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EQUIVALENT CIRCUIT FOR A VIBRATING BEAM WHICH INCLUDES SHEAR MOTIONS

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NAVAL RESEARCH LABORATORY

WASHINGTON, D.C.

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ABSTRACT

It has been demonstrated that the whipping modes of a ship are approximately the same as those computed for a tapered continuous beam, provided shear motions as well as bending motions are included in the computations. It is here shown that such a beam can be represented by an electric circuit made up of lumped passive components. Given such a circuit, it is then possible to determine experimentally the gross responses of the ship to any arbitrary excitation. Variation of responses with changes in the design of a ship can also be studied.

PROBLEM STATUS

This is an interim report on this problem; work is continuing.

AUTHORIZATION

NRL Problem P10-01R

EQUIVALENT CIRCUIT FOR A VIBRATING BEAM WHICH INCLUDES SHEAR MOTIONS

INTRODUCTION

The material contained in this report represents the results of a study pursued jointly by the Vibration Section of the David Taylor Model Basin and the Naval Research Laboratory. In fact, the impetus behind the work arose from theoretical and experimental studies initiated by the first activity. A few background comments are therefore in order.

For some time the Model Basin has been studying the possibility of representing the gross motions of a ship at low frequencies as being equivalent to those produced in a nonuniform continuous beam when subjected to the same excitations. In other words, the ship is approximated by a free-free beam whose mass per unit length and stiffness vary smoothly along its length, the magnitudes of these parameters at specified positions being equal to the corresponding known values for the ship at the same positions. If such an equivalence can be established, then it becomes possible to study the whipping modes of a ship through calculations based on a continuous beam theory. It is difficult to overemphasize the importance of such calculations for estimating the stresses and strains produced in a ship by excitations such as wave motion and underwater noncontact explosions.

Theoretical aspects of this problem are being studied by a group at the University of Michigan under the direction of Professor Ormandroyd. Initial efforts of this group were based upon the classical assumptions that the equivalent beam is linear, that rotary-inertia effects are insignificant, and that all motions are ones of pure bending. The Michigan group was able, under these assumptions, to show that all motions can be resolved into a superposition of normal modes and that the specification of the response resulting from an arbitrary loading can be reduced to quadratures.

Later, extensive vibration measurements conducted by the Model Basin on the USS NIAGARA (APA 87) showed that this simple theory did not predict natural frequencies well. The agreement at the first mode was good, but errors became progressively larger as the order of mode was increased. Further theoretical studies by the Michigan group have led to the conclusion that a more exact theory should include shear as well as bending motions. Theory indicates that, under this assumption, the difference in frequency between consecutive normal modes approaches a constant value at the higher modes. This is in agreement with experimental findings. To check the assumption further, the Model Basin has, from the design drawings, evaluated the parameters of the APA 87 at twenty stations. Using a beam theory which includes shear motions, extensive computations of the natural frequencies of the ship have been carried out by the use of the IBM punched-card system. Agreement between theory and experimental findings was good.

Calculations of this nature are tedious and time consuming. Even so, the importance of making such computations during the design stages of a new-construction vessel may justify the effort. Fortunately, the analogous-circuit technique allows model experiments to be conducted with much less time and effort. Answers with engineering accuracy are to be expected, and once the equivalent circuit for a ship has been assembled, the responses to be expected from arbitrary excitations can be observed in a straightforward fashion.

The importance of being able to construct an all-electric circuit equivalent to a ship is clearly evident. In an earlier report,[1] an equivalent circuit was derived and analyzed for a vibrating beam involving bending motions only. The present report shows the modifications necessary to the aforementioned circuit in order to include shear motions. Since this involves only the addition of one inductance per section, a detailed analysis of the characteristics of the circuit is not included. Instead, the discussion is limited to a derivation of the circuit from fundamental principles.

DERIVATION OF THE BASIC EQUATIONS

The basic technique employed in establishing an equivalent circuit for a continuous medium is to divide the medium into a finite number of lumps, establish dynamical equations for each lump, devise a network equivalent to the equations, and then connect in tandem the networks so established. Obviously, the accuracy of the analog increases with the number of lumps taken. It has already been indicated[2] that for an error of less than 3 percent, there must be included at least ten lumps per wavelength at the highest frequency of interest. This can be used as a rough "rule-of-thumb" in all cases, for it is closely associated with the approximation to a sine curve achieved by specifying ten points in each period. The present derivation will follow the steps in the order given.

Figure 1 shows schematically a typical section of the beam. The lateral deflections of the ends, Y_1 and Y_2, are measured from the equilibrium position. The mean slope of the beam is given by

$$\text{Slope} = \frac{Y_2 - Y_1}{\Delta x} = \tan\theta. \qquad (1)$$

It is assumed that the angle θ is small, an assumption well justified by practice. In general, the section will experience both shear and bending motions. In the first type, cross sections which were essentially parallel remain so after the motion has taken place. In the second, cross sections are inclined with respect to each other as a result of the motion. In constructing Figure 1, use has been made of the mechanical principle that a system of forces can

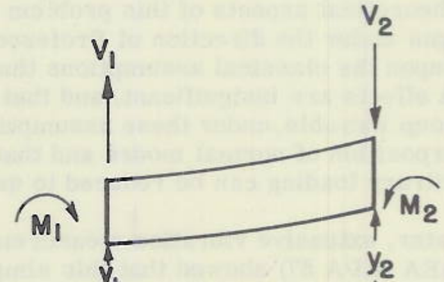

Fig. 1 - A typical section of the beam of length Δx

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[1] Trent, H. M., and E. S. Hoffer, "A Lumped-Constant Electrical Analog of a Beam Vibrating in Flexure," NRL Report P-3362, 1 Oct. 1948, Unclassified

[2] Trent and Hoffer, op.cit.

be reduced to a single force through a point and a couple. Such a reduction is made at each end section as shown. The single force is conventionally called the "shearing force," while the couple is called the "bending moment."

A universally accepted assumption is that the actual deflection at any section is the sum of two independent deflections - one due to shear motions alone, the other to bending alone. This assumption is equivalent to the relations,

$$\left.\begin{aligned} y_1 &= y_{1s} + y_{1b} \\ \text{and} \qquad y_2 &= y_{2s} + y_{2b}, \end{aligned}\right\} \tag{2}$$

where the subscript s stands for shear and b stands for bending. The slope then can be expressed as follows:

$$\tan\theta = \frac{y_2 - y_1}{\Delta x} = \frac{y_{2s} - y_{1s}}{\Delta x} + \frac{y_{2b} - y_{1b}}{\Delta x} = \tan\beta + \tan\gamma\,,$$

where

$$\tan\beta = \frac{y_{2s} - y_{1s}}{\Delta x}$$

and

$$\tan\gamma = \frac{y_{2b} - y_{1b}}{\Delta x}.$$

Under the assumption that β and γ are small (less than 0.1 radian), the following relations can be written:

$$\theta = \beta + \gamma = \frac{y_2 - y_1}{\Delta x}\,,$$

$$\beta = \frac{y_{2s} - y_{1s}}{\Delta x},$$

and

$$\gamma = \frac{y_{2b} - y_{1b}}{\Delta x}.$$

These equations are more useful in the form of their time derivatives, that is, as

$$\dot\beta = \frac{\dot y_{2s} - \dot y_{1s}}{\Delta x}$$

and

$$\dot\gamma = \frac{\dot y_{2b} - \dot y_{1b}}{\Delta x}\,, \tag{3}$$

which are equations of constraint that must be satisfied by the equivalent circuit.

Turning to dynamical considerations, it shall be required first that the net force on the section equal the product of its mass and its average acceleration. If the average mass per unit length is μ, then the total mass of the section will be $\mu\Delta x$. If $\bar{y}$ be the average deflection of the section, it follows that

$$V_1 - V_2 = \mu \Delta x \ddot{\bar{y}} = \mu \Delta x \, \frac{(\ddot{y}_1 + \ddot{y}_2)}{2},$$

or

$$V_1 - V_2 = \frac{\mu \Delta x}{2} \ddot{y}_1 + \frac{\mu \Delta x}{2} \ddot{y}_2. \tag{4}$$

Next, let it be required that the sum of the applied torques shall be equal to the moment of inertia times the angular acceleration of the section. Let ρ represent the radius of gyration of the lump about its center of mass. Then the moment of inertia of the section is

$$J = \mu \Delta x \rho^2.$$

If, now, moments are taken about the center of mass, which is approximately at the center of the section, it follows that

$$M_2 - M_1 = V_1 \frac{\Delta x}{2} + V_2 \frac{\Delta x}{2} + \mu \rho^2 \ddot{\gamma} \Delta x.$$

In formulating this expression, the conventional assumption is made that rotary-inertia effects are caused by bending motions alone. By virtue of equation (4), the above relation can be written

$$M_2 - M_1 = V_1 \Delta x - \mu \frac{\overline{\Delta x}^2}{4} (\ddot{y}_1 + \ddot{y}_2) + \mu \rho^2 \ddot{\gamma} \Delta x.$$

Under the assumption of small angles, the above expression can be written with little error as

$$M_2 - M_1 = (V_1 - \frac{\mu \Delta x}{2} \ddot{y}_1) \Delta x + \mu \rho^2 \ddot{\gamma} \Delta x. \tag{5}$$

To complete the set of basic relations, it is necessary to connect the elastic properties of the beam with the dynamic variables. Bending alone will be treated first. Standard treatments of beams point out that the average bending moment acting on the section is equal to EI times the rate of change of slope, where E is Young's modulus and I is the second moment of a cross section about its neutral axis. Obviously, the rate of change of slope applies only to the variable γ and is given by

$$\frac{\gamma_2 - \gamma_1}{\Delta x}.$$

It follows that,

$$\frac{M_2 + M_1}{2} = EI \frac{\gamma_2 - \gamma_1}{\Delta x},$$

or

$$\gamma_2 - \gamma_1 = \frac{\Delta x}{2EI} M_1 + \frac{\Delta x}{2EI} M_2.$$

The time derivative of the above expression is more useful for our purposes, that is,

$$\dot{\gamma}_2 - \dot{\gamma}_1 = \frac{\Delta x}{2EI} \dot{M}_1 + \frac{\Delta x}{2EI} \dot{M}_2. \tag{6}$$

In like fashion for shear effects, the average shear stress is equal to the shear modulus of the section (kG) times the shear strain. The factor k allows for the geometry of the cross section. The average shear stress is $(V_1 + V_2)/2A$ when A is the average cross-sectional area. The strain is given by $(y_{2S} - y_{1S})/\Delta x$. Hence,

$$-\frac{V_1 + V_2}{2A} = \frac{y_{2S} - y_{1S}}{\Delta x} kG.$$

Using a previous approximation, this relation can be written

$$(V_1 - \mu \frac{\Delta x}{2} \ddot{y}_1) \frac{\Delta x}{kAG} = y_{1S} - y_{2S}.$$

If, now, a time derivative is taken of this equation, the desired relation is obtained, namely,

$$\frac{d}{dt}(V_1 - \mu \frac{\Delta x}{2} \ddot{y}_1) \frac{\Delta x}{kAG} = \dot{y}_{1S} - \dot{y}_{2S}. \qquad (7)$$

It should be noted that equations (2) imply the relation

$$\dot{y}_2 - \dot{y}_1 = (\dot{y}_{2S} - \dot{y}_{1S}) + (\dot{y}_{2b} - \dot{y}_{1b}). \qquad (8)$$

The equivalent circuit must satisfy relations (1) through (8) if it is to be an exact analogy. It should be recognized that these relations are not all independent; they have been included in the presentation so that every aspect of the dynamics of the beam can easily be identified in the equivalent circuit.

DESCRIPTION OF THE ELECTRICAL ANALOG

The electrical analog to a section of the beam is shown in Figure 2, and it can be demonstrated that the circuit does in fact satisfy equations (1) through (8). At the outset, it must be pointed out that the mobility analogy has been used in establishing the network. According to this analogy, inertias are represented by capacitance, elastances by inductances, generalized forces by currents, generalized velocities by voltage drops, and couplings between generalized velocities as perfect transformers.

The circuit of a typical section is an 8-terminal network. It consists of two separate parts coupled by an ideal transformer. One part, the upper in Figure 2, is associated mainly with shear effects, the other with bending. Equation (4) says, in effect, that the shear current transmitted (V_2) by the section is less than that applied (V_1) by two terms which represent currents flowing to the reference bus, this bus representing an inertial reference. These shunt currents flow through the two condensers at the left and right ends of the network. Equation (5) states that the moment current applied at the right end (M_2) is greater than that transmitted at the left end (M_1). Again, there are two currents shunted

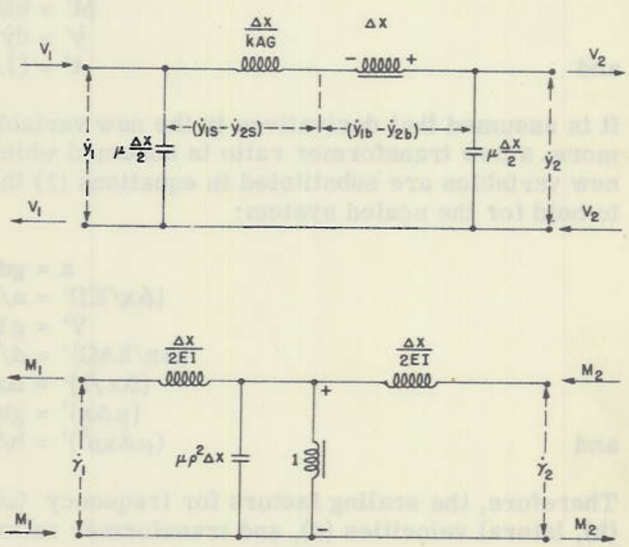

Fig. 2 - Equivalent circuit for a section of the beam

to the reference bus. One is the current flowing through a condenser representing the rotary inertia of the section; the other is proportional to the current through the series branch of the shear part of the network. Such a relation is established by a perfect transformer as shown.

Equation (6) states that the angular velocity at the right end is greater than that at the left end, the difference arising from a linear combination of time rates of changes of bending mement. This is suggestive of voltage drops across inductances and indeed gives rise to the two inductances appearing in the series branches of the bending part of the circuit. Equation (7) is like (6) in that it suggests an inductance. It must be recalled that the quantity in parentheses is the current through the series branch of the shear part of the network. Hence, an inductance to account for shear elastance must be inserted as shown. Equation (3) shows that the polarity markings on the transformer are as shown. Fundamentally, these markings only designate the relative winding sense of the transformer coils, and it should be noted also that equation (3) requires the same ratio for the transformer as that previously determined. Finally, it must be remarked that the circuit also satisfies relations (2) and (8). Hence, the circuit is an exact equivalent of the basic dynamic relationships.

SCALING PROCEDURE

Care must be exercised in computing the magnitudes of the components. The units of power employed should be the same in the mechanical system as in its electrical equivalent. Since practical electrical units are based upon the MKS system, it is suggested that mechanical computations be based upon these same units. In any case, it can be anticipated that the equivalent circuit will call for components and operating frequencies which are difficult to realize. Hence, scaling must be employed in order to realize a useful model experiment.

The general rules of scaling involve the definition of new variables (designated by primes):

$$\left.\begin{aligned} \dot{\gamma}' &= a\dot{\gamma}, \\ M' &= bM, \\ \dot{y}' &= d\dot{y}, \\ \text{and} \qquad t' &= (1/w)\,t. \end{aligned}\right\} \tag{9}$$

It is assumed that derivatives in the new variables are taken with respect to t′. Furthermore, a new transformer ratio is assumed which is $1/g$ times the original. When these new variables are substituted in equations (1) through (8), the following relations are found to hold for the scaled system:

$$\begin{aligned} a &= gd, \\ (\Delta x/EI)' &= a/bw\ \Delta x/EI, \\ V' &= gbV, \\ (\Delta x/kAG)' &= d/gbw\ \Delta x/kAG, \\ (\Delta x/1)' &= \Delta x/g, \\ (\mu\Delta x)' &= gb/dw\ \mu\Delta x, \\ \text{and} \qquad (\mu\Delta x\rho^2)' &= b/aw\ \mu\Delta x\rho^2. \end{aligned}$$

Therefore, the scaling factors for frequency (ω), bending strain rates (a), bending moments (b), lateral velocities (d), and transformer ratios can be assigned arbitrarily, subject only to the restriction, $a = gd$. This gives a lot of latitude in fixing the sizes of components and the characteristic frequencies of the system.

DISCUSSION

Figure 2, together with the rules of scaling, furnishes adequate means for establishing an electrical analog for a continuous beam, an analog which includes rotary inertia, bending, and shear effects. Frictional effects have not been included but would present no difficulties if they are viscous in nature. Unfortunately, a versatile nonlinear resistance has not been developed as yet.

Model studies on the whipping motions of ships through the use of electrical analogs appears promising in light of the success achieved in representing an actual ship by a continuous beam model. Useful answers can be obtained with a reasonable expenditure of time and money. Such a model is particularly useful when considering the results to be expected from an arbitrary type of excitation such as a change in the parameters of the ship. In light of these possibilities, the development of an electrical analog for each new-construction ship should receive serious consideration.

Two alternatives can be considered when planning a series of such model studies. In the first place, a collection of precise electrical components, each of which can be varied over wide limits, can be procured and permanently assembled in such form that component sizes can be varied easily and connections established quickly. Such an array becomes a general-purpose computer. This approach to the problem is appealing but at the same time expensive. The practicability of assembling an adequate supply of precision, variable components is questionable. Consider a model for a ship which is capable of representing up to the fifth whipping mode with an error of less than 3 percent. Such a model will require 100 inductances, 100 condensers, and 50 transformers. The fifth is not a very high mode, but already the number of components is large. Precision variable transformers are particularly expensive, bulky, and hard to procure.

Secondly and better, consideration can be given to the construction of a fixed model for each new-construction ship. Fixed condensers, such as are used in electronic circuits, are cheap and easy to obtain. If a winding machine is available, fixed coils can be fabricated easily. Transformers present more of a problem, but commercial developments now underway indicate that two conventional audio transformers in connection with a simple vacuum tube circuit can be made to approximate a perfect transformer with great precision. Here again, it becomes possible to construct a fixed component from readily available parts. Finally, the advisability cannot be denied of having a model available as long as a ship is in service. It is suggested therefore that, of the two alternatives, the latter embodying a collection of commercially available fixed components appears the more feasible.

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