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ANALYSIS AND DESIGN OF R-C PHASE-SHIFT OSCILLATOR NETWORKS

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ANALYSIS AND DESIGN OF R-C PHASE-SHIFT OSCILLATOR NETWORKS

INTRODUCTION

The phase-shift oscillator is a resistance-capacitance tuned oscillator normally employing three or more L-sections in cascade. The frequency of oscillation may be uniquely determined and expressed as a function of the network parameters.

In recent years various papers have been published concerning resistance-capacitance phase-shift oscillators, most of which discussed rather specific or novel circuits. Among the first papers only the basic circuit was discussed; its simplicity as a stable low-frequency sine-wave oscillator using only one tube was indicated. Other points brought out were compactness, lightness, and low expense of building such oscillators, since they contained no transformers and only a few resistors and capacitors. As time passed, the requirements for oscillators of this type increased and innovations of the basic network were investigated.

The purpose of this report is to analyze more generally the oscillator networks, specifically for use in designing frequency-modulated subcarrier oscillators (as used in FM/FM telemetering systems) and in general to correlate various scattered material for a better understanding of design procedure.

From general equations of the complex transfer characteristics (attenuation and phase shift) of the oscillator networks, important facts are uncovered governing oscillator stability, wave form, amplitude-modulation effects, and linearity of frequency deviation when resistance legs of the network are varied to cause frequency modulation. The effects of network "tapering," defined as the geometric increase or decrease of the impedance of each successive L-section with respect to its preceding section, are shown in families of curves for specific cases.

CRITERIA FOR OSCILLATION

If an oscillator is composed of a μ path and β path as shown in Figure 1, where μ denotes the amplification factor of the amplifier and β denotes the fraction of the amplifier's output that is fed back to its input, then for oscillations to occur:

$$\mu\beta \geq 1 . \qquad (1)$$

If $\qquad \mu = a + jb$

and $\qquad \beta = c + jd,$

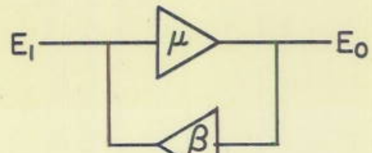


Figure 1

then $$(a + jb)\,(c + jd) \geqq 1$$

or the absolute value $$\left[(ac - bd)^2 + (ad + bc)^2\right]^{\frac{1}{2}} \geqq 1$$

and $$\arctan \frac{ad + bc}{ac - bd} = 0 \text{ or } \pi .$$

In practice, the imaginary part of the amplification can be considered equal to zero, owing to the negligible effect of electrode and interelectrode impedances at low frequencies. Thus the criterion for oscillation reduces to:

$$a(c^2 + d^2)^{\frac{1}{2}} \geqq 1 \tag{2}$$

$$\arctan \frac{d}{c} = 0 \text{ or } \pi , \tag{3}$$

but since $$\arctan \frac{d}{c} = 0 \text{ or } \pi ,$$

$$d = 0$$

Equation (2) may be further reduced to:

$$a \geqq \frac{1}{c} . \tag{4}$$

Equations (3) and (4) show that for oscillation to take place, the amplifier must have enough gain to overcome the loss through the feedback network and the network must be purely real. In order for the gain to be sufficient, feedback voltage must be additive in nature or the phase shift around the μ-β loop must be zero.

Thus if the amplifier produces 180 degrees phase shift, then the network must also shift the phase by 180 degrees. If the amplifier produces zero degree phase shift, so must the network. Since these two cases are the only ones, there can be only two types of phase-shifting networks, namely, the 180-degree phase-shift type and the zero-degree phase-shift type. The networks analyzed in this report are those shown below in Figures 2 to 6.

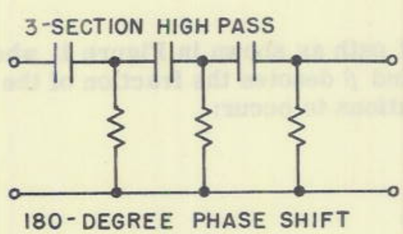


Figure 2

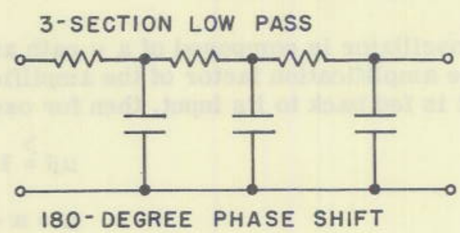


Figure 3

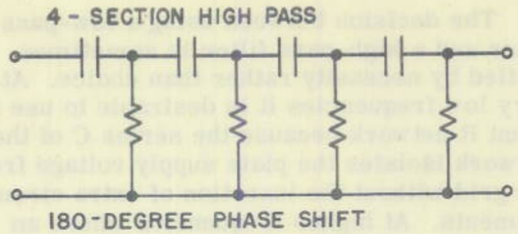


Figure 4

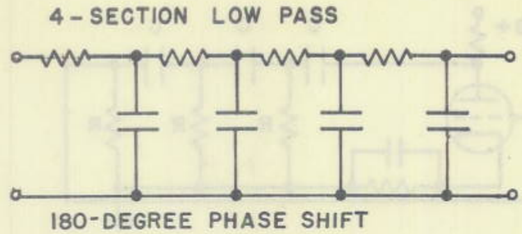


Figure 5

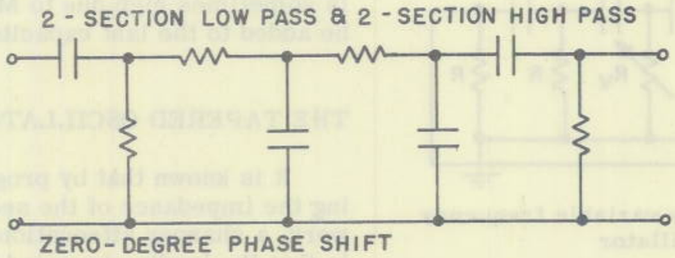


Figure 6

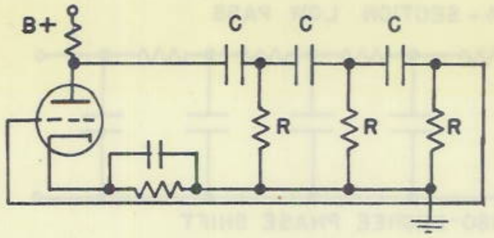
THE BASIC OSCILLATOR

Figure 7a shows the basic circuit of a resistance-capacitance phase-shift oscillator in which a three-section high-pass filter is used in conjunction with a single tube.

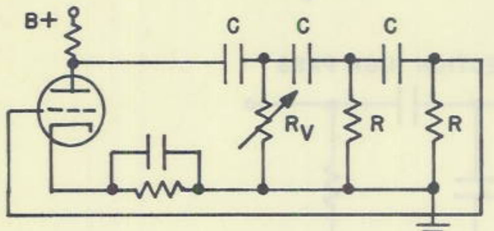
Figure 7b is the same oscillator as 7a except that the first fixed resistor in the phase-shifting network has been replaced by a variable resistor, thus making the frequency of oscillation variable according to mechanical position. Figure 7c is the same oscillator as 7a except that the first fixed resistor has been replaced by the resistance of a vacuum tube, thus making the frequency of oscillation variable according to electric potential.

Any one, any combination, or all the resistors or capacitors may be varied to change the frequency, and the linearity of the frequency deviation will be different depending on which circuit element or elements are varied. A linear change in any of the circuit elements will not produce a linear change in frequency, since frequency is inversely proportional to the resistance and capacitance of the network. This is not of great consequence, however, since most variable elements are not linearly variable with mechanical position or electric potential unless specifically so designed. This means that the frequency-controlling element may be designed to compensate for the nonlinearity in frequency deviation if desired.

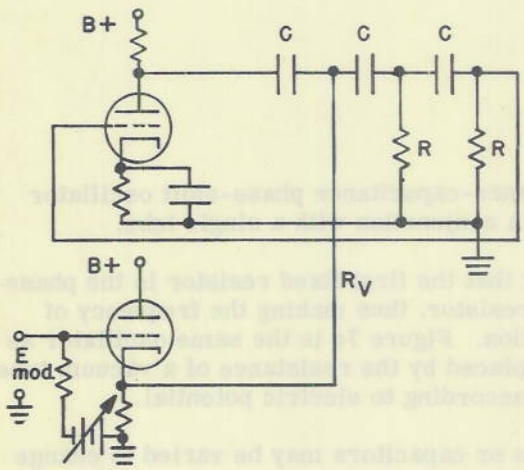
The network used in a phase-shift oscillator serves not only to supply a return voltage of the correct phase relationship but also as a filter to reduce the amplitude of harmonics generated in the tube. In the high-pass filter or shunt resistance network, harmonics occur at frequencies where the network attenuation is very low, and a reduced amplifier gain results, due to a large amount of feedback. In the low-pass filter or shunt capacitance network the harmonic voltages are attenuated much more than the fundamental, and a harmonic improvement at the filter output terminals results.



a - Fixed frequency oscillator



b - Mechanically variable frequency oscillator



c - Electronically variable frequency oscillator

Figure 7

The decision between using a low-pass filter and a high-pass filter is sometimes settled by necessity rather than choice. At very low frequencies it is desirable to use a shunt R network because the series C of the network isolates the plate supply voltage from the grid without the insertion of extra circuit elements. At higher frequencies where an isolating circuit may be added without affecting the frequency determining network, a shunt C network may be used, and to advantage, because the input capacitance of the tube (which is sometimes high due to Miller effect) may be added to the last capacitor of the network.

THE TAPERED OSCILLATOR

It is known that by progressively increasing the impedance of the sections along a network, a sharper attenuation curve may be had, in that the loading imposed by any one section on the previous section is reduced. This also results in a sharper phase-shift curve (assuming the curve to be a frequency plot). "Tapered-network" is a term that has been coined by other authors to describe such a network. Figure 8 shows the oscillator of Figure 7a again using a tapered network rather than the basic three-section high-pass filter.

It may be said that the network of Figure 8 has a taper factor of ten. It should be noted that the r-c time constants of each section of Figure 8 are still equal thereby retaining the simplicity of the design equations. The merits of such a design are investigated under the section of this report entitled "The Three-Section Low-Pass Filter."

METHOD OF ANALYSIS

In the analysis of any passive network, it is needless to say that the dependency of the network transfer characteristics upon its driving and terminating impedances cannot be overlooked. However, if the driving impedance can be assumed nearly equal to zero and the terminating impedance infinitely high, conditions which can usually be closely approximated in practice, the complexity of the network design is considerably reduced. This simplification is made use of throughout the network analysis of this report.

A great deal of information may be obtained from the transfer characteristics of a network in the normalized form. The easiest method of obtaining equations expressing attenuation and phase shift through cascaded networks is through the aid of matrix algebra.* After obtaining the expressions by this method, they are normalized with respect to a reference frequency that is closely related to the network parameters.† With the equations in this form, the networks may easily be analyzed to determine their relative merits.

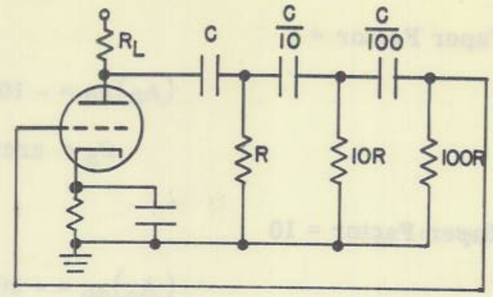


Figure 8

THE THREE-SECTION LOW-PASS FILTER

Harmonic-Transfer Characteristics

Using the methods of network analysis outlined in the previous section, the general equations for attenuation and phase shift through a three-section low-pass filter (Figure 9) are derived in Appendix III (Equations (217) and (218)), and are given below:

$$(A_n)\,\mathrm{db} = -10\log\left\{\left[1 - 2^{2n}\left(\frac{KM}{LT} + \frac{KN}{LT} + \frac{KM}{LS} + \frac{MN}{ST} + \frac{KN}{ST}\right)\right]^2 + \left[2^{3n}\left(\frac{KMN}{LST}\right) - 2^n\left(\frac{N}{T} + \frac{K}{L} + \frac{M}{S} + \frac{K}{S} + \frac{M}{T} + \frac{K}{T}\right)\right]^2\right\}, \quad (5)$$

$$\beta_n = \arctan \frac{2^{3n}\dfrac{KMN}{LST} - 2^n\left(\dfrac{N}{T} + \dfrac{K}{L} + \dfrac{M}{S} + \dfrac{K}{S} + \dfrac{M}{T} + \dfrac{K}{T}\right)}{1 - 2^{2n}\left(\dfrac{KM}{LT} + \dfrac{KN}{LT} + \dfrac{KM}{LS} + \dfrac{MN}{ST} + \dfrac{KN}{ST}\right)} \quad (6)$$

where

$$2^n = \frac{\omega}{\omega_0} \quad \text{and} \quad \omega_0 = \frac{1}{RC}.$$

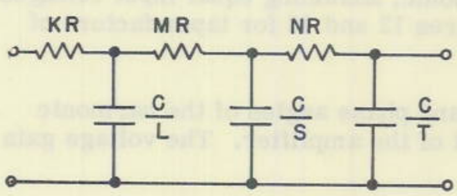


Figure 9

On reducing Equations (5) and (6) for taper factors of 0.1, 1, 10, and infinity, where taper factor equals M/K, N/M, S/L, and T/S, the following equations result:

Taper Factor = 0.1

$$(A_n)_{db} = -10\log\left(2^{6n} + 283\cdot 2^{4n} + 15083\cdot 2^{2n} + 1\right), \quad (7)$$

$$\beta_n = \arctan \frac{2^{3n} - 123\cdot 2^n}{1 - 23\cdot 2^{2n}}. \quad (8)$$

---

* Refer to Appendix I

† Refer to Appendix II

Taper Factor = 1

$$(A_n)_{db} = -10 \log \left(2^{6n} + 13 \cdot 2^{4n} + 26 \cdot 2^{2n} + 1\right), \tag{9}$$

$$\beta_n = \arctan \frac{2^{3n} - 6 \cdot 2^n}{1 - 5 \cdot 2^{2n}}. \tag{10}$$

Taper Factor = 10

$$(A_n)_{db} = -10 \log \left(2^{6n} + 3.82 \cdot 2^{4n} + 3.9041 \cdot 2^{2n} + 1\right), \tag{11}$$

$$\beta_n = \arctan \frac{2^{3n} - 3.21 \cdot 2^n}{1 - 3.2 \cdot 2^{2n}}. \tag{12}$$

Taper Factor = ∞

$$(A_n)_{db} = -10 \log \left(2^{6n} + 3 \cdot 2^{4n} + 3 \cdot 2^{2n} + 1\right), \tag{13}$$

$$\beta_n = \arctan \frac{2^{3n} - 3 \cdot 2^n}{1 - 3 \cdot 2^{2n}}. \tag{14}$$

A plot of attenuation, in decibels, on a logarithmic normalized frequency scale for these four different taper factors is shown in Figure 10 and its corresponding phase-shift curves in Figure 11. Since oscillation occurs when the phase shift through the network is 180 degrees, it is possible to locate the value of n on Figure 12 corresponding to this point and then find the value of attenuation corresponding to this value of n on the curve of Figure 11. Since the divisions of n are linear on the logarithmic paper, it is easier to use the ruling of the paper which actually represents 2^n, or by definition of the octave notation, ω/ω_0. Values of attenuation and phase shift corresponding to multiples of the value of ω/ω_0 at 180 degrees are the values of attenuation and phase shift of the harmonics of the oscillation frequency. If a vector diagram were drawn showing the relation between the output voltage of the fundamental, and second and third harmonic, assuming equal input voltages of zero phase angle, it would appear as shown in Figures 12 and 13 for taper factors of 1.0 and 10 respectively.

Figures 13 and 14 show the absolute magnitudes and phase angles of the harmonic voltages, neglecting the action of feedback to the input of the amplifier. The voltage gain of an amplifier with complex feedback is:

$$A' = \frac{\mu}{\left(1 + |\mu\beta|^2 - 2|\mu\beta| \cos\phi\right)^{\frac{1}{2}}} \tag{15}$$

where: A' = amplifier gain with complex feedback

μ = voltage amplification of amplifier without feedback

β = fraction of the amplifier output fed back to its input

ϕ = phase shift of the amplifier and feedback circuit at a given frequency.

A definite value of harmonic voltage is not known until a specific circuit is designed, but by comparing the change of gain due to feedback of various network types, plus the

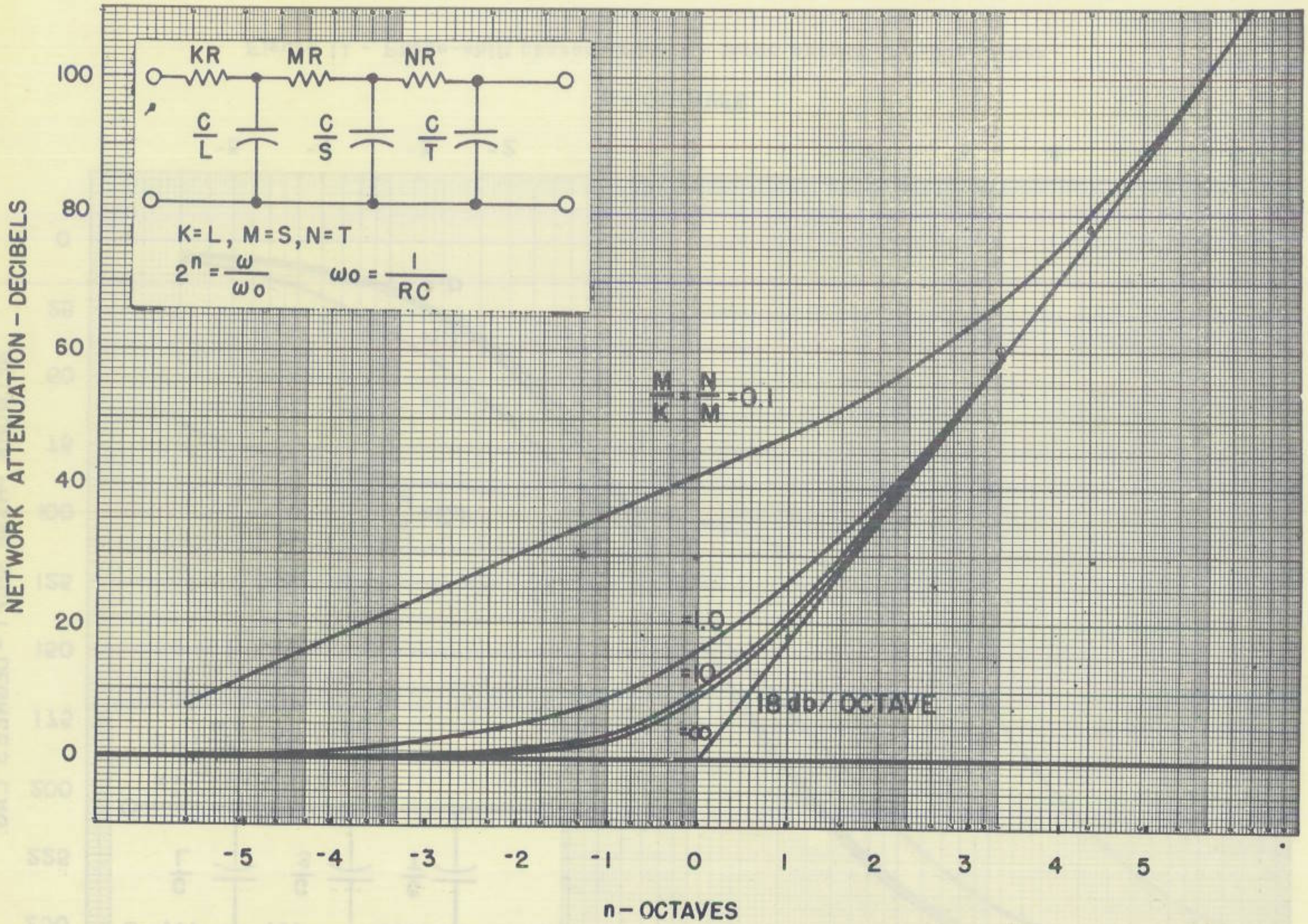


Figure 10 - Attenuation characteristics - three-section low-pass filter

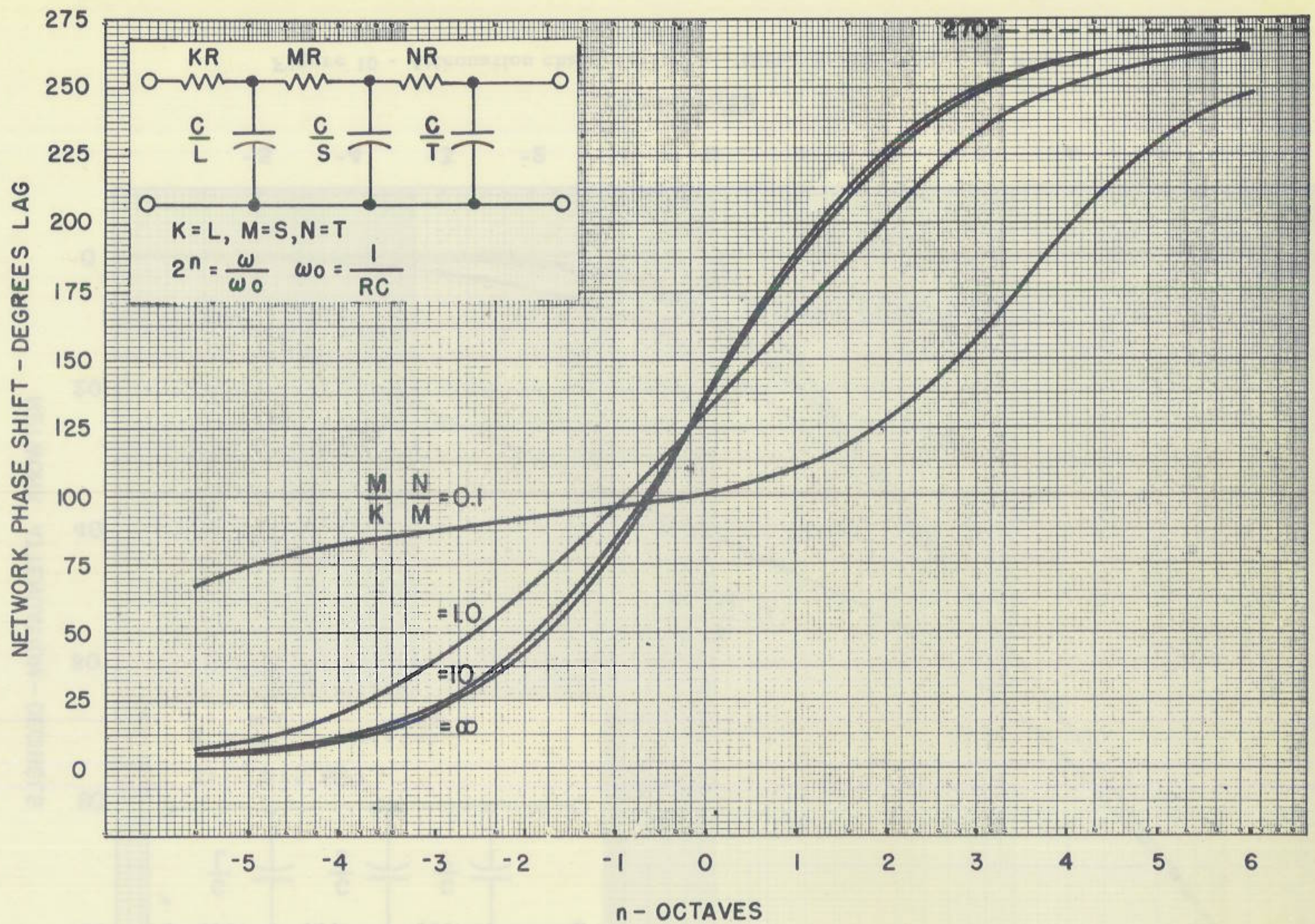


Figure 11 - Phase-shift characteristics - three-section low-pass filter

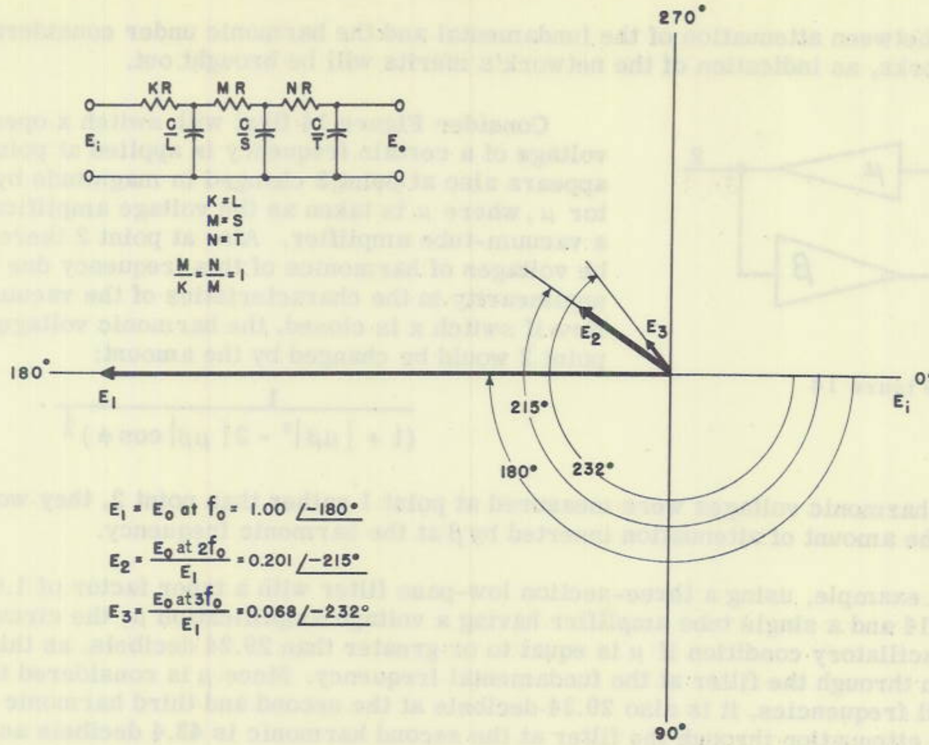


Figure 12 - Harmonic transfer of three-section low-pass network

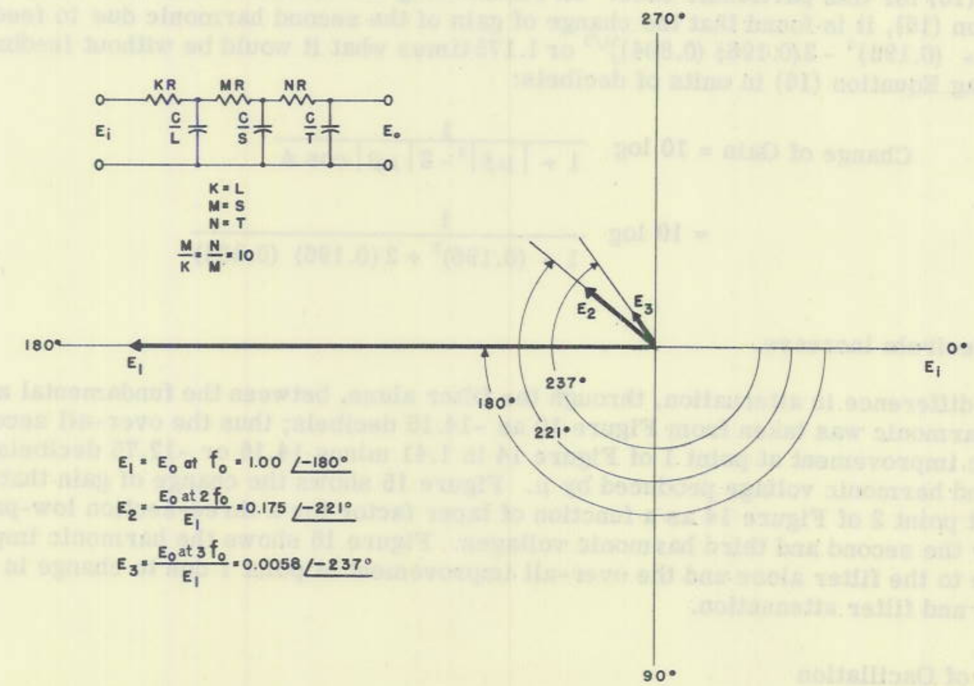


Figure 13 - Harmonic transfer of three-section low-pass tapered network

difference between attenuation of the fundamental and the harmonic under consideration for these networks, an indication of the network's merits will be brought out.

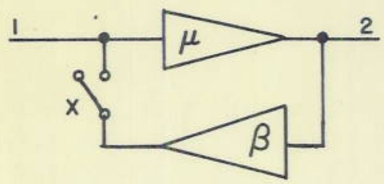


Figure 14

Consider Figure 14 first with switch x open. If a voltage of a certain frequency is applied at point 1, it appears also at point 2 changed in magnitude by a factor μ, where μ is taken as the voltage amplification of a vacuum-tube amplifier. Also at point 2 there would be voltages of harmonics of this frequency due to the nonlinearity in the characteristics of the vacuum tube. Now if switch x is closed, the harmonic voltages at point 2 would be changed by the amount:

$$\frac{1}{(1 + |\mu\beta|^2 - 2|\mu\beta| \cos\phi)^{\frac{1}{2}}} . \tag{16}$$

If the harmonic voltages were measured at point 1 rather than point 2, they would differ by the amount of attenuation inserted by β at the harmonic frequency.

As an example, using a three-section low-pass filter with a taper factor of 1.0 for β of Figure 14 and a single tube amplifier having a voltage amplification μ, the circuit would be in an oscillatory condition if μ is equal to or greater than 29.24 decibels, as this is the attenuation through the filter at the fundamental frequency. Since μ is considered to be the same at all frequencies, it is also 29.24 decibels at the second and third harmonic frequencies. The attenuation through the filter at the second harmonic is 43.4 decibels as taken from Figure 10. The product $\mu\beta$ is -14.16 decibels or a voltage ratio of 0.196. The phase shift through μ is 180 degrees and the phase shift through the network at the second harmonic is 216.5 degrees (Figure 11). The sum of these phase shifts ($180^0 + 216.5^0$) is ϕ of Equation (16) for this particular case. On substituting the values obtained for $\mu\beta$ and cos ϕ in Equation (16), it is found that the change of gain of the second harmonic due to feedback is: $1/[1 + (0.196)^2 - 2(0.196)(0.804)]^{1/2}$ or 1.175 times what it would be without feedback. Expressing Equation (16) in units of decibels:

$$\begin{aligned} \text{Change of Gain} &= 10 \log \frac{1}{1 + |\mu\beta|^2 - 2|\mu\beta| \cos\phi} \\ &= 10 \log \frac{1}{1 + (0.196)^2 + 2(0.196)(0.804)} \end{aligned} \tag{17}$$

or 1.41 decibels increase.

The difference in attenuation, through the filter alone, between the fundamental and second harmonic was taken from Figure 10 as -14.16 decibels; thus the over-all second harmonic improvement at point 1 of Figure 14 is 1.41 minus 14.16 or -12.75 decibels over the second harmonic voltage produced by μ. Figure 15 shows the change of gain that would appear at point 2 of Figure 14 as a function of taper factor for a three-section low-pass filter for the second and third harmonic voltages. Figure 16 shows the harmonic improvement due to the filter alone and the over-all improvement at point 1 due to change in amplifier gain and filter attenuation.

Stability of Oscillation

The degree of oscillator stability is determined mostly by the characteristics of the frequency-determining network, specifically its amplitude or phase characteristics.

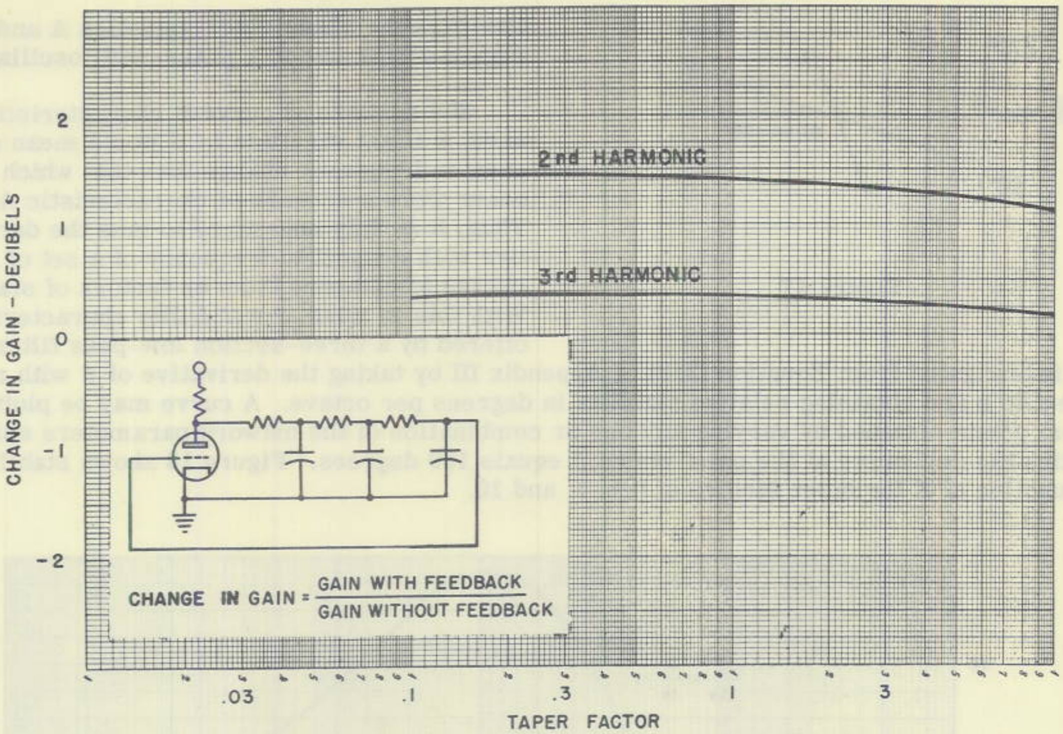


Figure 15 - Change in harmonic gain due to feedback - three-section low-pass filter

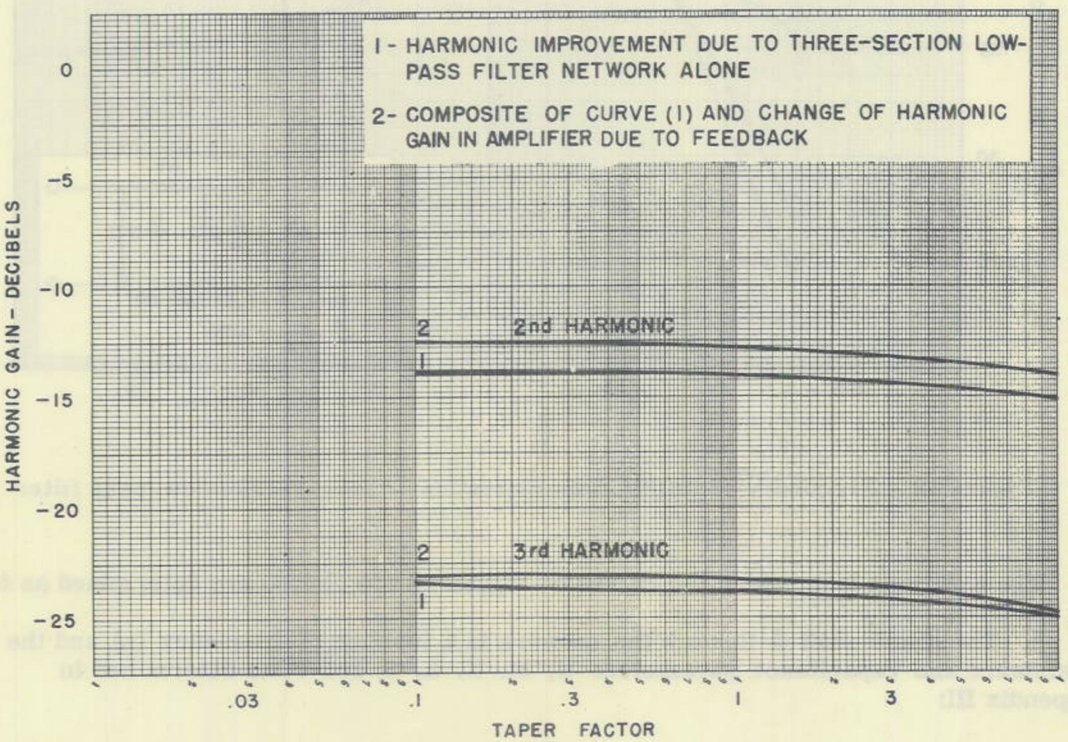


Figure 16 - Harmonic transfer characteristics - three-section low-pass filter

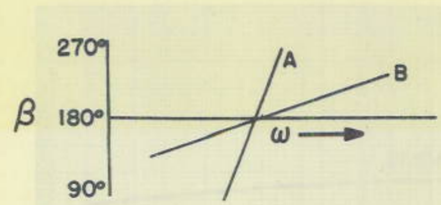


Figure 17

Compare the phase characteristics A and B of Figure 17 for use in a phase-shift oscillator.

If a network of curve B characteristic were used, a slight variation in β would mean a much greater frequency change than that which would occur were a network of characteristic A used. Thus, it is only necessary to take the derivative with respect to frequency of a set of phase curves to compare them as factors of stability. With this in mind, the stability characteristics offered by a three-section low-pass filter may be investigated from Equation (218) of Appendix III by taking the derivative of β with respect to n and obtaining relative stability in degrees per octave. A curve may be plotted from points obtained by varying any one or combination of the network parameters and taking the derivative at the point where β equals 180 degrees. Figure 18 shows stability as a function of K for taper factors of 0.1, 1, and 10.

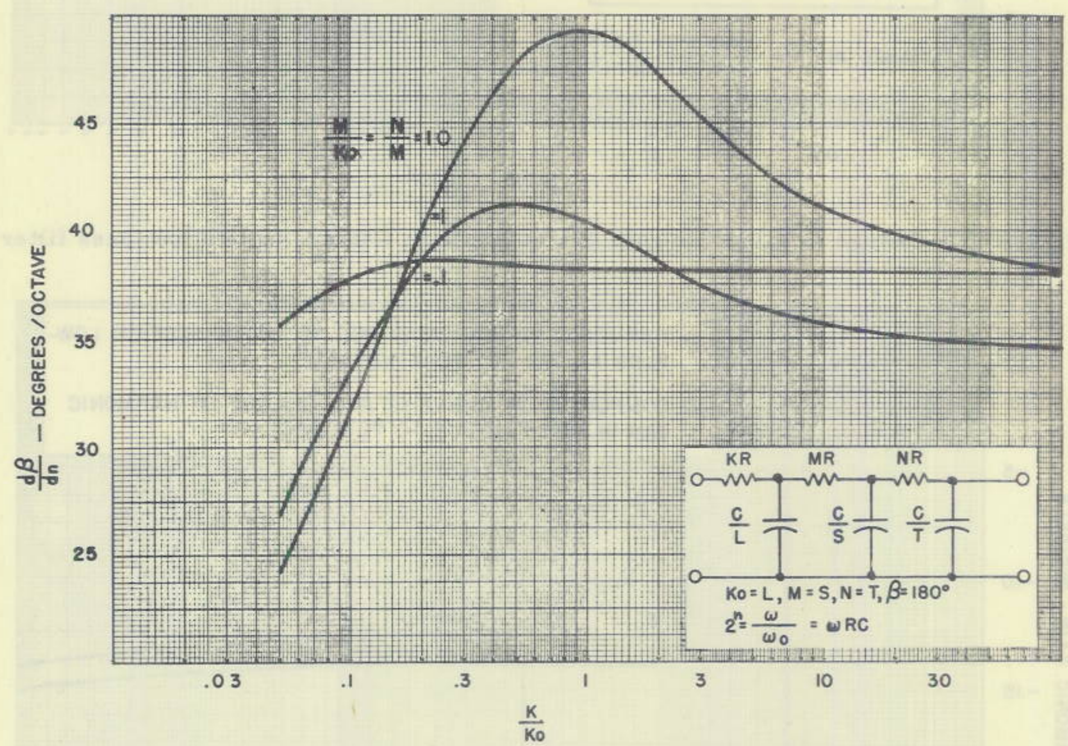


Figure 18 - Frequency-stability characteristics - three-section low-pass filter

The equations from which the curves of Figure 18 are plotted are determined as follows:

1. The phase shift β through the network is a function of frequency (n) and the resistance and capacitance parameters K, M, N, L, S, and T, as determined in Appendix III:

$$\beta = \arctan \frac{2^{3n}\,\frac{KMN}{LST} - 2^n\left(\frac{N}{T}+\frac{K}{L}+\frac{M}{S}+\frac{K}{S}+\frac{M}{T}+\frac{K}{T}\right)}{1 - 2^{2n}\left(\frac{KM}{LT}+\frac{KN}{LT}+\frac{KM}{LS}+\frac{MN}{ST}+\frac{KN}{ST}\right)} . \qquad (18)$$

The derivative of β with respect to frequency (n) is also a function of frequency and the parameters (see Appendix III) :

$$\frac{d\beta}{dn} = \frac{-\left[2^{4n}(XZ) + 2^{2n}(YZ - 3X) + Y\right] 2^n \; (39.72)}{2^{6n}X^2 + 2^{4n}(Z^2 - 2XY) + 2^{2n}(Y^2 - 2Z) + 1} \; \frac{\text{degrees}}{\text{octave}} \qquad (19)$$

where

$$X = \frac{KMN}{LST} \qquad (20)$$

$$Y = \frac{N}{T}+\frac{K}{L}+\frac{M}{S}+\frac{K}{S}+\frac{M}{T}+\frac{K}{T} \qquad (21)$$

$$Z = \frac{KM}{LT}+\frac{KN}{LT}+\frac{KM}{LS}+\frac{MN}{ST}+\frac{KN}{ST} . \qquad (22)$$

For β equal to 180 degrees, frequency is a function of the resistance and capacitance parameters [Appendix III, Equation (240)] :

$$2^n = \left(\frac{Y}{X}\right)^{\frac{1}{2}} \qquad (23)$$

where X and Y are the same as above.

Therefore the derivative of β with respect to frequency may be made a function of the parameters only, for β equal to 180 degrees [Appendix III, Equation (241)] :

$$\frac{d\beta}{dn} = \frac{-79.44\,Y\sqrt{XY}}{YZ - X} . \qquad (24)$$

This equation may be reduced for any taper factor in terms of any one or combination of parameters.

2. The following equations result when Equation (24) is reduced for taper factors of 0.1, 1, and 10 in terms of the resistance parameter K:

Taper Factor = 0.1

$$\frac{d\beta}{dn} = \frac{-79.44\left[111\frac{K}{K_0} + 12\right]\left[111\left(\frac{K}{K_0}\right)^2 + 12\,\frac{K}{K_0}\right]^{\frac{1}{2}}}{\left[2442\left(\frac{K}{K_0}\right)^2 + 374\,\frac{K}{K_0} + 12\right]} \qquad (25)$$

Taper Factor = 1

$$\frac{d\beta}{dn} = \frac{-79.44\left(3\,\frac{K}{K_0} + 3\right)\left[3\left(\frac{K}{K_0}\right)^2 + 3\,\frac{K}{K_0}\right]^{\frac{1}{2}}}{12\left(\frac{K}{K_0}\right)^2 + 14\left(\frac{K}{K_0}\right) + 3} \qquad (26)$$

Taper Factor = 10

$$\frac{d\beta}{dn} = \frac{-79.44\left(1.11\frac{K}{K_0} + 2.1\right)\left[1.11\left(\frac{K}{K_0}\right)^2 + 2.1\frac{K}{K_0}\right]^{\frac{1}{2}}}{2.442\left(\frac{K}{K_0}\right)^2 + 4.73\left(\frac{K}{K_0}\right) + 2.1} . \quad (27)$$

The parameter K_0 is a point around which K is varied and it is taken equal to its corresponding capacitance parameter L. An example of reducing the equations is given in Appendix III.

Amplitude-Modulation Effects

If all the network resistors or capacitors of Figure 9 were varied simultaneously to vary the network corner frequency, and therefore the frequency of oscillation, so that the corner frequencies of each section remained equal, no change in attenuation through the network would result. (Zero generator impedance and infinite load impedance are assumed.) Since this report deals with the variation of one- and two-resistor legs to vary frequency it is desirable to investigate the corresponding variations in network attenuation. The complex attenuation through a three-section low-pass filter at any frequency is (see Appendix III for derivation):

$$\alpha = \frac{1}{a - jb} \quad (28)$$

where

$$a = \left[1 - (\omega CR)^2\left(\frac{KM}{LT} + \frac{KN}{LT} + \frac{KM}{LS} + \frac{MN}{ST} + \frac{KN}{ST}\right)\right] \quad (29)$$

and

$$b = \left[(\omega CR)^3\frac{KMN}{LST} - (\omega CR)\left(\frac{N}{T} + \frac{K}{L} + \frac{M}{S} + \frac{K}{S} + \frac{M}{T} + \frac{K}{T}\right)\right]. \quad (30)$$

For oscillation to occur β equals 180 degrees, the imaginary part of the attenuation must be zero, and Equation (28) reduces to the following:

$$\alpha = \frac{1}{1 - (\omega CR)^2\left(\frac{KM}{LT} + \frac{KN}{LT} + \frac{KM}{LS} + \frac{MN}{ST} + \frac{KN}{ST}\right)} . \quad (31)$$

Since at β equal to 180 degrees, frequency is a function of the parameters K, M, N, L, S, and T, Equation (31) is reduced to attenuation (in decibels) as a function of the parameters (as given by Equation (248), Appendix III):

$$A_{db} = -20\log\left|1 - \frac{\left(\frac{N}{T} + \frac{K}{L} + \frac{M}{S} + \frac{K}{S} + \frac{M}{T} + \frac{K}{T}\right)\left(\frac{KM}{LT} + \frac{KN}{LT} + \frac{KM}{LS} + \frac{MN}{ST} + \frac{KN}{ST}\right)}{\frac{KMN}{LST}}\right|. \quad (32)$$

Further reduction of Equation (32) in terms of the resistance parameter K for taper factors of 0.1, 1, and 10 result in the following equations:

Taper Factor = 0.1

$$A_{db} = -20\log\left(2442\frac{K}{K_0} + 374 + 12\frac{K_0}{K}\right) \quad (33)$$

Taper Factor = 1

$$A_{db} = -20 \log\left(12\frac{K}{K_0} + 14 + \frac{3K_0}{K}\right) \tag{34}$$

Taper Factor = 10

$$A_{db} = -20 \log\left(2.442\frac{K}{K_0} + 4.73 + \frac{2.1K_0}{K}\right). \tag{35}$$

A plot of Equations 33-35 is shown in Figure 19. The attenuation at K equal to 1 has been taken equal to zero decibels or a voltage ratio of 1; the values of attenuation for other values of K are relative. The curves indicate the best point of operation for the least amount of amplitude modulation when the frequency of oscillation is varied by varying the resistance parameter K. It can be seen that the network tapering has little effect on the sharpness of the curves but has a marked effect on the best point of operation. The operating points for the least amount of amplitude modulation are seen to be at K equals 0.3, 0.5, and 1 for taper factors of 0.1, 1, and 10 respectively. The plotting of these curves with M or N as variable would result in curves similiar to those of Figure 19. If two of the resistance parameters were varied simultaneously, the resulting curves would be flatter than those for a single resistance variation.

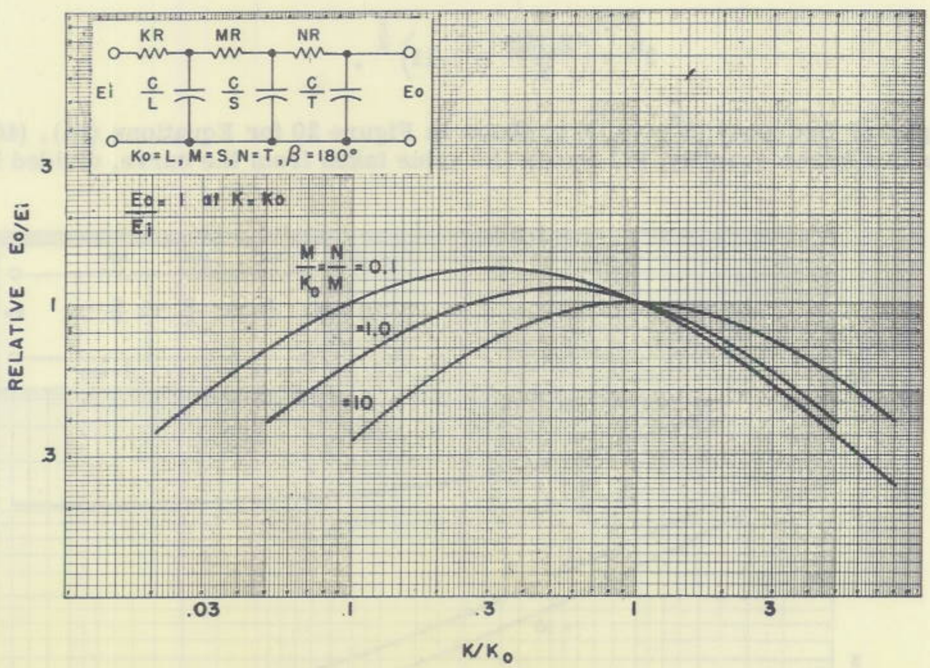


Figure 19 - Amplitude-modulation characteristics - three-section low-pass filter

Frequency-Deviation Characteristics

The equation of frequency in terms of the parameters K, M, N, L, S, and T is obtained by setting the network phase shift β equal to 180 degrees and solving for frequency:

$$2^n = \frac{\omega}{\omega_e} = \omega RC = \left(\frac{Y}{X}\right)^{\frac{1}{2}} \tag{36}$$

where

$$X = \frac{KMN}{LST} \tag{37}$$

$$Y = \left(\frac{N}{T} + \frac{K}{L} + \frac{M}{S} + \frac{K}{S} + \frac{M}{T} + \frac{K}{T}\right). \tag{38}$$

For taper factors of 0.1, 1, and 10, expressing frequency (2^n) as a function of the resistance parameter, K, the following equations result:

Taper Factor = 0.1

$$2^n = \left(\frac{12K_0}{K} + 111\right)^{\frac{1}{2}} \tag{39}$$

Taper Factor = 1

$$2^n = \left(\frac{3K_0}{K} + 3\right)^{\frac{1}{2}} \tag{40}$$

Taper Factor = 10

$$2^n = \left(\frac{2.1K_0}{K} + 1.11\right)^{\frac{1}{2}}. \tag{41}$$

A plot of frequency (2^n) vs. K is shown in Figure 20 for Equations (34), (40), and (41). Since ω is a known quantity, RC equals the value taken from the curve, divided by ω.

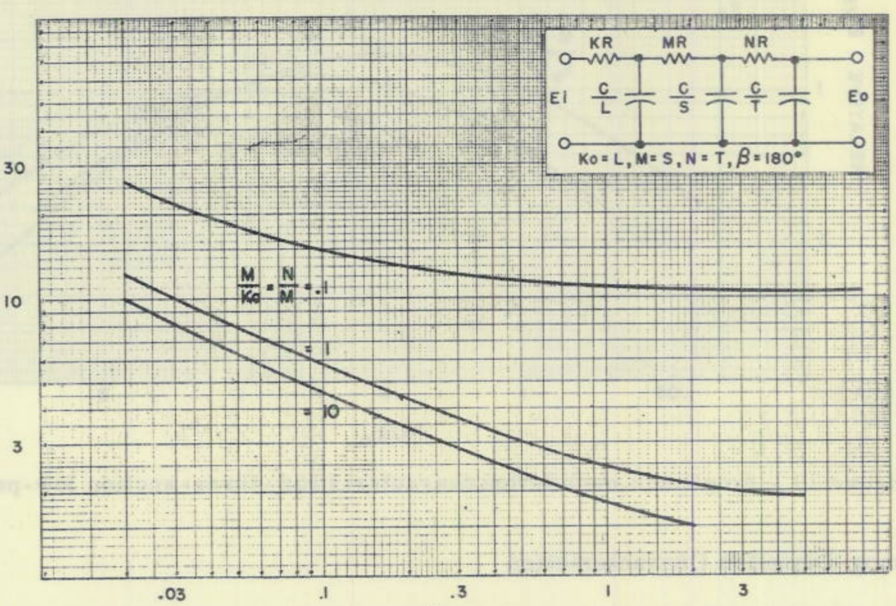


Figure 20 - Frequency-deviation characteristics— three-section low-pass filter

Theoretically, an infinite number of R and C combinations would work, but practical limitations prohibit this. Some of these are:

(1) Availability of precision capacitors (Mica, etc.) in large values

(2) Grid-resistance limitations of vacuum tubes

(3) Limit in approaching a zero-impedance driving source.

THE THREE-SECTION HIGH-PASS FILTER

Harmonic-Transfer Characteristics

The diagram of the three-section high-pass filter is shown in Figure 21.

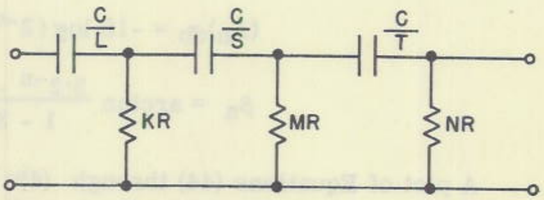


Figure 21

The general equations for attenuation and phase shift through this network are derived in Appendix IV, and are given by:

$$(A_n)_{db} = -10 \log \left\{ \left[1 - 2^{-2n} \left(\frac{LS}{KM} + \frac{ST}{MN} + \frac{LT}{KN} + \frac{LT}{MN} + \frac{LS}{KN} \right) \right]^2 + \left[+ 2^{-n} \left(\frac{L}{K} + \frac{S}{M} + \frac{L}{M} + \frac{T}{N} + \frac{S}{N} + \frac{L}{N} \right) - 2^{-3n} \frac{LST}{KMN} \right]^2 \right\}, \quad (42)$$

$$\beta_n = \arctan \frac{2^{-n} \left(\frac{S}{M} + \frac{L}{K} + \frac{L}{M} + \frac{T}{N} + \frac{S}{N} + \frac{L}{N} \right) - 2^{-3n} \frac{LST}{KMN}}{1 - 2^{-2n} \left(\frac{LS}{KM} + \frac{ST}{MN} + \frac{LT}{KN} + \frac{LT}{MN} + \frac{LS}{KN} \right)} \quad (43)$$

where $2^n = \omega / \omega_0$

and $\omega_0 = 1/RC$.

On reducing Equations (42) and (43) for taper factor of 0.1, 1, 10, and infinity the following equations result:

Taper Factor = 0.1

$$(A_n)_{db} = -10 \log (2^{-6n} + 283 \cdot 2^{-4n} + 15083 \cdot 2^{-2n} + 1), \quad (44)$$

$$\beta_n = \arctan \frac{123 \cdot 2^{-n} - 2^{-3n}}{1 - 23 \cdot 2^{-2n}}. \quad (45)$$

Taper Factor = 1

$$(A_n)_{db} = -10 \log (2^{-6n} + 13 \cdot 2^{-4n} + 26 \cdot 2^{-2n} + 1), \quad (46)$$

$$\beta_n = \arctan \frac{6 \cdot 2^{-n} - 2^{-3n}}{1 - 5 \cdot 2^{-2n}}. \quad (47)$$

Taper Factor = 10

$$(A_n)_{db} = -10 \log (2^{-6n} + 3.82 \cdot 2^{-4n} + 3.9041 \cdot 2^{-2n} + 1), \tag{48}$$

$$\beta_n = \arctan \frac{3.21 \cdot 2^{-n} - 2^{-3n}}{1 - 3.2 \cdot 2^{-2n}} \tag{49}$$

Taper Factor = ∞

$$(A_n)_{db} = -10 \log (2^{-6n} + 3 \cdot 2^{-4n} + 3 \cdot 2^{-2n} + 1), \tag{50}$$

$$\beta_n = \arctan \frac{3 \cdot 2^{-n} - 2^{-3n}}{1 - 3 \cdot 2^{-2n}}. \tag{51}$$

A plot of Equations (44) through (49) inclusive is shown in Figures 22 and 23. In the same manner as discussed in the preceding section, Figure 24 is obtained for the three-section high-pass filter for taper factor = 1, and shows the relation of phase shift and voltage transmission between the fundamental and second and third harmonics. On comparing Figure 24 with Figure 12, it is noted that the second and third harmonic components lag the fundamental for both the three-section low- and high-pass filters by about the same amount but their magnitude relationship is much different. In the low-pass filter the fundamental is much larger than the harmonics while in the high-pass network the harmonics are passed with much less attenuation than the fundamental. Because of this, the amount of feedback at harmonic frequencies is quite large for the high-pass network, thus producing a decrease of harmonic gain in the amplifier as shown in Figure 25.

Stability of Oscillation

The general equation relating frequency stability in terms of phase-shift curve slope in units of degrees per octave is as follows (see Appendix IV for derivation):

$$\frac{d\beta}{dn} = \frac{-79.44\,Y\,(XY)^{\frac{1}{2}}}{YZ - X}\,\frac{\text{degrees}}{\text{octave}} \tag{52}$$

where

$$X = \frac{LST}{KMN} \tag{53}$$

$$Y = \frac{S}{M} + \frac{L}{K} + \frac{L}{M} + \frac{T}{N} + \frac{S}{N} + \frac{L}{N} \tag{54}$$

$$Z = \frac{LS}{KM} + \frac{ST}{MN} + \frac{LT}{KN} + \frac{LT}{MN} + \frac{LS}{KN}. \tag{55}$$

Reduction of Equation (52) for taper factors of 0.1, 1, and 10 in terms of the resistance parameter K results in the following equations:

Taper Factor = 0.1

$$\frac{d\beta}{dn} = \frac{-79.44\left(122\frac{K}{K_0} + 1\right)^{3/2}}{1342\left(\frac{K}{K_0}\right)^2 + 1474\frac{K}{K_0} + 12} \tag{56}$$

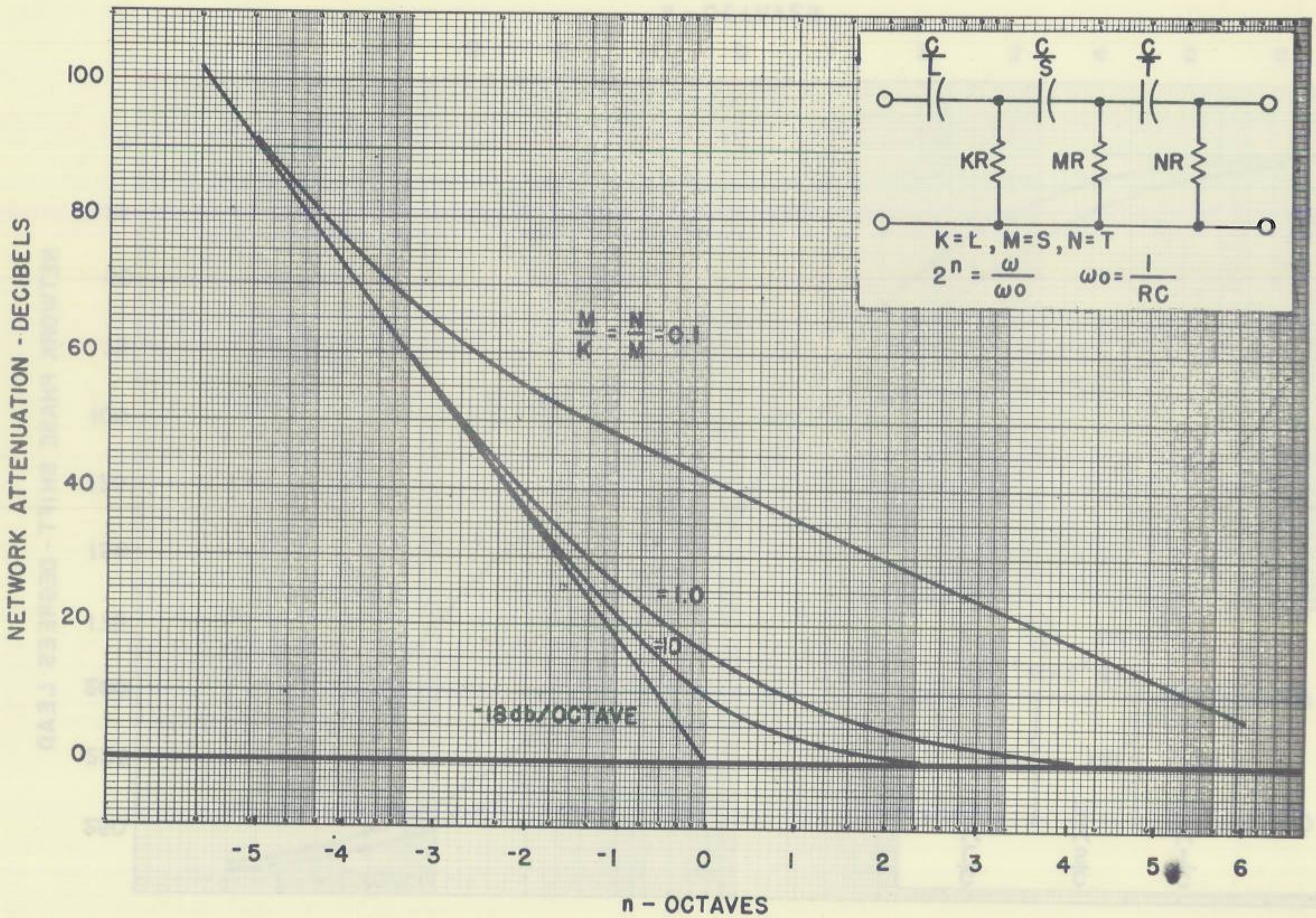


Figure 22 - Attenuation characteristics - three-section high-pass filter

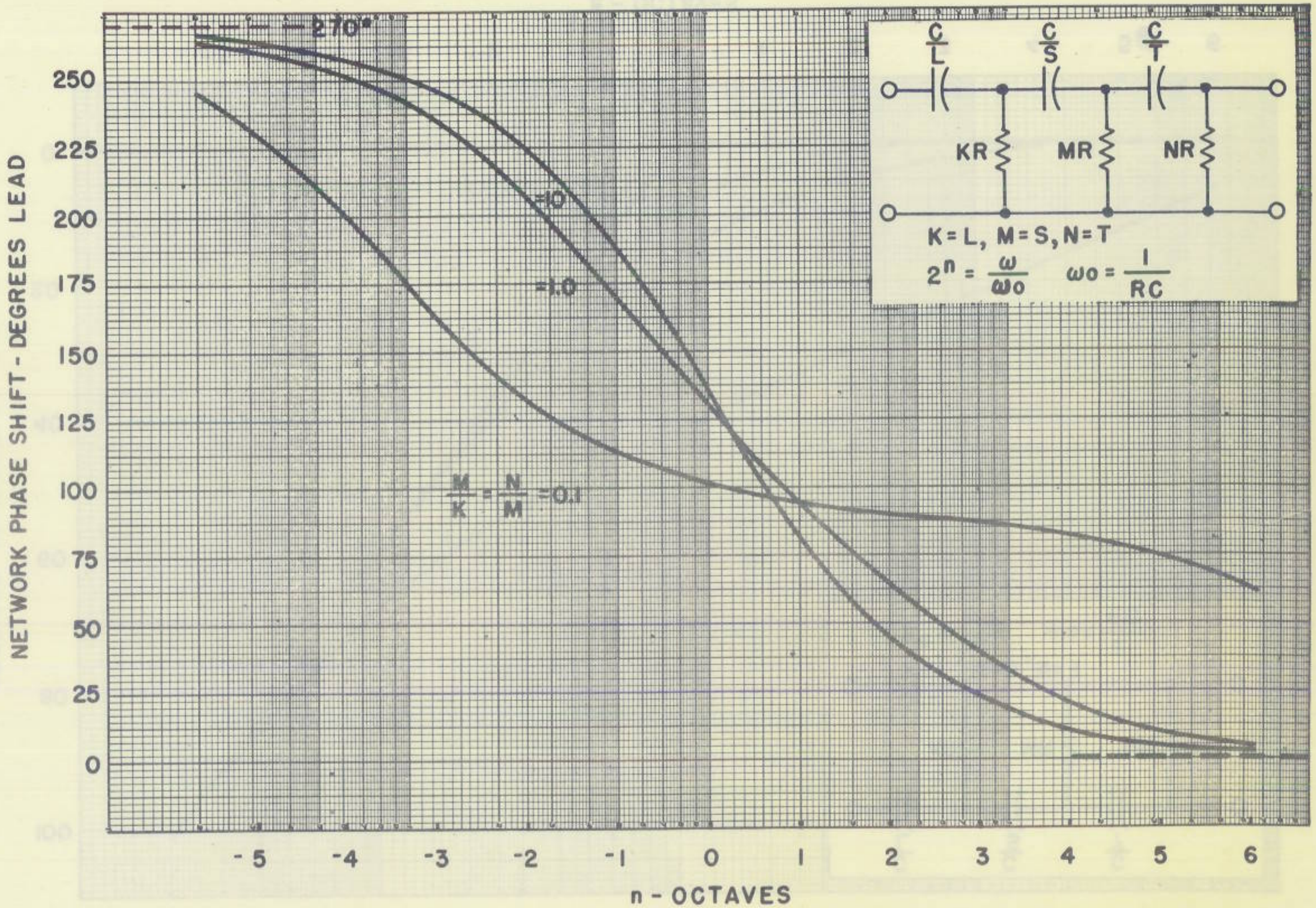


Figure 23 - Phase-shift characteristics - three-section high-pass filter

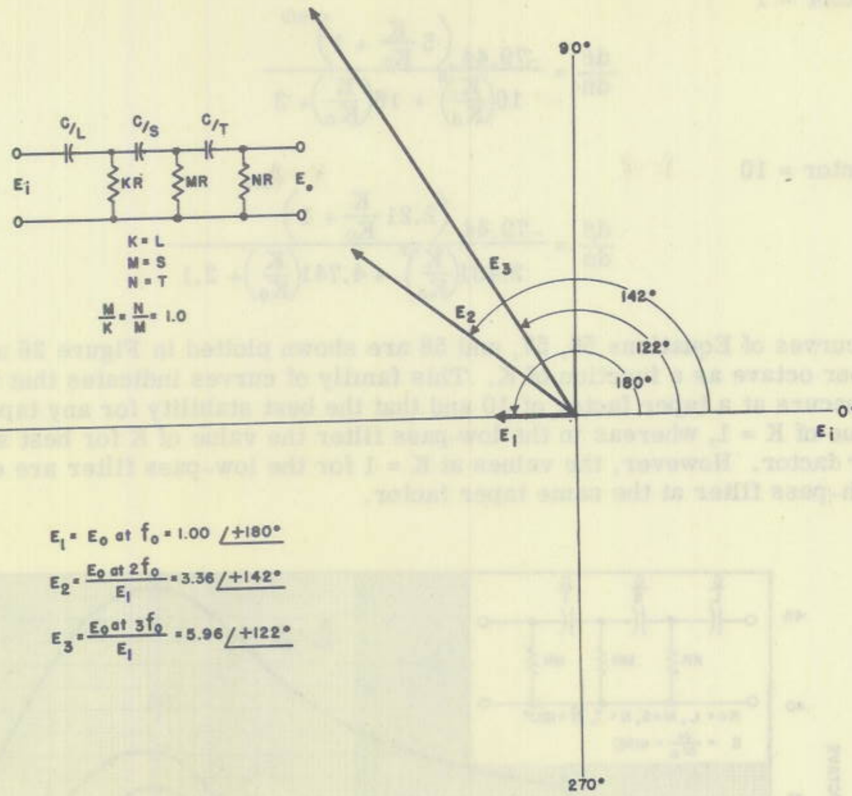


Figure 24 - Harmonic transfer of three-section high-pass network

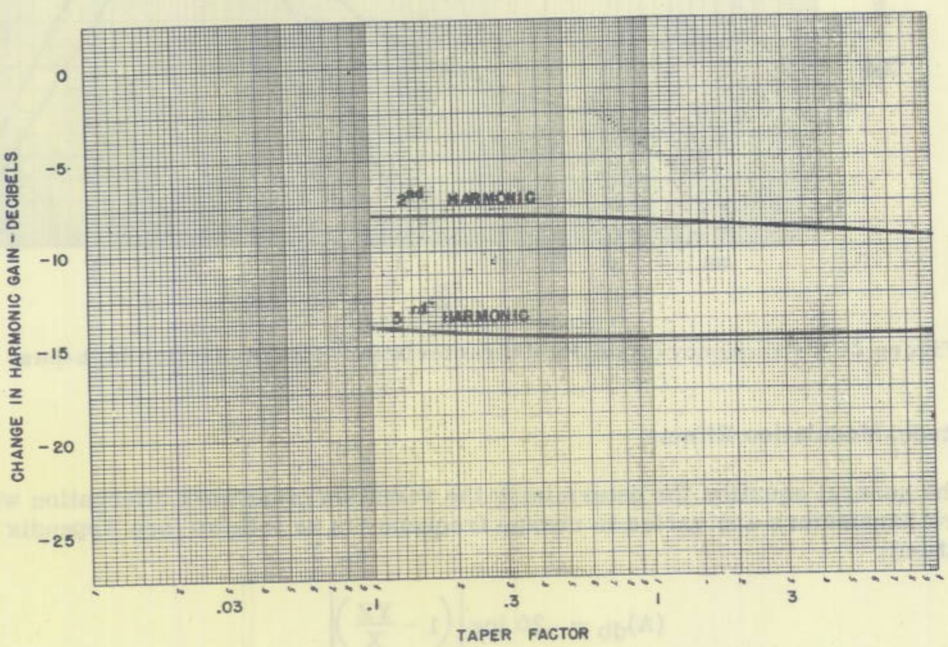


Figure 25 - Change in harmonic gain due to feedback - three-section high-pass filter

Taper Factor = 1

$$\frac{d\beta}{dn} = \frac{-79.44\left(5\frac{K}{K_0} + 1\right)^{3/2}}{10\left(\frac{K}{K_0}\right)^2 + 16\left(\frac{K}{K_0}\right) + 3} \tag{57}$$

Taper Factor = 10

$$\frac{d\beta}{dn} = \frac{-79.44\left(2.21\frac{K}{K_0} + 1\right)^{3/2}}{2.431\left(\frac{K}{K_0}\right)^2 + 4.741\left(\frac{K}{K_0}\right) + 2.1} \tag{58}$$

The curves of Equations 56, 57, and 58 are shown plotted in Figure 26 as slope in degrees per octave as a function of K. This family of curves indicates that the greatest stability occurs at a taper factor of 10 and that the best stability for any taper factor is at the value of K = 1, whereas in the low-pass filter the value of K for best stability varied with taper factor. However, the values at K = 1 for the low-pass filter are equal to those of the high-pass filter at the same taper factor.

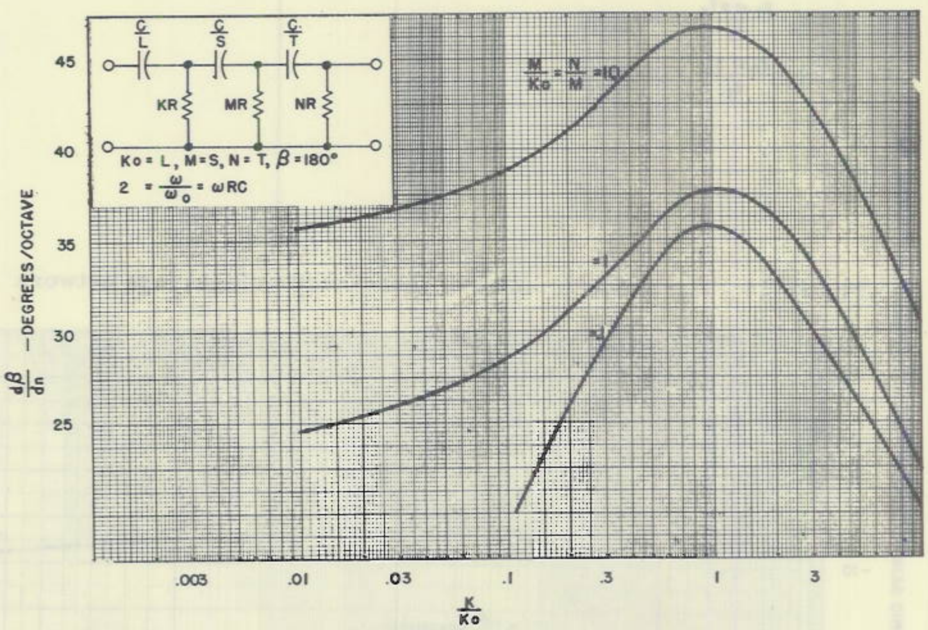


Figure 26 - Frequency-stability characteristics - three-section high-pass filter

Amplitude-Modulation Effects

The general equation for determining the variation of network attenuation when the network parameters are varied to change frequency is as follows (see Appendix IV for derivation):

$$(A)_{db} = -20 \log\left|\left(1 - \frac{YZ}{X}\right)\right| \tag{59}$$

where X, Y, and Z are the same as in Equations (53), (54), and (55) respectively. On reducing Equation (59) in terms of the resistance parameter K for taper factors of 0.1, 1, and 10, the following equations result:

Taper Factor = 0.1

$$(A)_{db} = -20 \log \left(1342 \frac{K}{K_0} + 1474 + \frac{12K_0}{K}\right) \tag{60}$$

Taper Factor = 1

$$(A)_{db} = -20 \log \left(10\frac{K}{K_0} + 16 + \frac{3K_0}{K}\right) \tag{61}$$

Taper Factor = 10

$$(A)_{db} = -20 \log \left(2.431 \frac{K}{K_0} + 4.741 + \frac{2.1K_0}{K}\right). \tag{62}$$

Figure 27 shows a plot of Equations (60), (61), and (62) in terms of voltage ratio with the largest ratio taken as 1 for ease in examining the curves. The broadness of the curves seems to vary inversely with taper factor. The taper factor of 10 introduces more amplitude modulation than the taper factor of 1, etc. The value of K at which the curves reach their maximum varies appreciably between the taper factor of 1 and taper factor of 0.1 curves.

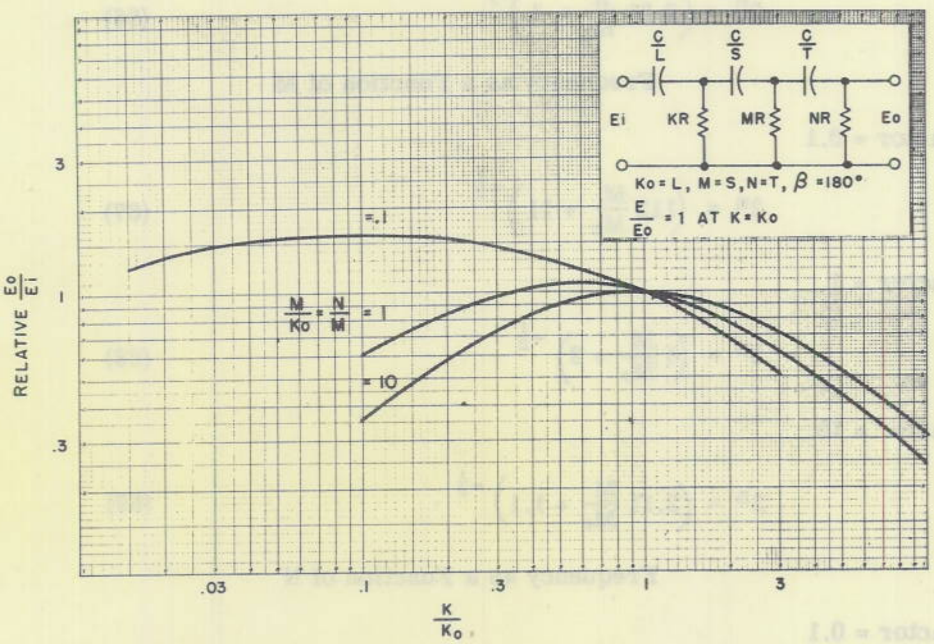


Figure 27 - Amplitude-modulation characteristics - three-section high-pass filter

Frequency-Deviation Characteristics

The general equation expressing the frequency of oscillation as a function of the network parameters is as follows (see Appendix IV for derivation):

$$2^{-n} = \left(\frac{Y}{X}\right)^{\frac{1}{2}} \quad \text{or} \quad 2^{n} = \left(\frac{X}{Y}\right)^{\frac{1}{2}} \tag{63}$$

where X and Y are the same as in Equations (53) and (54) respectively.

On reducing Equation (63) in terms of the resistance parameters K, M, and N for taper factors of 0.1, 1, and 10 the following equations result:

Frequency as a Function of K

Taper Factor = 0.1

$$2^{n} = \left(122\frac{K}{K_0} + 1\right)^{-\frac{1}{2}} \tag{64}$$

Taper Factor = 1

$$2^{n} = \left(5\frac{K}{K_0} + 1\right)^{-\frac{1}{2}} \tag{65}$$

Taper Factor = 10

$$2^{n} = \left(2.21\frac{K}{K_0} + 1\right)^{-\frac{1}{2}} \tag{66}$$

Frequency as a Function of M

Taper Factor = 0.1

$$2^{n} = \left(112\frac{M}{M_0} + 11\right)^{-\frac{1}{2}} \tag{67}$$

Taper Factor = 1

$$2^{n} = \left(4\frac{M}{M_0} + 2\right)^{-\frac{1}{2}} \tag{68}$$

Taper Factor = 10

$$2^{n} = \left(2.11\frac{M}{M_0} + 1.1\right)^{-\frac{1}{2}} \tag{69}$$

Frequency as a Function of N

Taper Factor = 0.1

$$2^{n} = \left(12\frac{N}{N_0} + 111\right)^{-\frac{1}{2}} \tag{70}$$

Taper Factor = 1

$$2^n = \left(3\frac{N}{N_0} + 3\right)^{-\frac{1}{2}} \tag{71}$$

Taper Factor = 10

$$2^n = \left(2.1\frac{N}{N_0} + 1.11\right)^{-\frac{1}{2}} \quad . \tag{72}$$

Figure 28 shows a plot of Equations (64), (65), and (66).

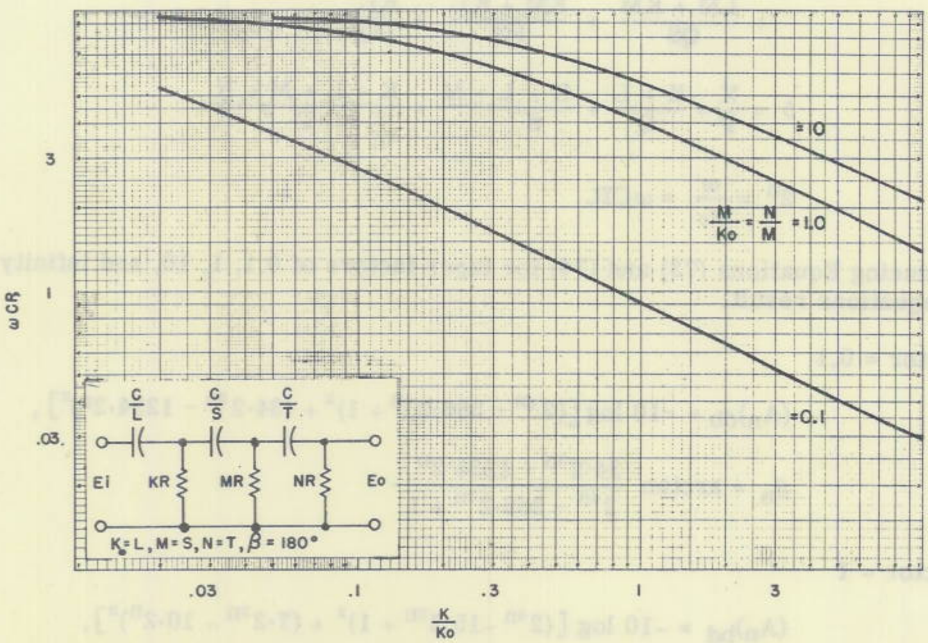


Figure 28 - Frequency-deviation characteristics three-section high-pass filter

THE FOUR-SECTION LOW-PASS FILTER

Harmonic-Transfer Characteristics

The diagram of the four-section low-pass filter is shown in Figure 29.

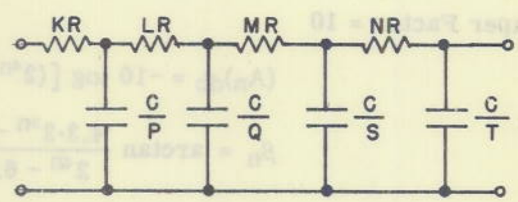


Figure 29

The general equations for gain and phase shift through this network are derived in Appendix V, and are given by Equations (73) and (74) respectively:

$$(A_n)_{db} = -10 \log \left[(2^{4n}X - 2^{2n}Z + 1)^2 + (2^{3n}Y - 2^n \delta)^2\right] \tag{73}$$

$$\beta_n = \arctan \frac{2^{3n} Y - 2^n \delta}{2^{4n} X - 2^{2n} Z + 1} \tag{74}$$

where

$$X = \frac{KLMN}{PQST} \tag{75}$$

$$Y = \frac{KLN}{PST} + \frac{KLN}{PQT} + \frac{KLM}{PQS} + \frac{KLM}{PQT} + \frac{LMN}{QST} + \frac{KMN}{PST} + \frac{KMN}{QST} \tag{76}$$

$$Z = \frac{LN + LM + KM + KN}{QT} + \frac{KN + KL + KM}{PT} + \frac{KN + LN + MN}{ST} + \frac{LM + KM}{QS} + \frac{KM + KL}{PS} + \frac{KL}{PQ} \tag{77}$$

$$\delta = \frac{K}{P} + \frac{K + L}{Q} + \frac{K + L + M}{S} + \frac{K + L + M + N}{T} \tag{78}$$

$$2^n = \frac{\omega}{\omega_0} = \omega CR. \tag{79}$$

On reducing Equations (73) and (74) for taper factors of 0.1, 1, 10, and infinity, the following equations result:

Taper Factor = 0.1

$$(A_n)_{db} = -10 \log \left[(2^{4n} - 366 \cdot 2^{2n} + 1)^2 + (34 \cdot 2^{3n} - 1234 \cdot 2^n)^2 \right], \tag{80}$$

$$\beta_n = \arctan \frac{34 \cdot 2^{3n} - 1234 \cdot 2^n}{2^{4n} - 366 \cdot 2^{2n} + 1}. \tag{81}$$

Taper Factor = 1

$$(A_n)_{bd} = -10 \log \left[(2^{4n} - 15 \cdot 2^{2n} + 1)^2 + (7 \cdot 2^{3n} - 10 \cdot 2^n)^2 \right], \tag{82}$$

$$\beta_n = \arctan \frac{7 \cdot 2^{3n} - 10 \cdot 2^n}{2^{4n} - 15 \cdot 2^{2n} + 1}. \tag{83}$$

Taper Factor = 10

$$(A_n)_{db} = -10 \log \left[(2^{4n} - 6.63 \cdot 2^{2n} + 1)^2 + (4.3 \cdot 2^{3n} - 4.321 \cdot 2^n)^2 \right], \tag{84}$$

$$\beta_n = \arctan \frac{4.3 \cdot 2^{3n} - 4.321 \cdot 2^n}{2^{4n} - 6.63 \cdot 2^{2n} + 1}. \tag{85}$$

Taper Factor = ∞

$$(A_n)_{db} = -10 \log \left[(2^{4n} - 6 \cdot 2^{2n} + 1)^2 + (4 \cdot 2^{3n} - 4 \cdot 2^n)^2 \right], \tag{86}$$

$$\beta_n = \arctan \frac{4 \cdot 2^{3n} - 4 \cdot 2^n}{2^{4n} - 6 \cdot 2^{2n} + 1}. \tag{87}$$

A plot of Equations (80) through (85) is given in Figures 30 and 31 for the four-section low-pass filter with taper factors of 0.1, 1, and 10. The curves show the variation with

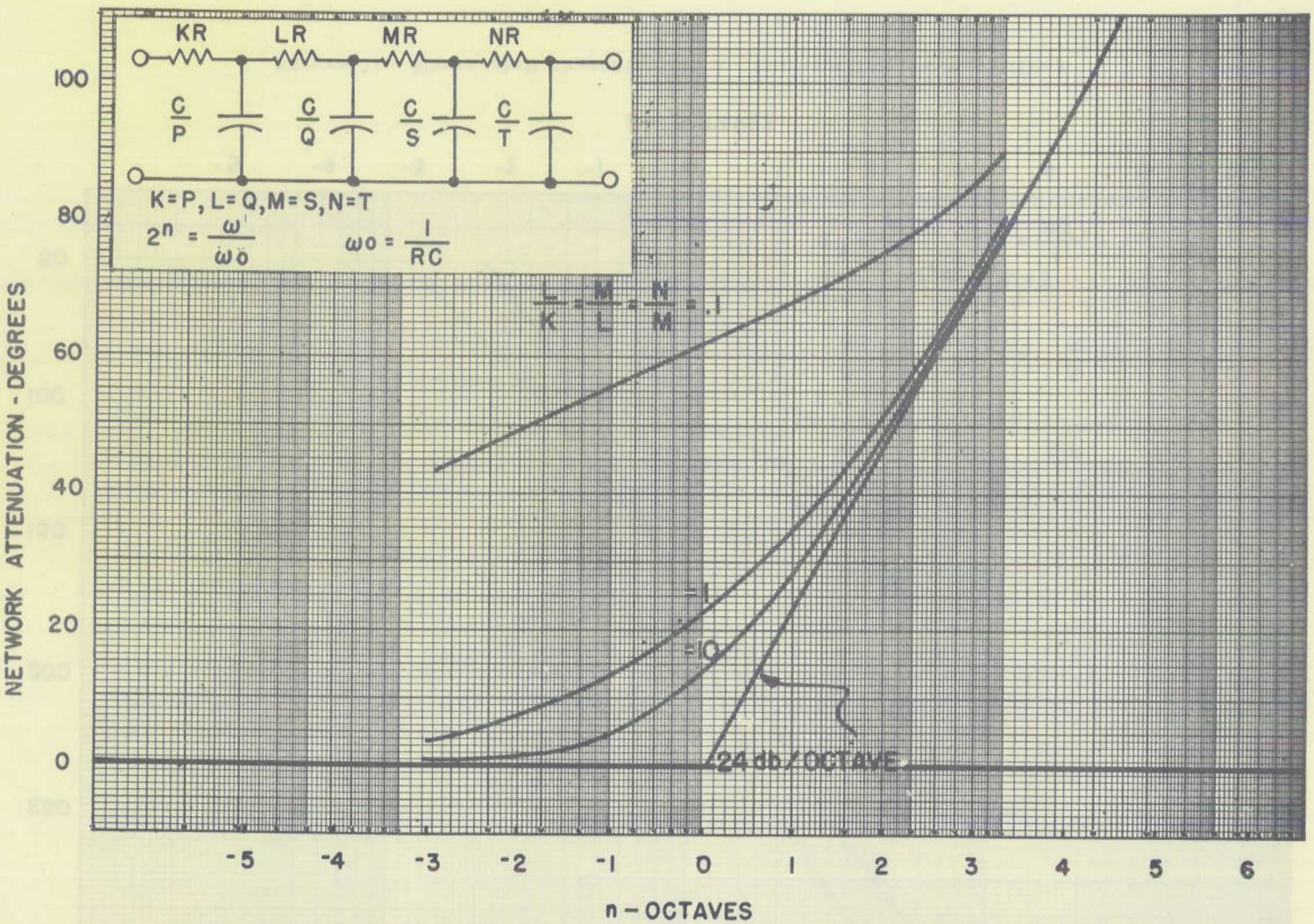


Figure 30 - Attenuation characteristics - four-section low-pass filter

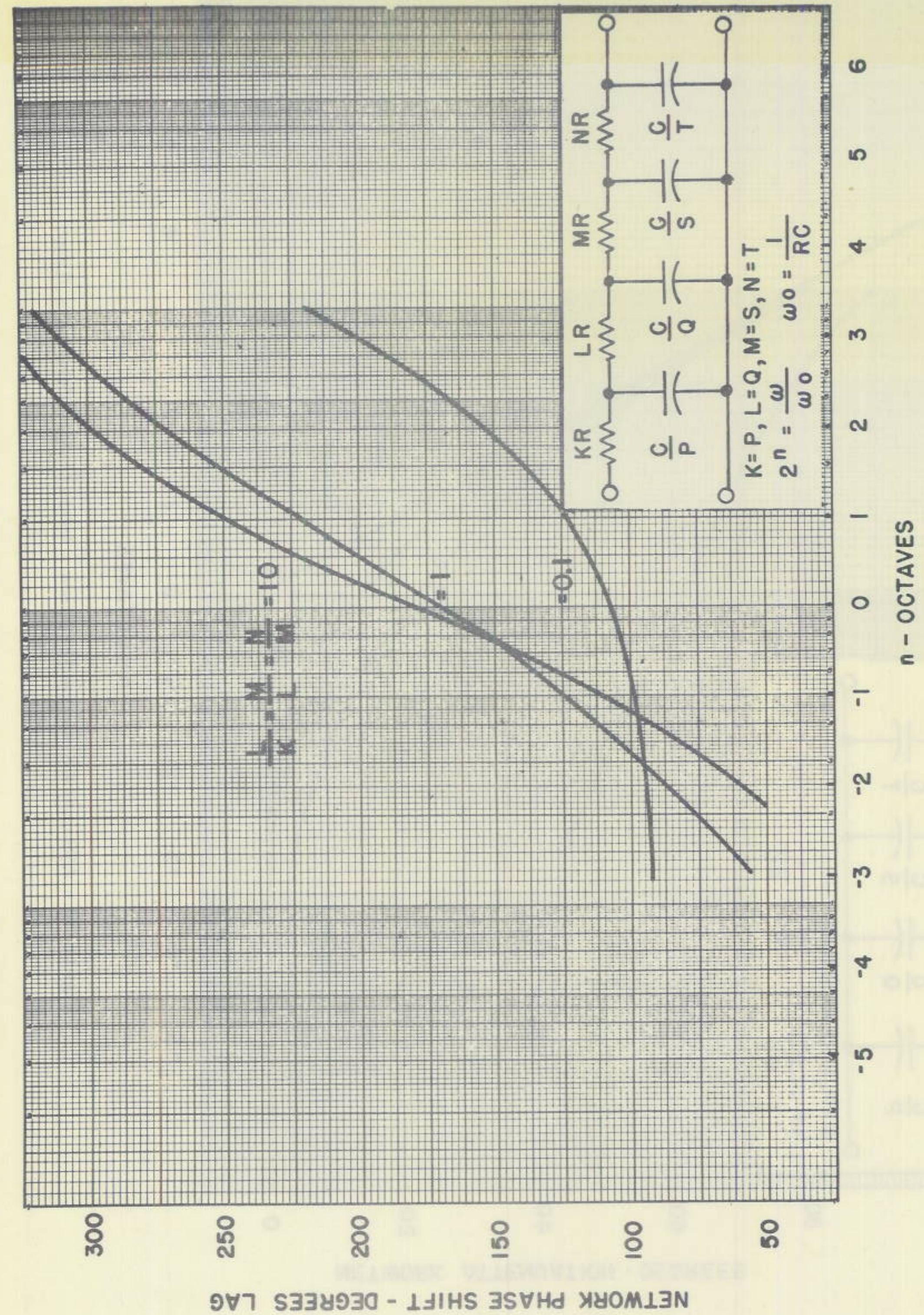


Figure 31 - Phase-shift characteristics - four-section low-pass filter

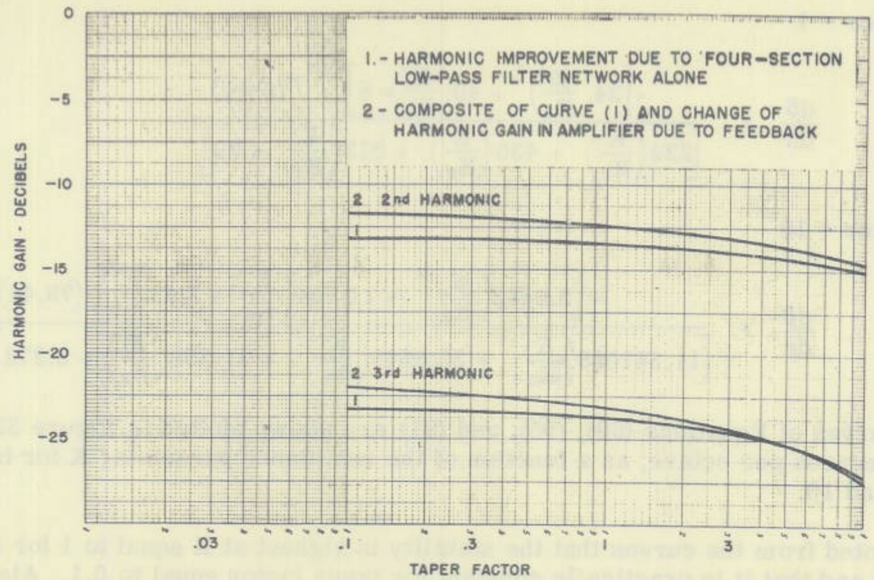


Figure 32 - Harmonic transfer characteristics - four-section low-pass filter

frequency of absolute attenuation in decibels and the phase shift in degrees through the network. From the curves of Figures 30 and 31, the curves of Figure 32 were obtained in the same manner as were the curves of Figure 16 obtained from the three-section low-pass filter. On comparing curves 2 of Figures 32 and 16 it is noted that the attenuation of second and third harmonics for the four-section network is about the same as for the three-section network. This means that one offers little or nothing over the other from the standpoint of passing harmonics into the output.

Stability of Oscillation

The general equation relating frequency stability and phase-shift curve slope in units of degrees per octave is derived in Appendix V:

$$\frac{d\beta}{dn} = \frac{-79.44\ (\delta Y)^{3/2}}{\delta YZ - X\delta^2 - Y^2} \tag{88}$$

where X, Y, Z, and δ are the same as in Equations (75), (76), (77), and (78) respectively.

On reducing Equation (88) in terms of K for taper factors of 0.1, 1, and 10, the following equations result:

Taper Factor = 0.1

$$\frac{d\beta}{dn} = \frac{-\left[36{,}663\left(\frac{K}{K_0}\right)^2 + 5{,}170\left(\frac{K}{K_0}\right) + 123\right]^{3/2} \quad (79.44)}{\left[11{,}341{,}088\left(\frac{K}{K_0}\right)^3 + 2{,}342{,}164\left(\frac{K}{K_0}\right)^2 + 145{,}904\left(\frac{K}{K_0}\right) + 2{,}828\right]} \tag{89}$$

Taper Factor = 1

$$\frac{d\beta}{dn} = \frac{-\left[24\left(\frac{K}{K_0}\right)^2 + 40\left(\frac{K}{K_0}\right) + 6\right]^{3/2}(79.44)}{\left[224\left(\frac{K}{K_0}\right)^3 + 436\left(\frac{K}{K_0}\right)^2 + 212\left(\frac{K}{K_0}\right) + 29\right]} \tag{90}$$

Taper Factor = 10

$$\frac{d\beta}{dn} = \frac{-\left[3.6663\left(\frac{K}{K_0}\right)^2 + 11.704\left(\frac{K}{K_0}\right) + 3.21\right]^{3/2}(79.44)}{\left[11.341088\left(\frac{K}{K_0}\right)^3 + 33.8546\left(\frac{K}{K_0}\right)^2 + 31.559\left(\frac{K}{K_0}\right) + 9.272\right]} \cdot \tag{91}$$

The curves of Equations (89), (90), and (91) are shown plotted in Figure 33 with stability, in degrees per octave, as a function of the resistance parameter K for taper factors of 0.1, 1, and 10.

It is noted from the curves that the stability is highest at K equal to 1 for taper factors of 1 and 10 and that it is practically constant for taper factor equal to 0.1. Also, it is seen from the curves that stability increases with taper factor and that the greatest rate of increase is for values of taper factor between 1 and 10.

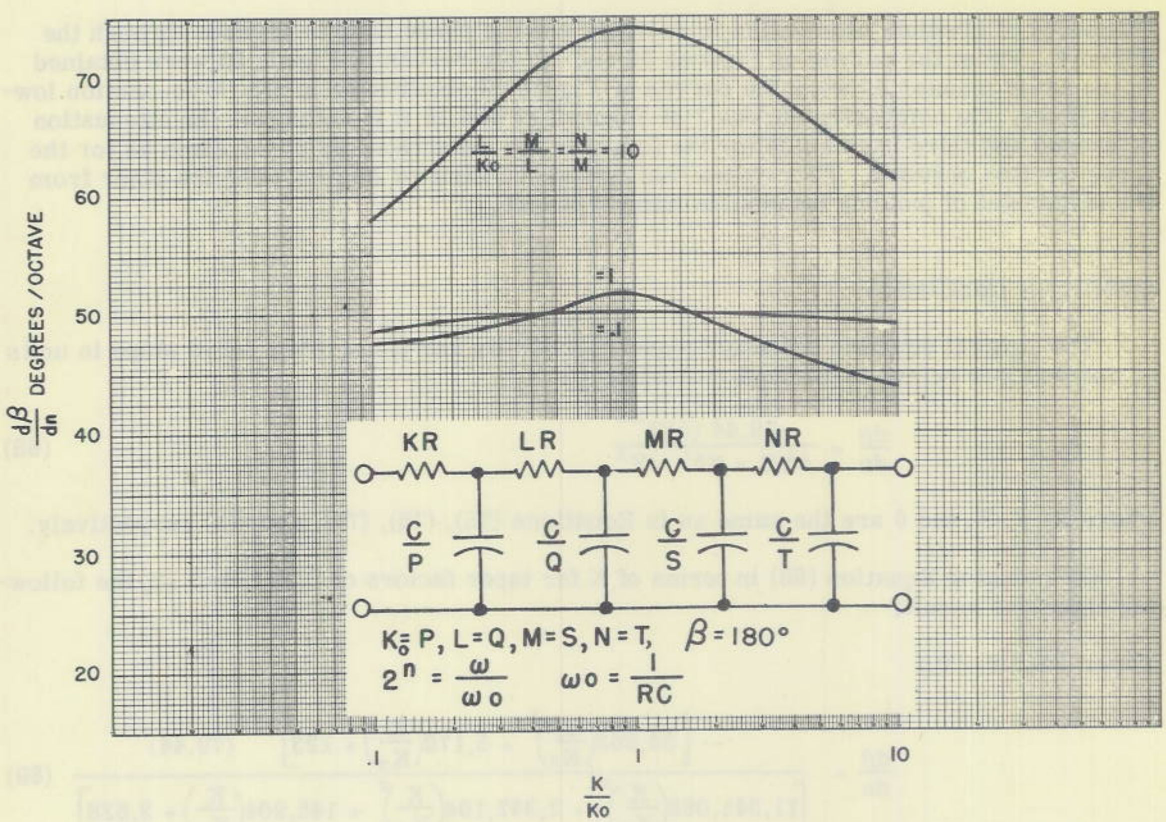


Figure 33 - Frequency-stability characteristics - four-section low-pass filter

Amplitude-Modulation Effects

The general equation for determining the variation of network attenuation when the network resistance or capacitance parameters are varied to change frequency is derived in Appendix V:

$$(A)_{db} = -20 \log \left[\left(\frac{\delta}{Y}\right)^2 X - \left(\frac{\delta}{Y}\right) Z + 1\right] \tag{92}$$

where X, Y, Z, and δ are given by Equations (75), (76), (77), and (78).

On reducing Equation (92) in terms of the resistance parameter K for taper factors of 0.1, 1, and 10, the following equations result:

Taper Factor = 0.1

$$(A)_{db} = -20 \log \left[\left(\frac{1111 \frac{K}{K_0} + 123}{33 \frac{K}{K_0} + 1}\right)^2 \frac{K}{K_0} - \left(\frac{1111 \frac{K}{K_0} + 123}{33 \frac{K}{K_0} + 1}\right)\left(343 \frac{K}{K_0} + 23\right) + 1\right] \tag{93}$$

Taper Factor = 1

$$(A)_{db} = -20 \log \left[\left(\frac{4 \frac{K}{K_0} + 6}{6 \frac{K}{K_0} + 1}\right)^2 \frac{K}{K_0} - \left(\frac{4 \frac{K}{K_0} + 6}{6 \frac{K}{K_0} + 1}\right)\left(10 \frac{K}{K_0} + 5\right) + 1\right] \tag{94}$$

Taper Factor = 10

$$(A)_{db} = -20 \log \left[\left(\frac{1.111 \frac{K}{K_0} + 3.21}{3.3 \frac{K}{K_0} + 1}\right)^2 \frac{K}{K_0} - \left(\frac{1.111 \frac{K}{K_0} + 3.21}{3.3 \frac{K}{K_0} + 1}\right)\left(3.43 \frac{K}{K_0} + 3.2\right) + 1\right]. \tag{95}$$

Reduction of Equation (92) in terms of the resistance parameters L, M, and N for taper factor = 1 results in the following equations:

As a Function of L:

$$(A)_{db} = -20 \log \left[\left(\frac{3 \frac{L}{L_0} + 7}{5 \frac{L}{L_0} + 2}\right)^2 \frac{L}{L_0} - \left(\frac{3 \frac{L}{L_0} + 7}{5 \frac{L}{L_0} + 2}\right)\left(7 \frac{L}{L_0} + 8\right) + 1\right] \tag{96}$$

As a Function of M:

$$(A)_{db} = -20 \log \left[\left(\frac{2 \frac{M}{M_0} + 8}{5 \frac{M}{M_0} + 2}\right)^2 \frac{M}{M_0} - \left(\frac{2 \frac{M}{M_0} + 8}{5 \frac{M}{M_0} + 2}\right)\left(7 \frac{M}{M_0} + 8\right) + 1\right] \tag{97}$$

As a Function of N:

$$(A)_{db} = -20 \log \left[\left(\frac{\frac{N}{N_0} + 9}{5\frac{N}{N_0} + 2} \right)^2 \frac{N}{N_0} - \left(\frac{\frac{N}{N_0} + 9}{5\frac{N}{N_0} + 2} \right) \left(6\frac{N}{N_0} + 9 \right) + 1 \right] . \tag{98}$$

Figure 34 shows a plot of Equations (94), (96), (97), and (98) where the relative network attenuation is given as a function of the resistance parameters with the network phase shift equal to 180 degrees. In Figures (35) and (36) similar curves are found with K plotted for taper factors of 0.1 and 10.

Frequency-Deviation Characteristics

The general equation expressing the frequency of oscillation as a function of the network parameters for the four-section low-pass filter is derived in Appendix V:

$$2^n = \frac{\omega}{\omega_0} = \omega CR = \left(\frac{\delta}{Y} \right)^{\frac{1}{2}} \tag{99}$$

where Y and δ are the same as in Equations (76) and (78) respectively.

On reducing Equation (99) in terms of the resistance parameters K, L, M, and N for taper factors of 0.1, 1, and 10, the following equations result:

Frequency as a Function of K:

Taper Factor = 0.1

$$2^n = \left(\frac{1111\frac{K}{K_0} + 123}{33\frac{K}{K_0} + 1} \right)^{\frac{1}{2}} \tag{100}$$

Taper Factor = 1

$$2^n = \left(\frac{4\frac{K}{K_0} + 6}{6\frac{K}{K_0} + 1} \right)^{\frac{1}{2}} \tag{101}$$

Taper Factor = 10

$$2^n = \left(\frac{1.111\frac{K}{K_0} + 3.21}{3.3\frac{K}{K_0} + 1} \right)^{\frac{1}{2}} \tag{102}$$

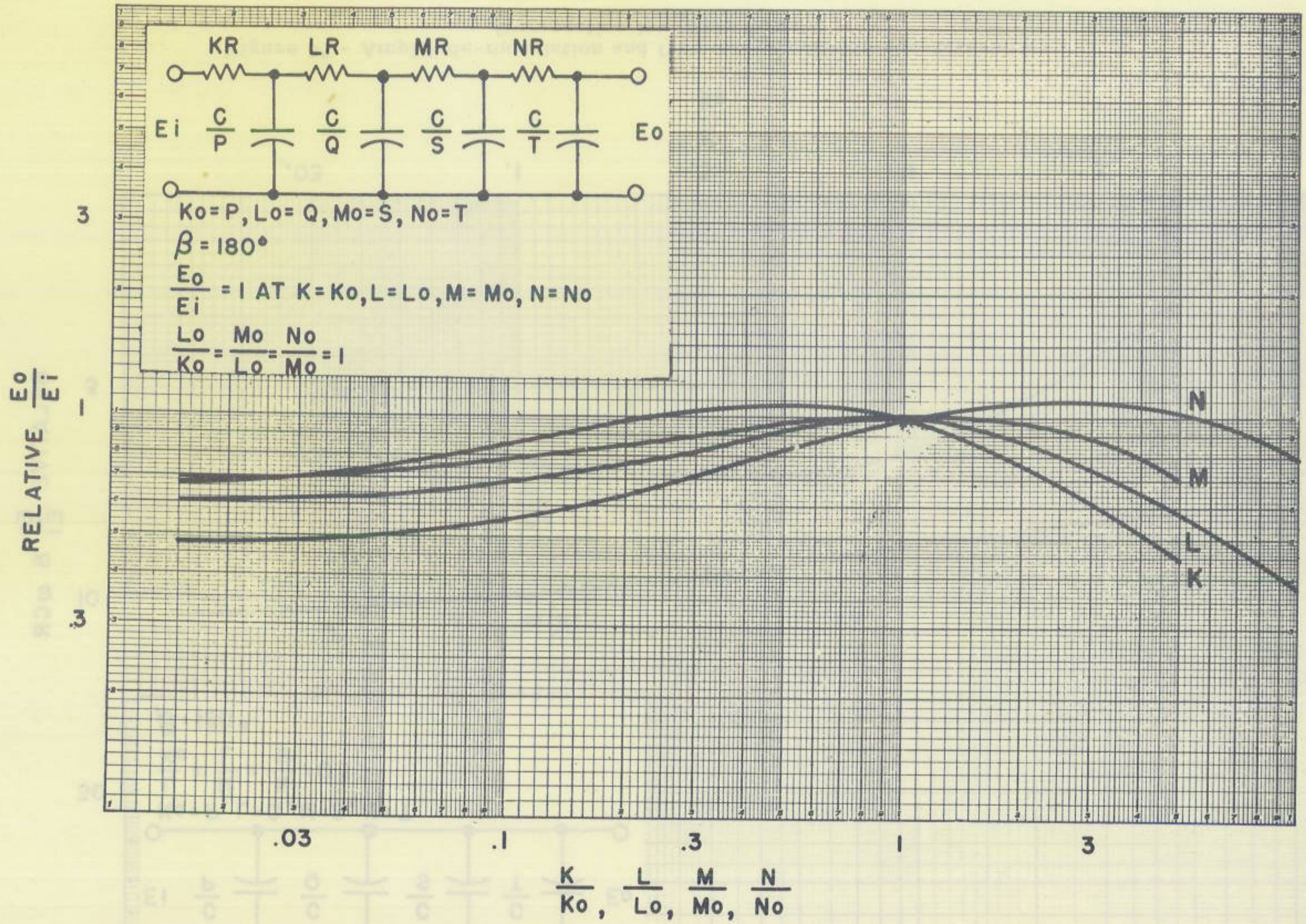


Figure 34 - Amplitude-modulation characteristics - four-section low-pass filter

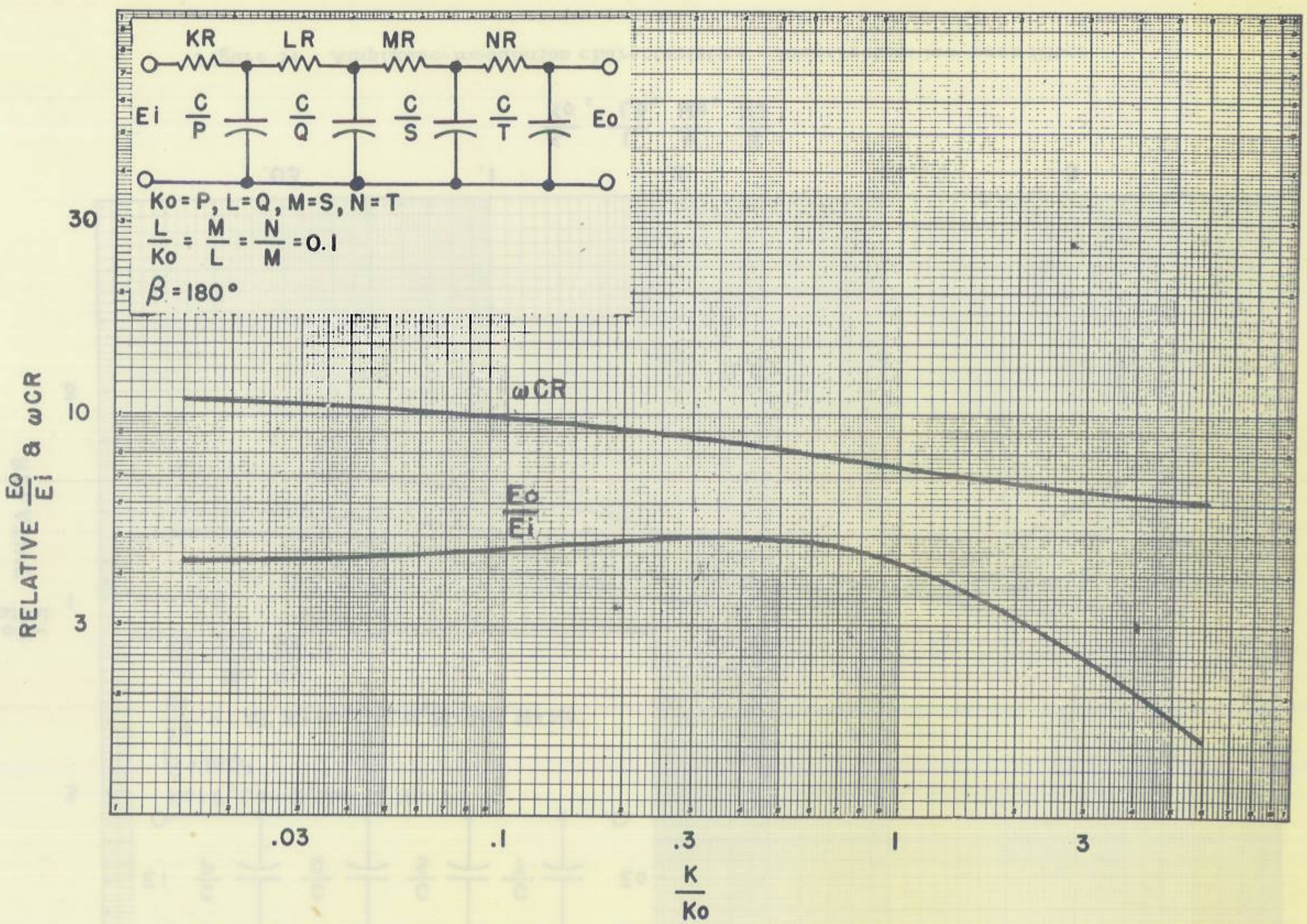


Figure 35 - Amplitude-modulation and frequency-deviation characteristics - four-section low-pass filter

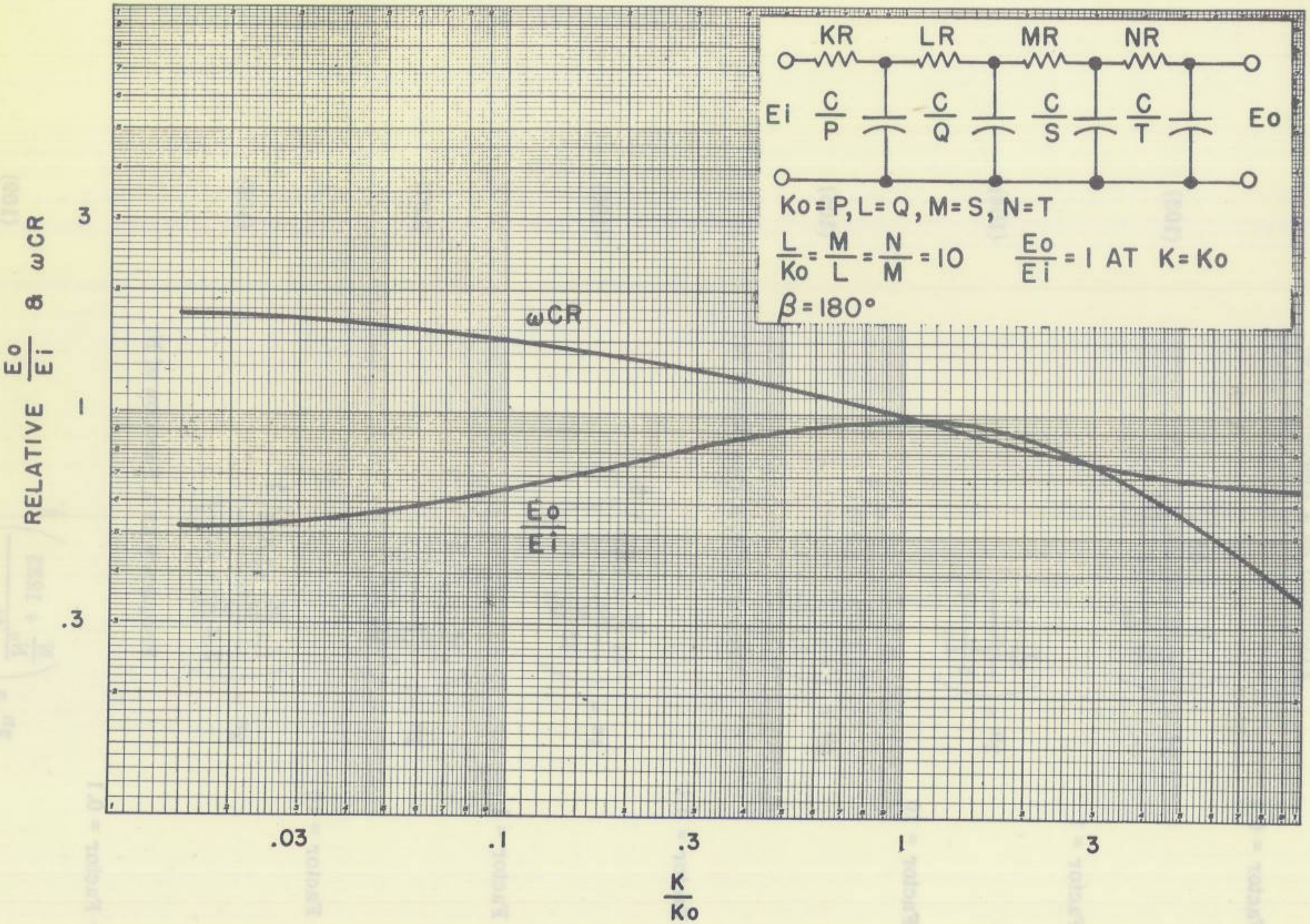


Figure 36 - Amplitude-modulation and frequency-deviation characteristics - four-section low-pass filter

Frequency as a Function of L:

Taper Factor = 0.1

$$2^n = \left(\frac{111\frac{L}{L_0} + 1123}{23\frac{L}{L_0} + 11} \right)^{\frac{1}{2}} \tag{103}$$

Taper Factor = 1

$$2^n = \left(\frac{3\frac{L}{L_0} + 7}{5\frac{L}{L_0} + 2} \right)^{\frac{1}{2}} \tag{104}$$

Taper Factor = 10

$$2^n = \left(\frac{1.11\frac{L}{L_0} + 3.211}{3.2\frac{L}{L_0} + 1.1} \right)^{\frac{1}{2}} . \tag{105}$$

Frequency as a Function of M:

Taper Factor = 0.1

$$2^n = \left(\frac{11\frac{M}{M_0} + 1223}{23\frac{M}{M_0} + 11} \right)^{\frac{1}{2}} \tag{106}$$

Taper Factor = 1

$$2^n = \left(\frac{2\frac{M}{M_0} + 8}{5\frac{M}{M_0} + 2} \right)^{\frac{1}{2}} \tag{107}$$

Taper Factor = 10

$$2^n = \left(\frac{1.1\frac{M}{M_0} + 3.221}{3.2\frac{M}{M_0} + 1.1} \right)^{\frac{1}{2}} \tag{108}$$

Frequency as a Function of N:

Taper Factor = 0.1

$$2^n = \left(\frac{\frac{N}{N_0} + 1233}{23\frac{N}{N_0} + 11} \right)^{\frac{1}{2}} \tag{109}$$

Taper Factor = 1

$$2^n = \left(\frac{\frac{N}{N_0} + 9}{5\frac{N}{N_0} + 2} \right)^{\frac{1}{2}} \tag{110}$$

Taper Factor = 10

$$2^n = \left(\frac{\frac{N}{N_0} + 3.321}{3.2\frac{N}{N_0} + 1.1} \right)^{\frac{1}{2}} . \tag{111}$$

A plot of Equations (101), (104), (107), and (110) is given in Figure 37 where the variation of frequency due to variations of the resistance parameters K, L, M, and N for taper factor = 1 is shown. Similar curves where K is varied for taper factors of 0.1, and 10 are given in Figures 35 and 36.

THE FOUR-SECTION HIGH-PASS FILTER

Harmonic-Transfer Characteristics

The diagram of the four-section high-pass filter is shown in Figure 38.

The general equations for attenuation and phase shift through this network are given by Equations (112) and (113) respectively:

$$(A_n)_{db} = -10 \log \left[(2^{-4n}X - 2^{-2n}Z + 1)^2 + (2^{-n}\delta - 2^{-3n}Y)^2 \right] , \tag{112}$$

$$\beta_n = \arctan \frac{2^{-n}\delta - 2^{-3n}Y}{2^{-4n}X - 2^{-2n}Z + 1} \tag{113}$$

where

$$X = \frac{PQST}{KLMN} \tag{114}$$

$$Y = \frac{PQT}{KMN} + \frac{PQT}{KLN} + \frac{PQS}{KLM} + \frac{PQS}{KLN} + \frac{QST}{LMN} + \frac{PST}{KMN} + \frac{PST}{LMN} \tag{115}$$

$$Z = \frac{QT + QS + PS + PT}{LN} + \frac{PQ + PS + PT}{KN} + \frac{PT + QT + ST}{MN} + \frac{QS + PS}{LM} + \frac{PS + PQ}{KM} + \frac{PQ}{KL} \tag{116}$$

$$\delta = \frac{P}{K} + \frac{P + Q}{L} + \frac{P + Q + S}{M} + \frac{P + Q + S + T}{N} \tag{117}$$

$$2^n = \frac{\omega}{\omega_0} = \omega CR.$$

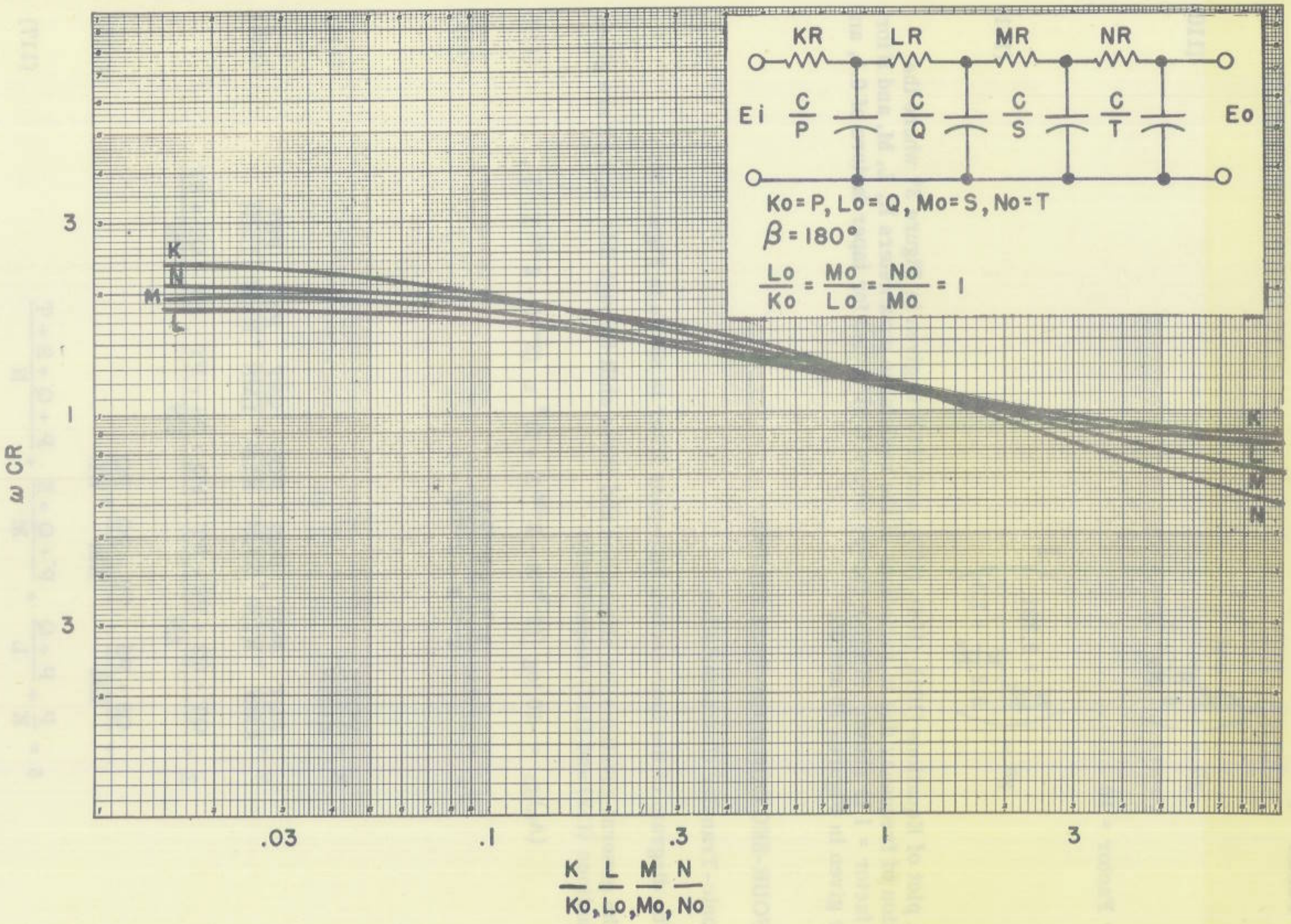


Figure 37 - Frequency-deviation characteristics - four-section low-pass filter

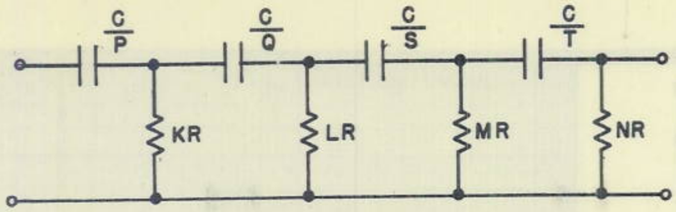


Figure 38

On reducing Equations (112) and (113) for taper factors of 0.1, 1, and 10, the following equations result:

Taper Factor = 0.1

$$(A_n)_{db} = -10 \log\left[(2^{-4n} - 366\cdot 2^{-2n} + 1)^2 + (1234\cdot 2^{-n} - 34\cdot 2^{-3n})^2\right], \tag{118}$$

$$\beta_n = \arctan\frac{1234\cdot 2^{-n} - 34\cdot 2^{-3n}}{2^{-4n} - 366\cdot 2^{-2n} + 1}\ . \tag{119}$$

Taper Factor = 1

$$(A_n)_{db} = -10 \log\left[(2^{-4n} - 15\cdot 2^{-2n} + 1)^2 + (10\cdot 2^{-n} - 7\cdot 2^{-3n})^2\right], \tag{120}$$

$$\beta_n = \arctan\frac{10\cdot 2^{-n} - 7\cdot 2^{-3n}}{2^{-4n} - 15\cdot 2^{-2n} + 1}\ . \tag{121}$$

Taper Factor = 10

$$(A_n)_{db} = -10 \log\left[(2^{-4n} - 6.63\cdot 2^{-2n} + 1)^2 + (4.321\cdot 2^{-n} - 4.3\cdot 2^{-3n})^2\right], \tag{122}$$

$$\beta_n = \arctan\frac{4.321\cdot 2^{-n} - 4.3\cdot 2^{-3n}}{2^{-4n} - 6.63\cdot 2^{-2n} + 1}\ . \tag{123}$$

A plot of Equations (118) through (123) is given in Figures 39 and 40 for the four-section high-pass filter with taper factors of 0.1, 1, and 10, and shows the variation with frequency of absolute attenuation in decibels and the phase shift in degrees through the network. From the curves of Figures 39 and 40, the curves of Figure 41 were obtained in the same manner as were the curves of Figure 16 obtained for the three-section low-pass filter. On comparing the curves of Figure 41 with those of Figure 25 (corresponding curves of the three-section high-pass filter), it is noted that the four-section network offers less harmonic attenuation than the three-section network. The reason for this is that the operating point of the four-section network (for oscillation) is on a flatter portion of its attenuation curve and therefore the difference in attenuation between fundamental and harmonics is less than for the three-section network.

Stability of Oscillation

The general equation relating frequency stability and phase-shift curve slope in units of degrees per octave is derived in Appendix VI:

$$\frac{d\beta}{dn} = \frac{-79.44\,(\delta Y)^{3/2}}{\delta YZ - X\delta^2 - Y^2} \tag{124}$$

where X, Y, Z, and δ are the same as in Equations (114), (115), (116), and (117) respectively.

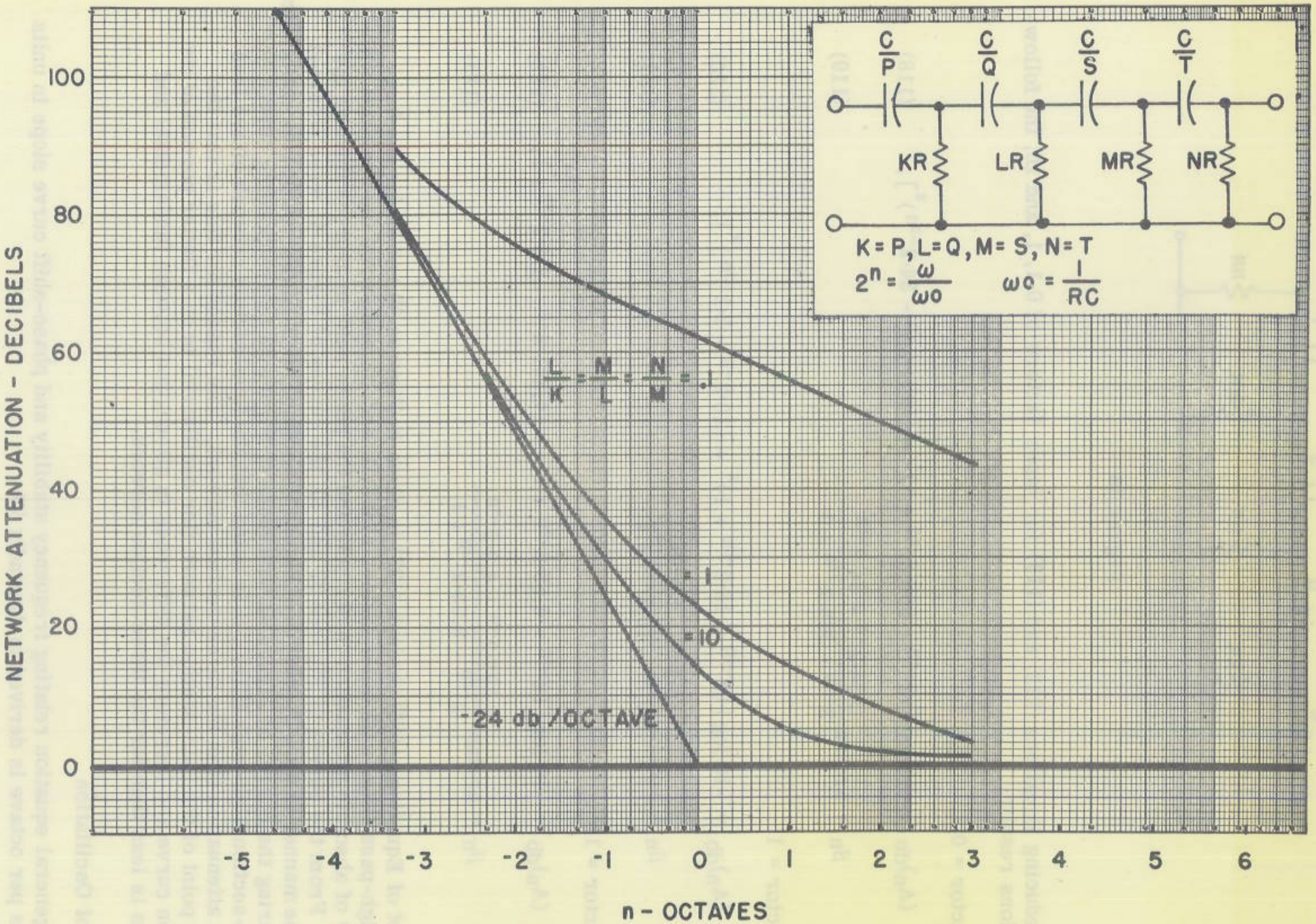


Figure 39 - Attenuation characteristics - four-section high-pass filter

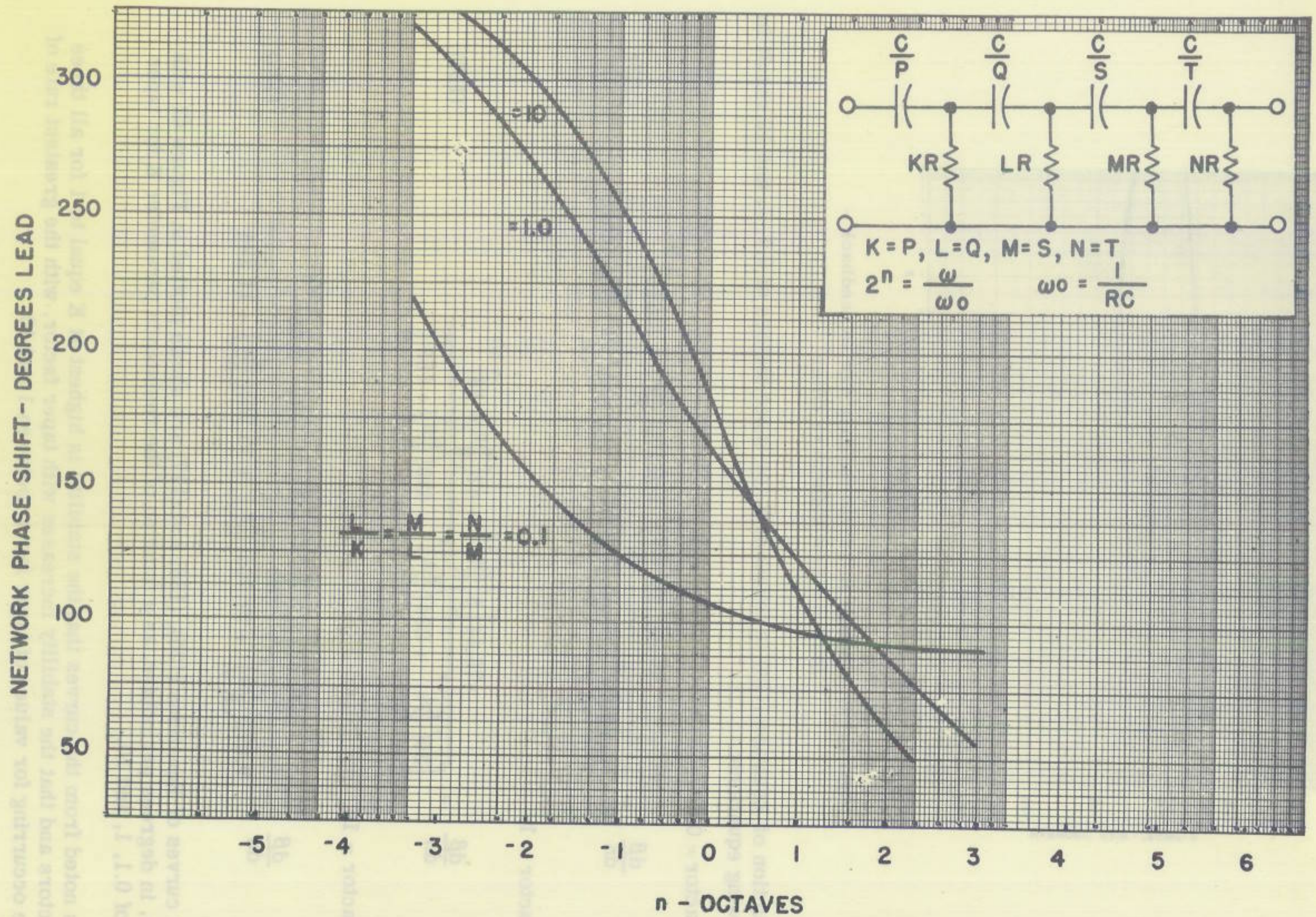


Figure 40 - Phase-shift characteristics - four-section high-pass filter

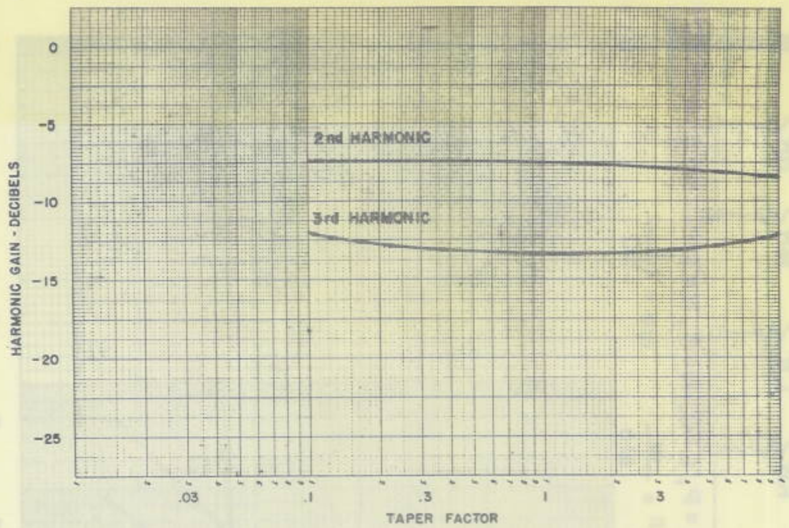


Figure 41 - Change in harmonic gain due to feedback - four-section high-pass filter

Reduction of Equation (124) in terms of K for taper factors of 0.1, 1, and 10 result in the following equations:

Taper Factor = 0.1

$$\frac{d\beta}{dn} = \frac{-79.44\left[23\left(\frac{K_0}{K}\right)^2 + 28370\left(\frac{K_0}{K}\right) + 13563\right]^{3/2}}{\left[2828\left(\frac{K_0}{K}\right)^3 + 3492104\left(\frac{K_0}{K}\right)^2 + 7041364\left(\frac{K_0}{K}\right) + 3295688\right]} \tag{125}$$

Taper Factor = 1

$$\frac{d\beta}{dn} = \frac{-79.44\left[5\left(\frac{K_0}{K}\right)^2 + 47\left(\frac{K_0}{K}\right) + 18\right]^{3/2}}{\left[29\left(\frac{K_0}{K}\right)^3 + 284\left(\frac{K_0}{K}\right)^2 + 430\left(\frac{K_0}{K}\right) + 158\right]} \tag{126}$$

Taper Factor = 10

$$\frac{d\beta}{dn} = \frac{-79.44\left[3.2\left(\frac{K_0}{K}\right)^2 + 11.73\left(\frac{K_0}{K}\right) + 3.65\right]^{3/2}}{9.272\left(\frac{K_0}{K}\right)^3 + 31.706\left(\frac{K_0}{K}\right)^2 + 33.764\left(\frac{K_0}{K}\right) + 11.284} \tag{127}$$

The curves of Equations (125), (126) and (127) are shown plotted in Figure 42 with stability, in degrees per octave, as a function of the resistance parameter K for taper factors of 0.1, 1, and 10.

It is noted from the curves that the stability is highest at K equal to 1 for all three taper factors and that the stability increases with taper factor, with the greatest rate of increase occurring for values of taper factor between 1 and 10.

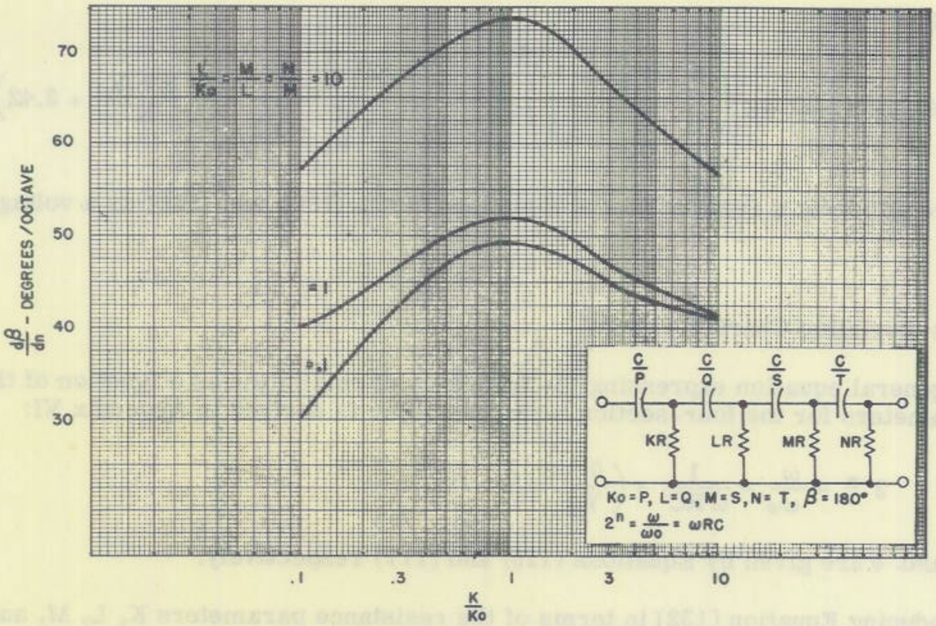


Figure 42 - Frequency-stability characteristics - four-section high-pass filter

Amplitude-Modulation Effects

The general equation for determining the variation of network attenuation when the network resistance or capacitance parameters are varied to change frequency is derived in Appendix VI:

$$(A_n)_{db} = -20 \log\left[\left(\frac{\delta}{Y}\right)^2 X - \left(\frac{\delta}{Y}\right) Z + 1\right] \tag{128}$$

where X, Y, Z, and δ are given by Equations (114), (115), (116), and (117).

On reducing Equation (128), in terms of the resistance parameter K for taper factors of 0.1, 1, and 10, the following equations result:

Taper Factor = 0.1

$$(A_n)_{db} = -20 \log\left[\left(\frac{1233\frac{K}{K_0} + 1}{11\frac{K}{K_0} + 23}\right)^2 \frac{K_0}{K} - \left(\frac{1233\frac{K}{K_0} + 1}{11\frac{K}{K_0} + 23}\right)\left(\frac{123K_0}{K} + 243\right) + 1\right]. \tag{129}$$

Taper Factor = 1

$$(A_n)_{db} = -20 \log\left[\left(\frac{9\frac{K}{K_0} + 1}{2\frac{K}{K_0} + 5}\right)^2 \frac{K_0}{K} - \left(\frac{9\frac{K}{K_0} + 1}{2\frac{K}{K_0} + 5}\right)\left(\frac{6K_0}{K} + 9\right) + 1\right]. \tag{130}$$

Taper Factor = 10

$$(A_n)_{db} = -20 \log \left[\left(\frac{3.321 \frac{K}{K_0} + 1}{1.1 \frac{K}{K_0} + 3.2} \right)^2 \frac{K_0}{K} - \left(\frac{3.321 \frac{K}{K_0} + 1}{1.1 \frac{K}{K_0} + 3.2} \right) \left(\frac{3.21 K_0}{K} + 3.42 \right) + 1 \right]. \quad (131)$$

Figure 43 shows a relative plot of Equations (129), (130), and (131) as a voltage ratio vs. K/K_0.

Frequency-Deviation Characteristics

The general equation expressing the frequency of oscillation as a function of the network parameters for the four-section high-pass filter is derived in Appendix VI:

$$2^{-n} = \frac{\omega}{\omega_0} = \frac{1}{\omega RC} = \left(\frac{\delta}{Y} \right)^{\frac{1}{2}} \quad (132)$$

where Y and δ are given by Equations (115) and (117) respectively.

On reducing Equation (132) in terms of the resistance parameters K, L, M, and N for taper factors of 0.1, 1, and 10, the following equations result:

Frequency as a Function of K

Taper Factor = 0.1

$$2^n = \left(\frac{11 \frac{K}{K_0} + 23}{1233 \frac{K}{K_0} + 1} \right)^{\frac{1}{2}} \quad (133)$$

Taper Factor = 1

$$2^n = \left(\frac{2 \frac{K}{K_0} + 5}{9 \frac{K}{K_0} + 1} \right)^{\frac{1}{2}} \quad (134)$$

Taper Factor = 10

$$2^n = \left(\frac{1.1 \frac{K}{K_0} + 3.2}{3.321 \frac{K}{K_0} + 1} \right)^{\frac{1}{2}}. \quad (135)$$

Frequency as a Function of L

Taper Factor = 0.1

$$2^n = \left(\frac{11 \frac{L}{L_0} + 23}{1223 \frac{L}{L_0} + 11} \right)^{\frac{1}{2}} \quad (136)$$

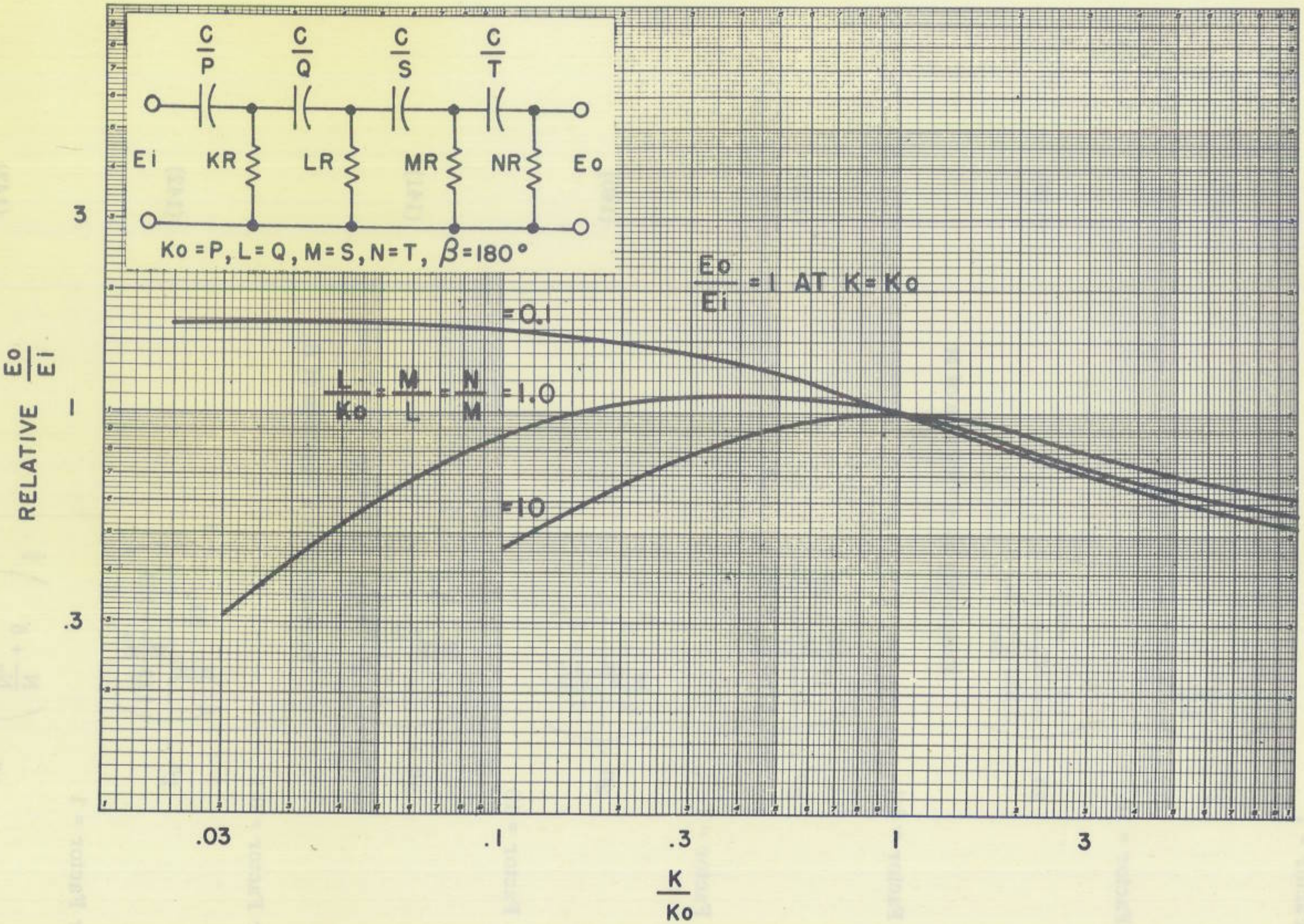


Figure 43 - Amplitude-modulation characteristics - four-section high-pass filter

Taper Factor = 1

$$2^n = \left(\frac{2 \frac{L}{L_0} + 5}{8 \frac{L}{L_0} + 2} \right)^{\frac{1}{2}} \tag{137}$$

Taper Factor = 10

$$2^n = \left(\frac{1.1 \frac{L}{L_0} + 3.2}{3.221 \frac{L}{L_0} + 1.1} \right)^{\frac{1}{2}} . \tag{138}$$

Frequency as a Function of M

Taper Factor = 0.1

$$2^n = \left(\frac{11 \frac{M}{M_0} + 23}{1123 \frac{M}{M_0} + 111} \right)^{\frac{1}{2}} \tag{139}$$

Taper Factor = 1

$$2^n = \left(\frac{2 \frac{M}{M_0} + 5}{7 \frac{M}{M_0} + 3} \right)^{\frac{1}{2}} \tag{140}$$

Taper Factor = 10

$$2^n = \left(\frac{1.1 \frac{M}{M_0} + 3.2}{3.211 \frac{M}{M_0} + 1.11} \right)^{\frac{1}{2}} . \tag{141}$$

Frequency as a Function of N

Taper Factor = 0.1

$$2^n = \left(\frac{\frac{N}{N_0} + 33}{123 \frac{N}{N_0} + 111} \right)^{\frac{1}{2}} \tag{142}$$

Taper Factor = 1

$$2^n = \left(\frac{\frac{N}{N_0} + 6}{6 \frac{N}{N_0} + 4} \right)^{\frac{1}{2}} \tag{143}$$

Taper Factor = 10

$$2^n = \left(\frac{\frac{N}{N_0} + 3.3}{3.21\frac{N}{N_0} + 1.11} \right)^{\frac{1}{2}} \tag{144}$$

Figure 44 shows a plot of Equations (133), (134), and (135).

THE ZERO-PHASE-SHIFT NETWORK

Harmonic-Transfer Characteristics

The diagram of the four-section zero-phase-shift network is shown in Figure 45.

The general equations for attenuation and phase shift through this network are derived in Appendix VII:

$$(A_n)_{db} = -10 \log\left\{(Z - 2^{-2n}\Sigma - 2^{2n}X)^2 + (2^{-n}\delta - 2^n Y)^2\right\}, \tag{145}$$

$$\beta_n = \arctan \frac{2^{-n}\delta - 2^n Y}{Z - 2^{-2n}\Sigma - 2^{2n}X} \tag{146}$$

where

$$X = \frac{LM}{QS} \tag{147}$$

$$Y = \frac{M}{S} + \frac{L}{Q} + \frac{LMT}{NQS} + \frac{LM}{QN} + \frac{PLM}{KQS} + \frac{PM}{QS} + \frac{L}{S} \tag{148}$$

$$Z = 1 + \frac{TM}{NS} + \frac{M}{N} + \frac{PL}{KQ} + \frac{PLTM}{KQNS} + \frac{PLM}{KQN} + \frac{P}{Q} + \frac{PTM}{NQS}$$
$$+ \frac{PM}{QN} + \frac{LT}{QN} + \frac{PM}{KS} + \frac{LT}{NS} + \frac{L}{N} + \frac{P}{S} + \frac{PL}{KS} \tag{149}$$

$$\delta = \frac{T}{N} + \frac{P}{K} + \frac{PTM}{KNS} + \frac{PM}{KN} + \frac{PLT}{KQN} + \frac{PT}{QN} + \frac{PLT}{KNS} + \frac{PL}{KN} + \frac{PT}{NS} + \frac{P}{N} \tag{150}$$

$$\Sigma = \frac{PT}{KN} \tag{151}$$

$$2^n = \frac{\omega}{\omega_0} = \omega CR.$$

On reducing Equations (145) and (146) for taper factor of 1 and 10 the following equations result:

Taper Factor = 1

$$(A_n)_{db} = -10 \log\left(2^{4n} + 19\cdot 2^{2n} + 87 + 70\cdot 2^{-2n} + 2^{-4n}\right), \tag{152}$$

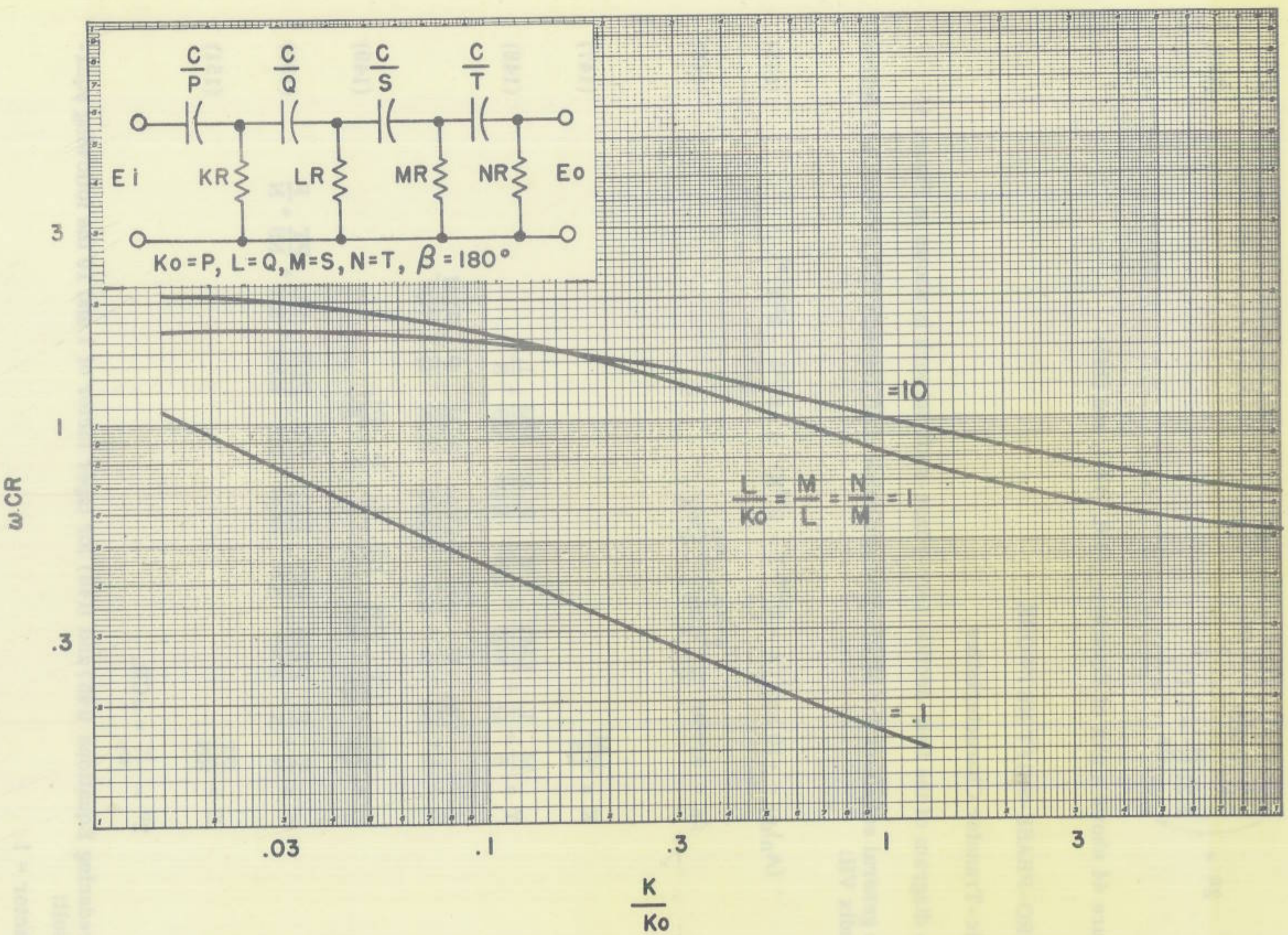


Figure 44 - Frequency-deviation characteristics - four-section high-pass filter

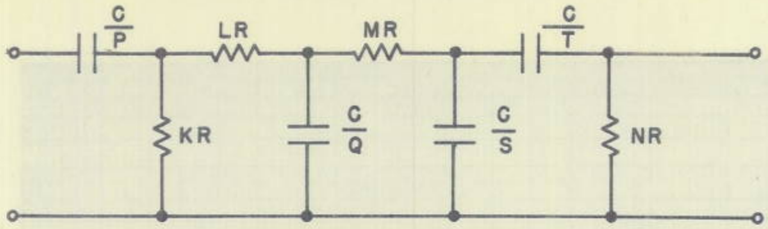


Figure 45

$$\beta_n = \arctan \frac{10 \cdot 2^{-n} - 7 \cdot 2^n}{-(2^{2n} - 15 + 2^{-2n})} . \tag{153}$$

Taper Factor = 10

$$(A_n)_{db} = -10 \log (2^{4n} + 5.41 \cdot 2^{2n} + 7.591 + 5.591041 \cdot 2^{-2n} + 2^{-4n}), \tag{154}$$

$$\beta_n = \arctan \frac{4.321 \cdot 2^{-n} - 4.3 \cdot 2^n}{-(2^{2n} - 6.54 + 2^{-2n})} . \tag{155}$$

A plot of Equations (152) through (155) is given in Figures 46 and 47 for the zero-phase-shift network with taper factors of 1 and 10, and shows the variation with frequency of absolute attenuation in decibels and the phase shift in degrees through the network. From the curves of Figures 46 and 47, the curves of Figure 48 were obtained in the same manner as were the curves of Figure 16 obtained for the three-section low-pass filter. Comparison of Figure 48 with like curves of any of the other filters shows that this network has the least desirable characteristics in this respect.

Stability of Oscillation

The general equation relating frequency stability in terms of phase shift curve slope in units of degrees per octave is derived in Appendix VI:

$$\frac{d\beta}{dn} = \frac{-79.44\,(\delta Y)^{3/2}}{\delta^2 X - \delta Y Z + Y^2 \Sigma} \tag{156}$$

where X, Y, Z, δ, and Σ are given by Equations (147) through (151). The stability in degrees per octave is the same as for the other four-section networks for any taper factor when the parameter against which stability is being plotted is equal to 1.

Amplitude-Modulation Effects

The general equation for determining the variation of network attenuation when the network parameters are varied to change frequency is derived in Appendix VI:

$$(A)_{db} = -20 \log \left| \left(Z - \frac{Y}{\delta} \Sigma - \frac{\delta}{Y} X \right) \right| . \tag{157}$$

where X, Y, Z, δ , and Σ are given by Equations (147) through (151) respectively. On reducing Equation (157) for taper factors of 1 and 10 in terms of K, the following equations result:

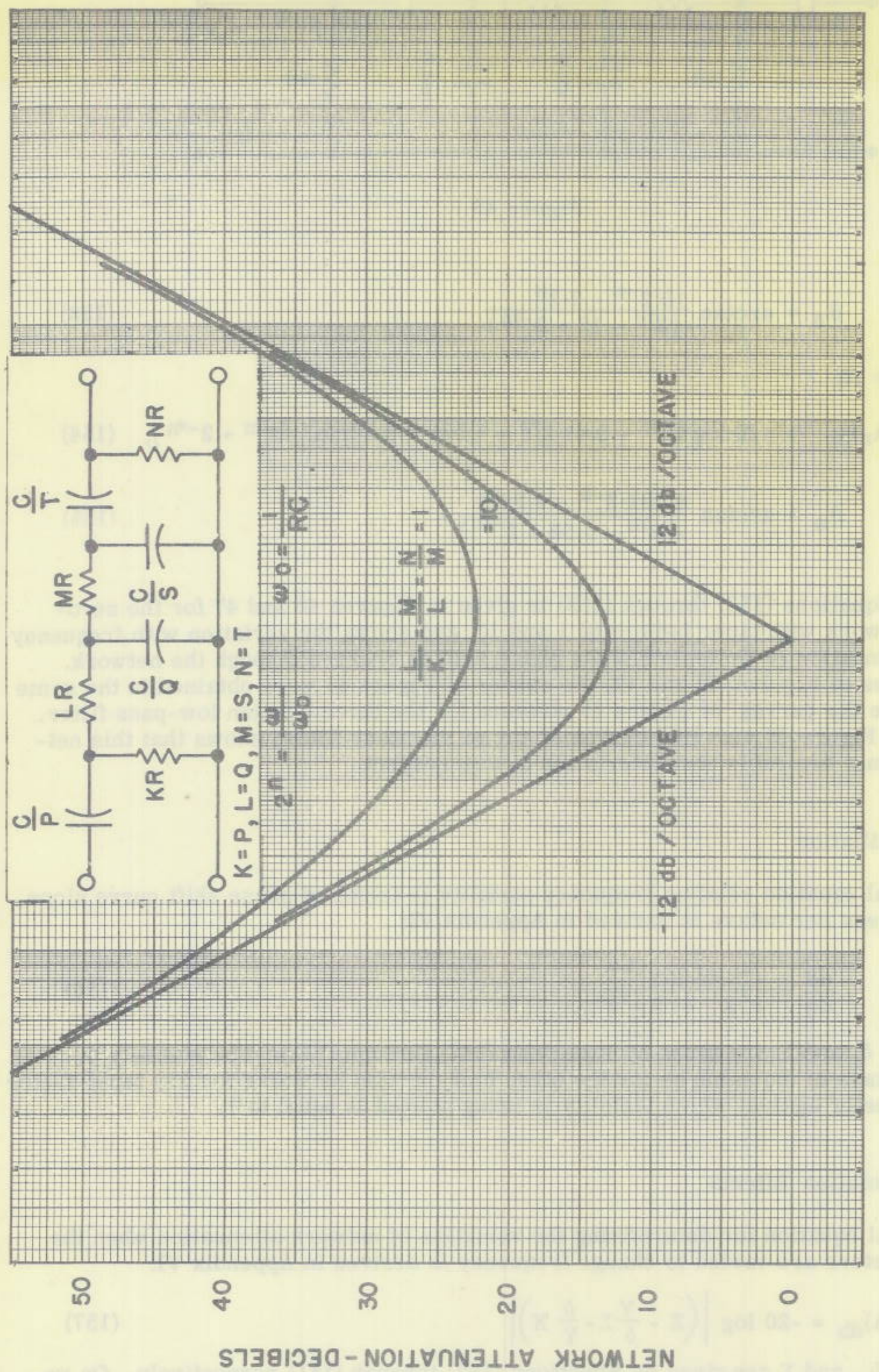


Figure 46 - Attenuation characteristics zero-phase-shift network

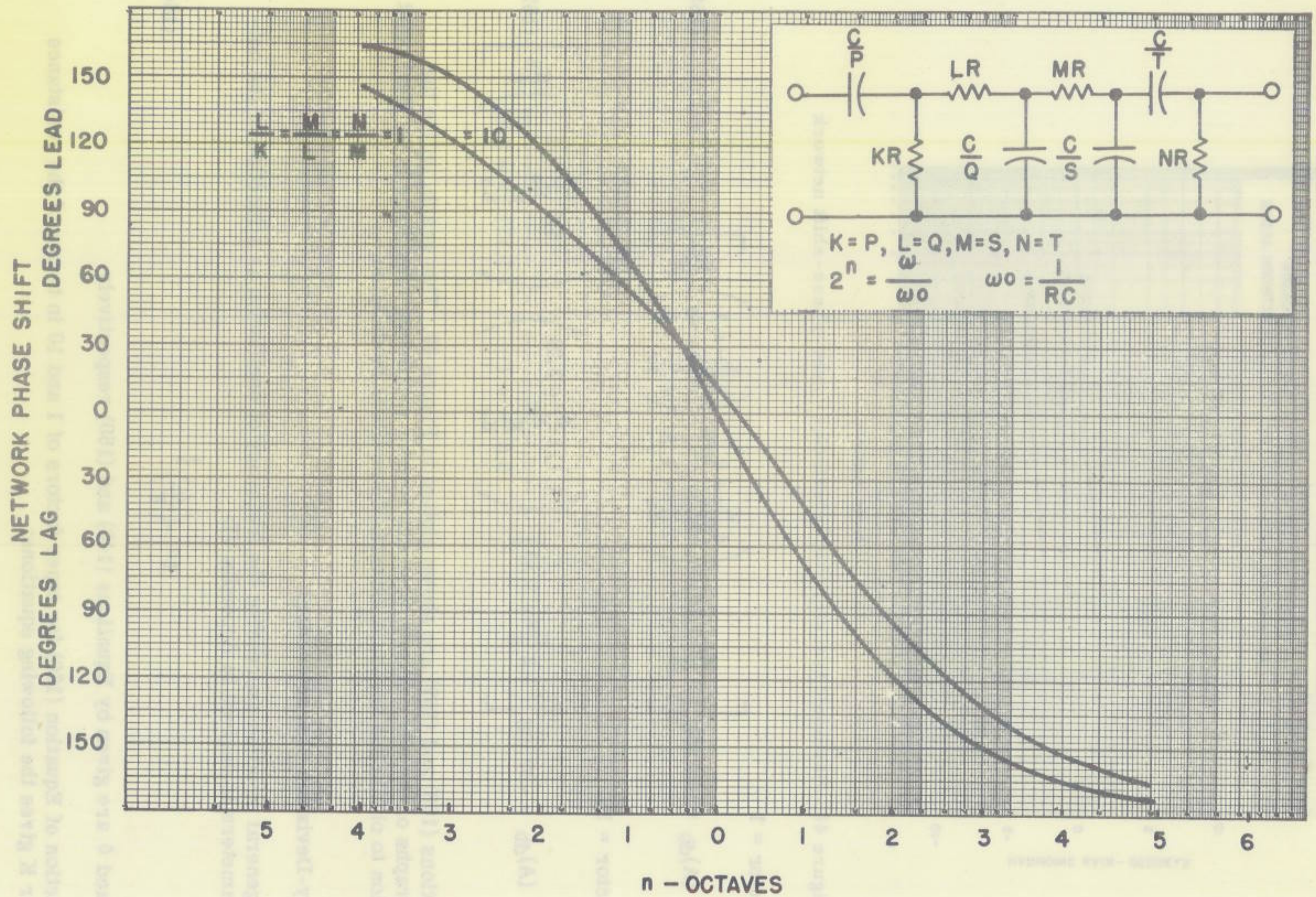


Figure 47 - Phase-shift characteristics - zero-phase-shift network

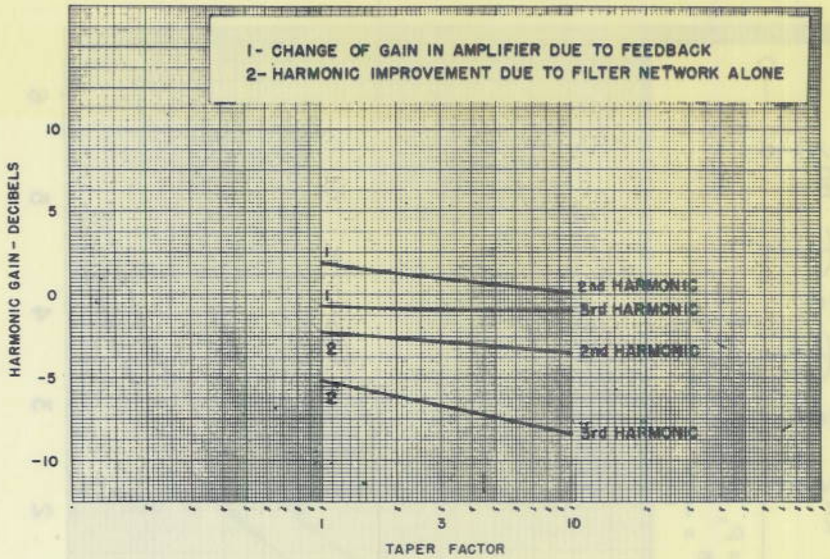


Figure 48 - Harmonic transfer characteristics - zero-phase-shift network

Taper Factor = 1

$$(A)_{db} = -20 \log \left[\left(5\frac{K_0}{K} + 10\right) - \left(\frac{4\frac{K}{K_0} + 6}{6\frac{K}{K_0} + 1}\right) - \left(\frac{6\frac{K}{K_0} + 1}{4\frac{K}{K_0} + 6}\right)\frac{K_0}{K}\right] \tag{158}$$

Taper Factor = 10

$$(A)_{db} = -20 \log \left[\left(3.2\frac{K_0}{K} + 3.43\right) - \left(\frac{1.111\frac{K}{K_0} + 3.21}{3.3\frac{K}{K_0} + 1}\right) - \left(\frac{3.3\frac{K}{K_0} + 1}{1.111\frac{K}{K_0} + 3.21}\right)\frac{K_0}{K}\right]. \tag{159}$$

Equations (158) and (159) are shown plotted in Figure 49. This graph, unlike the similar graphs of the other filters, does not reach a maximum; thus there is no ideal point of operation to obtain the minimum amount of amplitude modulation.

Frequency-Deviation Characteristics

The general equation expressing the frequency of oscillation as a function of the network parameters is derived in Appendix VI:

$$2^n = \left(\frac{\delta}{Y}\right)^{\frac{1}{2}} \tag{160}$$

where Y and δ are given by Equations (148) and (150) respectively.

Reduction of Equation (160) for taper factors of 1 and 10 in terms of the resistance parameter K gives the following equations:

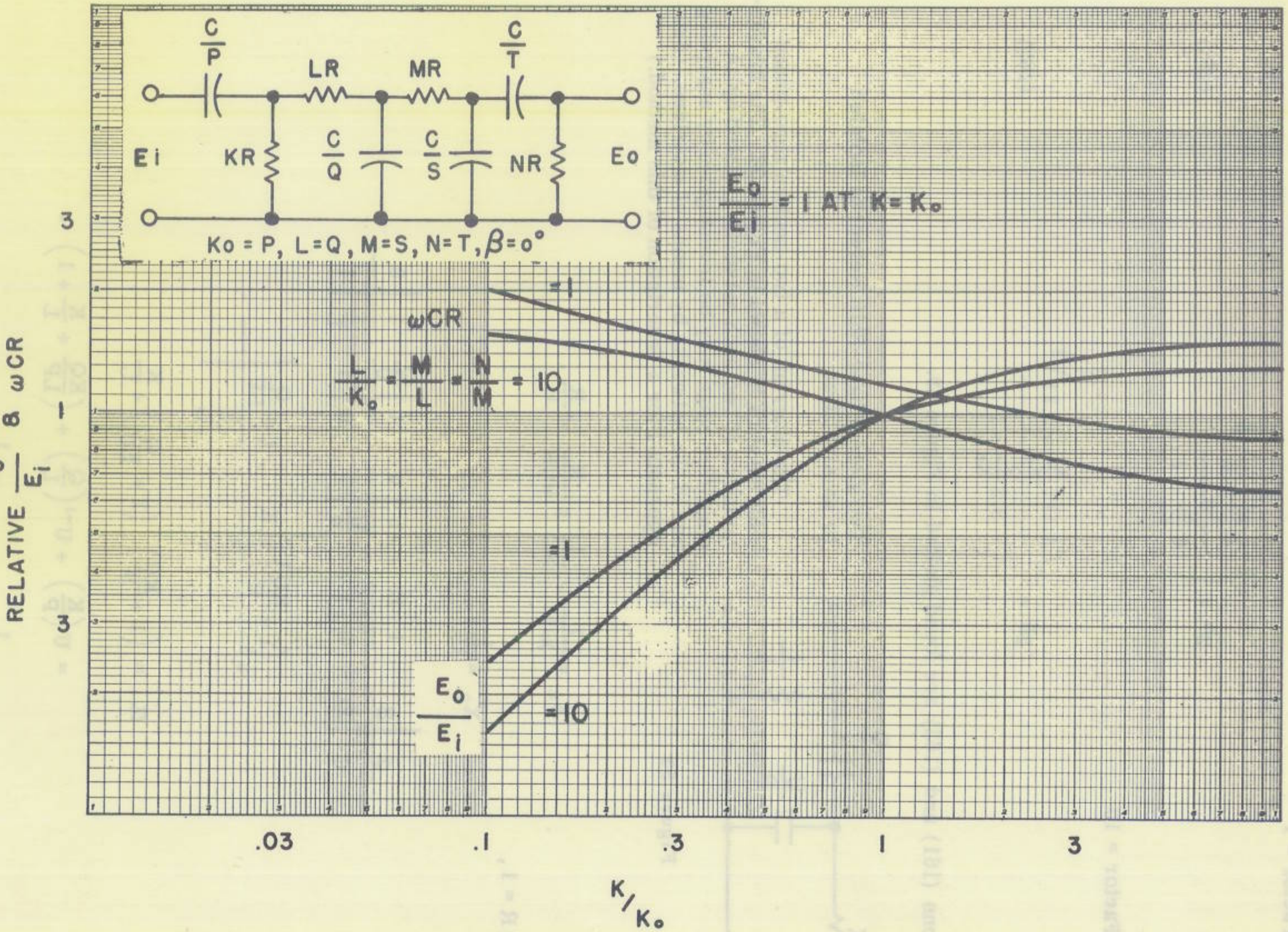


Figure 49 - Amplitude-modulation and frequency-deviation characteristics - zero-phase-shift network

Taper Factor = 1

$$2^n = \left(\frac{4\frac{K}{K_0} + 6}{6\frac{K}{K_0} + 1} \right)^{\frac{1}{2}} \tag{161}$$

Taper Factor = 10

$$2^n = \left(\frac{1.111\frac{K}{K_0} + 3.21}{3.3\frac{K}{K_0} + 1} \right)^{\frac{1}{2}} . \tag{162}$$

Equations (161) and (162) are found plotted in Figure 49.

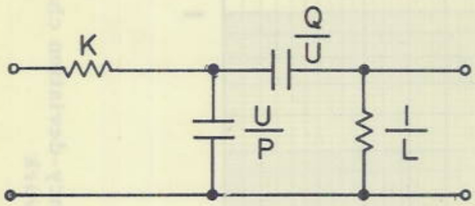


Figure 50

Two Zero-Phase-Shift Networks Isolated by an Amplifier

The diagram of a two-section zero-phase-shift network is shown in Figure 50. The characteristics of two such networks isolated by an amplifier stage may be investigated by analyzing one of the networks by means of matrices. (See Appendix I for a description of this method.)

$$X_c = \frac{1}{j\omega C} = \frac{1}{PC} = \frac{R}{PCR} = \frac{R}{U} .$$

Letting R = 1,

$$X_c = \frac{1}{U} .$$

$$\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array} = \begin{array}{c|c} 1 & K \\ \hline 0 & 1 \end{array} \cdot \begin{array}{c|c} 1 & 0 \\ \hline \frac{U}{P} & 1 \end{array} \cdot \begin{array}{c|c} 1 & \frac{Q}{U} \\ \hline 0 & 1 \end{array} \cdot \begin{array}{c|c} 1 & 0 \\ \hline \frac{1}{L} & 1 \end{array}$$

$$= \begin{array}{c|c} \left(1 + \frac{KU}{P}\right) & K \\ \hline & \end{array} \cdot \begin{array}{c|c} 1 + \frac{Q}{LU} & \\ \hline \frac{1}{L} & \end{array} .$$

$$\mathcal{A} = \left(1 + \frac{KU}{P}\right) \left(1 + \frac{Q}{LU}\right) + \frac{K}{L}$$

$$= U\left(\frac{K}{P}\right) + U^{-1}\left(\frac{Q}{L}\right) + \left(\frac{KQ}{LP} + \frac{K}{L} + 1\right) .$$

$$\alpha = \frac{1}{\mathcal{A}} = \frac{1}{U\left(\frac{K}{P}\right) + U^{-1}\left(\frac{Q}{L}\right) + \left(\frac{KQ}{LP} + \frac{K}{L} + 1\right)} ,$$

or substituting $U = j\omega CR$,

$$\alpha = \frac{1}{j\omega CR\left(\frac{K}{P}\right) - j\frac{1}{\omega CR}\left(\frac{Q}{L}\right) + \left(\frac{KQ}{LP} + \frac{K}{L} + 1\right)} . \tag{163}$$

The absolute attenuation (when used as an oscillator network) is:

$$A = |\alpha| = \frac{1}{\frac{KQ}{LP} + \frac{K}{L} + 1}$$

since the imaginary quantities are zero.

If $K = P$, $Q = L_0$, $Q/K = 1$, then

$$A = \frac{1}{2\frac{L_0}{L} + 1} \quad \text{as a function of L.}$$

Considering two such sections and expressing attenuation in terms of L,

$$A = \left(\frac{1}{2\frac{L_0}{L} + 1}\right)^2 = \left(2\frac{L_0}{L} + 1\right)^{-2},$$

or expressed in decibels,

$$A = +20 \log \left(2\frac{L_0}{L} + 1\right)^{-2} = -40 \log \left(2\frac{L_0}{L} + 1\right). \tag{164}$$

The frequency of oscillation, when two such networks are used, occurs when the imaginary part of the complex attenuation equals zero, or:

$$j\omega CR\left(\frac{K}{P}\right) - j\frac{1}{\omega CR}\left(\frac{Q}{L}\right) = 0$$

$$(\omega CR)^2 = \frac{QP}{KL}$$

$$\omega CR = \left(\frac{QP}{KL}\right)^{\frac{1}{2}}$$

which, when expressed in the rationalized form becomes:

$$2^n = \omega CR = \frac{\omega}{\omega_0} = \left(\frac{QP}{LK}\right)^{\frac{1}{2}},$$

or expressed as a function of L for taper factor = 1,

$$2^n = \left(\frac{L_0}{L}\right)^{\frac{1}{2}}. \tag{165}$$

Equations (164) and (165) are shown plotted in Figure 51.

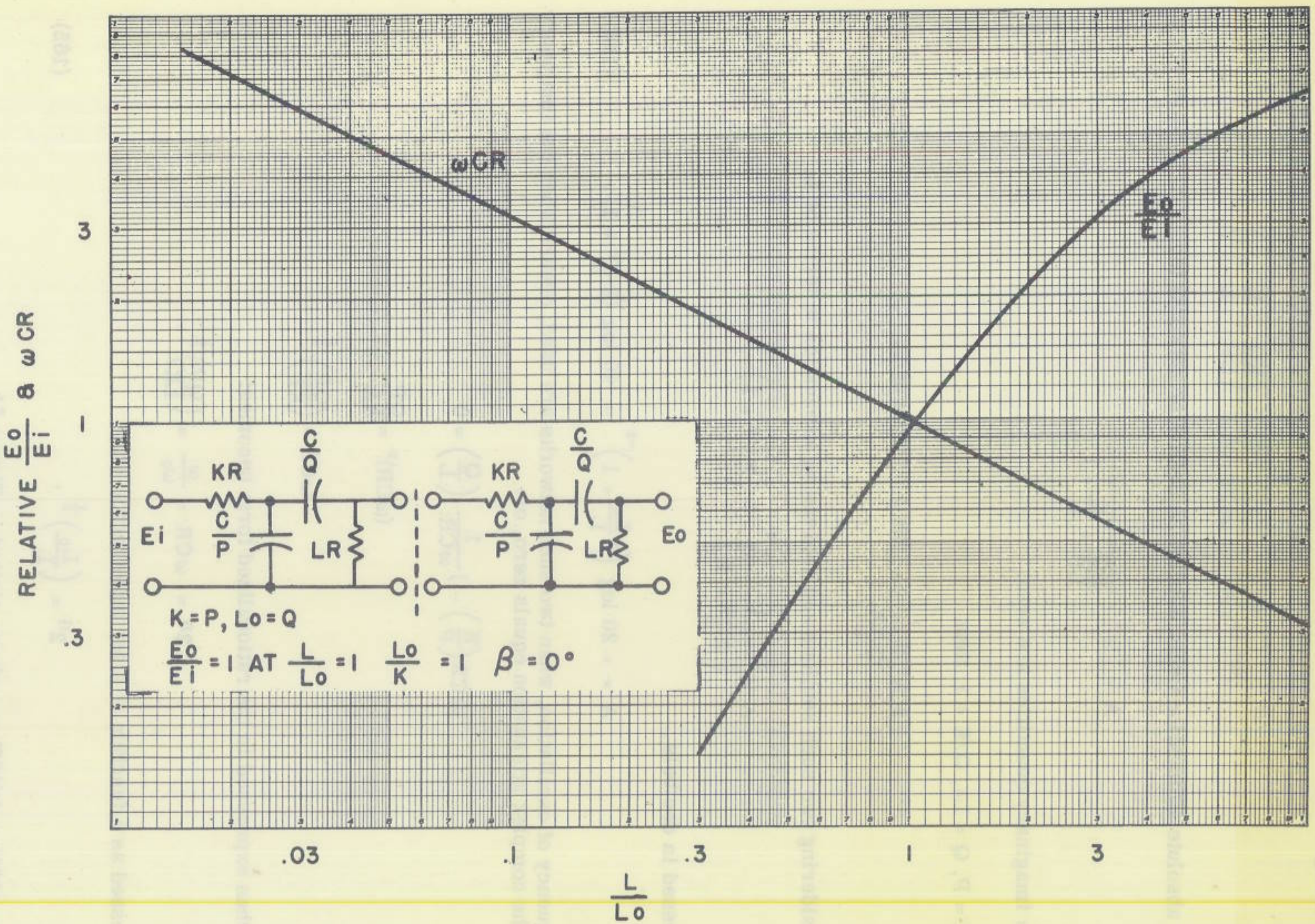


Figure 51 - Amplitude-modulation and frequency-deviation characteristics - zero-phase-shift network

DESIGN EXAMPLE

In order that the material of this report may be utilized more easily, the following example is given. Suppose it is desired to design an oscillator with center frequency of 3000 cycles per second which may be linearly deviated plus and minus 7.5% with a dc voltage. The characteristics to be considered in order of importance are:

1. Frequency stability
2. Amplitude modulation
3. Linearity of frequency deviation
4. Practicality of network in circuit
5. Harmonic content

Since stability is the prime factor, the three-section networks may be disregarded. Since amplitude modulation is of second importance, the zero-phase-shift network is eliminated. Therefore, the choice seems to be between the four-section low- and high-pass network. The frequency-deviation characteristics of the low- and high-pass networks are about the same and will be discussed later in the example. As for the practicality of the network in the circuit, the following items have to be considered:

1. Network driving impedance
2. DC isolation problems
3. Network load impedance
4. DC grid return problems

It is considered here that the network is driven and loaded by a vacuum-tube amplifier or cathode follower. The network driving source must have an impedance much lower than the input impedance of the network if it is not to affect the network design.

It is also necessary in most cases to have dc isolation between the driving source and the load. The high-pass network provides its own dc isolation by means of its series capacitors while the low-pass filter requires extra circuit elements. At high frequencies this isolating network can be made to have a negligible effect on the frequency-determining network by making its corner frequency much lower than that of the network.

The load placed across the output of the network must have an impedance much higher than the output impedance of the network if it is not to affect it. High input capacity of vacuum tube amplifiers due to Miller effect loads the high-pass network at high frequencies because of the low values of resistance and capacitance in the network, whereas with the low-pass network the Miller effect capacity may be subtracted from the last network capacitor without affecting the network characteristics.

The grid of the amplifier being driven by the network must have a dc return path. The high-pass network can provide this with its last shunt resistor while the low-pass network makes it necessary to provide a separate grid return (which tends to load the network) or use either the network series resistances and isolating network resistance, or the resistance of the driving source.

The four-section high-pass network has been chosen for this example, although if the oscillation frequency had been much higher it would have seemed profitable to use the low-pass network. Consider the block diagram of Figure 52.

A cathode follower is used to drive the network and thereby offer it a low source impedance while the network is loaded by the grid circuit of the oscillator, a very high

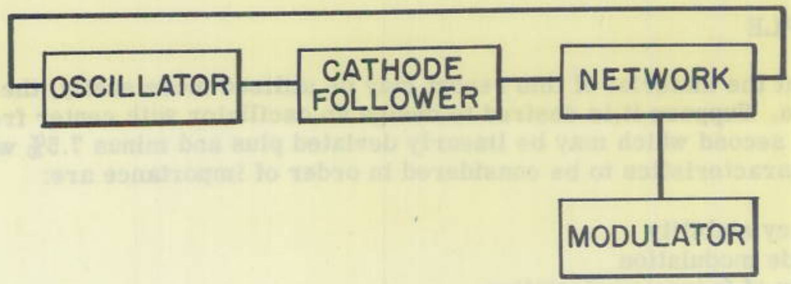


Figure 52

impedance. The use of a network with a taper factor of 10 would provide the highest stability but would be impractical for this simple circuit, since the first network resistor would have to be too low or the last resistor too high. The same reasoning can be followed for the network capacitors. This drawback could be overcome by separating the first two sections of the network from the second two by means of another cathode follower. However, a network with taper factor of three was decided upon for the example, and the actual circuit is shown in Figure 53. From the general equations for amplitude modulation (130) and frequency deviation (134) of a four-section high-pass filter, the following equations are obtained for taper factor of three in terms of the resistor parameter L:

$$\omega CR = 2^n = \left(\frac{1.33\,\frac{L}{L_0} + 3.67}{3.93\,\frac{L}{L_0} + 1.33}\right)^{\frac{1}{2}} \tag{166}$$

$$A_{db} = -20 \log \left[\frac{2^{-4n}}{\frac{L}{L_0}} - 2^{-2n}\left(\frac{4.11}{\frac{L}{L_0}} + 4.22\right) + 1\right]. \tag{167}$$

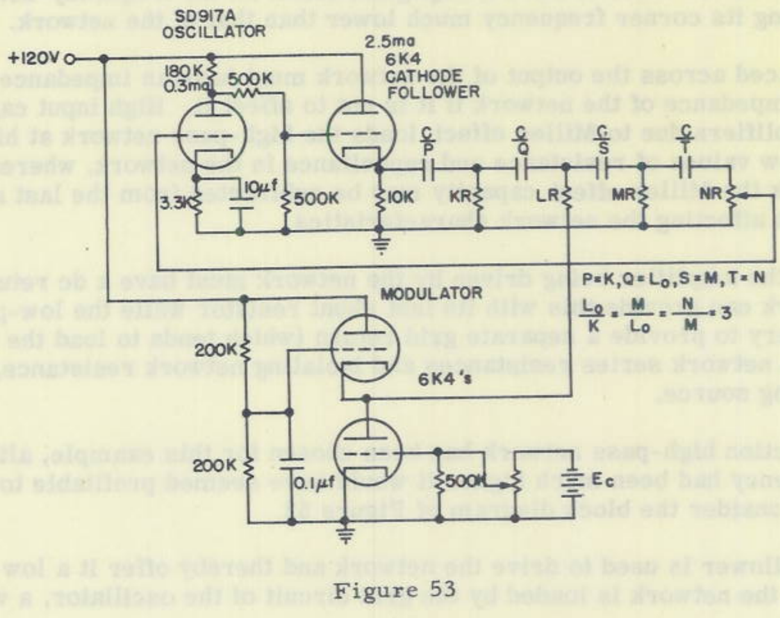


Figure 53

A plot of these curves is shown in Figure 54. From the attenuation vs. L/L_0 curve an operating frequency point corresponding to $L/L_0 = 1.15$ was chosen since the amplitude drops faster for L less than L_0 than for L greater than L_0. Substituting this value of L/L_0 in Equation (166) and solving for ωCR:

$$\omega CR = 0.9428$$

since $\omega = 2\pi(3000)$

$$CR = 0.9428/6000\pi = 5.002 \times 10^{-5}.$$

It is now necessary to determine what values of C and R to use in the network. If the resistance of any one leg is decided upon, the rest of the network must take on values according to taper factor. Since a linear variation of frequency with applied modulating voltage is desired, it is well to investigate the resistance characteristics of the modulator and determine the operating point around which linear deviation may be achieved. Artzt* gives a complete analysis of this type of modulator, but a reasonably good method of determining the operating point is to plot the modulator resistance vs. modulating voltage on log-log paper and operate on the linear portion of the curve. Since the frequency vs. resistance curve of Figure 54 is nearly linear on a log-log plot over the range being used, the outcome will be a linear variation of frequency with applied modulation voltage.

A plot of resistance vs. modulation voltage is shown in Figure 55 for the modulator of Figure 53. Over the resistance range of 10,000 ohms to 20,000 ohms, linear frequency characteristics may be obtained for linear deviations in modulation voltage. In order to vary the oscillator center frequency (3000 cps) by plus and minus 7.5% a resistance deviation of approximately 39% increase and 26% decrease of the center resistance value is required, as estimated from Figure 54 or calculated from Equation (166). From Figure 55 it is seen that this required deviation in resistance may easily be obtained. Using the value 15,000 ohms for the resistance leg LR:

$$L_0R = LR/1.15 = 15{,}000/1.15 = 13040 \text{ ohms}$$

$$KR = L_0R/3 = 4348 \text{ ohms}$$

$$MR = 3\,L_0R = 39130 \text{ ohms}$$

$$NR = 3\,MR = 117400 \text{ ohms}$$

since:

$$CR = 5.002\,(10^{-5})$$

$$C/K = 5.002\,(10^{-5})/KR = 11500 \ \mu\mu f$$

$$C/L_0 = 5.002\,(10^{-5})/L_0R = 3835 \ \mu\mu f$$

$$C/M = 5.002\,(10^{-5})/MR = 1278 \ \mu\mu f$$

$$C/N = 5.002\,(10^{-5})/NR = 426.1\,\mu\mu f.$$

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* Reference 5 in Bibliography

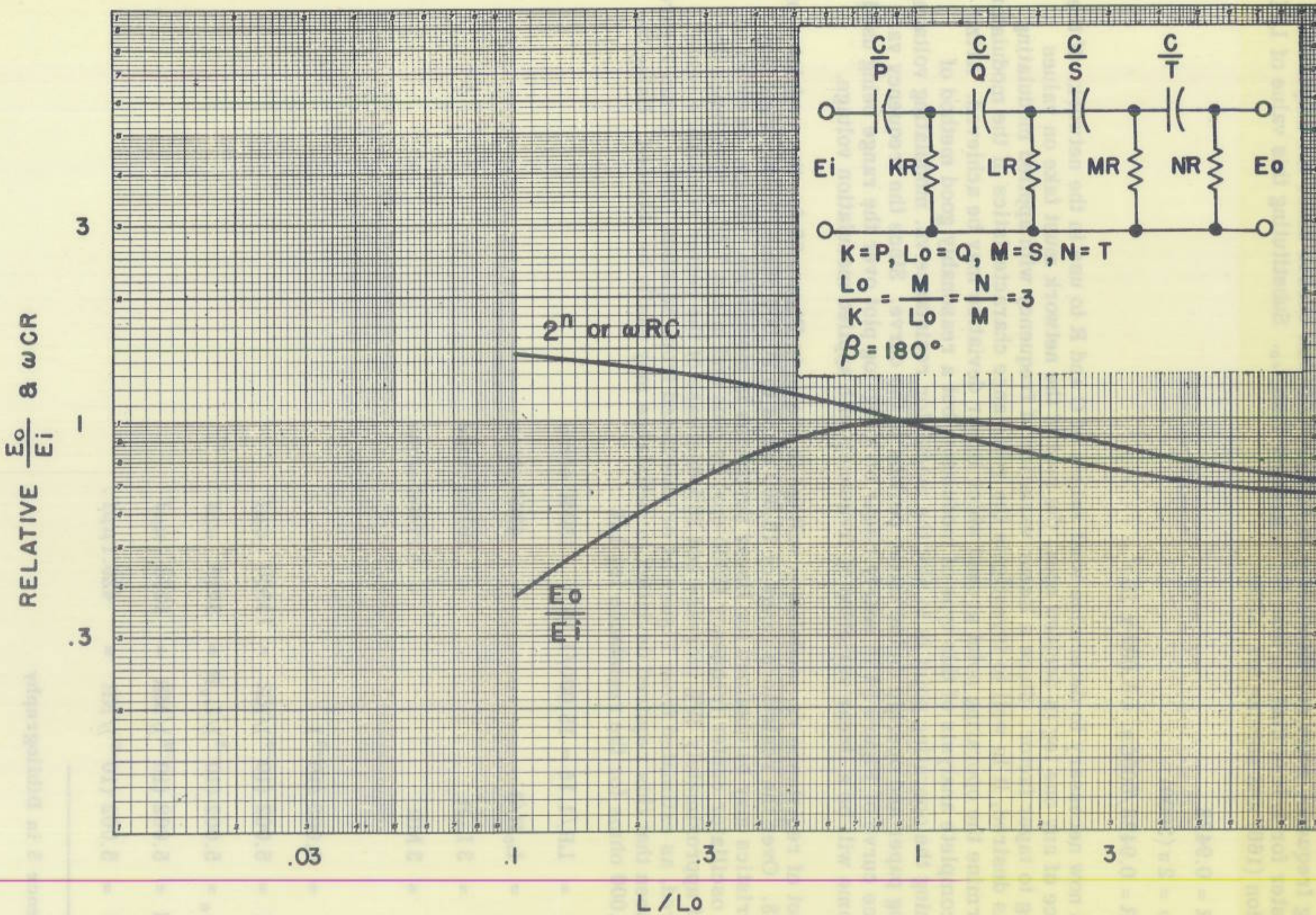

Figure 54 - Amplitude-modulation and frequency-deviation characteristics - four-section high-pass tapered network

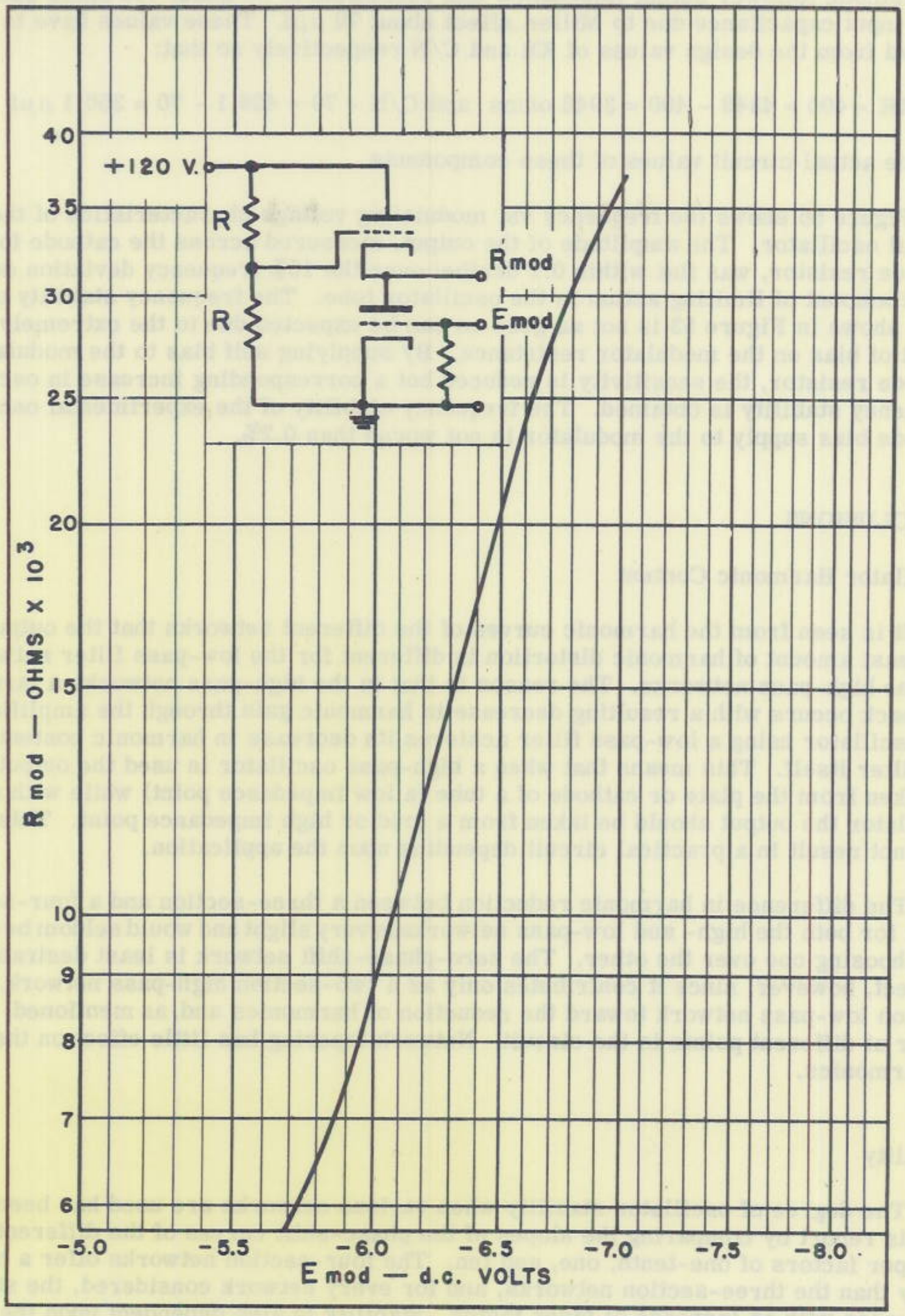

Figure 55 - Modulator resistance characteristics

The cathode follower output impedance was calculated to be about 400 ohms and the oscillator input capacitance due to Miller effect about 70 $\mu\mu$f. These values have to be subtracted from the design values of KR and C/N respectively so that:

$$KR - 400 = 4348 - 400 = 3948 \text{ ohms} \quad \text{and} \quad C/N - 70 = 426.1 - 70 = 356.1\ \mu\mu f$$

are the actual circuit values of these components.

Figure 56 shows the frequency vs. modulating voltage characteristics of the experimental oscillator. The amplitude of the output, measured across the cathode follower cathode resistor, was flat within 0.1 decibel over the 15% frequency deviation due to a slight amount of limiting action in the oscillator tube. The frequency stability of the oscillator shown in Figure 53 is not as good as can be expected due to the extremely sensitive effect of bias on the modulator resistance. By supplying self bias to the modulator with a cathode resistor, the sensitivity is reduced but a corresponding increase in oscillator frequency stability is obtained. The frequency stability of the experimental oscillator with cathode bias supply to the modulator is not worse than 0.2%.

CONCLUSIONS

Oscillator Harmonic Content

It is seen from the harmonic curves of the different networks that the output point for the least amount of harmonic distortion is different for the low-pass filter networks than for the high-pass networks. The reason is that in the high-pass networks a large amount of feedback occurs with a resulting decrease in harmonic gain through the amplifier, while the oscillator using a low-pass filter achieves its decrease in harmonic content through the filter itself. This means that when a high-pass oscillator is used the output should be taken from the plate or cathode of a tube (a low impedance point) while with a low-pass oscillator the output should be taken from a grid or high impedance point. This may or may not result in a practical circuit depending upon the application.

The difference in harmonic reduction between a three-section and a four-section network for both the high- and low-pass networks is very slight and would seldom be sufficient cause for choosing one over the other. The zero-phase-shift network is least desirable in this respect, however, since it contributes only as a two-section high-pass network, or a two-section low-pass network toward the reduction of harmonics and, as mentioned, these effects occur at different points in the circuit. Network tapering has little effect on the reduction of harmonics.

Stability

The degree of oscillator stability when various networks are used has been determined in this report by comparing the slopes of the phase-shift curves of the different networks at taper factors of one-tenth, one, and ten. The four-section networks offer a higher stability than the three-section networks, and for every network considered, the stability increases with an increase of taper factor. Stability is also dependent upon the quality of the parts used in the frequency-determining network and their temperature and humidity characteristics. Another factor affecting stability is the wave form produced by the oscillator, the greatest stability occurring when a pure sine wave is generated. This effect would be about the same for every filter network considered in this report since there is little difference in the harmonic filtering action of the networks.

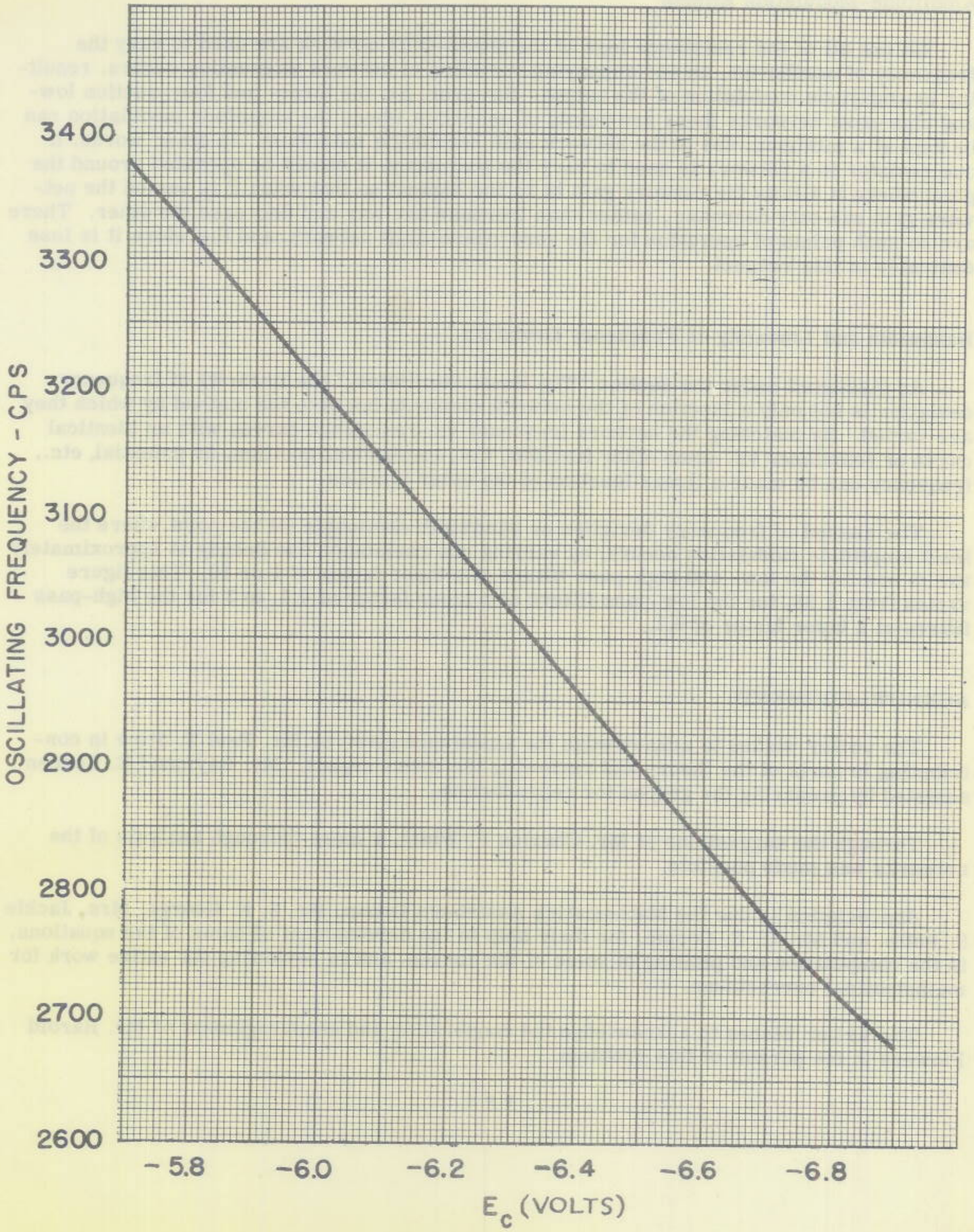

Figure 56 - Experimental oscillator frequency characteristics

Amplitude-Modulation Effects

Unless all of the resistance legs of the phase-shift network are used to vary the frequency of oscillation, an accompanying variation of network attenuation occurs, resulting in amplitude modulation of the output. However, for the three- and four-section low- and high-pass networks there are points of operation where the amplitude modulation can be kept at a minimum due to the network gain reaching a maximum. In other words, if one resistor in a network is used to vary the frequency, it should be operated around the point where it allows the network gain to be the highest so that when it is varied the network gain can only decrease, rather than increase one way and decrease the other. There are no such points of operation for the zero-phase-shift network, and therefore it is less desirable in this respect.

Bandwidth and Linearity of Frequency Deviation

As discussed under the section "The Basic Oscillator," the linearity of frequency deviation is not only a function of the network elements but also the method by which they are varied. By matching the network frequency vs. resistance curves with an identical curve of resistance vs. some other function, i.e., mechanical position, dc potential, etc., frequency can be made a linear function of the other function.

The amount of frequency deviation or bandwidth obtainable (at the point where the least amplitude modulation occurs) by varying one resistance one decade is approximately two to one for the low- and high-pass filters at a taper factor of 1 or 10. This figure varies from 1.35, for the low-pass filters at a taper factor of 0.1, to 3 for the high-pass filters at a taper factor of 0.1.

ACKNOWLEDGMENTS

The writer wishes to acknowledge the assistance given by Mr. Paul T. Stine in contributing to parts of the theory and reviewing the entire report. Mr. Raymond E. Koncen assisted by preparing the graphs for reproduction.

Through the suggestions of Mr. Charles F. White, a more thorough analysis of the networks was made possible.

Appreciation is due the Mathematics Assistants Group, Mr. S. F. George, Mrs. Jackie S. Potts, and Mr. C. E. Corum, for their help in the development of some of the equations, in the computation and plotting of some of the curves, and in reviewing the entire work for mathematical correctness.

The writer wishes to acknowledge the cooperation and encouragement of Mr. Harold Flowers in the pursuit of this problem.

* * *

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4. Sulzer, P. G., "Tapered Phase-Shift Oscillator," Proc. IRE, 36: 1302-1305, October 1948

5. Artzt, M., "Frequency Modulation of R-C Oscillators," Proc. IRE, 32: 409-414, July 1944

6. Ginzton, E. L. and Hollingsworth, L. M., "Phase-Shift Oscillators," Proc. IRE, 29: 43-49, February 1941

* * *

APPENDIX I
ANALYSIS OF FOUR-TERMINAL NETWORKS BY USE OF MATRIX ALGEBRA*

In any linear four-terminal network, there is, by definition, a linear relationship between the input voltage and current and output voltage and current:

$$E_1 = \mathcal{A} E_2 + \mathcal{B} I_2, \tag{168}$$

$$I_1 = \mathcal{C} E_2 + \mathcal{D} I_2. \tag{169}$$

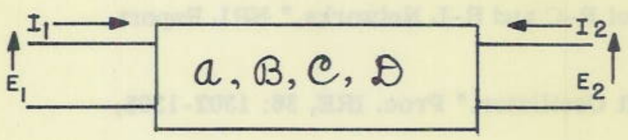

Figure 57

This may be placed in matrix form as follows:

$$\begin{Vmatrix} E_1 \\ I_1 \end{Vmatrix} = \begin{Vmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{Vmatrix} \cdot \begin{Vmatrix} E_2 \\ I_2 \end{Vmatrix}. \tag{170}$$

The values of $\mathcal{A}, \mathcal{B}, \mathcal{C}$, and $\mathcal{D}$ for several simple circuits are given in Table 1.

The advantage of this method appears when various networks are interconnected. In the case of cascade of several networks the rule is that the over-all matrix of the new network is merely the matrix product of the matrices for the individual networks taken in the order of connection. A large collection of formulas for cascade and other types of connections may be obtained.[1,2]

The procedure for multiplying two matrices to obtain a single matrix is as follows:

$$\begin{array}{c|c} a & b \\ \hline c & d \end{array} \cdot \begin{array}{c|c} e & f \\ \hline g & h \end{array} = \begin{array}{c|c} ae + bg & af + bh \\ \hline ce + dg & cf + dh \end{array}. \tag{171}$$

Equation (171) is obtained in four steps:

1. The first row of the first matrix is multiplied by the first column of the second matrix to obtain the upper left-hand square of the final matrix:

$$\begin{array}{c|c} a & b \\ \hline & \end{array} \cdot \begin{array}{c|c} e & \\ \hline g & \end{array} = \begin{array}{c|c} a \cdot e + b \cdot g & \\ \hline & \end{array}. \tag{172}$$

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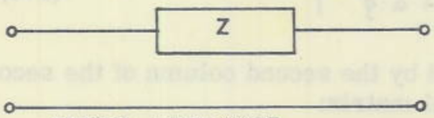
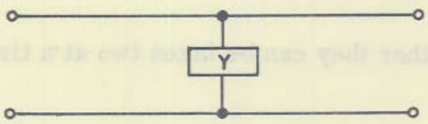
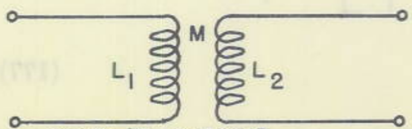
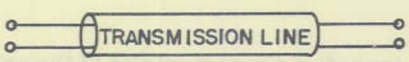
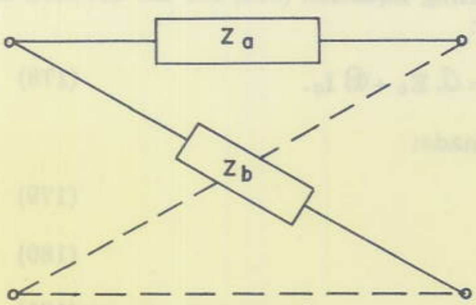
* Appendix I is derived, for the most part, from a paper entitled "Application of Matrix Algebra to Filter Theory," by Paul I. Richards, Proc. IRE, 34:145-150, March 1946.

[1] Pipes, L. A., "Matrix Theory of Four Terminal Networks," Phil. Mag., 30:370-395, November 1940

[2] Guillemin, E. A., Communication Networks, II:140, John Wiley and Sons, New York, N.Y., 1935

TABLE 1
Matrices for Simple Four-Terminal Networks

Network	Matrix
 SERIES IMPEDANCE	$\begin{array}{c\|c} 1 & Z \\ 0 & 1 \end{array}$
 SHUNT ADMITTANCE	$\begin{array}{c\|c} 1 & 0 \\ Y & 1 \end{array}$
 COUPLED CIRCUIT	$\begin{array}{c\|c} \frac{L_1}{M} & j\omega\left(\frac{L_1L_2 - M^2}{M}\right) \\ \hline \frac{1}{j\omega M} & \frac{L_2}{M} \end{array}$
 CHARACTERISTIC IMPEDANCE = Z_0 PROPAGATION CONSTANT = Γ	if $\Gamma = j\theta = \frac{j\omega L}{C} = j\,\frac{2\pi L}{\lambda}$ $\begin{array}{c\|c} \cos\theta & jZ_0\sin\theta \\ \hline j\,\frac{\sin\theta}{Z_0} & \cos\theta \end{array}$
 LATTICE (SYMMETRIC)	$\begin{array}{c\|c} \frac{Z_a + Z_b}{Z_b - Z_a} & \frac{2Z_aZ_b}{Z_b - Z_a} \\ \hline \frac{2}{Z_b - Z_a} & \frac{Z_a + Z_b}{Z_b - Z_a} \end{array}$

2. The first row of the first matrix is multiplied by the second column of the second matrix to obtain the upper right-hand square of the final matrix:

$$\begin{array}{c|c} a & b \\ \hline & \end{array} \cdot \begin{array}{c|c} & f \\ \hline & h \end{array} = \begin{array}{c|c} & a\cdot f + b\cdot h \\ \hline & \end{array} \cdot \qquad (173)$$

3. The second row of the first matrix is multiplied by the first column of the second matrix to obtain the lower left-hand square of the final matrix:

$$\begin{array}{c|c} & \\ \hline c & d \end{array} \cdot \begin{array}{c|c} e & \\ \hline g & \end{array} = \begin{array}{c|c} & \\ \hline c \cdot e + d \cdot g & \end{array} . \tag{174}$$

4. The second row of the first matrix is multiplied by the second column of the second matrix to obtain the lower right-hand square of the final matrix:

$$\begin{array}{c|c} & \\ \hline c & d \end{array} \cdot \begin{array}{c|c} & f \\ \hline & h \end{array} = \begin{array}{c|c} & \\ \hline & c \cdot f + d \cdot h \end{array} . \tag{175}$$

If more than two matrices are to be multiplied together they can be taken two at a time in any order, but none may be interchanged. That is:

$$\begin{array}{c|c} \mathcal{A} & \\ \hline & \end{array} \cdot \left[\begin{array}{c|c} \mathcal{B} & \\ \hline & \end{array} \cdot \begin{array}{c|c} \mathcal{C} & \\ \hline & \end{array} \right] = \left[\begin{array}{c|c} \mathcal{A} & \\ \hline & \end{array} \cdot \begin{array}{c|c} \mathcal{B} & \\ \hline & \end{array} \right] \cdot \begin{array}{c|c} \mathcal{C} & \\ \hline & \end{array} , \tag{176}$$

but

$$\begin{array}{c|c} \mathcal{A} & \\ \hline & \end{array} \cdot \begin{array}{c|c} \mathcal{B} & \\ \hline & \end{array} \neq \begin{array}{c|c} \mathcal{B} & \\ \hline & \end{array} \cdot \begin{array}{c|c} \mathcal{A} & \\ \hline & \end{array} . \tag{177}$$

As an illustration of the usefulness of matrices for determining propagation constants of cascaded filters and other four-terminal networks the complex transfer characteristic of the network shown in Figure 58 will be calculated.

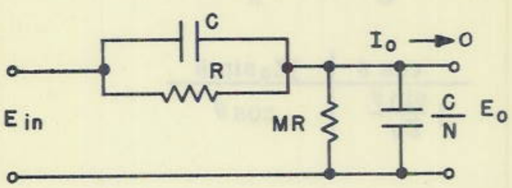


Figure 58

α = complex transfer characteristic of the network

A = $|\alpha|$, or the gain through the network

β = phase shift through the network

Repeating Equation (168) for the network of Figure 58:

$$E_{in} = \mathcal{A} E_o + \mathcal{B} I_o . \tag{178}$$

Since I_o equals zero, the following simplification is made:

$$E_{in} = \mathcal{A} E_o \tag{179}$$

or

$$E_o / E_{in} = 1/\mathcal{A} \tag{180}$$

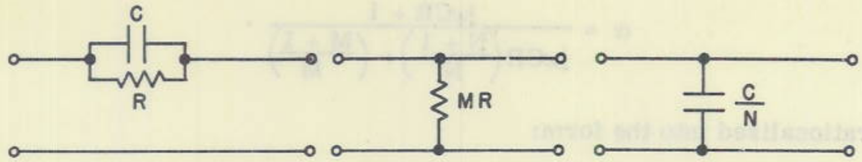
and since

$$\alpha = E_o / E_{in} \tag{181}$$

$$\alpha = 1/\mathcal{A} . \tag{182}$$

From Equation (182) both the absolute value of the network transfer characteristic A and phase shift β may be obtained in the usual manner.

On separating the network of Figure 58 into parts and writing its matrix (as taken from Table 1) beneath it, the following results:



$$\begin{array}{c|c} 1 & Z_{CR} \\ \hline 0 & 1 \end{array} \cdot \begin{array}{c|c} 1 & 0 \\ \hline Y_{MR} & 1 \end{array} \cdot \begin{array}{c|c} 1 & 0 \\ \hline Y_{\frac{C}{N}} & 1 \end{array}$$

Figure 59

Expanding the matrices of Figure 59 to obtain the network matrix:

$$\begin{array}{c|c} a & b \\ \hline c & d \end{array} = \begin{array}{c|c} 1 & Z_{RC} \\ \hline 0 & 1 \end{array} \cdot \begin{array}{c|c} 1 & 0 \\ \hline Y_{MR} & 1 \end{array} \cdot \begin{array}{c|c} 1 & 0 \\ \hline Y_{\frac{C}{N}} & 1 \end{array}$$

$$= \begin{array}{c|c} 1 + Z_{RC} Y_{MR} & Z_{RC} \\ \hline Y_{MR} & 1 \end{array} \cdot \begin{array}{c|c} 1 & 0 \\ \hline Y_{\frac{C}{N}} & 1 \end{array}$$

$$= \begin{array}{c|c} (1 + Z_{RC} Y_{MR}) + Z_{RC} \; Y_{\frac{C}{N}} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array} \quad , \tag{183}$$

and

$$\alpha = \frac{1}{a} = \frac{1}{1 + Z_{RC} Y_{MR} + Z_{RC} \; Y_{\frac{C}{N}}} \tag{184}$$

where

$$Z_{RC} = \frac{R\left(\frac{1}{j\omega C}\right)}{R + \frac{1}{j\omega C}} = \frac{R}{j\omega CR + 1} \; , \tag{185}$$

$$Y_{MR} = \frac{1}{MR} \; , \tag{186}$$

$$Y_{\frac{C}{N}} = \frac{j\omega C}{N} \; . \tag{187}$$

Substituting Equations (185), (186), and (187) into Equation (184):

$$\alpha = \frac{1}{1 + \left(\frac{R}{j\omega CR + 1}\right) \frac{1}{MR} + \left(\frac{R}{j\omega CR + 1}\right) \left(\frac{j\omega C}{N}\right)} \quad ,$$

$$\alpha = \frac{j\omega CR + 1}{j\omega CR\left(\frac{N+1}{N}\right) + \left(\frac{M+1}{M}\right)} \cdot \tag{188}$$

This may be rationalized into the form:

$$\alpha = a + jb \tag{189}$$

and the absolute value of the transfer function α, which is the actual gain, and the phase shift β are obtained as:

$$|\alpha| = A = \sqrt{a^2 + b^2} \ , \tag{190}$$

$$\beta = \arctan\frac{b}{a} \ . \tag{191}$$

* * *

APPENDIX II
ANALYSIS OF RESISTANCE-CAPACITANCE NETWORKS IN TERMS OF A REFERENCE FREQUENCY *

Transfer characteristics of resistance-capacitance filter networks may be plotted advantageously on a logarithmic scale since plots of attenuation in decibels on such a frequency scale approach straight-line low- and high-frequency characteristics. These straight lines are asymptotes of the curve at its outer extremities and may be extended to intersect each other. The point at which these asymptotic lines intersect is importantly related to the parameters of the network being analyzed, and may be conveniently used as a frequency reference point. To illustrate the procedure used in obtaining this reference frequency the following example is given:

The attenuation in decibels and phase shift in degrees through a single-section low-pass filter (Figure 60) are:

$$(A)_{db} = +10 \log \left[1 + (\omega CR)^2 \right], \qquad (192)$$

$$\beta = \arctan \omega CR. \qquad (193)$$

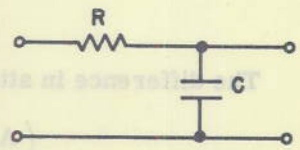


Figure 60

At very high values of ω, Equation (192) takes the form

$$(A)_{db} = 20 \log \omega CR. \qquad (194)$$

At very low values of ω, Equation (192) takes the form

$$(A)_{db} = +10 \log 1 = 0. \qquad (195)$$

If these lines are extended so that they intersect, the point of intersection must lie on the zero-attenuation axis or:

$$20 \log \omega CR = 0,$$

$$\omega CR = 1,$$

$$\omega = \frac{1}{RC}.$$

From this relationship a reference frequency, ω_0, may be conveniently defined as the reciprocal of product of network resistance and capacitance or:

$$\omega_0 = \frac{1}{RC}. \qquad (196)$$

Since a reference frequency has been defined, Equations (192) and (193) may be rewritten as follows:

---

* Appendix II is derived, for the most part, from an NRL Report R-2587 entitled "Resistance-Capacitance Low- and High-Pass Filters," by Charles F. White

$$(A)_{db} = +10 \log \left[1 + \left(\frac{\omega}{\omega_0}\right)^2 \right], \tag{197}$$

$$\beta = \arctan \frac{\omega}{\omega_0}. \tag{198}$$

Using ω/ω_0 in units of octaves where

$$\omega = \omega_n = 2^n \omega_0$$

Equations (197) and (198) may be rewritten in octave form as:

$$(A_n)_{db} = +10 \log (1 + 2^{2n}), \tag{199}$$

$$\beta_n = \arctan 2^n. \tag{200}$$

At very high frequencies Equation (199) takes the form:

$$(A_n)_{db} = 20 \log 2^n. \tag{201}$$

One octave below this, the attenuation may be expressed as:

$$(A_{n-1})_{db} = 20 \log 2^{(n-1)} . \tag{202}$$

The difference in attenuation per octave at high frequencies is therefore:

$$\left(A_n - A_{(n-1)} \right) = 20 \log 2^1 \cong 6 \text{ db/octave}. \tag{203}$$

In the same manner the attenuation at low frequencies may be found to be zero decibels per octave. A graph showing these asymptotic lines is given in Figure 61.

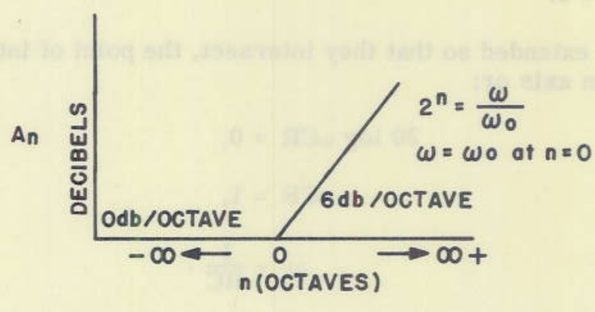


Figure 61

For the case of a filter with a number of like sections in cascade, the "corner frequency" of the filter is the geometric mean of the corner frequencies of each individual section taken separately. For a multisection low-pass filter such as shown in Figure 62, the attenuation at very high frequencies through each section is:

$$(A_{db})_1 = 20 \log \omega C_1 R_1 , \quad (A_{db})_2 = 20 \log \omega C_2 R_2 ,$$

$$(A_{db})_3 = 20 \log \omega C_3 R_3, \text{ ------ } (A_{db})_n = 20 \log \omega C_n R_n.$$

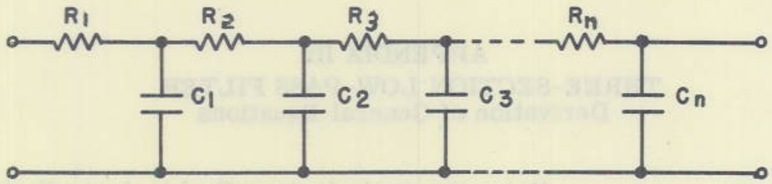


Figure 62

The attenuation in decibels through the complete network at very high frequencies is the sum of the attenuations of the sections composing the network, or:

$$A_{db} = 20 \log (\omega C_1 R_1)(\omega C_2 R_2)(\omega C_3 R_3)\text{----}(\omega C_n R_n). \tag{204}$$

At very low frequencies the attenuation in decibels through a low-pass filter section is zero by Equation (195). Since this is so, the attenuation at very low frequencies through n low-pass sections in cascade is also zero. The intersection of the two asymptotic lines occurs, therefore, when:

$$20 \log (\omega C_1 R_1)(\omega C_2 R_2)(\omega C_3 R_3)\text{---}(\omega C_n R_n) = 0, \tag{205}$$

$$(\omega C_1 R_1)(\omega C_2 R_2)(\omega C_3 R_3)\text{---}(\omega C_n R_n) = 1, \tag{206}$$

$$\omega^n = \frac{1}{C_1 R_1 \, C_2 R_2 \, C_3 R_3 \ldots\ldots C_n R_n},$$

$$\omega = (C_1 R_1 \, C_2 R_2 \, C_3 R_3 \ldots\ldots C_n R_n)^{-1/n}. \tag{207}$$

Since $1/C_1R_1$, $1/C_2R_2$, $1/C_3R_3$, ---, $1/C_nR_n$ are the corner frequencies of the first, second, third, and n^{th} sections respectively or (ω_{01}, ω_{02}, ω_{03}, ..., ω_{0n}), the corner frequency, ω_0, of the complete network is:

$$\omega_0 = (\omega_{01}\, \omega_{02}\, \omega_{03} \ldots\ldots \omega_{0n})^{1/n} \tag{208}$$

or the geometric mean of the individual section corner frequencies.

* * *

APPENDIX III

THREE-SECTION LOW-PASS FILTER

Derivation of General Equations

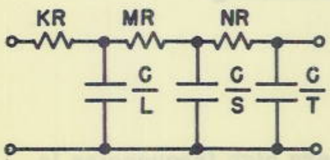


Figure 63

Using the methods described in Appendix I the matrix of Figure 63 is written down and expanded:

$$Z_R = KR,\ MR,\ NR,$$

$$Y_C = j\omega C;\ \frac{j\omega C}{L},\ \frac{j\omega C}{S},\ \frac{j\omega C}{T}.$$

$$\left.\frac{\mathcal{A}}{\mathcal{C}}\right|\frac{\mathcal{B}}{\mathcal{D}} = \begin{array}{c|c} 1 & KR \\ \hline 0 & 1 \end{array} \cdot \begin{array}{c|c} 1 & 0 \\ \hline j\frac{\omega C}{L} & 1 \end{array} \cdot \begin{array}{c|c} 1 & MR \\ \hline 0 & 1 \end{array} \cdot \begin{array}{c|c} 1 & 0 \\ \hline j\frac{\omega C}{S} & 1 \end{array} \cdot \begin{array}{c|c} 1 & NR \\ \hline 0 & 1 \end{array} \cdot \begin{array}{c|c} 1 & 0 \\ \hline j\frac{\omega C}{T} & 1 \end{array}$$

$$= \begin{array}{c|c} 1+\frac{K}{L}j\omega CR & KR \\ \hline & \end{array} \cdot \begin{array}{c|c} 1+\frac{M}{S}j\omega CR & MR \\ \hline j\frac{\omega C}{S} & 1 \end{array} \cdot \begin{array}{c|c} 1+\frac{N}{T}j\omega CR & \\ \hline j\frac{\omega C}{T} & \end{array}$$

$$= \begin{array}{c|c} \left(1+\frac{K}{L}j\omega CR\right)\left(1+\frac{M}{S}j\omega CR\right)+\frac{K}{S}j\omega CR & MR\left(1+\frac{K}{L}j\omega CR\right)+KR \\ \hline & \end{array}$$

$$\cdot \begin{array}{c|c} 1+\frac{N}{T}j\omega CR & \\ \hline \frac{\omega C}{T} & \end{array}$$

$$= \frac{\left[\left(1+\frac{K}{L}j\omega CR\right)\left(1+\frac{M}{S}j\omega CR\right)+\frac{K}{S}j\omega CR\right]\left[1+\frac{N}{T}j\omega CR\right]+\left[MR\left(1+\frac{K}{L}j\omega CR\right)+KR\right]\left[j\frac{\omega C}{T}\right]}{\mathcal{C}}\left|\frac{\mathcal{B}}{\mathcal{D}}\right. \cdot \quad (209)$$

Since the complex transfer characteristic, α, equals $1/\mathcal{A}$,

$$\alpha = \frac{1}{1+j\omega CR\left(\frac{N}{T}+\frac{K}{L}+\frac{M}{S}+\frac{K}{S}+\frac{M}{T}+\frac{K}{T}\right)-(\omega CR)^2\left(\frac{KM}{LT}+\frac{KN}{LT}+\frac{KM}{LS}+\frac{MN}{ST}+\frac{KN}{ST}\right)-j(\omega CR)^3\ \frac{KMN}{LST}} \cdot \quad (210)$$

Let

$$\alpha = \frac{1}{a - jb}$$

where

$$a = \left[1-(\omega CR)^2\left(\frac{KM}{LT}+\frac{KN}{LT}+\frac{KM}{LS}+\frac{MN}{ST}+\frac{KN}{ST}\right)\right], \quad (211)$$

and

$$b = \left[(\omega CR)^3 \ \frac{KMN}{LST} - (\omega CR)\left(\frac{N}{T} + \frac{K}{L} + \frac{M}{S} + \frac{K}{S} + \frac{M}{T} + \frac{K}{T}\right)\right]. \qquad (212)$$

Then

$$\alpha = \frac{a + jb}{a^2 + b^2} = \frac{a}{a^2 + b^2} + j\,\frac{b}{a^2 + b^2} \ . \qquad (213)$$

The absolute gain, $|\alpha|$, and phase shift are immediately found as:

$$(A) = \sqrt{\left(\frac{a}{a^2 + b^2}\right)^2 + \left(\frac{b}{a^2 + b^2}\right)^2} = (a^2 + b^2)^{-\frac{1}{2}} , \qquad (214)$$

$$\beta = \arctan \frac{b}{a} \ . \qquad (215)$$

The absolute gain in decibels is:

$$(A)_{db} = 20 \log (A) = -10 \log (a^2 + b^2). \qquad (216)$$

On expressing Equations (215) and (216) in terms of a reference frequency ω_0 where $\omega_0 = 1/RC$ and rewriting in octave form as described in Appendix II the following equations result:

$$(A_n)_{db} = -10 \log \left\{\left[1 - 2^{2n}\left(\frac{KM}{LT} + \frac{KN}{LT} + \frac{KM}{LS} + \frac{MN}{ST} + \frac{KN}{ST}\right)\right]^2 + \left[2^{3n}\ \frac{KMN}{LST} - 2^n\left(\frac{N}{T} + \frac{K}{L} + \frac{M}{S} + \frac{K}{S} + \frac{M}{T} + \frac{K}{T}\right)\right]^2\right\}, \qquad (217)$$

$$\beta_n = \arctan \left[\frac{2^{3n}\ \frac{KMN}{LST} - 2^n\left(\frac{N}{T} + \frac{K}{L} + \frac{M}{S} + \frac{K}{S} + \frac{M}{T} + \frac{K}{T}\right)}{1 - 2^{2n}\left(\frac{KM}{LT} + \frac{KN}{LT} + \frac{KM}{LS} + \frac{MN}{ST} + \frac{KN}{ST}\right)}\right] . \qquad (218)$$

Equations (217) and (218) are reduced for taper factors of 0.1, 1, 10, and infinity:

Taper Factor = 0.1, K = 1, M = 0.1, N = 0.01,
L = 1, S = 0.1, T = 0.01.

$$(A_n)_{db} = -10 \log [2^{6n} + 283 \cdot 2^{4n} + 15083 \cdot 2^{2n} + 1] , \qquad (219)$$

$$\beta_n = \arctan \left[\frac{2^{3n} - 123 \cdot 2^n}{1 - 23 \cdot 2^{2n}}\right] . \qquad (220)$$

Taper Factor = 1, K = M = N = L = S = T = 1.

$$(A_n)_{db} = -10 \log [2^{6n} + 13 \cdot 2^{4n} + 26 \cdot 2^{2n} + 1] , \qquad (221)$$

$$\beta_n = \arctan \left[\frac{2^{3n} - 6 \cdot 2^n}{1 - 5 \cdot 2^{2n}}\right] . \qquad (222)$$

Taper Factor = 10, $K = 1$, $M = 10$, $N = 100$,
$L = 1$, $S = 10$, $T = 100$.

$$(A_n)_{db} = -10 \log \left[2^{6n} + 3.82 \cdot 2^{4n} + 3.9041 \cdot 2^{2n} + 1\right], \tag{223}$$

$$\beta_n = \arctan \left[\frac{2^{3n} - 3.21 \cdot 2^n}{1 - 3.2 \cdot 2^{2n}}\right]. \tag{224}$$

Taper Factor $= \infty$, $M/K = N/M = \infty$,
$S/L = T/S = \infty$.

$$(A_n)_{db} = -10 \log \left[2^{6n} + 3 \cdot 2^{4n} + 3 \cdot 2^{2n} + 1\right], \tag{225}$$

$$\beta_n = \arctan \left[\frac{2^{3n} - 3 \cdot 2^n}{1 - 3 \cdot 2^{2n}}\right]. \tag{226}$$

Establishment of General Equations for Determining Oscillator Stability

Since the stability of an oscillator is a function of the phase-shift slope of its frequency-determining network, it is only necessary to take the derivative of β in Equation (218), and obtain a relative stability in degrees per octave. The following substitutions are made in Equation (218) before taking the derivative:

$$X = \frac{KMN}{LST}, \tag{227}$$

$$Y = \frac{N}{T} + \frac{K}{L} + \frac{M}{S} + \frac{K}{S} + \frac{M}{T} + \frac{K}{T}, \tag{228}$$

$$Z = \frac{KM}{LT} + \frac{KN}{LT} + \frac{KM}{LS} + \frac{MN}{ST} + \frac{KN}{ST}, \tag{229}$$

$$U = 2^{3n} X - 2^n Y, \tag{230}$$

$$V = 1 - 2^{2n} Z, \tag{231}$$

$$\Delta = \frac{U}{V}. \tag{232}$$

Equation (218) is now rewritten as:

$$\beta = \arctan \Delta \tag{233}$$

and the derivative of β with respect to n is:

$$\frac{d\beta}{dn} = \frac{1}{1 + \Delta^2} \cdot \frac{d\Delta}{dn} = \frac{1}{1 + \frac{U^2}{V^2}} \cdot \frac{V\frac{dU}{dn} - U\frac{dV}{dn}}{V^2}$$

$$\frac{d\beta}{dn} = \frac{V\frac{dU}{dn} - U\frac{dV}{dn}}{V^2 + U^2} \tag{234}$$

$$\frac{d\beta}{dn} = \frac{(1-2^{2n}\cdot Z)\,(3X\ln 2\cdot 2^{3n} - Y\ln 2\cdot 2^{n}) - (2^{3n}\cdot X - 2^{n}\cdot Y)\,(-2Z\ln 2\cdot 2^{2n})}{(2^{3n}\cdot X - 2^{n}\cdot Y)^2 + (1-2^{2n}\cdot Z)^2}$$

$$\frac{d\beta}{dn} = \frac{-[2^{4n}(XZ) + 2^{2n}(YZ-3X) + Y]\,2^{n}\ln 2}{2^{6n}\cdot X^2 + 2^{4n}(Z^2-2XY) + 2^{2n}(Y^2-2Z) + 1} \quad . \qquad (235)$$

Since this result is in units of radians per octave, it is necessary to multiply it by 57.3 to convert to degrees per octave. The ln 2 is a constant equal to 0.69315 and it may be combined with the 57.3 to give the following result:

$$\frac{d\beta}{dn} = \frac{-[2^{4n}(XZ) + 2^{2n}(YZ-3X) + Y]\,2^{n}\cdot(39.72)}{2^{6n}\cdot X^2 + 2^{4n}(Z^2-2XY) + 2^{2n}(Y^2-2Z) + 1}\ \text{degrees/octave.} \qquad (236)$$

To reduce Equations (227), (228), and (229) for any taper factor in terms of any one parameter the following procedure is followed:

$$\text{Taper Factor} = M/K = N/M = S/L = T/S.$$

In general $K = L$, $M = S$, $N = T$, but the separate symbols are retained for each parameter so that any one parameter may be varied independently.

The parameter in terms of which the equation is being reduced is given a zero subscript for an operating point around which it is to be varied, with the other parameters taking on relative values according to taper factor.

As an example, Equation (228) is reduced for a taper factor of 2 in terms of K where $K_0 = L$, $M = S$, $N = T$, and $M/K_0 = N/M = 2$:

$$Y = N/T + K/L + M/S + K/S + M/T + K/T$$

$$Y = 1 + K/K_0 + 1 + K/2K_0 + 2/4 + K/4K_0$$

$$Y = 1.75\,K/K_0 + 2.5.$$

As a second example Equation (229) is reduced for an infinite taper factor in terms of M where $K = L$, $M_0 = S$, $N = T$, and $M_0/K = N/M_0 = \infty$:

$$Z = KM/LT + KN/LT + KM/LS + MN/ST + KN/ST$$

$$Z = 0 + 1 + M/M_0 + M/M_0 + 0$$

$$Z = 2M/M_0 + 1.$$

Equation (236) gives the slope of the β vs. n curve for any value of β. Since it is desirable to solve for the slope only at the value of β equal to 180 degrees,

$$\beta = 180 \text{ degrees when } \arctan\Delta = \pi \qquad (237)$$

or

$$\Delta = \frac{2^{3n}\cdot X - 2^{n}\cdot Y}{1 - 2^{2n}\cdot Z} = 0 \qquad (238)$$

and

$$2^{3n}\cdot X - 2^{n}\cdot Y = 0, \qquad (239)$$

$$2^{2n} = Y/X,$$

$$2^n = \sqrt{Y/X}\ . \tag{240}$$

Substituting this value of 2^n into Equation (236) gives the 180-degree phase-shift slope for any values of the parameters K, M, N, L, S, and T:

$$\frac{d\beta}{dn} = \frac{-\left[(Y/X)^2 \cdot XZ + (Y/X)(YZ - 3X) + Y\right]\sqrt{Y/X}\,(39.72)}{(Y/X)^3 \cdot X^2 + (Y/X)^2 (Z^2 - 2XY) + (Y/X)(Y^2 - 2Z) + 1}$$

$$\frac{d\beta}{dn} = \frac{-\left[(XY^2 Z - X^2 Y)\right]\sqrt{Y/X}\,(79.42)}{(Y^2 Z^2 - 2XYZ + X^2)} = \frac{-Y\sqrt{XY}\,(79.44)}{YZ - X} \tag{241}$$

Equation (241) is reduced for taper factors of 0.1, 1.0, 10, and infinity in terms of the resistance parameter, K (Table 2 gives the values of X, Y, and Z):

Taper Factor = 0.1

$$\frac{d\beta}{dn} = \frac{-79.44\left(111\frac{K}{K_0} + 12\right)\sqrt{111\left(\frac{K}{K_0}\right)^2 + 12\frac{K}{K_0}}}{2442\left(\frac{K}{K_0}\right)^2 + 374\frac{K}{K_0} + 12} \tag{242}$$

Taper Factor = 1.0

$$\frac{d\beta}{dn} = \frac{-79.44\left(3\frac{K}{K_0} + 3\right)\left(3\frac{K^2}{K_0^2} + 3\frac{K}{K_0}\right)^{\frac{1}{2}}}{12\left(\frac{K}{K_0}\right)^2 + 14\frac{K}{K_0} + 3} \tag{243}$$

Taper Factor = 10

$$\frac{d\beta}{dn} = \frac{-79.44\left(1.11\frac{K}{K_0} + 2.1\right)\left(1.11\frac{K^2}{K_0^2} + 2.1\frac{K}{K_0}\right)^{\frac{1}{2}}}{2.442\left(\frac{K}{K_0}\right)^2 + 4.73\frac{K}{K_0} + 2.1} \tag{244}$$

Taper Factor = ∞

$$\frac{d\beta}{dn} = \frac{-79.44\left(\frac{K}{K_0} + 2\right)\left(\frac{K^2}{K_0^2} + 2\frac{K}{K_0}\right)^{\frac{1}{2}}}{2\left(\frac{K}{K_0}\right)^2 + 4\frac{K}{K_0} + 2}\ . \tag{245}$$

The functions X, Y, and Z of Equations (227), (228), and (229), tabulated at taper factors of 0.1, 1.0, 10 and infinity in terms of the resistance parameters K, M, and N, are shown in Table 2.

TABLE 2*

X, Y, and Z as a Function of K

Taper Factor	0.1	1.0	10.0	∞
X	$\frac{K}{K_0}$	$\frac{K}{K_0}$	$\frac{K}{K_0}$	$\frac{K}{K_0}$
Y	$111\frac{K}{K_0}+12$	$3\frac{K}{K_0}+3$	$1.11\frac{K}{K_0}+2.1$	$\frac{K}{K_0}+2$
Z	$22\frac{K}{K_0}+1$	$4\frac{K}{K_0}+1$	$2.2\frac{K}{K_0}+1$	$2\frac{K}{K_0}+1$

X, Y, and Z as a Function of M

Taper Factor	0.1	1.0	10.0	∞
X	$\frac{M}{M_0}$	$\frac{M}{M_0}$	$\frac{M}{M_0}$	$\frac{M}{M_0}$
Y	$11\frac{M}{M_0}+112$	$2\frac{M}{M_0}+4$	$1.1\frac{M}{M_0}+2.11$	$\frac{M}{M_0}+2$
Z	$12\frac{M}{M_0}+11$	$3\frac{M}{M_0}+2$	$2.1\frac{M}{M_0}+1.1$	$2\frac{M}{M_0}+1$

X, Y, and Z as a Function of N

Taper Factor	0.1	1.0	10.0	∞
X	$\frac{N}{N_0}$	$\frac{N}{N_0}$	$\frac{N}{N_0}$	$\frac{N}{N_0}$
Y	$\frac{N}{N_0}+122$	$\frac{N}{N_0}+5$	$\frac{N}{N_0}+2.21$	$\frac{N}{N_0}+2$
Z	$12\frac{N}{N_0}+11$	$3\frac{N}{N_0}+2$	$2.1\frac{N}{N_0}+1.1$	$2\frac{N}{N_0}+1$

* X = KMN/LST

Y = N/T + K/L + M/S + K/S + M/T + K/T

Z = KM/LT + KN/LT + KM/LS + MN/ST + KN/ST

Establishment of General Equations for Determining Amplitude-Modulation Effects

The oscillator output voltage is dependent on the attenuation through the frequency-determining network; thus, it is necessary to find the network attenuation as a function of the resistance parameters since they are used to vary the oscillator frequency.

The second bracketed term of Equation (217) drops out, because it is the imaginary part of the complex gain (Equation (210)), leaving:

$$(A)_{db} = -10 \log (1-2^{2n} \cdot Z)^2 \tag{246}$$

where Z is the same as in Equation (229).

Since oscillations occur when $$2^n = \sqrt{\frac{Y}{X}} \tag{247}$$

$$(A)_{db} = -20 \log \left| \left(1 - \frac{YZ}{X} \right) \right| \tag{248}$$

where X, Y, and Z are given by Equations (227), (228), and (229).

Equation (248) is reduced for taper factors of 0.1, 1.0, 10, and infinity in terms of the resistance parameter K (values of X, Y, and Z taken from Table 2):

Taper Factor = 0.1

$$(A)_{db} = -20 \log\left(2442 \frac{K}{K_0} + 374 + 12 \frac{K_0}{K} \right) \tag{249}$$

Taper Factor = 1.0

$$(A)_{db} = -20 \log\left(12 \frac{K}{K_0} + 14 + 3 \frac{K_0}{K} \right) \tag{250}$$

Taper Factor = 10

$$(A)_{db} = -20 \log\left(2.442 \frac{K}{K_0} + 4.73 + 2.1 \frac{K_0}{K} \right) \tag{251}$$

Taper Factor = ∞

$$(A)_{db} = -20 \log\left(2 \frac{K}{K_0} + 4 + 2 \frac{K_0}{K} \right) . \tag{252}$$

Establishment of Equations for Frequency in Terms of the Resistance Parameters K, M, and N

The relation between frequency of oscillation and the resistance parameters has already been given by Equation (240) in the octave form, as:

$$2^n = \left(\frac{Y}{X} \right)^{\frac{1}{2}} = \frac{\omega}{\omega_0} = \omega CR. \tag{253}$$

This equation is reduced for taper factors of 0.1, 1.0, 10, and infinity in terms of the resistance parameters K, M, and N (values of X and Y are taken from Table 2).

Frequency in Terms of K

Taper Factor = 0.1

$$2^n = \left(12\frac{K_0}{K} + 111\right)^{\frac{1}{2}} \tag{254}$$

Taper Factor = 1.0

$$2^n = \left(3\frac{K_0}{K} + 3\right)^{\frac{1}{2}} \tag{255}$$

Taper Factor = 10

$$2^n = \left(2.1\frac{K_0}{K} + 1.11\right)^{\frac{1}{2}} \tag{256}$$

Taper Factor = ∞

$$2^n = \left(2\frac{K_0}{K} + 1\right)^{\frac{1}{2}} . \tag{257}$$

Frequency in Terms of M

Taper Factor = 0.1

$$2^n = \left(112\frac{M_0}{M} + 11\right)^{\frac{1}{2}} \tag{258}$$

Taper Factor = 1.0

$$2^n = \left(4\frac{M_0}{M} + 2\right)^{\frac{1}{2}} \tag{259}$$

Taper Factor = 10

$$2^n = \left(2.11\frac{M_0}{M} + 1.1\right)^{\frac{1}{2}} \tag{260}$$

Taper Factor = ∞

$$2^n = \left(2\frac{M_0}{M} + 1\right)^{\frac{1}{2}} . \tag{261}$$

Frequency in Terms of N

Taper Factor = 0.1

$$2^n = \left(122\frac{N_0}{N} + 1\right)^{\frac{1}{2}} \tag{262}$$

Taper Factor = 1.0

$$2^n = \left(5\frac{N_0}{N} + 1\right)^{\frac{1}{2}} \tag{263}$$

Taper Factor = 10

$$2^n = \left(2.21\frac{N_0}{N} + 1\right)^{\frac{1}{2}} \tag{264}$$

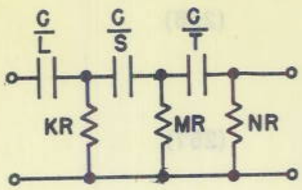
Taper Factor = ∞

$$2^n = \left(2\frac{N_0}{N} + 1\right)^{\frac{1}{2}} . \tag{265}$$

* * *

APPENDIX IV
THREE-SECTION HIGH-PASS FILTER
Derivation of General Equations

Using the methods described in Appendix I the matrix of Figure 64 is written down and expanded:



$$X_C = \frac{L}{j\omega C},\ \frac{S}{j\omega C},\ \frac{T}{j\omega C}$$

$$Y_R = \frac{1}{KR},\ \frac{1}{MR},\ \frac{1}{NR}\,.$$

Figure 64

$$\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array} = \begin{array}{c|c} 1 & \frac{L}{j\omega C} \\ \hline 0 & 1 \end{array} \cdot \begin{array}{c|c} 1 & 0 \\ \hline \frac{1}{KR} & 1 \end{array} \cdot \begin{array}{c|c} 1 & \frac{S}{j\omega C} \\ \hline 0 & 1 \end{array} \cdot \begin{array}{c|c} 1 & 0 \\ \hline \frac{1}{MR} & 1 \end{array} \cdot \begin{array}{c|c} 1 & \frac{T}{j\omega C} \\ \hline 0 & 1 \end{array} \cdot \begin{array}{c|c} 1 & 0 \\ \hline \frac{1}{NR} & 1 \end{array}$$

$$= \begin{array}{c|c} 1 + \frac{L}{K}\ \frac{1}{j\omega CR} & \frac{L}{j\omega C} \\ \hline \frac{1}{KR} & 1 \end{array} \cdot \begin{array}{c|c} 1 + \frac{S}{M}\ \frac{1}{j\omega CR} & \frac{S}{j\omega C} \\ \hline \frac{1}{MR} & 1 \end{array} \cdot \begin{array}{c|c} 1 + \frac{T}{N}\ \frac{1}{j\omega CR} & \frac{T}{j\omega C} \\ \hline \frac{1}{NR} & 1 \end{array}$$

$$= \begin{array}{c|c} \left(1 + \frac{L}{j\omega CRK}\right)\left(1 + \frac{S}{j\omega CRM}\right) + \frac{L}{j\omega CRM} & \left(1 + \frac{L}{j\omega CRK}\right)\frac{S}{j\omega C} + \frac{L}{j\omega C} \\ \hline & \end{array} \cdot \begin{array}{c|c} 1 + \frac{T}{j\omega CRN} & \\ \hline \frac{1}{NR} & \end{array}$$

$$= \begin{array}{c|c} \left[\left(1+\frac{L}{j\omega CRK}\right)\left(1+\frac{S}{j\omega CRM}\right) + \frac{L}{j\omega CRM}\right]\left[1+\frac{T}{j\omega CRN}\right] + \left[\left(1+\frac{L}{j\omega CRK}\right)\frac{S}{j\omega C} + \frac{L}{j\omega C}\right]\frac{1}{NR} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array} . \tag{266}$$

Since the complex transfer characteristic $\alpha = 1/\mathcal{A}$,

$$\alpha = \frac{1}{a - jb} = \frac{a}{a^2 + b^2} + j\,\frac{b}{a^2 + b^2} \tag{267}$$

where

$$a = \left[1 - (\omega CR)^{-2}\ \left(\frac{LS}{KM} + \frac{ST}{MN} + \frac{LT}{KN} + \frac{LT}{MN} + \frac{LS}{KN}\right)\right], \tag{268}$$

$$b = \left[(\omega CR)^{-1}\ \left(\frac{S}{M} + \frac{L}{K} + \frac{L}{M} + \frac{T}{N} + \frac{S}{N} + \frac{L}{N}\right) - (\omega CR)^{-3}\ \frac{LST}{KMN}\right]. \tag{269}$$

The absolute gain and phase shift are immediately found as:

$$A = \sqrt{\frac{a^2}{(a^2+b^2)^2} + \frac{b^2}{(a^2+b^2)^2}} = (a^2+b^2)^{-\frac{1}{2}}, \tag{270}$$

$$\beta = \arctan \frac{b}{a}. \tag{271}$$

The absolute gain in decibels is:

$$(A)_{db} = 20 \log (a^2+b^2)^{-\frac{1}{2}} = -10 \log (a^2+b^2). \tag{272}$$

Expressing Equations (271) and (272) in terms of a reference frequency ω_0 where ω_0 = 1/RC, and rewriting in octave form as described in Appendix II, the following equations result:

$$(A_n)_{db} = -10 \log \left\{ \left[1 - 2^{-2n}\left(\frac{LS}{KM} + \frac{ST}{MN} + \frac{LT}{KN} + \frac{LT}{MN} + \frac{LS}{KN}\right)\right]^2 + \left[2^{-n}\left(\frac{S}{M} + \frac{L}{K} + \frac{L}{M} + \frac{T}{N} + \frac{S}{N} + \frac{L}{N}\right) - 2^{-3n}\frac{LST}{KMN}\right]^2 \right\} \tag{273}$$

$$\beta_n = \arctan \frac{2^{-n}\left(\frac{S}{M} + \frac{L}{K} + \frac{L}{M} + \frac{T}{N} + \frac{S}{N} + \frac{L}{N}\right) - 2^{-3n}\frac{LST}{KMN}}{1 - 2^{-2n}\left(\frac{LS}{KM} + \frac{ST}{MN} + \frac{LT}{KN} + \frac{LT}{MN} + \frac{LS}{KN}\right)}. \tag{274}$$

Equations (273) and (274) are reduced for taper factors of 0.1, 1, 10, and infinity:

Taper Factor = 0.1, K = 1, M = 0.1, N = 0.01,
L = 1, S = 0.1, T = 0.01.

$$(A_n)_{db} = -10 \log [2^{-6n} + 283 \cdot 2^{-4n} + 15083 \cdot 2^{-2n} + 1], \tag{275}$$

$$\beta_n = \arctan \frac{123 \cdot 2^{-n} - 2^{-3n}}{1 - 23 \cdot 2^{-2n}}. \tag{276}$$

Taper Factor = 1.0, K = M = N = L = S = T = 1.

$$(A_n)_{db} = -10 \log [2^{-6n} + 13 \cdot 2^{-4n} + 26 \cdot 2^{-2n} + 1], \tag{277}$$

$$\beta_n = \arctan \frac{6 \cdot 2^{-n} - 2^{-3n}}{1 - 5 \cdot 2^{-2n}}. \tag{278}$$

Taper Factor = 10, K = 1, M = 10, N = 100,
L = 1, S = 10, T = 100.

$$(A_n)_{db} = -10 \log [2^{-6n} + 3.82 \cdot 2^{-4n} + 3.9041 \cdot 2^{-2n} + 1], \tag{279}$$

$$\beta_n = \arctan \frac{3.21 \cdot 2^{-n} - 2^{-3n}}{1 - 3.2 \cdot 2^{-2n}}. \tag{280}$$

Taper Factor = ∞, M/K = N/M = ∞
S/L = T/S = ∞.

$$(A_n)_{db} = -10 \log [2^{-6n} + 3\cdot 2^{-4n} + 3\cdot 2^{-2n} + 1], \tag{281}$$

$$\beta_n = \arctan \frac{3\cdot 2^{-n} - 2^{-3n}}{1 - 3\cdot 2^{-2n}}\,. \tag{282}$$

Establishment of General Equations for Determining Oscillator Stability

Since the stability of an oscillator is a function of the phase-shift slope of its frequency-determining network it is only necessary to take the derivative of β in Equation (274) and obtain a relative stability in degrees per octave.

The following substitutions are made in Equation (274) before taking the derivative:

$$X = LST/KMN, \tag{283}$$

$$Y = S/M + L/K + L/M + T/N + S/N + L/N, \tag{284}$$

$$Z = LS/KM + ST/MN + LT/KN + LT/MN + LS/KN, \tag{285}$$

$$U = 2^{-n} Y - 2^{-3n} X, \tag{286}$$

$$V = 1 - 2^{-2n} Z, \tag{287}$$

$$\Delta = U/V. \tag{288}$$

Equation (274) is now rewritten as:

$$\beta = \arctan \Delta \tag{289}$$

and the derivative of β with respect to n is:

$$\frac{d\beta}{dn} = \frac{1}{1+\Delta^2} \cdot \frac{d\Delta}{dn} = \frac{1}{1+\left(\frac{U}{V}\right)^2} \cdot \frac{V\frac{dU}{dn} - U\frac{dV}{dn}}{V^2} = \frac{V\frac{dU}{dn} - U\frac{dV}{dn}}{V^2 + U^2} \tag{290}$$

$$\frac{d\beta}{dn} = \frac{(1-2^{-2n}Z)(-Y\ln 2\cdot 2^{-n} + 3X\ln 2\cdot 2^{-3n}) - (2^{-n}Y - 2^{-3n}X)(2Z\ln 2\cdot 2^{-2n})}{(1-2^{-2n}Z)^2 + (2^{-n}Y - 2^{-3n}X)^2}$$

$$= \frac{-[2^{-4n}XZ + 2^{-2n}(YZ - 3X) + Y]\, 2^{-n}\ln 2}{2^{-6n}X^2 + 2^{-4n}(Z^2 - 2XY) + 2^{-2n}(Y^2 - 2Z) + 1}\,. \tag{291}$$

Since this result is in units of radians per octave, it is necessary to multiply it by 57.3 to convert to degrees per octave. The ln 2 is a constant equal to 0.69315 and it may be combined with the 57.3 to give the following result:

$$\frac{d\beta}{dn} = \frac{-[2^{-4n}XZ + 2^{-2n}(YZ - 3X) + Y]\, 2^{-n}(39.72)}{2^{-6n}X^2 + 2^{-4n}(Z^2 - 2XY) + 2^{-2n}(Y^2 - 2Z) + 1} \text{ degrees/octave.} \tag{292}$$

To reduce Equations (283), (284), and (285) for any taper factor in terms of any one parameter, the following procedure is followed:

Taper Factor = M/K = N/M = S/L = T/S.

In general K = L, M = S, N = T, but the separate symbols are retained for each parameter so that any one parameter may be varied independently.

The parameter in terms of which the equation is being reduced is given a zero subscript for an operating point around which it is to be varied with the other parameters taking on relative values, according to taper factor.

As an example, Equation (284) is reduced for a taper factor of 2 in terms of K where $K_0 = L$, $M = S$, $N = T$, and $M/K_0 = N/M = 2$:

$$Y = S/M + L/K + L/M + T/N + S/N + L/N$$

$$Y = 1 + \frac{K_0}{K} + .5 + 1 + .5 + .25 = \frac{K_0}{K} + 3.25.$$

As a second example, Equation (285) is reduced for an infinite taper factor in terms of M where $K = L$, $M_0 = S$, $N = T$, and $M_0/K = N/M_0 = \infty$:

$$Z = LS/KM + ST/MN + LT/KN + LT/MN + LS/KN$$

$$Z = \frac{M_0}{M} + 1 + \frac{M_0}{M} + 0 + 0 = 2\frac{M_0}{M} + 1.$$

Equation (292) gives the slope of the β vs. n curve for any value of β. Since it is desirable to solve for the slope only at the value of β equal to 180 degrees,

$$\beta = 180 \text{ degrees when } \arctan \Delta = \pi$$

$$\Delta = \frac{2^{-n}Y - 2^{-3n}X}{1 - 2^{-2n}Z} = 0$$

$$2^{-n}Y - 2^{-3n}X = 0$$

$$2^{-2n} = \frac{Y}{X}$$

$$2^{-n} = \sqrt{\frac{Y}{X}}\,. \tag{293}$$

Substituting this value of 2^{-n} in Equation (292) gives the 180-degree phase-shift slope for any values of the parameters K, L, M, N, S, and T.

$$\frac{d\beta}{dn} = \frac{-\left[\left(\frac{Y}{X}\right)^2 XZ + \left(\frac{Y}{X}\right)(YZ - 3X) + Y\right]\sqrt{\frac{Y}{X}}\ (39.72)}{\left(\frac{Y}{X}\right)^3 X^2 + \left(\frac{Y}{X}\right)^2 (Z^2 - 2XY) + \left(\frac{Y}{X}\right)(Y^2 - 2Z) + 1}$$

$$= \frac{-(XY^2Z - X^2Y)\sqrt{\frac{Y}{X}}\ (79.44)}{Y^2Z^2 - 2XYZ + X^2}$$

$$= \frac{-Y\sqrt{XY}\,(79.44)}{YZ - X}\,. \tag{294}$$

Equation (294) is reduced for taper factors of 0.1, 1.0, 10, and infinity in terms of the resistance parameter K (Table 3 gives the values of X, Y, and Z):

Taper Factor = 0.1

$$\frac{d\beta}{dn} = \frac{-79.44\left(122\frac{K}{K_0} + 1\right)^{3/2}}{1342\frac{K^2}{K_0^2} + 1474\frac{K}{K_0} + 12} \tag{295}$$

Taper Factor = 1.0

$$\frac{d\beta}{dn} = \frac{-79.44\left(5\frac{K}{K_0} + 1\right)^{3/2}}{10\frac{K^2}{K_0^2} + 16\frac{K}{K_0} + 3} \tag{296}$$

Taper Factor = 10

$$\frac{d\beta}{dn} = \frac{-79.44\left(2.21\frac{K}{K_0} + 1\right)^{3/2}}{2.431\left(\frac{K}{K_0}\right)^2 + 4.741\frac{K}{K_0} + 2.1} \tag{297}$$

Taper Factor = ∞

$$\frac{d\beta}{dn} = \frac{-79.44\left(2\frac{K}{K_0} + 1\right)^{3/2}}{2\left(\frac{K}{K_0} + 1\right)^2}\,. \tag{298}$$

The functions X, Y, and Z of Equations (283), (284), and (285) tabulated at taper factors of 0.1, 1, 10, and infinity in terms of the resistance parameters K, M, and N are shown in Table 3.

Establishment of General Equations for Determining Amplitude-Modulation Effects

The oscillator output voltage is dependent on the gain through the frequency-determining network; thus, it is necessary to find the network gain as a function of the resistance parameters since they are used to vary the oscillator frequency.

The second bracketed term of Equation (273) drops out because it is the imaginary part of the complex gain (Equation (267)) leaving:

$$(A_n)_{db} = -10\log\left(1 - 2^{-2n}Z\right)^2 \tag{299}$$

where Z is the same as in Equation (285). Since by Equation (293) oscillation occurs when $2^{-n} = \sqrt{Y/X}$,

TABLE 3*

X, Y, and Z as a Function of K

Taper Factor	0.1	1	10	∞
X	$\frac{K_0}{K}$	$\frac{K_0}{K}$	$\frac{K_0}{K}$	$\frac{K_0}{K}$
Y	$\frac{K_0}{K} + 122$	$\frac{K_0}{K} + 5$	$\frac{K_0}{K} + 2.21$	$\frac{K_0}{K} + 2$
Z	$12\frac{K_0}{K} + 11$	$3\frac{K_0}{K} + 2$	$2.1\frac{K_0}{K} + 1.1$	$2\frac{K_0}{K} + 1$

X, Y, and Z as a Function of M

Taper Factor	0.1	1	10	∞
X	$\frac{M_0}{M}$	$\frac{M_0}{M}$	$\frac{M_0}{M}$	$\frac{M_0}{M}$
Y	$11\frac{M_0}{M} + 112$	$2\frac{M_0}{M} + 4$	$1.1\frac{M_0}{M} + 2.11$	$\frac{M_0}{M} + 2$
Z	$12\frac{M_0}{M} + 11$	$3\frac{M_0}{M} + 2$	$2.1\frac{M_0}{M} + 1.1$	$2\frac{M_0}{M} + 1$

X, Y, and Z as a Function of N

Taper Factor	0.1	1	10	∞
X	$\frac{N_0}{N}$	$\frac{N_0}{N}$	$\frac{N_0}{N}$	$\frac{N_0}{N}$
Y	$111\frac{N_0}{N} + 12$	$3\frac{N_0}{N} + 3$	$1.11\frac{N_0}{N} + 2.1$	$\frac{N_0}{N} + 2$
Z	$22\frac{N_0}{N} + 1$	$4\frac{N_0}{N} + 1$	$2.2\frac{N_0}{N} + 1$	$2\frac{N_0}{N} + 1$

* X = LST/KMN

Y = S/M + L/K + L/M + T/N + S/N + L/N

Z = LS/KM + ST/MN + LT/KN + LT/MN + LS/KN.

$$(A_n)_{db} = -20 \log \left| 1 - \frac{YZ}{X} \right|. \tag{300}$$

Equation (300) is reduced for taper factors of 0.1, 1, 10, and infinity in terms of the resistance parameter K (values of X, Y, and Z are taken from Table 3):

Taper Factor = 0.1

$$(A_n)_{db} = -20 \log\left(1342 \frac{K}{K_0} + 1474 + 12 \frac{K_0}{K}\right) \tag{301}$$

Taper Factor = 1.0

$$(A_n)_{db} = -20 \log\left(10 \frac{K}{K_0} + 16 + 3 \frac{K_0}{K}\right) \tag{302}$$

Taper Factor = 10

$$(A_n)_{db} = -20 \log\left(2.431 \frac{K}{K_0} + 4.741 + 2.1 \frac{K_0}{K}\right) \tag{303}$$

Taper Factor = ∞

$$(A_n)_{db} = -20 \log\left(2 \frac{K}{K_0} + 4 + 2 \frac{K_0}{K}\right). \tag{304}$$

Reduction of Equation (300) in terms of the resistance parameters, L, M, and N at a taper factor of 1 results in the following equations:

As a Function of L:

$$(A_n)_{db} = -20 \log \left[\left(\frac{3 \frac{L}{L_0} + 7}{5 \frac{L}{L_0} + 2} \right)^2 \frac{L}{L_0} - \left(\frac{3 \frac{L}{L_0} + 7}{5 \frac{L}{L_0} + 2} \right) \left(7 \frac{L}{L_0} + 8\right) + 1 \right]. \tag{305}$$

As a Function of M:

$$(A_n)_{db} = -20 \log \left[\left(\frac{2 \frac{M}{M_0} + 8}{5 \frac{M}{M_0} + 2} \right)^2 \frac{M}{M_0} - \left(\frac{2 \frac{M}{M_0} + 8}{5 \frac{M}{M_0} + 2} \right) \left(7 \frac{M}{M_0} + 8\right) + 1 \right]. \tag{306}$$

As a Function of N:

$$(A_n)_{db} = -20 \log \left[\left(\frac{\frac{N}{N_0} + 9}{5 \frac{N}{N_0} + 2} \right)^2 \frac{N}{N_0} \left(\frac{\frac{N}{N_0} + 9}{5 \frac{N}{N_0} + 2} \right) \left(6 \frac{N}{N_0} + 9\right) + 1 \right]. \tag{307}$$

Establishment of Equations for Frequency in Terms of the Resistance Parameters K, M, and N

The relation between frequency of oscillation and the resistance parameters has already been given by Equation (293) in the octave form as:

$$2^{-n} = \frac{\omega}{\omega_0} = \omega CR = \left(\frac{Y}{X}\right)^{\frac{1}{2}}.$$

This equation is reduced for taper factors of 0.1, 1.0, 10, and infinity in terms of the resistance parameters K, M, and N (values of X and Y are taken from Table 3):

Frequency in Terms of K

Taper Factor = 0.1

$$2^n = \left(122\frac{K}{K_0} + 1\right)^{-\frac{1}{2}} \tag{308}$$

Taper Factor = 1.0

$$2^n = \left(5\frac{K}{K_0} + 1\right)^{-\frac{1}{2}} \tag{309}$$

Taper Factor = 10

$$2^n = \left(2.21\frac{K}{K_0} + 1\right)^{-\frac{1}{2}} \tag{310}$$

Taper Factor = ∞

$$2^n = \left(2\frac{K}{K_0} + 1\right)^{-\frac{1}{2}}. \tag{311}$$

Frequency in Terms of M

Taper Factor = 0.1

$$2^n = \left(112\frac{M}{M_0} + 11\right)^{-\frac{1}{2}} \tag{312}$$

Taper Factor = 1.0

$$2^n = \left(4\frac{M}{M_0} + 2\right)^{-\frac{1}{2}} \tag{313}$$

Taper Factor = 10

$$2^n = \left(2.11\frac{M}{M_0} + 1.1\right)^{-\frac{1}{2}} \tag{314}$$

Taper Factor = ∞

$$2^n = \left(2\frac{M}{M_0} + 1\right)^{-\frac{1}{2}}. \tag{315}$$

Frequency in Terms of N

Taper Factor = 0.1

$$2^n = \left(12\frac{N}{N_0} + 111\right)^{-\frac{1}{2}} \tag{316}$$

Taper Factor = 1.0

$$2^n = \left(3\frac{N}{N_0} + 3\right)^{-\frac{1}{2}} \tag{317}$$

Taper Factor = 10

$$2^n = \left(2.1\frac{N}{N_0} + 1.11\right)^{-\frac{1}{2}} \tag{318}$$

Taper Factor = ∞

$$2^n = \left(2\frac{N}{N_0} + 1\right)^{-\frac{1}{2}}. \tag{319}$$

* * *

APPENDIX V
FOUR-SECTION LOW-PASS FILTER
Derivation of General Equations

Using the methods described in Appendix I, the matrix of Figure 65 is written down and expanded:

$X_C = 1/j\omega C = 1/pC = R/pCR = R/U.$

Letting R equal 1.0 ,

$X_C = 1/U;\ P/U,\ Q/U,\ S/U,\ T/U$

$Y_C = U/P,\ U/Q,\ U/S,\ U/T$

$Z_R = K,\ L,\ M,\ N.$

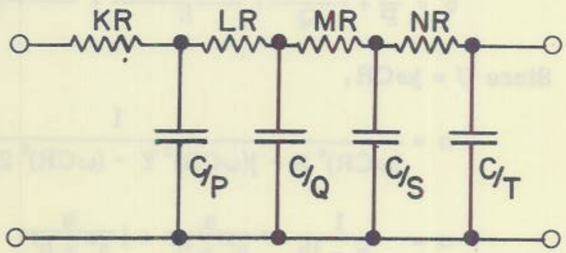


Figure 65

$$\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array} = \begin{array}{c|c} 1 & K \\ \hline 0 & 1 \end{array} \cdot \begin{array}{c|c} 1 & 0 \\ \hline \frac{U}{P} & 1 \end{array} \cdot \begin{array}{c|c} 1 & L \\ \hline 0 & 1 \end{array} \cdot \begin{array}{c|c} 1 & 0 \\ \hline \frac{U}{Q} & 1 \end{array} \cdot \begin{array}{c|c} 1 & M \\ \hline 0 & 1 \end{array} \cdot \begin{array}{c|c} 1 & 0 \\ \hline \frac{U}{S} & 1 \end{array} \cdot \begin{array}{c|c} 1 & N \\ \hline 0 & 1 \end{array} \cdot \begin{array}{c|c} 1 & 0 \\ \hline \frac{U}{T} & 1 \end{array}$$

$$= \begin{array}{c|c} 1+\frac{KU}{P} & K \\ \hline \frac{U}{P} & 1 \end{array} \cdot \begin{array}{c|c} 1+\frac{LU}{Q} & L \\ \hline \frac{U}{Q} & 1 \end{array} \cdot \begin{array}{c|c} 1+\frac{MU}{S} & M \\ \hline \frac{U}{S} & 1 \end{array} \cdot \begin{array}{c|c} 1+\frac{NU}{T} & N \\ \hline \frac{U}{T} & 1 \end{array}$$

$$= \begin{array}{c|c} \left(1+\frac{KU}{P}\right)\left(1+\frac{LU}{Q}\right)+\frac{KU}{Q} & \left(1+\frac{KU}{P}\right)L+K \\ \hline \left(1+\frac{LU}{Q}\right)\frac{U}{P}+\frac{U}{Q} & \frac{LU}{P}+1 \end{array} \cdot \begin{array}{c|c} \left(1+\frac{MU}{S}\right)\left(1+\frac{NU}{T}\right)+\frac{MU}{T} & \left(1+\frac{MU}{S}\right)N+M \\ \hline \left(1+\frac{NU}{T}\right)\frac{U}{S}+\frac{U}{T} & \frac{NU}{S}+1 \end{array}$$

$$\mathcal{A} = \left[\left(1+\frac{KU}{P}\right)\left(1+\frac{LU}{Q}\right)+\frac{KU}{Q}\right]\left[\left(1+\frac{MU}{S}\right)\left(1+\frac{NU}{T}\right)+\frac{MU}{T}\right]$$
$$+\left[\left(1+\frac{KU}{P}\right)L+K\right]\left[\left(1+\frac{NU}{T}\right)\frac{U}{S}+\frac{U}{T}\right]. \qquad (320)$$

Since the complex attenuation $\alpha = 1/\mathcal{A}$,

$$\alpha = \frac{1}{U^4X+U^3Y+U^2Z+U\delta+1} \qquad (321)$$

where

$$X = \frac{KLMN}{PQST} \qquad (322)$$

$$Y = \frac{KLN}{PST} + \frac{KLN}{PQT} + \frac{KLM}{PQS} + \frac{LKM}{PQT} + \frac{LMN}{QST} + \frac{KMN}{PST} + \frac{KMN}{QST} \tag{323}$$

$$Z = \frac{LN + LM + KM + KN}{QT} + \frac{KN + KL + KM}{PT} + \frac{KN + LN + MN}{ST} + \frac{LM + KM}{QS} + \frac{KM + KL}{PS} + \frac{KL}{PQ} \tag{324}$$

$$\delta = \frac{K}{P} + \frac{K + L}{Q} + \frac{K + L + M}{S} + \frac{K + L + M + N}{T} . \tag{325}$$

Since $U = j\omega CR$,

$$\alpha = \frac{1}{(\omega CR)^4 X - j(\omega CR)^3 Y - (\omega CR)^2 Z + j(\omega CR)\delta + 1} \tag{326}$$

$$\alpha = \frac{1}{a - jb} = \frac{a}{a^2 + b^2} + j\frac{b}{a^2 + b^2} \tag{327}$$

where $a = (\omega CR)^4 X - (\omega CR)^2 Z + 1$ (328)

$$b = (\omega CR)^3 Y - (\omega CR)\delta . \tag{329}$$

The absolute gain and phase shift are immediately found as:

$$(A) = \sqrt{\frac{a^2}{(a^2 + b^2)^2} + \frac{b^2}{(a^2 + b^2)^2}} = (a^2 + b^2)^{-\frac{1}{2}}, \tag{330}$$

$$\beta = \arctan\frac{b}{a} . \tag{331}$$

The absolute gain in decibels is:

$$(A)_{db} = 20 \log (a^2 + b^2)^{-\frac{1}{2}} = -10 \log (a^2 + b^2) \tag{332}$$

where a and b are given by Equations (328) and (329).

On expressing Equations (331) and (332) in terms of a reference frequency ω_0 (where $\omega_0 = 1/RC$) and rewriting in octave form as described in Appendix II, the following equations result:

$$(A_n)_{db} = -10 \log\left\{[2^{4n}X - 2^{2n}Z + 1]^2 + [2^{3n}Y - 2^n \delta]^2\right\}, \tag{333}$$

$$\beta_n = \arctan\frac{2^{3n}Y - 2^n\delta}{2^{4n}X - 2^{2n}Z + 1} . \tag{334}$$

Equations (333) and (334) are reduced for taper factors of 0.1, 1, 10, and infinity:

Taper Factor = 0.1

$$(A_n)_{db} = -10 \log[(2^{4n} - 2^{2n}\cdot 366 + 1)^2 + (34\cdot 2^{3n} - 1234\cdot 2^n)^2], \tag{335}$$

$$\beta_n = \arctan \frac{34\cdot 2^{3n} - 1234\cdot 2^n}{2^{4n} - 366\cdot 2^{2n} + 1} \; . \tag{336}$$

Taper Factor = 1.0

$$(A_n)_{db} = -10 \log \left[(2^{4n} - 15\cdot 2^{2n} + 1)^2 + (7\cdot 2^{3n} - 10\cdot 2^n)^2 \right], \tag{337}$$

$$\beta_n = \arctan \frac{7\cdot 2^{3n} - 10\cdot 2^n}{2^{4n} - 15\cdot 2^{2n} + 1} \; . \tag{338}$$

Taper Factor = 10

$$(A_n)_{db} = -10 \log \left[(2^{4n} - 6.63\cdot 2^{2n} + 1)^2 + (4.3\cdot 2^{3n} - 4.321\cdot 2^n)^2 \right], \tag{339}$$

$$\beta_n = \arctan \frac{4.3\cdot 2^{3n} - 4.321\cdot 2^n}{2^{4n} - 6.63\cdot 2^{2n} + 1} \; . \tag{340}$$

Taper Factor = ∞

$$(A_n)_{db} = -10 \log \left[(2^{4n} - 6\cdot 2^{2n} + 1)^2 + (4\cdot 2^{3n} - 4\cdot 2^n)^2 \right], \tag{341}$$

$$\beta_n = \arctan \frac{4\cdot 2^{3n} - 4\cdot 2^n}{2^{4n} - 6\cdot 2^{2n} + 1} \; . \tag{342}$$

Establishment of General Equations for Determining Oscillator Stability

Since the stability of an oscillator is a function of the phase-shift slope of its frequency-determining network it is only necessary to take the derivative of β in Equation (334) with respect to n and obtain a relative stability in degrees per octave:

$$\beta = \arctan \frac{2^{3n}Y - 2^n \delta}{2^{4n}X - 2^{2n}Z + 1} = \arctan \frac{U}{V} \tag{343}$$

$$\frac{d\beta}{dn} = \frac{V\dfrac{dU}{dn} - U\dfrac{dV}{dn}}{U^2 + V^2} \; . \tag{344}$$

$$\frac{d\beta}{dn} = \frac{(2^{4n}X - 2^{2n}Z + 1)\,(3Y\ln 2\cdot 2^{3n} - \delta \ln 2\cdot 2^n) - (2^{3n}Y - 2^n\delta)\,(4X\ln 2\cdot 2^{4n} - 2Z\ln 2\cdot 2^{2n})}{(2^{3n}Y - 2^n \delta)^2 + (2^{4n}X - 2^{2n}Z + 1)^2}$$

$$\frac{d\beta}{dn} = \frac{-\left[2^{6n}XY + 2^{4n}(YZ - 3X\delta) + 2^{2n}(Z\delta - 3Y) + \delta\right]2^n \ln 2}{2^{8n}X^2 + 2^{6n}(Y^2 - 2XZ) + 2^{4n}(Z^2 + 2X - 2Y\delta) + 2^{2n}(\delta^2 - 2Z) + 1} \; . \tag{345}$$

Since this result is in units of radians per octave, it is necessary to multiply it by 57.3 to convert to degrees per octave. The ln 2 is a constant equal to 0.69315 and it may be combined with the 57.3 to give the following result:

$$\frac{d\beta}{dn} = \frac{-\left[2^{6n}(XY) + 2^{4n}(YZ - 3X\delta) + 2^{2n}(Z\delta - 3Y) + \delta\right]2^n (39.71)}{2^{8n}X^2 + 2^{6n}(Y^2 - 2XZ) + 2^{4n}(Z^2 + 2X - 2Y\delta) + 2^{2n}(\delta^2 - 2Z) + 1}\,\frac{\text{degrees}}{\text{octave}} \; . \tag{346}$$

Equation (346) gives the slope of the β vs. n curve for any value of β. Since it is desirable to solve for the slope only at the value of β equal to 180 degrees,

$$\beta = 180^0 \quad \text{when} \quad \arctan \frac{U}{V} = \pi$$

$$\frac{U}{V} = 0$$

$$U = 2^{3n} Y - 2^n \delta = 0$$

$$2^n = (\delta/Y)^{\frac{1}{2}} . \tag{347}$$

Substituting this value of 2^n in Equation (346) gives the 180-degree phase-shift slope for any values of the parameters K, L, M, N, P, Q, S, and T.

$$\frac{d\beta}{dn} = \frac{-\left[\left(\frac{\delta}{Y}\right)^3 XY + \left(\frac{\delta}{Y}\right)^2 (YZ - 3X\delta) + \left(\frac{\delta}{Y}\right)(\delta Z - 3Y) + \delta\right]\left(\frac{\delta}{Y}\right)^{\frac{1}{2}}(39.72)}{\left(\frac{\delta}{Y}\right)^4 X^2 + \left(\frac{\delta}{Y}\right)^3 (Y^2 - 2XZ) + \left(\frac{\delta}{Y}\right)^2 (Z^2 + 2X - 2Y\delta) + \left(\frac{\delta}{Y}\right)(\delta^2 - 2Z) + 1}$$

$$= \frac{-79.44\,(\delta Y)^{3/2}}{\delta YZ - X\delta^2 - Y^2} . \tag{348}$$

Equation (348) is reduced for taper factors of 0.1, 1, 10, and infinity in terms of the resistance parameter K (Table 4 gives the values of X, Y, Z, and δ):

Taper Factor = 0.1

$$\frac{d\beta}{dn} = \frac{-\left[36663\left(\frac{K}{K_0}\right)^2 + 5170\left(\frac{K}{K_0}\right) + 123\right]^{3/2} (79.44)}{\left[11341088\left(\frac{K}{K_0}\right)^3 + 2342164\left(\frac{K}{K_0}\right)^2 + 145904\left(\frac{K}{K_0}\right) + 2828\right]} \tag{349}$$

Taper Factor = 1.0

$$\frac{d\beta}{dn} = \frac{-\left[24\left(\frac{K}{K_0}\right)^2 + 40\left(\frac{K}{K_0}\right) + 6\right]^{3/2} (79.44)}{\left[224\left(\frac{K}{K_0}\right)^3 + 436\left(\frac{K}{K_0}\right)^2 + 212\left(\frac{K}{K_0}\right) + 29\right]} \tag{350}$$

Taper Factor = 10

$$\frac{d\beta}{dn} = \frac{-\left[3.6663\left(\frac{K}{K_0}\right)^2 + 11.704\left(\frac{K}{K_0}\right) + 3.21\right]^{3/2} (79.44)}{11.341088\left(\frac{K}{K_0}\right)^3 + 33.8546\left(\frac{K}{K_0}\right)^2 + 31.559\left(\frac{K}{K_0}\right) + 9.272} \tag{351}$$

Taper Factor = ∞

$$\frac{d\beta}{dn} = \frac{-\left[3\left(\frac{K}{K_0}\right)^2 + 10\left(\frac{K}{K_0}\right) + 3\right]^{3/2} (79.44)}{8\left(\frac{K}{K_0} + 1\right)^3} . \tag{352}$$

The functions X, Y, Z, and δ of Equations (322) through (325) tabulated at taper factors of 0.1, 1, 10, and infinity in terms of the resistance parameters K, L, M, and N are shown in Table 4.

TABLE 4

X, Y, Z, and δ as a Function of K

Taper Factor	0.1	1.0	10	∞
X	$\frac{K}{K_0}$	$\frac{K}{K_0}$	$\frac{K}{K_0}$	$\frac{K}{K_0}$
Y	$33\frac{K}{K_0}+1$	$6\frac{K}{K_0}+1$	$3.3\frac{K}{K_0}+1$	$3\frac{K}{K_0}+1$
Z	$343\frac{K}{K_0}+23$	$10\frac{K}{K_0}+5$	$3.43\frac{K}{K_0}+3.2$	$3\frac{K}{K_0}+3$
δ	$1111\frac{K}{K_0}+123$	$4\frac{K}{K_0}+6$	$1.111\frac{K}{K_0}+3.21$	$\frac{K}{K_0}+3$

X, Y, Z, and δ as a Function of L

Taper Factor	0.1	1.0	10	∞
X	$\frac{L}{L_0}$	$\frac{L}{L_0}$	$\frac{L}{L_0}$	$\frac{L}{L_0}$
Y	$23\frac{L}{L_0}+11$	$5\frac{L}{L_0}+2$	$3.2\frac{L}{L_0}+1.1$	$3\frac{L}{L_0}+1$
Z	$133\frac{L}{L_0}+233$	$7\frac{L}{L_0}+8$	$3.31\frac{L}{L_0}+3.32$	$3\frac{L}{L_0}+3$
δ	$111\frac{L}{L_0}+1123$	$3\frac{L}{L_0}+7$	$1.11\frac{L}{L_0}+3.211$	$\frac{L}{L_0}+3$

X, Y, Z, and δ as a Function of M

Taper Factor	0.1	1.0	10	∞
X	$\frac{M}{M_0}$	$\frac{M}{M_0}$	$\frac{M}{M_0}$	$\frac{M}{M_0}$
Y	$23\frac{M}{M_0}+11$	$5\frac{M}{M_0}+2$	$3.2\frac{M}{M_0}+1.1$	$3\frac{M}{M_0}+1$
Z	$133\frac{M}{M_0}+233$	$7\frac{M}{M_0}+8$	$3.31\frac{M}{M_0}+3.32$	$3\frac{M}{M_0}+3$
δ	$11\frac{M}{M_0}+1223$	$2\frac{M}{M_0}+8$	$1.1\frac{M}{M_0}+3.221$	$\frac{M}{M_0}+3$

X, Y, Z, and δ as a Function of N

Taper Factor	0.1	1.0	10	∞
X	$\frac{N}{N_0}$	$\frac{N}{N_0}$	$\frac{N}{N_0}$	$\frac{N}{N_0}$
Y	$23\frac{N}{N_0}+11$	$5\frac{N}{N_0}+2$	$3.2\frac{N}{N_0}+1.1$	$3\frac{N}{N_0}+1$
Z	$123\frac{N}{N_0}+243$	$6\frac{N}{N_0}+9$	$3.21\frac{N}{N_0}+3.42$	$3\frac{N}{N_0}+3$
δ	$\frac{N}{N_0}+1233$	$\frac{N}{N_0}+9$	$\frac{N}{N_0}+3.321$	$\frac{N}{N_0}+3$

Establishment of Equations for Frequency in Terms of the Resistance Parameters K, L, M, and N

The relation between frequency of oscillation and the resistance parameters has already been given by Equation (347) in the octave form as:

$$2^n = \frac{\omega}{\omega_0} = \omega CR = \left(\frac{\delta}{Y}\right)^{\frac{1}{2}}$$

This equation is reduced for taper factors of 0.1, 1, 10, and infinity in terms of the resistance parameters K, L, M, and N (values of Y and δ are taken from Table 4):

Frequency in Terms of K

Taper Factor = 0.1

$$2^n = \left(\frac{1111\frac{K}{K_0} + 123}{33\frac{K}{K_0} + 1}\right)^{\frac{1}{2}} \tag{353}$$

Taper Factor = 1.0

$$2^n = \left(\frac{4\frac{K}{K_0} + 6}{6\frac{K}{K_0} + 1}\right)^{\frac{1}{2}} \tag{354}$$

Taper Factor = 10

$$2^n = \left(\frac{1.111\frac{K}{K_0} + 3.21}{3.3\frac{K}{K_0} + 1}\right)^{\frac{1}{2}} \tag{355}$$

Taper Factor = ∞

$$2^n = \left(\frac{\frac{K}{K_0} + 3}{3\frac{K}{K_0} + 1}\right)^{\frac{1}{2}} \tag{356}$$

Frequency in Terms of L

Taper Factor = 0.1

$$2^n = \left(\frac{111\frac{L}{L_0} + 1123}{23\frac{L}{L_0} + 11}\right)^{\frac{1}{2}} \tag{357}$$

Taper Factor = 1.0

$$2^n = \left(\frac{3\frac{L}{L_0} + 7}{5\frac{L}{L_0} + 2}\right)^{\frac{1}{2}} \tag{358}$$

Taper Factor = 10

$$2^n = \left(\frac{1.11 \frac{L}{L_o} + 3.211}{3.2 \frac{L}{L_o} + 1.1} \right)^{\frac{1}{2}} \quad (359)$$

Taper Factor = ∞

$$2^n = \left(\frac{\frac{L}{L_o} + 3}{3 \frac{L}{L_o} + 1} \right)^{\frac{1}{2}} . \quad (360)$$

Frequency in Terms of M

Taper Factor = 0.1

$$2^n = \left(\frac{11 \frac{M}{M_o} + 1223}{23 \frac{M}{M_o} + 11} \right)^{\frac{1}{2}} \quad (361)$$

Taper Factor = 1.0

$$2^n = \left(\frac{2 \frac{M}{M_o} + 8}{5 \frac{M}{M_o} + 2} \right)^{\frac{1}{2}} \quad (362)$$

Taper Factor = 10

$$2^n = \left(\frac{1.1 \frac{M}{M_o} + 3.221}{3.2 \frac{M}{M_o} + 1.1} \right)^{\frac{1}{2}} \quad (363)$$

Taper Factor = ∞

$$2^n = \left(\frac{\frac{M}{M_o} + 3}{3 \frac{M}{M_o} + 1} \right)^{\frac{1}{2}} . \quad (364)$$

Frequency in Terms of N

Taper Factor = 0.1

$$2^n = \left(\frac{\frac{N}{N_o} + 1233}{23 \frac{N}{N_o} + 11} \right)^{\frac{1}{2}} \quad (365)$$

Taper Factor = 1.0

$$2^n = \left(\frac{\frac{N}{N_0} + 9}{5\frac{N}{N_0} + 2} \right)^{\frac{1}{2}} \quad (366)$$

Taper Factor = 10

$$2^n = \left(\frac{\frac{N}{N_0} + 3.321}{3.2\frac{N}{N_0} + 1.1} \right)^{\frac{1}{2}} \quad (367)$$

Taper Factor = ∞

$$2^n = \left(\frac{\frac{N}{N_0} + 3}{3\frac{N}{N_0} + 1} \right)^{\frac{1}{2}} \quad (368)$$

Establishment of General Equations for
Determining Amplitude-Modulation Effects

The oscillator output voltage is dependent on the gain through the frequency-determining network; thus, it is necessary to find the network gain as a function of the resistance parameters, since they are used to vary the oscillator frequency.

The second bracketed term of Equation (333) drops out because it is the imaginary part of the complex gain (Equation (326)) leaving:

$$(A_n)_{db} = -10 \log (2^{4n}X - 2^{2n}Z + 1)^2 \quad (369)$$

where X and Z are the same as in Equations (322) and (324).

Since oscillation occurs when

$$2^n = \sqrt{\frac{\delta}{Y}},$$

$$(A_n)_{db} = -20 \log \left| \left[\left(\frac{\delta}{Y}\right)^2 X - \left(\frac{\delta}{Y}\right) Z + 1 \right] \right| . \quad (370)$$

Equation (370) is reduced for taper factors of 0.1, 1, 10, and infinity in terms of the resistance parameter K (values of X, Y, Z, and δ are taken from Table 4):

Taper Factor = 0.1

$$(A)_{db} = -20 \log \left| \left\{ \frac{K}{K_0} \left(\frac{1111\frac{K}{K_0} + 123}{33\frac{K}{K_0} + 1} \right)^2 - \left(\frac{1111\frac{K}{K_0} + 123}{33\frac{K}{K_0} + 1} \right) \left(343\frac{K}{K_0} + 23 \right) + 1 \right\} \right| \quad (371)$$

Taper Factor = 1.0

$$(A)_{db} = -20 \log \left| \left\{ \left(\frac{4\frac{K}{K_0} + 6}{6\frac{K}{K_0} + 1} \right)^2 \frac{K}{K_0} - \left(\frac{4\frac{K}{K_0} + 6}{6\frac{K}{K_0} + 1} \right) \left(10\frac{K}{K_0} + 5 \right) + 1 \right\} \right| \tag{372}$$

Taper Factor = 10

$$(A)_{db} = -20 \log \left[\left(\frac{1.111\frac{K}{K_0} + 3.21}{3.3\frac{K}{K_0} + 1} \right)^2 \left(\frac{K}{K_0} \right) - \left(\frac{1.111\frac{K}{K_0} + 3.21}{3.3\frac{K}{K_0} + 1} \right) \left(3.43\frac{K}{K_0} + 3.2 \right) + 1 \right] \tag{373}$$

Taper Factor = ∞

$$(A)_{db} = -20 \log \left[\left(\frac{\frac{K}{K_0} + 3}{3\frac{K}{K_0} + 1} \right)^2 \left(\frac{K}{K_0} \right) - \left(\frac{\frac{K}{K_0} + 3}{3\frac{K}{K_0} + 1} \right) \left(3\frac{K}{K_0} + 3 \right) + 1 \right] . \tag{374}$$

* * *

APPENDIX VI
FOUR-SECTION HIGH-PASS FILTER
Derivation of General Equations

Using the methods described in Appendix I the matrix of Figure 66 is written down and expanded:

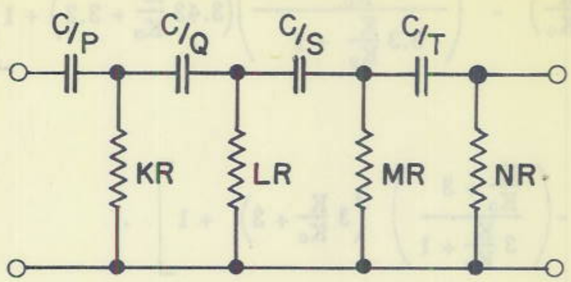


Figure 66

$$X_C = \frac{1}{j\omega C} = \frac{1}{PC} = \frac{R}{PCR} = \frac{R}{U} .$$

Letting R = 1 ,

$$X_C = \frac{1}{U} ;\ \frac{P}{U}, \frac{Q}{U}, \frac{S}{U}, \frac{T}{U}$$

$$Y_R = \frac{1}{K}, \frac{1}{L}, \frac{1}{M}, \frac{1}{N} .$$

$$\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array} = \begin{array}{c|c} 1 & \frac{P}{U} \\ \hline 0 & 1 \end{array} \cdot \begin{array}{c|c} 1 & 0 \\ \hline \frac{1}{K} & 1 \end{array} \cdot \begin{array}{c|c} 1 & \frac{Q}{U} \\ \hline 0 & 1 \end{array} \cdot \begin{array}{c|c} 1 & 0 \\ \hline \frac{1}{L} & 1 \end{array} \cdot \begin{array}{c|c} 1 & \frac{S}{U} \\ \hline 0 & 1 \end{array} \cdot \begin{array}{c|c} 1 & 0 \\ \hline \frac{1}{M} & 1 \end{array} \cdot \begin{array}{c|c} 1 & \frac{T}{U} \\ \hline 0 & 1 \end{array} \cdot \begin{array}{c|c} 1 & 0 \\ \hline \frac{1}{N} & 1 \end{array}$$

$$= \begin{array}{c|c} 1 + \frac{P}{KU} & \frac{P}{U} \\ \hline \frac{1}{K} & 1 \end{array} \cdot \begin{array}{c|c} 1 + \frac{Q}{LU} & \frac{Q}{U} \\ \hline \frac{1}{L} & 1 \end{array} \cdot \begin{array}{c|c} 1 + \frac{S}{MU} & \frac{S}{U} \\ \hline \frac{1}{M} & 1 \end{array} \cdot \begin{array}{c|c} 1 + \frac{T}{NU} & \frac{T}{U} \\ \hline \frac{1}{N} & 1 \end{array}$$

$$= \begin{array}{c|c} U^{-2}\frac{PQ}{KL} + U^{-1}\left(\frac{P}{K}+\frac{P}{L}+\frac{Q}{L}\right) + 1 & U^{-2}\frac{PQ}{K} + U^{-1}\ (P+Q) \\ \hline & \end{array} \cdot \begin{array}{c|} U^{-2}\frac{ST}{MN} + U^{-1}\left(\frac{S}{M}+\frac{T}{N}+\frac{S}{N}\right) + 1 \\ \hline U^{-1}\left(\frac{T}{MN}\right) + \left(\frac{1}{M}+\frac{1}{N}\right) \end{array}$$

$$\mathcal{A} = \left[U^{-2}\ \frac{PQ}{KL} + U^{-1}\left(\frac{P}{K}+\frac{P}{L}+\frac{Q}{L}\right) + 1\right]\left[U^{-2}\ \frac{ST}{MN} + U^{-1}\left(\frac{S}{M}+\frac{T}{N}+\frac{S}{N}\right) + 1\right]$$

$$+ \left[U^{-2}\frac{PQ}{K} + U^{-1}\ (P+Q)\right]\left[U^{-1}\ \frac{T}{MN} + \left(\frac{1}{M}+\frac{1}{N}\right)\right] . \tag{375}$$

Since the complex gain (α) through the network is $1/\mathcal{A}$,

$$\alpha = \frac{1}{U^{-4}X + U^{-3}Y + U^{-2}Z + U^{-1}\delta + 1} \tag{376}$$

where

$$X = \frac{PQST}{KLMN} , \tag{377}$$

$$Y = \frac{PQT}{KMN} + \frac{PQT}{KLN} + \frac{PQS}{KLM} + \frac{PQS}{KLN} + \frac{QST}{LMN} + \frac{PST}{KMN} + \frac{PST}{LMN} , \tag{378}$$

$$Z = \frac{QT + QS + PS + PT}{LN} + \frac{PT + PQ + PS}{KN} + \frac{PT + QT + ST}{MN} + \frac{QS + PS}{LM} + \frac{PS + PQ}{KM} + \frac{PQ}{KL}, \quad (379)$$

$$\delta = \frac{P}{K} + \frac{P + Q}{L} + \frac{P + Q + S}{M} + \frac{P + Q + S + T}{N}. \quad (380)$$

Since $U = j\omega CR$

$$\alpha = \frac{1}{(\omega CR)^{-4}X + j(\omega CR)^{-3}Y - (\omega CR)^{-2}Z - j(\omega CR)^{-1}\delta + 1} \quad (381)$$

$$\alpha = \frac{1}{a - jb} = \frac{a}{a^2 + b^2} + j\frac{b}{a^2 + b^2}. \quad (382)$$

The absolute gain and phase shift are:

$$(A) = (a^2 + b^2)^{-\frac{1}{2}} \quad (383)$$

$$\beta = \arctan \frac{b}{a} \quad (384)$$

where $a = (\omega CR)^{-4}X - (\omega CR)^{-2}Z + 1$ (385)

$b = (\omega CR)^{-1}\delta - (\omega CR)^{-3}Y.$ (386)

The absolute gain in decibels is:

$$(A)_{db} = 20 \log (a^2 + b^2)^{-\frac{1}{2}} = -10 \log (a^2 + b^2). \quad (387)$$

On expressing Equations (384) and (387) in terms of a reference frequency (ω_0) where ω_0 equals $1/RC$ and rewriting in octave form as described in Appendix II, the following equations result:

$$(A_n)_{db} = -10 \log \left\{ \left[(2^{-4n}X - 2^{-2n}Z + 1)^2\right] + \left[(2^{-n}\delta - 2^{-3n}Y)^2\right] \right\}, \quad (388)$$

$$\beta_n = \arctan \frac{2^{-n}\delta - 2^{-3n}Y}{2^{-4n}X - 2^{-2n}Z + 1}. \quad (389)$$

Equations (388) and (389) are reduced for taper factors of 0.1, 1, 10, and infinity:

Taper Factor = 0.1, $L/K = M/L = N/M = 0.1,$
$Q/P = S/Q = T/S = 0.1.$

$$(A_n)_{db} = -10 \log \left[(2^{-4n} - 366 \cdot 2^{-2n} + 1)^2 + (1234 \cdot 2^{-n} - 34 \cdot 2^{-3n})^2\right], \quad (390)$$

$$\beta_n = \arctan \frac{1234 \cdot 2^{-n} - 34 \cdot 2^{-3n}}{2^{-4n} - 366 \cdot 2^{-2n} + 1} \quad (391)$$

Taper Factor = 1.0, $L/K = M/L = N/M = 1.0,$
$Q/P = S/Q = T/S = 1.0.$

$$(A_n)_{db} = -10 \log \left[(2^{-4n} - 15 \cdot 2^{-2n} + 1)^2 + (10 \cdot 2^{-n} - 7 \cdot 2^{-3n})^2\right], \quad (392)$$

$$\beta_n = \arctan \frac{10 \cdot 2^{-n} - 7 \cdot 2^{-3n}}{2^{-4n} - 15 \cdot 2^{-2n} + 1} . \tag{393}$$

Taper Factor = 10, L/K = M/L = N/M = 10,
Q/P = S/Q = T/S = 10.

$$(A_n)_{db} = -10 \log[(2^{-4n} - 6.63 \cdot 2^{-2n} + 1)^2 + (4.321 \cdot 2^{-n} - 4.3 \cdot 2^{-3n})^2], \tag{394}$$

$$\beta_n = \arctan \frac{4.321 \cdot 2^{-n} - 4.3 \cdot 2^{-3n}}{2^{-4n} - 6.63 \cdot 2^{-2n} + 1} . \tag{395}$$

Taper Factor = ∞, L/K = M/L = N/M = ∞,
Q/P = S/Q = T/S = ∞.

$$(A_n)_{db} = -10 \log[(2^{-4n} - 6 \cdot 2^{-2n} + 1)^2 + (4 \cdot 2^{-n} - 4 \cdot 2^{-3n})^2], \tag{396}$$

$$\beta_n = \arctan \frac{4 \cdot 2^{-n} - 4 \cdot 2^{-3n}}{2^{-4n} - 6 \cdot 2^{-2n} + 1} . \tag{397}$$

Establishment of General Equations for Determining Oscillator Stability

Since the stability of an oscillator is a function of the phase-shift slope of its frequency-determining network, it is only necessary to take the derivative of β_n (Equation (389)) with respect to n and obtain a relative stability in degrees per octave.

$$\beta_n = \arctan \frac{2^{-n}\delta - 2^{-3n} Y}{2^{-4n} X - 2^{-2n} Z + 1} = \arctan \frac{U}{V} , \tag{398}$$

$$\frac{d\beta}{dn} = \frac{V\frac{dU}{dn} - U\frac{dV}{dn}}{U^2 + V^2} \tag{399}$$

$$\frac{d\beta}{dn} = \frac{(2^{-4n}X - 2^{-2n}Z + 1)(-\delta \cdot 2^{-n}\ln 2 + 3Y \cdot 2^{-3n}\ln 2) - (2^{-n}\delta - 2^{-3n}Y)(-4X \cdot 2^{-4n}\ln 2 + 2Z \cdot 2^{-2n}\ln 2)}{(2^{-2n}\delta^2 - 2\delta Y \cdot 2^{-4n} + 2^{-6n}Y^2) + (2^{-8n}X^2 + 2^{-4n}Z^2 + 1 - 2XZ \cdot 2^{-6n} + 2X \cdot 2^{-4n} - 2Z \cdot 2^{-2n})} \tag{400}$$

$$= \frac{-[2^{-6n} XY + 2^{-4n}(YZ - 3\delta X) + 2^{-2n}(\delta Z - 3Y) + \delta] 2^{-n} \ln 2}{2^{-8n}X^2 + 2^{-6n}(Y^2 - 2XZ) + 2^{-4n}(Z^2 + 2X - 2\delta Y) + 2^{-2n}(\delta^2 - 2Z) + 1} . \tag{401}$$

Since this result is in units of radians per octave it is necessary to multiply it by 57.3 to convert to degrees per octave. The ln 2 is a constant equal to 0.69315 and it may be combined with the 57.3 to give the following result:

$$\frac{d\beta}{dn} = \frac{-[2^{-6n}XY + 2^{-4n}(YZ - 3\delta X) + 2^{-2n}(\delta Z - 3Y) + \delta] 2^{-n}(39.72)}{2^{-8n}X^2 + 2^{-6n}(Y^2 - 2XZ) + 2^{-4n}(Z^2 + 2X - 2\delta Y) + 2^{-2n}(\delta^2 - 2Z) + 1} \frac{\text{degrees}}{\text{octave}} . \tag{402}$$

Equation (402) gives the slope of the β vs. n curve for any value of β. Since it is desirable to solve for the slope only at the value of β equal to 180 degrees:

$$\beta = 180^0 \text{ when } \arctan \frac{U}{V} = \pi$$

$$\frac{U}{V} = 0$$

$$U = 2^{-n}\delta - 2^{-3n}Y = 0$$

$$2^{-n} = \sqrt{\delta / Y}\,. \tag{403}$$

Substituting this value of 2^{-n} into Equation (402) gives the 180-degree phase-shift slope for any value of the parameters K, L, M, N, P, Q, S, and T.

$$\frac{d\beta}{dn} = \frac{-\left[\left(\frac{\delta}{Y}\right)^3 XY + \left(\frac{\delta}{Y}\right)^2 (YZ - 3\delta X) + \left(\frac{\delta}{Y}\right)(\delta Z - 3Y) + \delta\right]\sqrt{\frac{\delta}{Y}}\ (39.71)}{\left(\frac{\delta}{Y}\right)^4 X^2 + \left(\frac{\delta}{Y}\right)^3 (Y^2 - 2XZ) + \left(\frac{\delta}{Y}\right)^2 (Z^2 + 2X - 2\delta Y) + \left(\frac{\delta}{Y}\right)(\delta^2 - 2Z) + 1}$$

$$= \frac{-79.44\,(Y\delta)^{3/2}}{\delta YZ - X\delta^2 - Y^2}\,. \tag{404}$$

Equation (404) is reduced for taper factors of 0.1, 1, 10, and infinity in terms of the resistance parameter K (Table 5 gives the values of X, Y, Z, and δ):

Taper Factor = 0.1

$$\frac{d\beta}{dn} = \frac{-79.44\left[23\left(\frac{K_0}{K}\right)^2 + 28370\left(\frac{K_0}{K}\right) + 13563\right]^{3/2}}{\left[2828\left(\frac{K_0}{K}\right)^3 + 3492104\left(\frac{K_0}{K}\right)^2 + 7041364\frac{K_0}{K} + 3295688\right]} \tag{405}$$

Taper Factor = 1.0

$$\frac{d\beta}{dn} = \frac{-79.44\left[5\left(\frac{K_0}{K}\right)^2 + 47\left(\frac{K_0}{K}\right) + 18\right]^{3/2}}{\left[29\left(\frac{K_0}{K}\right)^3 + 284\left(\frac{K_0}{K}\right)^2 + 430\left(\frac{K_0}{K}\right) + 158\right]} \tag{406}$$

Taper Factor = 10

$$\frac{d\beta}{dn} = \frac{-79.44\left[3.2\left(\frac{K_0}{K}\right)^2 + 11.73\left(\frac{K_0}{K}\right) + 3.65\right]^{3/2}}{\left[9.272\left(\frac{K_0}{K}\right)^3 + 31.706\left(\frac{K_0}{K}\right)^2 + 33.764\left(\frac{K_0}{K}\right) + 11.284\right]} \tag{407}$$

Taper Factor = ∞

$$\frac{d\beta}{dn} = \frac{-79.44\left[3\left(\frac{K_0}{K}\right)^2 + 10\left(\frac{K_0}{K}\right) + 3\right]^{3/2}}{8\left(\frac{K_0}{K} + 1\right)^3}\,. \tag{408}$$

The functions X, Y, Z, and δ of Equations (377) through (380), tabulated at taper factors of 0.1, 1, 10, and infinity in terms of the resistance parameters K, L, M, and N are shown in Table 5.

TABLE 5

X, Y, Z, and δ as a Function of K				
Taper Factor	0.1	1	10	∞
X	$\frac{K_0}{K}$	$\frac{K_0}{K}$	$\frac{K_0}{K}$	$\frac{K_0}{K}$
Y	$23\frac{K_0}{K} + 11$	$5\frac{K_0}{K} + 2$	$3.2\frac{K_0}{K} + 1.1$	$3\frac{K_0}{K} + 1$
Z	$123\frac{K_0}{K} + 243$	$6\frac{K_0}{K} + 9$	$3.21\frac{K_0}{K} + 3.42$	$3\frac{K_0}{K} + 3$
δ	$\frac{K_0}{K} + 1233$	$\frac{K_0}{K} + 9$	$\frac{K_0}{K} + 3.321$	$\frac{K_0}{K} + 3$
X, Y, Z, and δ as a Function of L				
Taper Factor	0.1	1	10	∞
X	$\frac{L_0}{L}$	$\frac{L_0}{L}$	$\frac{L_0}{L}$	$\frac{L_0}{L}$
Y	$23\frac{L_0}{L} + 11$	$5\frac{L_0}{L} + 2$	$3.2\frac{L_0}{L} + 1.1$	$3\frac{L_0}{L} + 1$
Z	$133\frac{L_0}{L} + 233$	$7\frac{L_0}{L} + 8$	$3.31\frac{L_0}{L} + 3.32$	$3\frac{L_0}{L} + 3$
δ	$11\frac{L_0}{L} + 1223$	$2\frac{L_0}{L} + 8$	$1.1\frac{L_0}{L} + 3.221$	$\frac{L_0}{L} + 3$
X, Y, Z, and δ as a Function of M				
Taper Factor	0.1	1	10	∞
X	$\frac{M_0}{M}$	$\frac{M_0}{M}$	$\frac{M_0}{M}$	$\frac{M_0}{M}$
Y	$23\frac{M_0}{M} + 11$	$5\frac{M_0}{M} + 2$	$3.2\frac{M_0}{M} + 1.1$	$3\frac{M_0}{M} + 1$
Z	$133\frac{M_0}{M} + 233$	$7\frac{M_0}{M} + 8$	$3.31\frac{M_0}{M} + 3.32$	$3\frac{M_0}{M} + 3$
δ	$111\frac{M_0}{M} + 1123$	$3\frac{M_0}{M} + 7$	$1.11\frac{M_0}{M} + 3.211$	$\frac{M_0}{M} + 3$
X, Y, Z, and δ as a Function of N				
Taper Factor	0.1	1	10	∞
X	$\frac{N_0}{N}$	$\frac{N_0}{N}$	$\frac{N_0}{N}$	$\frac{N_0}{N}$
Y	$33\frac{N_0}{N} + 1$	$6\frac{N_0}{N} + 1$	$3.3\frac{N_0}{N} + 1$	$3\frac{N_0}{N} + 1$
Z	$343\frac{N_0}{N} + 23$	$10\frac{N_0}{N} + 5$	$3.43\frac{N_0}{N} + 3.2$	$3\frac{N_0}{N} + 3$
δ	$1111\frac{N_0}{N} + 123$	$4\frac{N_0}{N} + 6$	$1.111\frac{N_0}{N} + 3.21$	$\frac{N_0}{N} + 3$

The relation between frequency of oscillation and the resistance parameters has already been given by Equation (403) in the octave form as :

$$2^{-n} = \frac{1}{\frac{\omega}{\omega_0}} = \frac{1}{\omega CR} = \sqrt{\frac{\delta}{Y}}\,. \qquad (409)$$

Equation (409) is reduced for taper factors of 0.1, 1.0, 10, and infinity in terms of the resistance parameters K, L, M, and N (values of Y and δ are taken from Table 5) :

Frequency in Terms of K

Taper Factor = 0.1

$$2^n = \left(\frac{11\frac{K}{K_0} + 23}{1233\frac{K}{K_0} + 1}\right)^{\frac{1}{2}} \qquad (410)$$

Taper Factor = 1.0

$$2^n = \left(\frac{2\frac{K}{K_0} + 5}{9\frac{K}{K_0} + 1}\right)^{\frac{1}{2}} \qquad (411)$$

Taper Factor = 10

$$2^n = \left(\frac{1.1\frac{K}{K_0} + 3.2}{3.321\frac{K}{K_0} + 1}\right)^{\frac{1}{2}} \qquad (412)$$

Taper Factor = ∞

$$2^n = \left(\frac{\frac{K}{K_0} + 3}{3\frac{K}{K_0} + 1}\right)^{\frac{1}{2}} \qquad (413)$$

Frequency in Terms of L

Taper Factor = 0.1

$$2^n = \left(\frac{11\frac{L}{L_0} + 23}{1223\frac{L}{L_0} + 11}\right)^{\frac{1}{2}} \qquad (414)$$

Taper Factor = 1.0

$$2^n = \left(\frac{2\frac{L}{L_0} + 5}{8\frac{L}{L_0} + 2}\right)^{\frac{1}{2}} \qquad (415)$$

Taper Factor = 10

$$2^n = \left(\frac{1.1\frac{L}{L_0} + 3.2}{3.221\frac{L}{L_0} + 1.1}\right)^{\frac{1}{2}} \tag{416}$$

Taper Factor = ∞

$$2^n = \left(\frac{\frac{L}{L_0} + 3}{3\frac{L}{L_0} + 1}\right)^{\frac{1}{2}}. \tag{417}$$

Frequency in Terms of M

Taper Factor = 0.1

$$2^n = \left(\frac{11\frac{M}{M_0} + 23}{1123\frac{M}{M_0} + 111}\right)^{\frac{1}{2}} \tag{418}$$

Taper Factor = 1.0

$$2^n = \left(\frac{2\frac{M}{M_0} + 5}{7\frac{M}{M_0} + 3}\right)^{\frac{1}{2}} \tag{419}$$

Taper Factor = 10

$$2^n = \left(\frac{1.1\frac{M}{M_0} + 3.2}{3.211\frac{M}{M_0} + 1.11}\right)^{\frac{1}{2}} \tag{420}$$

Taper Factor = ∞

$$2^n = \left(\frac{\frac{M}{M_0} + 3}{3\frac{M}{M_0} + 1}\right)^{\frac{1}{2}}. \tag{421}$$

Frequency in Terms of N

Taper Factor = 0.1

$$2^n = \left(\frac{\frac{N}{N_0} + 33}{123\frac{N}{N_0} + 1111}\right)^{\frac{1}{2}} \tag{422}$$

Taper Factor = 1.0

$$2^n = \left(\frac{\frac{N}{N_0} + 6}{6\frac{N}{N_0} + 4}\right)^{\frac{1}{2}} \tag{423}$$

Taper Factor = 10

$$2^n = \left(\frac{\frac{N}{N_0} + 3.3}{3.21\frac{N}{N_0} + 1.111} \right)^{\frac{1}{2}} \tag{424}$$

Taper Factor = ∞

$$2^n = \left(\frac{\frac{N}{N_0} + 3}{3\frac{N}{N_0} + 1} \right)^{\frac{1}{2}}. \tag{425}$$

Establishment of General Equations for Determining Amplitude-Modulation Effects

The oscillator output voltage is dependent on the gain through the frequency-determining network; thus, it is necessary to find the network gain as a function of the resistance parameters since they are used to vary the oscillator frequency.

The second bracketed term of Equation (388) drops out because it is the imaginary part of the complex gain (Equation (381)) leaving:

$$(A_n)_{db} = -10 \log (2^{-4n}X - 2^{-2n}Z + 1)^2 \tag{426}$$

where X and Z are the same as in Equations (377) and (379).

Since oscillation occurs when

$$2^{-n} = \sqrt{\frac{\delta}{Y}}, \tag{427}$$

$$(A_n)_{db} = -20 \log \left| \left[\left(\frac{\delta}{Y}\right)^2 X - \left(\frac{\delta}{Y}\right) Z + 1 \right] \right|. \tag{428}$$

Equation (428) is reduced for taper factors of 0.1, 1.0, 10, and infinity in terms of the resistance parameter K (values of X, Y, Z, and δ are taken from Table 5).

Taper Factor = 0.1

$$(A)_{db} = -20 \log \left\{ \left(\frac{\frac{K_0}{K} + 1233}{23\frac{K_0}{K} + 11} \right)^2 \frac{K_0}{K} - \left(\frac{\frac{K_0}{K} + 1233}{23\frac{K_0}{K} + 11} \right) \left(123 \frac{K_0}{K} + 243 \right) + 1 \right\} \tag{429}$$

Taper Factor = 1.0

$$(A)_{db} = -20 \log \left\{ \left(\frac{\frac{K_0}{K} + 9}{5\frac{K_0}{K} + 2} \right)^2 \frac{K_0}{K} - \left(\frac{\frac{K_0}{K} + 9}{5\frac{K_0}{K} + 2} \right) \left(6\frac{K_0}{K} + 9 \right) + 1 \right\} \tag{430}$$

Taper Factor = 10

$$(A)_{db} = -20 \log \left\{ \left(\frac{\frac{K_0}{K} + 3.321}{3.2\frac{K_0}{K} + 1.1} \right)^2 \frac{K_0}{K} - \left(\frac{\frac{K_0}{K} + 3.321}{3.2\frac{K_0}{K} + 1.1} \right) \left(3.21\frac{K_0}{K} + 3.42 \right) + 1 \right\} \quad (431)$$

Taper Factor = ∞

$$(A)_{db} = -20 \log \left\{ \left(\frac{\frac{K_0}{K} + 3}{3\frac{K_0}{K} + 1} \right)^2 \frac{K_0}{K} - \left(\frac{\frac{K_0}{K} + 3}{3\frac{K_0}{K} + 1} \right) \left(3\frac{K_0}{K} + 3 \right) + 1 \right\} . \quad (432)$$

* * *

APPENDIX VII
ZERO-PHASE-SHIFT NETWORK
Derivation of General Equations

Using the methods described in Appendix I the matrix of Figure 67 is written down and expanded:

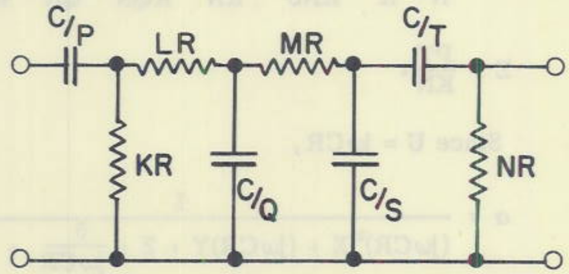


Figure 67

$X_C = 1/j\omega C = 1/pC = R/pCR = R/U.$

Letting $R = 1$,

$$X_C = \frac{1}{U}\ ;\ \frac{P}{U}\ ,\ \frac{T}{U}$$

$$Y_C = U\ ;\ \frac{U}{Q}\ ,\ \frac{U}{S}$$

$$Z_R = L\ ,\ M$$

$$Y_R = \frac{1}{K}\ ,\ \frac{1}{N}.$$

$$\begin{vmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{vmatrix} = \begin{vmatrix} 1 & \frac{P}{U} \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 \\ \frac{1}{K} & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & L \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 \\ \frac{U}{Q} & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & M \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 \\ \frac{U}{S} & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & \frac{T}{U} \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 \\ \frac{1}{N} & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 + \frac{P}{KU} & \frac{P}{U} \\ \frac{1}{K} & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 + \frac{LU}{Q} & L \\ \frac{U}{Q} & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 + \frac{MU}{S} & M \\ \frac{U}{S} & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 + \frac{T}{NU} & \frac{T}{U} \\ \frac{1}{N} & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \left(1 + \frac{P}{KU}\right)\left(1 + \frac{LU}{Q}\right) + \left(\frac{P}{U}\right)\left(\frac{U}{Q}\right) & \left(1 + \frac{P}{KU}\right)L + \frac{P}{U} \\ & \end{vmatrix} \begin{vmatrix} \left(1 + \frac{MU}{S}\right)\left(1 + \frac{T}{NU}\right) + \frac{M}{N} & \\ \left(1 + \frac{T}{NU}\right)\left(\frac{U}{S}\right) + \frac{1}{N} & \end{vmatrix}$$

$$\mathcal{A} = \left[\left(1 + \frac{P}{KU}\right)\left(1 + \frac{LU}{Q}\right) + \frac{P}{Q}\right]\left[\left(1 + \frac{MU}{S}\right)\left(1 + \frac{T}{NU}\right) + \frac{M}{N}\right] + \left[\left(1 + \frac{P}{KU}\right)L + \frac{P}{U}\right] \left[\left(1 + \frac{T}{NU}\right)\frac{U}{S} + \frac{1}{N}\right]. \tag{433}$$

Since the complex attenuation $\alpha = 1/\mathcal{A}$,

$$\alpha = \frac{1}{U^2 X + UY + Z + U^{-1}\delta + U^{-2}\Sigma} \tag{434}$$

where

$$X = \frac{LM}{QS} \tag{435}$$

$$Y = \frac{M}{S} + \frac{L}{Q} + \frac{LMT}{NQS} + \frac{LM}{QN} + \frac{PLM}{KQS} + \frac{PM}{QS} + \frac{L}{S} \tag{436}$$

$$Z = 1 + \frac{TM}{NS} + \frac{M}{N} + \frac{PL}{KQ} + \frac{PLTM}{KQNS} + \frac{PLM}{KQN} + \frac{P}{Q} + \frac{PTM}{NQS} + \frac{PM}{QN} + \frac{LT}{QN} + \frac{PM}{KS} + \frac{LT}{NS} + \frac{L}{N} + \frac{P}{S} + \frac{PL}{KS} \tag{437}$$

$$\delta = \frac{T}{N} + \frac{P}{K} + \frac{PTM}{KNS} + \frac{PM}{KN} + \frac{PLT}{KQN} + \frac{PT}{QN} + \frac{PLT}{KNS} + \frac{PL}{KN} + \frac{PT}{NS} + \frac{P}{N} \tag{438}$$

$$\Sigma = \frac{PT}{KN}. \tag{439}$$

Since $U = j\omega CR$,

$$\alpha = \frac{1}{(j\omega CR)^2 X + (j\omega CR)Y + Z + \frac{\delta}{j\omega CR} + \frac{\Sigma}{(j\omega CR)^2}}$$

$$\alpha = \frac{1}{\left[Z - \frac{\Sigma}{(\omega CR)^2} - (\omega CR)^2 X \right] + j\left[(\omega CR)Y - \frac{\delta}{\omega CR} \right]}. \tag{440}$$

On expressing Equation (440) in terms of a reference frequency ω_0 (where $\omega_0 = 1/RC$) and rewriting in octave form as described in Appendix II the following equation results:

$$\alpha = \frac{1}{[Z - 2^{-2n}\Sigma - 2^{2n}X] - j[2^{-n}\delta - 2^{n}Y]} \tag{441}$$

$$\alpha = \frac{1}{a - jb} = \frac{a}{a^2 + b^2} + j\frac{b}{a^2 + b^2} \tag{442}$$

where

$$a = (Z - 2^{-2n}\Sigma - 2^{2n}X) \tag{443}$$

$$b = (2^{-n}\delta - 2^{n}Y). \tag{444}$$

The equations for phase shift and absolute attenuation are immediately found as:

$$(A) = \left[\left(\frac{a}{a^2 + b^2}\right)^2 + \left(\frac{b}{a^2 + b^2}\right)^2\right]^{\frac{1}{2}} = (a^2 + b^2)^{-\frac{1}{2}}, \tag{445}$$

$$\beta = \arctan\frac{b}{a}. \tag{446}$$

The absolute attenuation may be expressed in decibels as:

$$(A_n)_{db} = -10 \log (a^2 + b^2). \tag{447}$$

Replacing the values for a and b found in Equations (443) and (444) into the phase-shift and attenuation Equations (445) and (446),

$$(A_n)_{db} = -10 \log\left\{(Z - 2^{-2n}\Sigma - 2^{2n}X)^2 + (2^{-n}\delta - 2^{n}Y)^2\right\}, \tag{448}$$

$$\beta_n = \arctan \frac{2^{-n}\delta - 2^{+n}Y}{Z - 2^{-2n}\Sigma - 2^{2n}X}. \tag{449}$$

Equations (448) and (449) are reduced for taper factors of 1, 10, and infinity:

Taper Factor = 1

$$(A_n)_{db} = -10 \log\ (2^{4n} + 19{\cdot}2^{2n} + 87 + 70{\cdot}2^{-2n} + 2^{-4n}), \tag{450}$$

$$\beta_n = \arctan \frac{10{\cdot}2^{-n} - 7{\cdot}2^{n}}{-(2^{2n} - 15 + 2^{-2n})}. \tag{451}$$

Taper Factor = 10

$$(A_n)_{db} = -10 \log (2^{4n} + 5.41{\cdot}2^{2n} + 7.591 + 5.591041{\cdot}2^{-2n} + 2^{-4n}), \tag{452}$$

$$\beta_n = \arctan \frac{4.321{\cdot}2^{-n} - 4.3{\cdot}2^{n}}{-(2^{-2n} - 6.54 + 2^{2n})} \tag{453}$$

Taper Factor = ∞

$$(A_n)_{db} = -10 \log (2^{4n} + 4{\cdot}2^{2n} + 6 + 4{\cdot}2^{-2n} + 2^{-4n}) \tag{454}$$

$$\beta_n = \arctan \frac{4{\cdot}2^{-n} - 4{\cdot}2^{n}}{-(2^{2n} - 6 + 2^{-2n})}. \tag{455}$$

Establishment of General Equations for Determining Oscillator Stability

$$\beta_n = \arctan \frac{2^{-n}\delta - 2^{n}Y}{Z - 2^{-2n}\Sigma - 2^{2n}X} = \arctan \frac{U}{V} \tag{456}$$

$$\frac{d\beta}{dn} = \frac{VdU - UdV}{U^2 + V^2} \tag{457}$$

$$\frac{d\beta}{dn} = \frac{(Z - 2^{-2n}\Sigma - 2^{2n}X)\ (-\delta \ln 2{\cdot}2^{-n} - Y \ln 2{\cdot}2^{n}) - (2^{-n}\delta - 2^{n}Y)\ (2\Sigma \ln 2{\cdot}2^{-2n} - 2X \ln 2{\cdot}2^{2n})}{(2^{-n}\delta - 2^{n}Y)^2 + (Z - 2^{-2n}\Sigma - 2^{2n}X)^2}$$

$$\frac{d\beta}{dn} = \frac{-[2^{3n}(XY) + 2^{n}(YZ - 3X\delta) + 2^{-n}(\delta Z - 3\Sigma Y) + 2^{-3n}(\Sigma\delta)] \ln 2 (57.3)}{2^{4n}(X^2) + 2^{2n}(Y^2 - 2XZ) + (Z^2 + 2X\Sigma - 2Y\delta) + 2^{-2n}(\delta^2 - 2Z\Sigma) + 2^{-4n}(\Sigma)^2}. \tag{458}$$

The last equation has the factor (57.3) in order that it be in units of degrees per octave rather than radians per octave.

Equation (458) gives the slope of the β vs. n curve for any value of β. Since it is desirable to solve for the slope only at the value of β equal to 180 degrees,

$$\beta = 0^{\circ} \text{ when } \arctan U/V = 0$$

$$\frac{U}{V} = 0$$

$$U = 2^{-n}\delta - 2^{n}Y = 0$$

$$2^{n} = \left(\frac{\delta}{Y}\right)^{\frac{1}{2}}. \qquad (459)$$

Substitution of this value of 2^n in Equation (458) gives the zero-degree phase-shift slope for any values of the parameters K, L, M, N, P, Q, S, and T.

$$\frac{d\beta}{dn} = \frac{-\left[\left(\frac{\delta}{Y}\right)^{3/2}(XY) + \left(\frac{\delta}{Y}\right)^{1/2}(YZ - 3X\delta) + \left(\frac{\delta}{Y}\right)^{-1/2}(\delta Z - 3\Sigma Y) + \left(\frac{\delta}{Y}\right)^{-3/2}\Sigma\delta\right](\ln 2)(57.3)}{\left(\frac{\delta}{Y}\right)^{2}X^{2} + \left(\frac{\delta}{Y}\right)(Y^{2} - 2XZ) + (Z^{2} + 2X\Sigma - 2Y\delta) + \left(\frac{\delta}{Y}\right)^{-1}(\delta^{2} - 2Z\Sigma) + \left(\frac{\delta}{Y}\right)^{-2}\Sigma^{2}} \qquad (460)$$

Establishment of Equations for Frequency in Terms of the Resistance Parameters K, L, M, and N

The relation between frequency of oscillation and the resistance parameters has already been established by Equation (459) in the octave form as:

$$2^{n} = \frac{\omega}{\omega_0} = \omega CR = \left(\frac{\delta}{Y}\right)^{\frac{1}{2}}.$$

This equation is reduced for taper factors of 1 and 10 in terms of the resistance parameters K, L, M, and N (values of Y and δ in terms of the resistance parameters are given in Table 6):

Frequency in Terms of K

Taper Factor = 1

$$2^{n} = \left(\frac{4\frac{K}{K_0} + 6}{6\frac{K}{K_0} + 1}\right)^{\frac{1}{2}} \qquad (461)$$

Taper Factor = 10

$$2^{n} = \left(\frac{1.111\frac{K}{K_0} + 3.21}{3.3\frac{K}{K_0} + 1}\right)^{\frac{1}{2}}. \qquad (462)$$

Frequency in Terms of L

Taper Factor = 1

$$2^{n} = \left(\frac{3\frac{L}{L_0} + 7}{5\frac{L}{L_0} + 2}\right)^{\frac{1}{2}} \qquad (463)$$

Taper Factor = 10

$$2^n = \left(\frac{1.11 \frac{L}{L_0} + 3.211}{3.2 \frac{L}{L_0} + 1.1} \right)^{\frac{1}{2}} . \qquad (464)$$

Frequency in Terms of M

Taper Factor = 1

$$2^n = \left(\frac{2 \frac{M}{M_0} + 8}{5 \frac{M}{M_0} + 2} \right)^{\frac{1}{2}} \qquad (465)$$

Taper Factor = 10

$$2^n = \left(\frac{1.1 \frac{M}{M_0} + 3.221}{3.2 \frac{M}{M_0} + 1.1} \right)^{\frac{1}{2}} . \qquad (466)$$

Frequency in Terms of N

Taper Factor = 1

$$2^n = \left(\frac{\frac{N}{N_0} + 9}{5 \frac{N}{N_0} + 2} \right)^{\frac{1}{2}} \qquad (467)$$

Taper Factor = 10

$$2^n = \left(\frac{\frac{N}{N_0} + 3.321}{3.2 \frac{N}{N_0} + 1.1} \right)^{\frac{1}{2}} . \qquad (468)$$

The functions X, Y, Z, δ, and Σ, of Equations (435) through (439), tabulated at taper factors of 1, 10, and infinity in terms of resistance parameters K, L, M, and N are shown in Table 6.

Establishment of General Equations for Determining Amplitude-Modulation Effects

The oscillator output voltage is dependent on the attenuation through the frequency-determining network; thus, it is necessary to find the network attenuation as a function of the resistance parameters, since they are used to vary the oscillator frequency.

The second bracketed term of Equation (448) drops out because it is the imaginary part of the complex attenuation, leaving:

$$\alpha = \frac{1}{Z - 2^{-2n}\Sigma - 2^{2n}X}$$

TABLE 6

X, Y, Z, δ, and Σ as a Function of K

Taper Factor	X	Y	Z	δ	Σ
1	1	$\frac{K_0}{K} + 6$	$5\frac{K_0}{K} + 10$	$6\frac{K_0}{K} + 4$	$\frac{K_0}{K}$
10	1	$\frac{K_0}{K} + 3.3$	$3.2\frac{K_0}{K} + 3.43$	$3.21\frac{K_0}{K} + 1.111$	$\frac{K_0}{K}$
∞	1	$\frac{K_0}{K} + 3$	$3\frac{K_0}{K} + 3$	$3\frac{K_0}{K} + 1$	$\frac{K_0}{K}$

X, Y, Z, δ, and Σ as a Function of L

Taper Factor	X	Y	Z	δ	Σ
1	$\frac{L}{L_0}$	$5\frac{L}{L_0} + 2$	$7\frac{L}{L_0} + 8$	$3\frac{L}{L_0} + 7$	1
10	$\frac{L}{L_0}$	$3.2\frac{L}{L_0} + 1.1$	$3.31\frac{L}{L_0} + 3.32$	$1.11\frac{L}{L_0} + 3.211$	1
∞	$\frac{L}{L_0}$	$3\frac{L}{L_0} + 1$	$3\frac{L}{L_0} + 3$	$\frac{L}{L_0} + 3$	1

X, Y, Z, δ, and Σ as a Function of M

Taper Factor	X	Y	Z	δ	Σ
1	$\frac{M}{M_0}$	$5\frac{M}{M_0} + 2$	$7\frac{M}{M_0} + 8$	$2\frac{M}{M_0} + 8$	1
10	$\frac{M}{M_0}$	$3.2\frac{M}{M_0} + 1.1$	$3.31\frac{M}{M_0} + 3.32$	$1.1\frac{M}{M_0} + 3.221$	1
∞	$\frac{M}{M_0}$	$3\frac{M}{M_0} + 1$	$3\frac{M}{M_0} + 3$	$\frac{M}{M_0} + 3$	1

X, Y, Z, δ, and Σ as a Function of N

Taper Factor	X	Y	Z	δ	Σ
1	1	$2\frac{N_0}{N} + 5$	$9\frac{N_0}{N} + 6$	$9\frac{N_0}{N} + 1$	$\frac{N_0}{N}$
10	1	$1.1\frac{N_0}{N} + 3.2$	$3.42\frac{N_0}{N} + 3.21$	$3.321\frac{N_0}{N} + 1$	$\frac{N_0}{N}$
∞	1	$\frac{N_0}{N} + 3$	$3\frac{N_0}{N} + 3$	$3\frac{N_0}{N} + 1$	$\frac{N_0}{N}$

where X, Z, and Σ are the same as in Equations (435), (437), and (439). Since oscillation occurs when:

$$2^n = \left(\frac{\delta}{Y}\right)^{\frac{1}{2}}$$

$$A_{db} = -20 \log \left| Z - \frac{Y}{\delta}\Sigma - \frac{\delta}{Y} X \right|. \qquad (469)$$

Equation (469) is reduced for taper factors of 1 and 10 in terms of the resistance parameter K (values of X, Y, Z, δ, and Σ are taken from Table 6):

Taper Factor = 1

$$(A)_{db} = -20 \log \left| \left[\left(5\frac{K_0}{K} + 10\right) - \frac{4\frac{K}{K_0} + 6}{6\frac{K}{K_0} + 1} - \left(\frac{6\frac{K}{K_0} + 1}{4\frac{K}{K_0} + 6}\right) 1\frac{K_0}{K} \right] \right| \qquad (470)$$

Taper Factor = 10

$$(A)_{db} = -20 \log \left| \left[\left(3.2\frac{K_0}{K} + 3.43\right) - \frac{1.111\frac{K}{K_0} + 3.21}{3.3\frac{K}{K_0} + 1} - \left(\frac{3.3\frac{K}{K_0} + 1}{1.111\frac{K}{K_0} + 3.21}\right) 1\frac{K_0}{K} \right] \right| \cdot \qquad (471)$$

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