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ANTENNA WAVEFRONT PROBLEMS

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CONTENTS

ABSTRACT

By means of a vector notation for surfaces, relations are derived among an incident wavefront, reflector, and reflected wavefront. A method is introduced for evaluating the deviation of a wavefront surface from a plane. A number of problems, including an analysis of the wavefront from a Foster Scanner antenna, are included in order to indicate the simplicity and utility of the analysis.

PROBLEM STATUS

This is an interim report; work on the problem is continuing.

AUTHORIZATION

NRL Problem R02-23R
NR 502-230

ANTENNA WAVEFRONT PROBLEMS

INTRODUCTION

This report outlines a method of analysis of three-dimensional antenna wavefronts. An application is made to the Foster Scanner antenna as well as to several other designs of similar nature. In a previous paper,[1] the author considered the problem of a two-dimensional wavefront, which was a curve orthogonal to rays from a general reflecting curve. The present investigation is an extension of the former treatment to include wavefronts and reflectors which are general, nonsingular surfaces. In particular, this analysis relates an incident wavefront, a reflector, and a reflected wavefront so that given any two of these surfaces, the third can be determined.

The methods used are based on a vector notation for the surfaces. Instead of the familiar representation

$$z = f(x, y), \tag{1}$$

the surface will be written as a vector with rectangular components

$$A = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}. \tag{2}$$

Although such a notation is quite generally used,[2] the reader unfamiliar with it may refer to Figure 1 and the following explanation.

Each point on the surface in Figure 1a has a certain value of (x, y, z) corresponding to it. The same point in space could be reached, as in Figure 1b by following the three vectors, $x\mathbf{i}$, $y\mathbf{j}$, $z\mathbf{k} = f(x, y)\mathbf{k}$, whose sum is A. It should be noted that the vector representing any surface is defined by the use of only two variables. This is evident for any surface which can be written in the form of Equation (1). However the more general surface vector would be written

$$X(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k},$$

where u and v are arbitrary parameters.

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[1] *Kelleher, K. S., "Analysis of Antenna Wavefronts," NRL Report R-3329, July 29, 1948*

[2] *Blaschke, W., "Differential-Geometrie," New York, Dover Publications, 1945*

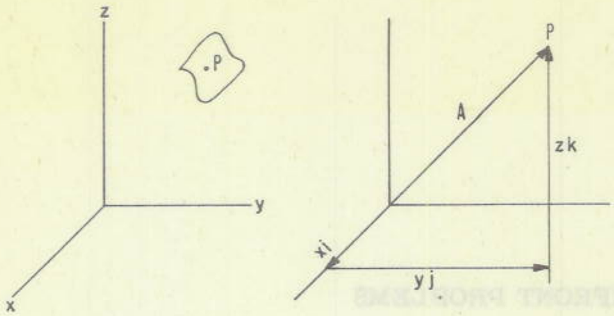

Figure 1 - Vector notation for surfaces

THE REFLECTED WAVEFRONT

Given an incident wavefront surface, X(u, v) (Figure 2) and a reflector R(s, t), the reflected wavefront Y(u, v) can be obtained by using the wavefront definition that all points of Y(u, v) lie at equal ray path length from X(u, v), together with the fact that rays are orthogonal to the wavefront surfaces. From Figure 2, two values of the general reflected ray can be obtained and equated.

$$Y - R = (W - |R - X|)\,\xi, \tag{3}$$

where W is the total optical path length from Y to X, |R-X| is the length of the incident ray, and ξ is a unit normal to the reflected wavefront, Y. The unit vector, ξ , is determined from the fact that incident and reflected angles at the reflector surface are equal.

$$\xi = \frac{R - X}{|R - X|} - 2\,n\left\langle n.\,\frac{R - X}{R - X}\right\rangle, \tag{4}$$

where n is the unit normal to the reflector at the general point (s, t). The proof of this expression, due to Silberstein,[3] is included in Appendix I.

Substituting Equation (4) in Equation (3) and reducing, there results

$$Y = W\xi + X + 2n\,[n\cdot(R-X)]\,. \tag{5}$$

For most practical purposes, it is convenient to consider the reflected wavefront surface at zero optical distance (W = 0) from the incident wavefront. This fictitious wavefront is a parallel surface[4] to any actual reflected wavefront and can therefore be used interchangeably in many problems.

With W = 0, Equation (5) reduces to

$$Y(u, v, s, t) = X(u, v) + 2\,n(s, t)\left[\,n(s, t)\cdot\left\{R(s, t) - X(u, v)\right\}\right]. \tag{5a}$$

The parentheses here are used to indicate the dependence of the functions on the various parameters involved. In order that Y be a function of two parameters only, a relation between (u, v) and (s, t) is required. This auxiliary expression is obtained from the condition that an incident ray be normal to the incident wavefront. Therefore the scalar product of a ray and a tangent vector vanishes. Analytically there results,

$$(R-X)\cdot X_u = 0 \text{ and } (R-X)\cdot X_v = 0, \tag{6}$$

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[3] *Silberstein, L., "Simplified Method of Tracing Rays Through Any Optical System," p. 1, Longmans, Green and Company, 1918.*

[4] *Eisenhart, L. P., "An Introduction to Differential Geometry," p. 272, Princeton, Princeton University Press, 1940*

where the subscript denotes partial differentiation. In general we may solve these equations for $s = s(u, v)$ and $t = t(u, v)$ which can be substituted into Equation (5a) to give the desired wavefront surface $Y(u, v)$. Note that when the incident wavefront is spherical, that is, produced by a point source, the defining vector X is a constant and Equation (5a) can be used directly to give $Y(s, t)$.

Since the incident and reflected wavefronts could be interchanged, without changing the above treatment, an expression similar to Equation (5a) yields the incident wavefront when the reflector and reflected wavefront surfaces are known.

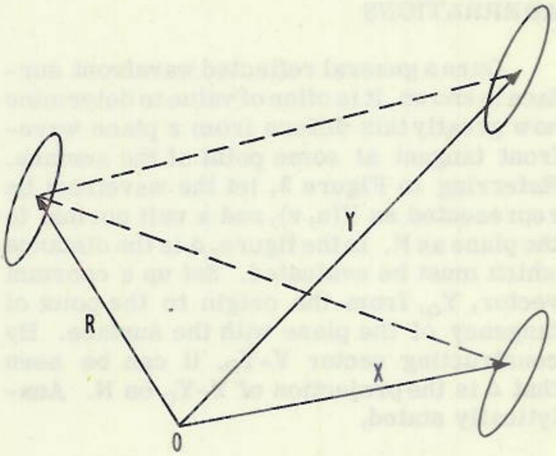

Figure 2 - Geometry of the wavefront problem (dotted lines indicate ray path)

THE REFLECTOR

The remaining problem in relating our three surfaces is that of determining the reflector which transforms a given incident wavefront into a given reflected wavefront. In order to do this, use is made of Equation (3) with the sign of unit normal, ξ , reversed:

$$R = Y + (W - |R-X|)\xi \ , \ X = X(p, q). \tag{7}$$

In order to evaluate $|R-X|$, X should be subtracted from both sides of Equation (7) after which both sides are squared. The result reduces to

$$|R - X| = \frac{W^2 + (Y-X)^2 + 2W(Y-X)\cdot\xi}{2\,[W + \xi\cdot(Y-X)]} \ .$$

Upon substitution of this, Equation (7) becomes

$$R = Y + \frac{W^2 - (Y-X)^2}{2\,[W + \xi\cdot(Y-X)]}\xi = Y + G\xi \ . \tag{7a}$$

In order to obtain R as a function of u and v alone, use is made of the condition that $R-X$ is normal to the surface X.

$$(R-X)\cdot X_u = (R-X)\cdot X_v = 0$$

or

$$(Y-X + G\xi)\cdot X_u = 0 \text{ and } (Y-X + G\xi)\cdot X_v = 0\,\cdot$$

These two expressions can be solved for p and q in terms of u and v so that Equation (7a) defines the reflector R as a function of u and v alone. As before, if one of the wavefronts is spherical, emanating from a point source, its defining vector is a constant, therefore Equation (7a) gives R in terms of only two parameters.

ABERRATIONS

Once a general reflected wavefront surface is known, it is often of value to determine how greatly this differs from a plane wavefront tangent at some point of the surface. Referring to Figure 3, let the wavefront be represented as $Y(u, v)$, and a unit normal to the plane as N. In the figure, Δ is the distance which must be evaluated. Set up a constant vector, Y_O, from the origin to the point of tangency of the plane with the surface. By constructing vector $Y-Y_O$, it can be seen that Δ is the projection of $Y-Y_O$ on N. Analytically stated,

$$\Delta(u, v) = |Y - Y_O| \cos\beta = (Y - Y_O) \cdot N. \quad (8)$$

Therefore for any point (u, v) of the surface, the aberration can be determined.

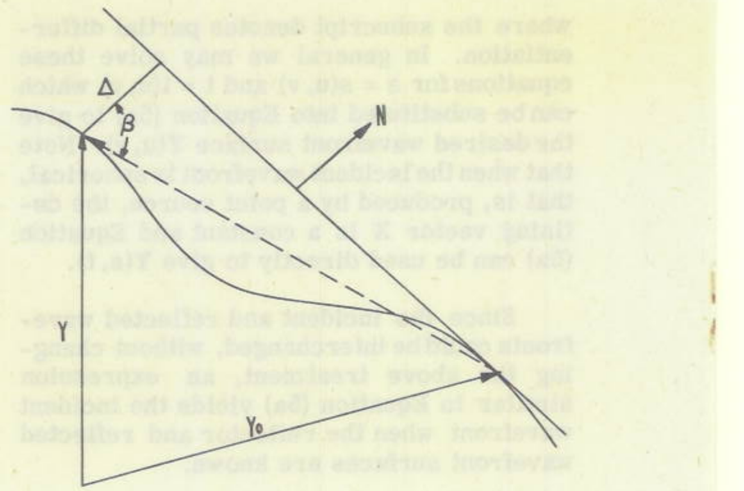

Figure 3 - Geometry of the aberration problem

If it is difficult to determine the value of a Y_O vector, any value, such as $Y_O = 0$, may be chosen. The Δ function will then have some constant phase error which may be subtracted out. Any constant in the phase function can be eliminated since such a subtraction merely shifts the position of the plane so that it more closely fits the surface.

APPLICATION

Cylindrical Parabola Reflector

The remainder of this report will be devoted to applications of the theory in an effort to demonstrate its simplicity and utility.

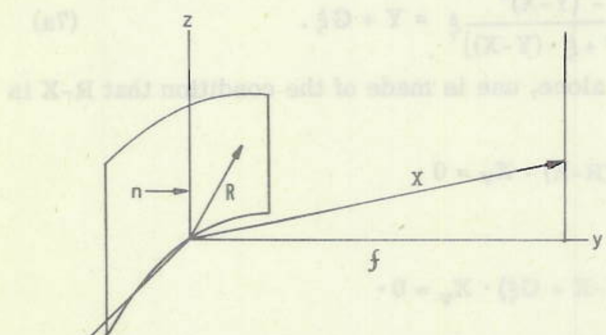

FIG 4

Figure 4 - Cylindrical parabola with line source

One of the simplest type of reflectors is a parabolic cylinder. Using the previous notation, its defining vector and unit normal are (Figure 4)

$$R = si + \frac{s^2}{4f} j + zk,$$

$$n = \frac{si - 2fj}{\sqrt{s^2 + 4f^2}}.$$

<u>Line Source</u>—One possible feed for this reflector consists of a line source parallel to the cylinder elements and placed at the focus. Its equation is (Figure 4)

$$X = fj + vk.$$

The reflected wavefront can be obtained from

$$Y = X + 2n[\, n \cdot (R - X)] \tag{5a}$$

by substituting the above values of X, R, and n. After some reduction this becomes

$$Y = si - fj + vk$$

which requires no auxiliary relation since it is a function of two parameters only. This is a plane perpendicular to the y axis at the point -f, which may be verified by showing that the deviation of this surface from the plane is zero.

In order to do this, use is made of Equation (8) with N = j and the constant vector, $Y_O = -fj$. Then the expression for the deviation is

$$\Delta = (Y - Y_O) \cdot N;$$

$$\Delta = (si + vk) \cdot j = 0.$$

Point Source—If the feed is a point source located on the focal line, the reflected wavefront can be obtained from the previous example. The feed vector, X, is now fj. This means that the previous expression for X can be used with v = 0. Placing v = 0 in the previous reflected wavefront, there results

$$Y = si - fj$$

which is a line intersecting the y axis at -f and parallel to the x axis. The parallel surfaces to this wavefront are circular cylinders, therefore a point source feeding a parabolic cylinder gives a cylindrical wavefront.

Phased Line Source—In order to determine the wavefront from the Foster Scanner antenna, a line source, positioned as before along the focus of the parabolic cylinder, should be considered. If this source has a linear phase distribution, a conical wavefront is produced (Appendix II). In such case, the incident wavefront (Figure 5) is given by

$$X = \left[(h-v)^2 \tan^2\alpha - (u-f)^2\right]^{\frac{1}{2}} i + uj + vk$$

$$X = Fi + uj + vk.$$

Upon substitution of X, R, and n into Equation (5a) on page 2, the reflected wavefront is

$$Y = \left[\frac{s(s^2 + 4uf) - (s^2 - 4f^2)F}{s^2 + 4f^2}\right] i + \left[\frac{s^2(u - 2f) - 4uf^2 + 4fsF}{s^2 + 4f^2}\right] j + vk.$$

In this case, an auxiliary relation is required and may be obtained from the fact that $(R - X) \cdot X_u = 0$. This relation turns out to be

$$(s^2 - 4f^2)F = 4fs(u - f).$$

When it is substituted, the expression for Y becomes

$$Y = si + \left(\frac{f - u}{\cos\theta} - f\right) j + vk$$

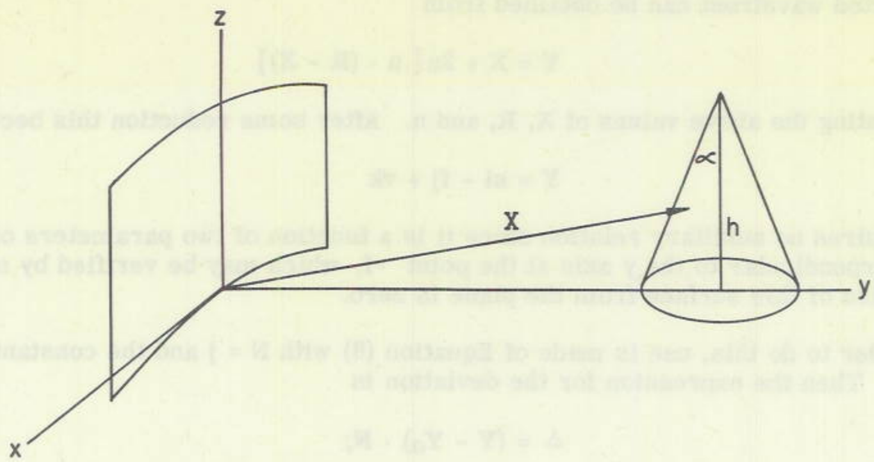

Figure 5 - Cylindrical parabola with conical wavefront

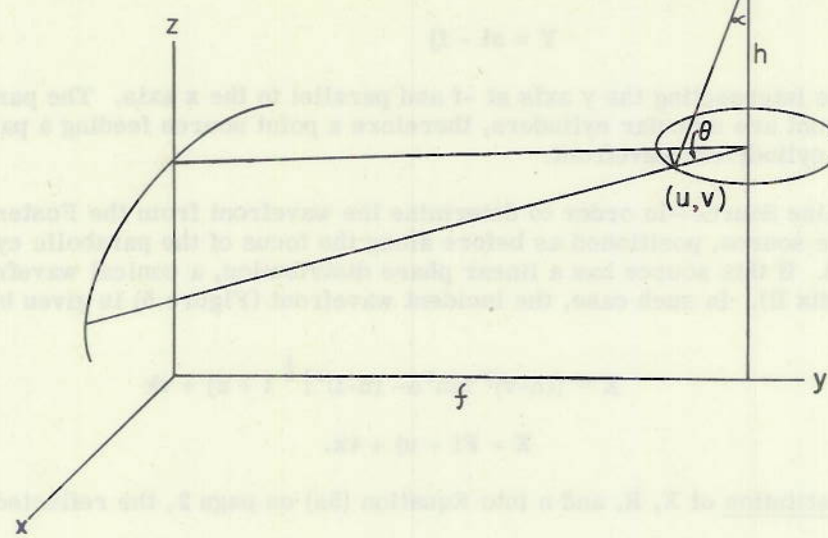

Figure 6 - Relation between reflector angle, θ, and cone angle, α

where $\cos\theta = [f - (s^2/4f)]/[f + (s^2/4f)]$. Figure 6 indicates the relation between $\cos\theta$ and the parameters which are being used. From the figure, it can be determined that

$$\frac{f - u}{\cos\theta} = (h - v)\tan\alpha,$$

and therefore the reflected wavefront can be written

$$Y = si - [(f - h\tan\alpha) + v\tan\alpha]j + vk.$$

This defines a plane which intersects the x - y plane in a line $y = (f - h\tan\alpha)$ and at an angle $\pi - \alpha$.

As before, in order to prove that this wavefront surface is a plane, it must be demonstrated that the deviation is zero. The plane inclined at an angle $\pi - \alpha$ has a unit normal

$$N = (\cos\alpha)j + (\sin\alpha)k.$$

The constant vector to a point on the plane is

$$Y_O = -(f - h\tan\alpha)j.$$

The deviation can now be found from

$$\Delta = (Y - Y_O)\cdot N.$$

$$\Delta = (is - jv\tan\alpha + kv)\cdot(j\cos\alpha + k\sin\alpha),$$

$$\Delta = -v\tan\alpha\cos\alpha + v\sin\alpha = 0.$$

Therefore, the reflected wavefront is a plane inclined at an angle which is the supplement of the cone angle, α.

Tilted Line Source—For some purpose, it may be desirable to rotate the previously described simple line source about some point on the focal line (Figure 7). In such a case, the feed vector is

$$X = (f + v\tan\alpha)j + vk.$$

The reflected wavefront is

$$Y = (s + v\tan\alpha\sin\theta)i - (f + v\tan\alpha\cos\theta)j + vk.$$

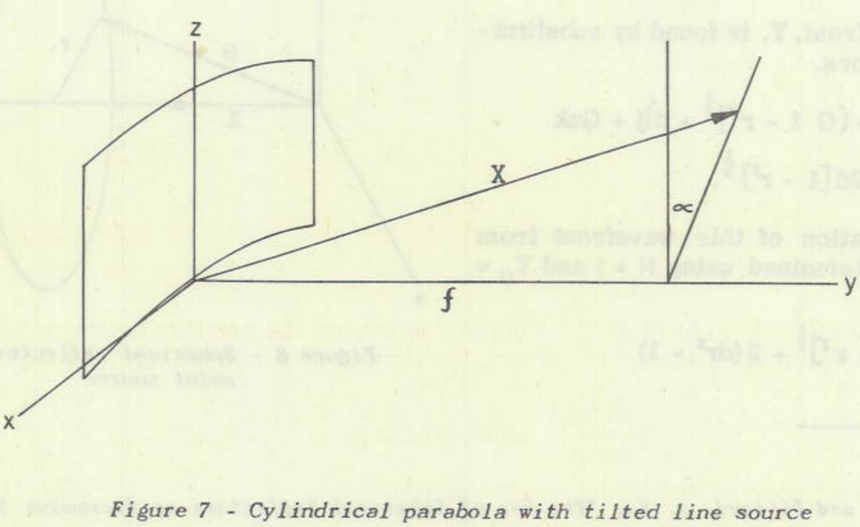

Figure 7 - Cylindrical parabola with tilted line source

Here θ is the same angle as shown in Figure 6. It is related to the f/D ratio ($\tan\theta/2 = D/4f$) and will be retained in order to visualize aberration as a function of f/D. The aberration will be expressed as deviation of this surface from the same plane used in the previous example. It has unit normal,

$$N = j\cos\alpha + k\sin\alpha.$$

In this case, the plane will be positioned by the constant vector $Y_O = fj$.

With such a choice, the deviation is

$$\Delta = v\sin\alpha\,(1 - \cos\theta).$$

As a practical example, for f/D ratio of 0.6, $\alpha = \pm 10^\circ$, and source length 10λ ; the error is about $\pm\lambda/4$ as a maximum.

Spherical Reflector

When a point source is used to feed a spherical reflector, it is possible to obtain a nearly plane wavefront characteristic by placing the feed near the half-radius point of the sphere. Ashmead and Pippard[5] investigated this problem using the classical analysis. The method of this paper enables one to obtain more accurate results.

Figure 8 is a sketch of a spherical reflector with center at the origin. For convenience all points are normalized to the radius of the sphere.

With the same notation as before, the reflector, unit normal, and feed vectors may be written

$$R = xi + [1 - r^2]^{\frac{1}{2}} j + zk$$

$$u = R$$

$$X = jd$$

where $r^2 = x^2 + z^2$.

The wavefront, Y, is found by substituting these vectors.

$$Y = Gxi + \left(G\,1 - r^2]^{\frac{1}{2}} + d\right)j + Gzk$$

where $G = 2 - 2d[1 - r^2]^{\frac{1}{2}}$.

The deviation of this wavefront from a plane is determined using $N = j$ and $Y_O = (2 - d)j$.

$$\Delta = 2[1 - r^2]^{\frac{1}{2}} + 2(dr^2 - 1)$$

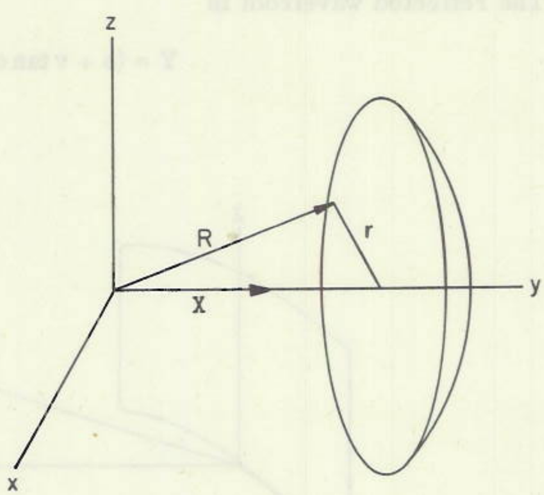

Figure 8 - Spherical reflector with point source

[5] *Ashmead, J. and Pippard, A. B., "The Use of Spherical Reflectors as Microwave Scanning Aerials" JIEE, March-May 1946*

It is evident that $\Delta(0) = 0$, but Δ will also vanish at another value of r if d, the feed position, is properly chosen. Setting $\Delta = 0$ and solving for d, there results

$$d = \frac{1 - (1 - r^2)^{\frac{1}{2}}}{r^2}.$$

Therefore, given a value of r for which it is desired to make the deviation zero, d can be easily calculated. Several values are tabulated below:

r	d
0.2	.503
0.4	.522
0.6	.556
0.8	.625

It is evident that as r increases, the feed position moves nearer the reflector. For a shallow reflector, which corresponds to a large f/D ratio, the feed is placed near the half-radius point, while as the f/D ratio decreases, the feed moves in toward the reflector.

Once the value of r and d have been established, it is possible to evaluate the maximum value of $\Delta(r)$ in the usual manner. The result for a reflector of unit radius is

$$\Delta_{max} = \frac{2}{d}(d - 0.5)^2,$$

and for a reflector of radius, R

$$\Delta_{max} = \frac{2}{d}(d - 0.5)^2 R.$$

As a practical example, for f/D ratio of 0.6, r is approximately 0.4 and d = 0.522. The maximum deviation is then

$$\Delta_{max} = \frac{2}{0.522}(.022)^2 R = 0.0018R.$$

In order to place a value on R, note that r = 0.4R, therefore,

$$R = \frac{r}{0.4} = \frac{D}{0.8}$$

where D is the reflector aperature. For an aperture of 20 wavelengths, the deviation from a plane is

$$\Delta_{max} = (0.0018)\frac{20\lambda}{0.8} = 0.045\lambda.$$

The value of this deviation, given by Ashmead and Pippard was

$$\Delta_{max} = \frac{1}{(f/D)^3}\frac{D}{2000} = 0.046\lambda.$$

Reflectors Which Yield Virtual Point Sources

A catalogue of the reflectors which will produce a spherical wavefront from a point source can be obtained from Equation (7a). In terms of this equation, two point sources are given and it is desired to determine the reflectors which focus rays from one point to the other. By choosing the ray path length, W, to be less than the distance between the sources, other reflectors are obtained which produce virtual sources.

From the analysis, the reflector desired is

$$R = Y + \frac{W^2 - (Y - X)\cdot(Y - X)}{2\left[W + \xi \cdot (Y - X)\right]}\,\xi\,. \tag{7a}$$

X and Y, the incident and reflected wavefronts, are in this case points. Let $X = 0$ and $Y = ai$. The unit normal to the Y wavefront is

$$\xi = (x - a)i + yj + zk$$

where $y = \left[1 - z^2 - (x - a)^2\right]^{\frac{1}{2}}$.

Substituting these values, the reflector is

$$R = r_1 i + r_2 j + r_3 k$$

$$R = \frac{\left[(x - a)(W^2 + a^2) + 2aW\right]i + y(W^2 - a^2)j + z(W^2 - a^2)k.}{2\left[W + a(x - a)\right]}$$

It is possible to eliminate the parameters here, and obtain

$$\frac{(r_1 - a/2)^2}{W^2} + \frac{r_2^2 + r_3^2}{W^2 - a^2} = \frac{1}{4}\,.$$

From inspection of this equation, the following tabulation may be made. For $W > a$, the reflector is an ellipsoid, unless $a = 0$, in which case it is a sphere of radius $W/2$.

For $W < a$, the reflector is a hyperboloid of revolution, unless $W = 0$, in which case it is a plane, $r_1 = a/2$.

For $W = a$, the result is a point reflector at the origin, the position of the second source.

* * *

APPENDIX I
The Reflected Unit Vector[6]

From geometrical optics, the incident ray, normal to the reflector, and reflected ray all lie in the same plane (Figure 9). Since the incident and reflected angles are equal, it can be seen that

$$n \cdot r = - n \cdot \xi$$

$$\xi \times n = r \times n.$$

Taking the vector product of the last equation by n this results,

$$n \times (\xi \times n) = n \times (r \times n)$$

or

$$\xi (n \cdot n) - n(n \cdot \xi) = r (n \cdot n) - n(n \cdot r).$$

Substituting for $n \cdot \xi$ and transposing,

$$\xi = r - 2n(n \cdot r).$$

Since r in the case under discussion is

$$\frac{R - X}{|R - X|},$$

equation (4) of the text is valid.

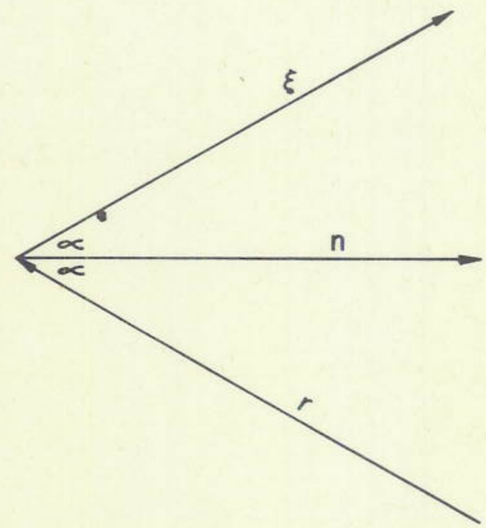

Figure 9 - Unit vectors at point of reflection

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[6] *Silberstein, op cit*

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APPENDIX II
Conical Wavefront from Phased Line Source

A line source can be interpreted physically as an infinite number of point sources, each of whose wavefronts has a radius proportional to its phase. With linear phase distribution, the radii vary linearly with position on the line. Figure 10 shows the line source in the y - z plane. If point b lags point a by $\Delta\lambda$, then bc, a ray from b perpendicular to the wavefront, has length $\Delta\lambda = (b - a)\sin\alpha$, and this is the radius of the sphere with center at b. Analytically, this general spherical wavefront is

$$x^2 + y^2 + (z - b)^2 = (b - a)^2 \sin^2\alpha.$$

As b is varied, there results a one-parameter family of spheres, one for each point source. The wavefront is the envelope of this family, which is obtained in the usual manner:

$$x^2 + y^2 + (z - b)^2 - (a - b)^2 \sin^2\alpha = 0;$$

and

$$(z - b) = (a - b)\sin^2\alpha$$

are the equations from which b must be eliminated. The first equation may be rewritten,

$$x^2 + y^2 - \cos^2\alpha(a - b)^2 \sin^2\alpha = 0$$

where

$$b = \frac{z - a\sin^2\alpha}{\cos^2\alpha} \quad .$$

Substituting the value of b, there results,

$$x^2 + y^2 - \cos^2\alpha \sin^2\alpha\left(\frac{a\cos^2\alpha - z + a\sin^2\alpha}{\cos^2\alpha}\right)^2 = 0,$$

or

$$x^2 + y^2 = (a - z)^2 \tan^2\alpha\,,$$

which is a cone of height, a, and angle, α.

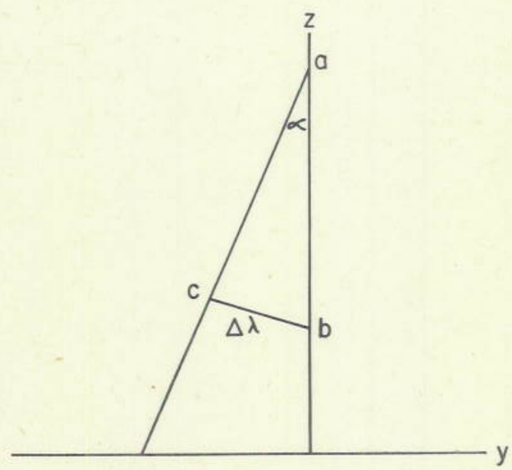

Figure 10 - Cross section of conical wavefront

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