NAVAL POSTGRADUATE SCHOOL

MONTEREY, CALIFORNIA

THESIS

REVISITING ENGEL'S VERIFICATION OF LANCHESTER'S SQUARE LAW USING BATTLE OF IWO JIMA DATA

by

Matthew G. Stymfal

September 2022

Thesis Advisor: Thomas W. Lucas
Second Reader: Jeffrey A. Appleget

THIS PAGE INTENTIONALLY LEFT BLANK

REPORT DOCUMENTATION PAGE			*Form Approved OMB No. 0704-0188*
Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instruction, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188) Washington, DC, 20503.			
1. AGENCY USE ONLY ***(Leave blank)***	**2. REPORT DATE** September 2022	**3. REPORT TYPE AND DATES COVERED** Master's thesis	
4. TITLE AND SUBTITLE REVISITING ENGEL'S VERIFICATION OF LANCHESTER'S SQUARE LAW USING BATTLE OF IWO JIMA DATA			**5. FUNDING NUMBERS**
6. AUTHOR(S) Matthew G. Stymfal			
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Naval Postgraduate School Monterey, CA 93943-5000			**8. PERFORMING ORGANIZATION REPORT NUMBER**
9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES) N/A			**10. SPONSORING / MONITORING AGENCY REPORT NUMBER**
11. SUPPLEMENTARY NOTES The views expressed in this thesis are those of the author and do not reflect the official policy or position of the Department of Defense or the U.S. Government.			
12a. DISTRIBUTION / AVAILABILITY STATEMENT Approved for public release. Distribution is unlimited.			**12b. DISTRIBUTION CODE** A

13. ABSTRACT (maximum 200 words)

Since Engel's 1954 verification of Lanchester's square law using Battle of Iwo Jima data, the homogenous Lanchester square law has been widely used as the default to approximate aggregate-level attrition in large modern battles. While this may work in some cases (particularly when forces are concentrated), there has been a death of efforts to assess the applicability of using other equations to fit the time-phased Iwo Jima battle data (for example the linear and log laws). Not all battles conform to a square law exponential attrition curve. Some battles (due to the nature or nuance of the scenario, battlefield, or engagement type) may lend themselves to being fitted better to linear, logarithmic, or several other equations. A better fit may lead to an improved understanding of future scenarios, which can help decision makers assign allocations for budgeting and deployment sizes and locations with better analytical support for approval of those decisions. This analysis revisits and replicates Engel's approach to fitting Lanchester's square law equation, extends it to other models, and finds/compares best fits. Using the attrition data from the Battle of Iwo Jima, we test the fitness (using R-squared values) of various Lanchester equations. Among many other discoveries, we find that Engel's model (and many other models) fits the data very well.

14. SUBJECT TERMS			**15. NUMBER OF PAGES**
Lanchester, Engel, force attrition, combat model, Iwo Jima			85
			16. PRICE CODE
17. SECURITY CLASSIFICATION OF REPORT Unclassified	**18. SECURITY CLASSIFICATION OF THIS PAGE** Unclassified	**19. SECURITY CLASSIFICATION OF ABSTRACT** Unclassified	**20. LIMITATION OF ABSTRACT** UU

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)
Prescribed by ANSI Std. 239-18

THIS PAGE INTENTIONALLY LEFT BLANK

Approved for public release. Distribution is unlimited.

REVISITING ENGEL'S VERIFICATION OF LANCHESTER'S SQUARE LAW USING BATTLE OF IWO JIMA DATA

Matthew G. Stymfal
Lieutenant Commander, United States Navy
BS, University of Arizona, 2008

Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN OPERATIONS RESEARCH

from the

NAVAL POSTGRADUATE SCHOOL
September 2022

Approved by: Thomas W. Lucas
Advisor

Jeffrey A. Appleget
Second Reader

W. Matthew Carlyle
Chair, Department of Operations Research

THIS PAGE INTENTIONALLY LEFT BLANK

ABSTRACT

Since Engel's 1954 verification of Lanchester's square law using Battle of Iwo Jima data, the homogenous Lanchester square law has been widely used as the default to approximate aggregate-level attrition in large modern battles. While this may work in some cases (particularly when forces are concentrated), there has been a death of efforts to assess the applicability of using other equations to fit the time-phased Iwo Jima battle data (for example the linear and log laws). Not all battles conform to a square law exponential attrition curve. Some battles (due to the nature or nuance of the scenario, battlefield, or engagement type) may lend themselves to being fitted better to linear, logarithmic, or several other equations. A better fit may lead to an improved understanding of future scenarios, which can help decision makers assign allocations for budgeting and deployment sizes and locations with better analytical support for approval of those decisions. This analysis revisits and replicates Engel's approach to fitting Lanchester's square law equation, extends it to other models, and finds/compares best fits. Using the attrition data from the Battle of Iwo Jima, we test the fitness (using R-squared values) of various Lanchester equations. Among many other discoveries, we find that Engel's model (and many other models) fits the data very well.

THIS PAGE INTENTIONALLY LEFT BLANK

TABLE OF CONTENTS

I. INTRODUCTION..1
A. OVERVIEW..1
B. BACKGROUND..1
C. LITERATURE REVIEW OF PAST ANALYSES..3

II. HISTORICAL OVERVIEW AND DATA SUMMARY..7
A. HISTORICAL OVERVIEW OF THE BATTLE OF IWO JIMA........7
B. DATA SUMMARY..10

III. VERIFICATION AND EXTENSION..13
A. VERIFICATION..13
1. Engel's Process..13
2. Improving upon Engel's Model..14
B. EXTENSIONS..16
1. Other Lanchester Models..16
2. Unconstrained *p* and *q* (Bracken)..19
3. Changing Attrition Coefficient..21

IV. CONCLUSIONS AND RECOMMENDATIONS..25
A. CONCLUSIONS..25
1. Results Summary..25
2. Discussion..26
B. RECOMMENDATIONS FOR FUTURE WORK..29

APPENDIX A. SQUARE LAW MODEL CONTOUR PLOTS..31

APPENDIX B. LINEAR LAW MODEL CONTOUR PLOTS..35

APPENDIX C. LOGARITHMIC LAW MODEL CONTOUR PLOTS........................39

APPENDIX D. BRACKEN MODEL CONTOUR PLOTS..41

APPENDIX E. R PROGRAM CODE..45

LIST OF REFERENCES..59

INITIAL DISTRIBUTION LIST..61

THIS PAGE INTENTIONALLY LEFT BLANK

LIST OF FIGURES

Figure 1. Map of Iwo Jima showing U.S. forces daily progress. Source: Morehouse (1946). 8

Figure 2. Landing beach as seen from Mt. Surabachi. Source: Office of Public and International Affairs (n.d.). 10

Figure 3. Actual daily attrition data for U.S. forces. Source: Morehouse (1946). 11

Figure 4. Plot of U.S. troops in action at the Battle of Iwo Jima. Source: Engel (1954). 14

Figure 5. Contour plot of R-squared fitness of attrition coefficient *a*-*b* pairs for the square law model (no final Japanese force level restriction) 15

Figure 6. Plot of actual active U.S. troops in battle with U.S. and Japan approximate values generated from the square law model 16

Figure 7. Plot of actual active U.S. troops in battle with U.S. and Japan approximate values generated from linear law model 18

Figure 8. Plot of actual active U.S. troops in battle with U.S. and Japan approximate values generated from the logarithmic law model 19

Figure 9. Plot of actual active U.S. troops in battle with U.S. and Japan approximate values generated from the Bracken model 20

Figure 10. Contour plot of R-squared fitness of *p*-*q* pairs from Bracken model 21

Figure 11. Attrition coefficient, *a*, calculated daily with linear trendline 22

Figure 12. Attrition coefficient, *a*, calculated daily with linear trendline (29 days) 23

Figure 13. Contour plot of R-squared fitness of p-q pairs from Bracken model (constant increments) 26

Figure 14. Contour plot of attrition coefficients, *a*, for best-fit Bracken models 28

Figure 15. Contour plot of attrition coefficients, *b*, for best-fit Bracken models 28

Figure 16. Contour plot of R-squared fitness for the square law model (final Japanese force level restriction $y(36) = 2000$) 31

Figure 17. Contour plot of R-squared fitness for the square law model (final Japanese force level restriction $y(36) = 1000$) 31

Figure 18. Contour plot of R-squared fitness for the square law model (final Japanese force level restriction $y(36) = 200$) 32

Figure 19. Contour plot of R-squared fitness for the square law model (final Japanese force level restriction $y(36) = 50$) 32

Figure 20. Contour plot of R-squared fitness for the square law model (final Japanese force level restriction $y(36) = 1$) 33

Figure 21. Contour plot of R-squared fitness for the linear law model (final Japanese force level restriction $y(36) = 20000$) 35

Figure 22. Contour plot of R-squared fitness for the linear law model (final Japanese force level restriction $y(36) = 2000$) 35

Figure 23. Contour plot of R-squared fitness for the linear law model (final Japanese force level restriction $y(36) = 1000$) 36

Figure 24. Contour plot of R-squared fitness for the linear law model (final Japanese force level restriction $y(36) = 200$) 36

Figure 25. Contour plot of R-squared fitness for the linear law model (final Japanese force level restriction $y(36) = 50$) 37

Figure 26. Contour plot of R-squared fitness for the logarithmic law model (final Japanese force level restriction $y(36) = 2000$) 39

Figure 27. Contour plot of R-squared fitness for the logarithmic law model (final Japanese force level restriction $y(36) = 1000$) 39

Figure 28. Contour plot of R-squared fitness for the logarithmic law model (final Japanese force level restriction $y(36) = 200$) 40

Figure 29. Contour plot of R-squared fitness for the logarithmic law model (final Japanese force level restriction $y(36) = 50$) 40

Figure 30. Contour plot (no model labels) of R-squared fitness of *p-q* pairs from the Bracken model 41

Figure 31. Contour plot (no model labels) of R-squared fitness of *p-q* pairs from the Bracken model (expanded high-fitness) 41

Figure 32. Contour plot (no model labels) of R-squared fitness of *p-q* pairs from the Bracken model (constant increments) 42

Figure 33. Contour plot (no model labels) of attrition coefficients, *a*, for the best-fit Bracken models 42

Figure 34. Contour plot (no model labels) of attrition coefficients, *b*, for the best-fit Bracken models 43

THIS PAGE INTENTIONALLY LEFT BLANK

LIST OF TABLES

Table 1. Model fitness and results summary..25

THIS PAGE INTENTIONALLY LEFT BLANK

EXECUTIVE SUMMARY

The purpose of this study is to explore the assertions of Engel (1954) in his verification of Lanchester's square law combat attrition equations, using historical data from the Battle of Iwo Jima during World War II. His work is replicated using the same original data set and analyzed using modern fitness test techniques. Furthermore, modern computing power is utilized to fit Lanchester models billions of times with various parameter values dictating the shape of the models.

The two models Lanchester (1916) conceived are called the linear law and the square law. The linear law is described as a series of one-on-one duels on the battlefield, most applicable to ancient warfare or area fire. In this scenario, the attrition ratio of the two forces is independent of the force ratio. The square law, associated with modern warfare, describes combat where multiple units of a force can focus their aimed fire onto single targets. In this concentrated fire model, the attrition that a force suffers is proportional to the number of enemies.

The linear law equations are:

$$\frac{dx}{dt} = -axy \tag{1}$$

$$\frac{dy}{dt} = -bxy \tag{2}$$

and the square law equations are:

$$\frac{dx}{dt} = -ay \tag{3}$$

$$\frac{dy}{dt} = -bx \tag{4}$$

where x and y are the force levels for each belligerent at time t, and a and b are the attrition coefficients for forces x and y, respectively. The attrition coefficients describe the rate of attrition suffered as a function of the forces. For example, in the square law equation (3), a

is the rate at which one y attrits the x force. For a time-step implementation, the x force suffers a losses for each y force per unit time.

The above equations can be solved to give the force levels of each side as a function of time. In addition, exponent variables, p and q, can be added to the force level variables, x and y, to give a more generalized equation (Bracken 1995):

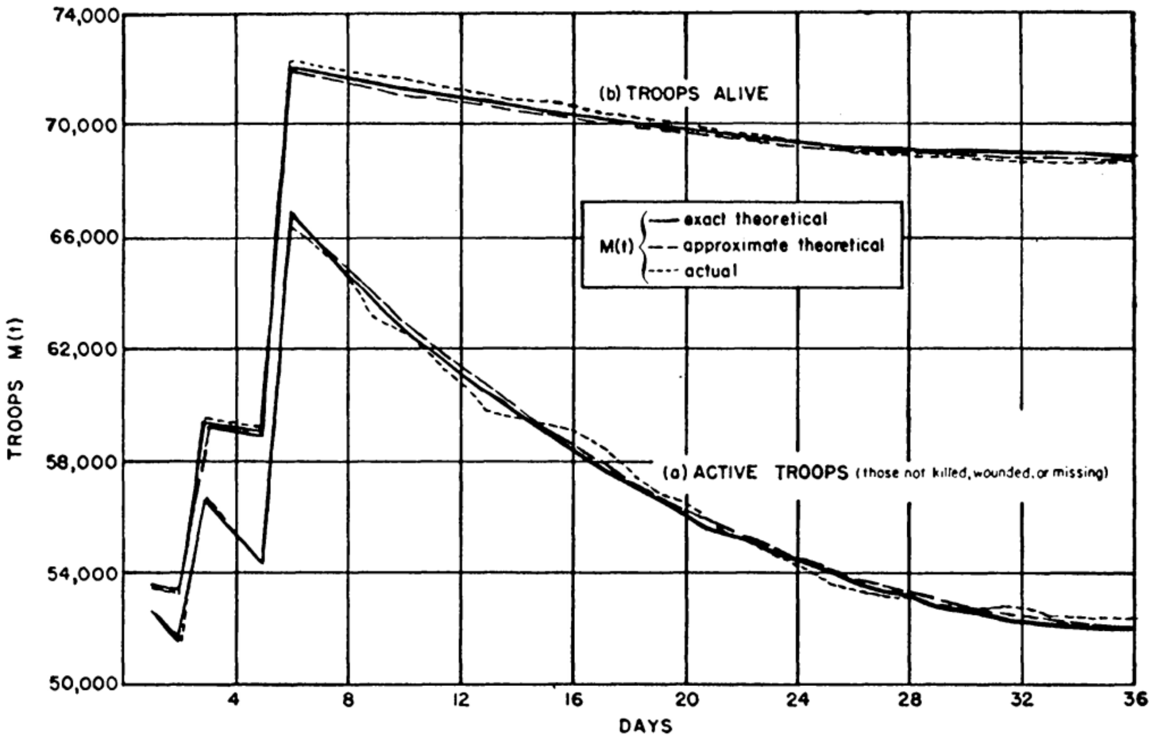
$$\frac{dx}{dt} = -ax^q y^p \quad (5)$$

$$\frac{dy}{dt} = -bx^p y^q \quad (6)$$

Here, q is the exponent variable for the force being attritted, and p is the exponent variable for the opposing force doing the attritting. Within Bracken's model, one can produce the linear law equation by setting p and q equal to 1 and the square law when p = 1 and q = 0.

With the square law equations (3) and (4) and the daily U.S. casualty data for the Battle of Iwo Jima, Engel could plot fitted theoretical values against the actual recorded number of U.S. troops in action (Figure 1).

Engel (1954) only looked at the square law and found his best-fit model analytically, and although he only used eyeball approximation to qualitatively assess fitness, he felt he had a "good" fit. When one uses an R-squared fitness test (between the historical force levels and the fitted force levels assuming a square law) to quantitatively assess the fitness of Engel's square law fit, one gets a value of 0.9937. This is indeed a very good fitness value that supports Engel's qualitative claim.



Plot of "(b) Troops Alive" includes U.S. forces WIA and MIA and can be disregarded for the purposes of this paper.

Figure 1. Plot of U.S. troops in action at the Battle of Iwo Jima. Source: Engel (1954)

In this thesis, best-fit models are found numerically through successive iteration. To do this, the exponent variables, p and q, corresponding to the model type (square law, etc.) were first specified. Next, a pair of values for the attrition coefficients, a and b, were selected and used in equations (5) and (6) iteratively and sequentially from $t = 0$ to $t = 36$ days—the length of the battle. Essentially, a time-phased battle was reconstructed using the selected attrition coefficients, the known initial force levels of $x(0) = 54000$ and $y(0) = 21500$, and the known reinforcement values of 6,000 and 13,000 troops on the 3rd and 6th day, respectively. This generated a fitted time-phased series of force levels for the U.S. forces that could then be compared to the original force level data set using an R-squared fitness test—see Figure 2.

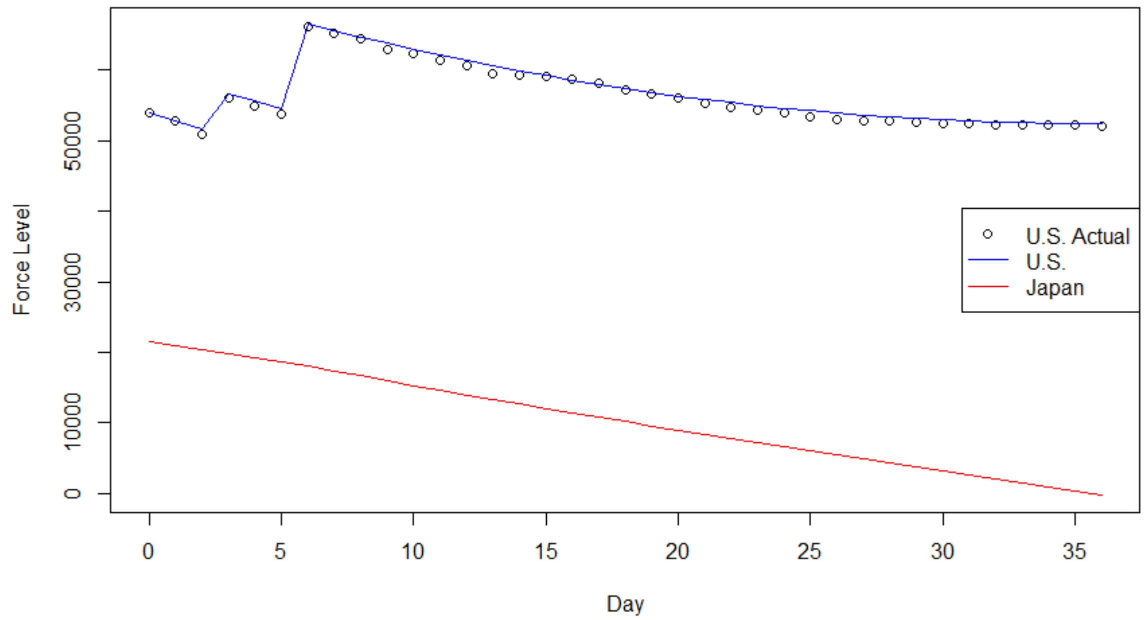


Figure 2. Plot of actual active U.S. troops in battle with U.S. and Japan approximate values generated from the square law model

This was repeated for a wide range of attrition coefficients, *a* and *b,* with each pair of coefficients having a calculated R-squared fitness associated with it. The model with the best (greatest) R-squared value was selected as the best fit and identified by its corresponding attrition coefficients, *a* and *b*.

The process of determining the attrition coefficients, *a* and *b,* for the best-fit model was repeated for a wide range of exponent variables, *p* and *q*. This set of models with non-binary exponent variables is referred to as the Bracken model in this paper (Bracken 1995). The general form equations (5) and (6) were used to calculate daily attrition with values of the exponent variables, *p* and *q*, ranging from $p, q = 0$ to $p, q = 1.5$.

The final extension explored was the concept of attrition coefficients that change throughout the course of the battle. The actual recorded number of U.S. troops in action, *x*, along with the theoretical approximation time-phased Japanese force level, *y* (generated from Engel's model with attrition coefficients, *a* and *b,* of 0.0544 and 0.0106, respectively), are used in equation (3) to solve for the time-phased attrition coefficient for the U.S. forces, *a*, for each day of the battle. As expected, and contrary to Lanchester's assumption of a constant attrition coefficient, the attrition coefficient has a high variance when calculated daily in this manner. In addition, an overall downward trend is observed.

In conclusion, it was found that not only did Engel's (1954) model provide a very good fit quantitatively speaking ($R^2 = 0.9937$), but performed just as well (and with nearly the same parameters) as the calculated square law and Bracken models. The tabulated results (Table 1) summarize the findings.

Table 1. Model Fitness and Results Summary

Model	a	b	p	q	R^2
Engel	0.0544	0.0106	1	0	0.9937
Square	0.0532	0.0105	1	0	0.9944
Linear	2.30×10^{-6}	2.28×10^{-6}	1	1	0.9027
Logarithmic	0.0108	0.516	0	1	0.9414
Bracken	0.0331	0.00611	1.05	0.00	0.9946

The table shows that the best-fit Bracken model has exponent variables, p and q, very similar to the square law model ($p = 1$, $q = 0$). Also of note is the similar effectiveness ratio found in the Engel, square law, and Bracken models ($\frac{a}{b} \cong 5$). This makes sense as the exponent variables, p and q, are very similar across all three models.

While the linear and logarithmic law models did not perform as well as the other models, they still showed R-squared fitness values above 0.90. Furthermore, when analyzing the contour plot of the fitness values of the various Bracken models, one sees a relatively large range of models with exponent variables, p and q, that give R-squared fitness values greater than 0.99. This finding rejects the notion that the square law-like models are the only ones with a very good fit.

References

Bracken J (1995) Lanchester models of the Ardennes campaign, *Naval Research Logistics*, 42: 559–577.

Engel JH (1954) A verification of Lanchester's Law, *J. of the ORSA*, 2(2): 163–171, INFORMS.

Lanchester FW (1916) *Aircraft in Warfare; the Dawn of the Fourth Arm* (Constable and Company, Ltd., London).

ACKNOWLEDGMENTS

A special thanks goes to Professor Lucas for the immeasurable wisdom, expertise, and guidance through the thesis-writing process. The many thorough theoretical discussions along with the frequent motivating pushes to dive deeper made for a more complete and enjoyable work.

Thanks also goes to Colonel Appleget for asking the right questions with regards to the big picture concepts. He was also very helpful in the editing process. His input made for a better end product.

THIS PAGE INTENTIONALLY LEFT BLANK

I. INTRODUCTION

A. OVERVIEW

Military research has extensively used combat models since the start of World War II. Over the years, computers have been able to significantly expand the application and the depth of such models. One of the most fundamental components of most combat models deals with attrition, or the loss of forces. While many aggregate-level combat attrition models are rooted in some manner on the basic Lanchester equations, several have attempted to depict combat more accurately with the use of expansions or other modifications to those equations (Hartley and Helmbold 1995).

The purpose of this study is to explore the assertions of Engel (1954) in his verification of Lanchester's (1916) square law, using the historical data from the Battle of Iwo Jima during World War II. Lanchester's work is replicated using the same original data set and analyzed using modern fitness test techniques. Furthermore, modern computing power is used to fit Lanchester models billions of times for added insight and sensitivity analysis.

B. BACKGROUND

F. W. Lanchester's (1916) observations of air combat during World War I and the naval battle at Trafalgar led to his derivation of the theory of quantifying the effects of force concentration and force ratios on attrition in battle. The two models he conceived as a result are called the linear law and the square law (Lanchester 1916). The linear law is described as a series of one-on-one duels on the battlefield, most applicable to ancient warfare or area fire. In this scenario, the attrition ratio of the two forces is independent of the force ratio. The square law, associated with modern warfare, describes combat where multiple units of a force can focus their aimed fire onto single targets. In this concentrated fire model, the attrition that a force suffers is proportional to the number of enemies.

The linear law equations are:

$$\frac{dx}{dt} = -axy \tag{1}$$

$$\frac{dy}{dt} = -bxy \tag{2}$$

and the square law equations are:

$$\frac{dx}{dt} = -ay \tag{3}$$

$$\frac{dy}{dt} = -bx \tag{4}$$

where x and y are the force levels for each belligerent at time t, and a and b are the attrition coefficients. The attrition coefficients describe the rate of attrition suffered as a function of the forces. For example, in the square law equation (3), a is the rate at which one y attrits the x force. For a time step implementation, the x force suffer a losses for each y force per unit time. An important assumption in Lanchester's (1916) work is that these attrition coefficients remain constant during the battle.

In addition, when evaluating numerically, the frequency that the force levels are incrementally evaluated at must also remain constant throughout the battle in order to maintain constant attrition coefficients; however, the time steps can be any length of time desired (with larger steps causing more approximation error). For example, this paper examines attrition in the battle of Iwo Jima in time steps of one day. This matches the resolution of the data, which consists of daily totals and casualties.

The above equations can be solved to give the force levels of each side as a function of time. In addition, exponent variables, p and q, can be added to the force level variables, x and y, to give a more generalized equation (Bracken 1995):

$$\frac{dx}{dt} = -ax^q y^p \tag{5}$$

$$\frac{dy}{dt} = -bx^p y^q \tag{6}$$

Here, q is the exponent variable for the force being attritted, and p is the exponent variable for the opposing force doing the attritting. Within Bracken's model, one can produce the linear law equation by setting p and q equal to 1 and the square law when $p = 1$ and $q = 0$.

Another notable model is when $p = 0$ and $q = 1$—commonly called the logarithmic law (Peterson 1967). This model describes battles where the attrition of a force is a function of its own force level and is independent of the opposing force level. Examples with attrition in this form include battles with extremely destructive fires/explosions (think cannon fodder) and battles where non-combat related attrition dominates (e.g., disease).

C. LITERATURE REVIEW OF PAST ANALYSES

(1) Lanchester's Aircraft in Warfare

Lanchester explores aircraft battles and attrition data from World War I. In his work, he derives his theories behind the relationship between force concentration and combat attrition. He uses the resulting square law and linear law as a means towards his recommendations for the relatively new concept of air warfare. He also extends the theory to include heterogeneous force composition, naval warfare, and other battle types (Lanchester 1916).

(2) Engle on Lanchester

Engel (1954) verifies Lanchester's (1916) attrition model theory using real combat data. Using the U.S. casualty data from the battle at Iwo Jima, he explores the utilization of the square law model to fit the U.S. combat attrition. Although he mentions other models and their possible applicability, he does not extend his analysis to other attrition equations, such as the linear or logarithmic laws. The work fits data for all U.S. troops, as well as active U.S. troops (not killed, wounded, or missing). The fitness test used was simply a qualitative eyeball approximation. His recommendation is to use the equation that fits the battle type, then adjust the attrition coefficients depending on the factors of the battle (time, defensive position, etc.). Engel also notes that "the value of such analyses increases when repeated often enough to permit general conclusions to be drawn" (Engel 1954, p. 163).

(3) Robert Samz on Engel

Samz (1972) verifies Engel's analytical work on the battle of Iwo Jima, but then considers the effects of using alternate data sources for the battle. The primary difference in the data sets is the timing of the first-day landing forces during the battle. Samz explores what he considers to be a more typical and realistic sequence of reinforcements landing in such an amphibious assault. He replicates Engel's process with the revised force and reinforcement data. However, the conclusion was that the revised data fit Lanchester's square law attrition equations just as well. Fitness test used for this work was a modified version of Theil's inequality coefficient (Samz 1972 p.51).

(4) Hartley and Helmbold on Lanchester and Soeul

Lanchester's square law is tested against the data from the Inchon-Seoul campaign of the Korean War. Linear regression was used for fitting and testing of the square law model as well as their extension. Their extension effectively split the duration of the battle into three smaller durations. Fitness was tested separately for each of these "phases" of the battle, where the attrition coefficients were allowed to change between the different phases. Hartley and Helmbold's (1995) conclusions are that the square law model does not sufficiently fit the data, nor did the time-phased extension. They also put emphasis on the point that, while the square law model may fit some situations, it does not fit this one and should not be used as a general-purpose attrition model for all warfare.

(5) Bracken on Lanchester and Ardennes

The objective of Bracken's (1995) research was to fit the Lanchester equations to the Ardennes campaign attrition data set. An extension used in the analysis was using an additional "tactical" factor, d, to account for attacking and defending postures. This factor was essentially a daily variable signifying that one side was in a defensive mode during that day of the battle. The factor was multiplied along with the attrition coefficient, as applicable. The sum of squared residuals (SSR) was used to test for fitness of the different models. The analysis showed that the linear law model best fit the daily Ardennes campaign data. In addition, the results showed a clear advantage to the attacker.

(6) Fricker on Bracken and Ardennes

This paper revises and extends the analysis by Bracken of the Ardennes campaign of World War II, using logarithmic transforms of the Lanchester equations. In addition, Fricker (1998) calls into question the applicability and sensibility of certain parameter values, particularly negative exponent parameters. He found that neither the Lanchester linear nor square laws fit the data well. Using SSR for fitness testing of all models, he found the best fit to be a model equation where $p = 0$ and $q = 3$.

(7) Chen and Chu on Ardennes

Chen and Chu (2001) explore the viability of changing the equations used to calculate force attrition from defensive to offensive (and vise-versa) throughout the battle. Using this idea of "shift time," an analytical method was used to locate the optimal solution (Chen and Chu 2001, p. 653). They then applied this solution to the Ardennes data set and compared their results with Bracken's. The primary finding is that the concept improves fitness to Ardennes (compared to Bracken) and can be applied to other campaigns/battles. Their research, however, considers only the linear law (with tactical factor, *d*) and the fitness test used was SSR.

(8) Dinges on Lanchester and Kursk

This NPS master's thesis explores the validation of Lanchester equations as models of the attrition process for the newly released casualty data for the Battle of Kursk in World War II. His research found that the best-fitting model is the linear law model. In addition, he found that the extensive optimization of force weights (determining the weights for tanks, armored vehicles, and artillery that maximize R-squared) does not significantly improve the fit (Dinges 2001).

(9) Bonder Lessons Learned

Bonder (2002) introduces a hybrid analytic/simulation model to analyze the effects of attack speed and force concentration on attrition in battle. His research and experience reject the general use of Lanchester equations due to their "holistic" nature. In addition, he recommends against the use of "firepower scores" as a hypothetical construct for modelling (Bonder 2002, p. 26).

(10) Lucas and Turkes on Lanchester and Kursk and Ardennes

This article re-examines Turkes's (2000) thesis to fit Lanchester models to the datasets of the battles of Kursk and Ardennes. Defender parameters were used in all the models, and the battles were broken up into phases for some of the analysis. Using R-squared for test of fitness, Lucas and Turkes' conclusion was that none of the basic Lanchester laws fit the data well, and none consistently outperformed the others (2004).

(11) Lucas and Dinges on Kursk

Lucas and Dinges (2004) explore the expanded Kursk dataset and break the data into the battle's natural (based on historical accounts) phases before fitting attrition equations. Using R-squared for their test of fitness, they found that the linear law fit best, that only combat unit data should be used, and that the battle should be broken into phases as applicable.

(12) Nigel Perry on Lanchester

Nigel Perry (2007) analyzes the applicability of the various Lanchester equations to different spaces of the battlefield and uses a fractal model to express the distribution of forces and the confined battle space at all scales. Specifically, the paper seeks to address the assumption made by previous works: No spatial distribution of forces, no variation of combat effectiveness, deterministic attrition, and no representation of battle termination conditions. His paper produces logical equations to be used to address these extensions and assumptions but does not test or apply them to any real-world data.

II. HISTORICAL OVERVIEW AND DATA SUMMARY

A. HISTORICAL OVERVIEW OF THE BATTLE OF IWO JIMA

> "Victory here will be obtained by a slow, pulverizing pressure on the Japanese defenses."
>
> —Major General Harry Schmidt, USMC (Morehouse 1946, p. vi).

The battle was towards the end of the war and the Pacific Theater campaign. The data in this paper covers the force attrition from 19 February through 26 March 1945. Japan was on the defensive at the strategic as well as the operational levels. The island was used by the Japanese military throughout the war as a waypoint and relay for communications, aircraft, and supplies between the Japan mainland and the rest of the Southwest Pacific.

While some portions of the island are generally flat (allowing for the construction of airfields), the degree of roughness of the terrain in many places was underestimated. The natural ridges and gorges allowed for "extremely effective military defenses" for the Japanese. The civilian population was estimated to be small, but at time of invasion there turned out to be no actual civilian population. The fortress was very strong, defensive tactics were advanced, and tenacity was "beyond all anticipation." (Morehouse 1946, pp. 5–8)

Throughout December and January, American land-based aircraft, like B-29's, attacked Iwo daily; however, air raids occurred for eight months leading up to amphibious assault. The Japanese island defenders were quite hardened to aerial attacks and bombardment by the time D-day occurred. The Japanese had several months to specifically prepare for an impending Allied amphibious assault. Nearly every conceivable defensive position, mineable ground, and playbook strategy was implemented and rehearsed well ahead of time (Morehouse 1946, pp. 1–4).

For the Americans, forces were relatively refreshed from their previous engagements in the Pacific, and their experience levels varied. There were three main combat divisions used in the assault. Previous campaigns had given the 3rd & 4th Divisions battle experience. Although 5th division had much less experience, it was able to train and practice the assault in Hawaii in January (Morehouse 1946, pp. 9–14).

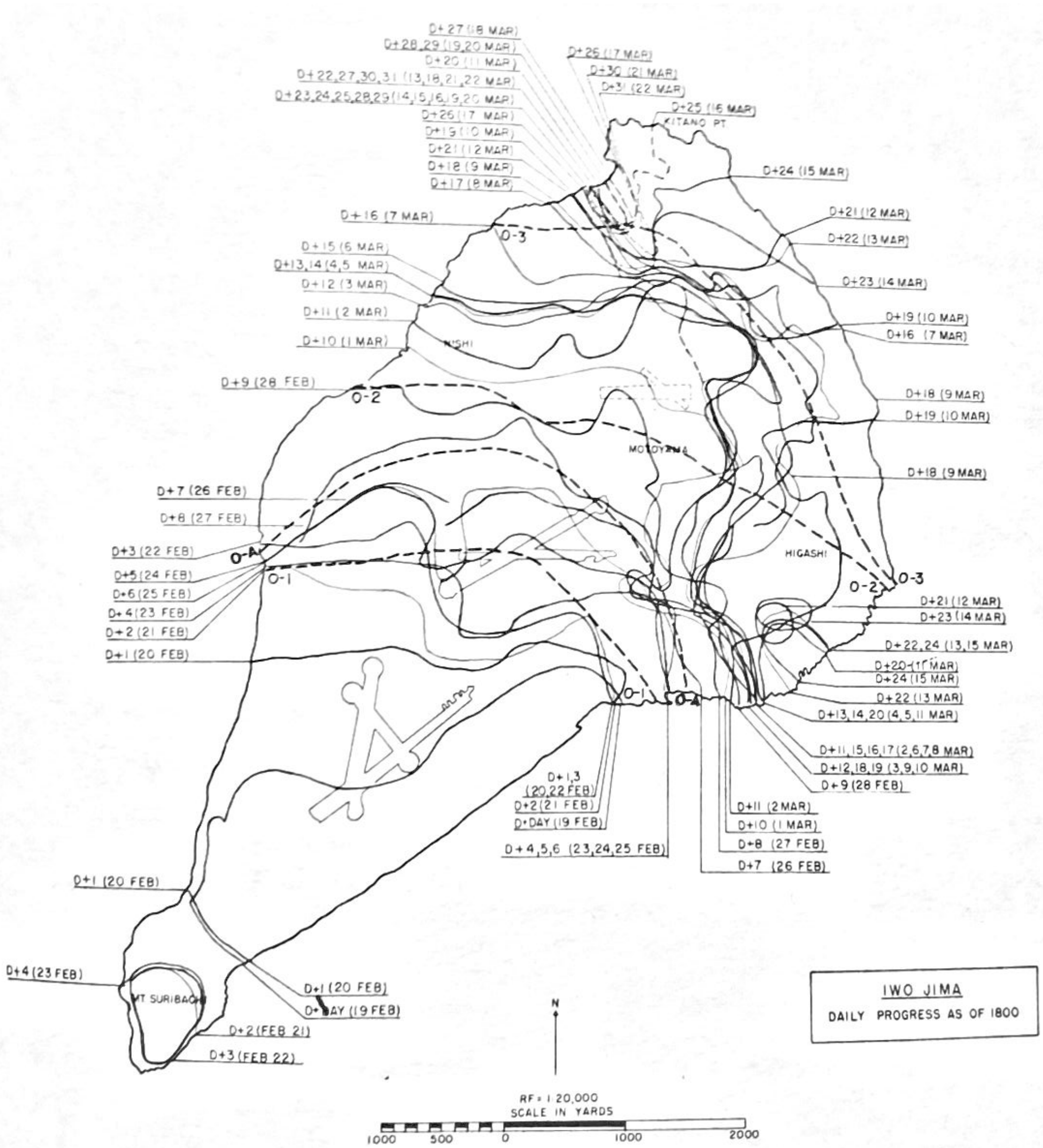


Figure 1. Map of Iwo Jima showing U.S. forces daily progress. Source: Morehouse (1946).

Weather on D-day was ideal for an amphibious assault. The Japanese were well prepared with mortars zeroed-in; there were high casualties for the landing forces on D-day and the following week. D-day had split Japanese forces into two areas—the smaller of which was Mt. Suribachi to the South. It was defended primarily via caves and concrete structures (like the main part of island) (Morehouse 1946, pp. 21–26). However, the split attention of the allied forces during the first four days did not negatively affect combat performance due to the limited space available on both fronts. In other words, not having the southern forces available to help the northern forces for the first four days did not matter because there was not enough space and front line to put them to good use.

Counterattacks were not a significant part of the Japanese plan. While there were some very minor night raids and nightly infiltration attempts, much of the Japanese doctrine on the island was to "dig-in" and defend (with the occasional withdraw to the next defensive layer) (Morehouse 1946, pp. 31–41). Seizure of the airfields did not greatly impact the flow of the battle (from an aviation standpoint). The airfields allowed for some emergency landings of allied aircraft in the area, but for the most part aerial attacks on the island were unchanged because of capturing the airfields.

U.S. artillery and naval gunfire was used consistently on deep targets throughout the battle. U.S. tanks and Japanese anti-tank emplacements were constantly matched against each other throughout the course of the island progression. Finally, cave positions were defended and cleared at a consistent rate all the way to D+35 (Morehouse 1946, pp. 43–50).

Figure 2. Landing beach as seen from Mt. Surabachi. Source: Office of Public and International Affairs (n.d.).

B. DATA SUMMARY

The source of the data is the daily casualty recordings from the historical accounts and official records from the battle of Iwo Jima. This information was compiled during the operation by the U.S. Marine Corps Historical Division and published in the book *The Iwo Jima Operation* by Capt. Clifford P. Morehouse, USMCR (1946). This is the same data source used by Engel (1954) in his verification of Lanchester (1916).

The casualty data from the book is broken down in detail by division, group, company, and by day. This paper only considers the U.S. force casualties by days, as taken from the action reports from Task Force 51, Joint Expeditionary Forces. This data set lists the aggregate U.S. forces killed in action (KIA), wounded in action (WIA), and missing in action (MIA). This paper ultimately uses the number of active U.S. forces each day as the data set used for analysis. As such, the known daily total casualties of those killed, wounded, or missing in action (KIA, WIA, and MIA, respectively) are subtracted from the known landing forces and reinforcement forces to get a running total of active U.S. troops each day of battle. The errors and assumptions in this data generation methodology are negligible, and is the same method used by Engel in his 1954 paper.

Day	Date	KIA	WIA	MIA	Cumulative Total	Daily Total
D-day	19 Feb	76	1,080	5	1,161	1,161
D+1	20 Feb	264	1,562	481	3,055	1,894
D+2	21 Feb	426	2,116	355	3,969	919
D+3 and 4	22, 23 Feb	870	4,711	670	6,251	2,282
D+5	24 Feb	1,021	5,284	537	6,845	594
D+6	25 Feb	1,195	6,006	549	7,750	905
D+7	26 Feb	1,347	6,791	484	8,622	872
D+8	27 Feb	1,556	7,984	586	10,126	1,504
D+9	28 Feb	1,616	8,418	575	10,661	535
D+10	1 Mar	1,845	9,150	599	11,595	934
D+11	2 Mar	2,005	9,780	545	12,333	738
D+12	3 Mar	2,278	10,632	541	13,451	1,118
D+13	4 Mar	2,468	10,556	631	13,655	204
D+14	5 Mar	2,620	10,753	546	13,919	264
D+15	6 Mar	2,715	11,054	452	14,221	302
D+16	7 Mar	2,869	11,505	430	14,804	583
D+17	8 Mar	3,055	12,251	435	15,741	937
D+18	9 Mar	3,191	12,723	445	16,359	618
D+19	10 Mar	3,315	13,203	421	18,204	581
D+20	11 Mar	3,488	13,785	419	17,692	752
D+21	12 Mar	3,653	14,130	421	18,204	512
D+22	13 Mar	3,765	14,403	434	18,692	398
D+23	14 Mar	3,878	14,750	434	19,062	460
D+24	15 Mar	4,112	15,102	437	19,653	591
D+25	16 Mar	4,206	15,290	423	19,928	275
D+26	17 Mar	4,305	15,474	417	20,196	268
D+27	18 Mar	4,357	15,511	397	20,265	69
D+28	19 Mar	4,403	15,598	391	20,392	127
D+29	20 Mar	4,457	15,677	359	20,493	101
D+30	21 Mar	4,503	15,732	353	20,538	95
D+31	22 Mar	4,540	15,820	336	20,696	108
D+32	23 Mar	4,590	15,954	301	20,845	149
D+33	24 Mar	4,590	15,954	301	20,845	0
D+34	25 Mar	4,590	15,954	301	20,845	0
D+35	26 Mar	4,590	15,969	301	20,860	15

* Action Report, Task Force 51, Joint Expeditionary Forces.

Figure 3. Actual daily attrition data for U.S. forces. Source: Morehouse (1946).

Such detailed casualty data for the Japanese forces is not available, however, and the only usable data regarding Japanese force size over time is that there were 21,500 at the start of the battle (D+0) and approximately zero at the end of the recorded fighting (D+35). D+28 is the official end of the battle as declared by the operational U.S. Marine command at the time; and, although there were some residual Japanese forces deep within buried tunnels and bunkers that were exposed over the following days, weeks, months, and even years, they were relatively low in number and are not considered for the scope of this analysis (Morehouse 1946).

There is found to be no statistically significant difference between the fitness performance of the models used in this paper when using either D+28 or D+35 as the last day of battle. Therefore, apart from verifying Engel's calculations and model fitness, this paper uses casualty data through D+35.

III. VERIFICATION AND EXTENSION

A. VERIFICATION

1. Engel's Process

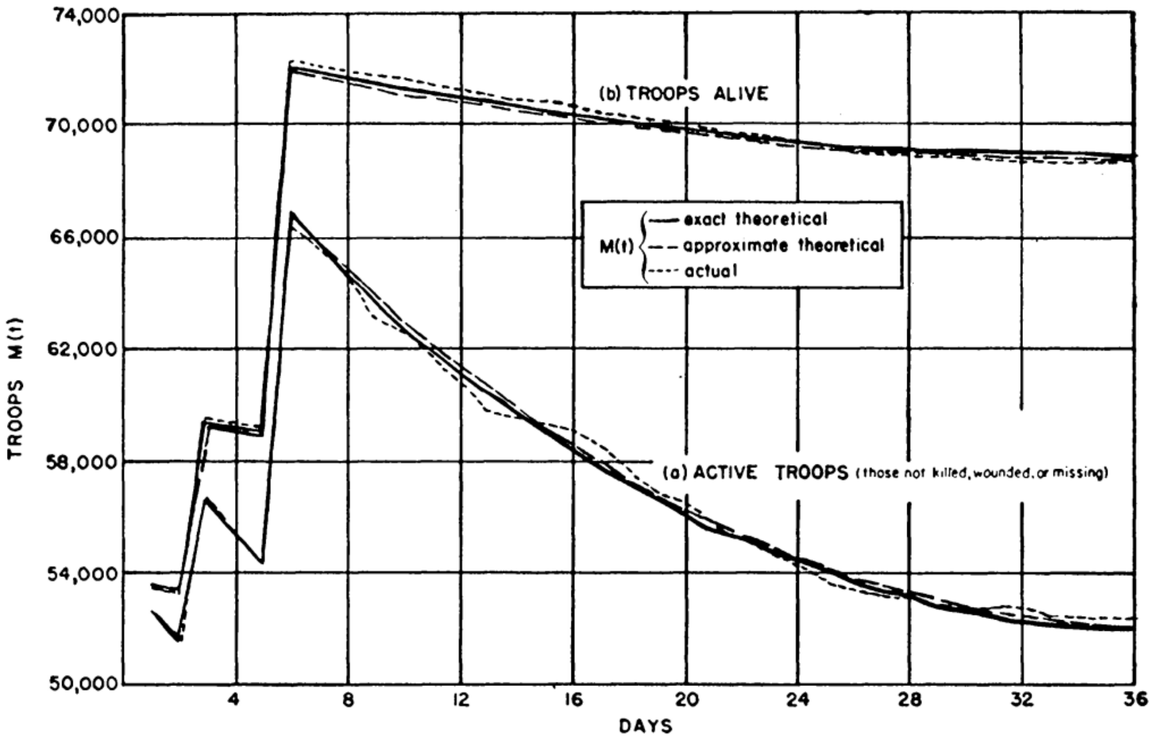
Before one can verify Engel's (1954) work on the Lanchester (1916) square law equation fitting for the Battle of Iwo Jima, one must first understand what he did and how he did it. His first step was to calculate the Japanese forces' attrition coefficient. He calculated this by dividing the total Japanese losses by the sum of the daily U.S. force levels across all 36 days, to include U.S. reinforcements. He incorporated American reinforcements by simply adding the reinforcement value (if any) to that day's force level change. This has the effect of modifying equation (3) in the following way:

$$\frac{dx}{dt} = \mathrm{P} - ay \tag{7}$$

where *P* is the number of reinforcements for that day. Again, he assumed and approximated that there were zero Japanese forces remaining on the last day. This gives a Japanese attrition coefficient, *b,* of 0.0106 (Engel 1954). That is, the rate at which Japanese soldiers were attritted was *b* multiplied by the American force level.

Next, he used the Lanchester square law equation (4) and the calculated value of *b* from above to generate a set of approximate theoretical values of the Japanese force levels, *y,* across the first 29 days. Engel chose to use only the first 29 days because although fighting continued and casualties were recorded to the 36th day, the island was officially declared secure on the 29th day. Engel (1954) "felt" that fighting after this declaration would be different enough as to cause a significant change to the attrition coefficients, *a* and *b*.

Engel (1954) then found the attrition coefficient of the U.S. forces, *a*, using the same technique as for finding *b*, but instead using a timeframe of 29 says instead of 36. Using this calculated value of 0.0544, he was able to calculate the set of fitted theoretical values of the U.S. force levels, *x,* across the first 29 days. With this, he can plot against the actual recorded number of U.S. troops in action (Figure 4).



Plot of "(b) Troops Alive" includes U.S. forces WIA and MIA and can be disregarded for the purposes of this paper.

Figure 4. Plot of U.S. troops in action at the Battle of Iwo Jima. Source: Engel (1954).

Although Engel (1954) only used eyeball approximation to qualitatively assess fitness, he found he had a good fit. When one uses an R-squared fitness test to quantitatively assess the fitness of Engel's square law model versus actual force levels, one gets a value of 0.9937. This is a very good fitness value that supports Engel's (1954) qualitative claim.

2. Improving upon Engel's Model

Engel (1954) found his best-fit model analytically, but in this section a best-fit model is found numerically through successive iteration. To do this, a pair of values for the attrition coefficients, a and b, were selected and used in equations (3) and (4) iteratively and sequentially from $t = 0$ to $t = 36$ days—the length of the battle. Essentially, a time-phased battle was reconstructed using the selected attrition coefficients, the known initial force levels of $x(0) = 54000$ and $y(0) = 21500$, the known final Japanese force level of $y(36) = 0$, and known reinforcement values, P, of 6000 and 13000 troops on the 3rd and 6th day, respectively. This process generated a time-phased series of force levels for the U.S. forces that could then be compared to the original force level data set using an R-squared fitness test.

This process was then repeated for a wide range of attrition coefficients, *a* and *b,* with each pair of coefficients having a calculated R-squared fitness associated with it. Due to there being no daily Japanese casualty data, one cannot determine the best-fit attrition coefficients, *a* and *b,* using a regression through the origin method, as in Fricker (1998). Therefore, a simple approach of systematic testing and checking many combinations was used. Initially, a broad range of attrition coefficient values were used to develop insight into the behavior of the model's performance as a function of *a* and *b* (Figure 5).

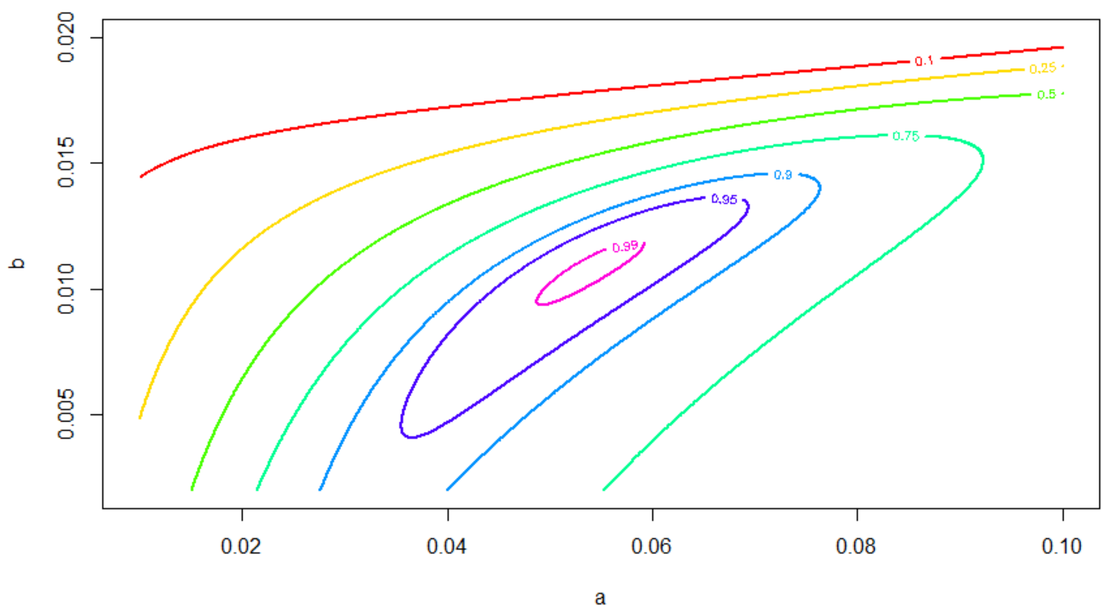


Figure 5. Contour plot of R-squared fitness of attrition coefficient *a*-*b* pairs for the square law model (no final Japanese force level restriction)

Once the general shape, trend, and convexity was clear, the number and range of attrition coefficients used was adjusted until the best (greatest) R-squared values among the set were no less than four significant figures different. In addition, the final Japanese force level constraint was introduced (being previously lifted to provide Figure 5) gradually towards the value of zero. This graduation is shown in Appendix A. square law model contour plots. The best (greatest) R-squared value among the set was then used to determine the attrition coefficients, *a* and *b,* that created the best-fit model. Figure 6 plots the actual U.S. force level versus the best fit obtained by this method.

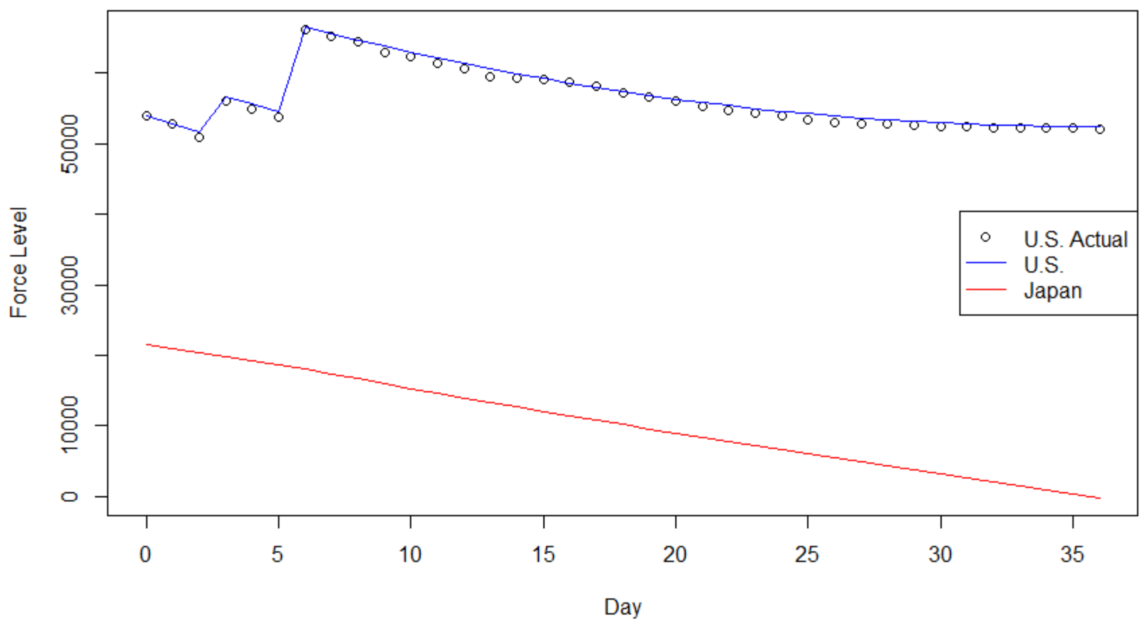


Figure 6. Plot of actual active U.S. troops in battle with U.S. and Japan approximate values generated from the square law model

After running through this process, it was found that the best-fit square law model has attrition coefficients, *a* and *b,* of 0.053150 and 0.010516, respectively. These are quite close to what Engle (1954) found. This model has a R-squared fitness of 0.9944—a slight improvement over Engel's model.

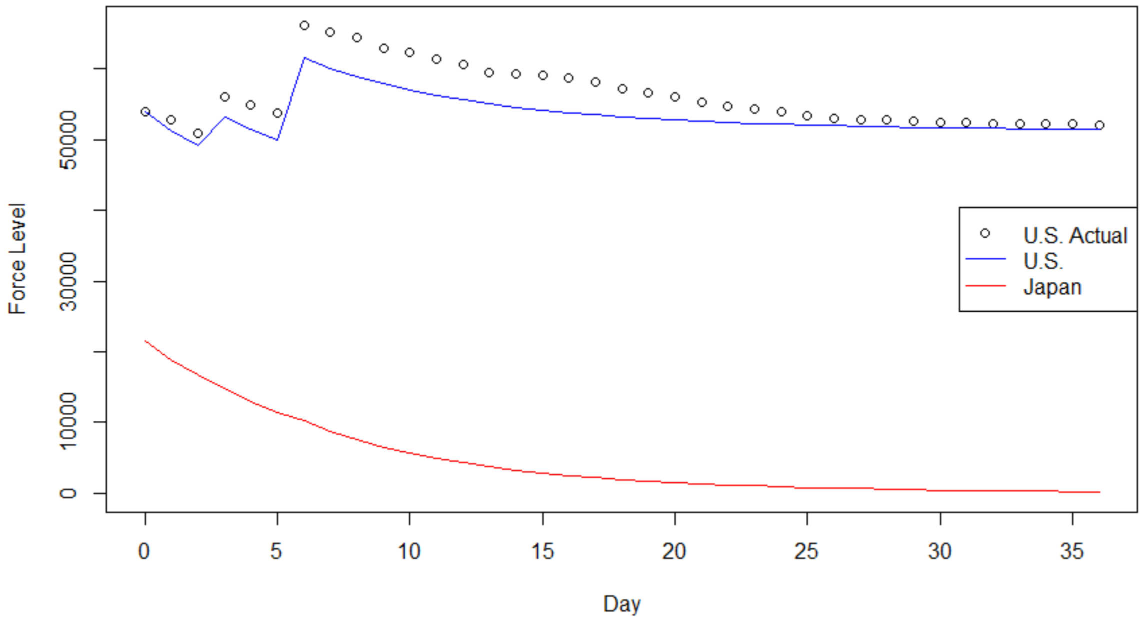
B. EXTENSIONS

1. Other Lanchester Models

a. linear law

Finding a best-fit linear law model numerically was set up the same way as was done with the square law above, except the equations used to calculate daily attrition are equations (1) and (2). Again, time-phased battles were reconstructed using a selected pair of attrition coefficients, a and b, from $t = 0$ to $t = 36$. The same known initial force levels, $x(0) = 54000$ and $y(0) = 21500$, and known reinforcement values, $P(3) = 6000$ and $P(6) = 13000$ were used in the reconstruction. However, the final Japanese force level of $y(36) = 0$ could not be used as a constraint, because models with an exponent variable, q, greater than zero cannot reach a force level of zero—they can only reach it *asymptotically*. Therefore, for the remainder of the models constructed in this paper, the final Japanese force level used as a known was $y(36) = 200$, or roughly 1% of the initial Japanese force level. This process still generated a time-phased series of force levels for the U.S. forces that could then be compared to the original force level data set using an R-squared fitness test.

This process was repeated for a wide range of attrition coefficients, a and b, with each pair of coefficients having either a calculated R-squared fitness associated with it or a value of zero (in the case of no possible solution that fit the final Japanese force level constraint). Again, due to there being no daily Japanese casualty data, one cannot determine the best-fit attrition coefficients, a and b, using a regression through the origin method. Therefore, a simple approach of systematic testing and checking many combinations was used. The same numerical approach as before was used to narrow the range of values of attrition coefficients tested. The associated contour plots are found in Appendix B. linear law model contour plots. The best (greatest) R-squared value among the set was then used to determine the attrition coefficients, a and b, that created the best-fit model—see Figure 7 for a plot of the fit.



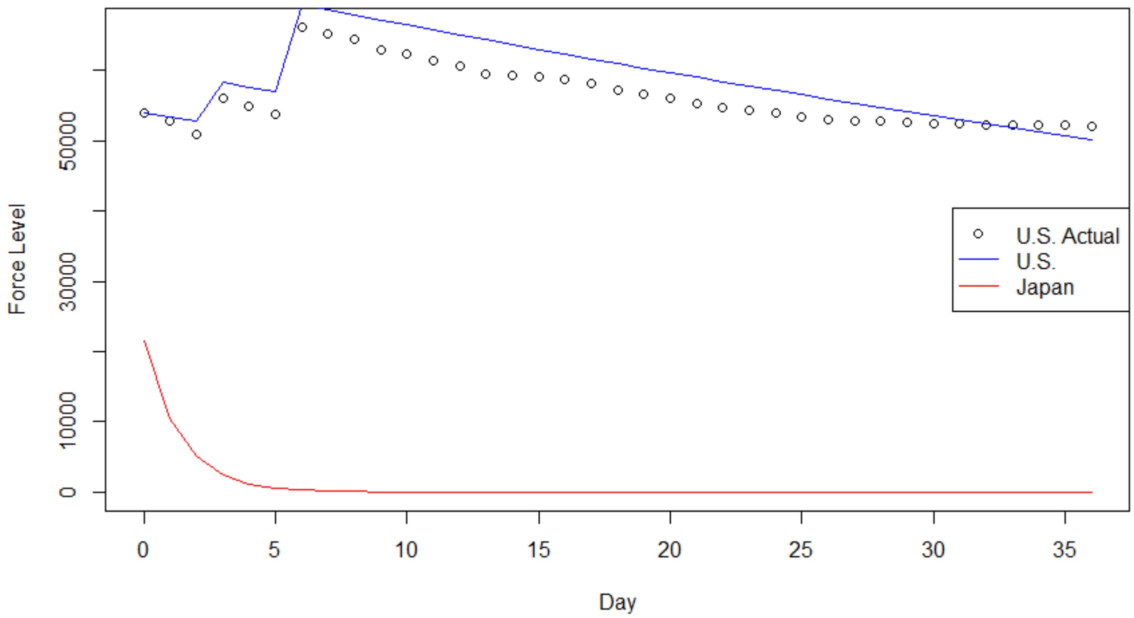
The best-fit linear law models show a more pronounced exponential curve for the Japanese forces than the square law models. Also note the U.S. forces fit is lower than the actual U.S. forces.

Figure 7. Plot of actual active U.S. troops in battle with U.S. and Japan approximate values generated from linear law model

After running through this process, it was found that the best-fit linear law model has attrition coefficients, *a* and *b,* of 2.304×10^{-6} and 2.280×10^{-6}, respectively. This model has a R-squared fitness of 0.9027—not as good as the square law models above, but still a good fit by most standards.

b. logarithmic law

Finding a best-fit logarithmic law model numerically was set up the same way as was done with the linear law above, except the equations used to calculate daily attrition were equations (5) and (6) when $p = 0$ and $q = 1$ (Fricker 1998). Again, time-phased battles were reconstructed using a selected pair of attrition coefficients, *a* and *b,* from $t = 0$ to $t = 36$. The same known initial force levels, $x(0) = 54000$ and $y(0) = 21500$, and known reinforcement values, $P(3) = 6000$ and $P(6) = 13000$ were used in the reconstruction. Just as in the linear law, because the exponent variable, q, is greater than zero, the final Japanese force level used as a known was $y(36) = 200$.



The best-fit logarithmic law models show an even more pronounced exponential curve for the Japanese forces than the linear law models. Also note the U.S. forces fit is mostly higher than the actual U.S. forces.

Figure 8. Plot of actual active U.S. troops in battle with U.S. and Japan approximate values generated from the logarithmic law model

After running through this process, it was found that the best-fit logarithmic law model has attrition coefficients, a and b, of 0.01075 and 0.5160, respectively. This model has a R-squared fitness of 0.9414—not as good as the square law models, but surprisingly better than the linear law model.

2. Unconstrained *p* and *q* (Bracken)

Next, the above process of determining the attrition coefficients, a and b, for the best-fit model was done repeatedly for a wide range of exponent variables, p and q. This model will be referred to as the Bracken model in this paper. The general form equations

(5) and (6) were used to calculate daily attrition with values of the exponent variables, *p* and *q*, ranging from *p, q* = 0 to *p, q* = 1.5.

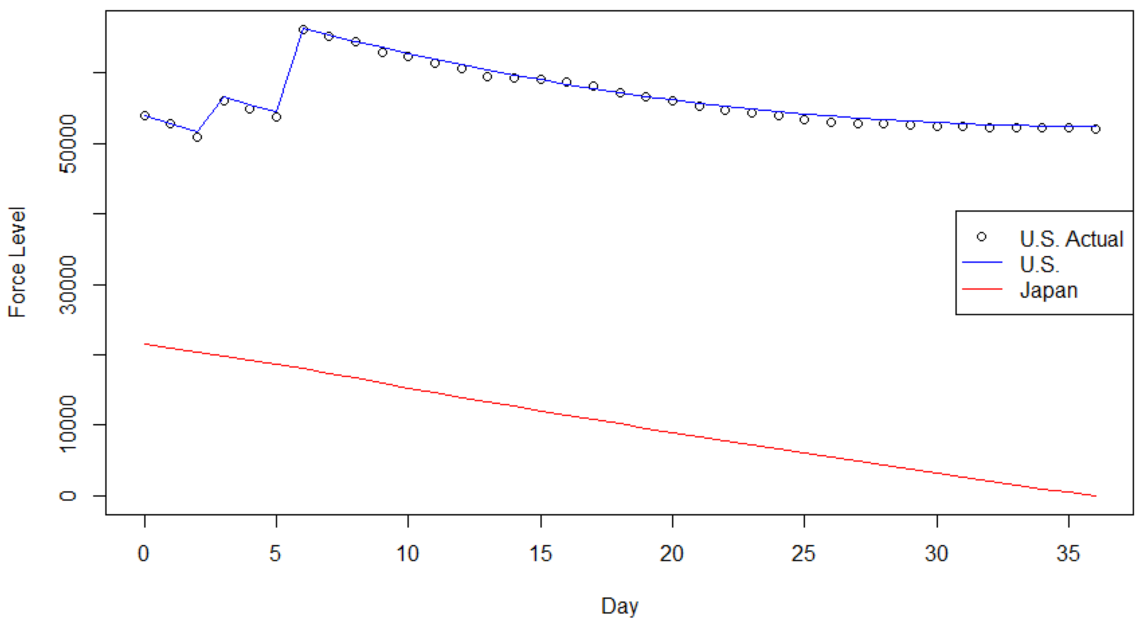


Figure 9. Plot of actual active U.S. troops in battle with U.S. and Japan approximate values generated from the Bracken model

The result is a matrix of *p-q* pairs, each with a numerically calculated best-fit model, R-squared value, and attrition coefficients, *a* and *b.* The numerical calculation method used was very similar to the other model processes, except for the added behavior of the attrition coefficients, *a* and *b*, for each *p-q* pair. The selection of the range of *a*'s and *b*'s used for a given *p-q* pair changed several orders of magnitude depending on the *p-q* pair, as shown in Appendix D. Bracken model contour plots. The best-fit model calculated was for exponent variables of *p* = 1.05 and *q* = 0, having an R-squared fitness of 0.9946 and attrition coefficients, *a* and *b,* of 0.03311 and 0.006109, respectively. The best fit is very close to the square law.

The contour plot of fitness values for the various *p-q* pairs (Figure 10) shows two interesting things. First, there is a significantly large range of models that can claim a very good fit (R-squared greater than 0.99.). Second, the optimum model is likely to have a negative exponent variable, *q*.

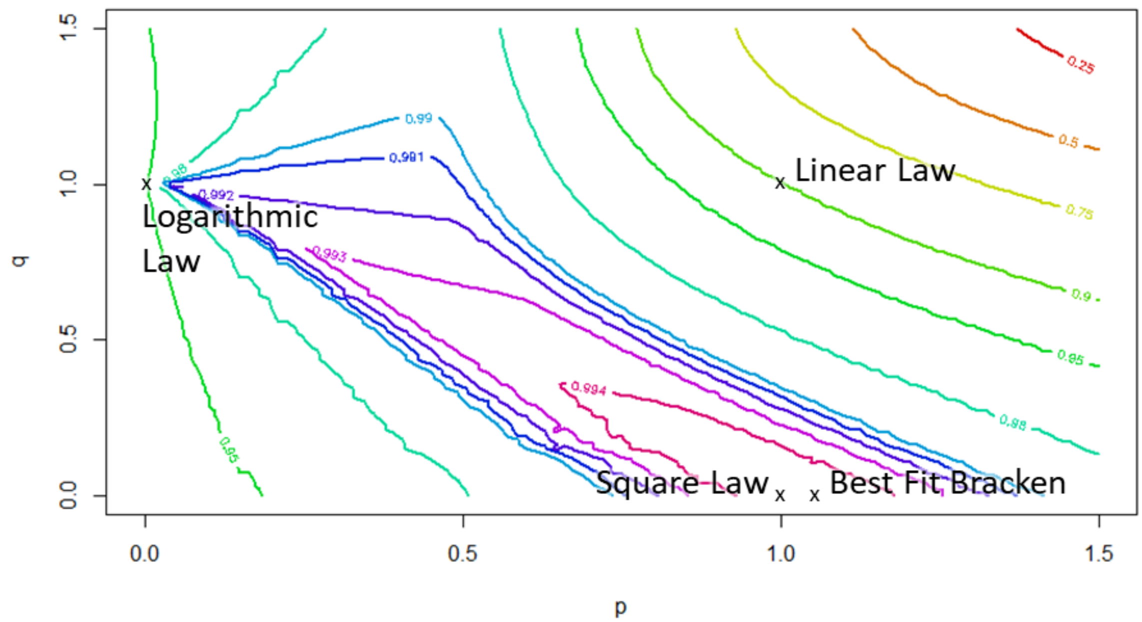


Figure 10. Contour plot of R-squared fitness of *p*-*q* pairs from Bracken model

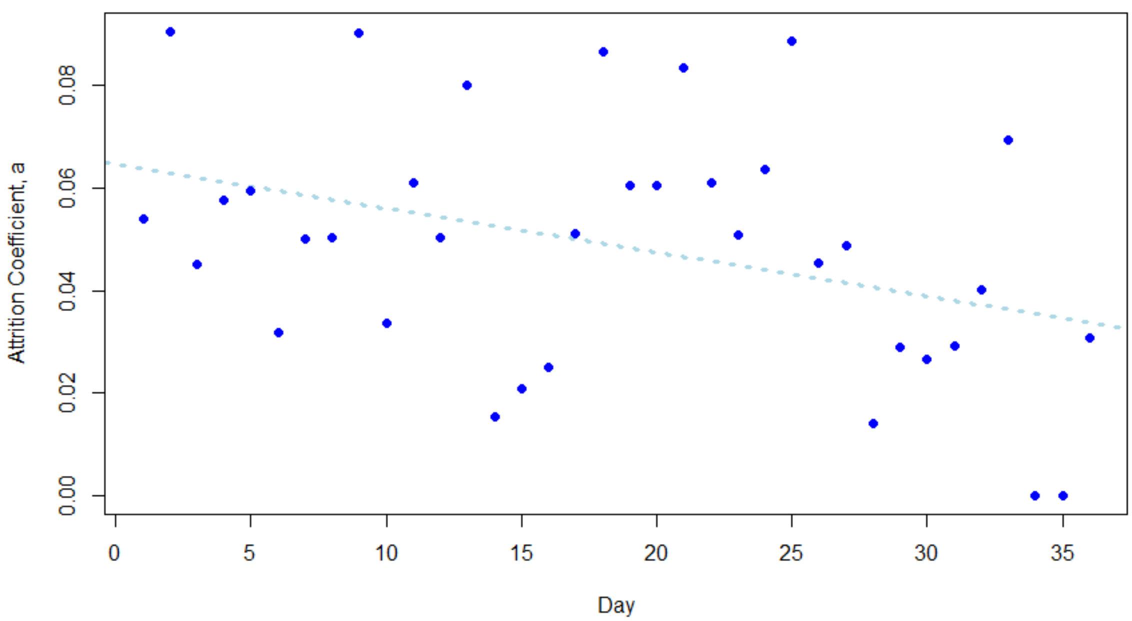
This contour plot is the result of over 100 *p*'s and 100 *q*'s for a total of over 10,000 models. For each of these, thousands of attrition coefficients were used to obtain a fit. One should also carefully note the R-squared fitness levels chosen for this contour plot. These levels show the general trend of performance across good (greater than 0.99) and bad (less than 0.99) models. For a constant level step contour plot, see Appendix D. Bracken model contour plots. Of note is the wide range of possible *p* and *q* values that yield good fits.

3. Changing Attrition Coefficient

The final extension explored was the concept of attrition coefficients that change throughout the course of the battle—even daily. The two main limitations encountered when exploring this extension with the Iwo Jima data set stem from the issue of only having time-phased data for one of the belligerents. The first limitation was that only the time-phased attrition coefficient for the U.S. forces, *a*, could be usefully calculated and analyzed.

The second was that the time-phased Japanese force level data, *y*, was limited to a theoretical approximation generated from a model. The model used to generate the theoretical approximation for this extension was the square law model created by Engel (1954).

The actual recorded number of U.S. troops in action, *x*, along with the theoretical approximation time-phased Japanese force level, *y* (generated from Engel's model with attrition coefficients, *a* and *b,* of 0.0544 and 0.0106, respectively), were used in equation (3) to solve for the time-phased attrition coefficient for the U.S. forces, *a*, for each day of the battle. As expected, the attrition coefficient has a high variance when calculated daily in this manner. This is in stark contrast to the fits assuming constant attrition coefficients throughout the battle. We also observe an overall downward trend, which is highlighted by the simple linear regression (Figure 11).

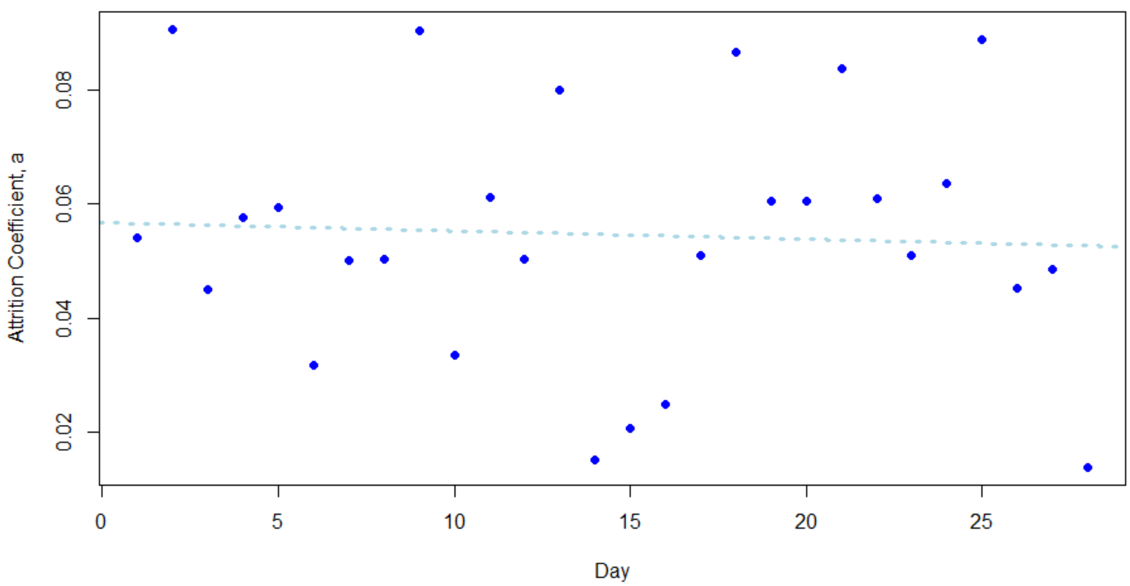


The negative slope, β, of this downward trend is statistically significant with a p-value < 0.0132.

Figure 11. Attrition coefficient, *a*, calculated daily with linear trendline

The downward trend of the U.S. attrition coefficient, a, over time indicates either an increase in U.S. combat efficiency ($^{b}/_{a}$) over time or a decrease in battle intensity (ab) over time. However, since the actual Japanese force levels are not known, the actual Japanese attrition coefficient, b, is also not known. Therefore, while neither efficiency nor intensity can be confirmed, this finding is consistent with the concept of a battle having diminishing intensity as the battle progresses. Finally, it is worth noting that using a deterministic approximation to a stochastic variable may result in biased outcomes (Lucas 2000).

The process of fitting a linear regression to the time-phased attrition coefficient for the U.S. forces, a, for each day of the battle was repeated for a battle duration of 29 days vice 36. Figure 12 shows the resulting trendline for the truncated dataset.



The negative slope, β, of this downward trend is not statistically significant with a p-value < 0.7831.

Figure 12. Attrition coefficient, a, calculated daily with linear trendline (29 days)

The result this time was a regression line with still a negative slope; however, the slope was not significant at any meaningful level (p-value < 0.7831). This means that either

Engel's (1954) concerns about the additional seven days affecting the attrition coefficients were merited in this case or there are not enough data points to show overwhelming statistical support towards any hypothesis.

IV. CONCLUSIONS AND RECOMMENDATIONS

A. CONCLUSIONS

1. Results Summary

"The homogeneous Lanchester square law cannot be regarded as a proven attrition algorithm for warfare; however, the square law cannot be regarded as disproved either" (Hartley and Helmbold 1995).

Not only did Engel's (1954) model provide a very good fit quantitatively speaking ($R_{squared}$ = 0.9937), but it performed just as well (and with nearly the same parameters) as the best/optimal calculated square law and Bracken models. The tabulated results (Table 1) summarize the findings.

Table 1. Model fitness and results summary

Model	a	b	p	q	R^2
Engel	0.0544	0.0106	1	0	0.9937
Square	0.0532	0.0105	1	0	0.9944
Linear	2.30×10^{-6}	2.28×10^{-6}	1	1	0.9027
Logarithmic	0.0108	0.516	0	1	0.9414
Bracken	0.0331	0.00611	1.05	0.00	0.9946

The table shows that the best-fit Bracken model has exponent variables, *p* and *q*, very similar to the square law model ($p = 1$, $q = 0$). Also of note is the similar effectiveness ratio found in the Engel, square law, and Bracken models ($a/b \cong 5$). This makes sense as the exponent variables, *p* and *q*, are very similar across all three models.

While the linear and logarithmic law models did not perform as well as the other models, they still showed R-squared fitness values above 0.90. Furthermore, when analyzing the contour plot of the fitness values of the various Bracken models (Figure13), one sees a relatively large range of models with exponent variables, *p* and *q*, that give R-squared fitness values greater than 0.99.

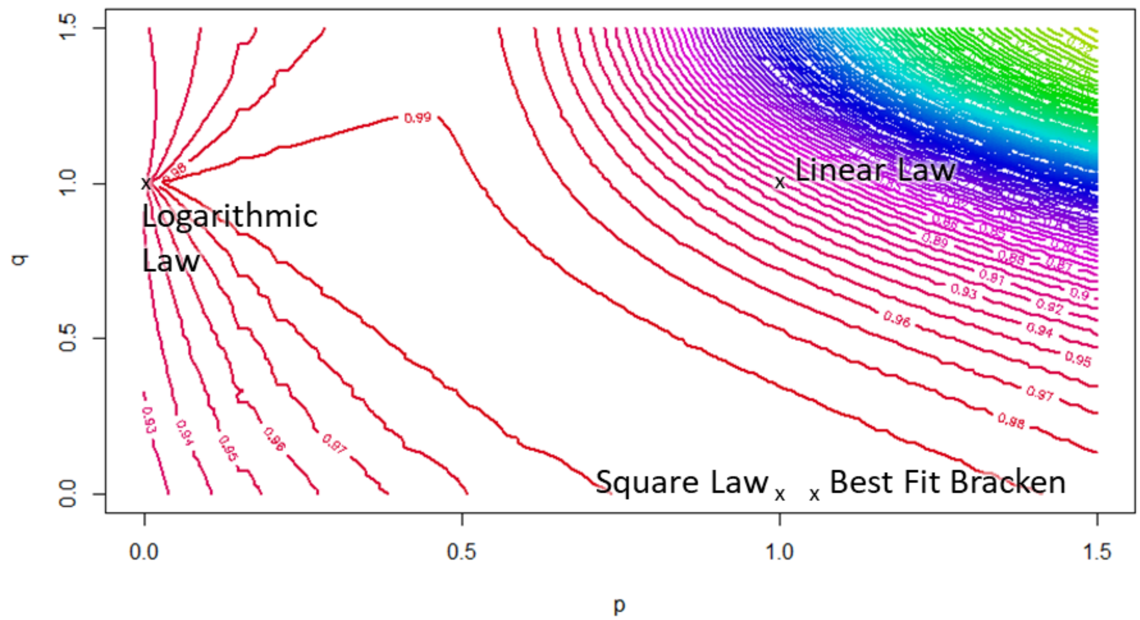


Figure 13. Contour plot of R-squared fitness of p-q pairs from Bracken model (constant increments)

The contour plot of R-squared fitness values at increments of 0.01 (Figure 13) highlights the relative size of the area of models with *p-q* pairs that can provide a fitness above 0.99. While the linear and logarithmic law models do not quite fit within this area, there are many other models that do. This finding rejects the notion that the square law-like models are the only ones with a very good fit.

2. Discussion

> To the extent that both sides concentrate their forces, the Lanchester square equations should be expected to hold. If, however, the campaign is best characterized as a collection of small engagements where the quantitatively superior side is not usually locally superior, the Lanchester linear equations should hold. (Bracken, 1995)

The large range of models that provide a very good fit may be explained by the fact that in this battle only one of the belligerent's data is known. This means that when a model is only judged on its fitness towards one belligerent (the U.S.), it has much more freedom with regards to the unknown and unjudged belligerent (the Japanese.) This freedom is seen very clearly between the square law and linear law forces plots (Figure 6 and Figure 7) and the differences of each's theoretical approximation of the Japanese forces (red line.) The models only need to satisfy the constraints of the first and last day for the Japanese force levels. If the Japanese daily attrition data was known, the fitness performance of the models would likely change drastically.

The circumstances of a battle matter. The type of battle seen at Iwo Jima is extremely rare. It was a fight to the death, there were no shifts in offense or defense, and it was consistent and continuous fighting and fighting style from start to finish (no massively decisive maneuvers or flanking involved.) The attrition models that fit the Battle of Iwo Jima will not necessarily fit other battles very well. The factors that must be considered range in the thousands (such as technology, terrain, weather, leadership, morale, political pressures, culmination points, etc.), and determine what kind of model is best to use.

Although the attrition coefficients for the best-fit Bracken models followed a general pattern as a function of the *p-q* pair (Figure 14 and Figure 15), the range of valid *a-b* pairs for a given *p-q* pair was often extremely narrow.

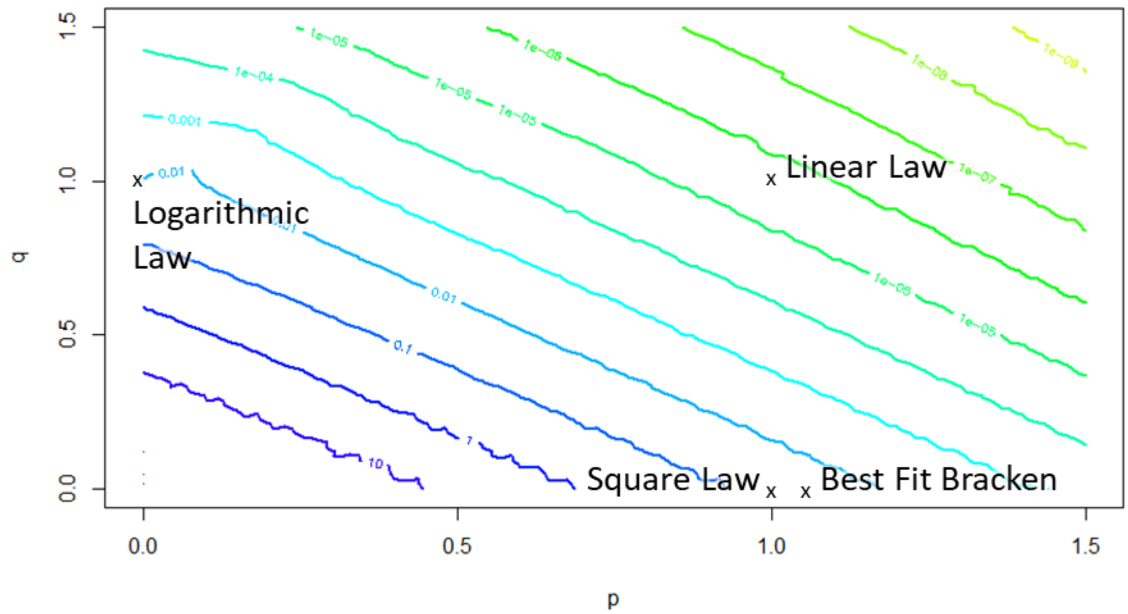


Figure 14. Contour plot of attrition coefficients, *a*, for best-fit Bracken models

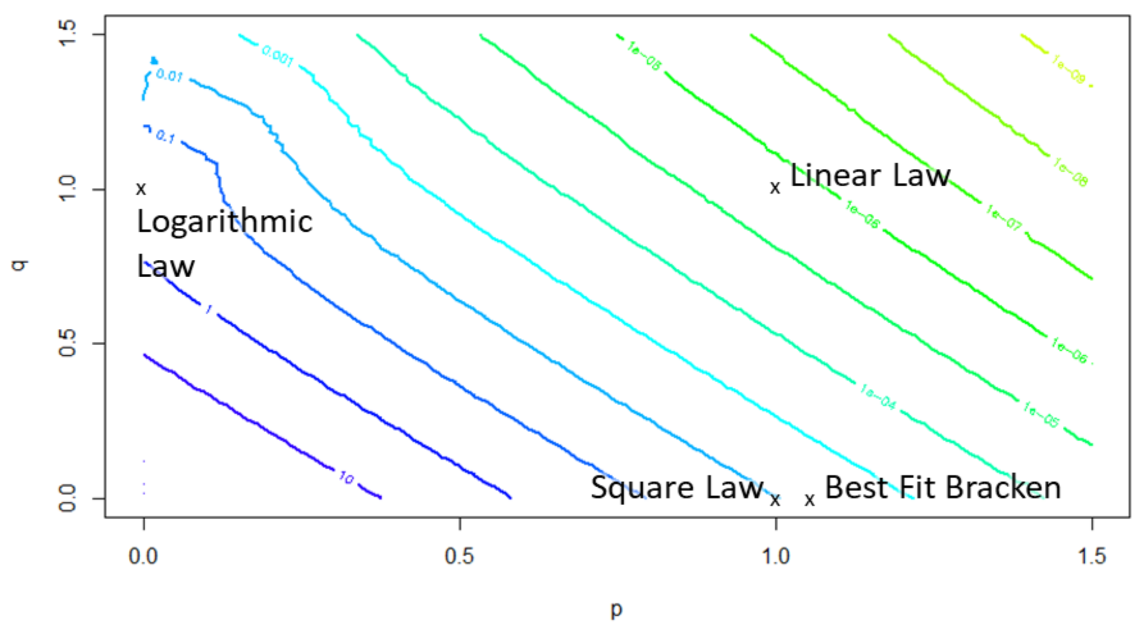


Figure 15. Contour plot of attrition coefficients, *b*, for best-fit Bracken models

In some instances, the range of valid *a*-*b* pairs required precision of five significant figures or more to detect. It was analogous to finding a needle in the Pacific Ocean or a speck of space debris orbiting the earth. The Bracken model contour plot (Figure 10), for example, was the result of over 3.2 billion iterations of battled simulated and tested for fitness. This was despite the knowledge of the *a*-*b* pairs following a general pattern.

B. RECOMMENDATIONS FOR FUTURE WORK

- More extensions could be of value. Specifically, having different equations for each belligerent (different set of exponent variables, *p* and *q*, for the U.S. and Japanese forces.) However, special care should be taken during analysis to address the fact that the Japanese attrition values are theoretical approximations and allow much more flexibility of all the models—potentially masking actual model performances.

- The extension of the Bracken model into the negative exponent variables, *p* and *q*. Specifically, the optimum Bracken model had $q = 0$, which is at the lower limit of the range of the variable tested. The optimum Bracken model could very well have a negative exponent variable, *q*. While not traditionally accepted as sensible, a negative *q* does have real-world application. A negative *q* value for a force essentially means that those forces attrite slower as their numbers increase and faster as their number decrease. This concept could be explained by combat performance affected by morale which, in turn, is affected by the apparent strength in numbers. Other examples include ancient warfare phalanx formations and modern warfare where cover and covering fire is paramount and dependent on force density (such as certain tank warfare or the Battle of Iwo Jima.)

- More work is needed with the concept of changing attrition coefficients, *a* and *b*. Possible extensions include performance of models with changing attrition coefficients, affects of changing the attrition coefficients at frequencies lower than daily, and exploring attrition coefficients changing in a non-linear fashion.

- The constraint of the final Japanese force level of 200 was chosen arbitrarily for the most part. There is value in exploring in more detail the merits of other constraint values and analyzing performance impacts on models, specifically the linear law model.
- Finally, the methods and models shown here, specifically the changing coefficients, have applicability to other battle data sets. Any battle where time-phased attrition data exists for both belligerents is of great interest.

APPENDIX A. SQUARE LAW MODEL CONTOUR PLOTS

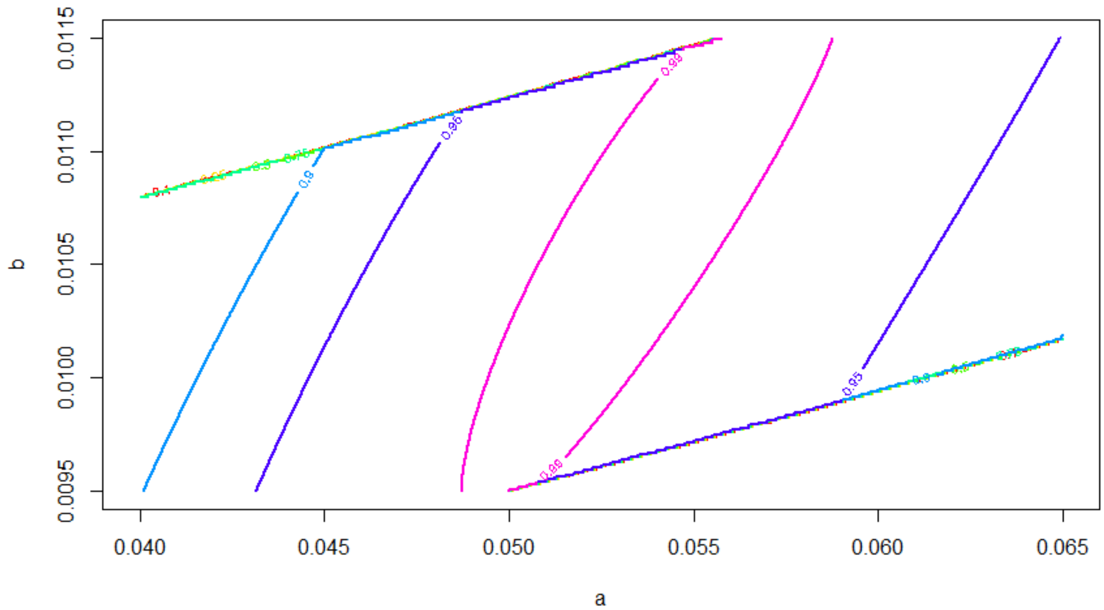


Figure 16. Contour plot of R-squared fitness for the square law model (final Japanese force level restriction $y(36) = 2000$)

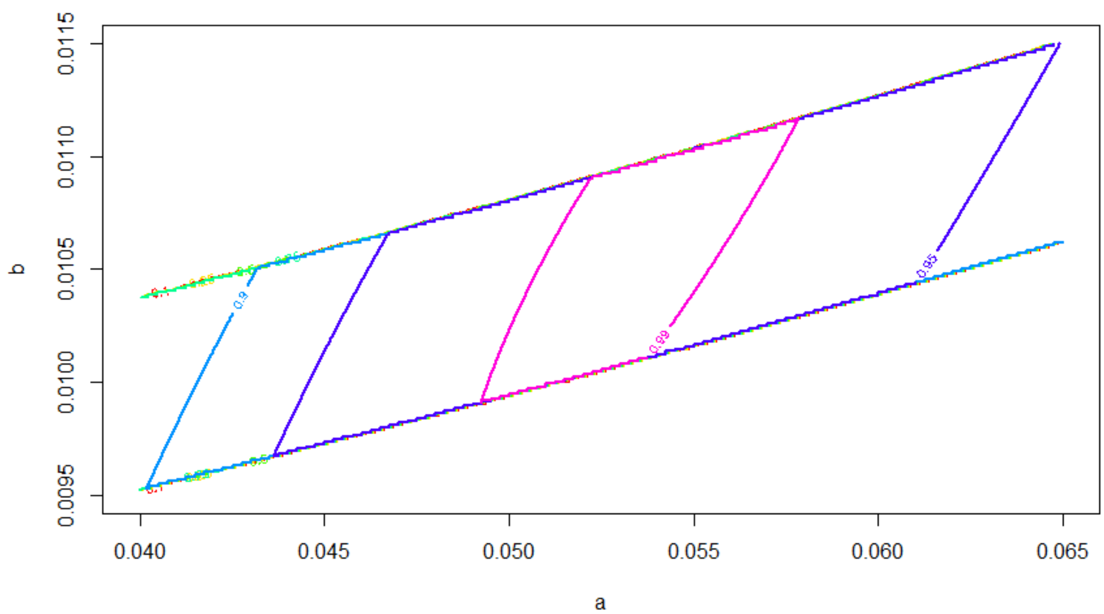


Figure 17. Contour plot of R-squared fitness for the square law model (final Japanese force level restriction $y(36) = 1000$)

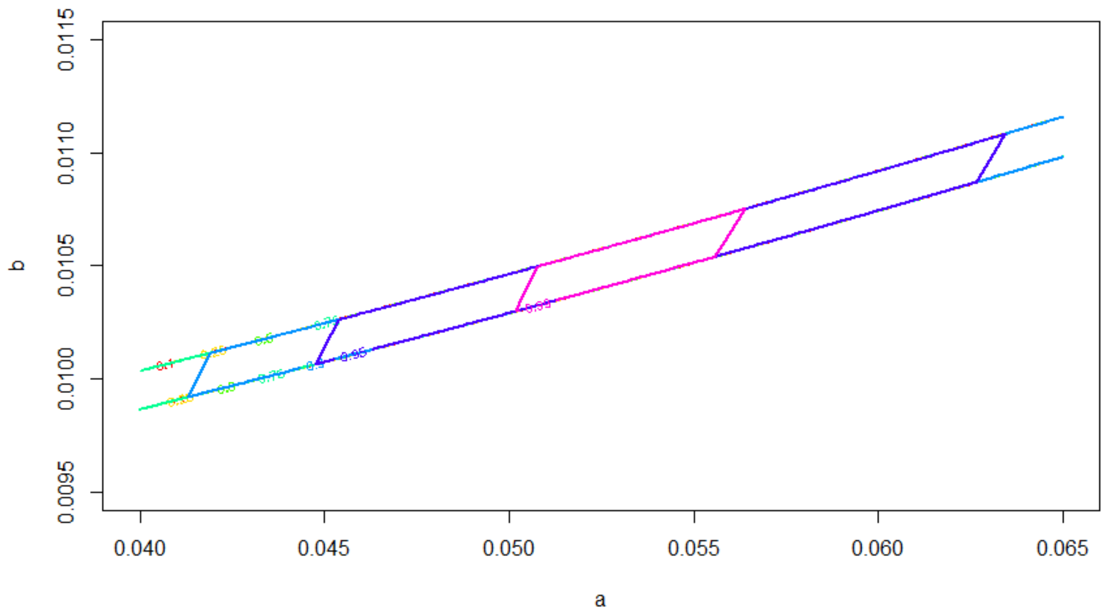


Figure 18. Contour plot of R-squared fitness for the square law model (final Japanese force level restriction $y(36) = 200$)

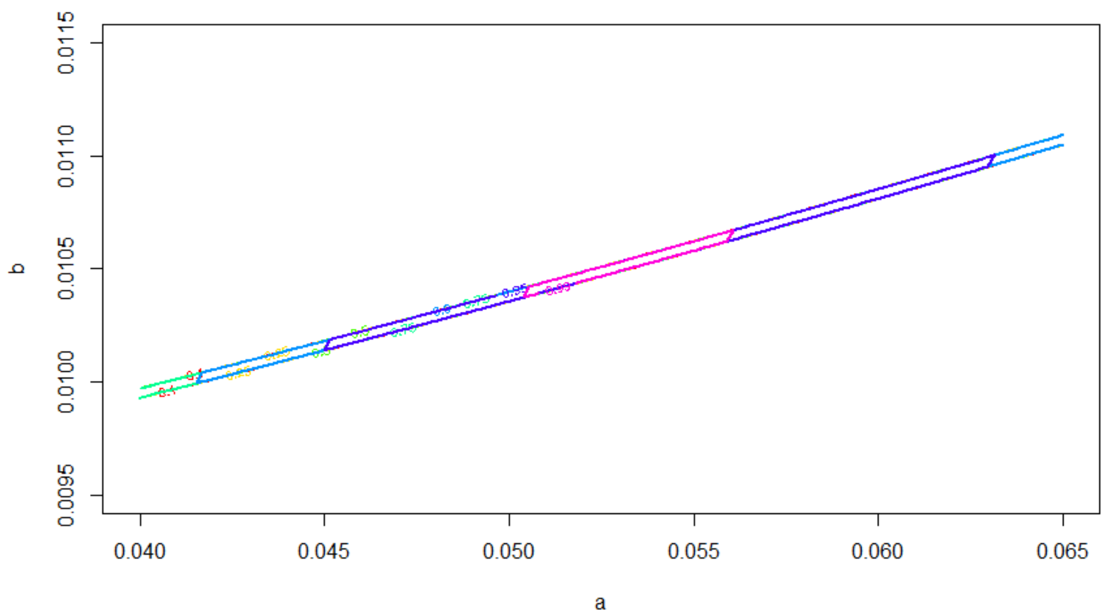


Figure 19. Contour plot of R-squared fitness for the square law model (final Japanese force level restriction $y(36) = 50$)

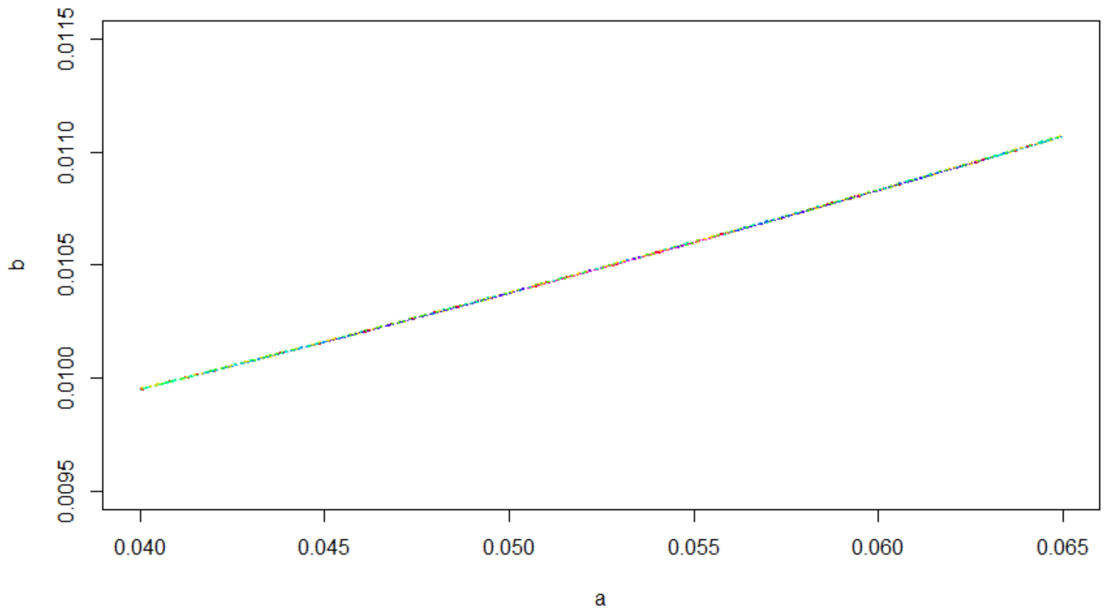


Figure 20. Contour plot of R-squared fitness for the square law model (final Japanese force level restriction $y(36) = 1$)

THIS PAGE INTENTIONALLY LEFT BLANK

APPENDIX B. LINEAR LAW MODEL CONTOUR PLOTS

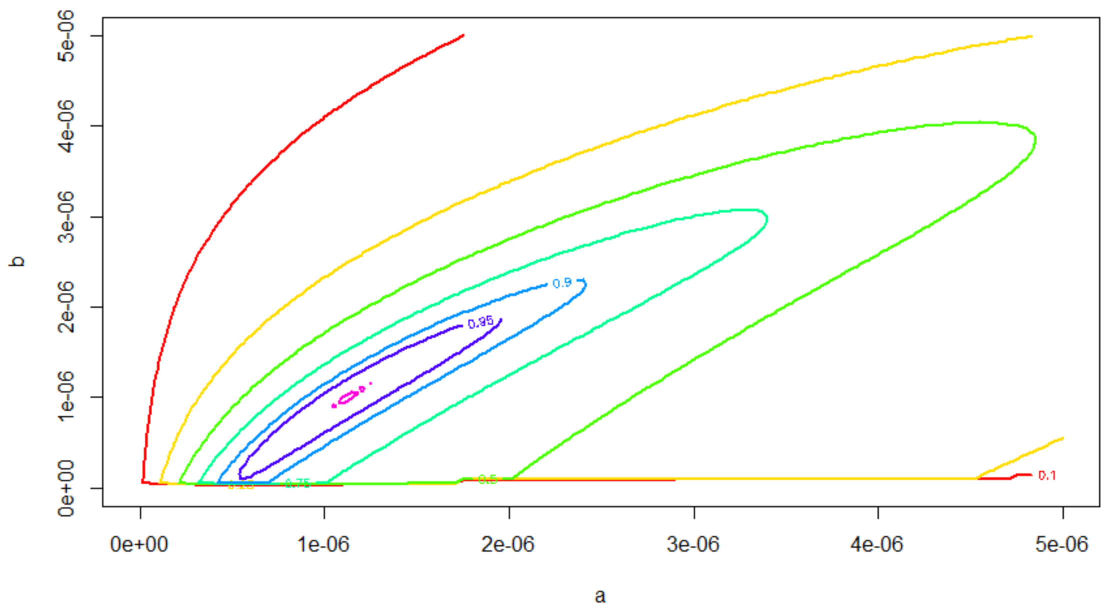


Figure 21. Contour plot of R-squared fitness for the linear law model (final Japanese force level restriction $y(36) = 20000$)

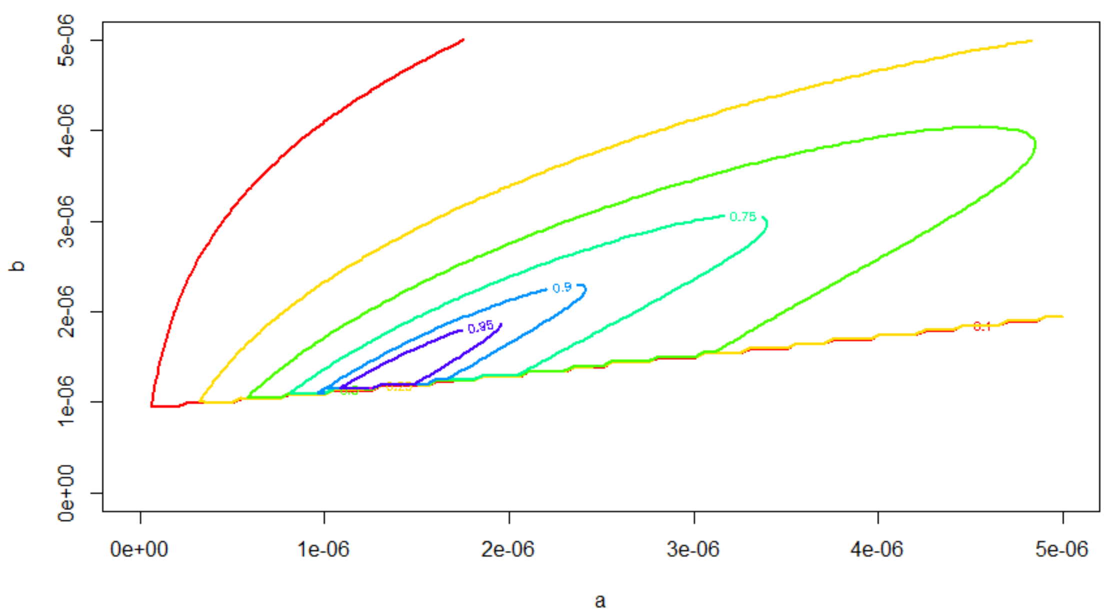


Figure 22. Contour plot of R-squared fitness for the linear law model (final Japanese force level restriction $y(36) = 2000$)

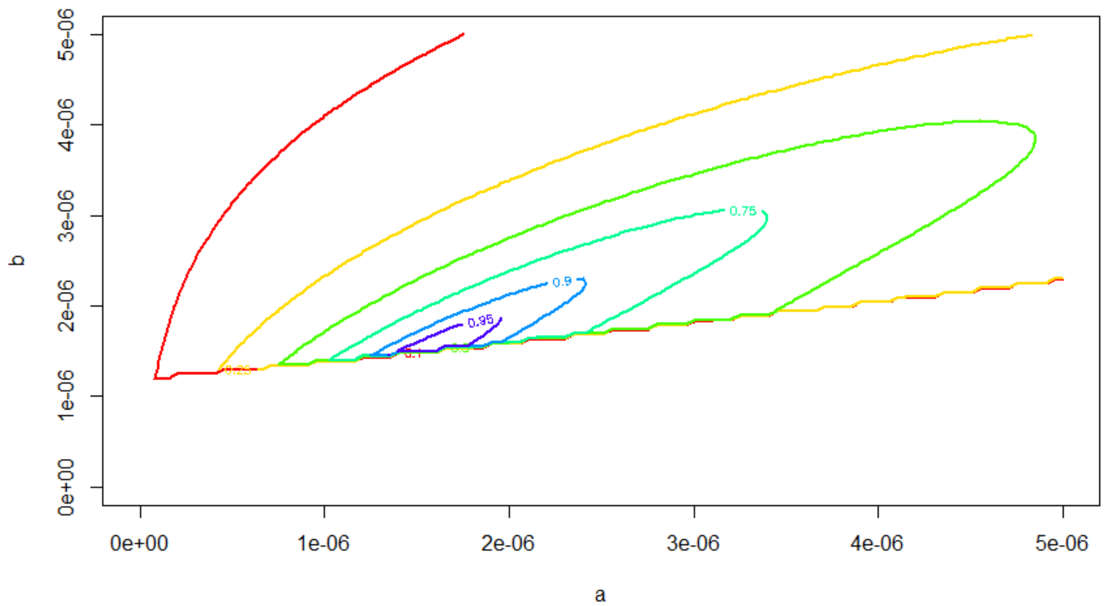


Figure 23. Contour plot of R-squared fitness for the linear law model (final Japanese force level restriction $y(36) = 1000$)

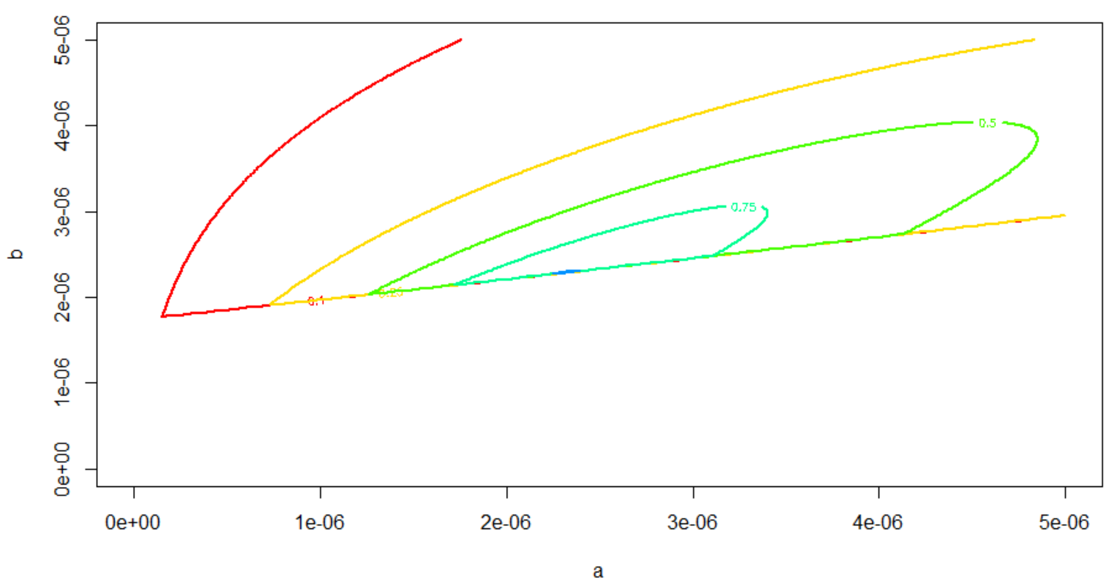


Figure 24. Contour plot of R-squared fitness for the linear law model (final Japanese force level restriction $y(36) = 200$)

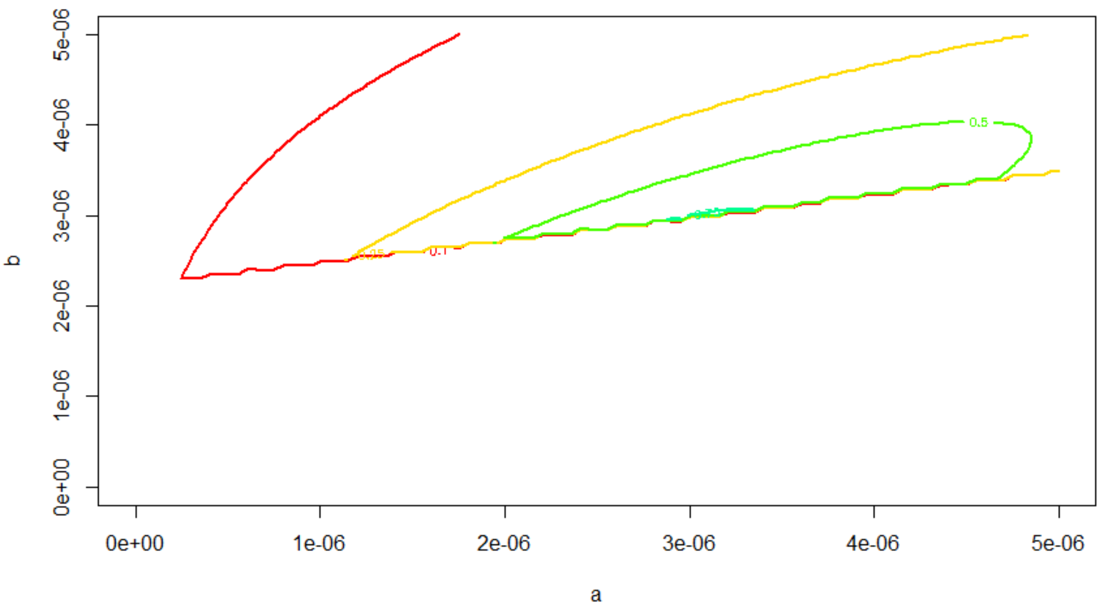


Figure 25. Contour plot of R-squared fitness for the linear law model (final Japanese force level restriction $y(36) = 50$)

THIS PAGE INTENTIONALLY LEFT BLANK

APPENDIX C. LOGARITHMIC LAW MODEL CONTOUR PLOTS

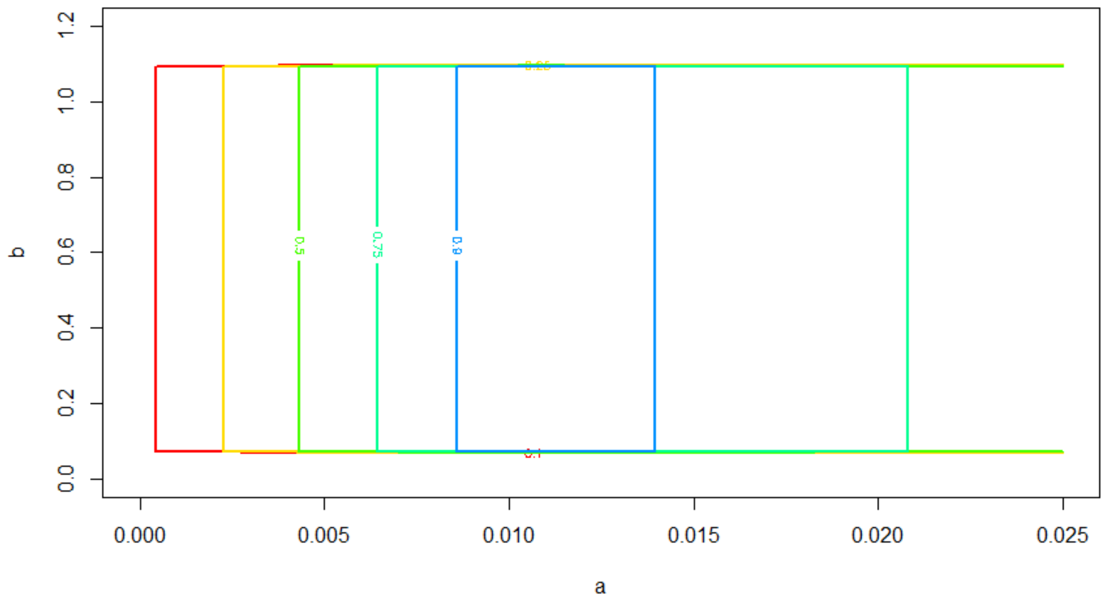


Figure 26. Contour plot of R-squared fitness for the logarithmic law model (final Japanese force level restriction $y(36) = 2000$)

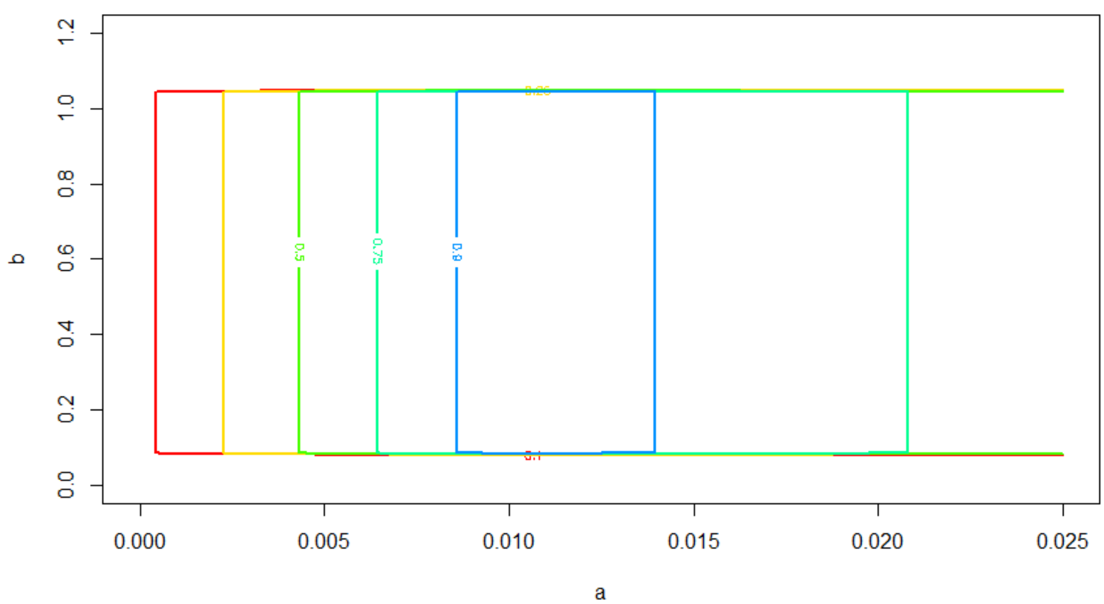


Figure 27. Contour plot of R-squared fitness for the logarithmic law model (final Japanese force level restriction $y(36) = 1000$)

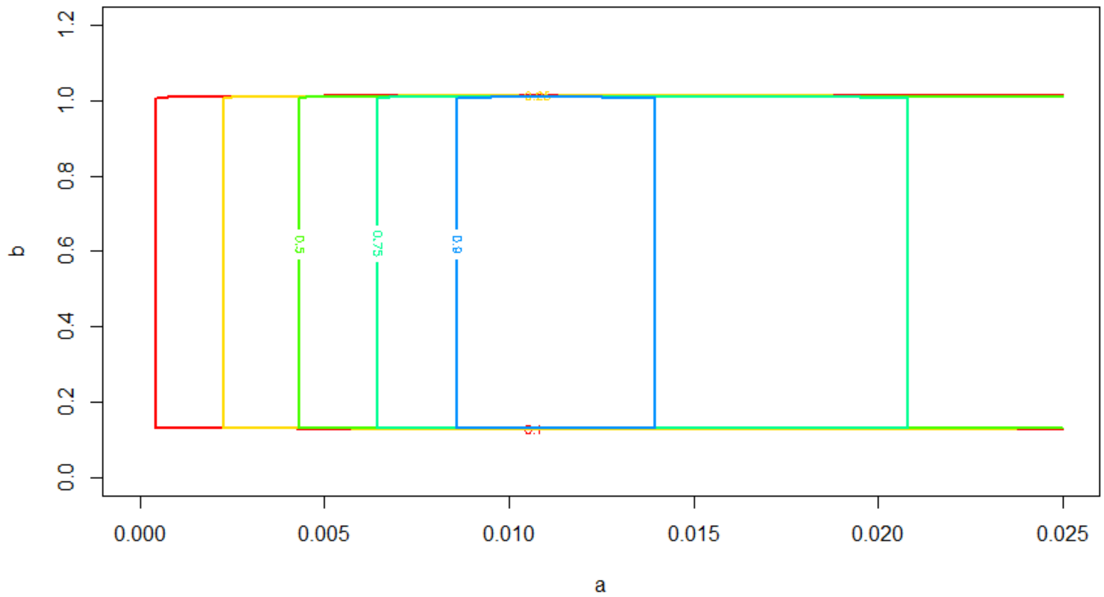


Figure 28. Contour plot of R-squared fitness for the logarithmic law model (final Japanese force level restriction $y(36) = 200$)

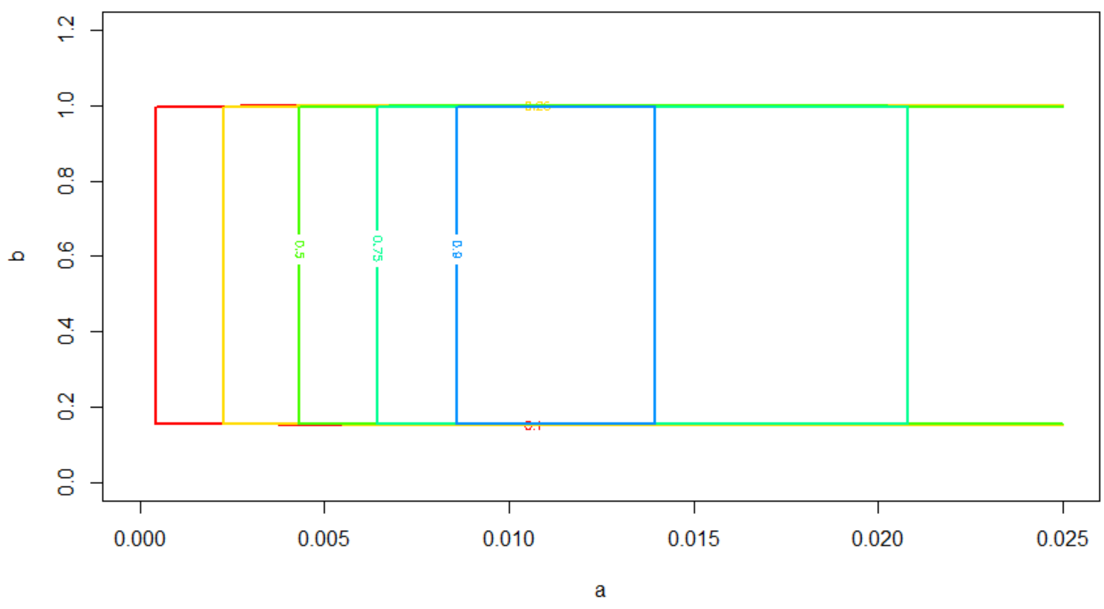


Figure 29. Contour plot of R-squared fitness for the logarithmic law model (final Japanese force level restriction $y(36) = 50$)

APPENDIX D. BRACKEN MODEL CONTOUR PLOTS

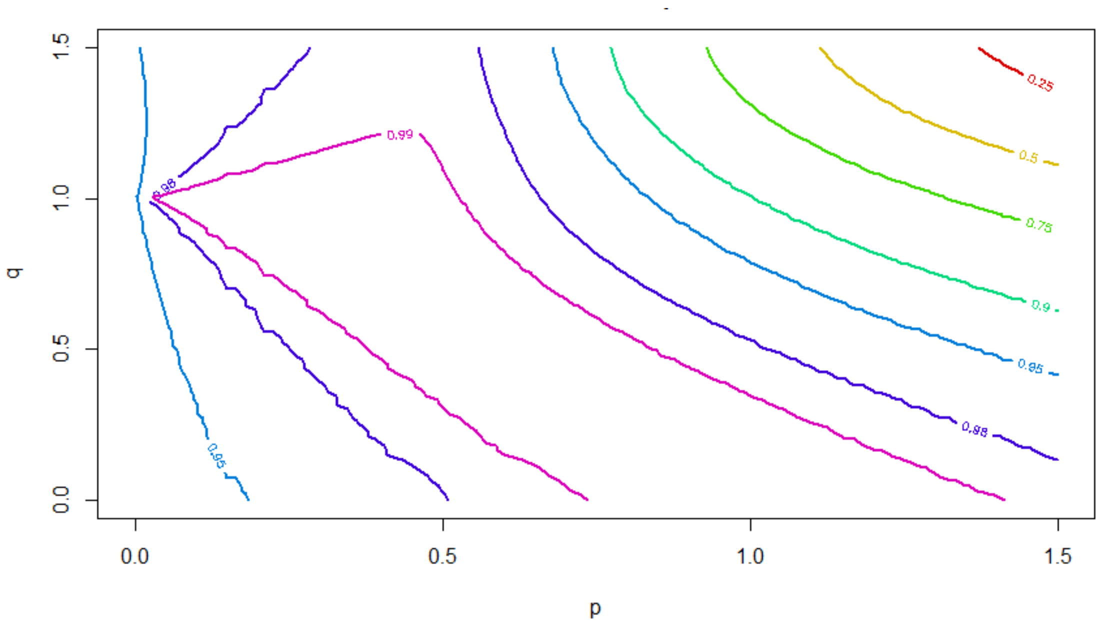


Figure 30. Contour plot (no model labels) of R-squared fitness of *p*-*q* pairs from the Bracken model

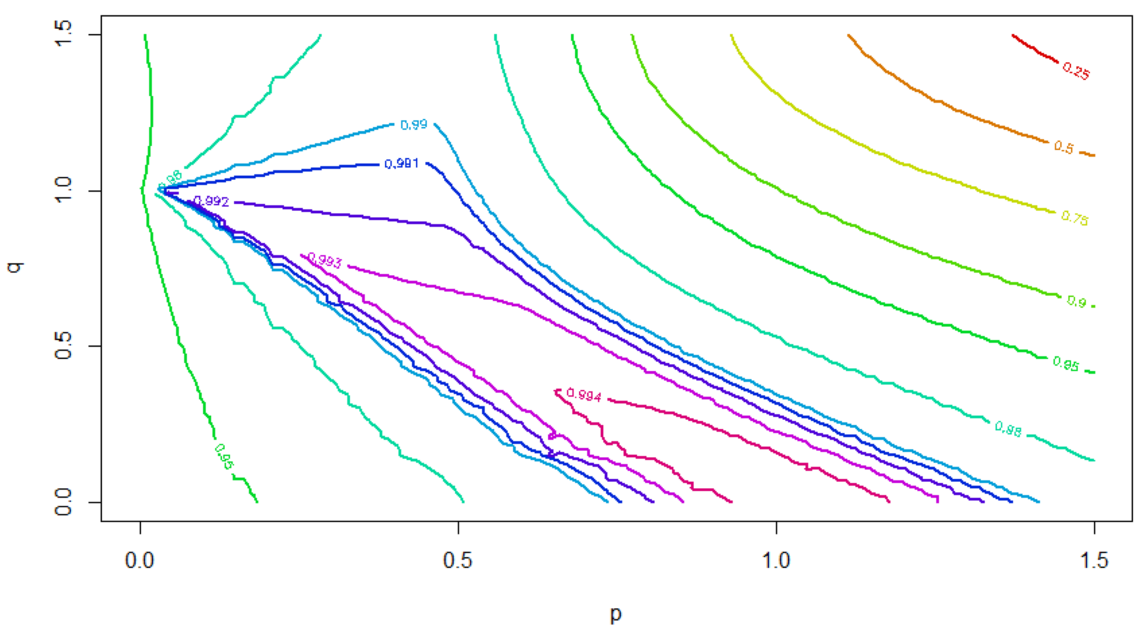


Figure 31. Contour plot (no model labels) of R-squared fitness of *p*-*q* pairs from the Bracken model (expanded high-fitness)

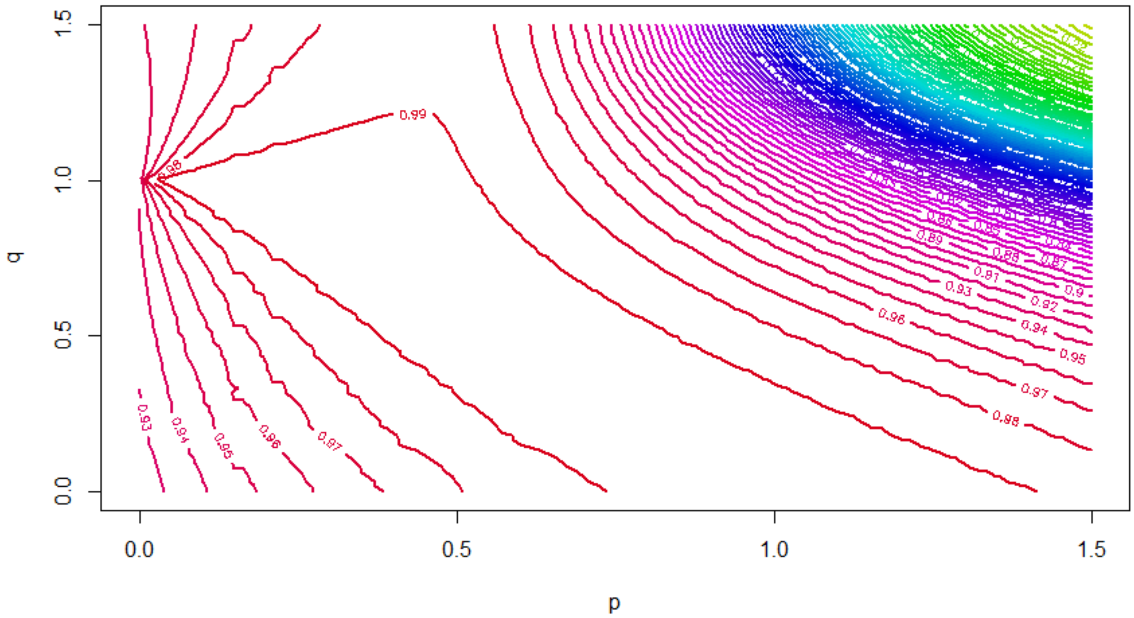


Figure 32. Contour plot (no model labels) of R-squared fitness of *p*-*q* pairs from the Bracken model (constant increments)

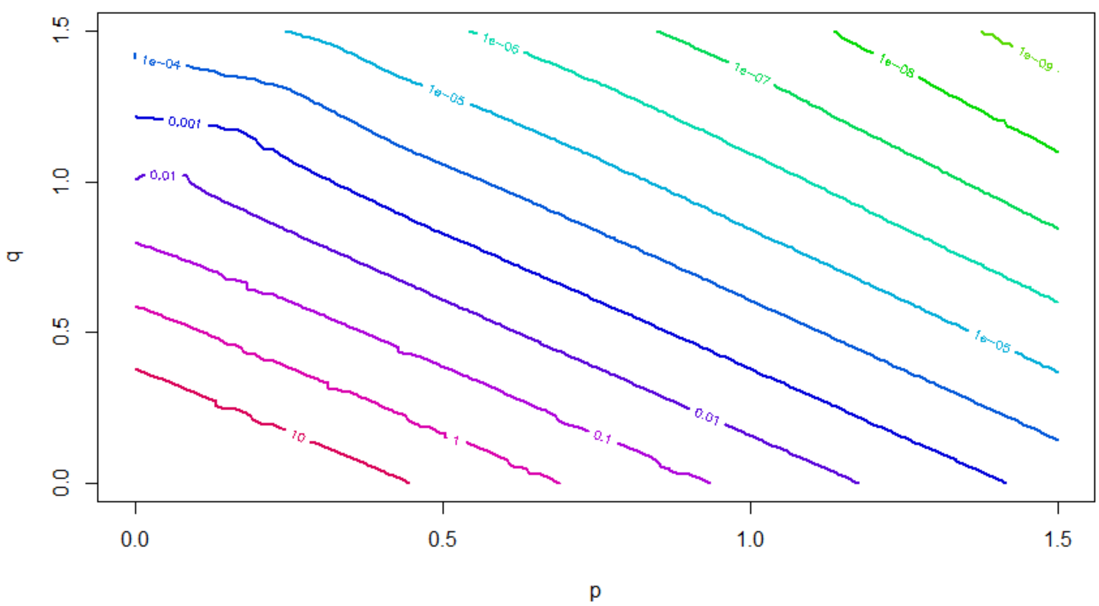


Figure 33. Contour plot (no model labels) of attrition coefficients, *a*, for the best-fit Bracken models

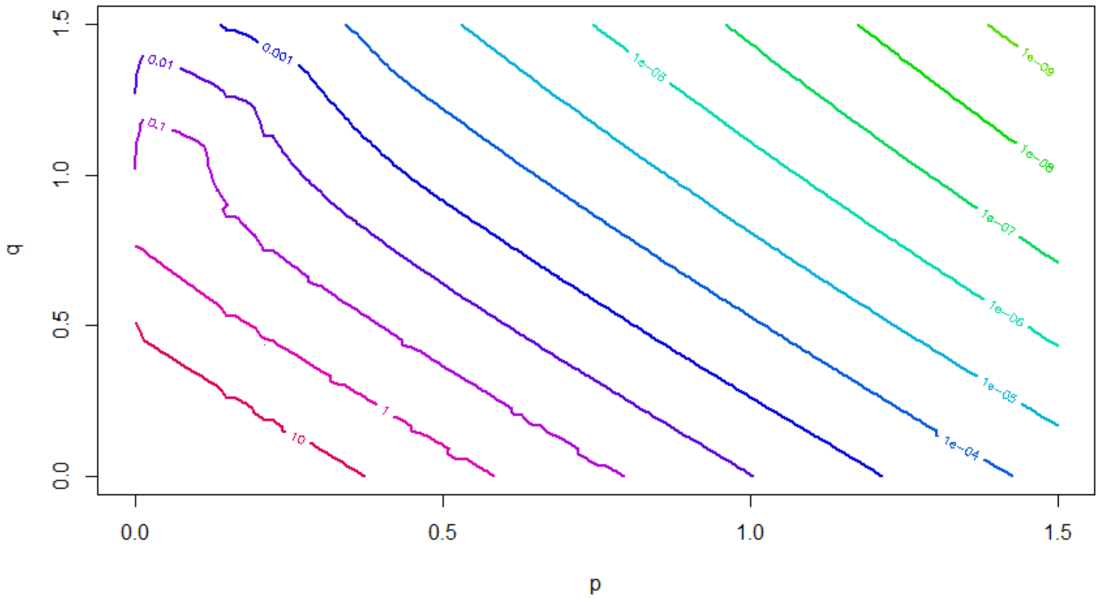


Figure 34. Contour plot (no model labels) of attrition coefficients, *b*, for the best-fit Bracken models

THIS PAGE INTENTIONALLY LEFT BLANK

APPENDIX E. R PROGRAM CODE

```
##### Givens #####

x_true = c(54000, 52839, 50945, 56026,
           54885, 53744, 66150, 65245,
           64373, 62869, 62334, 61400,
           60662, 59544, 59340, 59076,
           58774, 58191, 57254, 56636,
           56055, 55303, 54791, 54393,
           53933, 53342, 53067, 52799,
           52730, 52603, 52502, 52407,
           52299, 52150, 52150, 52150,
           52135)

x_add = c(0,  0,  0,  6000,  0,  0,
          13000,  0,  0,  0,  0,  0,
          0,  0,  0,  0,  0,  0,
          0,  0,  0,  0,  0,  0,
          0,  0,  0,  0,  0,  0,
          0,  0,  0,  0,  0,  0,
          0)

y_add = c(0,  0,  0,  0,  0,  0,
          0,  0,  0,  0,  0,  0,
          0,  0,  0,  0,  0,  0,
          0,  0,  0,  0,  0,  0,
          0,  0,  0,  0,  0,  0,
          0,  0,  0,  0,  0,  0,
          0)

x_0 = 54000
y_0 = 21500

days_total = 36
days = seq( 0, days_total )
# sequence of days during the battle

y_end_err = 200
# Y forces must end the battle with zero +/- this value

##### Engel calculations #####

x_test = c( 54000, 52830, 51692, 56584, 55506, 54460,
            66447, 65464, 64521, 63615, 62746, 61913,
```

```
            61117, 60357, 59632, 58942, 58286, 57664,
            57076, 56521, 55999, 55509, 55052, 54627,
            54233, 53871, 53540, 53241, 52972, 52734,
            52526, 52349, 52203, 52086, 51999, 51943,
            51916)

##### Functions #####

# Creating a "get R-squared" function
getR = function(test) {
  if (is.na(test)){
    return(-1)
# returning negative R-squared value if NA to help identify
  }
  return(summary(
    lm(data = data.frame(x_true,test)))$r.squared)
}
  # getR(x_test)

# Function to Generate X vector from input of a, b, p, q
gen_x = function(a, b, p, q) {
  x_new = x_0
  y_new = y_0
  for (i in seq( 2, length(days) )) {

    x_new[i] = x_new[i-1]
               - a * y_new[i-1]^p * x_new[i-1]^q
               + x_add[i]
    y_new[i] = y_new[i-1]
               - b * x_new[i-1]^p * y_new[i-1]^q

    if ((x_new[i] <= 0)
# if X forces are negative
      | (y_new[i] > y_0)
# if Y forces are more than start strength
      | (y_new[i] < -y_end_err)
# if Y forces go negative more than the tolerance
      | ((abs(y_new[i]) > y_end_err) & (i == length(days)))
# if Y forces are greater than the tolerance at last day
      | (is.nan(x_new[i]))
# if X values become not a number
      | (is.nan(y_new[i]))
# if Y values become not a number
      ){ # kill the battle and return NA
      return(NA)
    }
```

```
  }
  return (x_new)
}

# Function to Generate Y vector from input of a, b, p, q
# the only difference from the gen_x function is the return
value
gen_y = function(a, b, p, q) {
  x_new = x_0
  y_new = y_0
  for (i in seq( 2, length(days) )) {

    x_new[i] = x_new[i-1]
               - a * y_new[i-1]^p * x_new[i-1]^q
               + x_add[i]
    y_new[i] = y_new[i-1]
               - b * x_new[i-1]^p * y_new[i-1]^q

    if ((x_new[i] <= 0)
# if X forces are negative
      | (y_new[i] > y_0)
# if Y forces are more than start strength
      | (y_new[i] < -y_end_err)
# if Y forces go negative more than the tolerance
      | ((abs(y_new[i]) > y_end_err) & (i == length(days)))
# if Y forces are greater than the tolerance at last day
      | (is.nan(x_new[i]))
# if X values become not a number
      | (is.nan(y_new[i]))
# if Y values become not a number
      ){ # kill the battle and return NA
      return(NA)
    }
  }
  return (y_new)
}

# Function to Generate attrition vector, a, from input of
X, Y, p, q
gen_a = function(X, Y, p, q) {
  a_new = 0
  b_new = 0

  for (i in seq( 1, length(days)-1 )) {
    a_new[i] = ( X[i] - X[i+1] + x_add[i+1])
               / ( X[i]^q * Y[i]^p )
```

```
    b_new[i] = ( Y[i] - Y[i+1] + y_add[i+1])
               / ( Y[i]^q * X[i]^p )
  }

  return(a_new)
}
```

 # Function to Generate attrition vector, b, from input of X, Y, p, q
 # the only difference from the gen_a function is the return value

```
gen_b = function(X, Y, p, q) {
  a_new = 0
  b_new = 0

  for (i in seq( 1, length(days)-1 )) {
    a_new[i] = ( X[i] - X[i+1] + x_add[i+1])
               / ( X[i]^q * Y[i]^p )
    b_new[i] = ( Y[i] - Y[i+1] + y_add[i+1])
               / ( Y[i]^q * X[i]^p )
  }

  return(b_new)
}
```

 # Plot function - plots true X and both modeled forces

```
plot_single = function(x_forces, y_forces){
  plot( days, x_true,
        ylim= c(0,max(x_true)), xlim= c(days[1],max(days)),
        xlab = "Day", ylab = "Force Level",
        main = "U.S. Forces per day\nVisual Plot")
  legend("right", legend = c("U.S. Actual", "U.S.",
"Japan"), col= c("black", "blue","red"), pch=c(1,NA,NA),
lty=c(0,1,1))

  points(days,x_forces,type="l", col= "blue")

  points(days,y_forces,type="l", col= "red")
}
```

SQUARE LAW

```
p = 1
q = 0

a_best_sqr = 0
```

```
b_best_sqr = 0
r_best_sqr = 0

a_min = .040
a_max = .065
a_step = (a_max-a_min)/1000
a_seq = seq( a_min, a_max, a_step)

b_min = .0095
b_max = .0115
b_step = (b_max-b_min)/1000
b_seq = seq( b_min, b_max, b_step)

r_mat_sqr = a_seq%*%t(b_seq) *0
# inititalize the 2D R-squared array

for (a_i in seq(1, length(a_seq))) {
# a_i is the index for grabbing an a
  for (b_i in seq(1, length(b_seq))) {
# b_i is the index for grabbing a b
    r_new = getR(gen_x(a_seq[a_i], b_seq[b_i], p, q))
    r_mat_sqr[a_i, b_i] = r_new
    }
}
contour(a_seq,b_seq,r_mat_sqr,
        levels=c(.1, .25, .5, .75, .9, .95, .99),
        xlab = "a", ylab = "b",
        main = "Square Law\nContour Plot",
        col = rainbow(7),
        lwd = 2)
r_best_sqr = max(r_mat_sqr)
a_best_sqr = a_seq[ which(r_mat_sqr==r_best_sqr,
arr.ind=T)[1] ]
b_best_sqr = b_seq[ which(r_mat_sqr==r_best_sqr,
arr.ind=T)[2] ]

r_best_sqr
a_best_sqr
b_best_sqr

plot_single(gen_x(a_best_sqr, b_best_sqr, p, q),
gen_y(a_best_sqr, b_best_sqr, p, q))

##### LINEAR LAW #####

p = 1
```

```
q = 1

a_best_lin = 0
b_best_lin = 0
r_best_lin = 0

a_min = 0e-6
a_max = 5e-6
a_step = (a_max-a_min)/1000
a_seq = seq( a_min, a_max, a_step)

b_min = a_min
b_max = a_max
b_step = (b_max-b_min)/1000
b_seq = seq( b_min, b_max, b_step)

r_mat_lin = a_seq%*%t(b_seq) *0
# inititalize the 2D R-squared array

for (a_i in seq(1, length(a_seq))) {
# a_i is the index for grabbing an a
  for (b_i in seq(1, length(b_seq))) {
# b_i is the index for grabbing a b
    r_new = getR(gen_x(a_seq[a_i], b_seq[b_i], p, q))
    r_mat_lin[a_i, b_i] = r_new
  }
}
contour(a_seq,b_seq,r_mat_lin,
        levels=c(.1, .25, .5, .75, .9, .95, .99),
        xlab = "a", ylab = "b",
        main = "Linear Law\nContour Plot",
        col = rainbow(7),
        lwd = 2)
r_best_lin = max(r_mat_lin)
a_best_lin = a_seq[ which(r_mat_lin==r_best_lin,
arr.ind=T)[1] ]
b_best_lin = b_seq[ which(r_mat_lin==r_best_lin,
arr.ind=T)[2] ]

r_best_lin
a_best_lin
b_best_lin

plot_single(gen_x(a_best_lin, b_best_lin, p, q),
gen_y(a_best_lin, b_best_lin, p, q))
```

```
##### LOG LAW #####

p = 0
q = 1

#a_best_log = 0
#b_best_log = 0
#r_best_log = 0

a_min = 0
a_max = .025
a_step = (a_max-a_min)/1000
a_seq = seq( a_min, a_max, a_step)

b_min = 0
b_max = 1.2
b_step = (b_max-b_min)/1000
b_seq = seq( b_min, b_max, b_step)

r_mat_log = a_seq%*%t(b_seq) *0
# inititalize the 2D R-squared array

for (a_i in seq(1, length(a_seq))) {
# a_i is the index for grabbing an a
  for (b_i in seq(1, length(b_seq))) {
# b_i is the index for grabbing a b
    r_new = getR(gen_x(a_seq[a_i], b_seq[b_i], p, q))
    r_mat_log[a_i, b_i] = r_new
  }
}
contour(a_seq,b_seq,r_mat_log,
        levels=c(.1, .25, .5, .75, .9, .95, .99),
        xlab = "a", ylab = "b",
        main = "Log Law\nContour Plot",
        col = rainbow(7),
        lwd = 2)
r_best_log = max(r_mat_log)
a_best_log = a_seq[ which(r_mat_log==r_best_log,
arr.ind=T)[1] ]
b_best_log = b_seq[ which(r_mat_log==r_best_log,
arr.ind=T)[2] ]

r_best_log
a_best_log
b_best_log
```

```
plot_single(gen_x(a_best_log, b_best_log, p, q),
gen_y(a_best_log, b_best_log, p, q))

##### BRACKEN #####

battle_counter = 0
p_best = 0
q_best = 0
a_best = 0
b_best = 0
r_best = 0

p_min = p_step * 0
p_max = p_step * 100
p_step = (1.5)/100
p_seq = seq( p_min, p_max, p_step)

q_min = q_step * 0
q_max = q_step * 100
q_step = (1.5)/100
q_seq = seq( q_min, q_max, q_step)

r_mat = p_seq%*%t(q_seq) *0
# initialize the 2D R-squared array
a_mat = p_seq%*%t(q_seq) *0
# initialize the 2D R-squared array for best a
b_mat = p_seq%*%t(q_seq) *0
# initialize the 2D R-squared array for best b

bottom = 1e-2
# default starting a/b search range (2 orders of magnitude)

for ( p_i in p_seq ) {  # p_i is the value of the current p

  p_ind = round(p_i/p_step+1)
# p_ind is the index of the current p

  for ( q_i in q_seq ) {
# q_i is the value of the current q

    resolve = 500
# starting resolution each initial search within a p/q
    r_best_in = 0
# r squared of best a/b for current p/q (inner)
    q_ind = round(q_i/q_step+1)
# q_ind is the index of the current q
```

```
    while (resolve <= 500*2^0 & r_best_in <= 0){
# adjust exponent to increase number of deeper scanning
# exponent of zero means just the initial scan
# exponent of 1 means allow one additional scan
# if nothing found in first pass (doubling resolution)
      a_best_in = 0
      b_best_in = 0
      r_best_in = 0

      a_adjust = (( p_i + 1.2/1.12*q_i )
                  -.70
                  + log(bottom,10)/2/4) *-4
# adjustment based on p/q for a
      b_adjust = (( p_i + 1.0/1.45*q_i )
                  -.60
                  + log(bottom,10)/2/5) *-5
# adjustment based on p/q for b
# these equations are tailored specifically
# to this problem and the a/b behavior

      a_seq = 10^a_adjust
              / bottom ^ ( c( 1 : resolve )/resolve -1 )
      b_seq = 10^b_adjust
              / bottom ^ ( c( 1 : resolve )/resolve -1 )
# these create the sequences of a and b
# to be used for testing.
# They are essentially logarithmic sequences

      r_mat_in = a_seq%*%t(b_seq) *0
# initialize the 2D R-squared array

      for (a_i in seq(1, length(a_seq))) {
# a_i is the index for grabbing an a
        for (b_i in seq(1, length(b_seq))) {
# b_i is the index for grabbing a b

          r_mat_in[a_i, b_i] = getR( gen_x(a_seq[a_i],
b_seq[b_i], p_i, q_i) )
          battle_counter = battle_counter + 1
        } # for b
      } # for a

      r_best_in = max(r_mat_in)
      resolve = resolve *2
# double the resolution (if opted for above)
```

```
    } # while loop

    a_best_in = a_seq[ which(r_mat_in==r_best_in,
arr.ind=T)[1] ]
    b_best_in = b_seq[ which(r_mat_in==r_best_in,
arr.ind=T)[2] ]
# record the a and b that gave the best fit

    a_best_in_org = a_best_in
    b_best_in_org = b_best_in
    r_best_in_org = r_best_in
# saves copy of best model from initial scan
# for for this p/q

    r_best_in = 0 # reset best r
    if (max(r_mat_in) <= 0) {next}
# if no valid models so far for this p/q
# then skip to next p/q
# (leaving r as zero for this p/q)

    zoom_lvl = 1 # initialize zoom level
    while ( abs(( r_best_in - max(r_mat_in) )) / (r_best_in
+ max(r_mat_in)) > .0001 & zoom_lvl <= 10){
# adjust tolerance level between best
# and second-best r-squared values(ex: .0001)
# this loop keeps zooming in on the
# next best model until the tolerance
# is met.

      if (r_best_in < max(r_mat_in)){
        r_best_in = max(r_mat_in)
        a_best_in = a_seq[ which(r_mat_in==r_best_in,
arr.ind=T)[1] ]
        b_best_in = b_seq[ which(r_mat_in==r_best_in,
arr.ind=T)[2] ]
      }

      a_min = a_best_in
              * ( 1 + ( a_seq[1]/a_seq[2] - 1 ) / 3 )
      a_max = a_best_in
              * ( 1 + ( a_seq[2]/a_seq[1] - 1 ) / 3 )
      a_seq = c( a_min, a_best_in, a_max)

      b_min = b_best_in
              * ( 1 + ( b_seq[1]/b_seq[2] - 1 ) / 3 )
      b_max = b_best_in
```

```
            * ( 1 + ( b_seq[2]/b_seq[1] - 1 ) / 3 )
      b_seq = c( b_min, b_best_in, b_max)

      r_mat_in = a_seq%*%t(b_seq) *0
# initialize the 3x3 R-squared array

      for (a_i in seq(1, length(a_seq))) {
# a_i is the index for grabbing an a
        for (b_i in seq(1, length(b_seq))) {
# b_i is the index for grabbing a b

          r_mat_in[a_i, b_i] = getR( gen_x(a_seq[a_i],
b_seq[b_i], p_i, q_i) )
          battle_counter = battle_counter + 1
        } # for b
      } # for a

      if (max(r_mat_in) < 0) {r_mat_in = r_mat_in *0}

      zoom_lvl = zoom_lvl +1
    } # while loop

print(c(p_ind-1, q_ind-1, r_mat[p_ind, q_ind], r_best_in,
((which(p_i==p_seq)-
1)*length(q_seq)+which(q_i==q_seq))/length(p_seq%*%t(q_seq)
)*100))
# this print statement is used
# to display the current progress
# and status during long runs

    if (r_best_in < max(r_mat_in)){
      r_best_in = max(r_mat_in)
      a_best_in = a_seq[ which(r_mat_in==r_best_in,
arr.ind=T)[1] ]
      b_best_in = b_seq[ which(r_mat_in==r_best_in,
arr.ind=T)[2] ]
    }
      else {
        a_best_in = a_best_in_org
        b_best_in = b_best_in_org
        r_best_in = r_best_in_org
        }# picking best results
         # (or original input if timed-out)

    if (r_mat[p_ind, q_ind] < r_best_in){
      r_mat[p_ind, q_ind] = r_best_in
```

```
      a_mat[p_ind, q_ind] = a_best_in
      b_mat[p_ind, q_ind] = b_best_in
    } # update r_mat if new best is found at that p/q

    if (r_mat[p_ind, q_ind] > r_best){
      a_best = a_best_in
      b_best = b_best_in
      r_best = r_best_in
    } # if best r
  } # for q
} # for p

p_seq = seq( 0, 1.5, p_step)
q_seq = seq( 0, 1.5, q_step)
# reset p/q range (if changed)

cont_lvl = c(.25, .5, .75, .9, .95, .98, .99, .991, .992,
.993, .994)
# c(.25, .5, .75, .9, .95, .98, .99)
# seq(.25,1,.01)
# c(.25, .5, .75, .9, .95, .98, .99, .991, .992, .993,
.994)
# various contour levels to use
contour(p_seq,q_seq,r_mat,
        levels = cont_lvl,
        xlab = "p", ylab = "q",
        main = "Bracken\nContour Plot R-Squared",
        col = rainbow(length(cont_lvl), v=.85),
        lwd = 2)

p_best = p_seq[ which(r_mat==r_best, arr.ind=T)[1] ]
q_best = q_seq[ which(r_mat==r_best, arr.ind=T)[2] ]

p_best
q_best
r_best
a_best
b_best
battle_counter

plot_single(gen_x(a_best, b_best, p_best, q_best),
gen_y(a_best, b_best, p_best, q_best))

#### Changing attrition Coefficient ####

# a_cac vector will have one less length than X/Y vectors
```

```
days_cac = days[-1]

# input parameters used to generate y forces
a = a_best_sqr #0.0544
b = b_best_sqr #0.0106
p =  1
q =  0

x_cac = x_true              # given x forces
y_cac = gen_y( a, b, p, q ) # generate y forces

a_cac = gen_a( x_cac, y_cac, p, q )
# calculate vector of a's

model_cac = lm(a_cac~days_cac)
# store linear regression model of a's

summary(model_cac)

# Plot the chart
plot( days_cac ,a_cac,
      col = "blue",
      main = "Daily Changing Attrition Coefficient",
      abline( lm(a_cac~days_cac),
              col = "lightblue",
              lty = "dotted",
              lwd = 3),
      pch = 16,
      xlab = "Day",
      ylab = "Attrition Coefficient, a")

#### Changing attrition Coefficient 28-days####

# a_cac vector will have one less length than X/Y vectors
days_cac = days[1:29]
days_cac = days_cac[-1]

# input parameters used to generate y forces
a = 0.0544
b = 0.0106
p =  1
q =  0

x_cac = x_true[1:29]              # given x forces
y_cac = gen_y( a, b, p, q )[1:29] # generate y forces
```

```
a_cac = gen_a( x_cac, y_cac, p, q )[1:28]
# calculate vector of a's

model_cac = lm(a_cac~days_cac)
# store linear regression model of a's

summary(model_cac)

# Plot the chart
plot( days_cac ,a_cac,
      col = "blue",
      main = "Daily Changing Attrition Coefficient",
      abline( lm(a_cac~days_cac),
              col = "lightblue",
              lty = "dotted",
              lwd = 3),
      pch = 16,
      xlab = "Day",
      ylab = "Attrition Coefficient, a")

# end
```

LIST OF REFERENCES

Bonder S (2002) Army operations research: Historical perspectives and lessons learned, *Operations Research*, 50(1): 25–34, INFORMS.

Bracken J (1995) Lanchester models of the Ardennes campaign, *Naval Research Logistics*, 42: 559–577.

Chen PS, Chu P (2001) Applying Lanchester's Linear Law to model the Ardennes campaign, *Naval Research Logistics*, 48: 653–661.

Dinges JA (2001) Exploring the validation of Lanchester equations for the Battle of Kursk. Master's thesis, Department of Operations Research, Naval Postgraduate School, Monterey, CA.

Engel JH (1954) A verification of Lanchester's Law, *J. of the ORSA*, 2(2): 163–171, INFORMS.

Fricker RD (1998) Attrition models of the Ardennes campaign, *Naval Research Logistics*, 45: 1–22.

Hartley DS (2001) Topics in Operations Research: Predicting Combat Effects (INFORMS, Catonsville, MD).

Hartley DS, Helmbold RL (1995) Validating Lanchester's Square Law and other attrition models, *Naval Research Logistics*, 42: 609–633.

Lanchester FW (1916) *Aircraft in Warfare; the Dawn of the Fourth Arm* (Constable and Company, Ltd., London).

Lucas TW (2000) The Stochastic Versus Deterministic Argument for Combat Simulations: Tales of When the Average Won't Do!, *Military Operations Research: Special Issue on Warfare Analysis and Complexity—the New Sciences*, 5(3): 9–28.

Lucas TW, Dinges JA (2004) The effect of battle circumstances on fitting Lanchester equations to the Battle of Kursk, *Military Operations Research*, 9(2): 17–30.

Lucas TW, Turkes T (2004) Applying Lanchester's Linear Law to model the Ardennes campaign, *Naval Research Logistics*, 51: 95–116.

Morehouse CP (1946) *The Iwo Jima Operation* (USMCR, Historical Division, Headquarters U.S. Marine Corps).

Office of Public and International Affairs (n.d.) Photo 26-G-4140, U.S. Coast Guard, U.S. Department of Transportation.

Perry N (2009) Fractal effects in Lanchester models of combat. Technical Report TR-2331, DSTO Defence Science and Technology Organisation, Canberra, Australia, http://www.dsto.defence.gov.au/corporate/reports/DSTOTR-2331.pdf.

Peterson R (1967) On the Logarithmic Law of Combat and Its Application to Tank Combat, *Operations Research*, 15, 557–558.

Samz RW (1972) Some comments on Engel's "A verification of Lanchester's Law," *Operations Research*, 20(1): 49–52, INFORMS.

Turkes T (2000) Fitting Lanchester and other equations to the Battle of Kursk data. Master's thesis, Department of Operations Research, Naval Postgraduate School, Monterey, CA.

INITIAL DISTRIBUTION LIST

1. Defense Technical Information Center
   Ft. Belvoir, Virginia

2. Dudley Knox Library
   Naval Postgraduate School
   Monterey, California