

FR-3755

# AN AUTOMATIC ANTENNA-TUNING UNIT



**NAVAL RESEARCH LABORATORY**  
**WASHINGTON, D.C.**

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## ABSTRACT

An antenna-tuning system has been designed to match automatically the input impedance of a 35-foot whip antenna to a 50-ohm coaxial transmission line over the frequency range of 2 to 18 Mc. This unit has been designed specifically to operate with a Naval Communications Transmitter, type TCK (400-watt output), but the design is such that it is applicable to a wide variety of automatic impedance-matching problems with only minor variations. A satisfactory solution to the technical problems peculiar to such a system has been attained. Detailed consideration was limited to those components designed especially to fill the requirements of this problem. It is concluded that the principal limitations of the range of application of this system are of a physical nature, such as the maximum tuning ranges and voltage ratings available in continuously variable capacitors and inductors.

## PROBLEM STATUS

This is an interim report on this problem; work is continuing.

## AUTHORIZATION

NRL Problem R10-39R  
NE 121-001

## AN AUTOMATIC ANTENNA-TUNING UNIT

## INTRODUCTION

New naval-communication transmitters are designed to operate into a 50-ohm coaxial-transmission line. A matching device is required at the base of the antenna so that the input impedance to the antenna may be matched to the characteristic impedance of the transmission line.

Present matching devices are manually operated and may be remotely adjusted only to a fixed number of preset positions. Moreover, it is a complex and time consuming operation to preset these positions. Once set, no allowance is made for changes in input impedance to the antenna due to changing conditions, such as those encountered on a moving ship, where the antenna may be constantly changing its relative position and angle of mounting with respect to the ground plane. In view of these obvious disadvantages it is apparent that an improved matching system is desirable.

The following characteristics are necessary to make this new system completely satisfactory.

(a) The system must be capable of matching the antenna impedance to the transmission line at any frequency within the range of operation of the transmitter. This means that the matching system may be used at an infinite number of frequencies in the range with no band switching or manual adjustment of any kind.

(b) The impedance match between the antenna and the transmission line must be exact to such an extent that the transmitter can be loaded into a dummy antenna (50-ohm resistive load) then switched to the antenna without an appreciable change in the amount of power delivered by the transmitter. (A match such that the voltage standing-wave ratio on the line must never exceed 1.2 is now contemplated.)

(c) The entire matching operation must be completed in a minimum time. (Ten seconds is now considered a desirable maximum tune-up time.)

(d) The entire operation must be fully automatic, thereby reducing the complexity of the work of the operators.

(e) The matching system must operate at all times the transmitter is on and be continuously variable so that compensation will be made for variations in antenna impedance under operating conditions. For example, an antenna might be mounted over the side of the flight deck on an aircraft carrier, operating in an upright position normally, but operating in a horizontal position (extended over the water) when aircraft are taking off or landing on the flight deck. The proposed new system must make continuous compensation for the change in input impedance to an antenna so mounted.

(f) The tuning unit must be self-contained and of a type that may be housed and secured in a relatively inaccessible position and left without manual adjustment or maintenance for long periods of time. (A maximum of two years operation without adjustment or maintenance is now considered desirable.)

The specific purpose of the problem under discussion is to build an impedance-matching system which meets the above-listed requirements. Work has progressed to such an extent that it is now believed that a complete test model meeting these requirements will be completed in the near future. Design and construction details reported here are limited to those component parts developed especially for this problem.

### METHOD OF ATTACK

A careful study of the impedance characteristic of a 35-foot whip antenna (Figure 1) and a number of impedance-matching networks and error-detecting circuits led to the adoption of the generalized solution of the problem shown in the block diagram of the system (Figure 2). Although the cantilever-type matching network is more commonly used as a narrow-band matching device, it can, with certain modifications, be given the broad-band characteristics necessary in this application. Because the shunt arm of the cantilever network transforms the input impedance of the antenna to an impedance whose resistive component is equal to the characteristic resistance of the transmission line and the series arm of this network eliminates the reactive component from the parallel impedance of the antenna and the shunt arm, a sensing unit composed of a phase-angle detector and an impedance-magnitude detector was found desirable. The error voltage from the impedance-magnitude detector had to drive the shunt arm (through a servo system) and the error voltage from the phase-angle detector had to drive the series arm in a similar manner.

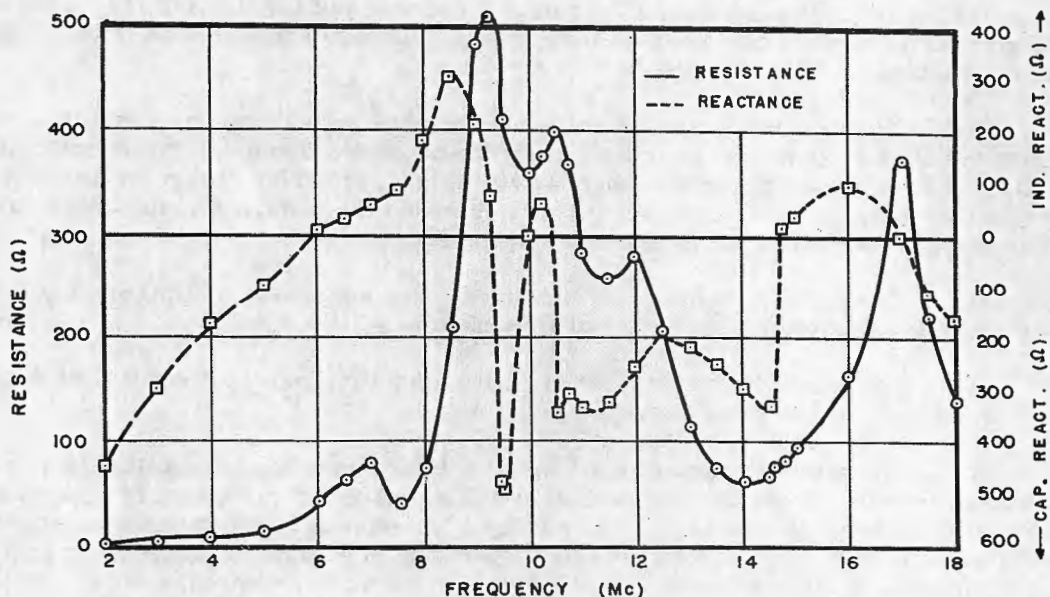


Figure 1 - Impedance characteristic of 35-foot whip antenna



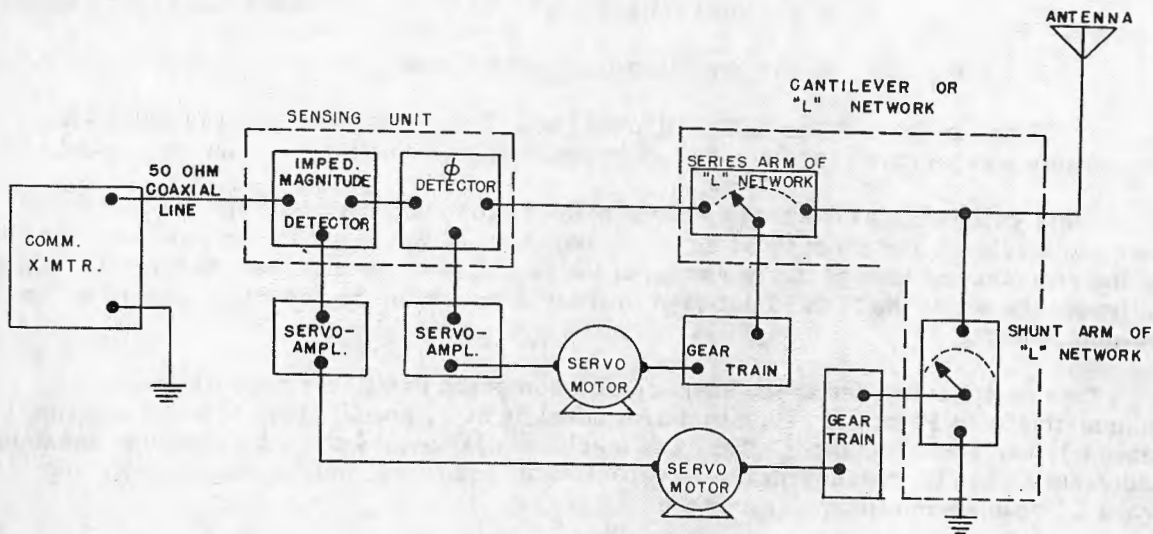
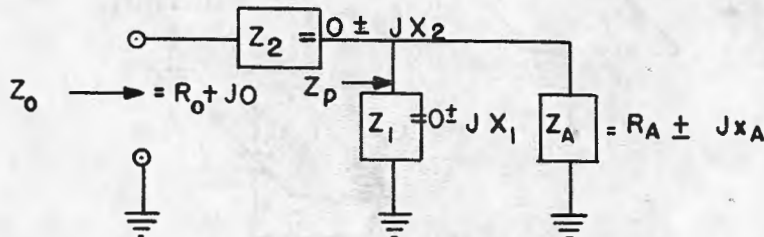


Figure 2 - Block diagram of impedance-matching system

### ANALYSIS OF THE CANTILEVER-MATCHING NETWORK

A mathematical analysis of the problem of matching the antenna impedance to the transmission line yielded the information that the match could be obtained by use of an "L" or cantilever network if the proper values of reactance could be obtained. The equations used are given below and are derived by application of ordinary circuit analysis (see Appendix).



$$(1) \quad X_1 = R_0 X_A \pm \frac{\sqrt{R_0 R_A (R_A^2 - R_0 R_A + X_A^2)}}{R_A - R_0}$$

$$(2) \quad X_2 = -X_p = -\frac{X_1 (R_A^2 + X_1 X_A + X_A^2)}{R_A^2 + (X_1 + X_A)^2}$$

where

$X_1$  = the reactance of the shunt arm of the cantilever network

$X_2$  = the reactance of the series arm of the cantilever network

$X_p$  = the reactive component of the parallel impedance of the antenna and the shunt arm of the cantilever network

$Z_0 = R_0 + j0$  = the input impedance to the cantilever network plus the antenna

$Z_A = R_A \pm jX_A$  = the input impedance to the antenna

Note: In the solution of Equations (1) and (2) the sign convention in which  $-X$  represents a capacitive reactance and  $+X$  represents an inductive reactance was used.

These calculations also revealed that neither of the two common types of cantilever circuits (series L and shunt C, or series C and shunt L) would suffice because the sign of the reactance of each of the two arms of the cantilever network must change with changes in frequency within the 2- to 18-Mc band in order to match the 35-foot whip antenna to the 50-ohm line.

This complication led to the adoption of a compound cantilever network shown schematically in Figure 3. The shunt arm consists of  $L_1$  and  $C_1$ , both of which are continuously variable reactances. They are mechanically coupled so that  $L_1$  reaches maximum inductance when  $C_1$  reaches maximum capacitance, and  $L_1$  reaches minimum inductance when  $C_1$  reaches minimum capacitance.

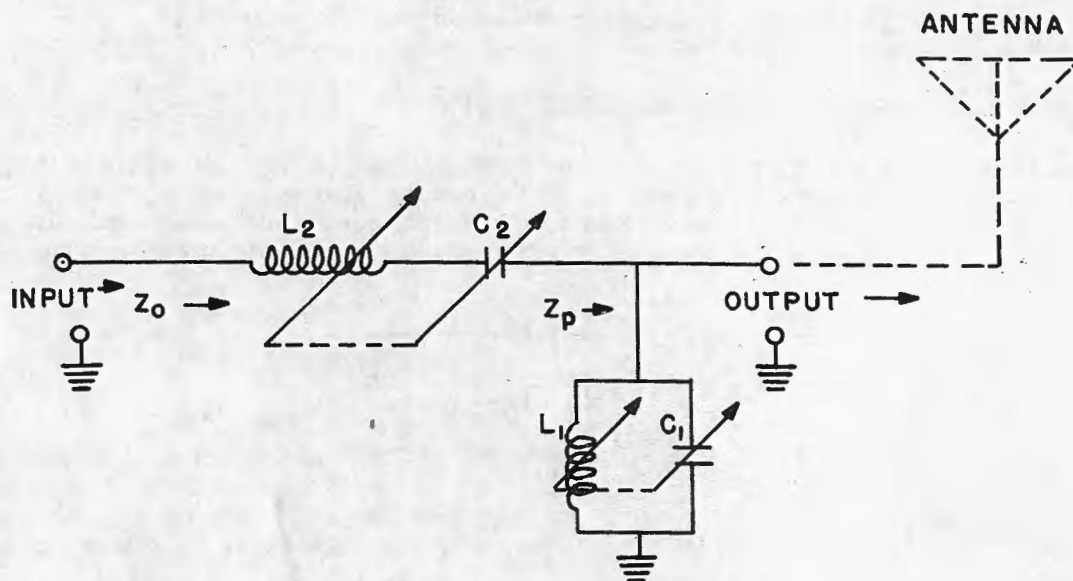


Figure 3 - Schematic diagram of cantilever-matching network

In Figure 4 the solid curves represent the limits of the inductive and capacitive reactances which may be obtained from this combination. Because  $L_1$  and  $C_1$  are electrically paralleled, any reactance lying "outside" these two solid curves may be obtained by rotating the rigidly coupled  $L_1$  and  $C_1$ .

On the same axes (Figure 4) the two solutions of the quadratic Equation (1) are plotted against frequency. Either of these curves represents a value of  $X_1$ , which will make  $Z_p$ , the parallel impedance of the antenna and the shunt arm of the cantilever network, equal to  $R_0 + jX_p$ . That is, the resistive component of  $Z_p$  is equal to the characteristic resistance of the transmission line when  $X_1$  takes either value represented by these two curves.



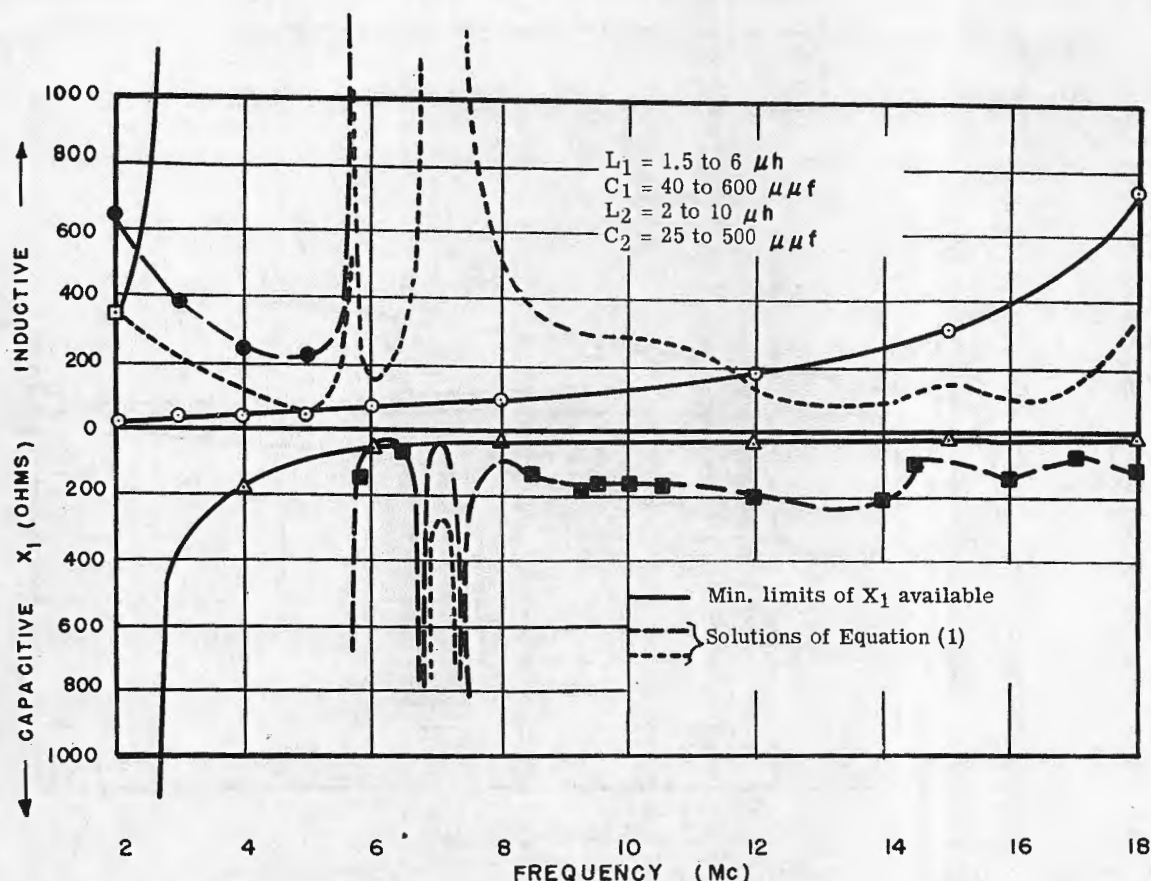


Figure 4 - Required and obtainable values of reactance for the shunt arm of the cantilever network

The series arm of the cantilever circuit consists of an inductor and a capacitor electrically in series and mechanically coupled so that  $L_2$  reaches maximum inductance when  $C_2$  reaches maximum capacitance, and  $L_2$  reaches minimum inductance when  $C_2$  reaches minimum capacitance. Both  $L_2$  and  $C_2$  are continuously variable components, similar in construction to  $L_1$  and  $C_1$ .

Figure 5 shows the limits of obtainable  $X_2$  versus frequency in solid curves. Because  $C_2$  and  $L_2$  are electrically in series, any value of reactance between the solid curves may be obtained by rotating the rigidly coupled  $L_2$  and  $C_2$ . On the same axes (Figure 5) the required values of  $X_2$  found necessary to make  $Z_0 = R_0 + j0$  are plotted against frequency. The two curves correspond to the two values of  $X_1$  used in the solution of Equation (2).

Now the criteria for the feasibility of the cantilever network can be determined and a check of the curves can determine whether or not this particular cantilever network is suitable for this problem.

The solutions of Equation (1) are plotted as the dashed and dotted curves of Figure 4. When the solution represented by the dashed curve (Figure 4) is substituted in Equation (2), the dashed curve of Figure 5 results. Similarly, the dotted curves of Figures 4 and 5 correspond.

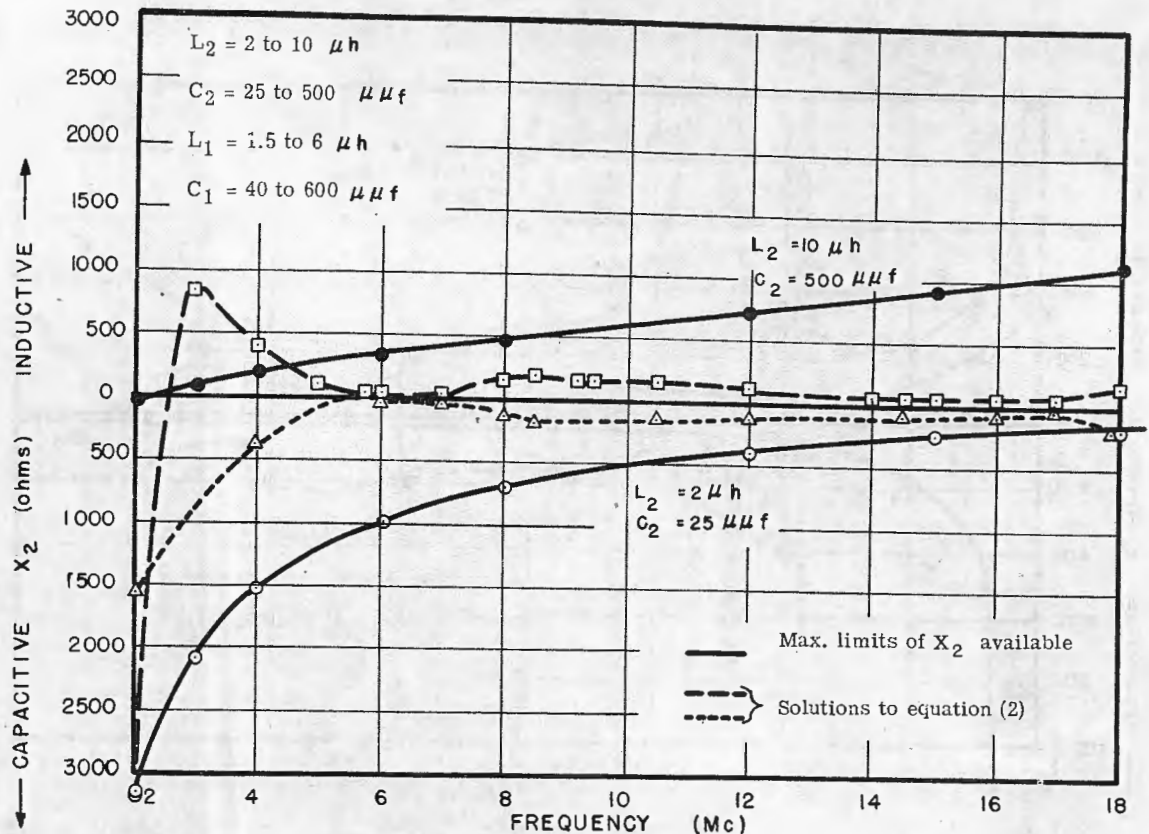


Figure 5 - Required and obtainable values of reactance for the series arm of the cantilever network

Careful consideration of these curves shows that the cantilever network described is suitable for matching the impedance of the antenna to the line at all frequencies where either or both of the following conditions exist: (a) the dashed curve of Figure 4 lies "outside" the two solid curves of Figure 4 and the dashed curve of Figure 5 lies between the two solid curves of Figure 5, and (b) the dotted curve of Figure 4 lies "outside" the two solid curves of Figure 4 and the dotted curve of Figure 5 lies between the two solid curves of Figure 5.

Examination of these curves reveals that this cantilever network provides sufficient ranges for  $X_1$  and  $X_2$  with considerable overlap over the entire frequency range of 2 to 18 Mc.

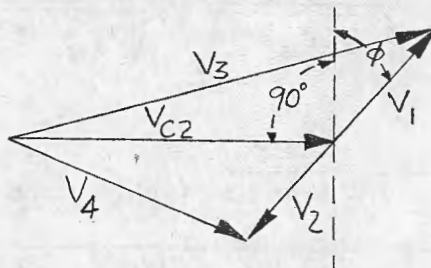
#### SENSING UNIT

In order to make the impedance-matching system fully automatic, two sensing or detecting devices are necessary: (a) a phase-angle detector which must produce an output voltage or current which is a function of the phase angle ( $\phi$ ) between line voltage and line current, and (b) a magnitude detector whose output must be a function of the ratio  $|Z_L|/|Z_0|$ , where  $Z_L$  is the input impedance to the sensing units plus the matching components plus the antenna, and  $Z_0$  is the characteristic impedance of the line.

## Phase-Angle Detector

Three conditions must be imposed upon this unit in order to make it satisfactory: (a) the output of the unit must be zero when the phase angle between line voltage and line current is zero, (b) the output of the unit must change in sign (or direction) with a change of sign of the phase angle, (c) the rate of change of output of the unit with respect to the phase angle must be greatest in the vicinity of zero phase angle so that the sharpest tuning effect may be obtained when the current and voltage of the line are approximately in phase.

An untuned discriminator circuit was found to possess all these characteristics, as shown by the following mathematical analysis. The circuit is shown schematically in Figure 6, in which point A is the centertap between points B and C, so that  $|V_1| = |V_2|$ . The following vector diagram is used to simplify the explanation.



$V_{C2}$  = the voltage across  $C_2$  (directly coupled in by the voltage divider composed of  $C_1$  and  $C_2$  in Figure 6)

$V_1$  = the voltage induced from A to B by the electromagnetic field produced by the line current

$V_2$  = the voltage induced from A to C by the electromagnetic field produced by the line current

$$V_3 = V_{C2} + V_1$$

$$V_4 = V_{C2} + V_2$$

$$V_0 = \text{the output voltage of the unit} = |V_3| - |V_4|$$

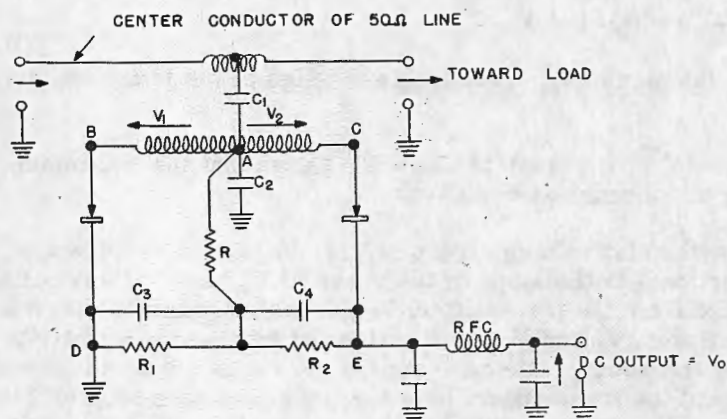


Figure 6 - Schematic diagram of the phase-angle detector



From the vector diagram:

$$|V_3| = \sqrt{(|V_{C_2}| + |V_1| \sin \phi)^2 + (|V_1| \cos \phi)^2}$$

$$|V_4| = \sqrt{(|V_{C_2}| - |V_1| \sin \phi)^2 + (|V_1| \cos \phi)^2}$$

$$(3) \quad V_0 = \sqrt{|V_{C_2}|^2 + |V_1|^2 + 2|V_1||V_{C_2}| \sin \phi} - \sqrt{|V_{C_2}|^2 + |V_1|^2 - 2|V_1||V_{C_2}| \sin \phi}$$

$$(4) \quad \frac{dV_0}{d\phi} = \frac{|V_1||V_{C_2}| \cos \phi}{\sqrt{|V_{C_2}|^2 + |V_1|^2 + 2|V_1||V_{C_2}| \sin \phi}} + \frac{|V_1||V_{C_2}| \cos \phi}{\sqrt{|V_{C_2}|^2 + |V_1|^2 - 2|V_1||V_{C_2}| \sin \phi}}$$

$$(5) \quad \frac{d^2V_0}{d\phi^2} = \frac{\sqrt{|V_{C_2}|^2 + |V_1|^2 + 2|V_1||V_{C_2}| \sin \phi} (-|V_1||V_{C_2}| \sin \phi) - \frac{|V_1|^2|V_{C_2}|^2 \cos^2 \phi}{\sqrt{|V_{C_2}|^2 + |V_1|^2 + 2|V_1||V_{C_2}| \sin \phi}}}{|V_{C_2}|^2 + |V_1|^2 + 2|V_1||V_{C_2}| \sin \phi}$$

$$+ \frac{\sqrt{|V_{C_2}|^2 + |V_1|^2 - 2|V_1||V_{C_2}| \sin \phi} (-|V_1||V_{C_2}| \sin \phi) + \frac{|V_1|^2|V_{C_2}|^2 \cos^2 \phi}{\sqrt{|V_{C_2}|^2 + |V_1|^2 - 2|V_1||V_{C_2}| \sin \phi}}}{|V_{C_2}|^2 + |V_1|^2 - 2|V_1||V_{C_2}| \sin \phi}$$

Setting the second derivative of  $V_0$  equal to zero and solving for  $\phi$  yields the result that  $\phi$  is equal to zero.

We now have mathematical confirmation for the statement that the discriminator circuit meets the above listed requirements.

Equation (3) shows that  $V_0 = 0$  when  $\phi = 0$ .

Equation (3) also shows that the sign of  $V_0$  reverses as the sign of the phase angle  $\phi$  reverses.

The solution to Equation (5) ( $d^2V_0/d\phi^2 = 0$  at  $\phi = 0$ ) shows that the maximum rate of change of  $V_0$  with respect to  $\phi$  occurs at  $\phi = 0$ .

The variation of sensitivity with relative magnitudes of  $V_{C_2}$ ,  $V_1$ , and  $V_2$  is shown in Figure 7, sensitivity being proportional to the slope of the curve of  $V_0$  vs.  $\phi$  at any point. Since, with a constant input voltage from the transmitter,  $V_{C_2}$  remains constant with frequency change and the induced voltages,  $V_1$  and  $V_2$ , are functions of frequency, sensitivity is also a function of the operating frequency. Adequate sensitivity can be obtained throughout the operating frequency range of the transmitter, however, by a judicious choice of the frequency at which the magnitude of  $V_{C_2}$  is made equal to the magnitudes of the induced voltages  $V_1$  and  $V_2$ .

### Impedance-Magnitude Detector

The requirements of this unit which make it satisfactory are: (a) the output voltage or current of the unit must be a function of the ratio  $|Z_L|/|Z_{ob}|$ , (b) the output must be zero when the load impedance,  $Z_L$ , is equal in magnitude to the characteristic impedance of the line, and (c) the sign of the output voltage or current must change when the ratio  $|Z_L|/|Z_o|$  changes from greater than one to less than one or vice versa.

A desirable but not a necessary characteristic of the unit is a sensitivity which is not a function of frequency.

The schematic diagram of the impedance magnitude detector (Figure 8) is used to explain the operation of this unit.

$V_1$  = the voltage across capacitor  $C_1$

$V_2$  = the voltage across capacitor  $C_2$

$V_{in}$  = the output voltage of the transmitter

$V_{R1}$  = the voltage across the resistor  $R_1$

$V_0$  = the output voltage of the unit (dc)

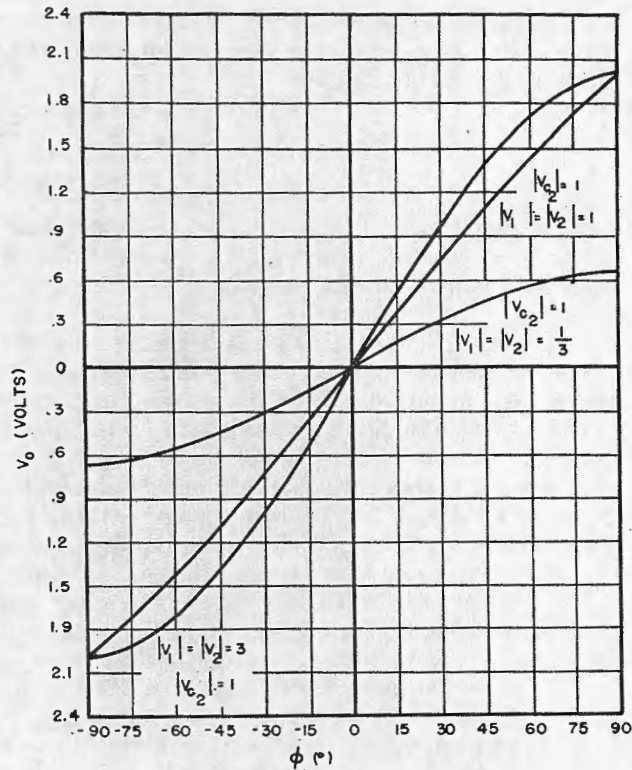


Figure 7 - Curve of output voltage vs. phase angle  $\phi$

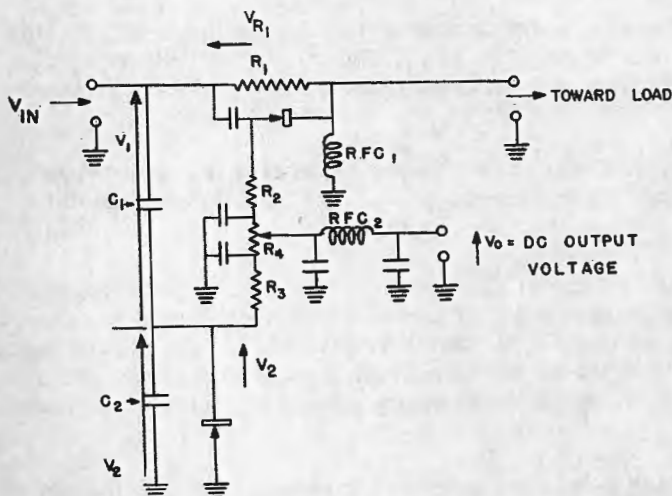


Figure 8 - Schematic diagram of the impedance-magnitude detector

The approximation is made, for the purpose of this discussion, that the output of the transmitter is a pure sine wave of constant amplitude. The error due to this approximation is small because

$$V = \sqrt{(V_1)^2 + (V_2)^2 + \dots + (V_n)^2}$$

and the magnitude of any harmonic voltage is at most only a few percent of the magnitude of the fundamental voltage. ( $V$  = the total output voltage of the transmitter and  $V_n$  = the  $n^{\text{th}}$  harmonic voltage in the output.)

$C_1$  and  $C_2$  (Figure 8) form a voltage divider so that  $V_2 = kV_1$ . The positive half of the waveform of  $V_2$  is rectified and applied to one end of a resistance combination  $R_2$ ,  $R_3$ , and  $R_4$ ; the negative half of the waveform of the voltage  $V_{R_1}$  is rectified and applied to the opposite end of this resistance combination.

Now, if  $|V_{R_1}| = |V_2|$ , the voltage at the center of the resistance combination (hence the output voltage,  $V_o$ ) is equal to zero. But, if  $|V_{R_1}|$  is equal to  $|V_2|$ , then  $R_1/|Z_L|$  must be equal to  $k = V_2/V_1 = C_1/C_2$ . Also, if  $|Z_L|$  increases so that  $R_1/|Z_L|$  is less than  $C_1/C_2$ ,  $|V_{R_1}|$  becomes less than  $|V_2|$  and the output of the unit is a positive dc voltage. Conversely, if  $|Z_L|$  decreases so that  $R_1/|Z_L|$  is greater than  $C_1/C_2$ , then  $|V_{R_1}|$  becomes greater than  $|V_2|$  and the output of the unit is a negative dc voltage.

As seen from the above analysis,  $|Z_L|$  at balance ( $V_o = 0$ ) can be determined by the ratio of  $C_1/C_2$  and by the fixed resistance  $R_1$ . For this particular problem  $C_1/C_2$  was chosen as  $1/49$  and the resistance of  $R_1$  was chosen as one ohm. Then, at balance  $|Z_L| = (C_2/C_1)(R_1) = (49)(1) = 49$  ohms. Assuming for the moment that the phase-angle detector has already been tuned so that  $\phi = 0$ , this makes  $Z_L = 49 + j0$ . Adding the resistance of  $R_1$  gives us the input impedance to the sensing units plus the tuning components, plus the antenna. This impedance, which is the terminal impedance of the transmission line, is  $50 + j0$  ohms, the characteristic impedance of the line.

The only approximation made here is neglecting the effects of the shunt impedance of  $C_1$  and  $C_2$  connected from line to ground on the input impedance to the unit. This error can be kept negligibly small by keeping the reactance of the combination of  $C_1$  and  $C_2$  relatively high compared to 50 ohms throughout the operating frequency range.

It has now been established that the unit under consideration meets the requirements set forth earlier, i.e., (a)  $V_o$  is a function of the ratio  $|Z_L|/|Z_o|$ , (b)  $V_o = 0$  when  $|Z_L|/|Z_o| = 1$ , (c) the sign of  $V_o$  changes as  $|Z_L|/|Z_o|$  changes from greater than one to less than one or vice versa.

The desirable feature of sensitivity insensitive to frequency is also attained because neither  $C_1/C_2$  nor  $V_1/V_2$  is a function of frequency and  $R_1$  is a pure resistance over this frequency range.

A note on sensitivity is in order at this point. The sensitivity of the unit can be varied by varying the resistance of  $R_1$  and the ratio  $C_1/C_2$ , increased sensitivity being obtained by increasing  $R_1$ , and at the same time increasing  $C_1/C_2$  accordingly, so that  $Z_L$  is still equal to  $Z_o$  at balance. This added sensitivity must be paid for in additional power dissipated in  $R_1$ . A compromise must therefore be made, maintaining adequate sensitivity with a minimum power dissipation in  $R_1$ .

In units designed to operate with high-power transmitters increased sensitivity can be obtained by using voltage doublers on  $V_2$  and  $V_{R_1}$  without increasing the power dissipation of  $R_1$ .



Although the present model does not employ voltage doublers, it is planned to install them before the complete engineering model of the matching system is completed.

This unit can readily be adapted to a wide variety of power outputs, characteristic line impedances, and antenna types by suitable manipulation of the resistance of  $R_1$  and the ratio  $C_1/C_2$ .

### SUMMARY OF WORK DONE

The system now under construction has been designed specifically to be used with a TCK transmitter (400-watt output) working into a 35-foot whip antenna through a 50-ohm coaxial line over the frequency range of 2 to 18 Mc.

The electrical characteristics of the tuning elements in the cantilever-matching network have been determined, design calculations have been made for the capacitors, and similar computations for the inductances are now under way. The two sensing units have been built, tested, and found satisfactory. In testing these detectors, a 35-foot whip antenna was used as a load at all frequencies where the tuning range of the available cantilever circuit permitted the sensing units to come to balance. At other frequencies in the 2- to 18-Mc band a dummy antenna was used. The impedance characteristic of this dummy was made such that the ranges of tuning obtainable with available tuning elements were sufficient to match the dummy load to the 50-ohm line. (The only components available at present are a variable capacitor and a variable inductance of insufficient tuning range to match the antenna to the line over the entire 2- to 18-Mc band. In the tests of the sensing unit, therefore, the cantilever network consisted of a series capacitor and a shunt inductor.)

The TCK transmitter was loaded into the antenna through the sensing units and the tuning elements and the entire matching system was allowed to come to balance. Then the transmitter was disconnected from the system and the input impedance to the sensing units plus the tuning elements plus the antenna was measured with the General Radio Impedance Bridge, type 916-A. (This impedance was the terminal impedance to the transmission line at balance.) The voltage standing-wave ratio on the line was then calculated from the terminal impedance and characteristic impedance of the line, and the procedure was repeated at close frequency intervals throughout the 2- to 18-Mc band.

The results are shown graphically in the form of a plot of voltage standing-wave ratio vs. frequency in Figure 9. The greatest mismatch occurs at 2 Mc, where the standing-wave ratio is approximately 1.2. This mismatch was caused primarily by an 8-ohm reactive component in the terminal impedance of the line. It is felt that this can be improved by increasing the sensitivity of the phase-angle detector in the lower half of 2- to 18-Mc band.

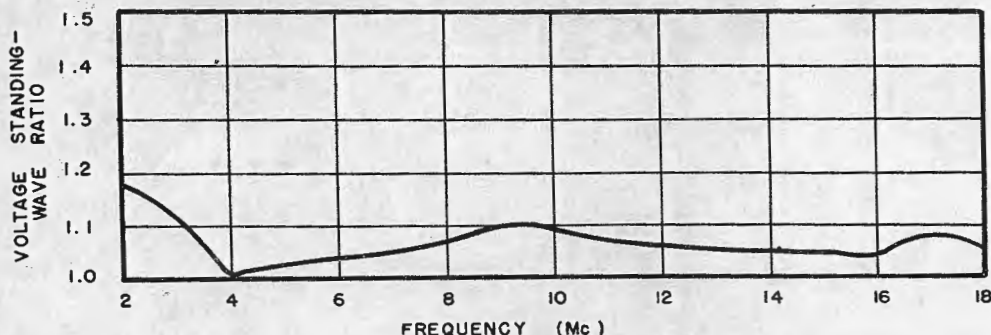


Figure 9 - Curve of voltage standing-wave ratio vs. frequency for balance conditions on the sensing unit

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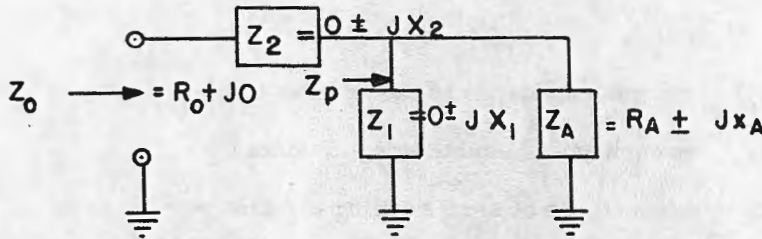
## CONCLUSIONS

On the basis of the work done to date it is concluded that, if the calculated ranges and voltage ratings of the components of the cantilever network can be mechanically realized, the completed unit will match a standard 35-foot whip antenna to a 50-ohm coaxial line with a resulting standing-wave ratio of not more than 1.1 at any frequency between 2 and 18 Mc. It is further concluded that comparable results will be obtainable in a wide variety of applications with relatively minor modifications of the original model and that the principal limitations on the range of applications are of a physical nature, such as the tuning ranges available in capacitors and inductances, voltage ratings obtainable with maximum tuning range, and other factors of a similar nature.

\* \* \*

# APPENDIX DERIVATION OF EQUATIONS (1) AND (2)

In the following diagram the antenna input impedance and the reactance of the two arms of the cantilever circuit are shown in block form. The sign convention in which +X denotes inductive reactance and -X denotes capacitive reactance is used.



$$Z_p = \frac{(jX_1)(R_A + jX_A)}{R_A + j(X_1 + X_A)}$$

Rationalizing:

$$Z_p = \frac{(jX_1)(R_A + jX_A)(R_A - j(X_1 + X_A))}{R_A^2 + (X_1 + X_A)^2}$$

$$Z_p = R_p + jX_p = \frac{R_A X_1^2}{R_A^2 + (X_1 + X_A)^2} + \frac{jX_1(R_A^2 + X_1 X_A + X_A^2)}{R_A^2 + (X_1 + X_A)^2}$$

Then

$$R_0 = R_p = \frac{R_A X_1^2}{R_A^2 + (X_1 + X_A)^2}$$

Rearranging:

$$R_A X_1^2 - R_0(X_1^2 + 2X_1 X_A + X_A^2) - R_0 R_A^2 = 0$$

or

$$(R_A - R_0) X_1^2 - (2X_A R_0) X_1 - R_0(X_A^2 + R_A^2) = 0$$

$$X_1 = \frac{2R_0 X_A \pm \sqrt{4R_0^2 X_A^2 + 4R_0(R_A^2 + X_A^2)(R_A - R_0)}}{2(R_A - R_0)}$$



Finally:

$$X_1 = \frac{R_0 X_A \pm \sqrt{R_0 R_A (R_A^2 - R_0 R_A + X_A^2)}}{(R_A - R_0)}$$

If  $Z_0 = R_0 + j0$ :

$$X_2 = X_p = - \frac{X_1 (R_A^2 + X_1 X_A + X_A^2)}{R_A^2 + (X_1 + X_A)^2}$$

where

$Z_A$  = input impedance to the antenna alone

$X_1$  = reactance of shunt-tuning element

$X_2$  = reactance of series-tuning element

$Z_p$  = parallel impedance of  $X_1$  and the antenna

$Z_0$  = 50 + j0 ohms

\* \* \*