REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. REPORT DATE *(DD-MM-YYYY)*
22-09-2022

2. REPORT TYPE
White Paper

3. DATES COVERED *(From - To)*

4. TITLE AND SUBTITLE
Improved Accuracy in Quantal Response Using Firth's Penalized Likelihood

5a. CONTRACT NUMBER

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)
Leonard Lombardo

5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
U.S. Army Aberdeen Test Center
ATTN: TEDT-ADR
Aberdeen Proving Ground, MD 21005-5068

8. PERFORMING ORGANIZATION REPORT NUMBER
22-ADA-047

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

10. SPONSOR/MONITOR'S ACRONYM(S)

11. SPONSORING/MONITORING AGENCY REPORT NUMBER

12. DISTRIBUTION AVAILABILITY STATEMENT

Approved for public release; distribution unlimited.

13. SUPPLEMENTARY NOTES

14. ABSTRACT
Sensitivity testing is a type of testing in which a stressor (independent variable) is continuous, and the response (dependent variable) is binary. Ballistic limit testing is a type of sensitivity testing where the stressor is the velocity of a kinetic energy threat, and the response is penetration (either a partial or complete penetration) of an armor target. During ballistic limit testing, both the threat velocity and the penetration response are recorded for each shot. Then, the data are analyzed to model the probability of complete penetration as a function of threat velocity.
In sensitivity testing, it is often desirable to use the location-scale parameterization instead of the linear parameterization. Maximum likelihood estimates are known to have small sample bias. Firth's logistic regression may be used to reduce this bias. Another advantage to Firth's logistic regression is it can be used to determine a unique solution when there is separation in the data. In ballistic limit testing, separation in the data is often described as there being no zone of mixed results (i.e., no overlap in partial and complete penetrations). Finally, penalized likelihood ratio tests, based on Firth's logistic regression, may be used to improve the accuracy of statistical tests which is the focus of this paper.

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF: a. REPORT	b. ABSTRACT	c. THIS PAGE	17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
Unclassified	Unclassified	Unclassified	UU	60	Leonard C. Lombardo
					19b. TELEPONE NUMBER (*Include area code*) 410-278-1286

RAM/ILS ENGINEERING & ANALYSIS DIVISION

AUTOMOTIVE DIRECTORATE

DATE: 22 SEPTEMBER 2022 TEST REPORT NO.: 22-ADA-047

WHITE PAPER

FOR THE

IMPROVED ACCURACY OF STATISTICAL TESTS IN QUANTAL RESPONSE

USING FIRTH'S PENALIZED LIKELIHOOD

PREPARED BY: LEONARD LOMBARDO

REVIEWED BY:	APPROVED BY:
//signed//	//signed//
AMANDA L. HUGHES	NELLIE M. DUPREY
Chief, Analytical Branch	Chief, RAM/ILS Engineering & Analysis Division

U.S. ARMY ABERDEEN TEST CENTER
ABERDEEN PROVING GROUND, MD 21005-5059

Overview

Sensitivity testing is a type of testing in which a stressor (independent variable) is continuous, and the response (dependent variable) is binary. Ballistic limit testing is a type of sensitivity testing where the stressor is the velocity of a kinetic energy threat, and the response is penetration (either a partial or complete penetration) of an armor target (ref. 1). During ballistic limit testing, both the threat velocity and the penetration response are recorded for each shot. Then, the data are analyzed to model the probability of complete penetration as a function of threat velocity (ref. 2).

A generalized linear model (GLM) is a generalization of an ordinary linear model that allows for dependent variables with errors that are not normally distributed. A GLM with a binary dependent variable attempts to model the response probability with a cumulative distribution function (CDF). Logistic regression models are a type of GLM that models the data using the logistic CDF. The link function for logistic regression is given by the logit function:

$$\ln \frac{p}{1-p} = X\beta \tag{1}$$

Where:

p is probability
X is the matrix form of the independent variables
β is the vector of linear parameters

Estimates for the linear parameters, known as maximum likelihood estimates (MLE), are determined by maximizing the likelihood function:

$$L = \prod_i p_i^{y_i}(1-p_i)^{1-y_i} \tag{2}$$

Where:

L is the likelihood
p_i is the i^{th} probability value
y_i is the i^{th} response either 0 or 1

In practice, it is often easier to maximize the log-likelihood function. The log-likelihood function for the logistic regression model combines equations 1 and 2 and is given by the following:

$$\ln L = \sum_i \left[y_i X_{i\bullet}\beta - \ln\left(1 + e^{X_{i\bullet}\beta}\right)\right] \tag{3}$$

Where, $X_{i\bullet}$ is the i^{th} row of the X matrix.

There is no closed form solution to maximizing the log-likelihood function. Therefore, the MLE are solved iteratively using Fisher scoring, an iteratively reweighted scoring algorithm. The first derivative of the log-likelihood function is given by:

$$\frac{\partial \ln L}{\partial \beta} = X^T S \tag{4}$$

S is known as the score function. The score function in logistic regression is given by the following:

$$S_i = y_i - p_i \tag{5}$$

Fisher scoring uses the negative expectation of the second derivative of the log-likelihood function, known as the Fisher information matrix, which is given by the following:

$$-E\left(\frac{\partial^2 \ln L}{\partial \beta^2}\right) = X^T W X \tag{6}$$

Where, W is a diagonal matrix of weights given by the following equation:

$$W_{ii} = p_i(1 - p_i) \tag{7}$$

The Fisher scoring algorithm is given by the following:

$$\beta^{\{t+1\}} = \beta^{\{t\}} + (X^T W X)^{-1} X^T S \tag{8}$$

Where, t denotes the iteration number. This algorithm is repeated until the change in the log-likelihood is sufficiently small.

The simplest model for ballistic limit testing is the univariate model. An example of the univariate model is shown in Figure 1. The data used to construct this plot are presented in Appendix A, Table A-1. Note that these and all data presented in this paper are simulated and do not represent the results of any military test of armor. The univariate logit function is given by the following:

$$\ln \frac{p}{1-p} = \beta_0 + \beta_1 x \tag{9}$$

Where:

β_0 is the intercept
β_1 is the slope
x is the velocity of the kinetic energy threat

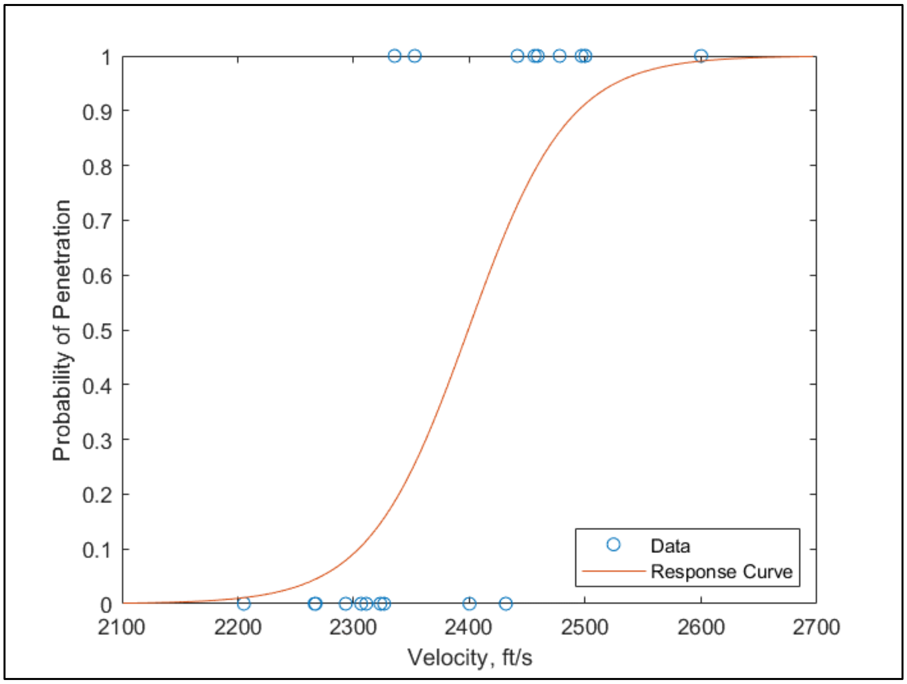

Figure 1. Example of the logistic regression model.

In sensitivity testing, it is often desirable to use the location-scale parameterization instead of the linear parameterization (ref. 3). The logit link function for the univariate problem using the location-scale parameterization is given by the following:

$$\ln \frac{p}{1-p} = \frac{x-\mu}{s} \tag{10}$$

Where:

μ is the location parameter
s is the scale parameter

The V50 (velocity at which there is a 50% probability of penetration) in ballistic limit testing is equivalent to μ for the logistic CDF (and other symmetric CDFs). Sigma, σ, is a reparameterization of s is given by the following:

$$\sigma = \frac{\pi}{\sqrt{3}} s \tag{11}$$

Maximum likelihood estimates are known to have small sample bias. Firth's logistic regression may be used to reduce this bias (ref. 4). Another advantage to Firth's logistic regression is it can be used to determine a unique solution when there is separation in the data. In ballistic limit testing, separation in the data is often described as there being no zone of mixed results (i.e., no overlap in partial and complete penetrations). Finally, penalized likelihood ratio tests, based on Firth's logistic regression, may be used to improve the accuracy of statistical tests which is the focus of this paper.

Firth's logistic regression penalizes the logistic regression score function to reduce the first order bias. The modified score function is given by the following (ref. 5):

$$S_i^* = S_i - h_{ii}(p_i - 1/2) \tag{12}$$

Where:

S_i^* is the i^th element of the modified score
h_{ii} is the i^th diagonal element of the hat matrix:

$$H = W^{1/2}X(X^TWX)^{-1}X^TW^{1/2} \tag{13}$$

This modification to the logistic regression score function is equivalent to penalizing the logistic regression log-likelihood function with Jeffrey's invariant prior (ref. 6).

$$\ln L^* = \sum_i \left[y_i X_{i\bullet}\beta - ln\left(1 + e^{X_{i\bullet}\beta}\right)\right] + \frac{1}{2} ln|I| \tag{14}$$

Where:

L^* is the penalized likelihood
$|I|$ is the determinant of the information matrix.

Due to the complexity of taking derivatives of the determinant of the information matrix, the first order information matrix (i.e., the information matrix used in logistic regression) is often used in the iteratively reweighted scoring algorithm in Firth's logistic regression. This algorithm is given by the following:

$$\beta^{t+1} = \beta^t + (X^TWX)^{-1}X^TS^* \tag{15}$$

Though tedious, a modified information matrix may be used for the univariate model. The iteratively reweighted scoring algorithm using the modified information matrix is given by the following:

$$\beta^{t+1} = \beta^t + (I^*)^{-1}X^TS^* \tag{16}$$

$$I^* = \begin{bmatrix} \sum W - \frac{1}{2}\frac{\partial^2 \ln|I|}{\partial\beta_0^2} & \sum Wx - \frac{1}{2}\frac{\partial^2 \ln|I|}{\partial\beta_0\partial\beta_1} \\ \sum Wx - \frac{1}{2}\frac{\partial^2 \ln|I|}{\partial\beta_0\partial\beta_1} & \sum Wx^2 - \frac{1}{2}\frac{\partial^2 \ln|I|}{\partial\beta_1^2} \end{bmatrix} \tag{17}$$

$$\frac{\partial^2 \ln|I|}{\partial\beta_0^2} = \frac{|I|\frac{\partial^2|I|}{\partial\beta_0^2} - \left(\frac{\partial|I|}{\partial\beta_0}\right)^2}{|I|^2} \tag{18}$$

$$\frac{\partial^2 \ln|I|}{\partial\beta_0\partial\beta_1} = \frac{|I|\frac{\partial^2|I|}{\beta_0\partial\beta_1} - \frac{\partial|I|}{\partial\beta_0}\frac{\partial|I|}{\partial\beta_1}}{|I|^2} \tag{19}$$

$$\frac{\partial^2 \ln|I|}{\partial \beta_1^2} = \frac{|I|\frac{\partial^2|I|}{\partial \beta_1^2} - \left(\frac{\partial |I|}{\partial \beta_1}\right)^2}{|I|^2} \tag{20}$$

$$\frac{\partial W_{ii}}{\partial \beta_0} = W_{ii}(1 - 2p_i) \tag{21}$$

$$\frac{\partial^2 W_{ii}}{\partial \beta_0^2} = W_{ii}(1 - 2p_i)^2 - 2W_{ii}^2 \tag{22}$$

$$\frac{\partial |I|}{\partial \beta_0} = \sum W_{ii} \sum \frac{\partial W_{ii}}{\partial \beta_0} x_i^2 - 2 \sum W_{ii} x_i \sum \frac{\partial W_{ii}}{\partial \beta_0} x_i + \sum W_{ii} x_i^2 \sum \frac{\partial W_{ii}}{\partial \beta_0} \tag{23}$$

$$\frac{\partial |I|}{\partial \beta_1} = \sum W_{ii} \sum \frac{\partial W_{ii}}{\partial \beta_0} x_i^3 - 2 \sum W_{ii} x_i \sum \frac{\partial W_{ii}}{\partial \beta_0} x_i^2 + \sum W_{ii} x_i^2 \sum \frac{\partial W_{ii}}{\partial \beta_0} x_i \tag{24}$$

$$\begin{aligned}\frac{\partial^2 |I|}{\partial \beta_0^2} &= \sum W_{ii} \sum \frac{\partial^2 W_{ii}}{\partial \beta_0^2} x_i^2 - 2 \sum W_{ii} x_i \sum \frac{\partial^2 W_{ii}}{\partial \beta_0^2} x_i + \sum W_{ii} x_i^2 \sum \frac{\partial^2 W_{ii}}{\partial \beta_0^2} \\ &+ 2 \sum \frac{\partial W_{ii}}{\partial \beta_0} \sum \frac{\partial W_{ii}}{\partial \beta_0} x_i^2 - 2 \left(\sum \frac{\partial W_{ii}}{\partial \beta_0} x_i\right)^2\end{aligned} \tag{25}$$

$$\begin{aligned}\frac{\partial^2 |I|}{\partial \beta_0 \partial \beta_1} &= \sum W_{ii} \sum \frac{\partial^2 W_{ii}}{\partial \beta_0^2} x_i^3 - 2 \sum W_{ii} x_i \sum \frac{\partial^2 W_{ii}}{\partial \beta_0^2} x_i^2 + \sum W_{ii} x_i^2 \sum \frac{\partial^2 W_{ii}}{\partial \beta_0^2} x_i \\ &+ \sum \frac{\partial W_{ii}}{\partial \beta_0} x_i^3 \sum \frac{\partial W_{ii}}{\partial \beta_0} - \sum \frac{\partial W_{ii}}{\partial \beta_0} x_i^2 \sum \frac{\partial W_{ii}}{\partial \beta_0} x_i\end{aligned} \tag{26}$$

$$\begin{aligned}\frac{\partial^2 |I|}{\partial \beta_1^2} &= \sum W_{ii} \sum \frac{\partial^2 W_{ii}}{\partial \beta_0^2} x_i^4 - 2 \sum W_{ii} x_i^3 \sum \frac{\partial^2 W_{ii}}{\partial \beta_0^2} x_i^2 + \sum W_{ii} x_i^2 \sum \frac{\partial^2 W_{ii}}{\partial \beta_0^2} \\ &+ 2 \sum \frac{\partial W_{ii}}{\partial \beta_0} x_i \sum \frac{\partial W_{ii}}{\partial \beta_0} x_i^3 - 2 \left(\sum \frac{\partial W_{ii}}{\partial \beta_0} x_i^2\right)^2\end{aligned} \tag{27}$$

Sequentially Optimal Test Methods

Sequential test methods are often used to select target velocities in ballistic limit testing. Early methods were very simple such that they could be easily conducted on range without the use of computers but were statistically inefficient. Newer methods including ones based on optimality criteria are more efficient but require the use of computers. Due to the increased availability of personal computers and laptops, the use of these more complex methods has become more prevalent. While many optimality criteria exist, Neyer's SenTest is software that provides methods based on D-optimality and c-optimality (ref. 7).

D-optimal methods select stressor values that maximize the determinant of the information matrix. This method is recommended when the goal of the test is to estimate both the V50 (velocity at which the probability of penetration is 50%) and sigma (slope of the response curve). Neyer's D-optimal method starts with a modified binary search to break separation. Then subsequent stressor levels are those that maximize the determinant of the information matrix (eq. 6) with the MLE for the model parameters recalculated between each shot. A flow chart for this method is shown in Figure 2. MuMin and MuMax are the initial parameters intended to bound the estimate of mu. SigmaGuess is the initial parameter for the estimate of sigma. MaxS is the maximum stressor level. MinS is the minimum stressor level. MinX is the minimum velocity resulting in a complete penetration. Max0 is the maximum velocity resulting in a partial penetration.

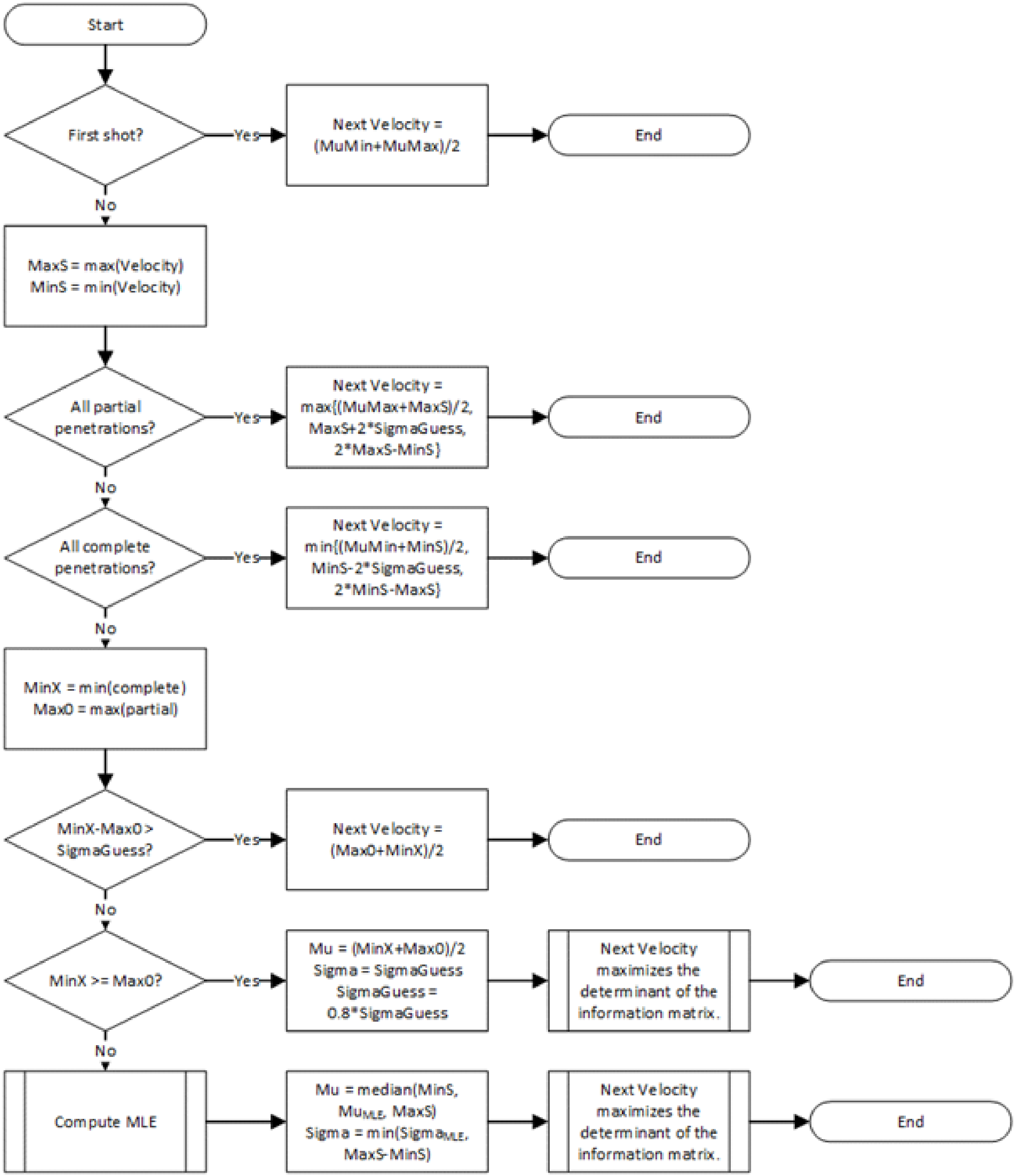

Figure 2. Flow chart for the Neyer D-Optimal Method.

A penalized D-optimal method is proposed in this paper. This method is recommended when Firth logistic regression is the planned analysis method and when the goal of the test is to estimate both the V50 and sigma. This test method uses the same modified binary search as the Neyer D-Optimal method to include the use of the first order information matrix before separation is broken (i.e., a zone of mixed results (ZMR) is achieved). The first order information matrix is initially used because points that maximize the modified information matrix tend to lie outside of those for the first order information matrix. Therefore, the use of the first order information matrix would be expected to break separation more quickly. After separation is broken, subsequent stressor levels are those that maximize the determinant of the modified information matrix (eq. 17). Additionally, different clipping rules are used. Clipping rules are intended to prevent extreme values for desired stressor levels when wild estimates for the model parameters are calculated. This can sometimes happen early in testing. The parameter estimates are then "clipped" and less extreme desired velocities are returned by the algorithm.

$$\left.\begin{array}{r}\sigma_{MPLE} < 0 \\ \sigma_{MPLE} > MaxS - MinS\end{array}\right\} \Rightarrow \sigma = MaxS - MinS \qquad (28)$$

$$\mu = median\left(MinS, \mu_{PMPLE|\sigma}, MaxS\right) \qquad (29)$$

Where:

σ_{MPLE} is the maximum penalized likelihood estimate for sigma
$\mu_{PMPLE|\sigma}$ is the profile maximum penalized likelihood estimate for μ given sigma

C-optimal methods attempt to minimize the variance of a linear combination of parameters. This method is recommended when the goal of the test is to estimate an extreme quantile, such as the V10 (velocity at which there is a 10% probability of penetration). Neyer's c-optimal method starts with a modified binary search to break separation. However, this method is based on the c-optimal algorithm instead of the D-optimal algorithm. Then subsequent stressor levels are those that minimize the variance of the extreme quantile of interest with the MLE for the model parameters recalculated between each shot. The variance of the extreme quantile, X_p, is given by the following:

$$Var\left(X_p\right) = \vec{X}(X^T W X)^{-1}\vec{X}^T \qquad (30)$$

$$\vec{X} = [1 \quad X_p] \qquad (31)$$

A flow chart for this method is shown in Figure 3.

A penalized c-optimal method is proposed in this paper. This method is recommended when Firth logistic regression is the planned analysis method and when the goal of the test is to estimate an extreme quantile, such as the V10. This test method uses the same modified binary search as the Neyer c-Optimal method to include the use of the first order information matrix before separation is broken (i.e., a ZMR is achieved). As with the D-optimal method, the first order information matrix is initially used, because points that minimize the variance using the modified information matrix tend to lie outside those for the first order information matrix. Therefore, the use of the first order information matrix would be expected to break separation more quickly. After separation is broken, subsequent stressor levels are those that minimize the variance of the extreme quantile using the modified information matrix (eq. 17). Additionally, the same clipping rules are used as in the penalized D-optimal method.

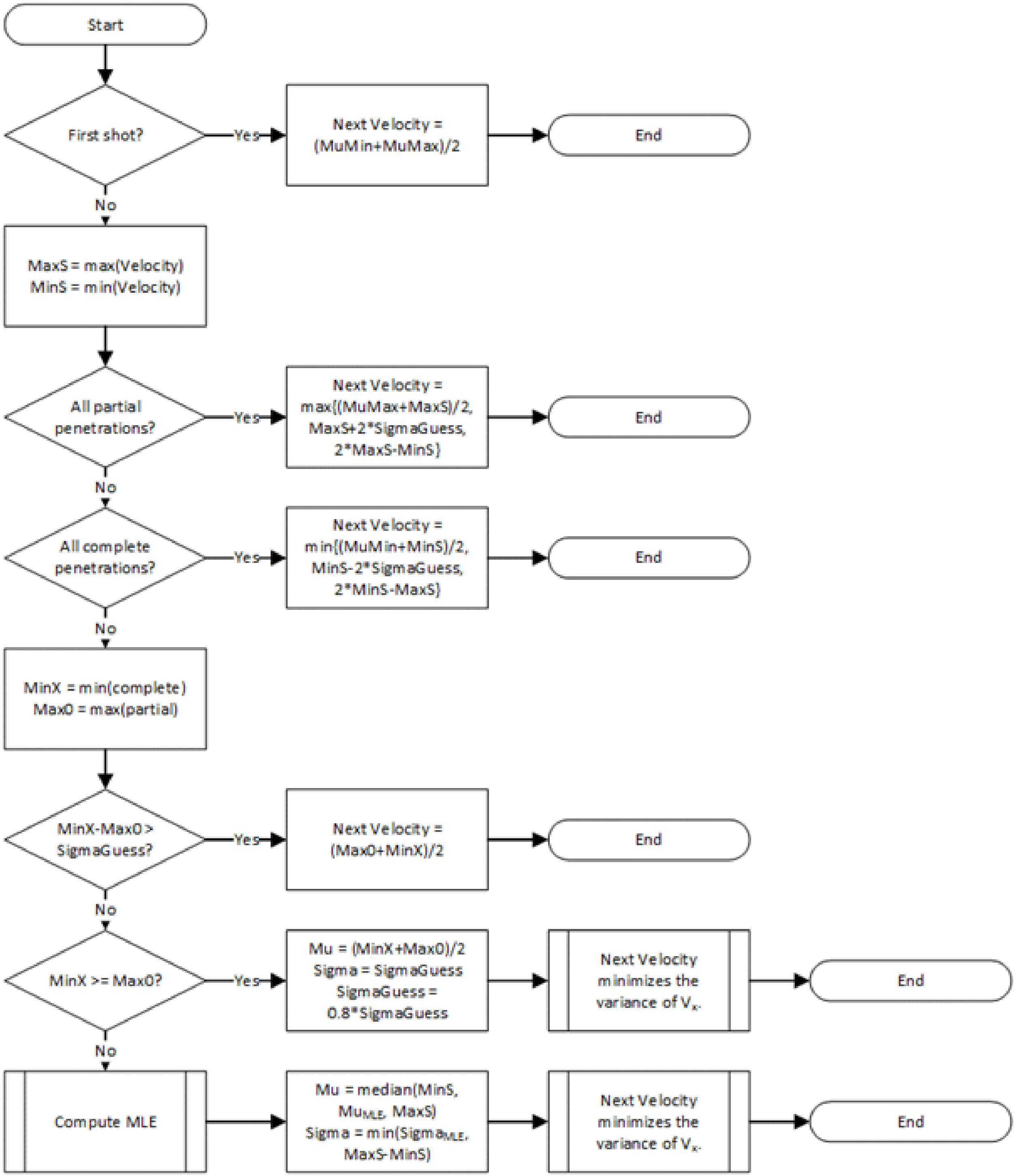

Figure 3. Flow chart for the Neyer c-Optimal Method.

Likelihood Ratio Test (LRT)

The likelihood ratio test is a statistical test comparing the likelihoods of two nested models. The null model is nested in the alternative model, meaning that the null model is similar to the alternative model but may be missing one or more parameters. Alternatively, one of the parameters in the null model is set to a constant or function of the other parameters as in profile likelihood. The test statistic for the likelihood ratio test, is given by the following:

$$\Delta = D_0 - D_1 \tag{32}$$

Where:

Δ is the test statistic for the likelihood ratio test which is approximately χ^2-distributed with degrees of freedom (df) equal to the difference in the number of parameters between the null and alternative models.
D_0 is the deviance for the null model.
D_1 is the deviance for the alternative model.

Deviance, D, is given by the following equation:

$$D = -2 \ln L \tag{33}$$

One-Sample LRT on V50

The null and alternative models are given by the following (ref. 8):

$$H_0\colon\ \beta(x - \mu_0),\ \ V50 = V50_0 \tag{34}$$

$$H_1\colon\ \beta_0 + \beta_1 x,\ \ V50 \neq V50_0 \tag{35}$$

Both the null and alternative models may be fit using standard logistic regression techniques. The test statistic is approximately χ^2-distributed with one df.

Figure 4 presents an example of the one-sample LRT on V50. The left plot displays the example data and response curves for the null and alternative models. The right plot displays the LRT test statistic as a function of V50 under the null model. For this example, the value for $V50_0$ is 2300 ft/s resulting in a Δ of 7.44 and p-value of 0.006. A copy of the data used for this example is presented in Table A-1.

Figure 5 presents an example of the one-sample LRT on V50 when there is separation in the data. The left plot displays the example data and response curve for the null model. A unique solution for the alternative model cannot be determined since any step function between the highest partial and lower complete would maximize the likelihood function. Instead, the rectangle indicates this gap between the highest velocity partial and lowest velocity complete penetration. The right plot displays the LRT test statistic as a function of V50 under the null model. For this example, the value for $V50_0$ is 2300 ft/s resulting in a Δ of 11.66 and p-value of 0.0006. Note the shape of the curve where there is a gap in the data. Again, since any step function in the gap maximizes the likelihood (minimizes the deviance), the profile likelihood is discontinuous at the highest partial and lowest complete penetration. For this reason, the Penalized Likelihood Ratio

Test (PLRT) is preferred for data with separation. A copy of the data used for this example is presented in Table A-2.

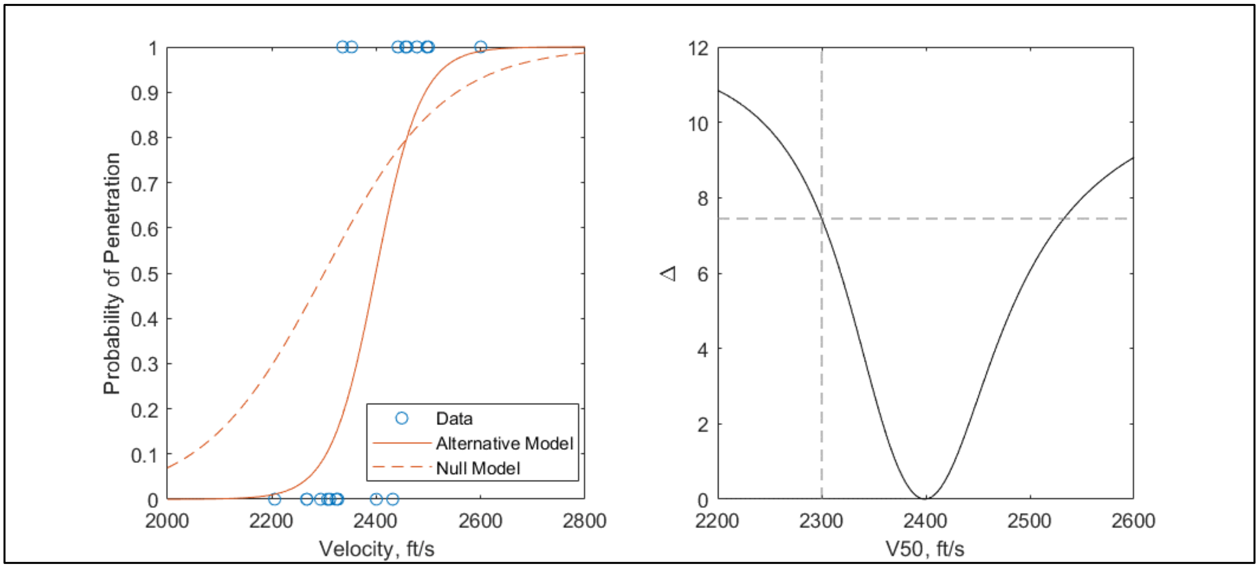

Figure 4. Example of the one-sample LRT on V50.

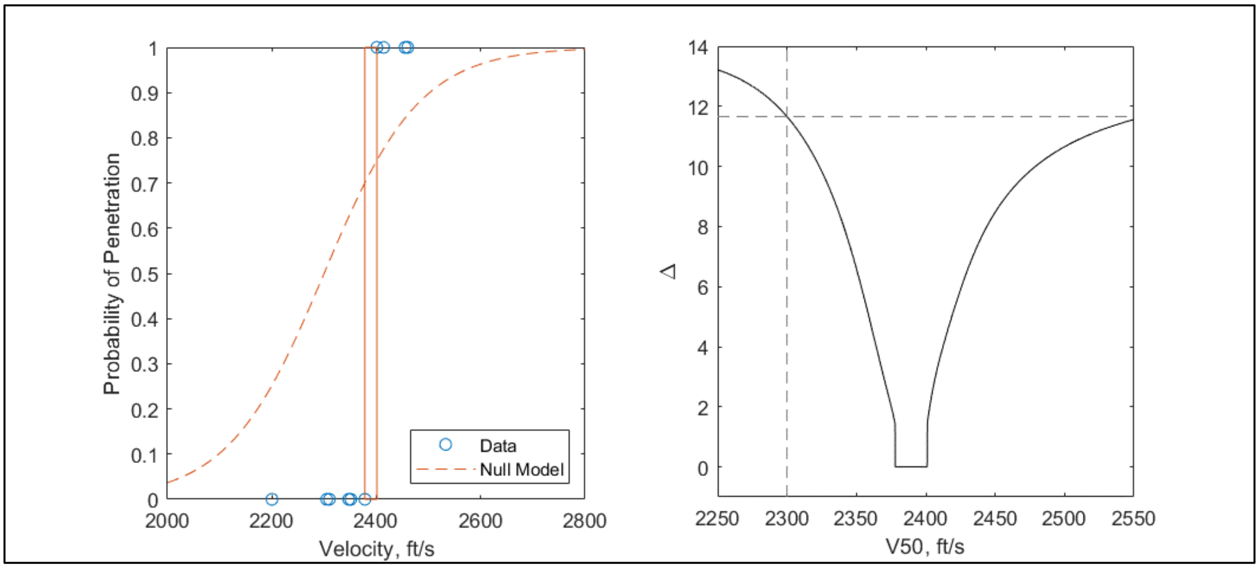

Figure 5. Example of the one-sample LRT on V50 with separation.

Two-Sample LRT on V50

The null and alternative models are given by the following (ref. 8 and 9):

$$H_0\text{: } \frac{x-\mu}{s+\varepsilon d},\ V50_1 = V50_2 \tag{36}$$

$$H_1\text{: } \frac{x-(\mu+\delta d)}{s+\varepsilon d},\ V50_1 \neq V50_2 \tag{37}$$

Where:

δ is the shift in V50 between the two samples under the alternative model
d is the design variable indicating which sample
ε is the shift in the scale parameter between the two samples

The alternative hypothesis may be presented in linear parameters and solved using standard logistic regression techniques.

$$H_1\text{: } \beta_0 + \beta_1 x + \beta_2 d + \beta_3 xd \tag{38}$$

The null model, however, is more challenging. Since the null model uses profile likelihood in the linear parameterization, it is possible for there to be local maxima and minima. The first step is to set μ_0 equal to values between $V50_1$ and $V50_2$ and solve for the profile maximum likelihood estimates (PMLEs) for s and ε given that value for μ_0. This approach is an application of profile likelihood. The iteratively reweighted scoring algorithm is given by the following:

$$\theta_0^{\{t+1\}} = \theta_0^{\{t\}} + \left(\left(\frac{\partial\eta}{\partial\theta_0}\right)^T W \frac{\partial\eta}{\partial\theta_0}\right)^{-1} \left(\frac{\partial\eta}{\partial\theta_0}\right)^T S \tag{39}$$

$$\theta_0 = \begin{bmatrix} s \\ \varepsilon \end{bmatrix} \tag{40}$$

$$\eta = \frac{x-\mu_0}{s+\varepsilon d} \tag{41}$$

$$\frac{\partial\eta}{\partial\theta_0} = \begin{bmatrix} \frac{\partial\eta}{\partial s} & \frac{\partial\eta}{\partial\varepsilon} \end{bmatrix} \tag{42}$$

$$\frac{\partial\eta_i}{\partial s} = -\frac{\eta_i}{s+\varepsilon d_i} \tag{43}$$

$$\frac{\partial\eta_i}{\partial\varepsilon} = -\frac{d_i\eta_i}{s+\varepsilon d_i} \tag{44}$$

The second step is to set the initial guesses for μ, s, and ε that maximize the log-likelihood from the first step. Then, iteratively solve for the PMLEs for μ, s, and ε using equation 39. The following equations should be used in support of the algorithm.

$$\theta_0 = \begin{bmatrix} \mu \\ s \\ \varepsilon \end{bmatrix} \tag{45}$$

$$\eta = \frac{x-\mu}{s+\varepsilon d} \tag{46}$$

$$\frac{\partial \eta}{\partial \theta_0} = \begin{bmatrix} \frac{\partial \eta}{\partial \mu} & \frac{\partial \eta}{\partial s} & \frac{\partial \eta}{\partial \varepsilon} \end{bmatrix} \tag{47}$$

$$\frac{\partial \eta_i}{\partial \mu} = -\frac{1}{s+\varepsilon d_i} \tag{48}$$

Note that equations 43 and 44 may still be used. The test statistic is approximately χ^2-distributed with one df. The method may be easily extended to one factor with multiple levels.

Figure 6 presents an example of the two-sample LRT on V50. The left plot displays the response curves for the null and alternative models for samples A and B. The right plot displays the LRT test statistic as a function of V50 under the null model. The resulting test statistic is 1.41and p-value is 0.234. A copy of the data used for this example is presented in Table A-3.

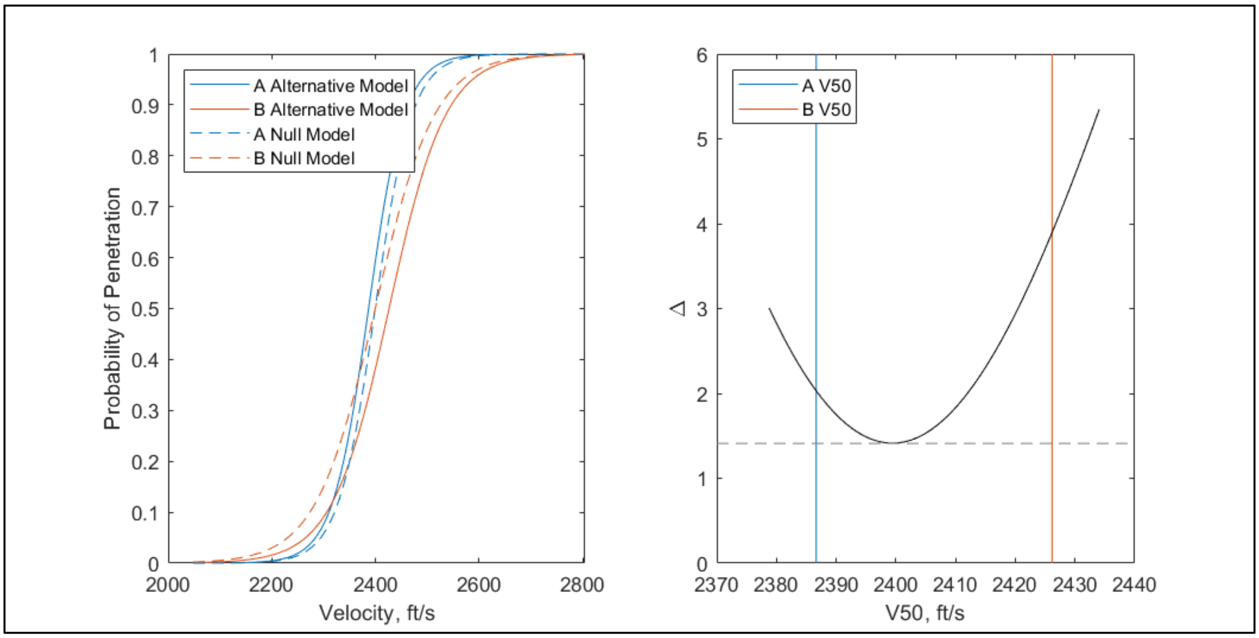

Figure 6. Example of the two-sample LRT on V50.

LRT on V50 with Multiple Factors

Although this method may be extendable to any number of factors and levels, the two factor each at two levels model is described here. The alternative model is given by the following (ref. 10):

$$\ln \frac{p}{1-p} = \frac{x-(\mu+\delta_1 d_1+\delta_2 d_2+\delta_{12} d_{12})}{s+\varepsilon_1 d_1+\varepsilon_2 d_2+\varepsilon_{12} d_{12}} \tag{49}$$

The alternative hypothesis may be presented in linear parameters and solved using standard logistic regression techniques.

$$\ln\frac{p}{1-p} = \beta_0 + \beta_1 d_1 + \beta_2 d_2 + \beta_{12} d_{12} + \beta_x x + \beta_{1x} d_1 x + \beta_{2x} d_2 x + \beta_{12x} d_{12} x \qquad (50)$$

There are three null models including the two main effects and the one interaction. These null models are given by the following:

$$\ln\frac{p}{1-p} = \frac{x-(\mu+\delta_2 d_2+\delta_{12} d_{12})}{s+\varepsilon_1 d_1+\varepsilon_2 d_2+\varepsilon_{12} d_{12}}, \delta_1 = 0 \qquad (51)$$

$$\ln\frac{p}{1-p} = \frac{x-(\mu+\delta_1 d_1+\delta_{12} d_{12})}{s+\varepsilon_1 d_1+\varepsilon_2 d_2+\varepsilon_{12} d_{12}}, \delta_2 = 0 \qquad (52)$$

$$\ln\frac{p}{1-p} = \frac{x-(\mu+\delta_1 d_1+\delta_2 d_2)}{s+\varepsilon_1 d_1+\varepsilon_2 d_2+\varepsilon_{12} d_{12}}, \delta_{12} = 0 \qquad (53)$$

The null models, however, are more challenging to solve. Since the null models use profile likelihood in the linear parameterization, it is possible for there to be local maxima and minima. There are two possible approaches. The first is the grid approach in which the log-likelihood for many combinations of parameters is evaluated and pick the best initial guess. However, this method is very time consuming and completely impractical for more complex models. The second approach is using a Latin hypercube with multiple starts. The Latin hypercube is a space filling design and is used to produce the initial guesses. The Fisher scoring algorithm is then performed for each initial guess. This method is preferred though is not without its challenges. Most initial guesses will not be good resulting in failure to converge on a solution. Therefore, computational issues will have to be addressed. For the first null model (δ_1=0) the following equations are used.

$$\theta_0^{\{t+1\}} = \theta_0^{\{t\}} + \left(\left(\frac{\partial\eta}{\partial\theta_0}\right)^T W \frac{\partial\eta}{\partial\theta_0}\right)^{-1} \left(\frac{\partial\eta}{\partial\theta_0}\right)^T S \qquad (39)$$

$$\theta_0^T = [\mu \quad \delta_2 \quad \delta_{12} \quad s \quad \varepsilon_1 \quad \varepsilon_2 \quad \varepsilon_{12}] \qquad (54)$$

$$\eta = \frac{x-(\mu+\delta_2 d_2+\delta_{12} d_{12})}{s+\varepsilon_1 d_1+\varepsilon_2 d_2+\varepsilon_{12} d_{12}} \qquad (55)$$

$$\frac{\partial\eta}{\partial\theta_0} = \left[\frac{\partial\eta}{\partial\mu} \quad \frac{\partial\eta}{\partial\delta_2} \quad \frac{\partial\eta}{\partial\delta_{12}} \quad \frac{\partial\eta}{\partial s} \quad \frac{\partial\eta}{\partial\varepsilon_1} \quad \frac{\partial\eta}{\partial\varepsilon_2} \quad \frac{\partial\eta}{\partial\varepsilon_{12}}\right] \qquad (56)$$

$$\frac{\partial\eta_i}{\partial\mu} = -\frac{1}{s+\varepsilon_1 d_{1,i}+\varepsilon_2 d_{2,i}+\varepsilon_{12} d_{12,i}} \qquad (57)$$

$$\frac{\partial\eta_i}{\partial\delta_2} = -\frac{d_{2,i}}{s+\varepsilon_1 d_{1,i}+\varepsilon_2 d_{2,i}+\varepsilon_{12} d_{12,i}} \qquad (58)$$

$$\frac{\partial\eta_i}{\partial\delta_{12}} = -\frac{d}{s+\varepsilon_1 d_{1,i}+\varepsilon_2 d_{2,i}+\varepsilon_{12} d_{12,i}} \qquad (59)$$

$$\frac{\partial\eta_i}{\partial s} = -\frac{\eta_i}{s+\varepsilon_1 d_{1,i}+\varepsilon_2 d_{2,i}+\varepsilon_{12} d_{12,i}} \qquad (60)$$

$$\frac{\partial\eta_i}{\partial\varepsilon_1} = -\frac{\eta_i d_{1,i}}{s+\varepsilon_1 d_{1,i}+\varepsilon_2 d_{2,i}+\varepsilon_{12} d_{12,i}} \qquad (61)$$

$$\frac{\partial\eta_i}{\partial\varepsilon_2} = -\frac{\eta_i d_{2,i}}{s+\varepsilon_1 d_{1,i}+\varepsilon_2 d_{2,i}+\varepsilon_{12} d_{12,i}} \quad (62)$$

$$\frac{\partial\eta_i}{\partial\varepsilon_{12}} = -\frac{\eta_i d_{12,i}}{s+\varepsilon_1 d_{1,i}+\varepsilon_2 d_{2,i}+\varepsilon_{12} d_{12,i}} \quad (63)$$

The test statistic is approximately χ^2-distributed with one df. This approach is repeated for design variable δ_2 and interaction δ_{12}.

An example of the likelihood ratio test table is present in Table 1. Figure 7 presents an example of the LRT on V50 with multiple factors for the first null model (δ_1=0). The top-left plot displays the response curves for the null and alternative models. The top-right plot displays the test statistic versus μ under the null model with PMLEs for parameters δ_2, δ_{12}, s, ε_1, ε_2, and ε_{12} calculated for each value of μ. Likewise, the bottom-left plot displays the test statistic versus δ_2 with PMLEs for parameters μ, δ_{12}, s, ε_1, ε_2, and ε_{12} calculated for each value of δ_2. Finally, the bottom-right plot displays the test statistic versus δ_{12} with PMLEs for parameters μ, δ_2, s, ε_1, ε_2, and ε_{12} calculated for each value of δ_{12}. The resulting test statistic for the first null model is 1.34 and p-value is 0.246. A copy of the data used for this example is presented in Table A-4.

TABLE 1. EXAMPLE LIKELIHOOD RATIO TEST TABLE

Factor	df	χ^2-Stat	P-Value
d_1	1	1.34	0.246
d_2	1	0.08	0.772
d_1*d_2	1	1.87	0.172

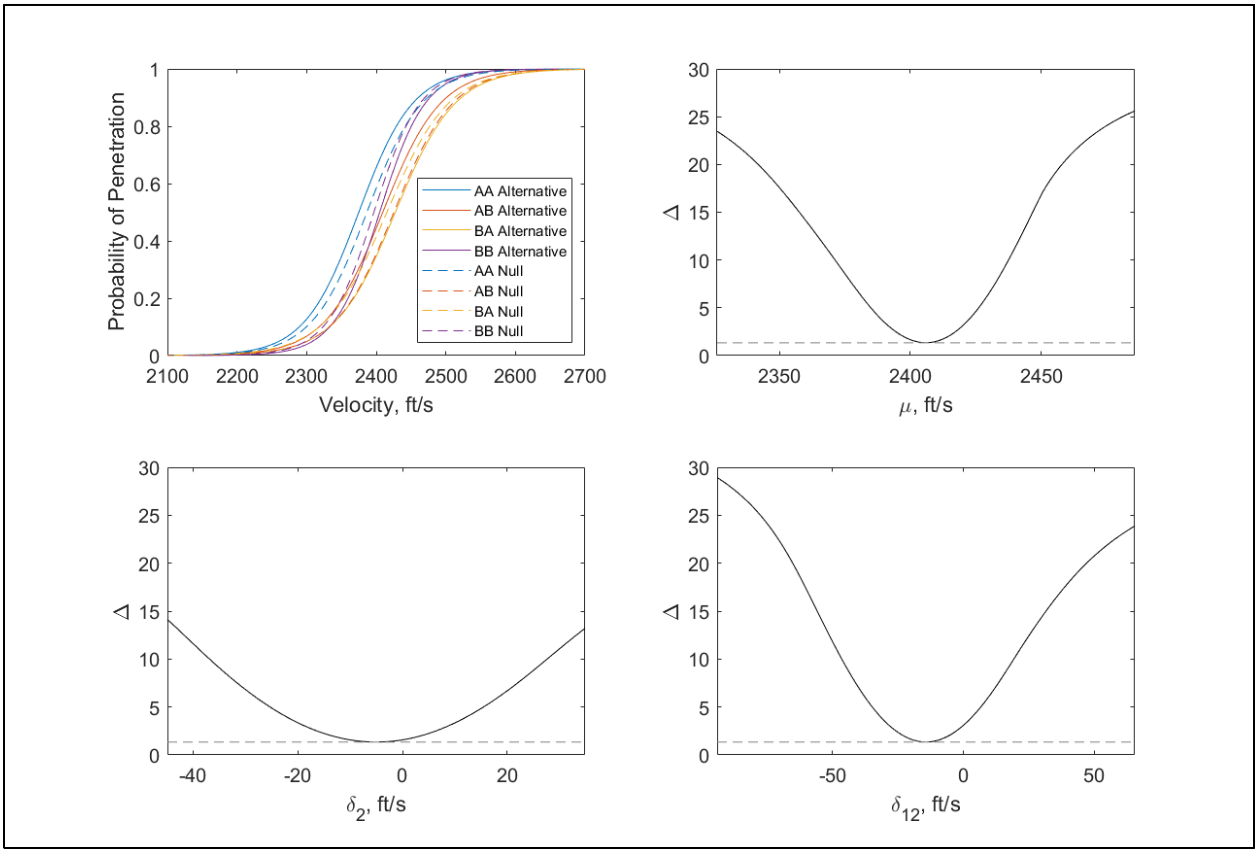

Figure 7. Example of the LRT on V50 with two factors each at two levels.

One-Sample LRT on Sigma

The null and alternative models are given by the following:

$$H_0 \colon \beta_0 + \frac{x}{s_0}, \;\; s = s_0 \tag{64}$$

$$H_1 \colon \beta_0 + \beta_1 x, \;\; s \neq s_0 \tag{65}$$

Both the null and alternative models may be fit using standard logistic regression techniques. The test statistic is approximately χ^2-distributed with one df.

Figure 8 presents an example of the one-sample LRT on sigma. The left plot displays the example data and response curves for the null and alternative models. The right plot displays the LRT test statistic as a function sigma under the null model. For this example, the value for σ_0 is 40 ft/s resulting in a Δ of 2.47 and p-value of 0.116. See eqn. 11 for the relationship between s and σ for logistic regression. A copy of the data used for this example is presented in Table A-5.

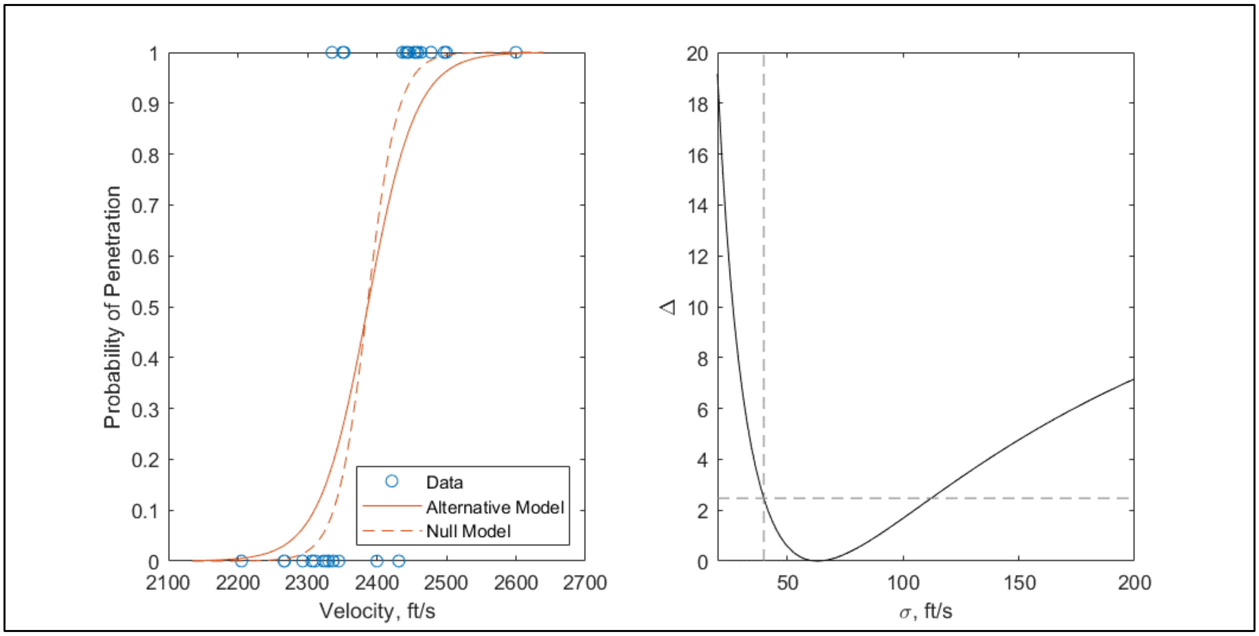

Figure 8. Example of one-sample LRT on sigma.

Two-Sample LRT on Sigma

The null and alternative models are given by the following:

$$H_0\colon \beta_0 + \beta_1 d + \beta_2 x,\;\; s_1 = s_2 \tag{66}$$

$$H_1\colon \beta_0 + \beta_1 d + \beta_2 x + \beta_3 dx,\;\; s_1 \neq s_2 \tag{67}$$

Both the null and alternative models may be fit using standard logistic regression techniques. The test statistic is approximately χ^2-distributed with one df.

Figure 9 presents an example of the two-sample LRT on sigma. The left plot displays the response curves for the null and alternative models for samples A and B. The right plot displays the LRT test statistic as a function of σ under the null model. The resulting test statistic is 0.21 and p-value is 0.643. A copy of the data used for this example is presented in Table A-6.

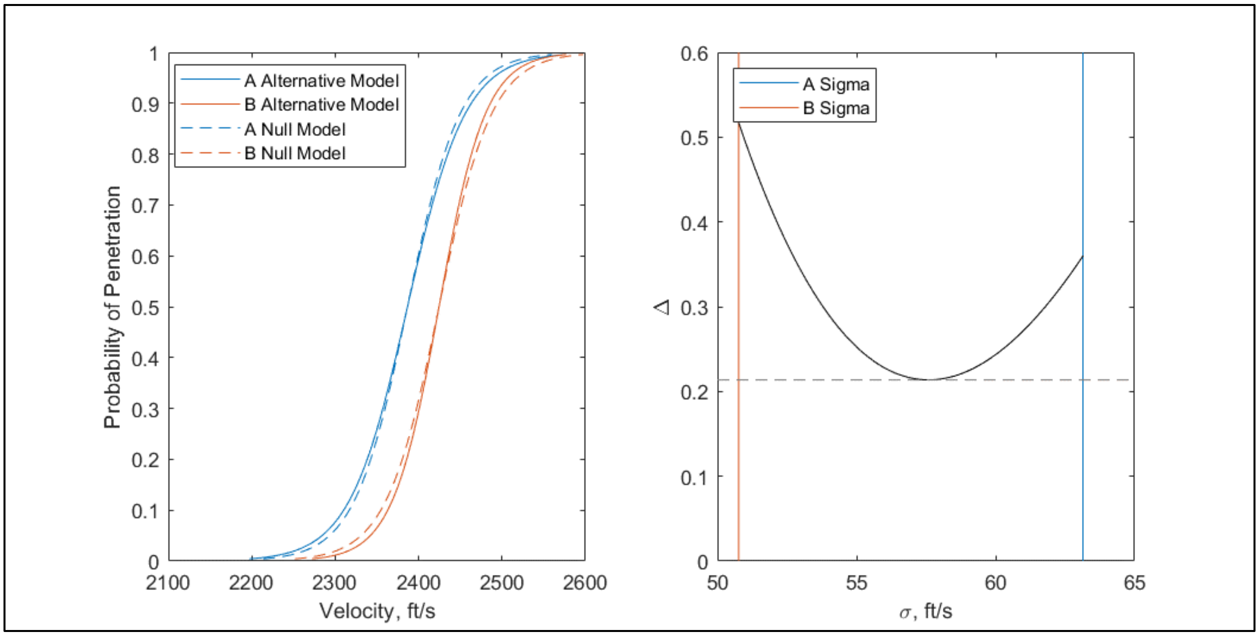

Figure 9. Example of two-sample LRT on sigma.

One-Sample LRT on V10

The null and alternative models are given by the following:

$$H_0 \colon q_0 + \beta(x - V10_0),\ V10 = V10_0 \tag{68}$$

$$H_1 \colon \beta_0 + \beta_1 x,\ V10 \neq V10_0 \tag{69}$$

$$q_0 = \ln\frac{p_0}{1-p_0} = \ln\frac{0.1}{0.9} \tag{70}$$

Both the null and alternative models may be fit using standard logistic regression techniques. The test statistic is approximately χ^2-distributed with one df.

Figure 10 presents an example of the one-sample LRT on V10. The left plot displays the example data and response curves for the null and alternative models. The right plot displays the LRT test statistic as a function V10 under the null model. For this example, the value for $V10_0$ is 2000 ft/s resulting in a Δ of 4.91 and p-value of 0.027. A copy of the data used for this example is presented in Table A-7.

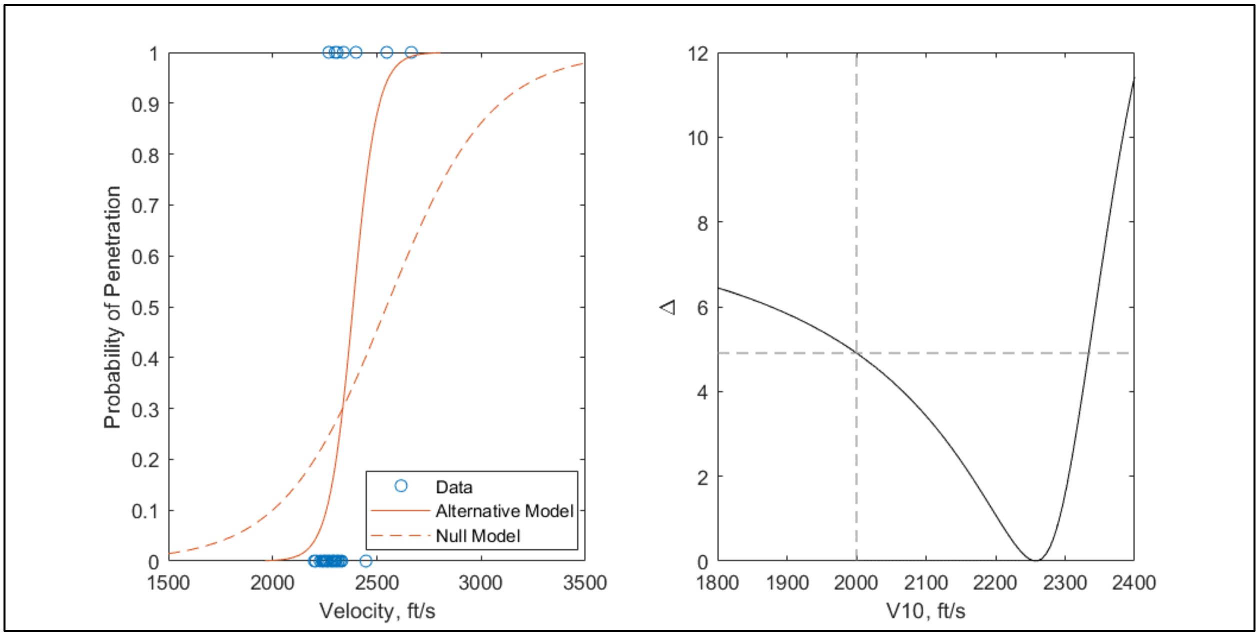

Figure 10. Example of one-sample LRT on V10.

Two-Sample LRT on V10

The null and alternative models are given by the following:

$$H_0: q_0 + \frac{x - V10}{s + \varepsilon d},\ \ V10_1 = V10_2 \tag{71}$$

$$H_1: q_0 + \frac{x - (V10 + \delta d)}{s + \varepsilon d},\ \ V10_1 \neq V10_2 \tag{72}$$

The alternative model may be presented in linear parameters and solved using standard logistic regression techniques. Recall equation 38:

$$H_1:\ \beta_0 + \beta_1 x + \beta_2 d + \beta_3 x d \tag{38}$$

The null model, however, is more challenging. Since the null model uses profile likelihood in the linear parameterization, it is possible for there to be local maxima and minima. The first step is to set $V10_0$ equal to values between $V10_1$ and $V10_2$ and solve for the PMLEs for s and ε given that value for $V10_0$. This approach is an application of profile likelihood. The iteratively reweighted scoring algorithm is given by equation 39.

$$\theta_0^{\{t+1\}} = \theta_0^{\{t\}} + \left(\left(\frac{\partial\eta}{\partial\theta_0}\right)^T W \frac{\partial\eta}{\partial\theta_0}\right)^{-1} \left(\frac{\partial\eta}{\partial\theta_0}\right)^T S \tag{39}$$

$$\theta_0 = \begin{bmatrix} S \\ \varepsilon \end{bmatrix} \tag{40}$$

$$\eta = q_0 + \frac{x - V10_0}{s + \varepsilon d} \tag{73}$$

$$\frac{\partial \eta}{\partial \theta_0} = \begin{bmatrix} \frac{\partial \eta}{\partial s} & \frac{\partial \eta}{\partial \varepsilon} \end{bmatrix} \tag{42}$$

$$\frac{\partial \eta_i}{\partial s} = -\frac{x_i - V10_0}{(s + \varepsilon d_i)^2} \tag{74}$$

$$\frac{\partial \eta_i}{\partial \varepsilon} = -\frac{d_i(x_i - V10_0)}{(s + \varepsilon d_i)^2} \tag{75}$$

The second step is to set the initial guesses for V10, s, and ε that maximize the log-likelihood from the first step. Then, iteratively solve for the PMLEs for V10, s, and ε using equation 39. However, the following equations should be used in support of the algorithm.

$$\theta_0 = \begin{bmatrix} V10 \\ s \\ \varepsilon \end{bmatrix} \tag{76}$$

$$\eta = q_0 + \frac{x - V10}{s + \varepsilon d} \tag{77}$$

$$\frac{\partial \eta}{\partial \theta_0} = \begin{bmatrix} \frac{\partial \eta}{\partial V10} & \frac{\partial \eta}{\partial s} & \frac{\partial \eta}{\partial \varepsilon} \end{bmatrix} \tag{78}$$

$$\frac{\partial \eta_i}{\partial V10} = -\frac{1}{s + \varepsilon d_i} \tag{79}$$

$$\frac{\partial \eta_i}{\partial s} = -\frac{x_i - V10}{(s + \varepsilon d_i)^2} \tag{80}$$

$$\frac{\partial \eta_i}{\partial \varepsilon} = -\frac{d_i(x_i - V10)}{(s + \varepsilon d_i)^2} \tag{81}$$

Figure 11 presents an example of the two-sample LRT on V10. The left plot displays the response curves for the null and alternative models for samples A and B. The right plot displays the LRT test statistic as a function of V10 under the null model. The resulting test statistic is 1.43 and p-value is 0.232. A copy of the data used for this example is presented in Table A-8.

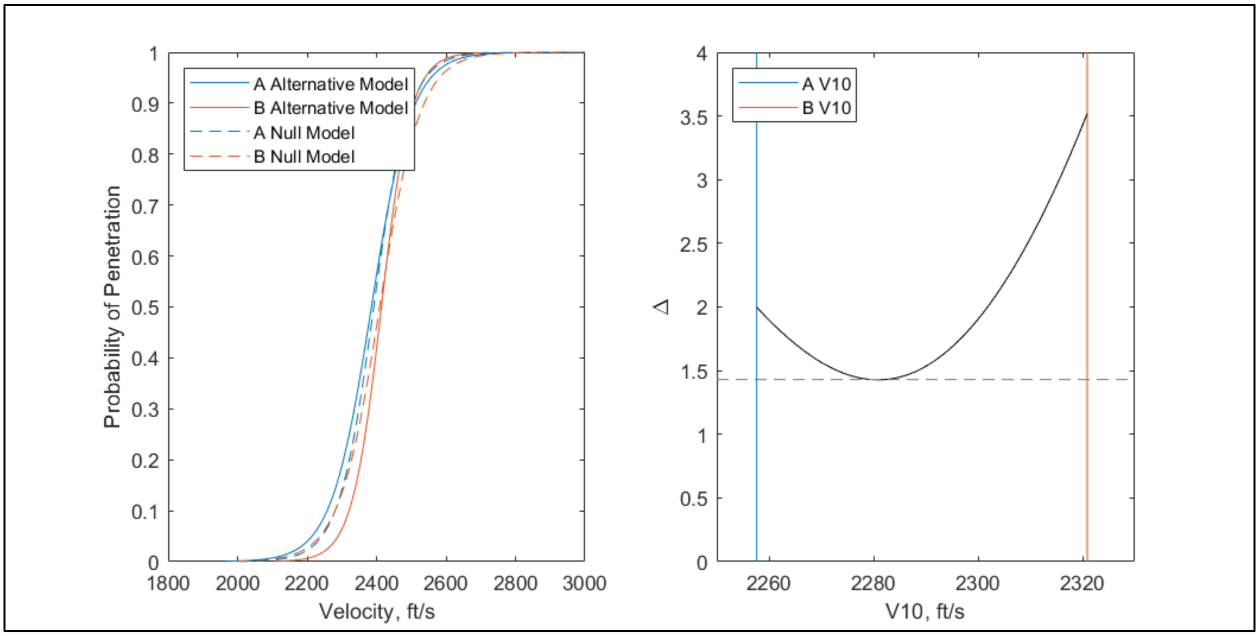

Figure 11. Example of two-sample LRT on V10.

Penalized Likelihood Ratio Test (PLRT)

The penalized likelihood ratio test is a statistical test comparing the penalized likelihoods of two nested models (ref. 11). The null model is nested in the alternative model, meaning that the null model is similar to the alternative model but may be missing one or more parameters. Alternatively, one of the parameters in the null model is set to a constant or function of the other parameters as in profile penalized likelihood. The test statistic for the penalized likelihood ratio test, is given by the following:

$$\Delta^* = D_0^* - D_1^* \tag{82}$$

Where:

Δ^* is the test statistic for the penalized likelihood ratio test which is approximately χ^2-distributed with degrees of freedom equal to the difference in the number of parameters between the null and alternative models.
D_0^* is the penalized deviance for the null model.
D_1^* is the penalized deviance for the alternative model.

Penalized deviance, D^*, is given by the following equation:

$$D^* = -2\ln L^* \tag{83}$$

One-Sample PLRT on V50

The null and alternative models are given by the following:

$$H_0\colon\ \beta(x - V50_0),\ \ V50 = V50_0 \tag{34}$$

$$H_1\colon \beta_0 + \beta_1 x,\ \ V50 \neq V50_0 \tag{35}$$

The alternative model may be fit using standard Firth logistic regression techniques. Since the penalized likelihood is in part a function of the information matrix, the dimensionality of the information matrix has a large impact on the penalized likelihood calculation. Therefore, care must be taken when fitting the null model for the purposes of the penalized likelihood ratio test. The iteratively reweighted scoring algorithm is given by the following:

$$\beta^{t+1} = \beta^t + (X_0^T W X_0)^{-1} X_0^T S^* \tag{84}$$

$$X_0 = x - V50_0 \tag{85}$$

$$S_i^* = S_i - h_{ii}(p_i - 1/2) \tag{12}$$

$$H = W^{1/2} X (X^T W X)^{-1} X^T W^{1/2} \tag{13}$$

The test statistic is approximately χ^2-distributed with one df.

Figure 12 presents an example of the one-sample PLRT on V50. The left plot displays the example data and response curves for the null and alternative models. The right plot displays the PLRT test statistic as a function of V50 under the null model. For this example, the value for $V50_0$ is 2200 ft/s resulting in a Δ^* of 4.28 and p-value of 0.039. A copy of the data used for this example is presented in Table A-9.

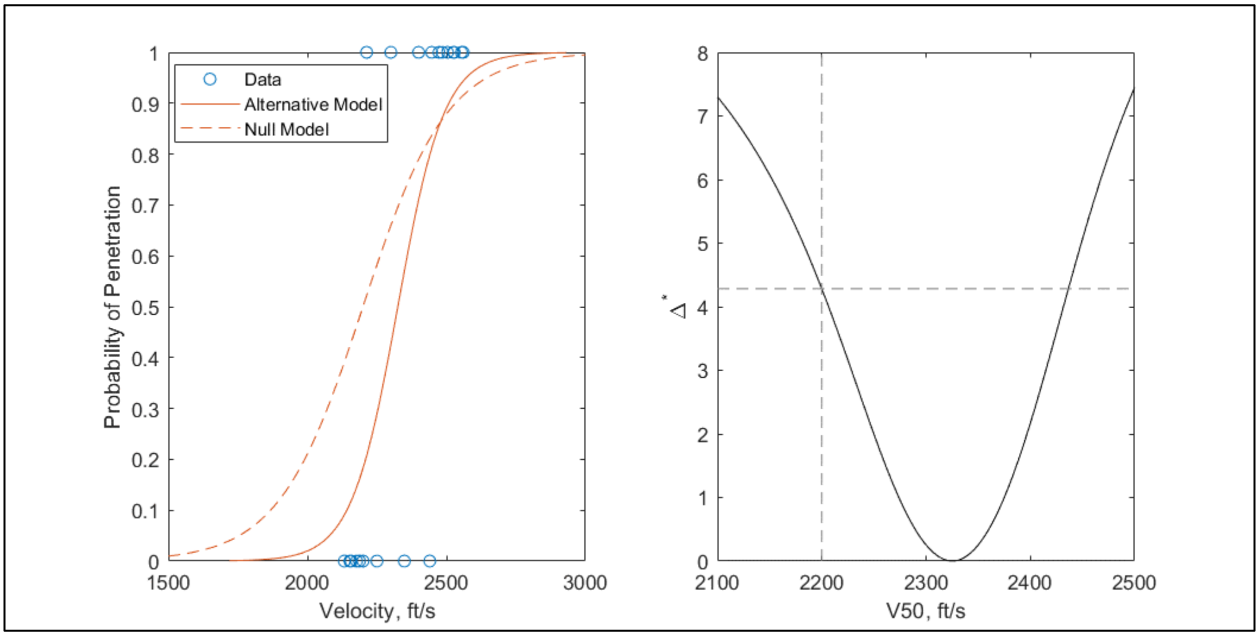

Figure 12. Example of one-sample PLRT on V50.

Two-Sample PLRT on V50

The null and alternative models are given by the following:

$$H_0\text{: } \frac{x-\mu}{s+\varepsilon d},\ V50_1 = V50_2 \tag{36}$$

$$H_1\text{: } \frac{x-(\mu+\delta d)}{s+\varepsilon d},\ V50_1 \neq V50_2 \tag{37}$$

The alternative hypothesis may be presented in linear parameters and solved using standard Firth logistic regression techniques.

$$H_1\text{: } \beta_0 + \beta_1 x + \beta_2 d + \beta_3 xd \tag{38}$$

The null model, however, is more challenging. Since the penalized likelihood is in part a function of the information matrix, the dimensionality of the information matrix has a large impact on the penalized likelihood calculation. Additionally, since the null model uses profile likelihood in the linear parameterization, it is possible for there to be local maxima and minima. The first step is to set μ_0 equal to values between $V50_1$ and $V50_2$ and solve for the profile maximum penalized likelihood estimates (PMPLEs) for s and ε given that value for μ_0. The iteratively reweighted scoring algorithm is given by the following:

$$\theta_0^{\{t+1\}} = \theta_0^{\{t\}} + \left(\left(\frac{\partial\eta}{\partial\theta_0}\right)^T W \frac{\partial\eta}{\partial\theta_0}\right)^{-1} \left(\frac{\partial\eta}{\partial\theta_0}\right)^T S^* \tag{86}$$

$$S_i^* = S_i - h_{ii}(p_i - 1/2) \tag{12}$$

$$H = W^{1/2}X(X^TWX)^{-1}X^TW^{1/2} \tag{13}$$

$$\theta_0 = \begin{bmatrix} s \\ \varepsilon \end{bmatrix} \tag{40}$$

$$\eta = \frac{x-\mu_0}{s+\varepsilon d} \tag{41}$$

$$\frac{\partial\eta}{\partial\theta_0} = \begin{bmatrix} \frac{\partial\eta}{\partial s} & \frac{\partial\eta}{\partial\varepsilon} \end{bmatrix} \tag{42}$$

$$\frac{\partial\eta_i}{\partial s} = -\frac{\eta_i}{s+\varepsilon d_i} \tag{43}$$

$$\frac{\partial\eta_i}{\partial\varepsilon} = -\frac{d_i\eta_i}{s+\varepsilon d_i} \tag{44}$$

The second step is to set the initial guesses for μ, s, and ε that maximize the penalized likelihood from the first step. Then, iteratively solve for the PMPLEs for μ, s, and ε using equation 86. However, the following equations should be used in support of the algorithm.

$$\theta_0 = \begin{bmatrix} \mu \\ s \\ \varepsilon \end{bmatrix} \tag{45}$$

$$\eta = \frac{x-\mu}{s+\varepsilon d} \tag{46}$$

$$\frac{\partial\eta}{\partial\theta_0} = \begin{bmatrix}\frac{\partial\eta}{\partial\mu} & \frac{\partial\eta}{\partial s} & \frac{\partial\eta}{\partial\varepsilon}\end{bmatrix} \tag{47}$$

$$\frac{\partial\eta_i}{\partial\mu} = -\frac{1}{s+\varepsilon d_i} \tag{48}$$

Note that equations 43 and 44 may still be used. The test statistic is approximately χ^2-distributed with one df. The method may be easily extended to one factor with multiple levels.

Figure 13 presents an example of the two-sample PLRT on V50. The left plot displays the response curves for the null and alternative models for samples A and B. The right plot displays the PLRT test statistic as a function of V50 under the null model. The resulting test statistic is 0.23 and p-value is 0.630. A copy of the data used for this example is presented in Table A-10.

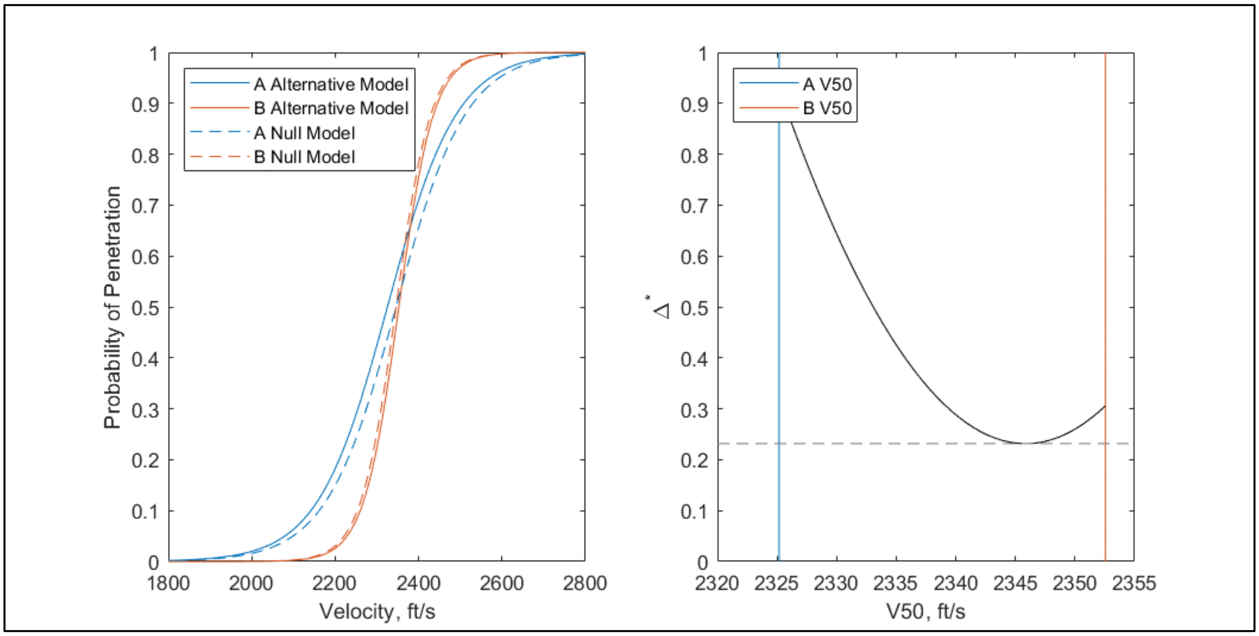

Figure 13. Example of two-sample PLRT on V50.

PLRT on V50 with Multiple Factors

Although this method may be easily extendable to any number of factors and levels, the two factor each at two levels model is described here. The alternative model is given by the following:

$$\ln\frac{p}{1-p} = \frac{x-(\mu+\delta_1 d_1+\delta_2 d_2+\delta_{12} d_{12})}{s+\varepsilon_1 d_1+\varepsilon_2 d_2+\varepsilon_{12} d_{12}} \tag{49}$$

The alternative hypothesis may be presented in linear parameters and solved using standard logistic regression techniques.

$$\ln \frac{p}{1-p} = \beta_0 + \beta_1 d_1 + \beta_2 d_2 + \beta_{12} d_{12} + \beta_x x + \beta_{1x} d_1 x + \beta_{2x} d_2 x + \beta_{12x} d_{12} x \quad (50)$$

There are three null models including the two main effects and the one interaction. These null models are given by the following:

$$\ln \frac{p}{1-p} = \frac{x-(\mu+\delta_2 d_2+\delta_{12} d_{12})}{s+\varepsilon_1 d_1+\varepsilon_2 d_2+\varepsilon_{12} d_{12}}, \delta_1 = 0 \quad (51)$$

$$\ln \frac{p}{1-p} = \frac{x-(\mu+\delta_1 d_1+\delta_{12} d_{12})}{s+\varepsilon_1 d_1+\varepsilon_2 d_2+\varepsilon_{12} d_{12}}, \delta_2 = 0 \quad (52)$$

$$\ln \frac{p}{1-p} = \frac{x-(\mu+\delta_1 d_1+\delta_2 d_2)}{s+\varepsilon_1 d_1+\varepsilon_2 d_2+\varepsilon_{12} d_{12}}, \delta_{12} = 0 \quad (53)$$

The null models, however, are more challenging. Since the penalized likelihood is in part a function of the information matrix, the dimensionality of the information matrix has a large impact on the penalized likelihood calculation. Additionally, since the null models use profile penalized likelihood in the linear parameterization, it is possible for there to be local maxima and minima. There are two possible approaches. The first is the grid approach in which the penalized likelihood for many combinations of parameters is evaluated and the best is selected as the initial guess. However, this method is very time consuming and completely impractical for more complex models. The second approach is using a Latin hypercube with multiple starts. The Latin hypercube is a space filling design and is used to produce the initial guesses. The Fisher scoring algorithm is then performed for each initial guess. This method is preferred though is not without its challenges. Most initial guesses will not be good and computational issues will have to be addressed. For the first null model (δ_1=0) the following equations are used.

$$\theta_0^{\{t+1\}} = \theta_0^{\{t\}} + \left(\left(\frac{\partial\eta}{\partial\theta_0}\right)^T W \frac{\partial\eta}{\partial\theta_0}\right)^{-1} \left(\frac{\partial\eta}{\partial\theta_0}\right)^T S^* \quad (86)$$

$$S_i^* = S_i - h_{ii}(p_i - 1/2) \quad (12)$$

$$H = W^{1/2} X (X^T W X)^{-1} X^T W^{1/2} \quad (13)$$

$$\theta_0^T = [\mu \quad \delta_2 \quad \delta_{12} \quad s \quad \varepsilon_1 \quad \varepsilon_2 \quad \varepsilon_{12}] \quad (54)$$

$$\eta = \frac{x-(\mu+\delta_2 d_2+\delta_{12} d_{12})}{s+\varepsilon_1 d_1+\varepsilon_2 d_2+\varepsilon_{12} d_{12}} \quad (55)$$

$$\frac{\partial\eta}{\partial\theta_0} = \left[\frac{\partial\eta}{\partial\mu} \quad \frac{\partial\eta}{\partial\delta_2} \quad \frac{\partial\eta}{\partial\delta_{12}} \quad \frac{\partial\eta}{\partial s} \quad \frac{\partial\eta}{\partial\varepsilon_1} \quad \frac{\partial\eta}{\partial\varepsilon_2} \quad \frac{\partial\eta}{\partial\varepsilon_{12}}\right] \quad (56)$$

$$\frac{\partial\eta_i}{\partial\mu} = -\frac{1}{s+\varepsilon_1 d_{1,i}+\varepsilon_2 d_{2,i}+\varepsilon_{12} d_{12,i}} \quad (57)$$

$$\frac{\partial\eta_i}{\partial\delta_2} = -\frac{d_{2,i}}{s+\varepsilon_1 d_{1,i}+\varepsilon_2 d_{2,i}+\varepsilon_{12} d_{12,i}} \quad (58)$$

$$\frac{\partial\eta_i}{\partial\delta_{12}} = -\frac{d}{s+\varepsilon_1 d_{1,i}+\varepsilon_2 d_{2,i}+\varepsilon_{12} d_{12,i}} \quad (59)$$

$$\frac{\partial \eta_i}{\partial s} = -\frac{\eta_i}{s+\varepsilon_1 d_{1,i}+\varepsilon_2 d_{2,i}+\varepsilon_{12} d_{12,i}} \tag{60}$$

$$\frac{\partial \eta_i}{\partial \varepsilon_1} = -\frac{\eta_i d_{1,i}}{s+\varepsilon_1 d_{1,i}+\varepsilon_2 d_{2,i}+\varepsilon_{12} d_{12,i}} \tag{61}$$

$$\frac{\partial \eta_i}{\partial \varepsilon_2} = -\frac{\eta_i d_{2,i}}{s+\varepsilon_1 d_{1,i}+\varepsilon_2 d_{2,i}+\varepsilon_{12} d_{12,i}} \tag{62}$$

$$\frac{\partial \eta_i}{\partial \varepsilon_{12}} = -\frac{\eta_i d_{12,i}}{s+\varepsilon_1 d_{1,i}+\varepsilon_2 d_{2,i}+\varepsilon_{12} d_{12,i}} \tag{63}$$

The test statistic is approximately χ^2-distributed with one df. This approach is repeated for design variable δ_2 and interaction δ_{12}.

An example of the penalized likelihood ratio test table is present in Table 2. Figure 14 presents an example of the PLRT on V50 with multiple factors. The top-left plot displays the response curves for the null and alternative models. The top-right plot displays a contour plot of δ_2 versus μ with color indicating the likelihood ratio test statistic and δ_{12} fixed at the PMPLE under the null model. Likewise, the bottom-left plot displays a contour plot of δ_{12} versus μ with color indicating the likelihood ratio test statistic and δ_2 fixed at the PMPLE under the null model. Finally, the bottom-right plot displays a contour plot of δ_{12} versus δ_2 with color indicating the likelihood ratio test statistic and μ fixed at the PMPLE under the null model. Additionally, PMPLE of parameters s, ε_1, ε_2, and ε_{12} were determined for each point in each of the contour plots. The resulting test statistic is 0.84 and p-value is 0.359. A copy of the data used for this example is presented in Table A-11.

TABLE 2. EXAMPLE PENALIZED LIKELIHOOD RATIO TEST TABLE

Factor	df	χ^2-Stat	P-Value
d_1	1	0.84	0.359
d_2	1	0.08	0.781
d_1*d_2	1	0.18	0.671

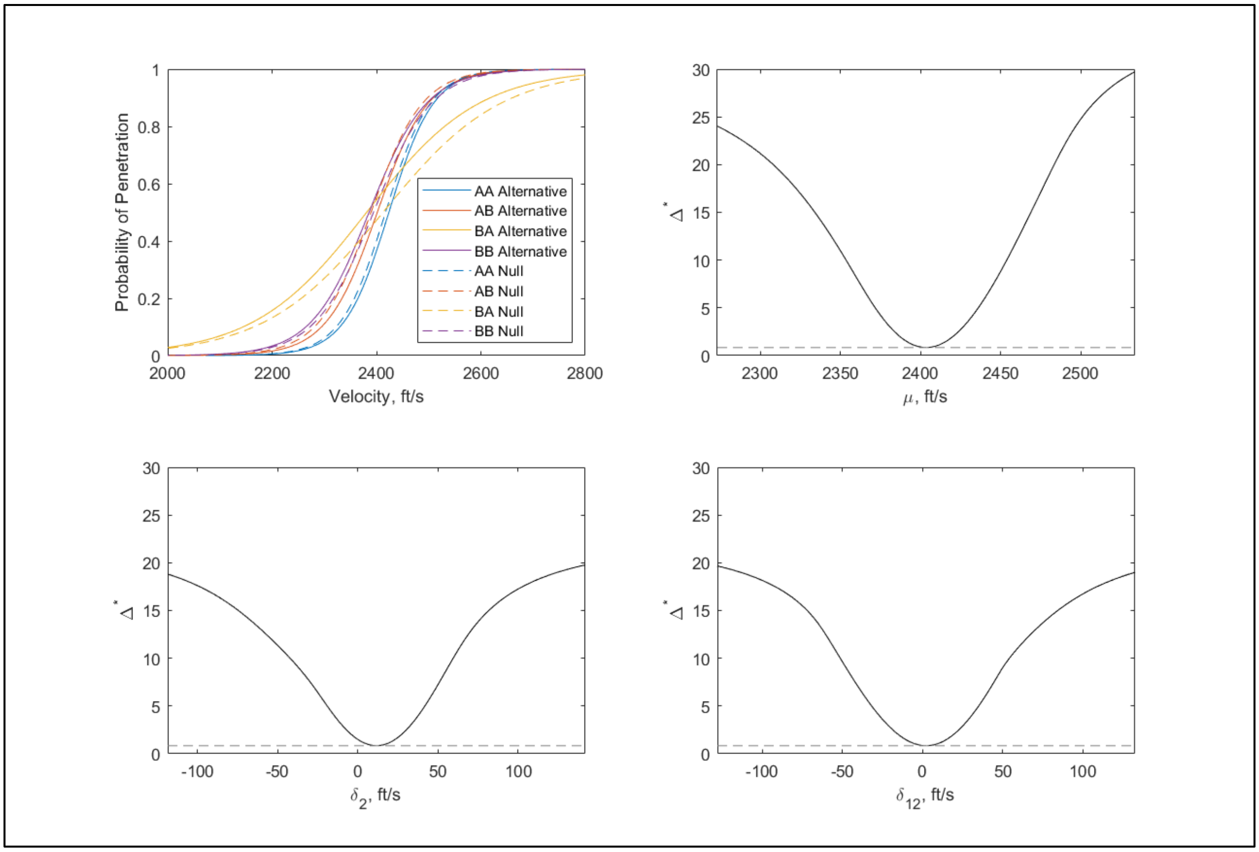

Figure 14. Example of the PLRT on V50 with two factors each at two levels.

One-Sample PLRT on Sigma

The null and alternative models are given by the following:

$$H_0\colon \beta_0 + \frac{x}{s_0},\ \ s = s_0 \tag{64}$$

$$H_1\colon \beta_0 + \beta_1 x,\ \ s \neq s_0 \tag{65}$$

The alternative model may be fit using standard Firth logistic regression techniques. As before, since the penalized likelihood is in part a function of the information matrix, the dimensionality of the information matrix has a large impact on the penalized likelihood calculation. The iteratively reweighted scoring algorithm is given by the following:

$$\beta^{t+1} = \beta^t + (\mathbf{1}^T W \mathbf{1})^{-1} \mathbf{1}^T S^* \tag{87}$$

$$S_i^* = S_i - h_{ii}(p_i - 1/2) \tag{12}$$

$$H = W^{1/2} X (X^T W X)^{-1} X^T W^{1/2} \tag{13}$$

Where, **1** is a vector of ones to solve for the intercept β_0. The test statistic is approximately χ^2-distributed with one df.

Figure 15 presents an example of the one-sample PLRT on sigma. The left plot displays the example data and response curves for the null and alternative models. The right plot displays the PLRT test statistic as a function of sigma under the null model. For this example, the value for σ_0 is 100 ft/s resulting in a Δ^* of 2.51 and p-value of 0.113. A copy of the data used for this example is presented in Table A-12.

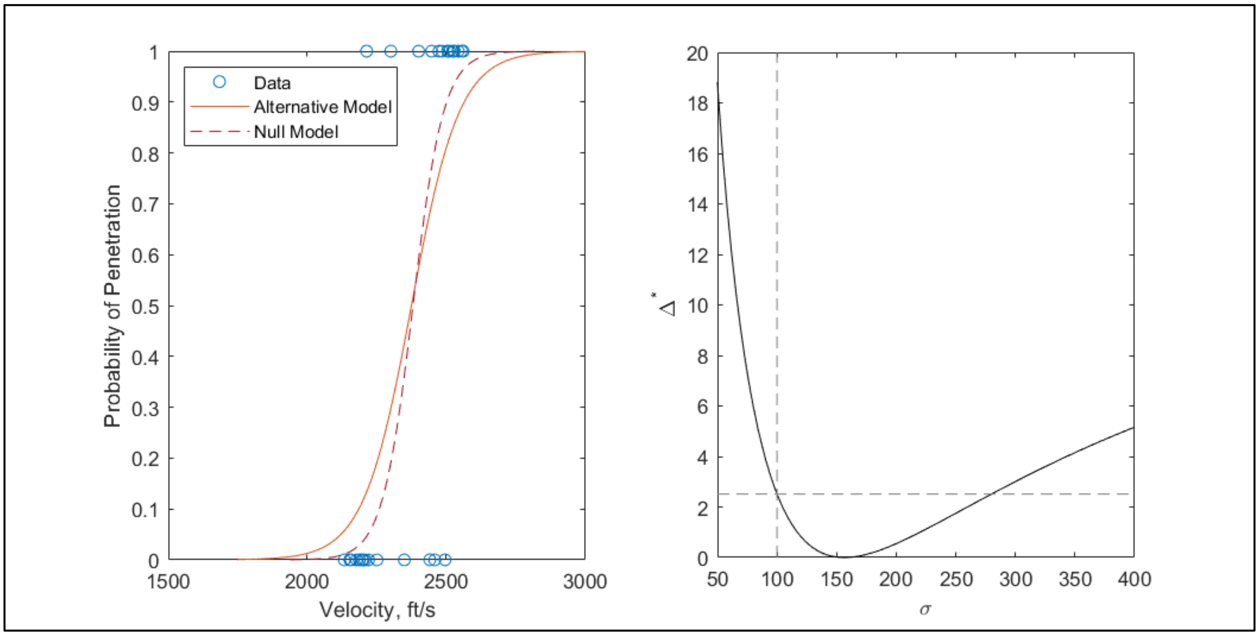

Figure 15. Example of one-sample PLRT on sigma.

Two-Sample PLRT on Sigma

The null and alternative models are given by the following:

$$H_0\colon \beta_0 + \beta_1 d + \beta_2 x, \;\; s_1 = s_2 \tag{66}$$

$$H_1\colon \beta_0 + \beta_1 d + \beta_2 x + \beta_3 dx, \;\; s_1 \neq s_2 \tag{67}$$

The alternative model may be fit using standard Firth logistic regression techniques. The following equations are used to solve for the null model.

$$\beta^{t+1} = \beta^t + (X_0^T W X_0)^{-1} X_0^T S^* \tag{84}$$

$$X_0 = [\mathbf{1} \quad \boldsymbol{d} \quad \boldsymbol{x}] \tag{88}$$

$$S_i^* = S_i - h_{ii}(p_i - 1/2) \tag{12}$$

$$H = W^{1/2} X (X^T W X)^{-1} X^T W^{1/2} \tag{13}$$

The test statistic is approximately χ^2-distributed with one df.

Figure 16 presents an example of the two-sample PLRT on sigma. The left plot displays the response curves for the null and alternative models for samples A and B. The right plot displays the PLRT test statistic as a function sigma under the null model. The resulting test statistic is 1.84 and p-value is 0.175. A copy of the data used for this example is presented in Table A-13.

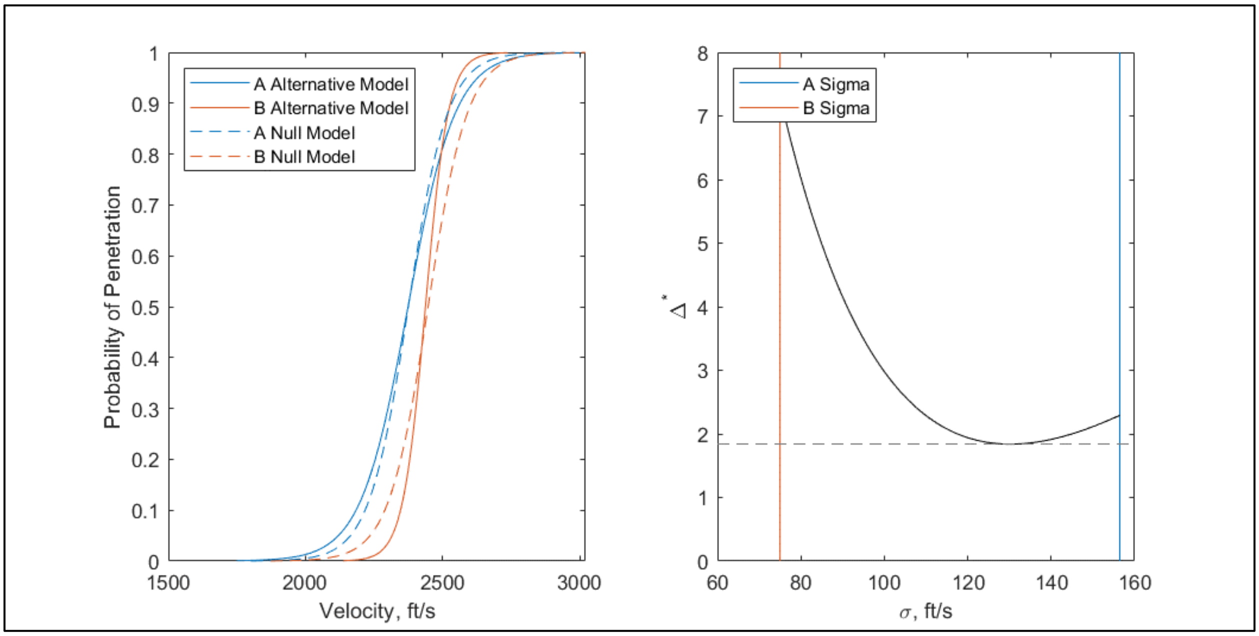

Figure 16. Example of two-sample PLRT on sigma.

One-Sample PLRT on V10

The null and alternative models are given by the following:

$$H_0\!: q_0 + \beta(x - V10_0),\ \ V10 = V10_0 \tag{68}$$

$$H_1\!: \beta_0 + \beta_1 x,\ \ V10 \neq V10_0 \tag{69}$$

$$q_0 = \ln\frac{p_0}{1-p_0} = \ln\frac{0.1}{0.9} \tag{70}$$

The alternative model may be fit using standard Firth logistic regression techniques. Again, since the penalized likelihood is in part a function of the information matrix, the dimensionality of the information matrix has a large impact on the penalized likelihood calculation. Therefore, care must be taken when fitting the null model for the purposes of the penalized likelihood ratio test. The iteratively reweighted scoring algorithm is given by the following:

$$\beta^{t+1} = \beta^t + (X_0^T W X_0)^{-1} X_0^T S^* \tag{84}$$

$$X_0 = x - V10_0 \tag{89}$$

$$S_i^* = S_i - h_{ii}(p_i - 1/2) \tag{12}$$

$$H = W^{1/2}X(X^TWX)^{-1}X^TW^{1/2} \tag{13}$$

The test statistic is approximately χ^2-distributed with one df.

Figure 17 presents an example of the one-sample PLRT on V10. The left plot displays the example data and response curves for the null and alternative models. The right plot displays the PLRT test statistic as a function of V10 under the null model. For this example, the value for $V10_0$ is 2200 ft/s resulting in a Δ^* of 2.85 and p-value of 0.091. A copy of the data used for this example is presented in Table A-14.

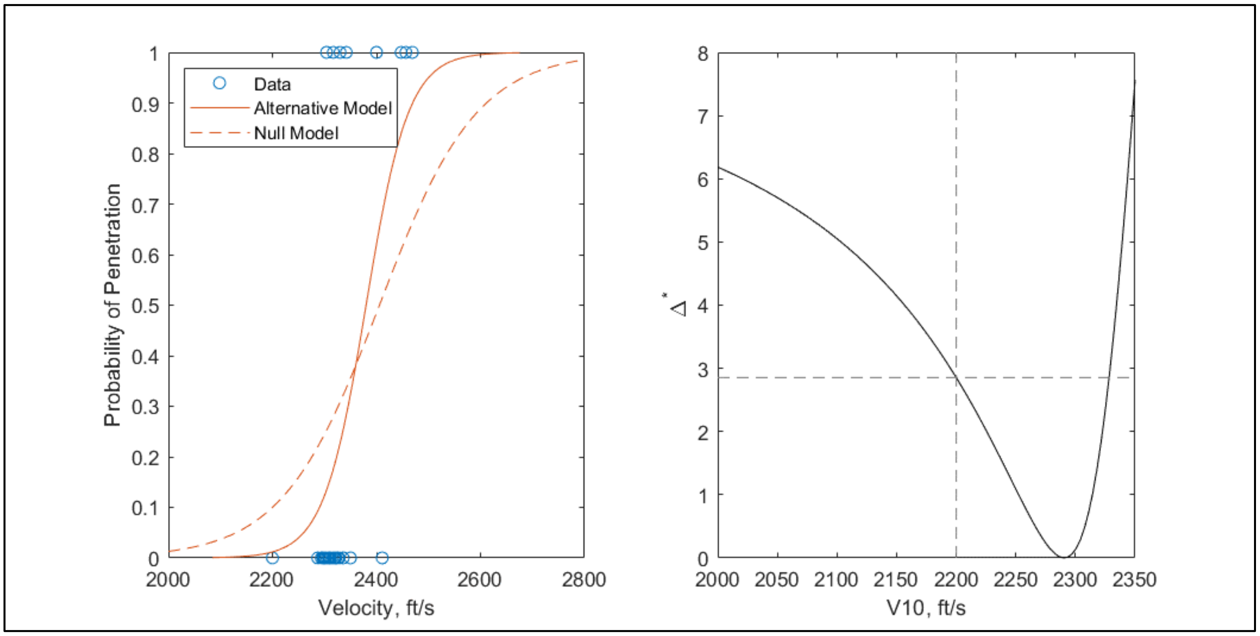

Figure 17. Example of one-sample PLRT on V10.

Two-Sample PLRT on V10

The null and alternative models are given by the following:

$$H_0: q_0 + \frac{x - V10}{s + \varepsilon d},\ V10_1 = V10_2 \tag{71}$$

$$H_1: q_0 + \frac{x - (V10 + \delta d)}{s + \varepsilon d},\ V10_1 \neq V10_2 \tag{72}$$

The alternative hypothesis may be presented in linear parameters and solved using standard logistic regression techniques.

$$H_1:\ \beta_0 + \beta_1 x + \beta_2 d + \beta_3 xd \tag{38}$$

As before, the null model is more challenging. Since the penalized likelihood is in part a function of the information matrix, the dimensionality of the information matrix has a large impact on the penalized likelihood calculation. Additionally, since the null model uses profile likelihood in the linear parameterization, it is possible for there to be local maxima and minima. The first step is to set $V10_0$ equal to values between $V10_1$ and $V10_2$ and solve for the PMPLEs for s and ε given that value for $V10_0$. The iteratively reweighted scoring algorithm is given by the following:

$$\theta_0^{\{t+1\}} = \theta_0^{\{t\}} + \left(\left(\frac{\partial\eta}{\partial\theta_0}\right)^T W \frac{\partial\eta}{\partial\theta_0}\right)^{-1} \left(\frac{\partial\eta}{\partial\theta_0}\right)^T S^* \quad (86)$$

$$S_i^* = S_i - h_{ii}(p_i - 1/2) \quad (12)$$

$$H = W^{1/2}X(X^TWX)^{-1}X^TW^{1/2} \quad (13)$$

$$\theta_0 = \begin{bmatrix} s \\ \varepsilon \end{bmatrix} \quad (40)$$

$$\eta = q_0 + \frac{x - V10_0}{s + \varepsilon d} \quad (73)$$

$$\frac{\partial\eta}{\partial\theta_0} = \begin{bmatrix} \frac{\partial\eta}{\partial s} & \frac{\partial\eta}{\partial\varepsilon} \end{bmatrix} \quad (42)$$

$$\frac{\partial\eta_i}{\partial s} = -\frac{x_i - V10_0}{(s + \varepsilon d_i)^2} \quad (74)$$

$$\frac{\partial\eta_i}{\partial\varepsilon} = -\frac{d_i(x_i - V10_0)}{(s + \varepsilon d_i)^2} \quad (75)$$

The second step is to set the initial guesses for V10, s, and ε that maximize the log-likelihood from the first step. Then, iteratively solve for the PMPLEs for V10, s, and ε using equation 86. However, the following equations should be used in support of the algorithm.

$$\theta_0 = \begin{bmatrix} V10 \\ s \\ \varepsilon \end{bmatrix} \quad (76)$$

$$\eta = q_0 + \frac{x - V10}{s + \varepsilon d} \quad (77)$$

$$\frac{\partial\eta}{\partial\theta_0} = \begin{bmatrix} \frac{\partial\eta}{\partial V10} & \frac{\partial\eta}{\partial s} & \frac{\partial\eta}{\partial\varepsilon} \end{bmatrix} \quad (78)$$

$$\frac{\partial\eta_i}{\partial V10} = -\frac{1}{s + \varepsilon d_i} \quad (79)$$

$$\frac{\partial\eta_i}{\partial s} = -\frac{x_i - V10}{(s + \varepsilon d_i)^2} \quad (80)$$

$$\frac{\partial\eta_i}{\partial\varepsilon} = -\frac{d_i(x_i - V10)}{(s + \varepsilon d_i)^2} \quad (81)$$

Note that equations 74 and 75 may still be used. The test statistic is approximately χ^2-distributed with one df. The method may be easily extended to one factor with multiple levels.

Figure 18 presents an example of the two-sample PLRT on V10. The left plot displays the response curves for the null and alternative models for samples A and B. The right plot displays the PLRT test statistic as a function of V10 under the null model. The resulting test statistic is 0.61 and p-value is 0.434. A copy of the data used for this example is presented in Table A-15.

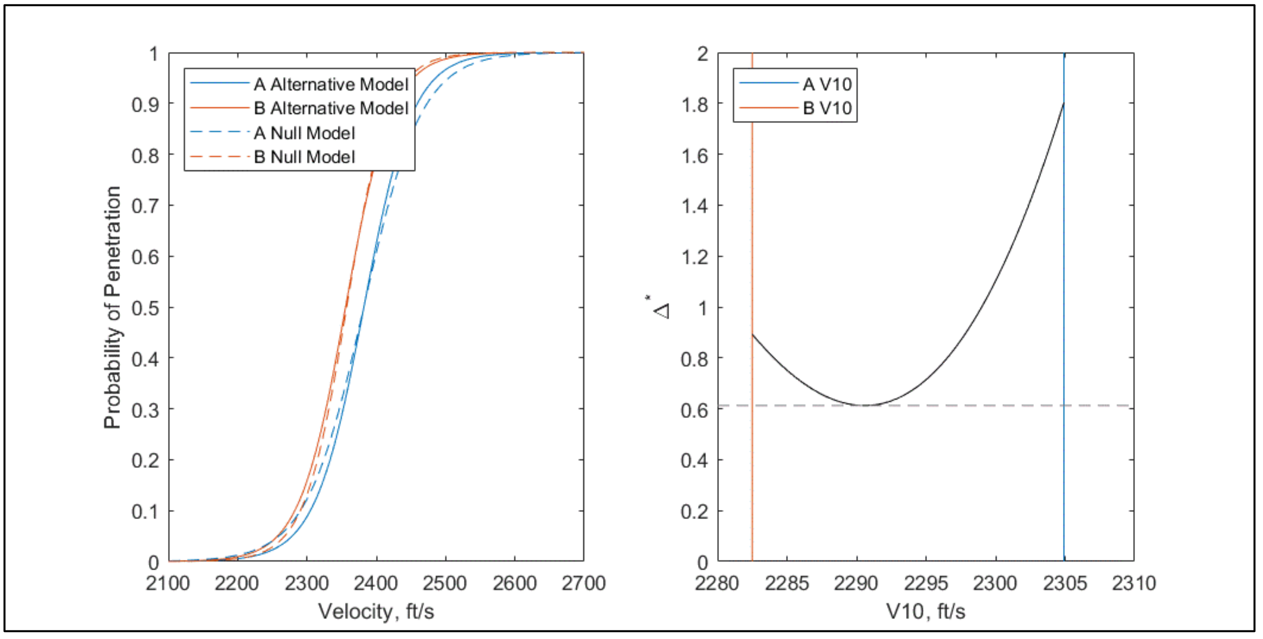

Figure 18. Example of two-sample PLRT on V10.

Simulation Setup

To generate data, four methods were used. To evaluate MLEs and LRTs for V50 and sigma, the Neyer D-Optimal Method was used as described in Figure 2. A penalized version of the D-optimal method that uses the modified information matrix was used to evaluate PMLEs and PLRTs for V50 and sigma. To evaluate MLEs and LRTs for V10, the Neyer c-Optimal method was used as described in Figure 3 with one exception. The binary search method from the Neyer D-Optimal method was used until separation was broken because this method tended to outperform the binary search method from the Neyer c-Optimal method. To evaluate PMLEs and PLRTs for V10, a penalized version of the c-Optimal method was used. Again, the binary search method from the Neyer D-Optimal method was used. A total of 10,000 simulated runs were completed for each test method.

Outputs of the simulations include relative mean square error (MSE) for V50, sigma, and V10 and relative median bias for sigma and V10. The V50 is unbiased for balanced designs. Quantile-quantile (Q-Q) plots are used to evaluate the performance of the statistical tests. Both LRT and PLRT are approximately chi-square distributed. The Q-Q plots display the proportion of the simulated runs that result in p-values below the alpha level of the test as a function of alpha when the null hypothesis is true. An accurate statistical test would lie on the ideal line.

For each test, the sample size was varied between 20 and 100 in increments of 10. For each sample size for the one-sample tests, the null and alternative models were compared for all 10,000 datasets for both the statistical test methods (LRT and the PLRT) when the null model was true. For the two-sample tests, datasets were paired resulting in 5000 tests for each statistical test method and sample size. For the multiple factor tests on V50, the datasets were put into groups of 4 resulting in 2500 tests for each statistical test method and sample size.

For a sample size of 20, there were 7 instances using the Neyer D-optimal method that separation was not broken. For the one-sample tests, the test was conducted in general accordance with the example presented in Figure 5. For the two-sample tests, the test was conducted in general accordance with ARL-TR-7088 (ref. 8). For the LRT with multiple factors, the situation was sufficiently complex, such that no attempt was made to complete the test when one of the samples did not have a ZMR. Therefore, for the LRT with multiple factors, there were only 2493 tests conducted.

Simulation Results

Figure 19 presents the performance with respect to V50 of logistic regression and Firth logistic regression using Neyer's D-Optimal method and a penalized D-optimal method, respectively. Specifically, the plot shows the inverse of the relative MSE of V50 versus sample size. The inverse of the relative MSE is presented because the relationship is approximately linear with sample size. The result using logistic regression is labeled logit for the link function for logistic regression. The result for the Firth logistic regression is labeled Firth. As shown, the MSE on V50 for logistic regression is less than the MSE for Firth logistic regression for all sample sizes evaluated in the simulation.

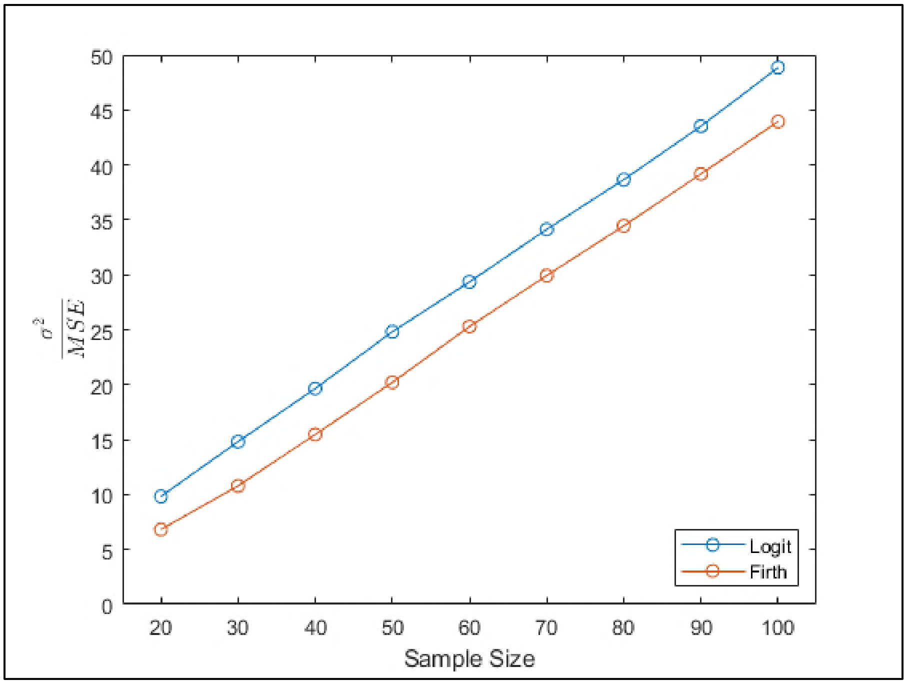

Figure 19. MSE for V50.

Figures 20 through 22 present Q-Q plots for the one-sample, two-sample, and multiple factor (two factors each at two levels) tests on V50. As shown, as sample size increases, both methods approach the ideal line. Additionally, the PLRT generally outperformed the LRT with respect to accuracy. Note that for the multiple factor tests on V50, the tests on the two main effects and the interaction each have one degree of freedom. Therefore, the results were combined in the Q-Q plot.

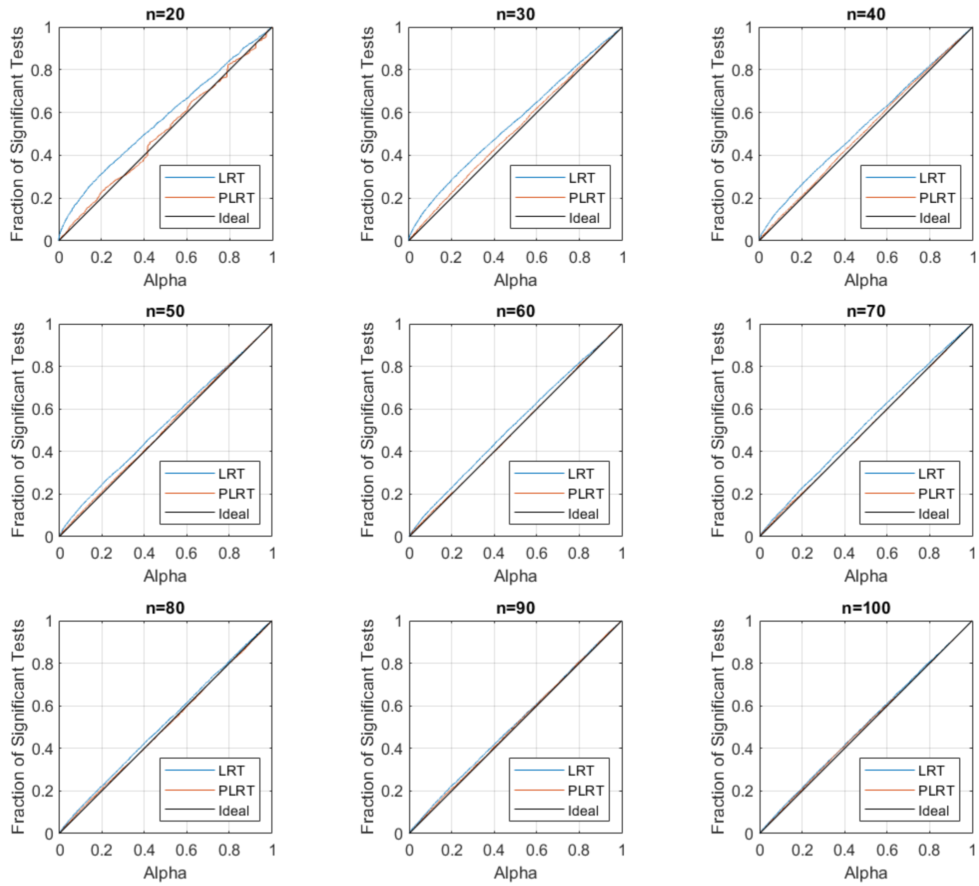

Figure 20. Q-Q plots for the one-sample test on V50.

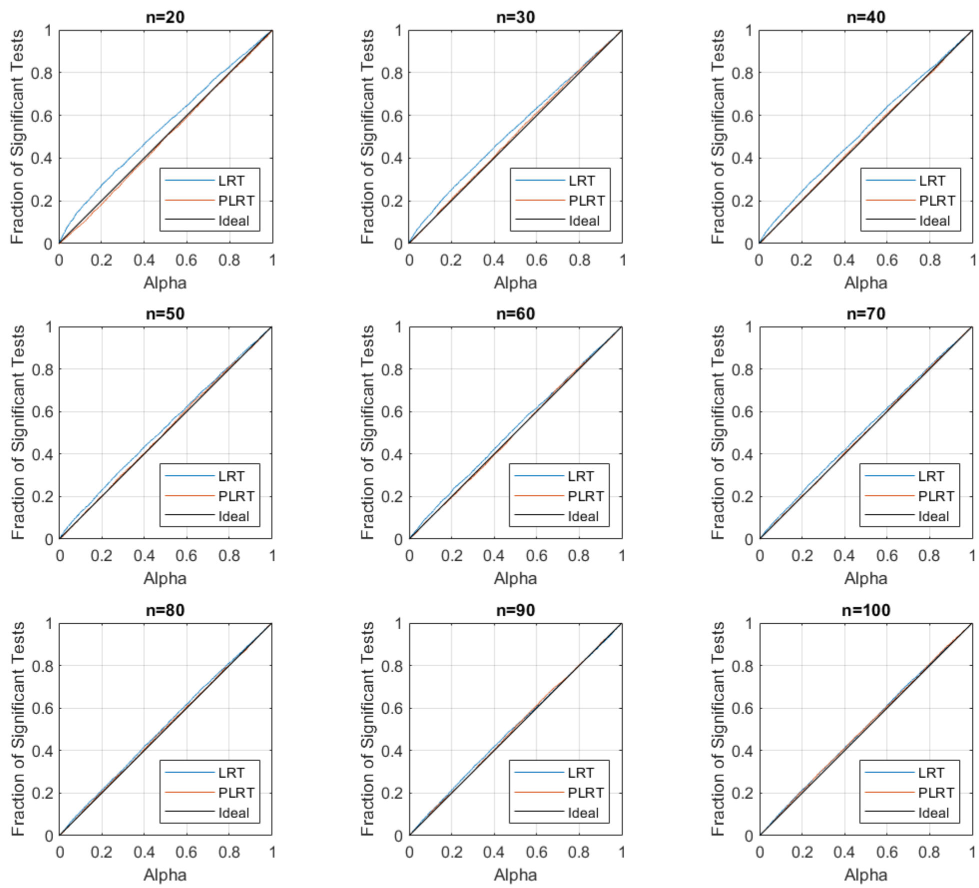

Figure 21.   Q-Q plots for the two-sample test on V50.

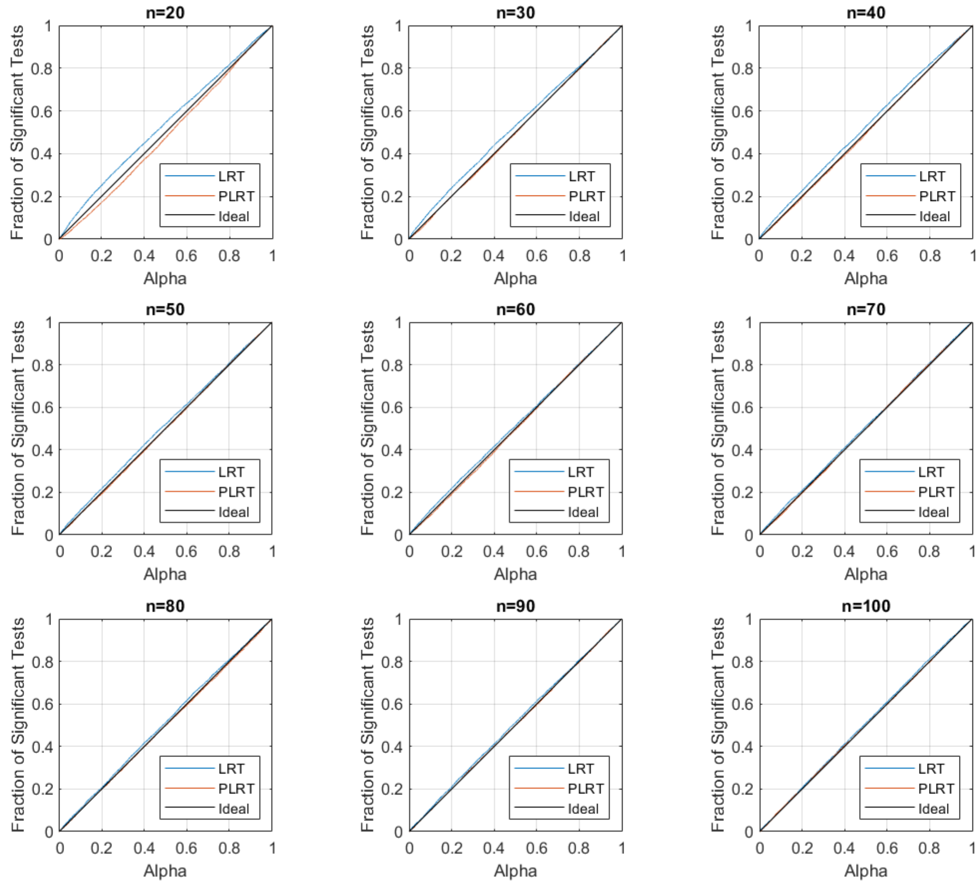

Figure 22. Q-Q plots for the test on V50 with multiple factors.

Figure 23 presents the performance with respect to sigma of logistic regression and Firth logistic regression using Neyer's D-Optimal method and a penalized D-optimal method, respectively. The left plot shows the relative median bias of sigma versus sample size. As shown, Firth logistic regression outperformed logistic regression with respect to relative median bias for the sample sizes investigated. The right plot shows the inverse of the MSE of sigma versus sample size. As shown, both methods performed similarly with respect to MSE.

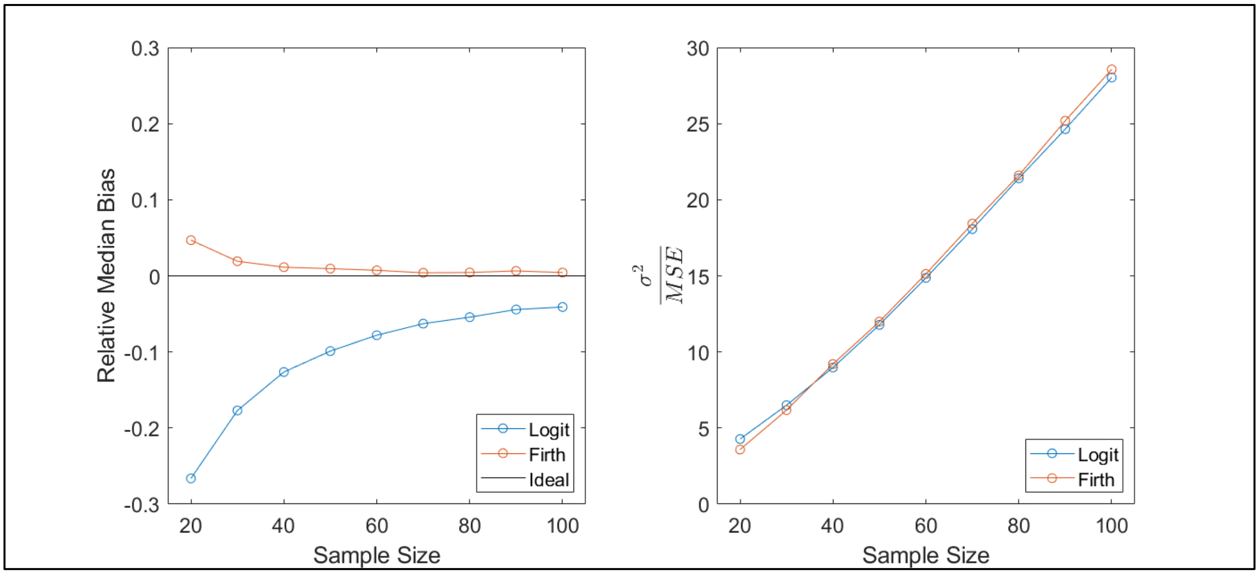

Figure 23. Relative median bias and MSE for sigma

Figures 24 and 25 present the Q-Q plots for the one-sample and two-sample tests on sigma, respectively. As shown, as sample size increases, the performance of both test methods approaches the ideal line. Additionally, the PLRT generally outperformed the LRT.

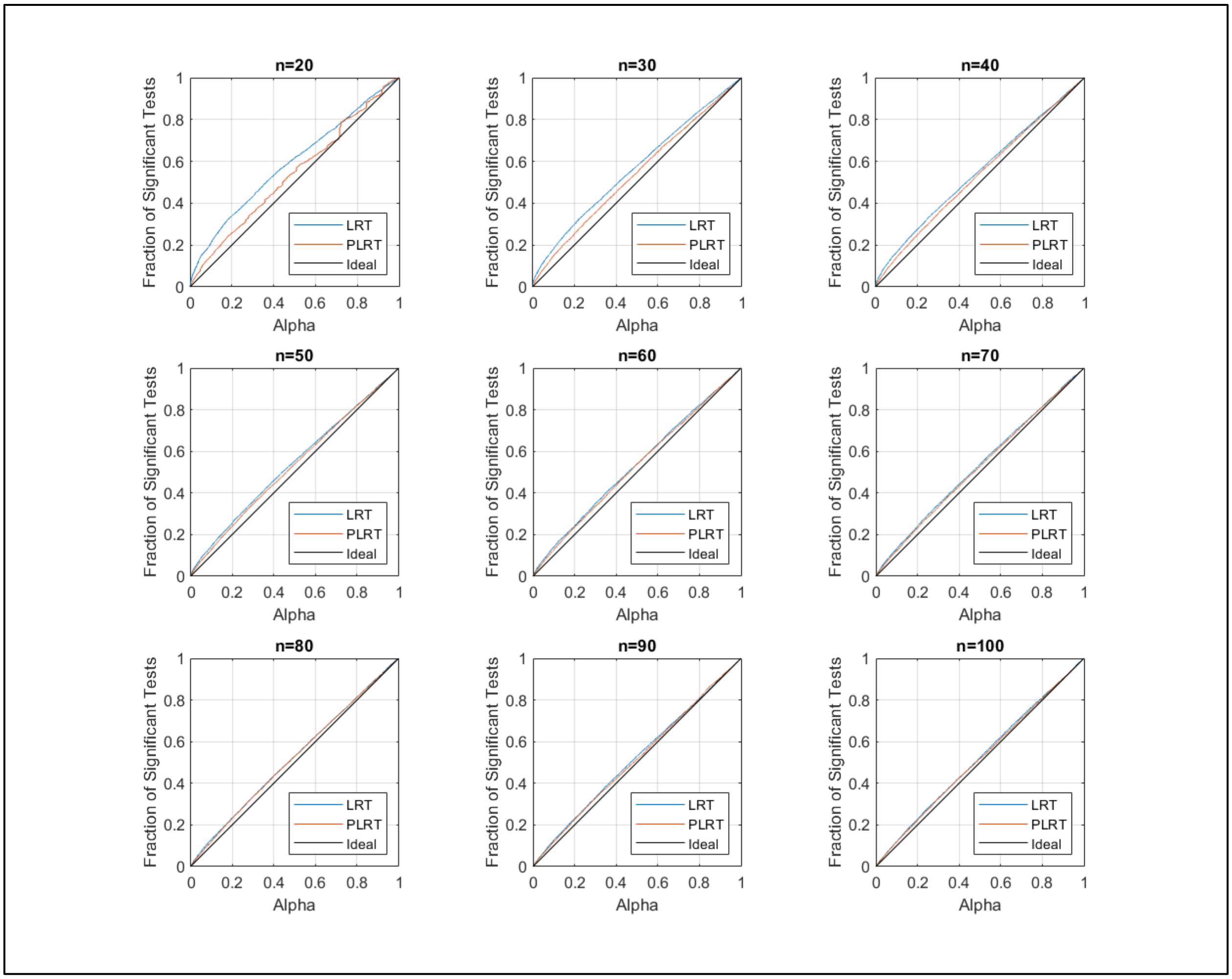

Figure 24. Q-Q plot for one-sample test on sigma.

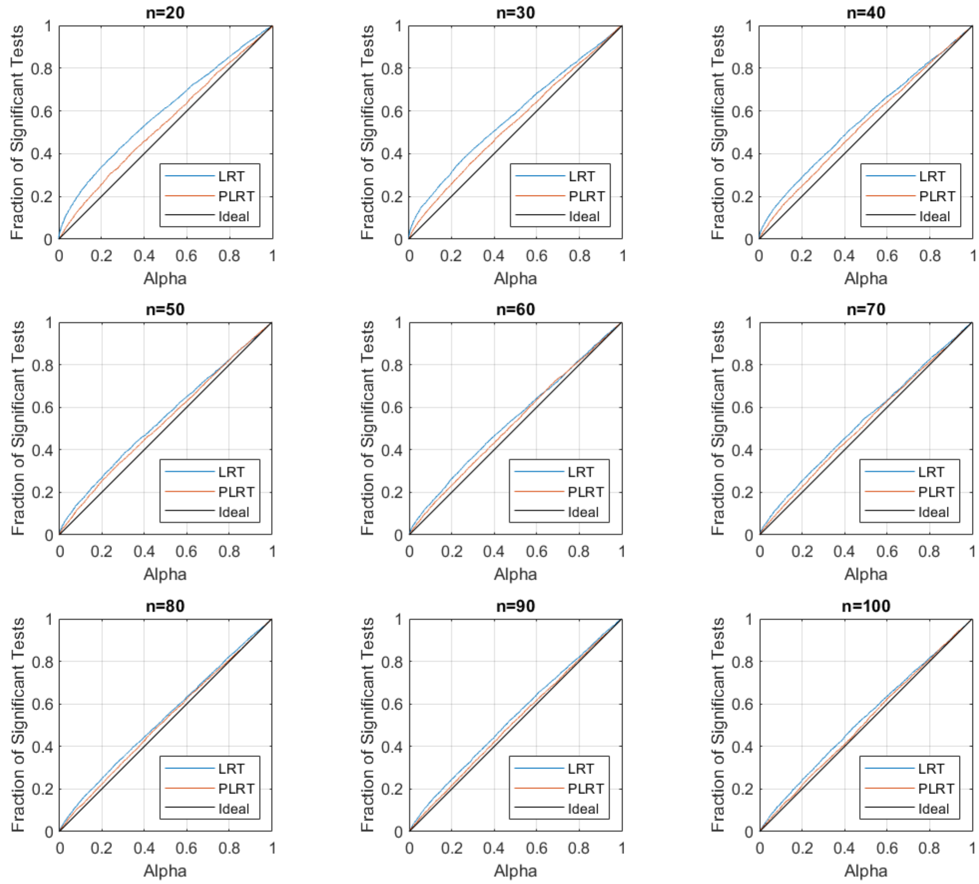

Figure 25. Q-Q plot for two-sample test on sigma.

Figure 26 presents the performance with respect to V10 of logistic regression and Firth logistic regression using Neyer's c-Optimal method and a penalized c-optimal method, respectively. The left plot shows the relative median bias of V10 versus sample size. As shown, Firth logistic regression outperformed logistic regression with respect to relative median bias for the sample sizes investigated. The right plot shows the inverse of the MSE of V10 versus sample size. As shown, logistic regression outperformed Firth logistic regression with respect to MSE.

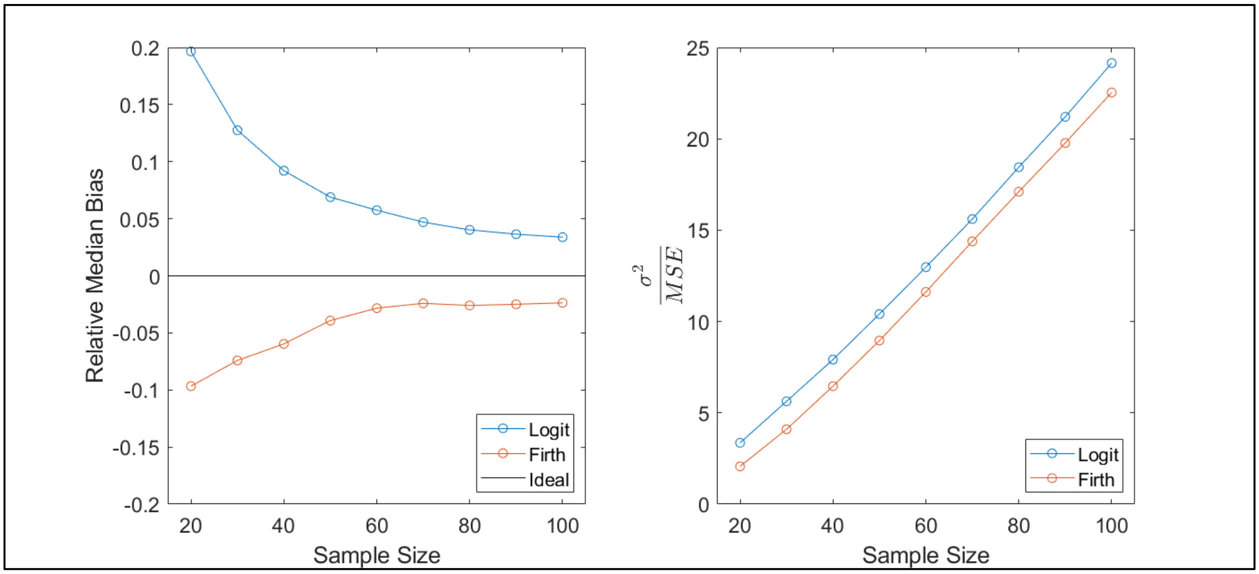

Figure 26. Relative median bias and MSE for V10.

Figures 27 and 28 present the Q-Q plots for the one-sample and two-sample tests on V10, respectively. As shown, as sample size increases, the performance of both test methods approaches the ideal line. Additionally, the PLRT generally outperformed the LRT.

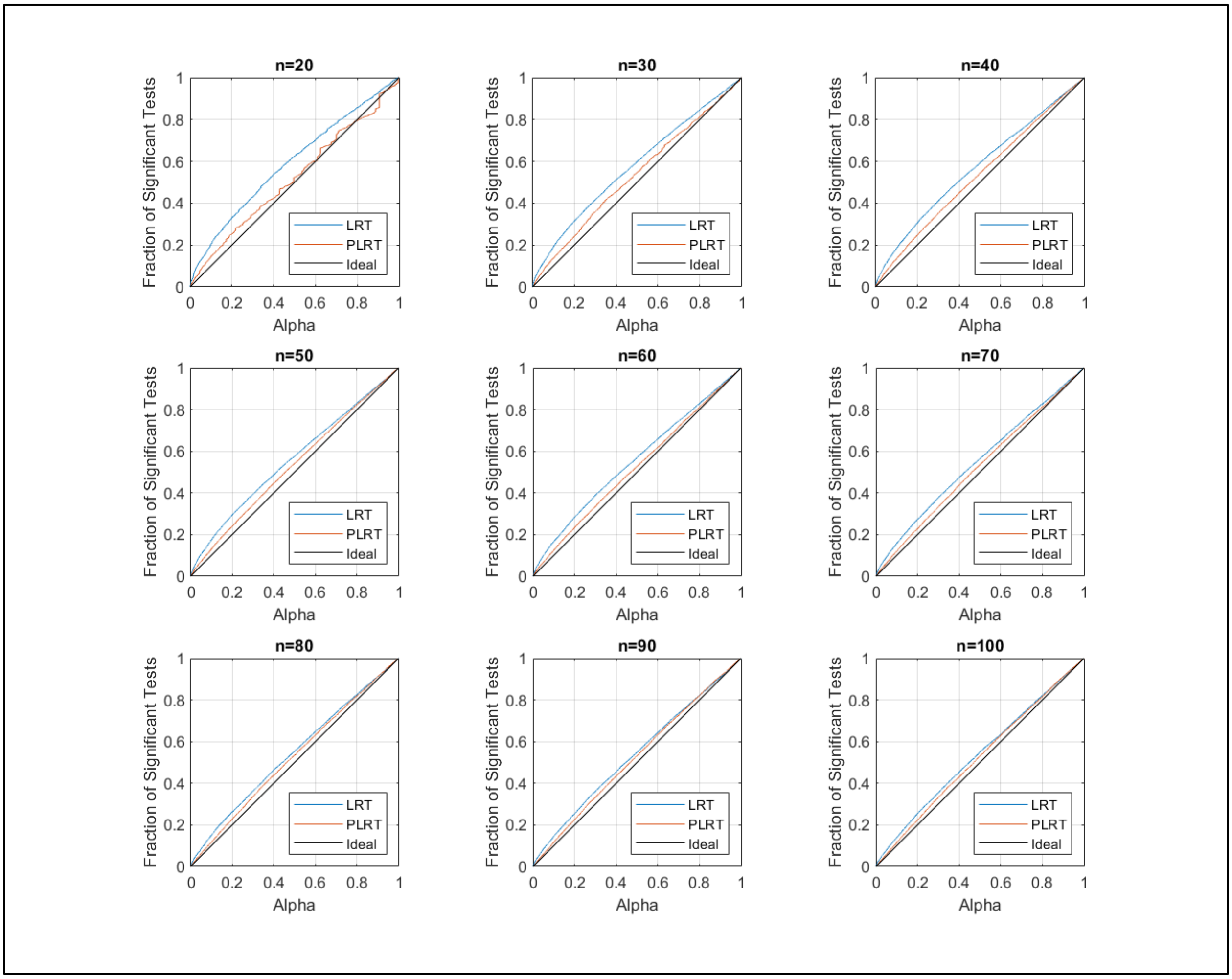

Figure 27. Q-Q plot for one-sample test on V10.

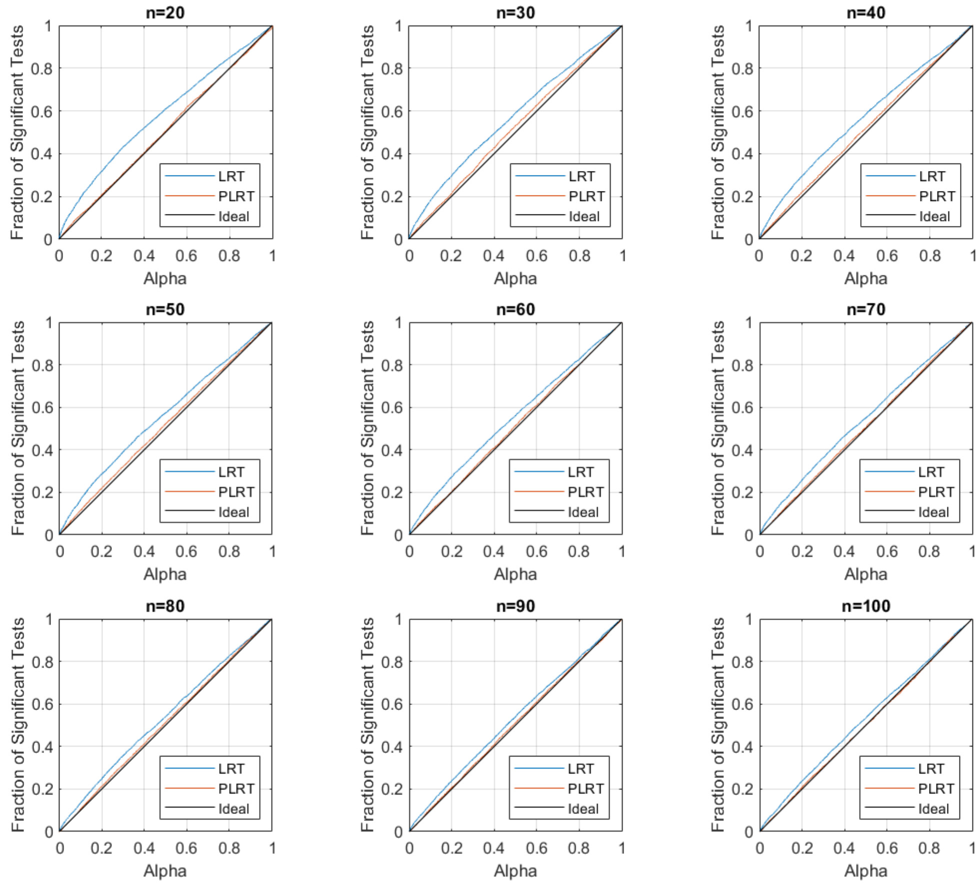

Figure 28. Q-Q plot for two-sample test on V10.

Conclusions

Firth's penalized likelihood reduces bias in the location-scale parametrization. Additionally, penalized likelihood has desirable properties when separation has not been broken. However, the MSE of the V50 and V10 were increased. Therefore, bias reduction may not always be desirable for point estimation.

For each statistical test, penalized likelihood ratio tests were more accurate than likelihood ratio tests. In practice, power analysis is recommended to determine the trade-offs between increased test accuracy and power.

References

1. MIL-STD-662F. Department of Defense Test Method Standard, V_{50} Ballistic Test for Armor. Aberdeen Proving Ground (MD): Army Research Laboratory (US); 1997 Dec.
2. Golub A, Grubbs F. On estimating ballistic limit and its precision. Aberdeen Proving Ground (MD): Ballistic Research Laboratory (US); 1950 Mar. Report No.: BRL-TN-151.
3. Collins J. Quantal response: practical sensitivity testing. Aberdeen Proving Ground (MD): Army Research Laboratory (US); 2012 Jun. Report No.: ARL-TR-6022.
4. Firth D. Bias reduction of maximum likelihood estimates. Biometrika, Vol. 80, No. 1, (Mar., 1993), pp. 27-38. http://www.jstor.org/stable/2336755
5. Cordeiro, G. M. & McCullagh, P. (1991). Bias correction in generalized linear models. J. R. Statist. Soc. B **53**, 629-43.
6. Jeffreys, H. (1946). An invariant form for the prior probability in estimation problems. Proc. R. Soc. A **186**, 453-61.
7. Neyer B. A D-Optimality Based Sensitivity Test, Technometrics, **36**, pp 61-70 (1994).
8. Collins J. Quantal response: estimation and inference. Aberdeen Proving Ground (MD): Army Research Laboratory (US); 2014 Jun. Report No.: ARL-TR-7088.
9. L. Lombardo and M. O'Connell. Two-Sample Statistical Inference for Ballistic Limit Testing of Body Armor. Proceedings of the Personal Armour Systems Symposium 2018. http://ipac-pass.org/
10. Lombardo L. Likelihood Ratio Test on V50 with Multiple Factors, Proceedings of the 58th Army Operations Research Symposium, Aberdeen Proving Ground, MD, October 2020.
11. Heinze, Georg and Schemper, Michael. A solution to the problem of separation in logistic regression. *Statistics in Medicine*; **21**:2409-2419.

APPENDIX A. EXAMPLE DATA

TABLE A-1. EXAMPLE DATA FOR THE RESPONSE CURVE PRESENTED IN FIGURE 1 AND ONE-SAMPLE LRT ON V50 PRESENTED IN FIGURE 4.

Velocity, ft/s	Penetration
2400.0	0
2600.0	1
2500.0	1
2352.7	1
2205.2	0
2266.3	0
2458.9	1
2306.4	0
2431.4	0
2496.7	1
2335.3	1
2267.2	0
2477.8	1
2293.1	0
2456.2	1
2310.8	0
2441.4	1
2323.0	0
2431.4	0
2326.3	0

TABLE A-2. EXAMPLE DATA FOR THE ONE-SAMPLE LRT ON V50 FOR GAP DATA PRESENTED IN FIGURE 5.

Velocity, ft/s	Penetration
2200	0
2305	0
2310	0
2347	0
2351	0
2378	0
2401	1
2414	1
2455	1
2460	1

TABLE A-3. EXAMPLE DATA FOR THE TWO-SAMPLE LRT ON V50 PRESENTED IN FIGURE 6.

Sample	Velocity, ft/s	Penetration	Sample	Velocity, ft/s	Penetration
A	2400.0	0	B	2400.0	0
A	2600.0	1	B	2600.0	1
A	2500.0	1	B	2500.0	1
A	2352.7	1	B	2352.7	0
A	2205.2	0	B	2450.0	1
A	2266.3	0	B	2343.6	1
A	2458.9	1	B	2285.8	0
A	2306.4	0	B	2460.9	0
A	2431.4	0	B	2554.8	1
A	2496.7	1	B	2305.9	0
A	2335.3	1	B	2515.0	1
A	2267.2	0	B	2336.0	0
A	2477.8	1	B	2491.0	1
A	2293.1	0	B	2355.3	0
A	2456.2	1	B	2475.5	0
A	2310.8	0	B	2519.4	1
A	2441.4	1	B	2364.7	0
A	2323.0	0	B	2374.6	1
A	2431.4	0	B	2512.9	0
A	2326.3	0	B	2565.0	1
A	2463.4	1	B	2323.5	0
A	2337.2	0	B	2546.3	1
A	2453.7	1	B	2340.4	1
A	2345.2	0	B	2292.6	0
A	2446.0	1	B	2548.6	1
A	2351.0	1	B	2309.5	0
A	2443.9	1	B	2533.1	1
A	2330.5	0	B	2323.4	0
A	2436.9	1	B	2520.4	1
A	2336.7	0	B	2335.1	0

TABLE A-4. EXAMPLE DATA FOR THE LRT ON V50 WITH MULTIPLE FACTORS PRESENTED IN FIGURE 7.

d_1	d_2	Velocity, ft/s	Penetration
A	A	2400.0	1
A	A	2200.0	0
A	A	2300.0	0
A	A	2447.3	1
A	A	2350.0	0
A	A	2456.4	0
A	A	2514.2	1
A	A	2339.1	0
A	A	2479.3	1
A	A	2365.2	1
A	A	2312.3	1
A	A	2234.1	0
A	A	2252.1	0
A	A	2471.2	1
A	A	2276.9	0
A	A	2450.7	1
A	A	2294.3	0
A	A	2436.2	0
A	A	2477.6	1
A	A	2300.5	0
A	A	2463.0	1
A	A	2313.6	0
A	A	2451.6	1
A	A	2323.6	0
A	A	2442.5	1
A	A	2331.1	0
A	A	2435.5	1
A	A	2336.9	1
A	A	2434.9	1
A	A	2315.8	0
A	B	2400.0	1
A	B	2200.0	0
A	B	2300.0	0
A	B	2447.3	0
A	B	2594.8	1
A	B	2533.7	0
A	B	2724.1	1
A	B	2665.7	1
A	B	2353.7	0
A	B	2611.9	1
A	B	2394.6	1
A	B	2298.4	0
A	B	2583.3	1
A	B	2333.0	0
A	B	2555.1	1
A	B	2537.3	1

TABLE A-4 (CONT)

d_1	d_2	Velocity, ft/s	Penetration
A	B	2356.4	0
A	B	2521.9	1
A	B	2370.6	1
A	B	2319.4	0
A	B	2515.8	1
A	B	2505.4	1
A	B	2335.3	0
A	B	2494.1	1
A	B	2485.7	1
A	B	2344.7	0
A	B	2478.1	1
A	B	2352.6	0
A	B	2472.4	1
A	B	2466.1	1
B	A	2400.0	1
B	A	2200.0	0
B	A	2300.0	0
B	A	2447.3	0
B	A	2594.8	1
B	A	2533.7	1
B	A	2341.1	0
B	A	2493.6	1
B	A	2368.6	1
B	A	2303.3	0
B	A	2464.7	1
B	A	2326.3	0
B	A	2446.6	1
B	A	2433.9	0
B	A	2337.0	0
B	A	2469.8	1
B	A	2350.9	0
B	A	2458.1	0
B	A	2492.6	1
B	A	2356.0	0
B	A	2481.5	1
B	A	2365.9	0
B	A	2372.1	1
B	A	2477.0	1
B	A	2350.6	0
B	A	2468.2	1
B	A	2358.5	0
B	A	2461.1	0
B	A	2483.3	0
B	A	2509.0	1
B	B	2400.0	1
B	B	2200.0	0
B	B	2300.0	0
B	B	2447.3	1

TABLE A-4 (CONT)

d_1	d_2	Velocity, ft/s	Penetration
B	B	2350.0	0
B	B	2456.4	0
B	B	2514.2	1
B	B	2339.1	0
B	B	2479.3	1
B	B	2365.2	0
B	B	2460.0	1
B	B	2378.7	1
B	B	2342.4	1
B	B	2295.3	0
B	B	2457.8	1
B	B	2312.2	0
B	B	2442.8	0
B	B	2481.5	1
B	B	2317.3	0
B	B	2467.0	1
B	B	2329.9	0
B	B	2455.7	1
B	B	2339.7	0
B	B	2446.9	1
B	B	2347.3	0
B	B	2440.2	0
B	B	2347.8	0
B	B	2460.4	1
B	B	2354.8	0
B	B	2453.9	1

TABLE A-5. EXAMPLE DATA FOR THE ONE-SAMPLE LRT ON SIGMA PRESENTED IN FIGURE 8.

Velocity, ft/s	Penetration
2400.0	0
2600.0	1
2500.0	1
2352.7	1
2205.2	0
2266.3	0
2458.9	1
2306.4	0
2431.4	0
2496.7	1
2335.3	1
2267.2	0
2477.8	1
2293.1	0
2456.2	1
2310.8	0
2441.4	1
2323.0	0
2431.4	0
2326.3	0
2463.4	1
2337.2	0
2453.7	1
2345.2	0
2446.0	1
2351.0	1
2443.9	1
2330.5	0
2436.9	1
2336.7	0

TABLE A-6. EXAMPLE DATA FOR THE TWO-SAMPLE LRT ON SIGMA PRESENTED IN FIGURE 9.

Sample	Velocity, ft/s	Penetration	Sample	Velocity, ft/s	Penetration
A	2400.0	0	B	2400.0	1
A	2600.0	1	B	2200.0	0
A	2500.0	1	B	2300.0	0
A	2352.7	1	B	2447.3	1
A	2205.2	0	B	2350.0	0
A	2266.3	0	B	2456.4	1
A	2458.9	1	B	2319.7	0
A	2306.4	0	B	2411.1	0
A	2431.4	0	B	2379.2	0
A	2496.7	1	B	2427.3	0
A	2335.3	1	B	2453.0	1
A	2267.2	0	B	2392.4	1
A	2477.8	1	B	2366.2	0
A	2293.1	0	B	2444.2	1
A	2456.2	1	B	2375.9	0
A	2310.8	0	B	2436.2	0
A	2441.4	1	B	2457.7	1
A	2323.0	0	B	2379.9	0
A	2431.4	0	B	2450.0	0
A	2326.3	0	B	2471.7	0
A	2463.4	1	B	2503.2	1
A	2337.2	0	B	2496.5	1
A	2453.7	1	B	2491.1	1
A	2345.2	0	B	2374.7	0
A	2446.0	1	B	2483.2	1
A	2351.0	1	B	2382.2	0
A	2443.9	1	B	2476.8	1
A	2330.5	0	B	2388.3	1
A	2436.9	1	B	2371.2	0
A	2336.7	0	B	2475.4	1

TABLE A-7. EXAMPLE DATA FOR THE ONE-SAMPLE LRT ON V10 PRESENTED IN FIGURE 10.

Velocity, ft/s	Penetration
2400.0	1
2200.0	0
2300.0	0
2447.3	0
2237.4	0
2281.8	0
2310.1	1
2665.0	1
2207.3	0
2227.7	0
2244.8	0
2259.6	0
2272.8	0
2284.7	0
2295.5	0
2305.4	0
2314.6	0
2323.2	0
2331.5	0
2339.5	1
2285.3	0
2292.6	0
2299.6	1
2249.7	0
2256.5	0
2263.0	0
2269.3	1
2547.3	1
2237.1	0
2242.2	0

TABLE A-8. EXAMPLE DATA FOR THE TWO-SAMPLE LRT ON V10 PRESENTED IN FIGURE 11.

Sample	Velocity, ft/s	Penetration	Sample	Velocity, ft/s	Penetration
A	2400.0	1	B	2400.0	0
A	2200.0	0	B	2600.0	1
A	2300.0	0	B	2500.0	1
A	2447.3	0	B	2352.7	0
A	2237.4	0	B	2450.0	0
A	2281.8	0	B	2548.7	1
A	2310.1	1	B	2526.3	1
A	2665.0	1	B	2437.1	0
A	2207.3	0	B	2475.0	1
A	2227.7	0	B	2427.3	1
A	2244.8	0	B	2413.3	0
A	2259.6	0	B	2418.2	1
A	2272.8	0	B	2392.0	1
A	2284.7	0	B	2353.8	0
A	2295.5	0	B	2361.4	1
A	2305.4	0	B	2311.1	0
A	2314.6	0	B	2321.3	1
A	2323.2	0	B	2249.6	0
A	2331.5	0	B	2264.5	0
A	2339.5	1	B	2275.7	0
A	2285.3	0	B	2284.7	0
A	2292.6	0	B	2292.3	0
A	2299.6	1	B	2298.8	0
A	2249.7	0	B	2304.6	0
A	2256.5	0	B	2309.8	0
A	2263.0	0	B	2314.5	0
A	2269.3	1	B	2318.8	0
A	2547.3	1	B	2322.8	0
A	2237.1	0	B	2326.5	0
A	2242.2	0	B	2329.9	0

TABLE A-9. EXAMPLE DATA FOR THE ONE-SAMPLE PLRT ON V50 PRESENTED IN FIGURE 12.

Velocity, ft/s	Penetration
2400.0	1
2200.0	0
2300.0	1
2152.7	0
2250.0	0
2348.7	0
2447.0	1
2440.0	0
2560.0	1
2554.0	1
2187.0	0
2525.0	1
2213.0	1
2133.0	0
2528.0	1
2157.0	0
2504.0	1
2177.0	0
2485.0	1
2473.0	1

TABLE A-10. EXAMPLE DATA FOR THE TWO-SAMPLE PLRT ON V50 PRESENTED IN FIGURE 13.

Sample	Velocity, ft/s	Penetration	Sample	Velocity, ft/s	Penetration
A	2400.0	1	B	2400.0	1
A	2200.0	0	B	2200.0	0
A	2300.0	1	B	2300.0	1
A	2152.7	0	B	2152.7	0
A	2250.0	0	B	2250.0	0
A	2348.7	0	B	2348.7	0
A	2447.0	1	B	2447.0	1
A	2440.0	0	B	2440.0	1
A	2560.0	1	B	2209.0	0
A	2554.0	1	B	2419.0	1
A	2187.0	0	B	2229.0	0
A	2525.0	1	B	2403.0	0
A	2213.0	1	B	2229.0	0
A	2133.0	0	B	2439.0	1
A	2528.0	1	B	2246.0	0
A	2157.0	0	B	2427.0	1
A	2504.0	1	B	2258.0	0
A	2177.0	0	B	2267.0	0
A	2485.0	1	B	2416.0	1
A	2473.0	1	B	2276.0	0

TABLE A-11. EXAMPLE DATA FOR THE PLRT ON V50 WITH MULTIPLE FACTORS PRESENTED IN FIGURE 14.

d_1	d_2	Velocity, ft/s	Penetration
A	A	2400.0	0
A	A	2600.0	1
A	A	2500.0	1
A	A	2352.7	0
A	A	2450.0	0
A	A	2548.7	1
A	A	2526.3	1
A	A	2437.1	1
A	A	2376.0	1
A	A	2312.0	0
A	A	2318.0	0
A	A	2525.0	1
A	A	2334.0	0
A	A	2508.0	1
A	A	2347.0	1
A	A	2310.0	0
A	A	2504.0	1
A	A	2496.0	1
A	A	2325.0	0
A	A	2485.0	1
A	A	2334.0	0
A	A	2476.0	1
A	A	2470.0	1
A	A	2342.0	0
A	A	2463.0	0
A	A	2344.0	0
A	A	2349.0	0
A	A	2478.0	1
A	A	2354.0	0
A	A	2473.0	0
A	B	2400.0	1
A	B	2200.0	0
A	B	2300.0	0
A	B	2447.3	0
A	B	2646.0	1
A	B	2696.0	1
A	B	2680.0	1
A	B	2263.0	0
A	B	2640.0	1
A	B	2300.0	0
A	B	2610.0	1
A	B	2590.0	1
A	B	2328.0	1
A	B	2580.0	1
A	B	2273.0	0
A	B	2556.0	1

TABLE A-11 (CONT)

d_1	d_2	Velocity, ft/s	Penetration
A	B	2542.0	1
A	B	2291.0	0
A	B	2525.0	1
A	B	2513.0	1
A	B	2303.0	0
A	B	2500.0	1
A	B	2489.0	1
A	B	2311.0	0
A	B	2479.0	1
A	B	2470.0	1
A	B	2317.0	0
A	B	2462.0	1
A	B	2455.0	0
A	B	2324.0	0
B	A	2400.0	1
B	A	2200.0	0
B	A	2300.0	0
B	A	2447.3	1
B	A	2350.0	0
B	A	2456.4	1
B	A	2319.7	1
B	A	2235.0	0
B	A	2247.0	0
B	A	2426.0	0
B	A	2481.0	1
B	A	2252.0	0
B	A	2465.0	1
B	A	2269.0	0
B	A	2452.0	1
B	A	2282.0	1
B	A	2454.0	0
B	A	2498.0	1
B	A	2235.0	1
B	A	2179.0	0
B	A	2185.0	0
B	A	2503.0	1
B	A	2200.0	0
B	A	2489.0	0
B	A	2539.0	0
B	A	2610.0	1
B	A	2604.0	1
B	A	2597.0	1
B	A	2590.0	1
B	A	2179.0	0
B	B	2400.0	1
B	B	2200.0	0
B	B	2300.0	0
B	B	2447.3	1

TABLE A-11 (CONT)

d_1	d_2	Velocity, ft/s	Penetration
B	B	2350.0	0
B	B	2456.4	1
B	B	2319.7	0
B	B	2411.1	1
B	B	2375.0	0
B	B	2352.6	0
B	B	2412.6	0
B	B	2443.0	1
B	B	2355.0	1
B	B	2326.0	1
B	B	2282.0	0
B	B	2288.0	0
B	B	2450.0	0
B	B	2492.0	1
B	B	2487.0	1
B	B	2287.0	0
B	B	2476.0	1
B	B	2298.0	0
B	B	2467.0	0
B	B	2497.0	1
B	B	2493.0	1
B	B	2300.0	0
B	B	2484.0	1
B	B	2309.0	1
B	B	2282.0	0
B	B	2486.0	1

TABLE A-12. EXAMPLE DATA FOR THE ONE-SAMPLE PLRT ON SIGMA PRESENTED IN FIGURE 15.

Velocity, ft/s	Penetration
2400.0	1
2200.0	0
2300.0	1
2152.7	0
2250.0	0
2348.7	0
2447.0	1
2440.0	0
2560.0	1
2554.0	1
2187.0	0
2525.0	1
2213.0	1
2133.0	0
2528.0	1
2157.0	0
2504.0	1
2177.0	0
2485.0	1
2473.0	1
2198.0	0
2458.0	0
2197.0	0
2495.0	0
2542.0	1
2198.0	0
2207.0	0
2522.0	1
2220.0	0
2510.0	1

TABLE A-13. EXAMPLE DATA FOR THE TWO-SAMPLE PLRT ON SIGMA PRESENTED IN FIGURE 16.

Sample	Velocity, ft/s	Penetration	Sample	Velocity, ft/s	Penetration
A	2400.0	1	B	2400.0	1
A	2200.0	0	B	2200.0	0
A	2300.0	1	B	2300.0	0
A	2152.7	0	B	2447.3	1
A	2250.0	0	B	2350.0	0
A	2348.7	0	B	2456.4	1
A	2447.0	1	B	2319.7	0
A	2440.0	0	B	2411.1	0
A	2560.0	1	B	2319.0	0
A	2554.0	1	B	2452.0	1
A	2187.0	0	B	2337.0	0
A	2525.0	1	B	2345.0	0
A	2213.0	1	B	2438.0	0
A	2133.0	0	B	2462.0	1
A	2528.0	1	B	2351.0	0
A	2157.0	0	B	2357.0	0
A	2504.0	1	B	2453.0	0
A	2177.0	0	B	2475.0	1
A	2485.0	1	B	2360.0	0
A	2473.0	1	B	2468.0	1
A	2198.0	0	B	2368.0	0
A	2458.0	0	B	2373.0	0
A	2197.0	0	B	2377.0	0
A	2495.0	0	B	2461.0	1
A	2542.0	1	B	2382.0	1
A	2198.0	0	B	2460.0	1
A	2207.0	0	B	2457.0	0
A	2522.0	1	B	2470.0	0
A	2220.0	0	B	2487.0	0
A	2510.0	1	B	2509.0	1

TABLE A-14. EXAMPLE DATA FOR THE
ONE-SAMPLE PLRT ON V10
PRESENTED IN FIGURE 17.

Velocity, ft/s	Penetration
2400.0	1
2200.0	0
2300.0	0
2447.3	1
2350.0	0
2456.4	1
2319.7	0
2411.1	0
2318.0	0
2328.0	0
2336.0	0
2342.0	1
2309.0	0
2314.0	0
2318.0	0
2322.0	0
2326.0	0
2330.0	1
2305.0	0
2308.0	0
2311.0	0
2314.0	0
2317.0	1
2294.0	0
2297.0	0
2299.0	0
2302.0	0
2304.0	1
2469.0	1
2287.0	0

TABLE A-15. EXAMPLE DATA FOR THE TWO-SAMPLE PLRT ON V10 PRESENTED IN FIGURE 18.

Sample	Velocity, ft/s	Penetration	Sample	Velocity, ft/s	Penetration
A	2400.0	1	B	2400.0	1
A	2200.0	0	B	2200.0	0
A	2300.0	0	B	2300.0	0
A	2447.3	1	B	2447.3	1
A	2350.0	0	B	2350.0	0
A	2456.4	1	B	2456.4	1
A	2319.7	0	B	2319.7	0
A	2411.1	0	B	2411.1	1
A	2318.0	0	B	2375.0	1
A	2328.0	0	B	2329.9	0
A	2336.0	0	B	2384.7	0
A	2342.0	1	B	2330.0	1
A	2309.0	0	B	2295.0	0
A	2314.0	0	B	2300.0	0
A	2318.0	0	B	2304.0	1
A	2322.0	0	B	2271.0	0
A	2326.0	0	B	2275.0	0
A	2330.0	1	B	2279.0	0
A	2305.0	0	B	2282.0	0
A	2308.0	0	B	2286.0	0
A	2311.0	0	B	2289.0	0
A	2314.0	0	B	2292.0	0
A	2317.0	1	B	2295.0	0
A	2294.0	0	B	2297.0	0
A	2297.0	0	B	2300.0	0
A	2299.0	0	B	2303.0	1
A	2302.0	0	B	2285.0	0
A	2304.0	1	B	2287.0	0
A	2469.0	1	B	2290.0	0
A	2287.0	0	B	2292.0	0
A	2289.0	0	B	2294.0	0
A	2292.0	0	B	2296.0	0
A	2294.0	0	B	2298.0	0
A	2296.0	0	B	2300.0	0
A	2298.0	0	B	2302.0	0
A	2300.0	0	B	2304.0	0
A	2302.0	0	B	2306.0	1
A	2305.0	0	B	2292.0	0
A	2307.0	0	B	2294.0	1
A	2309.0	0	B	2280.0	0