CA, 26-30 September, 2022; IEEE, Control Systems Society Conference on Decision and Control, Cancun, December, 2022, Mean Field Games Session; Mean Field Games Invited Session, IFAC World Congress, Yokahama, Japan, July 2023.

Furthermore, at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively), the last appearing in the period covered by this report. In addition to the very rapid expansion of mathematical and theoretical investigations, applications of MFG theory have spread to a vast range of domains, a sample of the most recent being the optimization of decentralized electric vehicle charging (Foguen Tchuendom et al., 2019), algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Grover et al., 2018), AI reinforcement learning for MFG systems (Agarwal et al., 2022) and polymer science (Welch et al., 2019).

As is shown by this indicative list of publications, meetings and presentations, it is reasonable to assert that the influence of the AFOSR supported research carried out on Mean Field Games and, in particular, on Mean Field Games on networks, has had, and is having, an impact on fields ranging from the core area of systems and control science to economics, finance, energy generation and distribution, operations research, the life sciences and the physical sciences.

5 Changes

There were no changes in the project or its direction.

6 Technical Updates

There are no technical updates to report beyond the continuing research progress and dissemination of results described in report.

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New Directions in Mean Field Games: MFG Subpopulation Behaviours and Graphon-MFG Systems

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1 Overview of Research Objectives

1.1 Background and Overview

Very large networks linking dynamical agents are now ubiquitous and the need to analyse, design and control them is evident. The emergence of the graphon theory of large networks and their infinite limits has enabled the formulation of a theory of the centralized control of dynamical systems distributed on asymptotically infinite networks.

Large-scale systems of competing agents are frequently formulated as dynamic games, this however confronts the problem that for even small populations non-cooperative game theory is intractable. The 2006 breakthrough of Minyi Huang, Roland Malhamé and the PI of FA9550-19-1-0138, Peter Caines, was the introduction of Mean Field Game (MFG) theory (originally termed Nash Certainty Equivalence theory) which was also simultaneously discovered by J.M. Lasry and P.L. Lions.

Mean Field Game (MFG) theory analyzes the existence of Nash equilibria for dynamical games in terms of infinite populations of negligible agents forming the limits of the original finite population systems (Caines, 2020). The core of MFG theory is a pair of partial differential equations consisting of (i) the Hamilton-Jacobi-Bellman equation of optimal control for a generic agent, and (ii) the Fokker-Planck-Kolmogorov equation describing the controlled evolution of the state of a generic agent. These are linked by the generic agent's state

distribution, known as the system's mean field.

A fundamental aspect of MFG theory justifying its application in a vast range of cases is the ϵ-Nash property by which controls derived from the MFG equations yield near-equilibrium performance for finite systems for sufficiently large populations.

The MFG field has expanded tremendously since its inception. Google search results now run to 1.10 billion, citations to papers are in the thousands, conferences and workshops on MFG have proliferated and at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively). In addition to the explosion of mathematical and theoretical investigations over last 16 years, applications of MFG theory have spread to a vast range of domains including: optimization of decentralized EV charging (Ma et al., 2011; Zhu et al., 2016; Foguen Tchuendom et al., 2019); nonlinear system state estimation (Yang et al., 2013); systemic risk in banking (Carmona et al., 2013); cell phone energy management (Aziz and Caines, 2016); economics of energy production (Ludkovski and Sircar, 2015); macro-economics of growth (Gomes et al., 2016); cloud-based networks in information technology (Hanif et al., 2015); algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Nourian et al., 2011; Grover et al., 2018), AI reinforcement learning for MFG systems (Subramanian and Mahajan, 2019; Agarwal et al., 2022), deep learning applications to the solution of MFGs (Ruthotto et al., 2020) and polymer science (Welch et al., 2019).

In this context it is important to note that the research on MFG by the Principal Investigator has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, most recently, for the period 2022-2025.

1.2 Major Research Goals under Grant FA9550-19-1-0138

The program of research under FA9550-19-1-0138 is focused on the development of a new form of Mean Field Games called Graphon Mean Field Games (GMFGs); these are large population games defined on large, heterogeneous, networks and their asymptotic (graphon) limits.

In this setting, game theoretic equilibria are expressed in terms of the newly defined GMFG PDE equations, which are a sweeping generalization of the classical MFG PDEs.

In a closely related development, the theory of graphons is used as a basis for Graphon Linear Control (GLC) theory in which large populations of centrally controlled dynamical systems interacting over large, heterogeneous networks

may be analyzed and optimized in a tractable way.

Canonical Design Procedure for GMFG and GLC Systems

A major goal has been to generate large scale network game strategies and, respectively, control design procedures with the following standard pattern:

The first step is to consider the graphon limit of the network over which the individual systems are distributed and which constitutes the global system.

In the second step, the resulting graphon based GMFG equations, or, respectively, the graphon based infinite dimensional linear system optimization problems are specified. The GMFG equations are then solved to generate the Nash best response strategies or, respectively, the linear feedback controls are synthesized using the well established infinite dimensional linear systems control theory.

Finally, the best response strategies, or, respectively, the infinite dimensional linear feedback controls are approximated in such a way that they may be applied to the original complex finite systems.

The key feature of this strategy is the following: In both the network mean field game and the network control settings it is demonstrated that as the size of the network tends to infinity, the performance of the finite systems when they are subject to the infinite limit system best response game strategies, or, respectively, centralized control laws, yield pay-offs which converge to their Nash equilibrium, or, respectively, optimal, infinite system limit values.

1.3 Organization of the Sequence of Research Objectives

The work over the full reporting period has been organized with the following sequence of objectives:

(i) The establishment of a rigorous theory of the central results of Graphon Mean Field Game (GMFG) theory,

(ii) The generation of a specific Graphon Linear Control (GLC) theory for the control of large populations of dynamical linear system agents with quadratic performance functions on large, complex networks.

(iii) The development of an approximation theory of associated numerical methods, especially those based upon finite dimensional invariant subspaces of the graphon operator in the GLC case.

(iv) The pioneering development of the theory of GMFG systems embedded in Euclidean space; this underlies our currently developing variational analysis of agent performance in networks as a function of position, where this theory also includes sparse as well as dense networks.

(v) The development of an SIR epidemic model software which permits the simulation of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity, these models are constructed on a data base of real world US surface and air-traffic travel data.

2 Details of Accomplishments

2.1 Graphon Mean Field Game Theory: Foundations

The work in (Caines and Huang, 2021) (Caines and Huang, 2018), (Caines and Huang, 2019), established the foundations and central results for graphon mean field game theory for nonlinear systems, that is to say games of infinite populations on infinite heterogeneous networks as limits of the corresponding finite systems.

Specifically, in the conference papers (Caines and Huang, 2018), (Caines and Huang, 2019) Graphon Mean Field Games (GMFG) and the GMFG equations were introduced for the analysis of non-cooperative dynamical games on unbounded networks. In this formulation, asymptotically infinite sub-populations are located at the nodes of large heterogeneous networks, the asymptotic network limits then constitute a part of the analysis. In (Caines and Huang, 2021), the basic existence and uniqueness results were established for the GMFG equations, and a full derivation of the fundamental results was provided, including the key epsilon-Nash theory for GMFG systems which relates the infinite population equilibria on infinite networks to finite population equilibria on finite networks.

2.2 LQG GMFG Decomposition Methods

The theory of invariant spaces of linear operators developed in the papers (Gao and Caines, 2021) is applied in (Gao et al., 2021b), (Gao et al., 2021a) and (Gao et al., 2022) to yield a tractable methodology for generating decentralized game strategies for systems distributed over complex networks. The fundamental properties of the existence and uniqueness of solutions are established in (Gao et al., 2021b,a, 2022) in two principal ways: (i) Riccati equation decomposition and (ii) fixed point contraction theorem methods.

2.3 LQG GMFG Master Equation Analysis

First, using a probabilistic approach, solutions to the Linear Quadratic Graphon Mean Field Games are expressed as solutions to coupled Forward Backward Stochastic Differential Equations (FBSDEs) of McKean-Vlasov type. Next, the

existence of the so-called Master field, which allows for the decoupling of these FBSDEs is established. Finally, the infinite dimensional Partial Differential Equation (PDE), so-called Master Equation, is established for which the Master Field is a solution (see (Foguen Tchuendom et al., 2020)).

2.4 GMFG with Affine Dynamics

GMFG theory for nonlinear problems with individual affine dynamics has been established in Caines et al. (2022) and the Nash value properties for control affine systems has been analyzed in Caines et al. (2023).

2.5 Graphon Linear Systems Control with Approximation Theory

As explained above, the theory of graphons is further used as a basis upon which centrally controlled linear dynamical systems interacting over large, heterogeneous networks are analyzed in a tractable way.

The graphon control methodology is developed in (Gao and Caines, 2020, 2019a,b) to study control problems for arbitrary-size networks of linear dynamical systems. The complete treatment of problems involving well-posedness, controllability, approximation errors for state-to-state control and regulations is established in (Gao and Caines, 2020).

Scalable and tractable solution methods are established for a class of graphon control problem in (Gao and Caines, 2019a,b). Next, the graphon linear system control framework is extended to more general cases where the state of each individual agent on the network is multi-dimensional (Gao and Caines, 2021); moreover, this work provides a generalized solution method with an application to large-scale networks of coupled harmonic oscillators.

2.6 LQ Graphon Dynamical Systems and Control with Q-space Noise

The work in (Dunyak and Caines, 2022) extends the formulation and analysis of (Gao and Caines, 2021) to include stochastic processes in infinite dimensions as a random process source driving the network's controlled dynamical systems; this generalization involves the formulation of a form of graphon Q-space noise.

2.7 Critical Nodes for GMFG and Embedded GMFG

The works (Caines, 2022),(Foguen Tchuendom et al., 2021), (Foguen Tchuendom et al., 2022a), (Foguen Tchuendom et al., 2022b) and (Caines et al., 2023)

pioneer the theory of GMFG systems embedded in Euclidean space by constructing limit graphs and limit graphons for sequences of graphs sitting in Euclidean space. The crucial notion here is that for embedded graphs and graphons we may define the differentiation of functions on the embedded graph limits. In particular we formulate an extended version of mean field game systems, termed Embedded Graphon Mean Field Games (EGMFG), or Graphexon Mean Field Games (GXMFG), which are amenable to the application of the calculus to the optimization of node location in such systems, this being one of the main objectives of the new development. Moreover this formulation includes the case of sparse graph limits (see (Caines, 2022)) which is not covered by the standard graphon formulation.

2.8 MFG Theory and Related Stochastic Adaptive and Hybrid Control Theory

The paper (Pakniyat and Caines, 2021) constitutes the climax of several decades of work by my own group and many other researchers aimed at producing a rigorous, general, hybrid optimal systems control theory (HOCT). These and related results have great applicability in many domains of engineering and scientific modelling, and furthermore in economics and finance. In fact the paper (Firoozi et al., 2022) presents an extension of HOCT to stochastic MFG systems motivated by trading dynamics in finance.

In (Saude and Caines, 2022), we study the Markowitz dynamic portfolio problem, where an agent seeks to maximize its expected return while minimizing the variance of the return (risk). For the first time in this topic, the impact of the aggregate of the agents' trades is incorporated into the dynamics and costs using an extended mean-field games framework.

2.9 Data Based Protocols for Modelling Large Populations on Large Networks with Applications to SIR Problems

The initial investigation of applying graphon theory to the control of epidemic spread is proposed in (Gao and Caines, 2019a) where scalable solutions based on spectral approximation and linearization are established for controlling virus spreads on networks following a type of controlled network SIS model.
The work (Dunyak and Caines, 2021; Salii, 2021) includes the development of SIR epidemic models which permits the study of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity; the latter models were constructed on a data base of real world US surface and air-traffic travel data.

2.10 Centrality Analysis for Finite and Infinite Networks

This work advances the theory of centralities for dynamical systems on networks and its associated applications. In (Gao, 2022) a family of centralities characterized by fixed-points of networks is identified which includes the standard centralities (e.g. PageRank, Katz-Bonacich, eigen centralities) and equilbrirum cost for games on networks (e.g. Nash values of LQG GMFG). Variation bounds of centralities with respect to the network variations are established. In (Gao, 2021) network centralities have been utilized in the modelling of opinion dynamics on social networks and in identifying significant network partitions based upon opinion disagreement states.

3 Dissemination to Communities of Interest

Research findings and results have been published in the leading systems and control and networks journals including the SIAM Journal on Control and Optimization, IEEE Transactions on Automatic Control, Automatica, IEEE Transactions on Control of Network Systems, and in conferences proceedings including the IEEE Conference on Decision and Control.

3.1 Publications: Journals

S. Gao, P. E. Caines, and M. Huang, "LQG graphon mean field games: analysis via graphon invariant subspaces," Conditionally accepted by IEEE Transactions on Automatic Control, 2021. (Gao et al., 2022)

D. Firoozi, A. Pakniyat, and P. E. Caines, "A class of hybrid LQG mean field games with state-invariant switching and stopping strategies," Automatica, vol. 141, p. 110244, 2022. (Firoozi et al., 2022)

P. E. Caines, D. Ho, M. Huang, J. Jian, and Q. Song. On the graphon mean field game equations: Individual agent affine dynamics and mean field dependent performance functions. ESAIM: Control, Optimisation and Calculus of Variations, 28:24, 2022 (Caines et al., 2022)

P. E. Caines and M. Huang, "Graphon mean field games and their equations," SIAM Journal on Control and Optimization, vol. 59, no. 6, pp. 4373–4399, 2021. (Caines and Huang, 2021)

A. Pakniyat and P. E. Caines, "On the hybrid minimum principle: The Hamiltonian and adjoint boundary conditions," IEEE Transactions on Automatic Control, vol. 66, no. 3, pp. 1246–1253, 2021. (Pakniyat and Caines, 2021)

S. Gao and P. E. Caines, “Subspace decomposition for graphon LQR: Applications to VLSNs of harmonic oscillators,” IEEE Transactions on Control of Network Systems, vol. 8, no. 2, pp. 576–586, 2021. (Gao and Caines, 2021)

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S. Gao and P. E. Caines. Graphon control of large-scale networks of linear systems. IEEE Transactions on Automatic Control, 65(10):4090–4105, 2020 (Gao and Caines, 2020)

N. Sen and P. E. Caines. Mean field games with partial observation. SIAM Journal on Control and Optimization, 57(3):2064–2091, 2019 (Sen and Caines, 2019)

P. E. Caines, D. Ho, and Q. Song. The density evolution of the killed mckean–vlasov process. Stochastics, 92(4):642–657, 2020. (Caines et al., 2020)

3.2 Publications: Conference Proceedings

A. Dunyak and P. E. Caines, “Linear stochastic graphon systems with q- space noise,” in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 3926–3932. (Dunyak and Caines, 2022)

P. E. Caines, “Embedded vertexon-graphons and embedded GMFG systems,” Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022,

pp. 5550–5557. (Caines, 2022)

S. Gao, "Fixed-point centrality for networks," in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 1628–1635. (Gao, 2022).

R. Foguen Tchuendom, S. Gao, P. E. Caines, and M. Huang, "Optimal network location in infinite horizon LQG graphon mean field games," in Proceedings of the 61th IEEE Conference on Decision and Control, Cancun, Mexico, December 2022, pp. 5558–5565. (Foguen Tchuendom et al., 2022b)

R. Foguen Tchuendom, S. Gao, and P. E. Caines "Stationary cost nodes in infinite horizon LQG GMFGs," IFAC-PapersOnLine for the 25th International Symposium on Mathematical Theory of Networks and Systems (MTNS), vol. 55, no. 30, pp. 284–289, 2022. (Foguen Tchuendom et al., 2022a)

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R. Foguen Tchuendom, P. E. Caines, and M. Huang, "Critical nodes in graphon mean field games," Proceedings of the 60th IEEE Conference on Decision and Control (CDC). IEEE, 2021, pp. 166–170. (Foguen Tchuendom et al., 2021)

A. Dunyak and P. E. Caines, "Large scale systems and SIR models: A featured graphon approach," Proceedings of the 60th IEEE Conference on Decision and Control (CDC), 2021, pp. 6928–6933. (Dunyak and Caines, 2021)

S. Gao, "Centrality-weighted opinion dynamics: Disagreement and social network partition," Proceedings of the 60th IEEE Conference on Decision and Control (CDC), Austin, Texas, USA, December 2021, pp. 5496–5501. (Gao, 2021)

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S. Gao and P. E. Caines, Optimal and approximate solutions to linear quadratic regulation of a class of graphon dynamical systems, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 8359–8365, Nice, France, December 2019b. (Gao and Caines, 2019b)

S. Gao and P. E. Caines, Spectral representations of graphons in very large network systems control, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 5068–5075, Nice, France, December 2019a.(Gao and Caines, 2019a)

P. E. Caines and M. Huang. Graphon mean field games and the GMFG equations: ϵ-Nash equilibria, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 286–292, December 2019. (Caines and Huang, 2019)

3.3 Talks: April 1st, 2019 to December 31st, 2022

1. P. E. Caines,"Vertexons and Embedded Graphon Mean Field Games", the Simons Institute for the Theory of Computing, UC Berkeley, CA, Sept.29, 2022 (Invited talk) in the "Graph Limits, Nonparametric Models, and Estimation Workshop" 26 - 30 September, 2022.

2. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", (Invited talk) 29th August, 2022, in the Conference on Many Player Games and Applications, Humbolt University Berlin, 29 - 31 August 2022.

3. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Large Populations on Complex Networks", Canadian Applied and Industrial Mathematics Society (CAIMS), June 23rd, 2021 (Plenary talk).

4. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", the Financial Mathematics/Engineering Seminar Series, Hong Kong Polytechnic University, June 16th 2021 (Invited talk).

5. P. E. Caines, "Static and Time Varying Graphons in Economic and Pandemic GMFG Models" SIAM Financial Mathematics and Engineering (FM21), June 1, 2021 (Invited talk).

6. S. Gao, "Subspace Decompositions in Graphon Control and Graphon Mean Field Games", Informal Systems Seminar, Centre for Intelligent Machines,

McGill University, Groupe d'Etudes et de Recherche en Analyse des Decisions, Montreal, CA, Dec. 11, 2020 (Invited talk).

7. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Networked Systems", Invited Speaker at the "Conference on Mean Field Games and Related Topics - 5". 9th - 13th September, 2019

8. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for a Networked World", University of Saint Petersburg, Russia, 6th September, 2019 (Distinguished Speaker).

9. P. E. Caines, "Graphon Control and Graphon Mean Field Games", Department of Applied Mathemaics, Hong Kong Polythenic Universty, Hong Kong), 28th June, 2019 (Distinguished Lecturer).

3.4 Workshop Organization: April 1st, 2019 to December 31st, 2022

Name: **Mean Field Games on Networks Workshop**

Location: Vancouver, BC, Canada (Virtual Meeting)

Dates: October 26 - 29, 2021

Organizers: Peter E. Caines, Shuang Gao, Minyi Huang, and Rinel Foguen Tchuendom

This workshop (PIMS et al., 2021) brought together researchers in applied mathematics, mean field games, network science, network games, and systems and control theory to exchange ideas and to work on the extensions of mean field game theory to dynamic game problems on heterogeneous large-scale networks. The workshop offered a platform to current results and stimulate discussions on the open challenges in this emerging field, on mathematical theory, computational algorithms, and the applications methodologies of MFG on general networks.

Sponsors: Pacific Institute for the Mathematical Sciences (PIMS), Groupe d'études et de recherche en analyse des décisions (GERAD), and the Fields Institute for Research in Mathematical Sciences.

Participants: Peter E. Caines (PI) and PI's Postdoctoral Researchers: Rinel Foguen Tchuendom, Shuang Gao, Joao Saude, and Yaroslav Salii, plus 50 other international participants.

4 Impacts

As stated earlier, the research on MFG and related topics by the Principal Investigator of the grant FA9550-19-1-0138 has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, presently 2022 - 2025. Consequently, the impacts of research under the 2019 - 2022 grant, FA9550-19-1-0138, may be viewed as a continuation and development of those of the entire program.

Indeed, as explained in Section 1, the MFG field has expanded enormously since its inception. Google links now run to 1.10 billion, citations to papers are in the thousands and conferences and workshops on MFG have proliferated. A sample of the most recent workshops and conference sessions is the following, where we note that the PI and group members have been invited to participate in all of them: Workshop “Mean Field Games on Networks”, Pacific Institute for the Mathematical Sciences, Vancouver, BC, Canada, 26 - 29, October, 2021; “Conference on Mathematics and Financial Economics, “Many Player Games and Applications”, Humbolt University, Berlin, 29 - 31 August 2022; Workshop on Mean Field Games, Montreal: Centre de Recherches Mathématiques (CRM), 11-15 April, 2022; “Graph Limits, Nonparametric Models, and Estimation Workshop”, Simons Institute for the Theory of Computation, UCB, Berkeley, CA, 26-30 September, 2022; IEEE, Control Systems Society Conference on Decision and Control, Cancun, December, 2022, Mean Field Games Session; Mean Field Games Invited Session, IFAC World Congress, Yokahama, Japan, July 2023.

Furthermore, at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively), the last appearing in the period covered by this report. In addition to the very rapid expansion of mathematical and theoretical investigations, applications of MFG theory have spread to a vast range of domains, a sample of the most recent being the optimization of decentralized electric vehicle charging (Foguen Tchuendom et al., 2019), algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Grover et al., 2018), AI reinforcement learning for MFG systems (Agarwal et al., 2022) and polymer science (Welch et al., 2019).

As is shown by this indicative list of publications, meetings and presentations, it is reasonable to assert that the influence of the AFOSR supported research carried out on Mean Field Games and, in particular, on Mean Field Games on networks, has had, and is having, an impact on fields ranging from the core area of systems and control science to economics, finance, energy generation and distribution, operations research, the life sciences and the physical sciences.

5 Changes

There were no changes in the project or its direction.

6 Technical Updates

There are no technical updates to report beyond the continuing research progress and dissemination of results described in report.

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AFOSR Final Report 2023-03-30
Grant FA9550-19-1-0138
New Directions in Mean Field Games: MFG Subpopulation Behaviours and Graphon-MFG Systems

Peter E. Caines

Reporting period April 1st, 2019 to December 31st, 2022

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1 Overview of Research Objectives

1.1 Background and Overview

Very large networks linking dynamical agents are now ubiquitous and the need to analyse, design and control them is evident. The emergence of the graphon theory of large networks and their infinite limits has enabled the formulation of a theory of the centralized control of dynamical systems distributed on asymptotically infinite networks.

Large-scale systems of competing agents are frequently formulated as dynamic games, this however confronts the problem that for even small populations non-cooperative game theory is intractable. The 2006 breakthrough of Minyi Huang, Roland Malhamé and the PI of FA9550-19-1-0138, Peter Caines, was the introduction of Mean Field Game (MFG) theory (originally termed Nash Certainty Equivalence theory) which was also simultaneously discovered by J.M. Lasry and P.L. Lions.

Mean Field Game (MFG) theory analyzes the existence of Nash equilibria for dynamical games in terms of infinite populations of negligible agents forming the limits of the original finite population systems (Caines, 2020). The core of MFG theory is a pair of partial differential equations consisting of (i) the Hamilton-Jacobi-Bellman equation of optimal control for a generic agent, and (ii) the Fokker-Planck-Kolmogorov equation describing the controlled evolution of the state of a generic agent. These are linked by the generic agent's state

distribution, known as the system's mean field.

A fundamental aspect of MFG theory justifying its application in a vast range of cases is the ϵ-Nash property by which controls derived from the MFG equations yield near-equilibrium performance for finite systems for sufficiently large populations.

The MFG field has expanded tremendously since its inception. Google search results now run to 1.10 billion, citations to papers are in the thousands, conferences and workshops on MFG have proliferated and at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively). In addition to the explosion of mathematical and theoretical investigations over last 16 years, applications of MFG theory have spread to a vast range of domains including: optimization of decentralized EV charging (Ma et al., 2011; Zhu et al., 2016; Foguen Tchuendom et al., 2019); nonlinear system state estimation (Yang et al., 2013); systemic risk in banking (Carmona et al., 2013); cell phone energy management (Aziz and Caines, 2016); economics of energy production (Ludkovski and Sircar, 2015); macro-economics of growth (Gomes et al., 2016); cloud-based networks in information technology (Hanif et al., 2015); algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Nourian et al., 2011; Grover et al., 2018), AI reinforcement learning for MFG systems (Subramanian and Mahajan, 2019; Agarwal et al., 2022), deep learning applications to the solution of MFGs (Ruthotto et al., 2020) and polymer science (Welch et al., 2019).

In this context it is important to note that the research on MFG by the Principal Investigator has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, most recently, for the period 2022-2025.

1.2 Major Research Goals under Grant FA9550-19-1-0138

The program of research under FA9550-19-1-0138 is focused on the development of a new form of Mean Field Games called Graphon Mean Field Games (GMFGs); these are large population games defined on large, heterogeneous, networks and their asymptotic (graphon) limits.

In this setting, game theoretic equilibria are expressed in terms of the newly defined GMFG PDE equations, which are a sweeping generalization of the classical MFG PDEs.

In a closely related development, the theory of graphons is used as a basis for Graphon Linear Control (GLC) theory in which large populations of centrally controlled dynamical systems interacting over large, heterogeneous networks

may be analyzed and optimized in a tractable way.

Canonical Design Procedure for GMFG and GLC Systems

A major goal has been to generate large scale network game strategies and, respectively, control design procedures with the following standard pattern:

The first step is to consider the graphon limit of the network over which the individual systems are distributed and which constitutes the global system.

In the second step, the resulting graphon based GMFG equations, or, respectively, the graphon based infinite dimensional linear system optimization problems are specified. The GMFG equations are then solved to generate the Nash best response strategies or, respectively, the linear feedback controls are synthesized using the well established infinite dimensional linear systems control theory.

Finally, the best response strategies, or, respectively, the infinite dimensional linear feedback controls are approximated in such a way that they may be applied to the original complex finite systems.

The key feature of this strategy is the following: In both the network mean field game and the network control settings it is demonstrated that as the size of the network tends to infinity, the performance of the finite systems when they are subject to the infinite limit system best response game strategies, or, respectively, centralized control laws, yield pay-offs which converge to their Nash equilibrium, or, respectively, optimal, infinite system limit values.

1.3 Organization of the Sequence of Research Objectives

The work over the full reporting period has been organized with the following sequence of objectives:

(i) The establishment of a rigorous theory of the central results of Graphon Mean Field Game (GMFG) theory,

(ii) The generation of a specific Graphon Linear Control (GLC) theory for the control of large populations of dynamical linear system agents with quadratic performance functions on large, complex networks.

(iii) The development of an approximation theory of associated numerical methods, especially those based upon finite dimensional invariant subspaces of the graphon operator in the GLC case.

(iv) The pioneering development of the theory of GMFG systems embedded in Euclidean space; this underlies our currently developing variational analysis of agent performance in networks as a function of position, where this theory also includes sparse as well as dense networks.

(v) The development of an SIR epidemic model software which permits the simulation of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity, these models are constructed on a data base of real world US surface and air-traffic travel data.

2 Details of Accomplishments

2.1 Graphon Mean Field Game Theory: Foundations

The work in (Caines and Huang, 2021) (Caines and Huang, 2018), (Caines and Huang, 2019), established the foundations and central results for graphon mean field game theory for nonlinear systems, that is to say games of infinite populations on infinite heterogeneous networks as limits of the corresponding finite systems.

Specifically, in the conference papers (Caines and Huang, 2018), (Caines and Huang, 2019) Graphon Mean Field Games (GMFG) and the GMFG equations were introduced for the analysis of non-cooperative dynamical games on unbounded networks. In this formulation, asymptotically infinite sub-populations are located at the nodes of large heterogeneous networks, the asymptotic network limits then constitute a part of the analysis. In (Caines and Huang, 2021), the basic existence and uniqueness results were established for the GMFG equations, and a full derivation of the fundamental results was provided, including the key epsilon-Nash theory for GMFG systems which relates the infinite population equilibria on infinite networks to finite population equilibria on finite networks.

2.2 LQG GMFG Decomposition Methods

The theory of invariant spaces of linear operators developed in the papers (Gao and Caines, 2021) is applied in (Gao et al., 2021b), (Gao et al., 2021a) and (Gao et al., 2022) to yield a tractable methodology for generating decentralized game strategies for systems distributed over complex networks. The fundamental properties of the existence and uniqueness of solutions are established in (Gao et al., 2021b,a, 2022) in two principal ways: (i) Riccati equation decomposition and (ii) fixed point contraction theorem methods.

2.3 LQG GMFG Master Equation Analysis

First, using a probabilistic approach, solutions to the Linear Quadratic Graphon Mean Field Games are expressed as solutions to coupled Forward Backward Stochastic Differential Equations (FBSDEs) of McKean-Vlasov type. Next, the

existence of the so-called Master field, which allows for the decoupling of these FBSDEs is established. Finally, the infinite dimensional Partial Differential Equation (PDE), so-called Master Equation, is established for which the Master Field is a solution (see (Foguen Tchuendom et al., 2020)).

2.4 GMFG with Affine Dynamics

GMFG theory for nonlinear problems with individual affine dynamics has been established in Caines et al. (2022) and the Nash value properties for control affine systems has been analyzed in Caines et al. (2023).

2.5 Graphon Linear Systems Control with Approximation Theory

As explained above, the theory of graphons is further used as a basis upon which centrally controlled linear dynamical systems interacting over large, heterogeneous networks are analyzed in a tractable way.

The graphon control methodology is developed in (Gao and Caines, 2020, 2019a,b) to study control problems for arbitrary-size networks of linear dynamical systems. The complete treatment of problems involving well-posedness, controllability, approximation errors for state-to-state control and regulations is established in (Gao and Caines, 2020).

Scalable and tractable solution methods are established for a class of graphon control problem in (Gao and Caines, 2019a,b). Next, the graphon linear system control framework is extended to more general cases where the state of each individual agent on the network is multi-dimensional (Gao and Caines, 2021); moreover, this work provides a generalized solution method with an application to large-scale networks of coupled harmonic oscillators.

2.6 LQ Graphon Dynamical Systems and Control with Q-space Noise

The work in (Dunyak and Caines, 2022) extends the formulation and analysis of (Gao and Caines, 2021) to include stochastic processes in infinite dimensions as a random process source driving the network's controlled dynamical systems; this generalization involves the formulation of a form of graphon Q-space noise.

2.7 Critical Nodes for GMFG and Embedded GMFG

The works (Caines, 2022),(Foguen Tchuendom et al., 2021), (Foguen Tchuendom et al., 2022a), (Foguen Tchuendom et al., 2022b) and (Caines et al., 2023)

pioneer the theory of GMFG systems embedded in Euclidean space by constructing limit graphs and limit graphons for sequences of graphs sitting in Euclidean space. The crucial notion here is that for embedded graphs and graphons we may define the differentiation of functions on the embedded graph limits. In particular we formulate an extended version of mean field game systems, termed Embedded Graphon Mean Field Games (EGMFG), or Graphexon Mean Field Games (GXMFG), which are amenable to the application of the calculus to the optimization of node location in such systems, this being one of the main objectives of the new development. Moreover this formulation includes the case of sparse graph limits (see (Caines, 2022)) which is not covered by the standard graphon formulation.

2.8 MFG Theory and Related Stochastic Adaptive and Hybrid Control Theory

The paper (Pakniyat and Caines, 2021) constitutes the climax of several decades of work by my own group and many other researchers aimed at producing a rigorous, general, hybrid optimal systems control theory (HOCT). These and related results have great applicability in many domains of engineering and scientific modelling, and furthermore in economics and finance. In fact the paper (Firoozi et al., 2022) presents an extension of HOCT to stochastic MFG systems motivated by trading dynamics in finance.

In (Saude and Caines, 2022), we study the Markowitz dynamic portfolio problem, where an agent seeks to maximize its expected return while minimizing the variance of the return (risk). For the first time in this topic, the impact of the aggregate of the agents' trades is incorporated into the dynamics and costs using an extended mean-field games framework.

2.9 Data Based Protocols for Modelling Large Populations on Large Networks with Applications to SIR Problems

The initial investigation of applying graphon theory to the control of epidemic spread is proposed in (Gao and Caines, 2019a) where scalable solutions based on spectral approximation and linearization are established for controlling virus spreads on networks following a type of controlled network SIS model.
The work (Dunyak and Caines, 2021; Salii, 2021) includes the development of SIR epidemic models which permits the study of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity; the latter models were constructed on a data base of real world US surface and air-traffic travel data.

2.10 Centrality Analysis for Finite and Infinite Networks

This work advances the theory of centralities for dynamical systems on networks and its associated applications. In (Gao, 2022) a family of centralities characterized by fixed-points of networks is identified which includes the standard centralities (e.g. PageRank, Katz-Bonacich, eigen centralities) and equilbrirum cost for games on networks (e.g. Nash values of LQG GMFG). Variation bounds of centralities with respect to the network variations are established. In (Gao, 2021) network centralities have been utilized in the modelling of opinion dynamics on social networks and in identifying significant network partitions based upon opinion disagreement states.

3 Dissemination to Communities of Interest

Research findings and results have been published in the leading systems and control and networks journals including the SIAM Journal on Control and Optimization, IEEE Transactions on Automatic Control, Automatica, IEEE Transactions on Control of Network Systems, and in conferences proceedings including the IEEE Conference on Decision and Control.

3.1 Publications: Journals

S. Gao, P. E. Caines, and M. Huang, "LQG graphon mean field games: analysis via graphon invariant subspaces," Conditionally accepted by IEEE Transactions on Automatic Control, 2021. (Gao et al., 2022)

D. Firoozi, A. Pakniyat, and P. E. Caines, "A class of hybrid LQG mean field games with state-invariant switching and stopping strategies," Automatica, vol. 141, p. 110244, 2022. (Firoozi et al., 2022)

P. E. Caines, D. Ho, M. Huang, J. Jian, and Q. Song. On the graphon mean field game equations: Individual agent affine dynamics and mean field dependent performance functions. ESAIM: Control, Optimisation and Calculus of Variations, 28:24, 2022 (Caines et al., 2022)

P. E. Caines and M. Huang, "Graphon mean field games and their equations," SIAM Journal on Control and Optimization, vol. 59, no. 6, pp. 4373–4399, 2021. (Caines and Huang, 2021)

A. Pakniyat and P. E. Caines, "On the hybrid minimum principle: The Hamiltonian and adjoint boundary conditions," IEEE Transactions on Automatic Control, vol. 66, no. 3, pp. 1246–1253, 2021. (Pakniyat and Caines, 2021)

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3.2 Publications: Conference Proceedings

A. Dunyak and P. E. Caines, "Linear stochastic graphon systems with q- space noise," in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 3926–3932. (Dunyak and Caines, 2022)

P. E. Caines, "Embedded vertexon-graphons and embedded GMFG systems," Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022,

pp. 5550–5557. (Caines, 2022)

S. Gao, “Fixed-point centrality for networks,” in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 1628–1635. (Gao, 2022).

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3.3 Talks: April 1st, 2019 to December 31st, 2022

1. P. E. Caines,"Vertexons and Embedded Graphon Mean Field Games", the Simons Institute for the Theory of Computing, UC Berkeley, CA, Sept.29, 2022 (Invited talk) in the "Graph Limits, Nonparametric Models, and Estimation Workshop" 26 - 30 September, 2022.

2. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", (Invited talk) 29th August, 2022, in the Conference on Many Player Games and Applications, Humbolt University Berlin, 29 - 31 August 2022.

3. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Large Populations on Complex Networks", Canadian Applied and Industrial Mathematics Society (CAIMS), June 23rd, 2021 (Plenary talk).

4. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", the Financial Mathematics/Engineering Seminar Series, Hong Kong Polytechnic University, June 16th 2021 (Invited talk).

5. P. E. Caines, "Static and Time Varying Graphons in Economic and Pandemic GMFG Models" SIAM Financial Mathematics and Engineering (FM21), June 1, 2021 (Invited talk).

6. S. Gao, "Subspace Decompositions in Graphon Control and Graphon Mean Field Games", Informal Systems Seminar, Centre for Intelligent Machines,

McGill University, Groupe d'Etudes et de Recherche en Analyse des Decisions, Montreal, CA, Dec. 11, 2020 (Invited talk).

7. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Networked Systems", Invited Speaker at the "Conference on Mean Field Games and Related Topics - 5". 9th - 13th September, 2019

8. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for a Networked World", University of Saint Petersburg, Russia, 6th September, 2019 (Distinguished Speaker).

9. P. E. Caines, "Graphon Control and Graphon Mean Field Games", Department of Applied Mathemaics, Hong Kong Polythenic Universty, Hong Kong), 28th June, 2019 (Distinguished Lecturer).

3.4 Workshop Organization: April 1st, 2019 to December 31st, 2022

Name: **Mean Field Games on Networks Workshop**

Location: Vancouver, BC, Canada (Virtual Meeting)

Dates: October 26 - 29, 2021

Organizers: Peter E. Caines, Shuang Gao, Minyi Huang, and Rinel Foguen Tchuendom

This workshop (PIMS et al., 2021) brought together researchers in applied mathematics, mean field games, network science, network games, and systems and control theory to exchange ideas and to work on the extensions of mean field game theory to dynamic game problems on heterogeneous large-scale networks. The workshop offered a platform to current results and stimulate discussions on the open challenges in this emerging field, on mathematical theory, computational algorithms, and the applications methodologies of MFG on general networks.

Sponsors: Pacific Institute for the Mathematical Sciences (PIMS), Groupe d'études et de recherche en analyse des décisions (GERAD), and the Fields Institute for Research in Mathematical Sciences.

Participants: Peter E. Caines (PI) and PI's Postdoctoral Researchers: Rinel Foguen Tchuendom, Shuang Gao, Joao Saude, and Yaroslav Salii, plus 50 other international participants.

4 Impacts

As stated earlier, the research on MFG and related topics by the Principal Investigator of the grant FA9550-19-1-0138 has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, presently 2022 - 2025. Consequently, the impacts of research under the 2019 - 2022 grant, FA9550-19-1-0138, may be viewed as a continuation and development of those of the entire program.

Indeed, as explained in Section 1, the MFG field has expanded enormously since its inception. Google links now run to 1.10 billion, citations to papers are in the thousands and conferences and workshops on MFG have proliferated. A sample of the most recent workshops and conference sessions is the following, where we note that the PI and group members have been invited to participate in all of them: Workshop "Mean Field Games on Networks", Pacific Institute for the Mathematical Sciences, Vancouver, BC, Canada, 26 - 29, October, 2021; "Conference on Mathematics and Financial Economics, "Many Player Games and Applications", Humbolt University, Berlin, 29 - 31 August 2022; Workshop on Mean Field Games, Montreal: Centre de Recherches Mathématiques (CRM), 11-15 April, 2022; "Graph Limits, Nonparametric Models, and Estimation Workshop", Simons Institute for the Theory of Computation, UCB, Berkeley, CA, 26-30 September, 2022; IEEE, Control Systems Society Conference on Decision and Control, Cancun, December, 2022, Mean Field Games Session; Mean Field Games Invited Session, IFAC World Congress, Yokahama, Japan, July 2023.

Furthermore, at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively), the last appearing in the period covered by this report. In addition to the very rapid expansion of mathematical and theoretical investigations, applications of MFG theory have spread to a vast range of domains, a sample of the most recent being the optimization of decentralized electric vehicle charging (Foguen Tchuendom et al., 2019), algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Grover et al., 2018), AI reinforcement learning for MFG systems (Agarwal et al., 2022) and polymer science (Welch et al., 2019).

As is shown by this indicative list of publications, meetings and presentations, it is reasonable to assert that the influence of the AFOSR supported research carried out on Mean Field Games and, in particular, on Mean Field Games on networks, has had, and is having, an impact on fields ranging from the core area of systems and control science to economics, finance, energy generation and distribution, operations research, the life sciences and the physical sciences.

5 Changes

There were no changes in the project or its direction.

6 Technical Updates

There are no technical updates to report beyond the continuing research progress and dissemination of results described in report.

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AFOSR Final Report 2023-03-30
Grant FA9550-19-1-0138
New Directions in Mean Field Games: MFG Subpopulation Behaviours and Graphon-MFG Systems

Peter E. Caines

Reporting period April 1st, 2019 to December 31st, 2022

March 24, 2023

1 Overview of Research Objectives

1.1 Background and Overview

Very large networks linking dynamical agents are now ubiquitous and the need to analyse, design and control them is evident. The emergence of the graphon theory of large networks and their infinite limits has enabled the formulation of a theory of the centralized control of dynamical systems distributed on asymptotically infinite networks.

Large-scale systems of competing agents are frequently formulated as dynamic games, this however confronts the problem that for even small populations non-cooperative game theory is intractable. The 2006 breakthrough of Minyi Huang, Roland Malhamé and the PI of FA9550-19-1-0138, Peter Caines, was the introduction of Mean Field Game (MFG) theory (originally termed Nash Certainty Equivalence theory) which was also simultaneously discovered by J.M. Lasry and P.L. Lions.

Mean Field Game (MFG) theory analyzes the existence of Nash equilibria for dynamical games in terms of infinite populations of negligible agents forming the limits of the original finite population systems (Caines, 2020). The core of MFG theory is a pair of partial differential equations consisting of (i) the Hamilton-Jacobi-Bellman equation of optimal control for a generic agent, and (ii) the Fokker-Planck-Kolmogorov equation describing the controlled evolution of the state of a generic agent. These are linked by the generic agent's state

distribution, known as the system's mean field.

A fundamental aspect of MFG theory justifying its application in a vast range of cases is the ϵ-Nash property by which controls derived from the MFG equations yield near-equilibrium performance for finite systems for sufficiently large populations.

The MFG field has expanded tremendously since its inception. Google search results now run to 1.10 billion, citations to papers are in the thousands, conferences and workshops on MFG have proliferated and at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively). In addition to the explosion of mathematical and theoretical investigations over last 16 years, applications of MFG theory have spread to a vast range of domains including: optimization of decentralized EV charging (Ma et al., 2011; Zhu et al., 2016; Foguen Tchuendom et al., 2019); nonlinear system state estimation (Yang et al., 2013); systemic risk in banking (Carmona et al., 2013); cell phone energy management (Aziz and Caines, 2016); economics of energy production (Ludkovski and Sircar, 2015); macro-economics of growth (Gomes et al., 2016); cloud-based networks in information technology (Hanif et al., 2015); algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Nourian et al., 2011; Grover et al., 2018), AI reinforcement learning for MFG systems (Subramanian and Mahajan, 2019; Agarwal et al., 2022), deep learning applications to the solution of MFGs (Ruthotto et al., 2020) and polymer science (Welch et al., 2019).

In this context it is important to note that the research on MFG by the Principal Investigator has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, most recently, for the period 2022-2025.

1.2 Major Research Goals under Grant FA9550-19-1-0138

The program of research under FA9550-19-1-0138 is focused on the development of a new form of Mean Field Games called Graphon Mean Field Games (GMFGs); these are large population games defined on large, heterogeneous, networks and their asymptotic (graphon) limits.

In this setting, game theoretic equilibria are expressed in terms of the newly defined GMFG PDE equations, which are a sweeping generalization of the classical MFG PDEs.

In a closely related development, the theory of graphons is used as a basis for Graphon Linear Control (GLC) theory in which large populations of centrally controlled dynamical systems interacting over large, heterogeneous networks

may be analyzed and optimized in a tractable way.

Canonical Design Procedure for GMFG and GLC Systems

A major goal has been to generate large scale network game strategies and, respectively, control design procedures with the following standard pattern:

The first step is to consider the graphon limit of the network over which the individual systems are distributed and which constitutes the global system.

In the second step, the resulting graphon based GMFG equations, or, respectively, the graphon based infinite dimensional linear system optimization problems are specified. The GMFG equations are then solved to generate the Nash best response strategies or, respectively, the linear feedback controls are synthesized using the well established infinite dimensional linear systems control theory.

Finally, the best response strategies, or, respectively, the infinite dimensional linear feedback controls are approximated in such a way that they may be applied to the original complex finite systems.

The key feature of this strategy is the following: In both the network mean field game and the network control settings it is demonstrated that as the size of the network tends to infinity, the performance of the finite systems when they are subject to the infinite limit system best response game strategies, or, respectively, centralized control laws, yield pay-offs which converge to their Nash equilibrium, or, respectively, optimal, infinite system limit values.

1.3 Organization of the Sequence of Research Objectives

The work over the full reporting period has been organized with the following sequence of objectives:

(i) The establishment of a rigorous theory of the central results of Graphon Mean Field Game (GMFG) theory,

(ii) The generation of a specific Graphon Linear Control (GLC) theory for the control of large populations of dynamical linear system agents with quadratic performance functions on large, complex networks.

(iii) The development of an approximation theory of associated numerical methods, especially those based upon finite dimensional invariant subspaces of the graphon operator in the GLC case.

(iv) The pioneering development of the theory of GMFG systems embedded in Euclidean space; this underlies our currently developing variational analysis of agent performance in networks as a function of position, where this theory also includes sparse as well as dense networks.

(v) The development of an SIR epidemic model software which permits the simulation of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity, these models are constructed on a data base of real world US surface and air-traffic travel data.

2 Details of Accomplishments

2.1 Graphon Mean Field Game Theory: Foundations

The work in (Caines and Huang, 2021) (Caines and Huang, 2018), (Caines and Huang, 2019), established the foundations and central results for graphon mean field game theory for nonlinear systems, that is to say games of infinite populations on infinite heterogeneous networks as limits of the corresponding finite systems.

Specifically, in the conference papers (Caines and Huang, 2018), (Caines and Huang, 2019) Graphon Mean Field Games (GMFG) and the GMFG equations were introduced for the analysis of non-cooperative dynamical games on unbounded networks. In this formulation, asymptotically infinite sub-populations are located at the nodes of large heterogeneous networks, the asymptotic network limits then constitute a part of the analysis. In (Caines and Huang, 2021), the basic existence and uniqueness results were established for the GMFG equations, and a full derivation of the fundamental results was provided, including the key epsilon-Nash theory for GMFG systems which relates the infinite population equilibria on infinite networks to finite population equilibria on finite networks.

2.2 LQG GMFG Decomposition Methods

The theory of invariant spaces of linear operators developed in the papers (Gao and Caines, 2021) is applied in (Gao et al., 2021b), (Gao et al., 2021a) and (Gao et al., 2022) to yield a tractable methodology for generating decentralized game strategies for systems distributed over complex networks. The fundamental properties of the existence and uniqueness of solutions are established in (Gao et al., 2021b,a, 2022) in two principal ways: (i) Riccati equation decomposition and (ii) fixed point contraction theorem methods.

2.3 LQG GMFG Master Equation Analysis

First, using a probabilistic approach, solutions to the Linear Quadratic Graphon Mean Field Games are expressed as solutions to coupled Forward Backward Stochastic Differential Equations (FBSDEs) of McKean-Vlasov type. Next, the

existence of the so-called Master field, which allows for the decoupling of these FBSDEs is established. Finally, the infinite dimensional Partial Differential Equation (PDE), so-called Master Equation, is established for which the Master Field is a solution (see (Foguen Tchuendom et al., 2020)).

2.4 GMFG with Affine Dynamics

GMFG theory for nonlinear problems with individual affine dynamics has been established in Caines et al. (2022) and the Nash value properties for control affine systems has been analyzed in Caines et al. (2023).

2.5 Graphon Linear Systems Control with Approximation Theory

As explained above, the theory of graphons is further used as a basis upon which centrally controlled linear dynamical systems interacting over large, heterogeneous networks are analyzed in a tractable way.

The graphon control methodology is developed in (Gao and Caines, 2020, 2019a,b) to study control problems for arbitrary-size networks of linear dynamical systems. The complete treatment of problems involving well-posedness, controllability, approximation errors for state-to-state control and regulations is established in (Gao and Caines, 2020).

Scalable and tractable solution methods are established for a class of graphon control problem in (Gao and Caines, 2019a,b). Next, the graphon linear system control framework is extended to more general cases where the state of each individual agent on the network is multi-dimensional (Gao and Caines, 2021); moreover, this work provides a generalized solution method with an application to large-scale networks of coupled harmonic oscillators.

2.6 LQ Graphon Dynamical Systems and Control with Q-space Noise

The work in (Dunyak and Caines, 2022) extends the formulation and analysis of (Gao and Caines, 2021) to include stochastic processes in infinite dimensions as a random process source driving the network's controlled dynamical systems; this generalization involves the formulation of a form of graphon Q-space noise.

2.7 Critical Nodes for GMFG and Embedded GMFG

The works (Caines, 2022),(Foguen Tchuendom et al., 2021), (Foguen Tchuendom et al., 2022a), (Foguen Tchuendom et al., 2022b) and (Caines et al., 2023)

pioneer the theory of GMFG systems embedded in Euclidean space by constructing limit graphs and limit graphons for sequences of graphs sitting in Euclidean space. The crucial notion here is that for embedded graphs and graphons we may define the differentiation of functions on the embedded graph limits. In particular we formulate an extended version of mean field game systems, termed Embedded Graphon Mean Field Games (EGMFG), or Graphexon Mean Field Games (GXMFG), which are amenable to the application of the calculus to the optimization of node location in such systems, this being one of the main objectives of the new development. Moreover this formulation includes the case of sparse graph limits (see (Caines, 2022)) which is not covered by the standard graphon formulation.

2.8 MFG Theory and Related Stochastic Adaptive and Hybrid Control Theory

The paper (Pakniyat and Caines, 2021) constitutes the climax of several decades of work by my own group and many other researchers aimed at producing a rigorous, general, hybrid optimal systems control theory (HOCT). These and related results have great applicability in many domains of engineering and scientific modelling, and furthermore in economics and finance. In fact the paper (Firoozi et al., 2022) presents an extension of HOCT to stochastic MFG systems motivated by trading dynamics in finance.

In (Saude and Caines, 2022), we study the Markowitz dynamic portfolio problem, where an agent seeks to maximize its expected return while minimizing the variance of the return (risk). For the first time in this topic, the impact of the aggregate of the agents' trades is incorporated into the dynamics and costs using an extended mean-field games framework.

2.9 Data Based Protocols for Modelling Large Populations on Large Networks with Applications to SIR Problems

The initial investigation of applying graphon theory to the control of epidemic spread is proposed in (Gao and Caines, 2019a) where scalable solutions based on spectral approximation and linearization are established for controlling virus spreads on networks following a type of controlled network SIS model.
The work (Dunyak and Caines, 2021; Salii, 2021) includes the development of SIR epidemic models which permits the study of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity; the latter models were constructed on a data base of real world US surface and air-traffic travel data.

2.10 Centrality Analysis for Finite and Infinite Networks

This work advances the theory of centralities for dynamical systems on networks and its associated applications. In (Gao, 2022) a family of centralities characterized by fixed-points of networks is identified which includes the standard centralities (e.g. PageRank, Katz-Bonacich, eigen centralities) and equilbrirum cost for games on networks (e.g. Nash values of LQG GMFG). Variation bounds of centralities with respect to the network variations are established. In (Gao, 2021) network centralities have been utilized in the modelling of opinion dynamics on social networks and in identifying significant network partitions based upon opinion disagreement states.

3 Dissemination to Communities of Interest

Research findings and results have been published in the leading systems and control and networks journals including the SIAM Journal on Control and Optimization, IEEE Transactions on Automatic Control, Automatica, IEEE Transactions on Control of Network Systems, and in conferences proceedings including the IEEE Conference on Decision and Control.

3.1 Publications: Journals

S. Gao, P. E. Caines, and M. Huang, "LQG graphon mean field games: analysis via graphon invariant subspaces," Conditionally accepted by IEEE Transactions on Automatic Control, 2021. (Gao et al., 2022)

D. Firoozi, A. Pakniyat, and P. E. Caines, "A class of hybrid LQG mean field games with state-invariant switching and stopping strategies," Automatica, vol. 141, p. 110244, 2022. (Firoozi et al., 2022)

P. E. Caines, D. Ho, M. Huang, J. Jian, and Q. Song. On the graphon mean field game equations: Individual agent affine dynamics and mean field dependent performance functions. ESAIM: Control, Optimisation and Calculus of Variations, 28:24, 2022 (Caines et al., 2022)

P. E. Caines and M. Huang, "Graphon mean field games and their equations," SIAM Journal on Control and Optimization, vol. 59, no. 6, pp. 4373–4399, 2021. (Caines and Huang, 2021)

A. Pakniyat and P. E. Caines, "On the hybrid minimum principle: The Hamiltonian and adjoint boundary conditions," IEEE Transactions on Automatic Control, vol. 66, no. 3, pp. 1246–1253, 2021. (Pakniyat and Caines, 2021)

S. Gao and P. E. Caines, "Subspace decomposition for graphon LQR: Applications to VLSNs of harmonic oscillators," IEEE Transactions on Control of Network Systems, vol. 8, no. 2, pp. 576–586, 2021. (Gao and Caines, 2021)

S. Gao, R. Foguen Tchuendom, and P. E. Caines, "Linear quadratic graphon field games," Communications in Information and Systems, vol. 21, no. 3, pp. 341–369, 2021. (Gao et al., 2021b)

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P. E. Caines. Mean field game theory: A tractable methodology for large population problems. SIAM News, 53(3):5–6, 2020. (Caines, 2020)

D. Firoozi, S. Jaimungal, and P. E. Caines. Convex analysis for lqg systems with applications to major–minor LQG mean–field game systems. Systems & Control Letters, 142:104734, 2020. (Firoozi et al., 2020)

S. Gao and P. E. Caines. Graphon control of large-scale networks of linear systems. IEEE Transactions on Automatic Control, 65(10):4090–4105, 2020 (Gao and Caines, 2020)

N. Sen and P. E. Caines. Mean field games with partial observation. SIAM Journal on Control and Optimization, 57(3):2064–2091, 2019 (Sen and Caines, 2019)

P. E. Caines, D. Ho, and Q. Song. The density evolution of the killed mckean–vlasov process. Stochastics, 92(4):642–657, 2020. (Caines et al., 2020)

3.2 Publications: Conference Proceedings

A. Dunyak and P. E. Caines, "Linear stochastic graphon systems with q- space noise," in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 3926–3932. (Dunyak and Caines, 2022)

P. E. Caines, "Embedded vertexon-graphons and embedded GMFG systems," Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022,

pp. 5550–5557. (Caines, 2022)

S. Gao, “Fixed-point centrality for networks,” in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 1628–1635. (Gao, 2022).

R. Foguen Tchuendom, S. Gao, P. E. Caines, and M. Huang, “Optimal network location in infinite horizon LQG graphon mean field games,” in Proceedings of the 61th IEEE Conference on Decision and Control, Cancun, Mexico, December 2022, pp. 5558–5565. (Foguen Tchuendom et al., 2022b)

R. Foguen Tchuendom, S. Gao, and P. E. Caines “Stationary cost nodes in infinite horizon LQG GMFGs,” IFAC-PapersOnLine for the 25th International Symposium on Mathematical Theory of Networks and Systems (MTNS), vol. 55, no. 30, pp. 284–289, 2022. (Foguen Tchuendom et al., 2022a)

J. Saude and P. E. Caines, “Markowitz portfolio optimization extended quadratic mean- field games approach,” Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 5523–5528. (Saude and Caines, 2022)

Y. V. Salii, “Benchmarking optimal control for network dynamic systems with plausible epidemic models,” in International Conference on Complex Networks and Their Applications. Springer, 2021, pp. 194–206. (Salii, 2021)

S. Gao, P. E. Caines, and M. Huang, “LQG graphon mean field games: Graphon invariant subspaces,” Proceedings of the 60th IEEE Conference on Decision and Control, Austin, Texas, USA, December 2021, pp. 5253–5260. (Gao et al., 2021a)

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S. Gao, “Centrality-weighted opinion dynamics: Disagreement and social network partition,” Proceedings of the 60th IEEE Conference on Decision and Control (CDC), Austin, Texas, USA, December 2021, pp. 5496–5501. (Gao, 2021)

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S. Gao and P. E. Caines, Optimal and approximate solutions to linear quadratic regulation of a class of graphon dynamical systems, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 8359–8365, Nice, France, December 2019b. (Gao and Caines, 2019b)

S. Gao and P. E. Caines, Spectral representations of graphons in very large network systems control, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 5068–5075, Nice, France, December 2019a.(Gao and Caines, 2019a)

P. E. Caines and M. Huang. Graphon mean field games and the GMFG equations: ϵ-Nash equilibria, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 286–292, December 2019. (Caines and Huang, 2019)

3.3 Talks: April 1st, 2019 to December 31st, 2022

1. P. E. Caines,"Vertexons and Embedded Graphon Mean Field Games", the Simons Institute for the Theory of Computing, UC Berkeley, CA, Sept.29, 2022 (Invited talk) in the "Graph Limits, Nonparametric Models, and Estimation Workshop" 26 - 30 September, 2022.

2. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", (Invited talk) 29th August, 2022, in the Conference on Many Player Games and Applications, Humbolt University Berlin, 29 - 31 August 2022.

3. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Large Populations on Complex Networks", Canadian Applied and Industrial Mathematics Society (CAIMS), June 23rd, 2021 (Plenary talk).

4. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", the Financial Mathematics/Engineering Seminar Series, Hong Kong Polytechnic University, June 16th 2021 (Invited talk).

5. P. E. Caines, "Static and Time Varying Graphons in Economic and Pandemic GMFG Models" SIAM Financial Mathematics and Engineering (FM21), June 1, 2021 (Invited talk).

6. S. Gao, "Subspace Decompositions in Graphon Control and Graphon Mean Field Games", Informal Systems Seminar, Centre for Intelligent Machines,

McGill University, Groupe d'Etudes et de Recherche en Analyse des Decisions, Montreal, CA, Dec. 11, 2020 (Invited talk).

7. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Networked Systems", Invited Speaker at the "Conference on Mean Field Games and Related Topics - 5". 9th - 13th September, 2019

8. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for a Networked World", University of Saint Petersburg, Russia, 6th September, 2019 (Distinguished Speaker).

9. P. E. Caines, "Graphon Control and Graphon Mean Field Games", Department of Applied Mathemaics, Hong Kong Polythenic Universty, Hong Kong), 28th June, 2019 (Distinguished Lecturer).

3.4 Workshop Organization: April 1st, 2019 to December 31st, 2022

Name: **Mean Field Games on Networks Workshop**

Location: Vancouver, BC, Canada (Virtual Meeting)

Dates: October 26 - 29, 2021

Organizers: Peter E. Caines, Shuang Gao, Minyi Huang, and Rinel Foguen Tchuendom

This workshop (PIMS et al., 2021) brought together researchers in applied mathematics, mean field games, network science, network games, and systems and control theory to exchange ideas and to work on the extensions of mean field game theory to dynamic game problems on heterogeneous large-scale networks. The workshop offered a platform to current results and stimulate discussions on the open challenges in this emerging field, on mathematical theory, computational algorithms, and the applications methodologies of MFG on general networks.

Sponsors: Pacific Institute for the Mathematical Sciences (PIMS), Groupe d'études et de recherche en analyse des décisions (GERAD), and the Fields Institute for Research in Mathematical Sciences.

Participants: Peter E. Caines (PI) and PI's Postdoctoral Researchers: Rinel Foguen Tchuendom, Shuang Gao, Joao Saude, and Yaroslav Salii, plus 50 other international participants.

4 Impacts

As stated earlier, the research on MFG and related topics by the Principal Investigator of the grant FA9550-19-1-0138 has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, presently 2022 - 2025. Consequently, the impacts of research under the 2019 - 2022 grant, FA9550-19-1-0138, may be viewed as a continuation and development of those of the entire program.

Indeed, as explained in Section 1, the MFG field has expanded enormously since its inception. Google links now run to 1.10 billion, citations to papers are in the thousands and conferences and workshops on MFG have proliferated. A sample of the most recent workshops and conference sessions is the following, where we note that the PI and group members have been invited to participate in all of them: Workshop "Mean Field Games on Networks", Pacific Institute for the Mathematical Sciences, Vancouver, BC, Canada, 26 - 29, October, 2021; "Conference on Mathematics and Financial Economics, "Many Player Games and Applications", Humbolt University, Berlin, 29 - 31 August 2022; Workshop on Mean Field Games, Montreal: Centre de Recherches Mathématiques (CRM), 11-15 April, 2022; "Graph Limits, Nonparametric Models, and Estimation Workshop", Simons Institute for the Theory of Computation, UCB, Berkeley, CA, 26-30 September, 2022; IEEE, Control Systems Society Conference on Decision and Control, Cancun, December, 2022, Mean Field Games Session; Mean Field Games Invited Session, IFAC World Congress, Yokahama, Japan, July 2023.

Furthermore, at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively), the last appearing in the period covered by this report. In addition to the very rapid expansion of mathematical and theoretical investigations, applications of MFG theory have spread to a vast range of domains, a sample of the most recent being the optimization of decentralized electric vehicle charging (Foguen Tchuendom et al., 2019), algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Grover et al., 2018), AI reinforcement learning for MFG systems (Agarwal et al., 2022) and polymer science (Welch et al., 2019).

As is shown by this indicative list of publications, meetings and presentations, it is reasonable to assert that the influence of the AFOSR supported research carried out on Mean Field Games and, in particular, on Mean Field Games on networks, has had, and is having, an impact on fields ranging from the core area of systems and control science to economics, finance, energy generation and distribution, operations research, the life sciences and the physical sciences.

5 Changes

There were no changes in the project or its direction.

6 Technical Updates

There are no technical updates to report beyond the continuing research progress and dissemination of results described in report.

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New Directions in Mean Field Games: MFG Subpopulation Behaviours and Graphon-MFG Systems

Peter E. Caines

Reporting period April 1st, 2019 to December 31st, 2022

March 24, 2023

1 Overview of Research Objectives

1.1 Background and Overview

Very large networks linking dynamical agents are now ubiquitous and the need to analyse, design and control them is evident. The emergence of the graphon theory of large networks and their infinite limits has enabled the formulation of a theory of the centralized control of dynamical systems distributed on asymptotically infinite networks.

Large-scale systems of competing agents are frequently formulated as dynamic games, this however confronts the problem that for even small populations non-cooperative game theory is intractable. The 2006 breakthrough of Minyi Huang, Roland Malhamé and the PI of FA9550-19-1-0138, Peter Caines, was the introduction of Mean Field Game (MFG) theory (originally termed Nash Certainty Equivalence theory) which was also simultaneously discovered by J.M. Lasry and P.L. Lions.

Mean Field Game (MFG) theory analyzes the existence of Nash equilibria for dynamical games in terms of infinite populations of negligible agents forming the limits of the original finite population systems (Caines, 2020). The core of MFG theory is a pair of partial differential equations consisting of (i) the Hamilton-Jacobi-Bellman equation of optimal control for a generic agent, and (ii) the Fokker-Planck-Kolmogorov equation describing the controlled evolution of the state of a generic agent. These are linked by the generic agent's state

distribution, known as the system's mean field.

A fundamental aspect of MFG theory justifying its application in a vast range of cases is the ϵ-Nash property by which controls derived from the MFG equations yield near-equilibrium performance for finite systems for sufficiently large populations.

The MFG field has expanded tremendously since its inception. Google search results now run to 1.10 billion, citations to papers are in the thousands, conferences and workshops on MFG have proliferated and at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively). In addition to the explosion of mathematical and theoretical investigations over last 16 years, applications of MFG theory have spread to a vast range of domains including: optimization of decentralized EV charging (Ma et al., 2011; Zhu et al., 2016; Foguen Tchuendom et al., 2019); nonlinear system state estimation (Yang et al., 2013); systemic risk in banking (Carmona et al., 2013); cell phone energy management (Aziz and Caines, 2016); economics of energy production (Ludkovski and Sircar, 2015); macro-economics of growth (Gomes et al., 2016); cloud-based networks in information technology (Hanif et al., 2015); algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Nourian et al., 2011; Grover et al., 2018), AI reinforcement learning for MFG systems (Subramanian and Mahajan, 2019; Agarwal et al., 2022), deep learning applications to the solution of MFGs (Ruthotto et al., 2020) and polymer science (Welch et al., 2019).

In this context it is important to note that the research on MFG by the Principal Investigator has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, most recently, for the period 2022-2025.

1.2 Major Research Goals under Grant FA9550-19-1-0138

The program of research under FA9550-19-1-0138 is focused on the development of a new form of Mean Field Games called Graphon Mean Field Games (GMFGs); these are large population games defined on large, heterogeneous, networks and their asymptotic (graphon) limits.

In this setting, game theoretic equilibria are expressed in terms of the newly defined GMFG PDE equations, which are a sweeping generalization of the classical MFG PDEs.

In a closely related development, the theory of graphons is used as a basis for Graphon Linear Control (GLC) theory in which large populations of centrally controlled dynamical systems interacting over large, heterogeneous networks

may be analyzed and optimized in a tractable way.

Canonical Design Procedure for GMFG and GLC Systems

A major goal has been to generate large scale network game strategies and, respectively, control design procedures with the following standard pattern:

The first step is to consider the graphon limit of the network over which the individual systems are distributed and which constitutes the global system.

In the second step, the resulting graphon based GMFG equations, or, respectively, the graphon based infinite dimensional linear system optimization problems are specified. The GMFG equations are then solved to generate the Nash best response strategies or, respectively, the linear feedback controls are synthesized using the well established infinite dimensional linear systems control theory.

Finally, the best response strategies, or, respectively, the infinite dimensional linear feedback controls are approximated in such a way that they may be applied to the original complex finite systems.

The key feature of this strategy is the following: In both the network mean field game and the network control settings it is demonstrated that as the size of the network tends to infinity, the performance of the finite systems when they are subject to the infinite limit system best response game strategies, or, respectively, centralized control laws, yield pay-offs which converge to their Nash equilibrium, or, respectively, optimal, infinite system limit values.

1.3 Organization of the Sequence of Research Objectives

The work over the full reporting period has been organized with the following sequence of objectives:

(i) The establishment of a rigorous theory of the central results of Graphon Mean Field Game (GMFG) theory,

(ii) The generation of a specific Graphon Linear Control (GLC) theory for the control of large populations of dynamical linear system agents with quadratic performance functions on large, complex networks.

(iii) The development of an approximation theory of associated numerical methods, especially those based upon finite dimensional invariant subspaces of the graphon operator in the GLC case.

(iv) The pioneering development of the theory of GMFG systems embedded in Euclidean space; this underlies our currently developing variational analysis of agent performance in networks as a function of position, where this theory also includes sparse as well as dense networks.

(v) The development of an SIR epidemic model software which permits the simulation of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity, these models are constructed on a data base of real world US surface and air-traffic travel data.

2 Details of Accomplishments

2.1 Graphon Mean Field Game Theory: Foundations

The work in (Caines and Huang, 2021) (Caines and Huang, 2018), (Caines and Huang, 2019), established the foundations and central results for graphon mean field game theory for nonlinear systems, that is to say games of infinite populations on infinite heterogeneous networks as limits of the corresponding finite systems.

Specifically, in the conference papers (Caines and Huang, 2018), (Caines and Huang, 2019) Graphon Mean Field Games (GMFG) and the GMFG equations were introduced for the analysis of non-cooperative dynamical games on unbounded networks. In this formulation, asymptotically infinite sub-populations are located at the nodes of large heterogeneous networks, the asymptotic network limits then constitute a part of the analysis. In (Caines and Huang, 2021), the basic existence and uniqueness results were established for the GMFG equations, and a full derivation of the fundamental results was provided, including the key epsilon-Nash theory for GMFG systems which relates the infinite population equilibria on infinite networks to finite population equilibria on finite networks.

2.2 LQG GMFG Decomposition Methods

The theory of invariant spaces of linear operators developed in the papers (Gao and Caines, 2021) is applied in (Gao et al., 2021b), (Gao et al., 2021a) and (Gao et al., 2022) to yield a tractable methodology for generating decentralized game strategies for systems distributed over complex networks. The fundamental properties of the existence and uniqueness of solutions are established in (Gao et al., 2021b,a, 2022) in two principal ways: (i) Riccati equation decomposition and (ii) fixed point contraction theorem methods.

2.3 LQG GMFG Master Equation Analysis

First, using a probabilistic approach, solutions to the Linear Quadratic Graphon Mean Field Games are expressed as solutions to coupled Forward Backward Stochastic Differential Equations (FBSDEs) of McKean-Vlasov type. Next, the

existence of the so-called Master field, which allows for the decoupling of these FBSDEs is established. Finally, the infinite dimensional Partial Differential Equation (PDE), so-called Master Equation, is established for which the Master Field is a solution (see (Foguen Tchuendom et al., 2020)).

2.4 GMFG with Affine Dynamics

GMFG theory for nonlinear problems with individual affine dynamics has been established in Caines et al. (2022) and the Nash value properties for control affine systems has been analyzed in Caines et al. (2023).

2.5 Graphon Linear Systems Control with Approximation Theory

As explained above, the theory of graphons is further used as a basis upon which centrally controlled linear dynamical systems interacting over large, heterogeneous networks are analyzed in a tractable way.

The graphon control methodology is developed in (Gao and Caines, 2020, 2019a,b) to study control problems for arbitrary-size networks of linear dynamical systems. The complete treatment of problems involving well-posedness, controllability, approximation errors for state-to-state control and regulations is established in (Gao and Caines, 2020).

Scalable and tractable solution methods are established for a class of graphon control problem in (Gao and Caines, 2019a,b). Next, the graphon linear system control framework is extended to more general cases where the state of each individual agent on the network is multi-dimensional (Gao and Caines, 2021); moreover, this work provides a generalized solution method with an application to large-scale networks of coupled harmonic oscillators.

2.6 LQ Graphon Dynamical Systems and Control with Q-space Noise

The work in (Dunyak and Caines, 2022) extends the formulation and analysis of (Gao and Caines, 2021) to include stochastic processes in infinite dimensions as a random process source driving the network's controlled dynamical systems; this generalization involves the formulation of a form of graphon Q-space noise.

2.7 Critical Nodes for GMFG and Embedded GMFG

The works (Caines, 2022),(Foguen Tchuendom et al., 2021), (Foguen Tchuendom et al., 2022a), (Foguen Tchuendom et al., 2022b) and (Caines et al., 2023)

pioneer the theory of GMFG systems embedded in Euclidean space by constructing limit graphs and limit graphons for sequences of graphs sitting in Euclidean space. The crucial notion here is that for embedded graphs and graphons we may define the differentiation of functions on the embedded graph limits. In particular we formulate an extended version of mean field game systems, termed Embedded Graphon Mean Field Games (EGMFG), or Graphexon Mean Field Games (GXMFG), which are amenable to the application of the calculus to the optimization of node location in such systems, this being one of the main objectives of the new development. Moreover this formulation includes the case of sparse graph limits (see (Caines, 2022)) which is not covered by the standard graphon formulation.

2.8 MFG Theory and Related Stochastic Adaptive and Hybrid Control Theory

The paper (Pakniyat and Caines, 2021) constitutes the climax of several decades of work by my own group and many other researchers aimed at producing a rigorous, general, hybrid optimal systems control theory (HOCT). These and related results have great applicability in many domains of engineering and scientific modelling, and furthermore in economics and finance. In fact the paper (Firoozi et al., 2022) presents an extension of HOCT to stochastic MFG systems motivated by trading dynamics in finance.

In (Saude and Caines, 2022), we study the Markowitz dynamic portfolio problem, where an agent seeks to maximize its expected return while minimizing the variance of the return (risk). For the first time in this topic, the impact of the aggregate of the agents' trades is incorporated into the dynamics and costs using an extended mean-field games framework.

2.9 Data Based Protocols for Modelling Large Populations on Large Networks with Applications to SIR Problems

The initial investigation of applying graphon theory to the control of epidemic spread is proposed in (Gao and Caines, 2019a) where scalable solutions based on spectral approximation and linearization are established for controlling virus spreads on networks following a type of controlled network SIS model.
The work (Dunyak and Caines, 2021; Salii, 2021) includes the development of SIR epidemic models which permits the study of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity; the latter models were constructed on a data base of real world US surface and air-traffic travel data.

2.10 Centrality Analysis for Finite and Infinite Networks

This work advances the theory of centralities for dynamical systems on networks and its associated applications. In (Gao, 2022) a family of centralities characterized by fixed-points of networks is identified which includes the standard centralities (e.g. PageRank, Katz-Bonacich, eigen centralities) and equilbrirum cost for games on networks (e.g. Nash values of LQG GMFG). Variation bounds of centralities with respect to the network variations are established. In (Gao, 2021) network centralities have been utilized in the modelling of opinion dynamics on social networks and in identifying significant network partitions based upon opinion disagreement states.

3 Dissemination to Communities of Interest

Research findings and results have been published in the leading systems and control and networks journals including the SIAM Journal on Control and Optimization, IEEE Transactions on Automatic Control, Automatica, IEEE Transactions on Control of Network Systems, and in conferences proceedings including the IEEE Conference on Decision and Control.

3.1 Publications: Journals

S. Gao, P. E. Caines, and M. Huang, "LQG graphon mean field games: analysis via graphon invariant subspaces," Conditionally accepted by IEEE Transactions on Automatic Control, 2021. (Gao et al., 2022)

D. Firoozi, A. Pakniyat, and P. E. Caines, "A class of hybrid LQG mean field games with state-invariant switching and stopping strategies," Automatica, vol. 141, p. 110244, 2022. (Firoozi et al., 2022)

P. E. Caines, D. Ho, M. Huang, J. Jian, and Q. Song. On the graphon mean field game equations: Individual agent affine dynamics and mean field dependent performance functions. ESAIM: Control, Optimisation and Calculus of Variations, 28:24, 2022 (Caines et al., 2022)

P. E. Caines and M. Huang, "Graphon mean field games and their equations," SIAM Journal on Control and Optimization, vol. 59, no. 6, pp. 4373–4399, 2021. (Caines and Huang, 2021)

A. Pakniyat and P. E. Caines, "On the hybrid minimum principle: The Hamiltonian and adjoint boundary conditions," IEEE Transactions on Automatic Control, vol. 66, no. 3, pp. 1246–1253, 2021. (Pakniyat and Caines, 2021)

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S. Gao, R. Foguen Tchuendom, and P. E. Caines, "Linear quadratic graphon field games," Communications in Information and Systems, vol. 21, no. 3, pp. 341–369, 2021. (Gao et al., 2021b)

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N. Sen and P. E. Caines. Mean field games with partial observation. SIAM Journal on Control and Optimization, 57(3):2064–2091, 2019 (Sen and Caines, 2019)

P. E. Caines, D. Ho, and Q. Song. The density evolution of the killed mckean–vlasov process. Stochastics, 92(4):642–657, 2020. (Caines et al., 2020)

3.2 Publications: Conference Proceedings

A. Dunyak and P. E. Caines, "Linear stochastic graphon systems with q- space noise," in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 3926–3932. (Dunyak and Caines, 2022)

P. E. Caines, "Embedded vertexon-graphons and embedded GMFG systems," Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022,

pp. 5550–5557. (Caines, 2022)

S. Gao, "Fixed-point centrality for networks," in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 1628–1635. (Gao, 2022).

R. Foguen Tchuendom, S. Gao, P. E. Caines, and M. Huang, "Optimal network location in infinite horizon LQG graphon mean field games," in Proceedings of the 61th IEEE Conference on Decision and Control, Cancun, Mexico, December 2022, pp. 5558–5565. (Foguen Tchuendom et al., 2022b)

R. Foguen Tchuendom, S. Gao, and P. E. Caines "Stationary cost nodes in infinite horizon LQG GMFGs," IFAC-PapersOnLine for the 25th International Symposium on Mathematical Theory of Networks and Systems (MTNS), vol. 55, no. 30, pp. 284–289, 2022. (Foguen Tchuendom et al., 2022a)

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Y. V. Salii, "Benchmarking optimal control for network dynamic systems with plausible epidemic models," in International Conference on Complex Networks and Their Applications. Springer, 2021, pp. 194–206. (Salii, 2021)

S. Gao, P. E. Caines, and M. Huang, "LQG graphon mean field games: Graphon invariant subspaces," Proceedings of the 60th IEEE Conference on Decision and Control, Austin, Texas, USA, December 2021, pp. 5253–5260. (Gao et al., 2021a)

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S. Gao, "Centrality-weighted opinion dynamics: Disagreement and social network partition," Proceedings of the 60th IEEE Conference on Decision and Control (CDC), Austin, Texas, USA, December 2021, pp. 5496–5501. (Gao, 2021)

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S. Gao and P. E. Caines, Optimal and approximate solutions to linear quadratic regulation of a class of graphon dynamical systems, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 8359–8365, Nice, France, December 2019b. (Gao and Caines, 2019b)

S. Gao and P. E. Caines, Spectral representations of graphons in very large network systems control, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 5068–5075, Nice, France, December 2019a.(Gao and Caines, 2019a)

P. E. Caines and M. Huang. Graphon mean field games and the GMFG equations: ϵ-Nash equilibria, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 286–292, December 2019. (Caines and Huang, 2019)

3.3 Talks: April 1st, 2019 to December 31st, 2022

1. P. E. Caines,"Vertexons and Embedded Graphon Mean Field Games", the Simons Institute for the Theory of Computing, UC Berkeley, CA, Sept.29, 2022 (Invited talk) in the "Graph Limits, Nonparametric Models, and Estimation Workshop" 26 - 30 September, 2022.

2. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", (Invited talk) 29th August, 2022, in the Conference on Many Player Games and Applications, Humbolt University Berlin, 29 - 31 August 2022.

3. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Large Populations on Complex Networks", Canadian Applied and Industrial Mathematics Society (CAIMS), June 23rd, 2021 (Plenary talk).

4. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", the Financial Mathematics/Engineering Seminar Series, Hong Kong Polytechnic University, June 16th 2021 (Invited talk).

5. P. E. Caines, "Static and Time Varying Graphons in Economic and Pandemic GMFG Models" SIAM Financial Mathematics and Engineering (FM21), June 1, 2021 (Invited talk).

6. S. Gao, "Subspace Decompositions in Graphon Control and Graphon Mean Field Games", Informal Systems Seminar, Centre for Intelligent Machines,

McGill University, Groupe d'Etudes et de Recherche en Analyse des Decisions, Montreal, CA, Dec. 11, 2020 (Invited talk).

7. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Networked Systems", Invited Speaker at the "Conference on Mean Field Games and Related Topics - 5". 9th - 13th September, 2019
8. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for a Networked World", University of Saint Petersburg, Russia, 6th September, 2019 (Distinguished Speaker).
9. P. E. Caines, "Graphon Control and Graphon Mean Field Games", Department of Applied Mathemaics, Hong Kong Polythenic Universty, Hong Kong), 28th June, 2019 (Distinguished Lecturer).

3.4 Workshop Organization: April 1st, 2019 to December 31st, 2022

Name: **Mean Field Games on Networks Workshop**

Location: Vancouver, BC, Canada (Virtual Meeting)

Dates: October 26 - 29, 2021

Organizers: Peter E. Caines, Shuang Gao, Minyi Huang, and Rinel Foguen Tchuendom

This workshop (PIMS et al., 2021) brought together researchers in applied mathematics, mean field games, network science, network games, and systems and control theory to exchange ideas and to work on the extensions of mean field game theory to dynamic game problems on heterogeneous large-scale networks. The workshop offered a platform to current results and stimulate discussions on the open challenges in this emerging field, on mathematical theory, computational algorithms, and the applications methodologies of MFG on general networks.

Sponsors: Pacific Institute for the Mathematical Sciences (PIMS), Groupe d'études et de recherche en analyse des décisions (GERAD), and the Fields Institute for Research in Mathematical Sciences.

Participants: Peter E. Caines (PI) and PI's Postdoctoral Researchers: Rinel Foguen Tchuendom, Shuang Gao, Joao Saude, and Yaroslav Salii, plus 50 other international participants.

4 Impacts

As stated earlier, the research on MFG and related topics by the Principal Investigator of the grant FA9550-19-1-0138 has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, presently 2022 - 2025. Consequently, the impacts of research under the 2019 - 2022 grant, FA9550-19-1-0138, may be viewed as a continuation and development of those of the entire program.

Indeed, as explained in Section 1, the MFG field has expanded enormously since its inception. Google links now run to 1.10 billion, citations to papers are in the thousands and conferences and workshops on MFG have proliferated. A sample of the most recent workshops and conference sessions is the following, where we note that the PI and group members have been invited to participate in all of them: Workshop "Mean Field Games on Networks", Pacific Institute for the Mathematical Sciences, Vancouver, BC, Canada, 26 - 29, October, 2021; "Conference on Mathematics and Financial Economics, "Many Player Games and Applications", Humbolt University, Berlin, 29 - 31 August 2022; Workshop on Mean Field Games, Montreal: Centre de Recherches Mathématiques (CRM), 11-15 April, 2022; "Graph Limits, Nonparametric Models, and Estimation Workshop", Simons Institute for the Theory of Computation, UCB, Berkeley, CA, 26-30 September, 2022; IEEE, Control Systems Society Conference on Decision and Control, Cancun, December, 2022, Mean Field Games Session; Mean Field Games Invited Session, IFAC World Congress, Yokahama, Japan, July 2023.

Furthermore, at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively), the last appearing in the period covered by this report. In addition to the very rapid expansion of mathematical and theoretical investigations, applications of MFG theory have spread to a vast range of domains, a sample of the most recent being the optimization of decentralized electric vehicle charging (Foguen Tchuendom et al., 2019), algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Grover et al., 2018), AI reinforcement learning for MFG systems (Agarwal et al., 2022) and polymer science (Welch et al., 2019).

As is shown by this indicative list of publications, meetings and presentations, it is reasonable to assert that the influence of the AFOSR supported research carried out on Mean Field Games and, in particular, on Mean Field Games on networks, has had, and is having, an impact on fields ranging from the core area of systems and control science to economics, finance, energy generation and distribution, operations research, the life sciences and the physical sciences.

5 Changes

There were no changes in the project or its direction.

6 Technical Updates

There are no technical updates to report beyond the continuing research progress and dissemination of results described in report.

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AFOSR Final Report 2023-03-30
Grant FA9550-19-1-0138
New Directions in Mean Field Games: MFG Subpopulation Behaviours and Graphon-MFG Systems

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1 Overview of Research Objectives

1.1 Background and Overview

Very large networks linking dynamical agents are now ubiquitous and the need to analyse, design and control them is evident. The emergence of the graphon theory of large networks and their infinite limits has enabled the formulation of a theory of the centralized control of dynamical systems distributed on asymptotically infinite networks.

Large-scale systems of competing agents are frequently formulated as dynamic games, this however confronts the problem that for even small populations non-cooperative game theory is intractable. The 2006 breakthrough of Minyi Huang, Roland Malhamé and the PI of FA9550-19-1-0138, Peter Caines, was the introduction of Mean Field Game (MFG) theory (originally termed Nash Certainty Equivalence theory) which was also simultaneously discovered by J.M. Lasry and P.L. Lions.

Mean Field Game (MFG) theory analyzes the existence of Nash equilibria for dynamical games in terms of infinite populations of negligible agents forming the limits of the original finite population systems (Caines, 2020). The core of MFG theory is a pair of partial differential equations consisting of (i) the Hamilton-Jacobi-Bellman equation of optimal control for a generic agent, and (ii) the Fokker-Planck-Kolmogorov equation describing the controlled evolution of the state of a generic agent. These are linked by the generic agent's state

distribution, known as the system's mean field.

A fundamental aspect of MFG theory justifying its application in a vast range of cases is the ϵ-Nash property by which controls derived from the MFG equations yield near-equilibrium performance for finite systems for sufficiently large populations.

The MFG field has expanded tremendously since its inception. Google search results now run to 1.10 billion, citations to papers are in the thousands, conferences and workshops on MFG have proliferated and at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively). In addition to the explosion of mathematical and theoretical investigations over last 16 years, applications of MFG theory have spread to a vast range of domains including: optimization of decentralized EV charging (Ma et al., 2011; Zhu et al., 2016; Foguen Tchuendom et al., 2019); nonlinear system state estimation (Yang et al., 2013); systemic risk in banking (Carmona et al., 2013); cell phone energy management (Aziz and Caines, 2016); economics of energy production (Ludkovski and Sircar, 2015); macro-economics of growth (Gomes et al., 2016); cloud-based networks in information technology (Hanif et al., 2015); algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Nourian et al., 2011; Grover et al., 2018), AI reinforcement learning for MFG systems (Subramanian and Mahajan, 2019; Agarwal et al., 2022), deep learning applications to the solution of MFGs (Ruthotto et al., 2020) and polymer science (Welch et al., 2019).

In this context it is important to note that the research on MFG by the Principal Investigator has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, most recently, for the period 2022-2025.

1.2 Major Research Goals under Grant FA9550-19-1-0138

The program of research under FA9550-19-1-0138 is focused on the development of a new form of Mean Field Games called Graphon Mean Field Games (GMFGs); these are large population games defined on large, heterogeneous, networks and their asymptotic (graphon) limits.

In this setting, game theoretic equilibria are expressed in terms of the newly defined GMFG PDE equations, which are a sweeping generalization of the classical MFG PDEs.

In a closely related development, the theory of graphons is used as a basis for Graphon Linear Control (GLC) theory in which large populations of centrally controlled dynamical systems interacting over large, heterogeneous networks

may be analyzed and optimized in a tractable way.

Canonical Design Procedure for GMFG and GLC Systems

A major goal has been to generate large scale network game strategies and, respectively, control design procedures with the following standard pattern:

The first step is to consider the graphon limit of the network over which the individual systems are distributed and which constitutes the global system.

In the second step, the resulting graphon based GMFG equations, or, respectively, the graphon based infinite dimensional linear system optimization problems are specified. The GMFG equations are then solved to generate the Nash best response strategies or, respectively, the linear feedback controls are synthesized using the well established infinite dimensional linear systems control theory.

Finally, the best response strategies, or, respectively, the infinite dimensional linear feedback controls are approximated in such a way that they may be applied to the original complex finite systems.

The key feature of this strategy is the following: In both the network mean field game and the network control settings it is demonstrated that as the size of the network tends to infinity, the performance of the finite systems when they are subject to the infinite limit system best response game strategies, or, respectively, centralized control laws, yield pay-offs which converge to their Nash equilibrium, or, respectively, optimal, infinite system limit values.

1.3 Organization of the Sequence of Research Objectives

The work over the full reporting period has been organized with the following sequence of objectives:

(i) The establishment of a rigorous theory of the central results of Graphon Mean Field Game (GMFG) theory,

(ii) The generation of a specific Graphon Linear Control (GLC) theory for the control of large populations of dynamical linear system agents with quadratic performance functions on large, complex networks.

(iii) The development of an approximation theory of associated numerical methods, especially those based upon finite dimensional invariant subspaces of the graphon operator in the GLC case.

(iv) The pioneering development of the theory of GMFG systems embedded in Euclidean space; this underlies our currently developing variational analysis of agent performance in networks as a function of position, where this theory also includes sparse as well as dense networks.

(v) The development of an SIR epidemic model software which permits the simulation of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity, these models are constructed on a data base of real world US surface and air-traffic travel data.

2 Details of Accomplishments

2.1 Graphon Mean Field Game Theory: Foundations

The work in (Caines and Huang, 2021) (Caines and Huang, 2018), (Caines and Huang, 2019), established the foundations and central results for graphon mean field game theory for nonlinear systems, that is to say games of infinite populations on infinite heterogeneous networks as limits of the corresponding finite systems.

Specifically, in the conference papers (Caines and Huang, 2018), (Caines and Huang, 2019) Graphon Mean Field Games (GMFG) and the GMFG equations were introduced for the analysis of non-cooperative dynamical games on unbounded networks. In this formulation, asymptotically infinite sub-populations are located at the nodes of large heterogeneous networks, the asymptotic network limits then constitute a part of the analysis. In (Caines and Huang, 2021), the basic existence and uniqueness results were established for the GMFG equations, and a full derivation of the fundamental results was provided, including the key epsilon-Nash theory for GMFG systems which relates the infinite population equilibria on infinite networks to finite population equilibria on finite networks.

2.2 LQG GMFG Decomposition Methods

The theory of invariant spaces of linear operators developed in the papers (Gao and Caines, 2021) is applied in (Gao et al., 2021b), (Gao et al., 2021a) and (Gao et al., 2022) to yield a tractable methodology for generating decentralized game strategies for systems distributed over complex networks. The fundamental properties of the existence and uniqueness of solutions are established in (Gao et al., 2021b,a, 2022) in two principal ways: (i) Riccati equation decomposition and (ii) fixed point contraction theorem methods.

2.3 LQG GMFG Master Equation Analysis

First, using a probabilistic approach, solutions to the Linear Quadratic Graphon Mean Field Games are expressed as solutions to coupled Forward Backward Stochastic Differential Equations (FBSDEs) of McKean-Vlasov type. Next, the

existence of the so-called Master field, which allows for the decoupling of these FBSDEs is established. Finally, the infinite dimensional Partial Differential Equation (PDE), so-called Master Equation, is established for which the Master Field is a solution (see (Foguen Tchuendom et al., 2020)).

2.4 GMFG with Affine Dynamics

GMFG theory for nonlinear problems with individual affine dynamics has been established in Caines et al. (2022) and the Nash value properties for control affine systems has been analyzed in Caines et al. (2023).

2.5 Graphon Linear Systems Control with Approximation Theory

As explained above, the theory of graphons is further used as a basis upon which centrally controlled linear dynamical systems interacting over large, heterogeneous networks are analyzed in a tractable way.

The graphon control methodology is developed in (Gao and Caines, 2020, 2019a,b) to study control problems for arbitrary-size networks of linear dynamical systems. The complete treatment of problems involving well-posedness, controllability, approximation errors for state-to-state control and regulations is established in (Gao and Caines, 2020).

Scalable and tractable solution methods are established for a class of graphon control problem in (Gao and Caines, 2019a,b). Next, the graphon linear system control framework is extended to more general cases where the state of each individual agent on the network is multi-dimensional (Gao and Caines, 2021); moreover, this work provides a generalized solution method with an application to large-scale networks of coupled harmonic oscillators.

2.6 LQ Graphon Dynamical Systems and Control with Q-space Noise

The work in (Dunyak and Caines, 2022) extends the formulation and analysis of (Gao and Caines, 2021) to include stochastic processes in infinite dimensions as a random process source driving the network's controlled dynamical systems; this generalization involves the formulation of a form of graphon Q-space noise.

2.7 Critical Nodes for GMFG and Embedded GMFG

The works (Caines, 2022),(Foguen Tchuendom et al., 2021), (Foguen Tchuendom et al., 2022a), (Foguen Tchuendom et al., 2022b) and (Caines et al., 2023)

pioneer the theory of GMFG systems embedded in Euclidean space by constructing limit graphs and limit graphons for sequences of graphs sitting in Euclidean space. The crucial notion here is that for embedded graphs and graphons we may define the differentiation of functions on the embedded graph limits. In particular we formulate an extended version of mean field game systems, termed Embedded Graphon Mean Field Games (EGMFG), or Graphexon Mean Field Games (GXMFG), which are amenable to the application of the calculus to the optimization of node location in such systems, this being one of the main objectives of the new development. Moreover this formulation includes the case of sparse graph limits (see (Caines, 2022)) which is not covered by the standard graphon formulation.

2.8 MFG Theory and Related Stochastic Adaptive and Hybrid Control Theory

The paper (Pakniyat and Caines, 2021) constitutes the climax of several decades of work by my own group and many other researchers aimed at producing a rigorous, general, hybrid optimal systems control theory (HOCT). These and related results have great applicability in many domains of engineering and scientific modelling, and furthermore in economics and finance. In fact the paper (Firoozi et al., 2022) presents an extension of HOCT to stochastic MFG systems motivated by trading dynamics in finance.

In (Saude and Caines, 2022), we study the Markowitz dynamic portfolio problem, where an agent seeks to maximize its expected return while minimizing the variance of the return (risk). For the first time in this topic, the impact of the aggregate of the agents' trades is incorporated into the dynamics and costs using an extended mean-field games framework.

2.9 Data Based Protocols for Modelling Large Populations on Large Networks with Applications to SIR Problems

The initial investigation of applying graphon theory to the control of epidemic spread is proposed in (Gao and Caines, 2019a) where scalable solutions based on spectral approximation and linearization are established for controlling virus spreads on networks following a type of controlled network SIS model.
The work (Dunyak and Caines, 2021; Salii, 2021) includes the development of SIR epidemic models which permits the study of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity; the latter models were constructed on a data base of real world US surface and air-traffic travel data.

2.10 Centrality Analysis for Finite and Infinite Networks

This work advances the theory of centralities for dynamical systems on networks and its associated applications. In (Gao, 2022) a family of centralities characterized by fixed-points of networks is identified which includes the standard centralities (e.g. PageRank, Katz-Bonacich, eigen centralities) and equilbrirum cost for games on networks (e.g. Nash values of LQG GMFG). Variation bounds of centralities with respect to the network variations are established. In (Gao, 2021) network centralities have been utilized in the modelling of opinion dynamics on social networks and in identifying significant network partitions based upon opinion disagreement states.

3 Dissemination to Communities of Interest

Research findings and results have been published in the leading systems and control and networks journals including the SIAM Journal on Control and Optimization, IEEE Transactions on Automatic Control, Automatica, IEEE Transactions on Control of Network Systems, and in conferences proceedings including the IEEE Conference on Decision and Control.

3.1 Publications: Journals

S. Gao, P. E. Caines, and M. Huang, "LQG graphon mean field games: analysis via graphon invariant subspaces," Conditionally accepted by IEEE Transactions on Automatic Control, 2021. (Gao et al., 2022)

D. Firoozi, A. Pakniyat, and P. E. Caines, "A class of hybrid LQG mean field games with state-invariant switching and stopping strategies," Automatica, vol. 141, p. 110244, 2022. (Firoozi et al., 2022)

P. E. Caines, D. Ho, M. Huang, J. Jian, and Q. Song. On the graphon mean field game equations: Individual agent affine dynamics and mean field dependent performance functions. ESAIM: Control, Optimisation and Calculus of Variations, 28:24, 2022 (Caines et al., 2022)

P. E. Caines and M. Huang, "Graphon mean field games and their equations," SIAM Journal on Control and Optimization, vol. 59, no. 6, pp. 4373–4399, 2021. (Caines and Huang, 2021)

A. Pakniyat and P. E. Caines, "On the hybrid minimum principle: The Hamiltonian and adjoint boundary conditions," IEEE Transactions on Automatic Control, vol. 66, no. 3, pp. 1246–1253, 2021. (Pakniyat and Caines, 2021)

S. Gao and P. E. Caines, "Subspace decomposition for graphon LQR: Applications to VLSNs of harmonic oscillators," IEEE Transactions on Control of Network Systems, vol. 8, no. 2, pp. 576–586, 2021. (Gao and Caines, 2021)

S. Gao, R. Foguen Tchuendom, and P. E. Caines, "Linear quadratic graphon field games," Communications in Information and Systems, vol. 21, no. 3, pp. 341–369, 2021. (Gao et al., 2021b)

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D. Firoozi, S. Jaimungal, and P. E. Caines. Convex analysis for lqg systems with applications to major–minor LQG mean–field game systems. Systems & Control Letters, 142:104734, 2020. (Firoozi et al., 2020)

S. Gao and P. E. Caines. Graphon control of large-scale networks of linear systems. IEEE Transactions on Automatic Control, 65(10):4090–4105, 2020 (Gao and Caines, 2020)

N. Sen and P. E. Caines. Mean field games with partial observation. SIAM Journal on Control and Optimization, 57(3):2064–2091, 2019 (Sen and Caines, 2019)

P. E. Caines, D. Ho, and Q. Song. The density evolution of the killed mckean–vlasov process. Stochastics, 92(4):642–657, 2020. (Caines et al., 2020)

3.2 Publications: Conference Proceedings

A. Dunyak and P. E. Caines, "Linear stochastic graphon systems with q- space noise," in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 3926–3932. (Dunyak and Caines, 2022)

P. E. Caines, "Embedded vertexon-graphons and embedded GMFG systems," Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022,

pp. 5550–5557. (Caines, 2022)

S. Gao, "Fixed-point centrality for networks," in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 1628–1635. (Gao, 2022).

R. Foguen Tchuendom, S. Gao, P. E. Caines, and M. Huang, "Optimal network location in infinite horizon LQG graphon mean field games," in Proceedings of the 61th IEEE Conference on Decision and Control, Cancun, Mexico, December 2022, pp. 5558–5565. (Foguen Tchuendom et al., 2022b)

R. Foguen Tchuendom, S. Gao, and P. E. Caines "Stationary cost nodes in infinite horizon LQG GMFGs," IFAC-PapersOnLine for the 25th International Symposium on Mathematical Theory of Networks and Systems (MTNS), vol. 55, no. 30, pp. 284–289, 2022. (Foguen Tchuendom et al., 2022a)

J. Saude and P. E. Caines, "Markowitz portfolio optimization extended quadratic mean- field games approach," Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 5523–5528. (Saude and Caines, 2022)

Y. V. Salii, "Benchmarking optimal control for network dynamic systems with plausible epidemic models," in International Conference on Complex Networks and Their Applications. Springer, 2021, pp. 194–206. (Salii, 2021)

S. Gao, P. E. Caines, and M. Huang, "LQG graphon mean field games: Graphon invariant subspaces," Proceedings of the 60th IEEE Conference on Decision and Control, Austin, Texas, USA, December 2021, pp. 5253–5260. (Gao et al., 2021a)

R. Foguen Tchuendom, P. E. Caines, and M. Huang, "Critical nodes in graphon mean field games," Proceedings of the 60th IEEE Conference on Decision and Control (CDC). IEEE, 2021, pp. 166–170. (Foguen Tchuendom et al., 2021)

A. Dunyak and P. E. Caines, "Large scale systems and SIR models: A featured graphon approach," Proceedings of the 60th IEEE Conference on Decision and Control (CDC), 2021, pp. 6928–6933. (Dunyak and Caines, 2021)

S. Gao, "Centrality-weighted opinion dynamics: Disagreement and social network partition," Proceedings of the 60th IEEE Conference on Decision and Control (CDC), Austin, Texas, USA, December 2021, pp. 5496–5501. (Gao, 2021)

R. Foguen Tchuendom, P. E. Caines, and M. Huang. "On the master equation for linear quadratic graphon mean field games", Proceedings of the 59th

IEEE Conference on Decision and Control (CDC), pages 1026–1031, 2020. (Foguen Tchuendom et al., 2020)

S. Gao and P. E. Caines, Optimal and approximate solutions to linear quadratic regulation of a class of graphon dynamical systems, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 8359–8365, Nice, France, December 2019b. (Gao and Caines, 2019b)

S. Gao and P. E. Caines, Spectral representations of graphons in very large network systems control, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 5068–5075, Nice, France, December 2019a.(Gao and Caines, 2019a)

P. E. Caines and M. Huang. Graphon mean field games and the GMFG equations: ϵ-Nash equilibria, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 286–292, December 2019. (Caines and Huang, 2019)

3.3 Talks: April 1st, 2019 to December 31st, 2022

1. P. E. Caines,"Vertexons and Embedded Graphon Mean Field Games", the Simons Institute for the Theory of Computing, UC Berkeley, CA, Sept.29, 2022 (Invited talk) in the "Graph Limits, Nonparametric Models, and Estimation Workshop" 26 - 30 September, 2022.

2. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", (Invited talk) 29th August, 2022, in the Conference on Many Player Games and Applications, Humbolt University Berlin, 29 - 31 August 2022.

3. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Large Populations on Complex Networks", Canadian Applied and Industrial Mathematics Society (CAIMS), June 23rd, 2021 (Plenary talk).

4. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", the Financial Mathematics/Engineering Seminar Series, Hong Kong Polytechnic University, June 16th 2021 (Invited talk).

5. P. E. Caines, "Static and Time Varying Graphons in Economic and Pandemic GMFG Models" SIAM Financial Mathematics and Engineering (FM21), June 1, 2021 (Invited talk).

6. S. Gao, "Subspace Decompositions in Graphon Control and Graphon Mean Field Games", Informal Systems Seminar, Centre for Intelligent Machines,

McGill University, Groupe d'Etudes et de Recherche en Analyse des Decisions, Montreal, CA, Dec. 11, 2020 (Invited talk).

7. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Networked Systems", Invited Speaker at the "Conference on Mean Field Games and Related Topics - 5". 9th - 13th September, 2019

8. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for a Networked World", University of Saint Petersburg, Russia, 6th September, 2019 (Distinguished Speaker).

9. P. E. Caines, "Graphon Control and Graphon Mean Field Games", Department of Applied Mathemaics, Hong Kong Polythenic Universty, Hong Kong), 28th June, 2019 (Distinguished Lecturer).

3.4 Workshop Organization: April 1st, 2019 to December 31st, 2022

Name: **Mean Field Games on Networks Workshop**

Location: Vancouver, BC, Canada (Virtual Meeting)

Dates: October 26 - 29, 2021

Organizers: Peter E. Caines, Shuang Gao, Minyi Huang, and Rinel Foguen Tchuendom

This workshop (PIMS et al., 2021) brought together researchers in applied mathematics, mean field games, network science, network games, and systems and control theory to exchange ideas and to work on the extensions of mean field game theory to dynamic game problems on heterogeneous large-scale networks. The workshop offered a platform to current results and stimulate discussions on the open challenges in this emerging field, on mathematical theory, computational algorithms, and the applications methodologies of MFG on general networks.

Sponsors: Pacific Institute for the Mathematical Sciences (PIMS), Groupe d'études et de recherche en analyse des décisions (GERAD), and the Fields Institute for Research in Mathematical Sciences.

Participants: Peter E. Caines (PI) and PI's Postdoctoral Researchers: Rinel Foguen Tchuendom, Shuang Gao, Joao Saude, and Yaroslav Salii, plus 50 other international participants.

4 Impacts

As stated earlier, the research on MFG and related topics by the Principal Investigator of the grant FA9550-19-1-0138 has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, presently 2022 - 2025. Consequently, the impacts of research under the 2019 - 2022 grant, FA9550-19-1-0138, may be viewed as a continuation and development of those of the entire program.

Indeed, as explained in Section 1, the MFG field has expanded enormously since its inception. Google links now run to 1.10 billion, citations to papers are in the thousands and conferences and workshops on MFG have proliferated. A sample of the most recent workshops and conference sessions is the following, where we note that the PI and group members have been invited to participate in all of them: Workshop "Mean Field Games on Networks", Pacific Institute for the Mathematical Sciences, Vancouver, BC, Canada, 26 - 29, October, 2021; "Conference on Mathematics and Financial Economics, "Many Player Games and Applications", Humbolt University, Berlin, 29 - 31 August 2022; Workshop on Mean Field Games, Montreal: Centre de Recherches Mathématiques (CRM), 11-15 April, 2022; "Graph Limits, Nonparametric Models, and Estimation Workshop", Simons Institute for the Theory of Computation, UCB, Berkeley, CA, 26-30 September, 2022; IEEE, Control Systems Society Conference on Decision and Control, Cancun, December, 2022, Mean Field Games Session; Mean Field Games Invited Session, IFAC World Congress, Yokahama, Japan, July 2023.

Furthermore, at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively), the last appearing in the period covered by this report. In addition to the very rapid expansion of mathematical and theoretical investigations, applications of MFG theory have spread to a vast range of domains, a sample of the most recent being the optimization of decentralized electric vehicle charging (Foguen Tchuendom et al., 2019), algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Grover et al., 2018), AI reinforcement learning for MFG systems (Agarwal et al., 2022) and polymer science (Welch et al., 2019).

As is shown by this indicative list of publications, meetings and presentations, it is reasonable to assert that the influence of the AFOSR supported research carried out on Mean Field Games and, in particular, on Mean Field Games on networks, has had, and is having, an impact on fields ranging from the core area of systems and control science to economics, finance, energy generation and distribution, operations research, the life sciences and the physical sciences.

5 Changes

There were no changes in the project or its direction.

6 Technical Updates

There are no technical updates to report beyond the continuing research progress and dissemination of results described in report.

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New Directions in Mean Field Games: MFG Subpopulation Behaviours and Graphon-MFG Systems

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1 Overview of Research Objectives

1.1 Background and Overview

Very large networks linking dynamical agents are now ubiquitous and the need to analyse, design and control them is evident. The emergence of the graphon theory of large networks and their infinite limits has enabled the formulation of a theory of the centralized control of dynamical systems distributed on asymptotically infinite networks.

Large-scale systems of competing agents are frequently formulated as dynamic games, this however confronts the problem that for even small populations non-cooperative game theory is intractable. The 2006 breakthrough of Minyi Huang, Roland Malhamé and the PI of FA9550-19-1-0138, Peter Caines, was the introduction of Mean Field Game (MFG) theory (originally termed Nash Certainty Equivalence theory) which was also simultaneously discovered by J.M. Lasry and P.L. Lions.

Mean Field Game (MFG) theory analyzes the existence of Nash equilibria for dynamical games in terms of infinite populations of negligible agents forming the limits of the original finite population systems (Caines, 2020). The core of MFG theory is a pair of partial differential equations consisting of (i) the Hamilton-Jacobi-Bellman equation of optimal control for a generic agent, and (ii) the Fokker-Planck-Kolmogorov equation describing the controlled evolution of the state of a generic agent. These are linked by the generic agent's state

distribution, known as the system's mean field.

A fundamental aspect of MFG theory justifying its application in a vast range of cases is the ϵ-Nash property by which controls derived from the MFG equations yield near-equilibrium performance for finite systems for sufficiently large populations.

The MFG field has expanded tremendously since its inception. Google search results now run to 1.10 billion, citations to papers are in the thousands, conferences and workshops on MFG have proliferated and at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively). In addition to the explosion of mathematical and theoretical investigations over last 16 years, applications of MFG theory have spread to a vast range of domains including: optimization of decentralized EV charging (Ma et al., 2011; Zhu et al., 2016; Foguen Tchuendom et al., 2019); nonlinear system state estimation (Yang et al., 2013); systemic risk in banking (Carmona et al., 2013); cell phone energy management (Aziz and Caines, 2016); economics of energy production (Ludkovski and Sircar, 2015); macro-economics of growth (Gomes et al., 2016); cloud-based networks in information technology (Hanif et al., 2015); algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Nourian et al., 2011; Grover et al., 2018), AI reinforcement learning for MFG systems (Subramanian and Mahajan, 2019; Agarwal et al., 2022), deep learning applications to the solution of MFGs (Ruthotto et al., 2020) and polymer science (Welch et al., 2019).

In this context it is important to note that the research on MFG by the Principal Investigator has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, most recently, for the period 2022-2025.

1.2 Major Research Goals under Grant FA9550-19-1-0138

The program of research under FA9550-19-1-0138 is focused on the development of a new form of Mean Field Games called Graphon Mean Field Games (GMFGs); these are large population games defined on large, heterogeneous, networks and their asymptotic (graphon) limits.

In this setting, game theoretic equilibria are expressed in terms of the newly defined GMFG PDE equations, which are a sweeping generalization of the classical MFG PDEs.

In a closely related development, the theory of graphons is used as a basis for Graphon Linear Control (GLC) theory in which large populations of centrally controlled dynamical systems interacting over large, heterogeneous networks

may be analyzed and optimized in a tractable way.

Canonical Design Procedure for GMFG and GLC Systems

A major goal has been to generate large scale network game strategies and, respectively, control design procedures with the following standard pattern:

The first step is to consider the graphon limit of the network over which the individual systems are distributed and which constitutes the global system.

In the second step, the resulting graphon based GMFG equations, or, respectively, the graphon based infinite dimensional linear system optimization problems are specified. The GMFG equations are then solved to generate the Nash best response strategies or, respectively, the linear feedback controls are synthesized using the well established infinite dimensional linear systems control theory.

Finally, the best response strategies, or, respectively, the infinite dimensional linear feedback controls are approximated in such a way that they may be applied to the original complex finite systems.

The key feature of this strategy is the following: In both the network mean field game and the network control settings it is demonstrated that as the size of the network tends to infinity, the performance of the finite systems when they are subject to the infinite limit system best response game strategies, or, respectively, centralized control laws, yield pay-offs which converge to their Nash equilibrium, or, respectively, optimal, infinite system limit values.

1.3 Organization of the Sequence of Research Objectives

The work over the full reporting period has been organized with the following sequence of objectives:

(i) The establishment of a rigorous theory of the central results of Graphon Mean Field Game (GMFG) theory,

(ii) The generation of a specific Graphon Linear Control (GLC) theory for the control of large populations of dynamical linear system agents with quadratic performance functions on large, complex networks.

(iii) The development of an approximation theory of associated numerical methods, especially those based upon finite dimensional invariant subspaces of the graphon operator in the GLC case.

(iv) The pioneering development of the theory of GMFG systems embedded in Euclidean space; this underlies our currently developing variational analysis of agent performance in networks as a function of position, where this theory also includes sparse as well as dense networks.

(v) The development of an SIR epidemic model software which permits the simulation of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity, these models are constructed on a data base of real world US surface and air-traffic travel data.

2 Details of Accomplishments

2.1 Graphon Mean Field Game Theory: Foundations

The work in (Caines and Huang, 2021) (Caines and Huang, 2018), (Caines and Huang, 2019), established the foundations and central results for graphon mean field game theory for nonlinear systems, that is to say games of infinite populations on infinite heterogeneous networks as limits of the corresponding finite systems.

Specifically, in the conference papers (Caines and Huang, 2018), (Caines and Huang, 2019) Graphon Mean Field Games (GMFG) and the GMFG equations were introduced for the analysis of non-cooperative dynamical games on unbounded networks. In this formulation, asymptotically infinite sub-populations are located at the nodes of large heterogeneous networks, the asymptotic network limits then constitute a part of the analysis. In (Caines and Huang, 2021), the basic existence and uniqueness results were established for the GMFG equations, and a full derivation of the fundamental results was provided, including the key epsilon-Nash theory for GMFG systems which relates the infinite population equilibria on infinite networks to finite population equilibria on finite networks.

2.2 LQG GMFG Decomposition Methods

The theory of invariant spaces of linear operators developed in the papers (Gao and Caines, 2021) is applied in (Gao et al., 2021b), (Gao et al., 2021a) and (Gao et al., 2022) to yield a tractable methodology for generating decentralized game strategies for systems distributed over complex networks. The fundamental properties of the existence and uniqueness of solutions are established in (Gao et al., 2021b,a, 2022) in two principal ways: (i) Riccati equation decomposition and (ii) fixed point contraction theorem methods.

2.3 LQG GMFG Master Equation Analysis

First, using a probabilistic approach, solutions to the Linear Quadratic Graphon Mean Field Games are expressed as solutions to coupled Forward Backward Stochastic Differential Equations (FBSDEs) of McKean-Vlasov type. Next, the

existence of the so-called Master field, which allows for the decoupling of these FBSDEs is established. Finally, the infinite dimensional Partial Differential Equation (PDE), so-called Master Equation, is established for which the Master Field is a solution (see (Foguen Tchuendom et al., 2020)).

2.4 GMFG with Affine Dynamics

GMFG theory for nonlinear problems with individual affine dynamics has been established in Caines et al. (2022) and the Nash value properties for control affine systems has been analyzed in Caines et al. (2023).

2.5 Graphon Linear Systems Control with Approximation Theory

As explained above, the theory of graphons is further used as a basis upon which centrally controlled linear dynamical systems interacting over large, heterogeneous networks are analyzed in a tractable way.

The graphon control methodology is developed in (Gao and Caines, 2020, 2019a,b) to study control problems for arbitrary-size networks of linear dynamical systems. The complete treatment of problems involving well-posedness, controllability, approximation errors for state-to-state control and regulations is established in (Gao and Caines, 2020).

Scalable and tractable solution methods are established for a class of graphon control problem in (Gao and Caines, 2019a,b). Next, the graphon linear system control framework is extended to more general cases where the state of each individual agent on the network is multi-dimensional (Gao and Caines, 2021); moreover, this work provides a generalized solution method with an application to large-scale networks of coupled harmonic oscillators.

2.6 LQ Graphon Dynamical Systems and Control with Q-space Noise

The work in (Dunyak and Caines, 2022) extends the formulation and analysis of (Gao and Caines, 2021) to include stochastic processes in infinite dimensions as a random process source driving the network's controlled dynamical systems; this generalization involves the formulation of a form of graphon Q-space noise.

2.7 Critical Nodes for GMFG and Embedded GMFG

The works (Caines, 2022),(Foguen Tchuendom et al., 2021), (Foguen Tchuendom et al., 2022a), (Foguen Tchuendom et al., 2022b) and (Caines et al., 2023)

pioneer the theory of GMFG systems embedded in Euclidean space by constructing limit graphs and limit graphons for sequences of graphs sitting in Euclidean space. The crucial notion here is that for embedded graphs and graphons we may define the differentiation of functions on the embedded graph limits. In particular we formulate an extended version of mean field game systems, termed Embedded Graphon Mean Field Games (EGMFG), or Graphexon Mean Field Games (GXMFG), which are amenable to the application of the calculus to the optimization of node location in such systems, this being one of the main objectives of the new development. Moreover this formulation includes the case of sparse graph limits (see (Caines, 2022)) which is not covered by the standard graphon formulation.

2.8 MFG Theory and Related Stochastic Adaptive and Hybrid Control Theory

The paper (Pakniyat and Caines, 2021) constitutes the climax of several decades of work by my own group and many other researchers aimed at producing a rigorous, general, hybrid optimal systems control theory (HOCT). These and related results have great applicability in many domains of engineering and scientific modelling, and furthermore in economics and finance. In fact the paper (Firoozi et al., 2022) presents an extension of HOCT to stochastic MFG systems motivated by trading dynamics in finance.

In (Saude and Caines, 2022), we study the Markowitz dynamic portfolio problem, where an agent seeks to maximize its expected return while minimizing the variance of the return (risk). For the first time in this topic, the impact of the aggregate of the agents' trades is incorporated into the dynamics and costs using an extended mean-field games framework.

2.9 Data Based Protocols for Modelling Large Populations on Large Networks with Applications to SIR Problems

The initial investigation of applying graphon theory to the control of epidemic spread is proposed in (Gao and Caines, 2019a) where scalable solutions based on spectral approximation and linearization are established for controlling virus spreads on networks following a type of controlled network SIS model.
The work (Dunyak and Caines, 2021; Salii, 2021) includes the development of SIR epidemic models which permits the study of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity; the latter models were constructed on a data base of real world US surface and air-traffic travel data.

2.10 Centrality Analysis for Finite and Infinite Networks

This work advances the theory of centralities for dynamical systems on networks and its associated applications. In (Gao, 2022) a family of centralities characterized by fixed-points of networks is identified which includes the standard centralities (e.g. PageRank, Katz-Bonacich, eigen centralities) and equilbrirum cost for games on networks (e.g. Nash values of LQG GMFG). Variation bounds of centralities with respect to the network variations are established. In (Gao, 2021) network centralities have been utilized in the modelling of opinion dynamics on social networks and in identifying significant network partitions based upon opinion disagreement states.

3 Dissemination to Communities of Interest

Research findings and results have been published in the leading systems and control and networks journals including the SIAM Journal on Control and Optimization, IEEE Transactions on Automatic Control, Automatica, IEEE Transactions on Control of Network Systems, and in conferences proceedings including the IEEE Conference on Decision and Control.

3.1 Publications: Journals

S. Gao, P. E. Caines, and M. Huang, "LQG graphon mean field games: analysis via graphon invariant subspaces," Conditionally accepted by IEEE Transactions on Automatic Control, 2021. (Gao et al., 2022)

D. Firoozi, A. Pakniyat, and P. E. Caines, "A class of hybrid LQG mean field games with state-invariant switching and stopping strategies," Automatica, vol. 141, p. 110244, 2022. (Firoozi et al., 2022)

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P. E. Caines and M. Huang, "Graphon mean field games and their equations," SIAM Journal on Control and Optimization, vol. 59, no. 6, pp. 4373–4399, 2021. (Caines and Huang, 2021)

A. Pakniyat and P. E. Caines, "On the hybrid minimum principle: The Hamiltonian and adjoint boundary conditions," IEEE Transactions on Automatic Control, vol. 66, no. 3, pp. 1246–1253, 2021. (Pakniyat and Caines, 2021)

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N. Sen and P. E. Caines. Mean field games with partial observation. SIAM Journal on Control and Optimization, 57(3):2064–2091, 2019 (Sen and Caines, 2019)

P. E. Caines, D. Ho, and Q. Song. The density evolution of the killed mckean–vlasov process. Stochastics, 92(4):642–657, 2020. (Caines et al., 2020)

3.2 Publications: Conference Proceedings

A. Dunyak and P. E. Caines, “Linear stochastic graphon systems with q- space noise,” in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 3926–3932. (Dunyak and Caines, 2022)

P. E. Caines, “Embedded vertexon-graphons and embedded GMFG systems,” Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022,

pp. 5550–5557. (Caines, 2022)

S. Gao, “Fixed-point centrality for networks,” in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 1628–1635. (Gao, 2022).

R. Foguen Tchuendom, S. Gao, P. E. Caines, and M. Huang, “Optimal network location in infinite horizon LQG graphon mean field games,” in Proceedings of the 61th IEEE Conference on Decision and Control, Cancun, Mexico, December 2022, pp. 5558–5565. (Foguen Tchuendom et al., 2022b)

R. Foguen Tchuendom, S. Gao, and P. E. Caines “Stationary cost nodes in infinite horizon LQG GMFGs,” IFAC-PapersOnLine for the 25th International Symposium on Mathematical Theory of Networks and Systems (MTNS), vol. 55, no. 30, pp. 284–289, 2022. (Foguen Tchuendom et al., 2022a)

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Y. V. Salii, “Benchmarking optimal control for network dynamic systems with plausible epidemic models,” in International Conference on Complex Networks and Their Applications. Springer, 2021, pp. 194–206. (Salii, 2021)

S. Gao, P. E. Caines, and M. Huang, “LQG graphon mean field games: Graphon invariant subspaces,” Proceedings of the 60th IEEE Conference on Decision and Control, Austin, Texas, USA, December 2021, pp. 5253–5260. (Gao et al., 2021a)

R. Foguen Tchuendom, P. E. Caines, and M. Huang, “Critical nodes in graphon mean field games,” Proceedings of the 60th IEEE Conference on Decision and Control (CDC). IEEE, 2021, pp. 166–170. (Foguen Tchuendom et al., 2021)

A. Dunyak and P. E. Caines, “Large scale systems and SIR models: A featured graphon approach,” Proceedings of the 60th IEEE Conference on Decision and Control (CDC), 2021, pp. 6928–6933. (Dunyak and Caines, 2021)

S. Gao, “Centrality-weighted opinion dynamics: Disagreement and social network partition,” Proceedings of the 60th IEEE Conference on Decision and Control (CDC), Austin, Texas, USA, December 2021, pp. 5496–5501. (Gao, 2021)

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IEEE Conference on Decision and Control (CDC), pages 1026–1031, 2020. (Foguen Tchuendom et al., 2020)

S. Gao and P. E. Caines, Optimal and approximate solutions to linear quadratic regulation of a class of graphon dynamical systems, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 8359–8365, Nice, France, December 2019b. (Gao and Caines, 2019b)

S. Gao and P. E. Caines, Spectral representations of graphons in very large network systems control, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 5068–5075, Nice, France, December 2019a.(Gao and Caines, 2019a)

P. E. Caines and M. Huang. Graphon mean field games and the GMFG equations: ϵ-Nash equilibria, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 286–292, December 2019. (Caines and Huang, 2019)

3.3 Talks: April 1st, 2019 to December 31st, 2022

1. P. E. Caines,"Vertexons and Embedded Graphon Mean Field Games", the Simons Institute for the Theory of Computing, UC Berkeley, CA, Sept.29, 2022 (Invited talk) in the "Graph Limits, Nonparametric Models, and Estimation Workshop" 26 - 30 September, 2022.

2. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", (Invited talk) 29th August, 2022, in the Conference on Many Player Games and Applications, Humbolt University Berlin, 29 - 31 August 2022.

3. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Large Populations on Complex Networks", Canadian Applied and Industrial Mathematics Society (CAIMS), June 23rd, 2021 (Plenary talk).

4. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", the Financial Mathematics/Engineering Seminar Series, Hong Kong Polytechnic University, June 16th 2021 (Invited talk).

5. P. E. Caines, "Static and Time Varying Graphons in Economic and Pandemic GMFG Models" SIAM Financial Mathematics and Engineering (FM21), June 1, 2021 (Invited talk).

6. S. Gao, "Subspace Decompositions in Graphon Control and Graphon Mean Field Games", Informal Systems Seminar, Centre for Intelligent Machines,

McGill University, Groupe d'Etudes et de Recherche en Analyse des Decisions, Montreal, CA, Dec. 11, 2020 (Invited talk).

7. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Networked Systems", Invited Speaker at the "Conference on Mean Field Games and Related Topics - 5". 9th - 13th September, 2019

8. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for a Networked World", University of Saint Petersburg, Russia, 6th September, 2019 (Distinguished Speaker).

9. P. E. Caines, "Graphon Control and Graphon Mean Field Games", Department of Applied Mathemaics, Hong Kong Polythenic Universty, Hong Kong), 28th June, 2019 (Distinguished Lecturer).

3.4 Workshop Organization: April 1st, 2019 to December 31st, 2022

Name: **Mean Field Games on Networks Workshop**

Location: Vancouver, BC, Canada (Virtual Meeting)

Dates: October 26 - 29, 2021

Organizers: Peter E. Caines, Shuang Gao, Minyi Huang, and Rinel Foguen Tchuendom

This workshop (PIMS et al., 2021) brought together researchers in applied mathematics, mean field games, network science, network games, and systems and control theory to exchange ideas and to work on the extensions of mean field game theory to dynamic game problems on heterogeneous large-scale networks. The workshop offered a platform to current results and stimulate discussions on the open challenges in this emerging field, on mathematical theory, computational algorithms, and the applications methodologies of MFG on general networks.

Sponsors: Pacific Institute for the Mathematical Sciences (PIMS), Groupe d'études et de recherche en analyse des décisions (GERAD), and the Fields Institute for Research in Mathematical Sciences.

Participants: Peter E. Caines (PI) and PI's Postdoctoral Researchers: Rinel Foguen Tchuendom, Shuang Gao, Joao Saude, and Yaroslav Salii, plus 50 other international participants.

4 Impacts

As stated earlier, the research on MFG and related topics by the Principal Investigator of the grant FA9550-19-1-0138 has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, presently 2022 - 2025. Consequently, the impacts of research under the 2019 - 2022 grant, FA9550-19-1-0138, may be viewed as a continuation and development of those of the entire program.

Indeed, as explained in Section 1, the MFG field has expanded enormously since its inception. Google links now run to 1.10 billion, citations to papers are in the thousands and conferences and workshops on MFG have proliferated. A sample of the most recent workshops and conference sessions is the following, where we note that the PI and group members have been invited to participate in all of them: Workshop "Mean Field Games on Networks", Pacific Institute for the Mathematical Sciences, Vancouver, BC, Canada, 26 - 29, October, 2021; "Conference on Mathematics and Financial Economics, "Many Player Games and Applications", Humbolt University, Berlin, 29 - 31 August 2022; Workshop on Mean Field Games, Montreal: Centre de Recherches Mathématiques (CRM), 11-15 April, 2022; "Graph Limits, Nonparametric Models, and Estimation Workshop", Simons Institute for the Theory of Computation, UCB, Berkeley, CA, 26-30 September, 2022; IEEE, Control Systems Society Conference on Decision and Control, Cancun, December, 2022, Mean Field Games Session; Mean Field Games Invited Session, IFAC World Congress, Yokahama, Japan, July 2023.

Furthermore, at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively), the last appearing in the period covered by this report. In addition to the very rapid expansion of mathematical and theoretical investigations, applications of MFG theory have spread to a vast range of domains, a sample of the most recent being the optimization of decentralized electric vehicle charging (Foguen Tchuendom et al., 2019), algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Grover et al., 2018), AI reinforcement learning for MFG systems (Agarwal et al., 2022) and polymer science (Welch et al., 2019).

As is shown by this indicative list of publications, meetings and presentations, it is reasonable to assert that the influence of the AFOSR supported research carried out on Mean Field Games and, in particular, on Mean Field Games on networks, has had, and is having, an impact on fields ranging from the core area of systems and control science to economics, finance, energy generation and distribution, operations research, the life sciences and the physical sciences.

5 Changes

There were no changes in the project or its direction.

6 Technical Updates

There are no technical updates to report beyond the continuing research progress and dissemination of results described in report.

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AFOSR Final Report 2023-03-30
Grant FA9550-19-1-0138
New Directions in Mean Field Games: MFG Subpopulation Behaviours and Graphon-MFG Systems

Peter E. Caines

Reporting period April 1st, 2019 to December 31st, 2022

March 24, 2023

1 Overview of Research Objectives

1.1 Background and Overview

Very large networks linking dynamical agents are now ubiquitous and the need to analyse, design and control them is evident. The emergence of the graphon theory of large networks and their infinite limits has enabled the formulation of a theory of the centralized control of dynamical systems distributed on asymptotically infinite networks.

Large-scale systems of competing agents are frequently formulated as dynamic games, this however confronts the problem that for even small populations non-cooperative game theory is intractable. The 2006 breakthrough of Minyi Huang, Roland Malhamé and the PI of FA9550-19-1-0138, Peter Caines, was the introduction of Mean Field Game (MFG) theory (originally termed Nash Certainty Equivalence theory) which was also simultaneously discovered by J.M. Lasry and P.L. Lions.

Mean Field Game (MFG) theory analyzes the existence of Nash equilibria for dynamical games in terms of infinite populations of negligible agents forming the limits of the original finite population systems (Caines, 2020). The core of MFG theory is a pair of partial differential equations consisting of (i) the Hamilton-Jacobi-Bellman equation of optimal control for a generic agent, and (ii) the Fokker-Planck-Kolmogorov equation describing the controlled evolution of the state of a generic agent. These are linked by the generic agent's state

distribution, known as the system's mean field.

A fundamental aspect of MFG theory justifying its application in a vast range of cases is the ϵ-Nash property by which controls derived from the MFG equations yield near-equilibrium performance for finite systems for sufficiently large populations.

The MFG field has expanded tremendously since its inception. Google search results now run to 1.10 billion, citations to papers are in the thousands, conferences and workshops on MFG have proliferated and at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively). In addition to the explosion of mathematical and theoretical investigations over last 16 years, applications of MFG theory have spread to a vast range of domains including: optimization of decentralized EV charging (Ma et al., 2011; Zhu et al., 2016; Foguen Tchuendom et al., 2019); nonlinear system state estimation (Yang et al., 2013); systemic risk in banking (Carmona et al., 2013); cell phone energy management (Aziz and Caines, 2016); economics of energy production (Ludkovski and Sircar, 2015); macro-economics of growth (Gomes et al., 2016); cloud-based networks in information technology (Hanif et al., 2015); algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Nourian et al., 2011; Grover et al., 2018), AI reinforcement learning for MFG systems (Subramanian and Mahajan, 2019; Agarwal et al., 2022), deep learning applications to the solution of MFGs (Ruthotto et al., 2020) and polymer science (Welch et al., 2019).

In this context it is important to note that the research on MFG by the Principal Investigator has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, most recently, for the period 2022-2025.

1.2 Major Research Goals under Grant FA9550-19-1-0138

The program of research under FA9550-19-1-0138 is focused on the development of a new form of Mean Field Games called Graphon Mean Field Games (GMFGs); these are large population games defined on large, heterogeneous, networks and their asymptotic (graphon) limits.

In this setting, game theoretic equilibria are expressed in terms of the newly defined GMFG PDE equations, which are a sweeping generalization of the classical MFG PDEs.

In a closely related development, the theory of graphons is used as a basis for Graphon Linear Control (GLC) theory in which large populations of centrally controlled dynamical systems interacting over large, heterogeneous networks

may be analyzed and optimized in a tractable way.

Canonical Design Procedure for GMFG and GLC Systems

A major goal has been to generate large scale network game strategies and, respectively, control design procedures with the following standard pattern:

The first step is to consider the graphon limit of the network over which the individual systems are distributed and which constitutes the global system.

In the second step, the resulting graphon based GMFG equations, or, respectively, the graphon based infinite dimensional linear system optimization problems are specified. The GMFG equations are then solved to generate the Nash best response strategies or, respectively, the linear feedback controls are synthesized using the well established infinite dimensional linear systems control theory.

Finally, the best response strategies, or, respectively, the infinite dimensional linear feedback controls are approximated in such a way that they may be applied to the original complex finite systems.

The key feature of this strategy is the following: In both the network mean field game and the network control settings it is demonstrated that as the size of the network tends to infinity, the performance of the finite systems when they are subject to the infinite limit system best response game strategies, or, respectively, centralized control laws, yield pay-offs which converge to their Nash equilibrium, or, respectively, optimal, infinite system limit values.

1.3 Organization of the Sequence of Research Objectives

The work over the full reporting period has been organized with the following sequence of objectives:

(i) The establishment of a rigorous theory of the central results of Graphon Mean Field Game (GMFG) theory,

(ii) The generation of a specific Graphon Linear Control (GLC) theory for the control of large populations of dynamical linear system agents with quadratic performance functions on large, complex networks.

(iii) The development of an approximation theory of associated numerical methods, especially those based upon finite dimensional invariant subspaces of the graphon operator in the GLC case.

(iv) The pioneering development of the theory of GMFG systems embedded in Euclidean space; this underlies our currently developing variational analysis of agent performance in networks as a function of position, where this theory also includes sparse as well as dense networks.

(v) The development of an SIR epidemic model software which permits the simulation of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity, these models are constructed on a data base of real world US surface and air-traffic travel data.

2 Details of Accomplishments

2.1 Graphon Mean Field Game Theory: Foundations

The work in (Caines and Huang, 2021) (Caines and Huang, 2018), (Caines and Huang, 2019), established the foundations and central results for graphon mean field game theory for nonlinear systems, that is to say games of infinite populations on infinite heterogeneous networks as limits of the corresponding finite systems.

Specifically, in the conference papers (Caines and Huang, 2018), (Caines and Huang, 2019) Graphon Mean Field Games (GMFG) and the GMFG equations were introduced for the analysis of non-cooperative dynamical games on unbounded networks. In this formulation, asymptotically infinite sub-populations are located at the nodes of large heterogeneous networks, the asymptotic network limits then constitute a part of the analysis. In (Caines and Huang, 2021), the basic existence and uniqueness results were established for the GMFG equations, and a full derivation of the fundamental results was provided, including the key epsilon-Nash theory for GMFG systems which relates the infinite population equilibria on infinite networks to finite population equilibria on finite networks.

2.2 LQG GMFG Decomposition Methods

The theory of invariant spaces of linear operators developed in the papers (Gao and Caines, 2021) is applied in (Gao et al., 2021b), (Gao et al., 2021a) and (Gao et al., 2022) to yield a tractable methodology for generating decentralized game strategies for systems distributed over complex networks. The fundamental properties of the existence and uniqueness of solutions are established in (Gao et al., 2021b,a, 2022) in two principal ways: (i) Riccati equation decomposition and (ii) fixed point contraction theorem methods.

2.3 LQG GMFG Master Equation Analysis

First, using a probabilistic approach, solutions to the Linear Quadratic Graphon Mean Field Games are expressed as solutions to coupled Forward Backward Stochastic Differential Equations (FBSDEs) of McKean-Vlasov type. Next, the

existence of the so-called Master field, which allows for the decoupling of these FBSDEs is established. Finally, the infinite dimensional Partial Differential Equation (PDE), so-called Master Equation, is established for which the Master Field is a solution (see (Foguen Tchuendom et al., 2020)).

2.4 GMFG with Affine Dynamics

GMFG theory for nonlinear problems with individual affine dynamics has been established in Caines et al. (2022) and the Nash value properties for control affine systems has been analyzed in Caines et al. (2023).

2.5 Graphon Linear Systems Control with Approximation Theory

As explained above, the theory of graphons is further used as a basis upon which centrally controlled linear dynamical systems interacting over large, heterogeneous networks are analyzed in a tractable way.

The graphon control methodology is developed in (Gao and Caines, 2020, 2019a,b) to study control problems for arbitrary-size networks of linear dynamical systems. The complete treatment of problems involving well-posedness, controllability, approximation errors for state-to-state control and regulations is established in (Gao and Caines, 2020).

Scalable and tractable solution methods are established for a class of graphon control problem in (Gao and Caines, 2019a,b). Next, the graphon linear system control framework is extended to more general cases where the state of each individual agent on the network is multi-dimensional (Gao and Caines, 2021); moreover, this work provides a generalized solution method with an application to large-scale networks of coupled harmonic oscillators.

2.6 LQ Graphon Dynamical Systems and Control with Q-space Noise

The work in (Dunyak and Caines, 2022) extends the formulation and analysis of (Gao and Caines, 2021) to include stochastic processes in infinite dimensions as a random process source driving the network's controlled dynamical systems; this generalization involves the formulation of a form of graphon Q-space noise.

2.7 Critical Nodes for GMFG and Embedded GMFG

The works (Caines, 2022),(Foguen Tchuendom et al., 2021), (Foguen Tchuendom et al., 2022a), (Foguen Tchuendom et al., 2022b) and (Caines et al., 2023)

pioneer the theory of GMFG systems embedded in Euclidean space by constructing limit graphs and limit graphons for sequences of graphs sitting in Euclidean space. The crucial notion here is that for embedded graphs and graphons we may define the differentiation of functions on the embedded graph limits. In particular we formulate an extended version of mean field game systems, termed Embedded Graphon Mean Field Games (EGMFG), or Graphexon Mean Field Games (GXMFG), which are amenable to the application of the calculus to the optimization of node location in such systems, this being one of the main objectives of the new development. Moreover this formulation includes the case of sparse graph limits (see (Caines, 2022)) which is not covered by the standard graphon formulation.

2.8 MFG Theory and Related Stochastic Adaptive and Hybrid Control Theory

The paper (Pakniyat and Caines, 2021) constitutes the climax of several decades of work by my own group and many other researchers aimed at producing a rigorous, general, hybrid optimal systems control theory (HOCT). These and related results have great applicability in many domains of engineering and scientific modelling, and furthermore in economics and finance. In fact the paper (Firoozi et al., 2022) presents an extension of HOCT to stochastic MFG systems motivated by trading dynamics in finance.

In (Saude and Caines, 2022), we study the Markowitz dynamic portfolio problem, where an agent seeks to maximize its expected return while minimizing the variance of the return (risk). For the first time in this topic, the impact of the aggregate of the agents' trades is incorporated into the dynamics and costs using an extended mean-field games framework.

2.9 Data Based Protocols for Modelling Large Populations on Large Networks with Applications to SIR Problems

The initial investigation of applying graphon theory to the control of epidemic spread is proposed in (Gao and Caines, 2019a) where scalable solutions based on spectral approximation and linearization are established for controlling virus spreads on networks following a type of controlled network SIS model.
The work (Dunyak and Caines, 2021; Salii, 2021) includes the development of SIR epidemic models which permits the study of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity; the latter models were constructed on a data base of real world US surface and air-traffic travel data.

2.10 Centrality Analysis for Finite and Infinite Networks

This work advances the theory of centralities for dynamical systems on networks and its associated applications. In (Gao, 2022) a family of centralities characterized by fixed-points of networks is identified which includes the standard centralities (e.g. PageRank, Katz-Bonacich, eigen centralities) and equilbrirum cost for games on networks (e.g. Nash values of LQG GMFG). Variation bounds of centralities with respect to the network variations are established. In (Gao, 2021) network centralities have been utilized in the modelling of opinion dynamics on social networks and in identifying significant network partitions based upon opinion disagreement states.

3 Dissemination to Communities of Interest

Research findings and results have been published in the leading systems and control and networks journals including the SIAM Journal on Control and Optimization, IEEE Transactions on Automatic Control, Automatica, IEEE Transactions on Control of Network Systems, and in conferences proceedings including the IEEE Conference on Decision and Control.

3.1 Publications: Journals

S. Gao, P. E. Caines, and M. Huang, "LQG graphon mean field games: analysis via graphon invariant subspaces," Conditionally accepted by IEEE Transactions on Automatic Control, 2021. (Gao et al., 2022)

D. Firoozi, A. Pakniyat, and P. E. Caines, "A class of hybrid LQG mean field games with state-invariant switching and stopping strategies," Automatica, vol. 141, p. 110244, 2022. (Firoozi et al., 2022)

P. E. Caines, D. Ho, M. Huang, J. Jian, and Q. Song. On the graphon mean field game equations: Individual agent affine dynamics and mean field dependent performance functions. ESAIM: Control, Optimisation and Calculus of Variations, 28:24, 2022 (Caines et al., 2022)

P. E. Caines and M. Huang, "Graphon mean field games and their equations," SIAM Journal on Control and Optimization, vol. 59, no. 6, pp. 4373–4399, 2021. (Caines and Huang, 2021)

A. Pakniyat and P. E. Caines, "On the hybrid minimum principle: The Hamiltonian and adjoint boundary conditions," IEEE Transactions on Automatic Control, vol. 66, no. 3, pp. 1246–1253, 2021. (Pakniyat and Caines, 2021)

S. Gao and P. E. Caines, "Subspace decomposition for graphon LQR: Applications to VLSNs of harmonic oscillators," IEEE Transactions on Control of Network Systems, vol. 8, no. 2, pp. 576–586, 2021. (Gao and Caines, 2021)

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S. Gao and P. E. Caines. Graphon control of large-scale networks of linear systems. IEEE Transactions on Automatic Control, 65(10):4090–4105, 2020 (Gao and Caines, 2020)

N. Sen and P. E. Caines. Mean field games with partial observation. SIAM Journal on Control and Optimization, 57(3):2064–2091, 2019 (Sen and Caines, 2019)

P. E. Caines, D. Ho, and Q. Song. The density evolution of the killed mckean–vlasov process. Stochastics, 92(4):642–657, 2020. (Caines et al., 2020)

3.2 Publications: Conference Proceedings

A. Dunyak and P. E. Caines, "Linear stochastic graphon systems with q- space noise," in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 3926–3932. (Dunyak and Caines, 2022)

P. E. Caines, "Embedded vertexon-graphons and embedded GMFG systems," Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022,

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3.3 Talks: April 1st, 2019 to December 31st, 2022

1. P. E. Caines,"Vertexons and Embedded Graphon Mean Field Games", the Simons Institute for the Theory of Computing, UC Berkeley, CA, Sept.29, 2022 (Invited talk) in the "Graph Limits, Nonparametric Models, and Estimation Workshop" 26 - 30 September, 2022.

2. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", (Invited talk) 29th August, 2022, in the Conference on Many Player Games and Applications, Humbolt University Berlin, 29 - 31 August 2022.

3. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Large Populations on Complex Networks", Canadian Applied and Industrial Mathematics Society (CAIMS), June 23rd, 2021 (Plenary talk).

4. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", the Financial Mathematics/Engineering Seminar Series, Hong Kong Polytechnic University, June 16th 2021 (Invited talk).

5. P. E. Caines, "Static and Time Varying Graphons in Economic and Pandemic GMFG Models" SIAM Financial Mathematics and Engineering (FM21), June 1, 2021 (Invited talk).

6. S. Gao, "Subspace Decompositions in Graphon Control and Graphon Mean Field Games", Informal Systems Seminar, Centre for Intelligent Machines,

McGill University, Groupe d'Etudes et de Recherche en Analyse des Decisions, Montreal, CA, Dec. 11, 2020 (Invited talk).

7. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Networked Systems", Invited Speaker at the "Conference on Mean Field Games and Related Topics - 5". 9th - 13th September, 2019

8. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for a Networked World", University of Saint Petersburg, Russia, 6th September, 2019 (Distinguished Speaker).

9. P. E. Caines, "Graphon Control and Graphon Mean Field Games", Department of Applied Mathemaics, Hong Kong Polythenic Universty, Hong Kong), 28th June, 2019 (Distinguished Lecturer).

3.4 Workshop Organization: April 1st, 2019 to December 31st, 2022

Name: **Mean Field Games on Networks Workshop**

Location: Vancouver, BC, Canada (Virtual Meeting)

Dates: October 26 - 29, 2021

Organizers: Peter E. Caines, Shuang Gao, Minyi Huang, and Rinel Foguen Tchuendom

This workshop (PIMS et al., 2021) brought together researchers in applied mathematics, mean field games, network science, network games, and systems and control theory to exchange ideas and to work on the extensions of mean field game theory to dynamic game problems on heterogeneous large-scale networks. The workshop offered a platform to current results and stimulate discussions on the open challenges in this emerging field, on mathematical theory, computational algorithms, and the applications methodologies of MFG on general networks.

Sponsors: Pacific Institute for the Mathematical Sciences (PIMS), Groupe d'études et de recherche en analyse des décisions (GERAD), and the Fields Institute for Research in Mathematical Sciences.

Participants: Peter E. Caines (PI) and PI's Postdoctoral Researchers: Rinel Foguen Tchuendom, Shuang Gao, Joao Saude, and Yaroslav Salii, plus 50 other international participants.

4 Impacts

As stated earlier, the research on MFG and related topics by the Principal Investigator of the grant FA9550-19-1-0138 has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, presently 2022 - 2025. Consequently, the impacts of research under the 2019 - 2022 grant, FA9550-19-1-0138, may be viewed as a continuation and development of those of the entire program.

Indeed, as explained in Section 1, the MFG field has expanded enormously since its inception. Google links now run to 1.10 billion, citations to papers are in the thousands and conferences and workshops on MFG have proliferated. A sample of the most recent workshops and conference sessions is the following, where we note that the PI and group members have been invited to participate in all of them: Workshop "Mean Field Games on Networks", Pacific Institute for the Mathematical Sciences, Vancouver, BC, Canada, 26 - 29, October, 2021; "Conference on Mathematics and Financial Economics, "Many Player Games and Applications", Humbolt University, Berlin, 29 - 31 August 2022; Workshop on Mean Field Games, Montreal: Centre de Recherches Mathématiques (CRM), 11-15 April, 2022; "Graph Limits, Nonparametric Models, and Estimation Workshop", Simons Institute for the Theory of Computation, UCB, Berkeley, CA, 26-30 September, 2022; IEEE, Control Systems Society Conference on Decision and Control, Cancun, December, 2022, Mean Field Games Session; Mean Field Games Invited Session, IFAC World Congress, Yokahama, Japan, July 2023.

Furthermore, at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively), the last appearing in the period covered by this report. In addition to the very rapid expansion of mathematical and theoretical investigations, applications of MFG theory have spread to a vast range of domains, a sample of the most recent being the optimization of decentralized electric vehicle charging (Foguen Tchuendom et al., 2019), algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Grover et al., 2018), AI reinforcement learning for MFG systems (Agarwal et al., 2022) and polymer science (Welch et al., 2019).

As is shown by this indicative list of publications, meetings and presentations, it is reasonable to assert that the influence of the AFOSR supported research carried out on Mean Field Games and, in particular, on Mean Field Games on networks, has had, and is having, an impact on fields ranging from the core area of systems and control science to economics, finance, energy generation and distribution, operations research, the life sciences and the physical sciences.

5 Changes

There were no changes in the project or its direction.

6 Technical Updates

There are no technical updates to report beyond the continuing research progress and dissemination of results described in report.

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New Directions in Mean Field Games: MFG Subpopulation Behaviours and Graphon-MFG Systems

Peter E. Caines

Reporting period April 1st, 2019 to December 31st, 2022

March 24, 2023

1 Overview of Research Objectives

1.1 Background and Overview

Very large networks linking dynamical agents are now ubiquitous and the need to analyse, design and control them is evident. The emergence of the graphon theory of large networks and their infinite limits has enabled the formulation of a theory of the centralized control of dynamical systems distributed on asymptotically infinite networks.

Large-scale systems of competing agents are frequently formulated as dynamic games, this however confronts the problem that for even small populations non-cooperative game theory is intractable. The 2006 breakthrough of Minyi Huang, Roland Malhamé and the PI of FA9550-19-1-0138, Peter Caines, was the introduction of Mean Field Game (MFG) theory (originally termed Nash Certainty Equivalence theory) which was also simultaneously discovered by J.M. Lasry and P.L. Lions.

Mean Field Game (MFG) theory analyzes the existence of Nash equilibria for dynamical games in terms of infinite populations of negligible agents forming the limits of the original finite population systems (Caines, 2020). The core of MFG theory is a pair of partial differential equations consisting of (i) the Hamilton-Jacobi-Bellman equation of optimal control for a generic agent, and (ii) the Fokker-Planck-Kolmogorov equation describing the controlled evolution of the state of a generic agent. These are linked by the generic agent's state

distribution, known as the system's mean field.

A fundamental aspect of MFG theory justifying its application in a vast range of cases is the ϵ-Nash property by which controls derived from the MFG equations yield near-equilibrium performance for finite systems for sufficiently large populations.

The MFG field has expanded tremendously since its inception. Google search results now run to 1.10 billion, citations to papers are in the thousands, conferences and workshops on MFG have proliferated and at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively). In addition to the explosion of mathematical and theoretical investigations over last 16 years, applications of MFG theory have spread to a vast range of domains including: optimization of decentralized EV charging (Ma et al., 2011; Zhu et al., 2016; Foguen Tchuendom et al., 2019); nonlinear system state estimation (Yang et al., 2013); systemic risk in banking (Carmona et al., 2013); cell phone energy management (Aziz and Caines, 2016); economics of energy production (Ludkovski and Sircar, 2015); macro-economics of growth (Gomes et al., 2016); cloud-based networks in information technology (Hanif et al., 2015); algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Nourian et al., 2011; Grover et al., 2018), AI reinforcement learning for MFG systems (Subramanian and Mahajan, 2019; Agarwal et al., 2022), deep learning applications to the solution of MFGs (Ruthotto et al., 2020) and polymer science (Welch et al., 2019).

In this context it is important to note that the research on MFG by the Principal Investigator has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, most recently, for the period 2022-2025.

1.2 Major Research Goals under Grant FA9550-19-1-0138

The program of research under FA9550-19-1-0138 is focused on the development of a new form of Mean Field Games called Graphon Mean Field Games (GMFGs); these are large population games defined on large, heterogeneous, networks and their asymptotic (graphon) limits.

In this setting, game theoretic equilibria are expressed in terms of the newly defined GMFG PDE equations, which are a sweeping generalization of the classical MFG PDEs.

In a closely related development, the theory of graphons is used as a basis for Graphon Linear Control (GLC) theory in which large populations of centrally controlled dynamical systems interacting over large, heterogeneous networks

may be analyzed and optimized in a tractable way.

Canonical Design Procedure for GMFG and GLC Systems

A major goal has been to generate large scale network game strategies and, respectively, control design procedures with the following standard pattern:

The first step is to consider the graphon limit of the network over which the individual systems are distributed and which constitutes the global system.

In the second step, the resulting graphon based GMFG equations, or, respectively, the graphon based infinite dimensional linear system optimization problems are specified. The GMFG equations are then solved to generate the Nash best response strategies or, respectively, the linear feedback controls are synthesized using the well established infinite dimensional linear systems control theory.

Finally, the best response strategies, or, respectively, the infinite dimensional linear feedback controls are approximated in such a way that they may be applied to the original complex finite systems.

The key feature of this strategy is the following: In both the network mean field game and the network control settings it is demonstrated that as the size of the network tends to infinity, the performance of the finite systems when they are subject to the infinite limit system best response game strategies, or, respectively, centralized control laws, yield pay-offs which converge to their Nash equilibrium, or, respectively, optimal, infinite system limit values.

1.3 Organization of the Sequence of Research Objectives

The work over the full reporting period has been organized with the following sequence of objectives:

(i) The establishment of a rigorous theory of the central results of Graphon Mean Field Game (GMFG) theory,

(ii) The generation of a specific Graphon Linear Control (GLC) theory for the control of large populations of dynamical linear system agents with quadratic performance functions on large, complex networks.

(iii) The development of an approximation theory of associated numerical methods, especially those based upon finite dimensional invariant subspaces of the graphon operator in the GLC case.

(iv) The pioneering development of the theory of GMFG systems embedded in Euclidean space; this underlies our currently developing variational analysis of agent performance in networks as a function of position, where this theory also includes sparse as well as dense networks.

(v) The development of an SIR epidemic model software which permits the simulation of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity, these models are constructed on a data base of real world US surface and air-traffic travel data.

2 Details of Accomplishments

2.1 Graphon Mean Field Game Theory: Foundations

The work in (Caines and Huang, 2021) (Caines and Huang, 2018), (Caines and Huang, 2019), established the foundations and central results for graphon mean field game theory for nonlinear systems, that is to say games of infinite populations on infinite heterogeneous networks as limits of the corresponding finite systems.

Specifically, in the conference papers (Caines and Huang, 2018), (Caines and Huang, 2019) Graphon Mean Field Games (GMFG) and the GMFG equations were introduced for the analysis of non-cooperative dynamical games on unbounded networks. In this formulation, asymptotically infinite sub-populations are located at the nodes of large heterogeneous networks, the asymptotic network limits then constitute a part of the analysis. In (Caines and Huang, 2021), the basic existence and uniqueness results were established for the GMFG equations, and a full derivation of the fundamental results was provided, including the key epsilon-Nash theory for GMFG systems which relates the infinite population equilibria on infinite networks to finite population equilibria on finite networks.

2.2 LQG GMFG Decomposition Methods

The theory of invariant spaces of linear operators developed in the papers (Gao and Caines, 2021) is applied in (Gao et al., 2021b), (Gao et al., 2021a) and (Gao et al., 2022) to yield a tractable methodology for generating decentralized game strategies for systems distributed over complex networks. The fundamental properties of the existence and uniqueness of solutions are established in (Gao et al., 2021b,a, 2022) in two principal ways: (i) Riccati equation decomposition and (ii) fixed point contraction theorem methods.

2.3 LQG GMFG Master Equation Analysis

First, using a probabilistic approach, solutions to the Linear Quadratic Graphon Mean Field Games are expressed as solutions to coupled Forward Backward Stochastic Differential Equations (FBSDEs) of McKean-Vlasov type. Next, the

existence of the so-called Master field, which allows for the decoupling of these FBSDEs is established. Finally, the infinite dimensional Partial Differential Equation (PDE), so-called Master Equation, is established for which the Master Field is a solution (see (Foguen Tchuendom et al., 2020)).

2.4 GMFG with Affine Dynamics

GMFG theory for nonlinear problems with individual affine dynamics has been established in Caines et al. (2022) and the Nash value properties for control affine systems has been analyzed in Caines et al. (2023).

2.5 Graphon Linear Systems Control with Approximation Theory

As explained above, the theory of graphons is further used as a basis upon which centrally controlled linear dynamical systems interacting over large, heterogeneous networks are analyzed in a tractable way.

The graphon control methodology is developed in (Gao and Caines, 2020, 2019a,b) to study control problems for arbitrary-size networks of linear dynamical systems. The complete treatment of problems involving well-posedness, controllability, approximation errors for state-to-state control and regulations is established in (Gao and Caines, 2020).

Scalable and tractable solution methods are established for a class of graphon control problem in (Gao and Caines, 2019a,b). Next, the graphon linear system control framework is extended to more general cases where the state of each individual agent on the network is multi-dimensional (Gao and Caines, 2021); moreover, this work provides a generalized solution method with an application to large-scale networks of coupled harmonic oscillators.

2.6 LQ Graphon Dynamical Systems and Control with Q-space Noise

The work in (Dunyak and Caines, 2022) extends the formulation and analysis of (Gao and Caines, 2021) to include stochastic processes in infinite dimensions as a random process source driving the network's controlled dynamical systems; this generalization involves the formulation of a form of graphon Q-space noise.

2.7 Critical Nodes for GMFG and Embedded GMFG

The works (Caines, 2022),(Foguen Tchuendom et al., 2021), (Foguen Tchuendom et al., 2022a), (Foguen Tchuendom et al., 2022b) and (Caines et al., 2023)

pioneer the theory of GMFG systems embedded in Euclidean space by constructing limit graphs and limit graphons for sequences of graphs sitting in Euclidean space. The crucial notion here is that for embedded graphs and graphons we may define the differentiation of functions on the embedded graph limits. In particular we formulate an extended version of mean field game systems, termed Embedded Graphon Mean Field Games (EGMFG), or Graphexon Mean Field Games (GXMFG), which are amenable to the application of the calculus to the optimization of node location in such systems, this being one of the main objectives of the new development. Moreover this formulation includes the case of sparse graph limits (see (Caines, 2022)) which is not covered by the standard graphon formulation.

2.8 MFG Theory and Related Stochastic Adaptive and Hybrid Control Theory

The paper (Pakniyat and Caines, 2021) constitutes the climax of several decades of work by my own group and many other researchers aimed at producing a rigorous, general, hybrid optimal systems control theory (HOCT). These and related results have great applicability in many domains of engineering and scientific modelling, and furthermore in economics and finance. In fact the paper (Firoozi et al., 2022) presents an extension of HOCT to stochastic MFG systems motivated by trading dynamics in finance.

In (Saude and Caines, 2022), we study the Markowitz dynamic portfolio problem, where an agent seeks to maximize its expected return while minimizing the variance of the return (risk). For the first time in this topic, the impact of the aggregate of the agents' trades is incorporated into the dynamics and costs using an extended mean-field games framework.

2.9 Data Based Protocols for Modelling Large Populations on Large Networks with Applications to SIR Problems

The initial investigation of applying graphon theory to the control of epidemic spread is proposed in (Gao and Caines, 2019a) where scalable solutions based on spectral approximation and linearization are established for controlling virus spreads on networks following a type of controlled network SIS model.
The work (Dunyak and Caines, 2021; Salii, 2021) includes the development of SIR epidemic models which permits the study of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity; the latter models were constructed on a data base of real world US surface and air-traffic travel data.

2.10 Centrality Analysis for Finite and Infinite Networks

This work advances the theory of centralities for dynamical systems on networks and its associated applications. In (Gao, 2022) a family of centralities characterized by fixed-points of networks is identified which includes the standard centralities (e.g. PageRank, Katz-Bonacich, eigen centralities) and equilbrirum cost for games on networks (e.g. Nash values of LQG GMFG). Variation bounds of centralities with respect to the network variations are established. In (Gao, 2021) network centralities have been utilized in the modelling of opinion dynamics on social networks and in identifying significant network partitions based upon opinion disagreement states.

3 Dissemination to Communities of Interest

Research findings and results have been published in the leading systems and control and networks journals including the SIAM Journal on Control and Optimization, IEEE Transactions on Automatic Control, Automatica, IEEE Transactions on Control of Network Systems, and in conferences proceedings including the IEEE Conference on Decision and Control.

3.1 Publications: Journals

S. Gao, P. E. Caines, and M. Huang, "LQG graphon mean field games: analysis via graphon invariant subspaces," Conditionally accepted by IEEE Transactions on Automatic Control, 2021. (Gao et al., 2022)

D. Firoozi, A. Pakniyat, and P. E. Caines, "A class of hybrid LQG mean field games with state-invariant switching and stopping strategies," Automatica, vol. 141, p. 110244, 2022. (Firoozi et al., 2022)

P. E. Caines, D. Ho, M. Huang, J. Jian, and Q. Song. On the graphon mean field game equations: Individual agent affine dynamics and mean field dependent performance functions. ESAIM: Control, Optimisation and Calculus of Variations, 28:24, 2022 (Caines et al., 2022)

P. E. Caines and M. Huang, "Graphon mean field games and their equations," SIAM Journal on Control and Optimization, vol. 59, no. 6, pp. 4373–4399, 2021. (Caines and Huang, 2021)

A. Pakniyat and P. E. Caines, "On the hybrid minimum principle: The Hamiltonian and adjoint boundary conditions," IEEE Transactions on Automatic Control, vol. 66, no. 3, pp. 1246–1253, 2021. (Pakniyat and Caines, 2021)

S. Gao and P. E. Caines, "Subspace decomposition for graphon LQR: Applications to VLSNs of harmonic oscillators," IEEE Transactions on Control of Network Systems, vol. 8, no. 2, pp. 576–586, 2021. (Gao and Caines, 2021)

S. Gao, R. Foguen Tchuendom, and P. E. Caines, "Linear quadratic graphon field games," Communications in Information and Systems, vol. 21, no. 3, pp. 341–369, 2021. (Gao et al., 2021b)

D. Firoozi, and P. E. Caines, "ϵ-Nash Equilibria for Major-Minor LQG Mean Field Games with Partial Observations of All Agents", *IEEE Trans.on Automatic Control*, July, 2020, Vol 66, No. 6, pp 2778 - 2786. On-line: DOI: 10.1109/TAC.2020.3010129 (Firoozi and Caines, 2020)

P. E. Caines. Mean field games. In Encyclopedia of Systems and Control, pages 1197–1202. Springer, 2021. (Caines, 2021)

P. E. Caines. Mean field game theory: A tractable methodology for large population problems. SIAM News, 53(3):5–6, 2020. (Caines, 2020)

D. Firoozi, S. Jaimungal, and P. E. Caines. Convex analysis for lqg systems with applications to major–minor LQG mean–field game systems. Systems & Control Letters, 142:104734, 2020. (Firoozi et al., 2020)

S. Gao and P. E. Caines. Graphon control of large-scale networks of linear systems. IEEE Transactions on Automatic Control, 65(10):4090–4105, 2020 (Gao and Caines, 2020)

N. Sen and P. E. Caines. Mean field games with partial observation. SIAM Journal on Control and Optimization, 57(3):2064–2091, 2019 (Sen and Caines, 2019)

P. E. Caines, D. Ho, and Q. Song. The density evolution of the killed mckean–vlasov process. Stochastics, 92(4):642–657, 2020. (Caines et al., 2020)

3.2 Publications: Conference Proceedings

A. Dunyak and P. E. Caines, "Linear stochastic graphon systems with q- space noise," in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 3926–3932. (Dunyak and Caines, 2022)

P. E. Caines, "Embedded vertexon-graphons and embedded GMFG systems," Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022,

pp. 5550–5557. (Caines, 2022)

S. Gao, "Fixed-point centrality for networks," in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 1628–1635. (Gao, 2022).

R. Foguen Tchuendom, S. Gao, P. E. Caines, and M. Huang, "Optimal network location in infinite horizon LQG graphon mean field games," in Proceedings of the 61th IEEE Conference on Decision and Control, Cancun, Mexico, December 2022, pp. 5558–5565. (Foguen Tchuendom et al., 2022b)

R. Foguen Tchuendom, S. Gao, and P. E. Caines "Stationary cost nodes in infinite horizon LQG GMFGs," IFAC-PapersOnLine for the 25th International Symposium on Mathematical Theory of Networks and Systems (MTNS), vol. 55, no. 30, pp. 284–289, 2022. (Foguen Tchuendom et al., 2022a)

J. Saude and P. E. Caines, "Markowitz portfolio optimization extended quadratic mean- field games approach," Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 5523–5528. (Saude and Caines, 2022)

Y. V. Salii, "Benchmarking optimal control for network dynamic systems with plausible epidemic models," in International Conference on Complex Networks and Their Applications. Springer, 2021, pp. 194–206. (Salii, 2021)

S. Gao, P. E. Caines, and M. Huang, "LQG graphon mean field games: Graphon invariant subspaces," Proceedings of the 60th IEEE Conference on Decision and Control, Austin, Texas, USA, December 2021, pp. 5253–5260. (Gao et al., 2021a)

R. Foguen Tchuendom, P. E. Caines, and M. Huang, "Critical nodes in graphon mean field games," Proceedings of the 60th IEEE Conference on Decision and Control (CDC). IEEE, 2021, pp. 166–170. (Foguen Tchuendom et al., 2021)

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S. Gao, "Centrality-weighted opinion dynamics: Disagreement and social network partition," Proceedings of the 60th IEEE Conference on Decision and Control (CDC), Austin, Texas, USA, December 2021, pp. 5496–5501. (Gao, 2021)

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S. Gao and P. E. Caines, Optimal and approximate solutions to linear quadratic regulation of a class of graphon dynamical systems, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 8359–8365, Nice, France, December 2019b. (Gao and Caines, 2019b)

S. Gao and P. E. Caines, Spectral representations of graphons in very large network systems control, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 5068–5075, Nice, France, December 2019a.(Gao and Caines, 2019a)

P. E. Caines and M. Huang. Graphon mean field games and the GMFG equations: ϵ-Nash equilibria, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 286–292, December 2019. (Caines and Huang, 2019)

3.3 Talks: April 1st, 2019 to December 31st, 2022

1. P. E. Caines,"Vertexons and Embedded Graphon Mean Field Games", the Simons Institute for the Theory of Computing, UC Berkeley, CA, Sept.29, 2022 (Invited talk) in the "Graph Limits, Nonparametric Models, and Estimation Workshop" 26 - 30 September, 2022.

2. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", (Invited talk) 29th August, 2022, in the Conference on Many Player Games and Applications, Humbolt University Berlin, 29 - 31 August 2022.

3. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Large Populations on Complex Networks", Canadian Applied and Industrial Mathematics Society (CAIMS), June 23rd, 2021 (Plenary talk).

4. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", the Financial Mathematics/Engineering Seminar Series, Hong Kong Polytechnic University, June 16th 2021 (Invited talk).

5. P. E. Caines, "Static and Time Varying Graphons in Economic and Pandemic GMFG Models" SIAM Financial Mathematics and Engineering (FM21), June 1, 2021 (Invited talk).

6. S. Gao, "Subspace Decompositions in Graphon Control and Graphon Mean Field Games", Informal Systems Seminar, Centre for Intelligent Machines,

McGill University, Groupe d'Etudes et de Recherche en Analyse des Decisions, Montreal, CA, Dec. 11, 2020 (Invited talk).

7. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Networked Systems", Invited Speaker at the "Conference on Mean Field Games and Related Topics - 5". 9th - 13th September, 2019

8. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for a Networked World", University of Saint Petersburg, Russia, 6th September, 2019 (Distinguished Speaker).

9. P. E. Caines, "Graphon Control and Graphon Mean Field Games", Department of Applied Mathemaics, Hong Kong Polythenic Universty, Hong Kong), 28th June, 2019 (Distinguished Lecturer).

3.4 Workshop Organization: April 1st, 2019 to December 31st, 2022

Name: **Mean Field Games on Networks Workshop**

Location: Vancouver, BC, Canada (Virtual Meeting)

Dates: October 26 - 29, 2021

Organizers: Peter E. Caines, Shuang Gao, Minyi Huang, and Rinel Foguen Tchuendom

This workshop (PIMS et al., 2021) brought together researchers in applied mathematics, mean field games, network science, network games, and systems and control theory to exchange ideas and to work on the extensions of mean field game theory to dynamic game problems on heterogeneous large-scale networks. The workshop offered a platform to current results and stimulate discussions on the open challenges in this emerging field, on mathematical theory, computational algorithms, and the applications methodologies of MFG on general networks.

Sponsors: Pacific Institute for the Mathematical Sciences (PIMS), Groupe d'études et de recherche en analyse des décisions (GERAD), and the Fields Institute for Research in Mathematical Sciences.

Participants: Peter E. Caines (PI) and PI's Postdoctoral Researchers: Rinel Foguen Tchuendom, Shuang Gao, Joao Saude, and Yaroslav Salii, plus 50 other international participants.

4 Impacts

As stated earlier, the research on MFG and related topics by the Principal Investigator of the grant FA9550-19-1-0138 has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, presently 2022 - 2025. Consequently, the impacts of research under the 2019 - 2022 grant, FA9550-19-1-0138, may be viewed as a continuation and development of those of the entire program.

Indeed, as explained in Section 1, the MFG field has expanded enormously since its inception. Google links now run to 1.10 billion, citations to papers are in the thousands and conferences and workshops on MFG have proliferated. A sample of the most recent workshops and conference sessions is the following, where we note that the PI and group members have been invited to participate in all of them: Workshop "Mean Field Games on Networks", Pacific Institute for the Mathematical Sciences, Vancouver, BC, Canada, 26 - 29, October, 2021; "Conference on Mathematics and Financial Economics, "Many Player Games and Applications", Humbolt University, Berlin, 29 - 31 August 2022; Workshop on Mean Field Games, Montreal: Centre de Recherches Mathématiques (CRM), 11-15 April, 2022; "Graph Limits, Nonparametric Models, and Estimation Workshop", Simons Institute for the Theory of Computation, UCB, Berkeley, CA, 26-30 September, 2022; IEEE, Control Systems Society Conference on Decision and Control, Cancun, December, 2022, Mean Field Games Session; Mean Field Games Invited Session, IFAC World Congress, Yokahama, Japan, July 2023.

Furthermore, at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively), the last appearing in the period covered by this report. In addition to the very rapid expansion of mathematical and theoretical investigations, applications of MFG theory have spread to a vast range of domains, a sample of the most recent being the optimization of decentralized electric vehicle charging (Foguen Tchuendom et al., 2019), algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Grover et al., 2018), AI reinforcement learning for MFG systems (Agarwal et al., 2022) and polymer science (Welch et al., 2019).

As is shown by this indicative list of publications, meetings and presentations, it is reasonable to assert that the influence of the AFOSR supported research carried out on Mean Field Games and, in particular, on Mean Field Games on networks, has had, and is having, an impact on fields ranging from the core area of systems and control science to economics, finance, energy generation and distribution, operations research, the life sciences and the physical sciences.

5 Changes

There were no changes in the project or its direction.

6 Technical Updates

There are no technical updates to report beyond the continuing research progress and dissemination of results described in report.

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AFOSR Final Report 2023-03-30
Grant FA9550-19-1-0138
New Directions in Mean Field Games: MFG Subpopulation Behaviours and Graphon-MFG Systems

Peter E. Caines

Reporting period April 1st, 2019 to December 31st, 2022

March 24, 2023

1 Overview of Research Objectives

1.1 Background and Overview

Very large networks linking dynamical agents are now ubiquitous and the need to analyse, design and control them is evident. The emergence of the graphon theory of large networks and their infinite limits has enabled the formulation of a theory of the centralized control of dynamical systems distributed on asymptotically infinite networks.

Large-scale systems of competing agents are frequently formulated as dynamic games, this however confronts the problem that for even small populations non-cooperative game theory is intractable. The 2006 breakthrough of Minyi Huang, Roland Malhamé and the PI of FA9550-19-1-0138, Peter Caines, was the introduction of Mean Field Game (MFG) theory (originally termed Nash Certainty Equivalence theory) which was also simultaneously discovered by J.M. Lasry and P.L. Lions.

Mean Field Game (MFG) theory analyzes the existence of Nash equilibria for dynamical games in terms of infinite populations of negligible agents forming the limits of the original finite population systems (Caines, 2020). The core of MFG theory is a pair of partial differential equations consisting of (i) the Hamilton-Jacobi-Bellman equation of optimal control for a generic agent, and (ii) the Fokker-Planck-Kolmogorov equation describing the controlled evolution of the state of a generic agent. These are linked by the generic agent's state

distribution, known as the system's mean field.

A fundamental aspect of MFG theory justifying its application in a vast range of cases is the ϵ-Nash property by which controls derived from the MFG equations yield near-equilibrium performance for finite systems for sufficiently large populations.

The MFG field has expanded tremendously since its inception. Google search results now run to 1.10 billion, citations to papers are in the thousands, conferences and workshops on MFG have proliferated and at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively). In addition to the explosion of mathematical and theoretical investigations over last 16 years, applications of MFG theory have spread to a vast range of domains including: optimization of decentralized EV charging (Ma et al., 2011; Zhu et al., 2016; Foguen Tchuendom et al., 2019); nonlinear system state estimation (Yang et al., 2013); systemic risk in banking (Carmona et al., 2013); cell phone energy management (Aziz and Caines, 2016); economics of energy production (Ludkovski and Sircar, 2015); macro-economics of growth (Gomes et al., 2016); cloud-based networks in information technology (Hanif et al., 2015); algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Nourian et al., 2011; Grover et al., 2018), AI reinforcement learning for MFG systems (Subramanian and Mahajan, 2019; Agarwal et al., 2022), deep learning applications to the solution of MFGs (Ruthotto et al., 2020) and polymer science (Welch et al., 2019).

In this context it is important to note that the research on MFG by the Principal Investigator has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, most recently, for the period 2022-2025.

1.2 Major Research Goals under Grant FA9550-19-1-0138

The program of research under FA9550-19-1-0138 is focused on the development of a new form of Mean Field Games called Graphon Mean Field Games (GMFGs); these are large population games defined on large, heterogeneous, networks and their asymptotic (graphon) limits.

In this setting, game theoretic equilibria are expressed in terms of the newly defined GMFG PDE equations, which are a sweeping generalization of the classical MFG PDEs.

In a closely related development, the theory of graphons is used as a basis for Graphon Linear Control (GLC) theory in which large populations of centrally controlled dynamical systems interacting over large, heterogeneous networks

may be analyzed and optimized in a tractable way.

Canonical Design Procedure for GMFG and GLC Systems

A major goal has been to generate large scale network game strategies and, respectively, control design procedures with the following standard pattern:

The first step is to consider the graphon limit of the network over which the individual systems are distributed and which constitutes the global system.

In the second step, the resulting graphon based GMFG equations, or, respectively, the graphon based infinite dimensional linear system optimization problems are specified. The GMFG equations are then solved to generate the Nash best response strategies or, respectively, the linear feedback controls are synthesized using the well established infinite dimensional linear systems control theory.

Finally, the best response strategies, or, respectively, the infinite dimensional linear feedback controls are approximated in such a way that they may be applied to the original complex finite systems.

The key feature of this strategy is the following: In both the network mean field game and the network control settings it is demonstrated that as the size of the network tends to infinity, the performance of the finite systems when they are subject to the infinite limit system best response game strategies, or, respectively, centralized control laws, yield pay-offs which converge to their Nash equilibrium, or, respectively, optimal, infinite system limit values.

1.3 Organization of the Sequence of Research Objectives

The work over the full reporting period has been organized with the following sequence of objectives:

(i) The establishment of a rigorous theory of the central results of Graphon Mean Field Game (GMFG) theory,

(ii) The generation of a specific Graphon Linear Control (GLC) theory for the control of large populations of dynamical linear system agents with quadratic performance functions on large, complex networks.

(iii) The development of an approximation theory of associated numerical methods, especially those based upon finite dimensional invariant subspaces of the graphon operator in the GLC case.

(iv) The pioneering development of the theory of GMFG systems embedded in Euclidean space; this underlies our currently developing variational analysis of agent performance in networks as a function of position, where this theory also includes sparse as well as dense networks.

(v) The development of an SIR epidemic model software which permits the simulation of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity, these models are constructed on a data base of real world US surface and air-traffic travel data.

2 Details of Accomplishments

2.1 Graphon Mean Field Game Theory: Foundations

The work in (Caines and Huang, 2021) (Caines and Huang, 2018), (Caines and Huang, 2019), established the foundations and central results for graphon mean field game theory for nonlinear systems, that is to say games of infinite populations on infinite heterogeneous networks as limits of the corresponding finite systems.

Specifically, in the conference papers (Caines and Huang, 2018), (Caines and Huang, 2019) Graphon Mean Field Games (GMFG) and the GMFG equations were introduced for the analysis of non-cooperative dynamical games on unbounded networks. In this formulation, asymptotically infinite sub-populations are located at the nodes of large heterogeneous networks, the asymptotic network limits then constitute a part of the analysis. In (Caines and Huang, 2021), the basic existence and uniqueness results were established for the GMFG equations, and a full derivation of the fundamental results was provided, including the key epsilon-Nash theory for GMFG systems which relates the infinite population equilibria on infinite networks to finite population equilibria on finite networks.

2.2 LQG GMFG Decomposition Methods

The theory of invariant spaces of linear operators developed in the papers (Gao and Caines, 2021) is applied in (Gao et al., 2021b), (Gao et al., 2021a) and (Gao et al., 2022) to yield a tractable methodology for generating decentralized game strategies for systems distributed over complex networks. The fundamental properties of the existence and uniqueness of solutions are established in (Gao et al., 2021b,a, 2022) in two principal ways: (i) Riccati equation decomposition and (ii) fixed point contraction theorem methods.

2.3 LQG GMFG Master Equation Analysis

First, using a probabilistic approach, solutions to the Linear Quadratic Graphon Mean Field Games are expressed as solutions to coupled Forward Backward Stochastic Differential Equations (FBSDEs) of McKean-Vlasov type. Next, the

existence of the so-called Master field, which allows for the decoupling of these FBSDEs is established. Finally, the infinite dimensional Partial Differential Equation (PDE), so-called Master Equation, is established for which the Master Field is a solution (see (Foguen Tchuendom et al., 2020)).

2.4 GMFG with Affine Dynamics

GMFG theory for nonlinear problems with individual affine dynamics has been established in Caines et al. (2022) and the Nash value properties for control affine systems has been analyzed in Caines et al. (2023).

2.5 Graphon Linear Systems Control with Approximation Theory

As explained above, the theory of graphons is further used as a basis upon which centrally controlled linear dynamical systems interacting over large, heterogeneous networks are analyzed in a tractable way.

The graphon control methodology is developed in (Gao and Caines, 2020, 2019a,b) to study control problems for arbitrary-size networks of linear dynamical systems. The complete treatment of problems involving well-posedness, controllability, approximation errors for state-to-state control and regulations is established in (Gao and Caines, 2020).

Scalable and tractable solution methods are established for a class of graphon control problem in (Gao and Caines, 2019a,b). Next, the graphon linear system control framework is extended to more general cases where the state of each individual agent on the network is multi-dimensional (Gao and Caines, 2021); moreover, this work provides a generalized solution method with an application to large-scale networks of coupled harmonic oscillators.

2.6 LQ Graphon Dynamical Systems and Control with Q-space Noise

The work in (Dunyak and Caines, 2022) extends the formulation and analysis of (Gao and Caines, 2021) to include stochastic processes in infinite dimensions as a random process source driving the network's controlled dynamical systems; this generalization involves the formulation of a form of graphon Q-space noise.

2.7 Critical Nodes for GMFG and Embedded GMFG

The works (Caines, 2022),(Foguen Tchuendom et al., 2021), (Foguen Tchuendom et al., 2022a), (Foguen Tchuendom et al., 2022b) and (Caines et al., 2023)

pioneer the theory of GMFG systems embedded in Euclidean space by constructing limit graphs and limit graphons for sequences of graphs sitting in Euclidean space. The crucial notion here is that for embedded graphs and graphons we may define the differentiation of functions on the embedded graph limits. In particular we formulate an extended version of mean field game systems, termed Embedded Graphon Mean Field Games (EGMFG), or Graphexon Mean Field Games (GXMFG), which are amenable to the application of the calculus to the optimization of node location in such systems, this being one of the main objectives of the new development. Moreover this formulation includes the case of sparse graph limits (see (Caines, 2022)) which is not covered by the standard graphon formulation.

2.8 MFG Theory and Related Stochastic Adaptive and Hybrid Control Theory

The paper (Pakniyat and Caines, 2021) constitutes the climax of several decades of work by my own group and many other researchers aimed at producing a rigorous, general, hybrid optimal systems control theory (HOCT). These and related results have great applicability in many domains of engineering and scientific modelling, and furthermore in economics and finance. In fact the paper (Firoozi et al., 2022) presents an extension of HOCT to stochastic MFG systems motivated by trading dynamics in finance.

In (Saude and Caines, 2022), we study the Markowitz dynamic portfolio problem, where an agent seeks to maximize its expected return while minimizing the variance of the return (risk). For the first time in this topic, the impact of the aggregate of the agents' trades is incorporated into the dynamics and costs using an extended mean-field games framework.

2.9 Data Based Protocols for Modelling Large Populations on Large Networks with Applications to SIR Problems

The initial investigation of applying graphon theory to the control of epidemic spread is proposed in (Gao and Caines, 2019a) where scalable solutions based on spectral approximation and linearization are established for controlling virus spreads on networks following a type of controlled network SIS model.
The work (Dunyak and Caines, 2021; Salii, 2021) includes the development of SIR epidemic models which permits the study of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity; the latter models were constructed on a data base of real world US surface and air-traffic travel data.

2.10 Centrality Analysis for Finite and Infinite Networks

This work advances the theory of centralities for dynamical systems on networks and its associated applications. In (Gao, 2022) a family of centralities characterized by fixed-points of networks is identified which includes the standard centralities (e.g. PageRank, Katz-Bonacich, eigen centralities) and equilbrirum cost for games on networks (e.g. Nash values of LQG GMFG). Variation bounds of centralities with respect to the network variations are established. In (Gao, 2021) network centralities have been utilized in the modelling of opinion dynamics on social networks and in identifying significant network partitions based upon opinion disagreement states.

3 Dissemination to Communities of Interest

Research findings and results have been published in the leading systems and control and networks journals including the SIAM Journal on Control and Optimization, IEEE Transactions on Automatic Control, Automatica, IEEE Transactions on Control of Network Systems, and in conferences proceedings including the IEEE Conference on Decision and Control.

3.1 Publications: Journals

S. Gao, P. E. Caines, and M. Huang, "LQG graphon mean field games: analysis via graphon invariant subspaces," Conditionally accepted by IEEE Transactions on Automatic Control, 2021. (Gao et al., 2022)

D. Firoozi, A. Pakniyat, and P. E. Caines, "A class of hybrid LQG mean field games with state-invariant switching and stopping strategies," Automatica, vol. 141, p. 110244, 2022. (Firoozi et al., 2022)

P. E. Caines, D. Ho, M. Huang, J. Jian, and Q. Song. On the graphon mean field game equations: Individual agent affine dynamics and mean field dependent performance functions. ESAIM: Control, Optimisation and Calculus of Variations, 28:24, 2022 (Caines et al., 2022)

P. E. Caines and M. Huang, "Graphon mean field games and their equations," SIAM Journal on Control and Optimization, vol. 59, no. 6, pp. 4373–4399, 2021. (Caines and Huang, 2021)

A. Pakniyat and P. E. Caines, "On the hybrid minimum principle: The Hamiltonian and adjoint boundary conditions," IEEE Transactions on Automatic Control, vol. 66, no. 3, pp. 1246–1253, 2021. (Pakniyat and Caines, 2021)

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N. Sen and P. E. Caines. Mean field games with partial observation. SIAM Journal on Control and Optimization, 57(3):2064–2091, 2019 (Sen and Caines, 2019)

P. E. Caines, D. Ho, and Q. Song. The density evolution of the killed mckean–vlasov process. Stochastics, 92(4):642–657, 2020. (Caines et al., 2020)

3.2 Publications: Conference Proceedings

A. Dunyak and P. E. Caines, "Linear stochastic graphon systems with q- space noise," in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 3926–3932. (Dunyak and Caines, 2022)

P. E. Caines, "Embedded vertexon-graphons and embedded GMFG systems," Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022,

pp. 5550–5557. (Caines, 2022)

S. Gao, "Fixed-point centrality for networks," in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 1628–1635. (Gao, 2022).

R. Foguen Tchuendom, S. Gao, P. E. Caines, and M. Huang, "Optimal network location in infinite horizon LQG graphon mean field games," in Proceedings of the 61th IEEE Conference on Decision and Control, Cancun, Mexico, December 2022, pp. 5558–5565. (Foguen Tchuendom et al., 2022b)

R. Foguen Tchuendom, S. Gao, and P. E. Caines "Stationary cost nodes in infinite horizon LQG GMFGs," IFAC-PapersOnLine for the 25th International Symposium on Mathematical Theory of Networks and Systems (MTNS), vol. 55, no. 30, pp. 284–289, 2022. (Foguen Tchuendom et al., 2022a)

J. Saude and P. E. Caines, "Markowitz portfolio optimization extended quadratic mean- field games approach," Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 5523–5528. (Saude and Caines, 2022)

Y. V. Salii, "Benchmarking optimal control for network dynamic systems with plausible epidemic models," in International Conference on Complex Networks and Their Applications. Springer, 2021, pp. 194–206. (Salii, 2021)

S. Gao, P. E. Caines, and M. Huang, "LQG graphon mean field games: Graphon invariant subspaces," Proceedings of the 60th IEEE Conference on Decision and Control, Austin, Texas, USA, December 2021, pp. 5253–5260. (Gao et al., 2021a)

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S. Gao, "Centrality-weighted opinion dynamics: Disagreement and social network partition," Proceedings of the 60th IEEE Conference on Decision and Control (CDC), Austin, Texas, USA, December 2021, pp. 5496–5501. (Gao, 2021)

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S. Gao and P. E. Caines, Optimal and approximate solutions to linear quadratic regulation of a class of graphon dynamical systems, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 8359–8365, Nice, France, December 2019b. (Gao and Caines, 2019b)

S. Gao and P. E. Caines, Spectral representations of graphons in very large network systems control, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 5068–5075, Nice, France, December 2019a.(Gao and Caines, 2019a)

P. E. Caines and M. Huang. Graphon mean field games and the GMFG equations: ϵ-Nash equilibria, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 286–292, December 2019. (Caines and Huang, 2019)

3.3 Talks: April 1st, 2019 to December 31st, 2022

1. P. E. Caines,"Vertexons and Embedded Graphon Mean Field Games", the Simons Institute for the Theory of Computing, UC Berkeley, CA, Sept.29, 2022 (Invited talk) in the "Graph Limits, Nonparametric Models, and Estimation Workshop" 26 - 30 September, 2022.

2. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", (Invited talk) 29th August, 2022, in the Conference on Many Player Games and Applications, Humbolt University Berlin, 29 - 31 August 2022.

3. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Large Populations on Complex Networks", Canadian Applied and Industrial Mathematics Society (CAIMS), June 23rd, 2021 (Plenary talk).

4. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", the Financial Mathematics/Engineering Seminar Series, Hong Kong Polytechnic University, June 16th 2021 (Invited talk).

5. P. E. Caines, "Static and Time Varying Graphons in Economic and Pandemic GMFG Models" SIAM Financial Mathematics and Engineering (FM21), June 1, 2021 (Invited talk).

6. S. Gao, "Subspace Decompositions in Graphon Control and Graphon Mean Field Games", Informal Systems Seminar, Centre for Intelligent Machines,

McGill University, Groupe d'Etudes et de Recherche en Analyse des Decisions, Montreal, CA, Dec. 11, 2020 (Invited talk).

7. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Networked Systems", Invited Speaker at the "Conference on Mean Field Games and Related Topics - 5". 9th - 13th September, 2019

8. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for a Networked World", University of Saint Petersburg, Russia, 6th September, 2019 (Distinguished Speaker).

9. P. E. Caines, "Graphon Control and Graphon Mean Field Games", Department of Applied Mathemaics, Hong Kong Polythenic Universty, Hong Kong), 28th June, 2019 (Distinguished Lecturer).

3.4 Workshop Organization: April 1st, 2019 to December 31st, 2022

Name: **Mean Field Games on Networks Workshop**

Location: Vancouver, BC, Canada (Virtual Meeting)

Dates: October 26 - 29, 2021

Organizers: Peter E. Caines, Shuang Gao, Minyi Huang, and Rinel Foguen Tchuendom

This workshop (PIMS et al., 2021) brought together researchers in applied mathematics, mean field games, network science, network games, and systems and control theory to exchange ideas and to work on the extensions of mean field game theory to dynamic game problems on heterogeneous large-scale networks. The workshop offered a platform to current results and stimulate discussions on the open challenges in this emerging field, on mathematical theory, computational algorithms, and the applications methodologies of MFG on general networks.

Sponsors: Pacific Institute for the Mathematical Sciences (PIMS), Groupe d'études et de recherche en analyse des décisions (GERAD), and the Fields Institute for Research in Mathematical Sciences.

Participants: Peter E. Caines (PI) and PI's Postdoctoral Researchers: Rinel Foguen Tchuendom, Shuang Gao, Joao Saude, and Yaroslav Salii, plus 50 other international participants.

4 Impacts

As stated earlier, the research on MFG and related topics by the Principal Investigator of the grant FA9550-19-1-0138 has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, presently 2022 - 2025. Consequently, the impacts of research under the 2019 - 2022 grant, FA9550-19-1-0138, may be viewed as a continuation and development of those of the entire program.

Indeed, as explained in Section 1, the MFG field has expanded enormously since its inception. Google links now run to 1.10 billion, citations to papers are in the thousands and conferences and workshops on MFG have proliferated. A sample of the most recent workshops and conference sessions is the following, where we note that the PI and group members have been invited to participate in all of them: Workshop “Mean Field Games on Networks”, Pacific Institute for the Mathematical Sciences, Vancouver, BC, Canada, 26 - 29, October, 2021; “Conference on Mathematics and Financial Economics, “Many Player Games and Applications”, Humbolt University, Berlin, 29 - 31 August 2022; Workshop on Mean Field Games, Montreal: Centre de Recherches Mathématiques (CRM), 11-15 April, 2022; “Graph Limits, Nonparametric Models, and Estimation Workshop”, Simons Institute for the Theory of Computation, UCB, Berkeley, CA, 26-30 September, 2022; IEEE, Control Systems Society Conference on Decision and Control, Cancun, December, 2022, Mean Field Games Session; Mean Field Games Invited Session, IFAC World Congress, Yokahama, Japan, July 2023.

Furthermore, at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively), the last appearing in the period covered by this report. In addition to the very rapid expansion of mathematical and theoretical investigations, applications of MFG theory have spread to a vast range of domains, a sample of the most recent being the optimization of decentralized electric vehicle charging (Foguen Tchuendom et al., 2019), algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Grover et al., 2018), AI reinforcement learning for MFG systems (Agarwal et al., 2022) and polymer science (Welch et al., 2019).

As is shown by this indicative list of publications, meetings and presentations, it is reasonable to assert that the influence of the AFOSR supported research carried out on Mean Field Games and, in particular, on Mean Field Games on networks, has had, and is having, an impact on fields ranging from the core area of systems and control science to economics, finance, energy generation and distribution, operations research, the life sciences and the physical sciences.

5 Changes

There were no changes in the project or its direction.

6 Technical Updates

There are no technical updates to report beyond the continuing research progress and dissemination of results described in report.

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Grant FA9550-19-1-0138
New Directions in Mean Field Games: MFG Subpopulation Behaviours and Graphon-MFG Systems

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1 Overview of Research Objectives

1.1 Background and Overview

Very large networks linking dynamical agents are now ubiquitous and the need to analyse, design and control them is evident. The emergence of the graphon theory of large networks and their infinite limits has enabled the formulation of a theory of the centralized control of dynamical systems distributed on asymptotically infinite networks.

Large-scale systems of competing agents are frequently formulated as dynamic games, this however confronts the problem that for even small populations non-cooperative game theory is intractable. The 2006 breakthrough of Minyi Huang, Roland Malhamé and the PI of FA9550-19-1-0138, Peter Caines, was the introduction of Mean Field Game (MFG) theory (originally termed Nash Certainty Equivalence theory) which was also simultaneously discovered by J.M. Lasry and P.L. Lions.

Mean Field Game (MFG) theory analyzes the existence of Nash equilibria for dynamical games in terms of infinite populations of negligible agents forming the limits of the original finite population systems (Caines, 2020). The core of MFG theory is a pair of partial differential equations consisting of (i) the Hamilton-Jacobi-Bellman equation of optimal control for a generic agent, and (ii) the Fokker-Planck-Kolmogorov equation describing the controlled evolution of the state of a generic agent. These are linked by the generic agent's state

distribution, known as the system's mean field.

A fundamental aspect of MFG theory justifying its application in a vast range of cases is the ϵ-Nash property by which controls derived from the MFG equations yield near-equilibrium performance for finite systems for sufficiently large populations.

The MFG field has expanded tremendously since its inception. Google search results now run to 1.10 billion, citations to papers are in the thousands, conferences and workshops on MFG have proliferated and at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively). In addition to the explosion of mathematical and theoretical investigations over last 16 years, applications of MFG theory have spread to a vast range of domains including: optimization of decentralized EV charging (Ma et al., 2011; Zhu et al., 2016; Foguen Tchuendom et al., 2019); nonlinear system state estimation (Yang et al., 2013); systemic risk in banking (Carmona et al., 2013); cell phone energy management (Aziz and Caines, 2016); economics of energy production (Ludkovski and Sircar, 2015); macro-economics of growth (Gomes et al., 2016); cloud-based networks in information technology (Hanif et al., 2015); algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Nourian et al., 2011; Grover et al., 2018), AI reinforcement learning for MFG systems (Subramanian and Mahajan, 2019; Agarwal et al., 2022), deep learning applications to the solution of MFGs (Ruthotto et al., 2020) and polymer science (Welch et al., 2019).

In this context it is important to note that the research on MFG by the Principal Investigator has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, most recently, for the period 2022-2025.

1.2 Major Research Goals under Grant FA9550-19-1-0138

The program of research under FA9550-19-1-0138 is focused on the development of a new form of Mean Field Games called Graphon Mean Field Games (GMFGs); these are large population games defined on large, heterogeneous, networks and their asymptotic (graphon) limits.

In this setting, game theoretic equilibria are expressed in terms of the newly defined GMFG PDE equations, which are a sweeping generalization of the classical MFG PDEs.

In a closely related development, the theory of graphons is used as a basis for Graphon Linear Control (GLC) theory in which large populations of centrally controlled dynamical systems interacting over large, heterogeneous networks

may be analyzed and optimized in a tractable way.

Canonical Design Procedure for GMFG and GLC Systems

A major goal has been to generate large scale network game strategies and, respectively, control design procedures with the following standard pattern:

The first step is to consider the graphon limit of the network over which the individual systems are distributed and which constitutes the global system.

In the second step, the resulting graphon based GMFG equations, or, respectively, the graphon based infinite dimensional linear system optimization problems are specified. The GMFG equations are then solved to generate the Nash best response strategies or, respectively, the linear feedback controls are synthesized using the well established infinite dimensional linear systems control theory.

Finally, the best response strategies, or, respectively, the infinite dimensional linear feedback controls are approximated in such a way that they may be applied to the original complex finite systems.

The key feature of this strategy is the following: In both the network mean field game and the network control settings it is demonstrated that as the size of the network tends to infinity, the performance of the finite systems when they are subject to the infinite limit system best response game strategies, or, respectively, centralized control laws, yield pay-offs which converge to their Nash equilibrium, or, respectively, optimal, infinite system limit values.

1.3 Organization of the Sequence of Research Objectives

The work over the full reporting period has been organized with the following sequence of objectives:

(i) The establishment of a rigorous theory of the central results of Graphon Mean Field Game (GMFG) theory,

(ii) The generation of a specific Graphon Linear Control (GLC) theory for the control of large populations of dynamical linear system agents with quadratic performance functions on large, complex networks.

(iii) The development of an approximation theory of associated numerical methods, especially those based upon finite dimensional invariant subspaces of the graphon operator in the GLC case.

(iv) The pioneering development of the theory of GMFG systems embedded in Euclidean space; this underlies our currently developing variational analysis of agent performance in networks as a function of position, where this theory also includes sparse as well as dense networks.

(v) The development of an SIR epidemic model software which permits the simulation of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity, these models are constructed on a data base of real world US surface and air-traffic travel data.

2 Details of Accomplishments

2.1 Graphon Mean Field Game Theory: Foundations

The work in (Caines and Huang, 2021) (Caines and Huang, 2018), (Caines and Huang, 2019), established the foundations and central results for graphon mean field game theory for nonlinear systems, that is to say games of infinite populations on infinite heterogeneous networks as limits of the corresponding finite systems.

Specifically, in the conference papers (Caines and Huang, 2018), (Caines and Huang, 2019) Graphon Mean Field Games (GMFG) and the GMFG equations were introduced for the analysis of non-cooperative dynamical games on unbounded networks. In this formulation, asymptotically infinite sub-populations are located at the nodes of large heterogeneous networks, the asymptotic network limits then constitute a part of the analysis. In (Caines and Huang, 2021), the basic existence and uniqueness results were established for the GMFG equations, and a full derivation of the fundamental results was provided, including the key epsilon-Nash theory for GMFG systems which relates the infinite population equilibria on infinite networks to finite population equilibria on finite networks.

2.2 LQG GMFG Decomposition Methods

The theory of invariant spaces of linear operators developed in the papers (Gao and Caines, 2021) is applied in (Gao et al., 2021b), (Gao et al., 2021a) and (Gao et al., 2022) to yield a tractable methodology for generating decentralized game strategies for systems distributed over complex networks. The fundamental properties of the existence and uniqueness of solutions are established in (Gao et al., 2021b,a, 2022) in two principal ways: (i) Riccati equation decomposition and (ii) fixed point contraction theorem methods.

2.3 LQG GMFG Master Equation Analysis

First, using a probabilistic approach, solutions to the Linear Quadratic Graphon Mean Field Games are expressed as solutions to coupled Forward Backward Stochastic Differential Equations (FBSDEs) of McKean-Vlasov type. Next, the

existence of the so-called Master field, which allows for the decoupling of these FBSDEs is established. Finally, the infinite dimensional Partial Differential Equation (PDE), so-called Master Equation, is established for which the Master Field is a solution (see (Foguen Tchuendom et al., 2020)).

2.4 GMFG with Affine Dynamics

GMFG theory for nonlinear problems with individual affine dynamics has been established in Caines et al. (2022) and the Nash value properties for control affine systems has been analyzed in Caines et al. (2023).

2.5 Graphon Linear Systems Control with Approximation Theory

As explained above, the theory of graphons is further used as a basis upon which centrally controlled linear dynamical systems interacting over large, heterogeneous networks are analyzed in a tractable way.

The graphon control methodology is developed in (Gao and Caines, 2020, 2019a,b) to study control problems for arbitrary-size networks of linear dynamical systems. The complete treatment of problems involving well-posedness, controllability, approximation errors for state-to-state control and regulations is established in (Gao and Caines, 2020).

Scalable and tractable solution methods are established for a class of graphon control problem in (Gao and Caines, 2019a,b). Next, the graphon linear system control framework is extended to more general cases where the state of each individual agent on the network is multi-dimensional (Gao and Caines, 2021); moreover, this work provides a generalized solution method with an application to large-scale networks of coupled harmonic oscillators.

2.6 LQ Graphon Dynamical Systems and Control with Q-space Noise

The work in (Dunyak and Caines, 2022) extends the formulation and analysis of (Gao and Caines, 2021) to include stochastic processes in infinite dimensions as a random process source driving the network's controlled dynamical systems; this generalization involves the formulation of a form of graphon Q-space noise.

2.7 Critical Nodes for GMFG and Embedded GMFG

The works (Caines, 2022),(Foguen Tchuendom et al., 2021), (Foguen Tchuendom et al., 2022a), (Foguen Tchuendom et al., 2022b) and (Caines et al., 2023)

pioneer the theory of GMFG systems embedded in Euclidean space by constructing limit graphs and limit graphons for sequences of graphs sitting in Euclidean space. The crucial notion here is that for embedded graphs and graphons we may define the differentiation of functions on the embedded graph limits. In particular we formulate an extended version of mean field game systems, termed Embedded Graphon Mean Field Games (EGMFG), or Graphexon Mean Field Games (GXMFG), which are amenable to the application of the calculus to the optimization of node location in such systems, this being one of the main objectives of the new development. Moreover this formulation includes the case of sparse graph limits (see (Caines, 2022)) which is not covered by the standard graphon formulation.

2.8 MFG Theory and Related Stochastic Adaptive and Hybrid Control Theory

The paper (Pakniyat and Caines, 2021) constitutes the climax of several decades of work by my own group and many other researchers aimed at producing a rigorous, general, hybrid optimal systems control theory (HOCT). These and related results have great applicability in many domains of engineering and scientific modelling, and furthermore in economics and finance. In fact the paper (Firoozi et al., 2022) presents an extension of HOCT to stochastic MFG systems motivated by trading dynamics in finance.

In (Saude and Caines, 2022), we study the Markowitz dynamic portfolio problem, where an agent seeks to maximize its expected return while minimizing the variance of the return (risk). For the first time in this topic, the impact of the aggregate of the agents' trades is incorporated into the dynamics and costs using an extended mean-field games framework.

2.9 Data Based Protocols for Modelling Large Populations on Large Networks with Applications to SIR Problems

The initial investigation of applying graphon theory to the control of epidemic spread is proposed in (Gao and Caines, 2019a) where scalable solutions based on spectral approximation and linearization are established for controlling virus spreads on networks following a type of controlled network SIS model.
The work (Dunyak and Caines, 2021; Salii, 2021) includes the development of SIR epidemic models which permits the study of various centralized optimal control strategies to limit epidemic spread over networks at specifiable levels of granularity; the latter models were constructed on a data base of real world US surface and air-traffic travel data.

2.10 Centrality Analysis for Finite and Infinite Networks

This work advances the theory of centralities for dynamical systems on networks and its associated applications. In (Gao, 2022) a family of centralities characterized by fixed-points of networks is identified which includes the standard centralities (e.g. PageRank, Katz-Bonacich, eigen centralities) and equilbrirum cost for games on networks (e.g. Nash values of LQG GMFG). Variation bounds of centralities with respect to the network variations are established. In (Gao, 2021) network centralities have been utilized in the modelling of opinion dynamics on social networks and in identifying significant network partitions based upon opinion disagreement states.

3 Dissemination to Communities of Interest

Research findings and results have been published in the leading systems and control and networks journals including the SIAM Journal on Control and Optimization, IEEE Transactions on Automatic Control, Automatica, IEEE Transactions on Control of Network Systems, and in conferences proceedings including the IEEE Conference on Decision and Control.

3.1 Publications: Journals

S. Gao, P. E. Caines, and M. Huang, "LQG graphon mean field games: analysis via graphon invariant subspaces," Conditionally accepted by IEEE Transactions on Automatic Control, 2021. (Gao et al., 2022)

D. Firoozi, A. Pakniyat, and P. E. Caines, "A class of hybrid LQG mean field games with state-invariant switching and stopping strategies," Automatica, vol. 141, p. 110244, 2022. (Firoozi et al., 2022)

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P. E. Caines and M. Huang, "Graphon mean field games and their equations," SIAM Journal on Control and Optimization, vol. 59, no. 6, pp. 4373–4399, 2021. (Caines and Huang, 2021)

A. Pakniyat and P. E. Caines, "On the hybrid minimum principle: The Hamiltonian and adjoint boundary conditions," IEEE Transactions on Automatic Control, vol. 66, no. 3, pp. 1246–1253, 2021. (Pakniyat and Caines, 2021)

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S. Gao, R. Foguen Tchuendom, and P. E. Caines, "Linear quadratic graphon field games," Communications in Information and Systems, vol. 21, no. 3, pp. 341–369, 2021. (Gao et al., 2021b)

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S. Gao and P. E. Caines. Graphon control of large-scale networks of linear systems. IEEE Transactions on Automatic Control, 65(10):4090–4105, 2020 (Gao and Caines, 2020)

N. Sen and P. E. Caines. Mean field games with partial observation. SIAM Journal on Control and Optimization, 57(3):2064–2091, 2019 (Sen and Caines, 2019)

P. E. Caines, D. Ho, and Q. Song. The density evolution of the killed mckean–vlasov process. Stochastics, 92(4):642–657, 2020. (Caines et al., 2020)

3.2 Publications: Conference Proceedings

A. Dunyak and P. E. Caines, "Linear stochastic graphon systems with q- space noise," in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 3926–3932. (Dunyak and Caines, 2022)

P. E. Caines, "Embedded vertexon-graphons and embedded GMFG systems," Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022,

pp. 5550–5557. (Caines, 2022)

S. Gao, “Fixed-point centrality for networks,” in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 1628–1635. (Gao, 2022).

R. Foguen Tchuendom, S. Gao, P. E. Caines, and M. Huang, “Optimal network location in infinite horizon LQG graphon mean field games,” in Proceedings of the 61th IEEE Conference on Decision and Control, Cancun, Mexico, December 2022, pp. 5558–5565. (Foguen Tchuendom et al., 2022b)

R. Foguen Tchuendom, S. Gao, and P. E. Caines “Stationary cost nodes in infinite horizon LQG GMFGs,” IFAC-PapersOnLine for the 25th International Symposium on Mathematical Theory of Networks and Systems (MTNS), vol. 55, no. 30, pp. 284–289, 2022. (Foguen Tchuendom et al., 2022a)

J. Saude and P. E. Caines, “Markowitz portfolio optimization extended quadratic mean- field games approach,” Proceedings of the 61st IEEE Conference on Decision and Control (CDC), 2022, pp. 5523–5528. (Saude and Caines, 2022)

Y. V. Salii, “Benchmarking optimal control for network dynamic systems with plausible epidemic models,” in International Conference on Complex Networks and Their Applications. Springer, 2021, pp. 194–206. (Salii, 2021)

S. Gao, P. E. Caines, and M. Huang, “LQG graphon mean field games: Graphon invariant subspaces,” Proceedings of the 60th IEEE Conference on Decision and Control, Austin, Texas, USA, December 2021, pp. 5253–5260. (Gao et al., 2021a)

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IEEE Conference on Decision and Control (CDC), pages 1026–1031, 2020. (Foguen Tchuendom et al., 2020)

S. Gao and P. E. Caines, Optimal and approximate solutions to linear quadratic regulation of a class of graphon dynamical systems, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 8359–8365, Nice, France, December 2019b. (Gao and Caines, 2019b)

S. Gao and P. E. Caines, Spectral representations of graphons in very large network systems control, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 5068–5075, Nice, France, December 2019a.(Gao and Caines, 2019a)

P. E. Caines and M. Huang. Graphon mean field games and the GMFG equations: ϵ-Nash equilibria, Proceedings of the 58th IEEE Conference on Decision and Control (CDC), pages 286–292, December 2019. (Caines and Huang, 2019)

3.3 Talks: April 1st, 2019 to December 31st, 2022

1. P. E. Caines,"Vertexons and Embedded Graphon Mean Field Games", the Simons Institute for the Theory of Computing, UC Berkeley, CA, Sept.29, 2022 (Invited talk) in the "Graph Limits, Nonparametric Models, and Estimation Workshop" 26 - 30 September, 2022.

2. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", (Invited talk) 29th August, 2022, in the Conference on Many Player Games and Applications, Humbolt University Berlin, 29 - 31 August 2022.

3. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Large Populations on Complex Networks", Canadian Applied and Industrial Mathematics Society (CAIMS), June 23rd, 2021 (Plenary talk).

4. P. E. Caines, "Optimal Execution Problems in Single and Networked Markets: a Mean Field Game Formulation", the Financial Mathematics/Engineering Seminar Series, Hong Kong Polytechnic University, June 16th 2021 (Invited talk).

5. P. E. Caines, "Static and Time Varying Graphons in Economic and Pandemic GMFG Models" SIAM Financial Mathematics and Engineering (FM21), June 1, 2021 (Invited talk).

6. S. Gao, "Subspace Decompositions in Graphon Control and Graphon Mean Field Games", Informal Systems Seminar, Centre for Intelligent Machines,

McGill University, Groupe d'Etudes et de Recherche en Analyse des Decisions, Montreal, CA, Dec. 11, 2020 (Invited talk).

7. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for Networked Systems", Invited Speaker at the "Conference on Mean Field Games and Related Topics - 5". 9th - 13th September, 2019

8. P. E. Caines, "Graphon Mean Field Games: A Dynamical Equilibrium Theory for a Networked World", University of Saint Petersburg, Russia, 6th September, 2019 (Distinguished Speaker).

9. P. E. Caines, "Graphon Control and Graphon Mean Field Games", Department of Applied Mathemaics, Hong Kong Polythenic Universty, Hong Kong), 28th June, 2019 (Distinguished Lecturer).

3.4 Workshop Organization: April 1st, 2019 to December 31st, 2022

Name: **Mean Field Games on Networks Workshop**

Location: Vancouver, BC, Canada (Virtual Meeting)

Dates: October 26 - 29, 2021

Organizers: Peter E. Caines, Shuang Gao, Minyi Huang, and Rinel Foguen Tchuendom

This workshop (PIMS et al., 2021) brought together researchers in applied mathematics, mean field games, network science, network games, and systems and control theory to exchange ideas and to work on the extensions of mean field game theory to dynamic game problems on heterogeneous large-scale networks. The workshop offered a platform to current results and stimulate discussions on the open challenges in this emerging field, on mathematical theory, computational algorithms, and the applications methodologies of MFG on general networks.

Sponsors: Pacific Institute for the Mathematical Sciences (PIMS), Groupe d'études et de recherche en analyse des décisions (GERAD), and the Fields Institute for Research in Mathematical Sciences.

Participants: Peter E. Caines (PI) and PI's Postdoctoral Researchers: Rinel Foguen Tchuendom, Shuang Gao, Joao Saude, and Yaroslav Salii, plus 50 other international participants.

4 Impacts

As stated earlier, the research on MFG and related topics by the Principal Investigator of the grant FA9550-19-1-0138 has been supported by the AFOSR Dynamics and Control Program over the periods 2009 - 2011, 2012 - 2015, 2019 - 2022, and, presently 2022 - 2025. Consequently, the impacts of research under the 2019 - 2022 grant, FA9550-19-1-0138, may be viewed as a continuation and development of those of the entire program.

Indeed, as explained in Section 1, the MFG field has expanded enormously since its inception. Google links now run to 1.10 billion, citations to papers are in the thousands and conferences and workshops on MFG have proliferated. A sample of the most recent workshops and conference sessions is the following, where we note that the PI and group members have been invited to participate in all of them: Workshop "Mean Field Games on Networks", Pacific Institute for the Mathematical Sciences, Vancouver, BC, Canada, 26 - 29, October, 2021; "Conference on Mathematics and Financial Economics, "Many Player Games and Applications", Humbolt University, Berlin, 29 - 31 August 2022; Workshop on Mean Field Games, Montreal: Centre de Recherches Mathématiques (CRM), 11-15 April, 2022; "Graph Limits, Nonparametric Models, and Estimation Workshop", Simons Institute for the Theory of Computation, UCB, Berkeley, CA, 26-30 September, 2022; IEEE, Control Systems Society Conference on Decision and Control, Cancun, December, 2022, Mean Field Games Session; Mean Field Games Invited Session, IFAC World Congress, Yokahama, Japan, July 2023.

Furthermore, at least four monographs on MFG have been written (by Bensoussan et al. (2013), Gomes et al. (2014), Carmona and Delarue (2018a,b) and Cardaliaguet et al. (2019), respectively), the last appearing in the period covered by this report. In addition to the very rapid expansion of mathematical and theoretical investigations, applications of MFG theory have spread to a vast range of domains, a sample of the most recent being the optimization of decentralized electric vehicle charging (Foguen Tchuendom et al., 2019), algorithmic trading and optimal execution in finance (Casgrain and Jaimungal, 2020), MFG models of flocking (Grover et al., 2018), AI reinforcement learning for MFG systems (Agarwal et al., 2022) and polymer science (Welch et al., 2019).

As is shown by this indicative list of publications, meetings and presentations, it is reasonable to assert that the influence of the AFOSR supported research carried out on Mean Field Games and, in particular, on Mean Field Games on networks, has had, and is having, an impact on fields ranging from the core area of systems and control science to economics, finance, energy generation and distribution, operations research, the life sciences and the physical sciences.

5 Changes

There were no changes in the project or its direction.

6 Technical Updates

There are no technical updates to report beyond the continuing research progress and dissemination of results described in report.

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