AFRL-AFOSR-UK-TR-2022-0014

Multi-Level Robust Optimization: Theory, Algorithms and Practice

Akartunali, Kerem
UNIVERSITY OF STRATHCLYDE VIZ ROYAL COLLEGE OF SCIENCE & TECHNOLOGY
16 RICHMOND STREET
50 RICHMOND ST
GLASGOW, LANARKSHIRE, G1 1XT
GBR

01/12/2022
Final Technical Report

DISTRIBUTION A: Distribution approved for public release.

Air Force Research Laboratory
Air Force Office of Scientific Research
European Office of Aerospace Research and Development
Unit 4515 Box 14, APO AE 09421

REPORT DOCUMENTATION PAGE

PLEASE DO NOT RETURN YOUR FORM TO THE ABOVE ORGANIZATION.

1. REPORT DATE	**2. REPORT TYPE**	**3. DATES COVERED** START DATE	END DATE
20220112	Final	20180820	20210819

4. TITLE AND SUBTITLE
Multi-Level Robust Optimization: Theory, Algorithms and Practice

5a. CONTRACT NUMBER	**5b. GRANT NUMBER**	**5c. PROGRAM ELEMENT NUMBER**
	FA9550-18-1-7003	61102F
5d. PROJECT NUMBER	**5e. TASK NUMBER**	**5f. WORK UNIT NUMBER**

6. AUTHOR(S)
Kerem Akartunali

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
UNIVERSITY OF STRATHCLYDE VIZ ROYAL COLLEGE OF SCIENCE & TECHNOLOGY
16 RICHMOND STREET
50 RICHMOND ST
GLASGOW, LANARKSHIRE G1 1XT
GBR

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
EOARD
UNIT 4515
APO AE 09421-4515

10. SPONSOR/MONITOR'S ACRONYM(S)
AFRL/AFOSR IOE

11. SPONSOR/MONITOR'S REPORT NUMBER(S)
AFRL-AFOSR-UK-TR-2022-0014

12. DISTRIBUTION/AVAILABILITY STATEMENT
A Distribution Unlimited: PB Public Release

13. SUPPLEMENTARY NOTES

14. ABSTRACT
In practice, decision-problems which concern planning horizons spanning several months or years will typically involve various sources of uncertainty, and in many cases these uncertainties will impact decisions over differing time-scales and in different levels of the problem. Accurately accounting for these uncertainties in an optimization model is challenging with traditional approaches, as all uncertainties would generally be treated simultaneously. The proposed project aimed to develop novel optimization theory and methodologies to tackle multi-level robust optimization problems with efficient algorithmic approaches. A range of practical problems from the domains of manufacturing and health systems were identified as ideal candidates to enable the broader research goals: production planning under uncertainty (where uncertain parameters were novel, e.g., timing of a delivery rather than quantity), complex manufacturing problems that integrate often competing decisions in different levels (e.g., lot-sizing and cutting stock), and healthcare staff scheduling that contain many real-world constraints. The project successfully concluded with an extensive range of theoretical results including mathematical properties of specific problem structures or competing uncertainties, equivalence/strength of alternative formulations, and problem complexities. Moreover, a rich range of algorithms were proposed and evaluated, including dynamic programming algorithms that work in polynomial time in certain cases, sophisticated multi-stage algorithms to handle uncertainties in a systematic fashion, and heuristic algorithms for computationally challenging real-world settings. Three peer reviewed journal articles were developed and submitted from this research. At this time one of the submissions has been accepted.

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF: a. REPORT	b. ABSTRACT	c. THIS PAGE	**17. LIMITATION OF ABSTRACT**	**18. NUMBER OF PAGES**
U	U	U	SAR	24

19a. NAME OF RESPONSIBLE PERSON	**19b. PHONE NUMBER** *(Include area code)*
MARK FRIEND	314-235-6292

USAF OFFICE OF SCIENTIFIC RESEARCH

Final Performance Report on the Project Titled

Multi-Level Robust Optimization: Theory, Algorithms and Practice

PI:
Prof. Kerem Akartunali
Dept. of Management Science
University of Strathclyde
Glasgow G4 0GE, UK

Award No:
FA9550-18-1-7003

Period of Performance: 20 Aug 2018 - 19 Aug 2021

Contents

1 Accomplishments

In practice, decision-problems which concern planning horizons spanning several months or years will typically involve various sources of uncertainty, and in many cases these uncertainties will impact decisions over differing time-scales and in different levels of the problem. Accurately accounting for these uncertainties in an optimization model is challenging with traditional approaches, as all uncertainties would generally be treated simultaneously. Broadly speaking, the proposed project aimed to develop novel optimization theory and methodologies to tackle multi-level robust optimization problems in a PhD project over a period of 3 years, with efficient algorithmic approaches being a cornerstone of the project. We recruited a PhD student with impressive background in industrial engineering, Farzad Shams, to work fully on this project shortly after the official start date of the project.

In our proposal, we formally stated our research questions as follows:

Q1. Can we establish a thorough theoretical understanding of the structure of the robust multi-level optimization problems, including alternative reformulations of the problems and their theoretical strengths and weaknesses? Furthermore, can we extend such results to the case of multi-objective problems?

Q2. Can we exploit the statistical or structural properties of these problems in the design of efficient algorithms? How effective can such algorithms be, whether in regard to theoretical or computational limitations?

Q3. Can we apply these algorithms effectively to case studies representing a number of different practical decision-problems?

Though the objectives of the project were set ambitious and despite the dynamic nature of research, a clear plan of methodology helped to achieve most of the original objectives. This included extensive theoretical and methodological development (in particular, using robust and stochastic optimization), identification of real-world problems that would provide simple enough but also meaningful application of outcomes, and algorithmic development and empirical testing. We next discus specific accomplishments of the project.

The major activities of the project have had a nature of basic research, where in particular theoretical and methodological development were essential for success. We first revisit the specific research questions in order to discuss the accomplishments more in detail.

Q1. This research question aimed at establishing an extensive theoretical understanding on the structure of the general optimization problems under uncertainty at hand. In particular, effective model development was crucial. As we discuss later in more detail, the key outputs of the project present a broad range of theory that was particularly successful addressing problems with uncertainty, but also effective optimization theory for problems of practical interest. In order to address this objective, a range of practical problems from the domains of manufacturing and health systems were identified for investigation: production planning under uncertainty (where uncertain parameters were novel, e.g., timing of a delivery rather than quantity), complex manufacturing problems that integrate often competing decisions in different levels (e.g., lot-sizing and cutting stock), and healthcare staff scheduling that contain many real-world constraints. The established theory to address this research question range mathematical properties of specific problem structures and competing uncertainties, to equivalence/strength of alternative formulations.

Q2. There is an extensive range of algorithms developed throughout this project. These range from dynamic programming algorithms that may work in polynomial time in certain cases (but have exponential complexity in general cases) to sophisticated multi-stage algorithms to handle uncertainties in a systematic fashion, as well as heuristic algorithms for computationally challenging real-world settings where suboptimal solutions are still very valuable. The theory developed for Q1 have significantly input into this research question.

Q3. In comparison to the previous two research questions, which have been very thoroughly addressed (and at times going beyond the original proposed aims), this question was addressed rather in a more limited fashion, primarily due to the unavailability of real-world data from companies that have solely prioritized their operations during the Covid-19 pandemic. However, we have still accomplished valuable extensive computational testing based on some past data we had available, as well as various randomly generated realistic test instances.

The outcomes of the project were primarily disseminated through journal papers submitted/to be submitted to prestigious OR journals. Although our original proposal had a thorough plan of dissemination through national and international conferences, in particular in the final year, we had to revise this plan due to the ongoing pandemic (see Section 3 for further details.) Next, we list the publications

1. K. Akartunalı, S. Dauzère-Pérès. Lot Sizing with Stochastic Demand Timing. Conditionally accepted in *European Journal of Operational Research*, 2021.

2. E.M. Silva, G.M. Melega, K. Akartunalı, S. de Araujo. Formulations and Theoretical Analysis of the Multi-Period Cutting Stock Problem with Setups. Under second round of revision with *European Journal of Operational Research*, 2021.

3. V.A.P.A. Devesse, K. Akartunalı, M.S. Arantes, C.F.M. Toledo. Linear Approximations to Improve Lower Bounds of a Physician Scheduling Problem in Emergency Rooms. Revision submitted to *Journal of the Operational Research Society*, 2021.

4. F. Shams, A. Agra, K. Akartunalı, E. Barlow. Formulations for Multi-Stage Robust Production Planning with Quality Classes of Returns. Working paper, 2021.

5. F. Shams, A. Agra, K. Akartunalı, E. Barlow. A Computational Analysis of Multi-Stage Production Planning with Imperfect Returns. Working paper, 2021.

2 Impacts

In this section, we briefly discuss the potential impacts of the outcomes of the project, whether purely in academic sense or to the broader society. Due to the basic research nature of the project, such impacts are likely to realize in 5-10 years.

There were many practical concepts that were theoretically investigated in this project. For example, when there is uncertainty with respect to the timing of a delivery, rather than an uncertainty in its quantity, the existing methodologies are very limited to propose how to tackle such an uncertainty. Similarly, when there are competing uncertainties, e.g., some having a longer time horizon than others, it is not straightforward to employ existing methodologies to address such issues. The theoretical findings of the project, therefore, address a number of challenges stemming from practical realities, as to provide a better understanding, and primarily contribute to the discipline of Operations Research/Industrial Engineering. We firstly expect that some of our theoretical findings will be very likely extended to broader contexts from their current problem specific contexts. For example, the stochastic timing is a concept not only limited to the domain of production planning, and the theoretical understanding we have established, e.g., some interesting special cases being polynomially solvable, will find its application in other applications. Moreover, there is a wealth of algorithms developed in the project, which again can be extended to other applications of interest.

Albeit more limited, we also expect some impacts in other disciplines such as machine learning and artificial intelligence. Although these disciplines have a much stronger focus on handling massive amounts of data rather than optimizing results, there are some key overlaps in application domains (e.g., both OR and these domains address similar planning and scheduling problems). This is why there is already a strong synergy between these disciplines and OR, resulting in intriguing conferences such as CPAIOR.

In addition to employing a young researcher and developing him extensively in research skills, there is a clear scholarship impact of the project in terms of methodology developed. Although some theoretical results are not very accessible to the broader OR discipline, some of the outcomes are rather easy to interpret in a practical manner, and therefore, we plan to develop intriguing case studies for educational purposes that may be used in the broader business context as well as for practitioners in strongly related application areas. Moreover, we are already planning some extensions from this project's outcomes, for which we will be seeking funding over the next 3-5 years from a range of funding bodies, including EPSRC and AFOSR, in order to further contribute to the development of human resources.

In the longer term, we also expect outcomes in the broader society, in particular through improving economic conditions. As the project has particularly focused on tackling uncertainties that are not straightforward to address, there is a realistic expectation that the outcomes will result in improved decision making tools in some industries (e.g., better software for production planning or hospital scheduling).

3 Changes

The major challenge the project experienced has been the ongoing Covid-19 pandemic. Like in many other ongoing projects and organizations everywhere in the world, the extraordinary circumstances caused some disruptions and delays, however, these were mostly with minor nature and resolved quickly with replanning.

The only change for which we received an approval from the program officer was the re-purposing of the travel budget. Due to complete infeasibility of travel during the second half of the project, we have requested this travel budget to be re-purposed for staff time so that the PI could be much more significantly involved to deliver the expected outcomes of the project. This change resulted in some re-planning of activities, in particular the cancellation of a crucial research visit of the PhD student to an international collaborator (and hence some delay in those specific outputs) while new work of PI contributing to the scope of the project (and more than compensating with new outputs for the delay in other outputs.)

There have been no further major change in the project, whether in its scope, approach or impacts.

4 Technical Updates

In this section, we provide some exposure to the technical outcomes of the project. First, we present a simple model for a problem of interest, and how we can employ the techniques developed in the project, and then move onto a different problem to present some of the methodological results. We present here only a limited exposure just to provide the reader a methodological context, rather than a full detailed exposure which would be too long and heavily technical (and left to the papers to be published.)

4.1 Robust Multi-stage Lot-sizing with Multiple Quality Classes of Returns: A Deterministic Model

In this section, we present a deterministic model for the robust multi-stage lot-sizing problem, where the returned or collected products (cores) are classified into multiple quality classes (RML-MQ). A relevant practical example with such a setting would be the IBM's remanufacturing facility in Raleigh, N.C., where the firm receives end-of-lease or returned laptops from different sources and channels (Denizel et al. (2009)). Each returned laptop could be remanufactured to a certain acceptable quality level and configuration, before being put back to the market for sale. In a given inventory of returned laptops, the amount of effort for bringing any two random laptops to a like-new state can be very different. While one laptop may just require a thorough cleaning and formatting the hard drive, another laptop may require a few new parts, such as a new screen panel to replace a worn one, or a new memory card to replace a faulty one. The effort for bringing the latter laptop to the acceptable condition may require considerable effort as opposed to the effort for the former one. This directly effects the time and cost of remanufacturing, creating an important issue that has to be considered in the production planning.

We consider N finite time periods, where the manufacturer produces a single type of product, and has a random demand. In each time period, the manufacturer receives a random amount of returned products, which then are graded and grouped into Q different quality classes. We assume that the returned products are sorted before any decision has to be made. We consider cost of sorting activities to be a sunk cost, as they do not directly impact the problem structure, and including them is out of the scope of this study. The manufacturer has the option to remanufacture the available cores, to fulfill the demand or keep them in stock to be remanufactured in the future, and backlogging of demand is also allowed. This is a common approach in the literature for the lot sizing with remanufacturing option (Bienstock & Özbay (2008), See & Sim (2010), Tao & Zhou (2014), Attila et al. (2017)). The unit manufacturing cost is higher than the remanufacturing cost of any type of core (for remanufacturing to be a reasonable option), and graded cores can be disposed at a certain cost.

The problem is to find the optimal values for production of new products, and the amounts of different quality cores to remanufacture or salvage in each time period, to minimize the total costs of production, inventory/backlogging, and disposal. We assume that the customers are indifferent between manufactured and remanufactured products, which is valid for certain category of products such as printer cartridges and refillable containers (Zhou et al. (2011)). We call the ready-to-sell products "serviceable products" in line with the literature. Further, we consider the lead times for (re)manufacturing process and the inventory of each type of cores and serviceable products to be zero without loss of generality. We next present the deterministic model, following the notation used in our model.

Indices and sets:
$t \in N = \{1, 2, ..., N\}$: Number of periods;
$q \in Q = \{1, 2, ..., Q\}$: Number of quality classes;

Variables:
x_t^m: amount of items manufactured in period t,
x_t^q: amount of items of quality class q remanufactured in period t,
s_t^q: amount of items of quality class q salvaged in period t,

$$y_t := \begin{cases} 1 & \text{if there is a setup in period t} \\ 0 & \text{otherwise} \end{cases}$$

Parameters:

$\mathbf{c} = (c_0, c_1, ..., c_Q, c_{Q+1})$,
where c_0 is the cost of manufacturing per item, $c_1, ..., c_Q$ are costs of remanufacturing per item for returns in classes 1 through Q, and c_{Q+1} is the salvage cost per item for all the return classes,
d_t: demand for serviceable products in period t,
r_t^q: return amount in period t for quality class q,
h^s: inventory holding cost of serviceable products per item,
h^q: inventory holding cost of return products of quality class q per item,
b: backlogging cost per item,
f: joint setup cost for manufacturing and remanufacturing.

Then the objective is to minimize the total cost incurred due to setup, manufacturing, remanufacturing, disposal, inventory, and shortage costs as given below:

$$\sum_{t=1}^{N}(f_t y_t + c_0 x_t^m + \sum_{q=1}^{Q}(c_q x_t^q + c_{Q+1} s_t^q) + I_t^s + \sum_{q=1}^{Q} I_t^q) \tag{1}$$

where, I_t^s and I_t^qs capture the inventory and backlogging costs. Below we present the deterministic mixed integer model of the problem (DML-MQ):

$$\min \sum_{t=1}^{N}(f_t y_t + c_0 x_t^m + \sum_{q=1}^{Q}(c_q x_t^q + c_{Q+1} s_t^q) + I_t^s + \sum_{q=1}^{Q} I_t^q) \tag{2}$$

$$\text{s.t. } I_t^s \geq h^s \sum_{i=1}^{t}(x_i^m + \sum_{q=1}^{Q} x_i^q - d_i), \quad t \in N \tag{3}$$

$$I_t^s \geq -b \sum_{i=1}^{t}(x_i^m + \sum_{q=1}^{Q} x_i^q - d_i), \quad t \in N \tag{4}$$

$$I_t^q \geq h^q \sum_{i=1}^{t}(r_i^q - x_i^q - s_i^q), \quad q \in Q, t \in N \tag{5}$$

$$\sum_{i=1}^{t}(r_i^q - s_i^q - x_i^q) \geq 0 \quad q \in Q, t \in N \tag{6}$$

$$x_t^m + \sum_{q=1}^{Q} x_t^q \leq M_t y_t, \quad t \in N \tag{7}$$

$$y_t \in \{0,1\}, x_t^m, x_t^q, s_t^q \geq 0, \quad t \in N \tag{8}$$

where the demand and return amounts are assumed to be deterministic.

4.2 General Robust Multi-Stage Model

We first define our variables as functions of the past data, i.e., $d^t(d_1, ..., d_t)$ and $r^{q,t}(r_1^q, ..., r_t^q)$. Then, the adjustable multi-stage robust problem will be:

$$\min \quad F + \sum_{t=1}^{N}(f_t y_t) \tag{9}$$

$$\text{s.t. } F \geq \sum_{t=1}^{N}(c_0 x_t^m(d^{t-1}) + \sum_{q=1}^{Q}(c_q x_t^q(r^{q,t-1}) + c_{Q+1} s_t^q(r^{q,t-1})) + I_t^s + \sum_{q=1}^{Q} I_t^q), d_t \in D_t, r_t^q \in R_t^q \tag{10}$$

$$I_t^s(d^{i-1}, r^{q,i-1}) \geq h^s \sum_{i=1}^{t}(x_i^m(d^{i-1}) + \sum_{q=1}^{Q} x_i^q(r^{q,i-1}) - d_i), t \in N, d_t \in D_t, r_t^q \in R_t^q \tag{11}$$

$$I_t^s(d^{i-1}, r^{q,i-1}) \geq -b \sum_{i=1}^{t}(x_i^m(d^{i-1}) + \sum_{q=1}^{Q} x_i^q(r^{q,i-1}) - d_i), t \in N, d_t \in D_t, r_t^q \in R_t^q \tag{12}$$

$$\tag{13}$$

$$I_t^q(r^{q,i-1}) \geq h^q \sum_{i=1}^{t}(r_i^q - x_i^q(r^{q,i-1}) - s_i^q(r^{q,i-1})), q \in Q, t \in N, r_t^q \in R_t^q \tag{14}$$

$$\sum_{i=1}^{t}(r_i^q - s_i^q(r^{q,i-1}) - x_i^q(r^{q,i-1})) \geq 0, q \in Q, t \in N, r_t^q \in R_t^q \tag{15}$$

$$x_t^m(d^{t-1}) + \sum_{q=1}^{Q} x_t^q(r^{q,t-1}) \leq M_t y_t, t \in N, d_t \in D_t, r_t^q \in R_t^q \tag{16}$$

$$y_t \in \{0,1\}, t \in N \tag{17}$$

$$x_t^m(d^{t-1}), \geq 0, t \in N, d_t \in D_t \tag{18}$$

$$x_t^q(r^{q,t-1}), s_t^q(r^{q,t-1}), I_t^q(r^{q,i-1}) \geq 0, t \in N, q \in Q, r_t^q \in R_t^q \tag{19}$$

where now, variables $x_t^m(d^{t-1}), x_t^q(r^{q,t-1})$, and $s_t^q(r^{q,t-1})$ are defined as functions of the past data d^t and $r^{q,t}$.

4.3 Affinely Adjustable Model

Next, we present the robust multistage lot-sizing problem where we assume that we have adjustable manufacturing and remnaufacturing and disposal variables that are affine functions of the demand and return values, following the work of Ben-Tal et al. (2004). This means that the decisions on these variables can be adjusted to the revealed demand/return values up until the point of time of the decision. We utilise affine decision rules for the variables x_t^m, x_t^q, s_t^q, I_t^q, and I_t^s as follows:

$$x_t^m = x_{t,0}^m + \sum_{i=1}^{t-1} x_{t,i}^m d_i, \qquad t \in N, \tag{20}$$

$$x_t^q = x_{t,0}^q + \sum_{i=1}^{t-1} x_{t,i}^q r_i^q, \qquad t \in N, q \in Q, \tag{21}$$

$$s_t^q = s_{t,0}^q + \sum_{i=1}^{t-1} s_{t,i}^q r_i^q, \qquad t \in N, q \in Q, \tag{22}$$

$$I_t^q = I_{t,0}^q + \sum_{i=1}^{t-1} I_{t,i}^q r_i^q, \qquad t \in N, q \in Q, \tag{23}$$

$$I_t^s = I_{t,0}^s + \sum_{i=1}^{t-1} I_{t,i}^s d_i + \sum_{q=1}^{Q}\sum_{i=1}^{t-1} I_{t,i}^{s,q} r_i^q, \qquad t \in N, \tag{24}$$

where now, the demand and return values belong to box uncertainty sets D and R, $d_t \in D_t, r_t^q \in R_t^q$ with $D_t = [\overline{d}_t - \alpha\overline{d}_t, \overline{d}_t + \alpha\overline{d}_t]$, $R_t^q = [\overline{r}_t^q - \beta\overline{r}_t^q, \overline{r}_t^q + \beta\overline{r}_t^q]$ and $\alpha, \beta \in [0,1]$. $\overline{d}_t$ and $\overline{r}_t^q$ are nominal demand and return values in period t. Note that we assume the deviation values α and β are the same throughout the planning horizon. Inserting these new decision rules in the deterministic model and writing the model in the epigraph form, we obtain:

$$min \quad F + \sum_{t=1}^{N}(f_t y_t) \tag{25}$$

$$st.\ F \geq \sum_{t=1}^{N}(c_0(x^m_{t,0} + \sum_{i=1}^{t-1} x^m_{t,i} d_i) + \sum_{q=1}^{Q}(c_q(x^q_{t,0} + \sum_{i=1}^{t-1} x^q_{t,i} r^q_i) + c_{Q+1}(s^q_{t,0} + \sum_{i=1}^{t-1} s^q_{t,i} r^q_i))$$

$$+ (I^s_{t,0} + \sum_{i=1}^{t-1} I^s_{t,i} d_i + \sum_{q=1}^{Q}\sum_{i=1}^{t-1} I^{s,q}_{t,i} r^q_i) + \sum_{q=1}^{Q}(I^q_{t,0} + \sum_{i=1}^{t-1} I^q_{t,i} r^q_i)), \quad d_t \in D_t, r^q_t \in R^q_t \tag{26}$$

$$(I^s_{t,0} + \sum_{i=1}^{t-1} I^s_{t,i} d_i + \sum_{q=1}^{Q}\sum_{i=1}^{t-1} I^{s,q}_{t,i} r^q_i) \geq h^s \sum_{i=1}^{t}((x^m_{i,0} + \sum_{j=1}^{i-1} x^m_{i,j} d_j) + \sum_{q=1}^{Q}(x^q_{i,0} + \sum_{j=1}^{i-1} x^q_{i,j} r^q_j) - d_i),$$
$$t \in N, d_t \in D_t, r^q_t \in R^q_t \tag{27}$$

$$(I^s_{t,0} + \sum_{i=1}^{t-1} I^s_{t,i} d_i + \sum_{q=1}^{Q}\sum_{i=1}^{t-1} I^{s,q}_{t,i} r^q_i) \geq -b \sum_{i=1}^{t}((x^m_{i,0} + \sum_{j=1}^{i-1} x^m_{i,j} d_j) + \sum_{q=1}^{Q}(x^q_{i,0} + \sum_{j=1}^{i-1} x^q_{i,j} r^q_j) - d_i),$$
$$t \in N, d_t \in D_t, r^q_t \in R^q_t \tag{28}$$

$$(I^q_{t,0} + \sum_{i=1}^{t-1} I^q_{t,i} r^q_i) \geq h^q \sum_{i=1}^{t}(r^q_i - (x^q_{i,0} + \sum_{j=1}^{i-1} x^q_{i,j} r^q_j) - (s^q_{i,0} + \sum_{j=1}^{i-1} s^q_{i,j} r^q_j)), \quad t \in N, q \in Q, r^q_t \in R^q_t \tag{29}$$

$$\sum_{i=1}^{t}(r^q_i - (s^q_{i,0} + \sum_{j=1}^{i-1} s^q_{i,j} r^q_j) - (x^q_{i,0} + \sum_{j=1}^{i-1} x^q_{i,j} r^q_j)) \geq 0, q \in Q, t \in N, r^q_t \in R^q_t \tag{30}$$

$$(x^m_{t,0} + \sum_{i=1}^{t-1} x^m_{t,i} d_i) + \sum_{q=1}^{Q}(x^q_{t,0} + \sum_{i=1}^{t-1} x^q_{t,i} r^q_i) \leq M_t y_t, \quad t \in N, d_t \in D_t, r^q_t \in R^q_t, \tag{31}$$

$$x^m_{t,0} + \sum_{i=1}^{t-1} x^m_{t,i} d_i \geq 0, \quad t \in N, , d_t \in D_t, \tag{32}$$

$$x^q_{t,0} + \sum_{i=1}^{t-1} x^q_{t,i} r^q_i \geq 0, \quad t \in N, q \in Q, d_t \in D_t, r^q_t \in R^q_t, \tag{33}$$

$$s^q_{t,0} + \sum_{i=1}^{t-1} s^q_{t,i} r^q_i \geq 0, \quad t \in N, q \in Q, d_t \in D_t, r^q_t \in R^q_t, \tag{34}$$

$$I^q_{t,0} + \sum_{i=1}^{t-1} I^q_{t,i} r^q_i \geq 0, \quad t \in N, q \in Q, d_t \in D_t, r^q_t \in R^q_t, \tag{35}$$

$$y_t \in \{0,1\}, \quad t \in N. \tag{36}$$

The above program is a semi-infinite model. In order to derive a tractable AARC model we need to reformulate the constraints.

We will generate the constraints one by one. Constraint (26) becomes:

$$0 \geq -F + \sum_{t=1}^{N}[c_0 x_{t,0}^m + I_{t,0}^s + \sum_{q=1}^{Q}(c_q x_{t,0}^q + c_{Q+1} s_{t,0}^q + I_{t,0}^q)]$$
$$+ \sum_{t=1}^{N}\sum_{i=1}^{t-1}(c_0 x_{t,i}^m + I_{t,i}^s)d_i + \sum_{t=1}^{N}\sum_{i=1}^{t-1}\sum_{q=1}^{Q}(c_q x_{t,i}^q + c_{Q+1} s_{t,i}^q + I_{t,i}^{s,q} + I_{t,i}^q)r_i^q,$$

By change of summation:

$$0 \geq -F + \sum_{t=1}^{N}[c_0 x_{t,0}^m + I_{t,0}^s + \sum_{q=1}^{Q}(c_q x_{t,0}^q + c_{Q+1} s_{t,0}^q + I_{t,0}^q)]$$
$$+ \sum_{i=1}^{N-1}\sum_{t=i+1}^{N}(c_0 x_{t,i}^m + I_{t,i}^s)d_i + \sum_{i=1}^{N-1}\sum_{q=1}^{Q}\sum_{t=i+1}^{N}(c_q x_{t,i}^q + c_{Q+1} s_{t,i}^q + I_{t,i}^{s,q} + I_{t,i}^q)r_i^q,$$

Using the following change of variables:

$$\eta_i = \sum_{t=i+1}^{N}(c_0 x_{t,i}^m + I_{t,i}^s) \qquad \delta_i^q = \sum_{t=i+1}^{N}(c_q x_{t,i}^q + c_{Q+1} s_{t,i}^q + I_{t,i}^{s,q} + I_{t,i}^q)$$

we have:

$$0 \geq -F + \sum_{t=1}^{N}[c_0 x_{t,0}^m + I_{t,0}^s + \sum_{q=1}^{Q}(c_q x_{t,0}^q + c_{Q+1} s_{t,0}^q + I_{t,0}^q)] + \sum_{i=1}^{N-1}\eta_i d_i + \sum_{i=1}^{N-1}\sum_{q=1}^{Q}\delta_i^q r_i^q,$$

then the constraint is equivalent to:

$$0 \geq -F + \sum_{t=1}^{N}[c_0 x_{t,0}^m + I_{t,0}^s + \sum_{q=1}^{Q}(c_q x_{t,0}^q + c_{Q+1} s_{t,0}^q + I_{t,0}^q)] + \sum_{i=1}^{N-1}(\eta_i \bar{d}_i + \alpha \bar{d}_i \lambda_i^1) + \sum_{i=1}^{N-1}\sum_{q=1}^{Q}(\delta_i^q \bar{r}_i^q + \beta \bar{r}_i^q \lambda_i^{2,q}),$$
$$\eta_i = \sum_{t=i+1}^{N}(c_0 x_{t,i}^m + I_{t,i}^s), \quad i \in \{1, ..., N-1\},$$
$$\delta_i^q = \sum_{t=i+1}^{N}(c_q x_{t,i}^q + c_{Q+1} s_{t,i}^q + I_{t,i}^{s,q} + I_{t,i}^q), \quad i \in \{1, ..., N-1\}, q \in Q,$$
$$-\lambda_i^1 \leq \eta_i \leq \lambda_i^1, \quad i \in \{1, ..., N-1\}$$
$$-\lambda_i^{2,q} \leq \delta_i^q \leq \lambda_i^{2,q}, \quad i \in \{1, ..., N-1\}, q \in \{1, ..., Q\}$$

Next, constraint (27):

$$0 \geq -I^s_{t,0} + h^s \sum_{i=1}^{t} [x^m_{i,0} + \sum_{q=1}^{Q} x^q_{i,0}] + \sum_{i=1}^{t} \sum_{j=1}^{i-1} (h^s x^m_{i,j} d_j - h^s d_i) - \sum_{i=1}^{t-1} I^s_{t,i} d_i$$
$$+ \sum_{i=1}^{t} \sum_{j=1}^{i-1} \sum_{q=1}^{Q} h^s x^q_{i,j} r^q_j - \sum_{q=1}^{Q} \sum_{i=1}^{t-1} I^{s,q}_{t,i} r^q_i,$$

By change of summation :

$$0 \geq -I^s_{t,0} + h^s \sum_{i=1}^{t} [x^m_{i,0} + \sum_{q=1}^{Q} x^q_{i,0}] + \sum_{j=1}^{t-1} [(\sum_{i=j+1}^{t} (h^s x^m_{i,j} - h^s)) - I^s_{t,j}] d_j +$$
$$\sum_{j=1}^{t-1} \sum_{q=1}^{Q} [(\sum_{i=j+1}^{t} (h^s x^q_{i,j})) - I^{s,q}_{t,j}] r^q_j - h^s d_t,$$

Using the following change of variables:

$$\xi_{t,j} = (\sum_{i=j+1}^{t} (h^s x^m_{i,j} - h^s)) - I^s_{t,j} \qquad \theta^q_{t,j} = (\sum_{i=j+1}^{t} (h^s x^q_{i,j})) - I^{s,q}_{t,j}$$

we will have the following:

$$0 \geq -I^s_{t,0} + h^s \sum_{i=1}^{t} [x^m_{i,0} + \sum_{q=1}^{Q} x^q_{i,0}] + \sum_{j=1}^{t-1} \xi_{t,j} d_j + \sum_{j=1}^{t-1} \sum_{q=1}^{Q} \theta^q_{t,j} r^q_j - h^s d_t,$$

then the constraint is equivalent to:

$$0 \geq -I^s_{t,0} + h^s \sum_{i=1}^{t} [x^m_{i,0} + \sum_{q=1}^{Q} x^q_{i,0}] + \sum_{j=1}^{t-1} (\xi_{t,j} \bar{d}_j + \alpha \bar{d}_j \lambda^3_{t,j}) + \sum_{j=1}^{t-1} \sum_{q=1}^{Q} (\theta^q_{t,j} \bar{r}^q_j + \beta \bar{r}^q_j \lambda^{4,q}_j) - h^s \bar{d}_t + h^s \alpha \bar{d}_t,$$
$$- \lambda^3_{t,j} \leq \xi_{t,j} \leq \lambda^3_{t,j}, \quad j \in \{1, ..., t-1\}, t \in N$$
$$- \lambda^{4,q}_{t,j} \leq \theta^q_{t,j} \leq \lambda^{4,q}_{t,j}, \quad j \in \{1, ..., t-1\}, q \in Q, t \in N$$

Next, constraint (28):

$$0 \geq -I^s_{t,0} - \sum_{i=1}^{t} b(x^m_{i,0} + \sum_{q=1}^{Q} x^q_{i,0}) - \sum_{i=1}^{t} \sum_{j=1}^{i-1} (b x^m_{i,j} d_j - b d_i) - \sum_{i=1}^{t-1} I^s_{t,i} d_i - \sum_{i=1}^{t} \sum_{j=1}^{i-1} \sum_{q=1}^{Q} b x^q_{i,j} r^q_j$$
$$- \sum_{q=1}^{Q} \sum_{i=1}^{t-1} I^{s,q}_{t,i} r^q_i, \quad t \in N,$$

By change of summation :

$$0 \geq -I^s_{t,0} - \sum_{i=1}^{t} b(x^m_{i,0} + \sum_{q=1}^{Q} x^q_{i,0}) + \sum_{j=1}^{t-1}[(\sum_{i=j+1}^{t} (b - bx^m_{i,j})) - I^s_{t,j}]d_j + bd_t$$
$$+ \sum_{j=1}^{t-1}\sum_{q=1}^{Q}[(\sum_{i=j+1}^{t} -bx^q_{i,j}) - I^{s,q}_{t,j})]r^q_j,$$

Using the following change of variables:

$$\epsilon_{t,j} = (\sum_{i=j+1}^{t} (b - bx^m_{i,j})) - I^s_{t,j} \qquad \mu^q_{t,j} = (\sum_{i=j+1}^{t} -bx^q_{i,j}) - I^{s,q}_{t,j}$$

we can write the constraint as follows:

$$0 \geq -I^s_{t,0} - \sum_{i=1}^{t} b(x^m_{i,0} + \sum_{q=1}^{Q} x^q_{i,0}) + \sum_{j=1}^{t-1}(\epsilon_{t,j}\bar{d}_j + \alpha\bar{d}_j\lambda^5_{t,j}) + b(\bar{d}_t + \alpha\bar{d}_t) + \sum_{j=1}^{t-1}\sum_{q=1}^{Q}(\mu^q_{t,j}\bar{r}^q_j + \beta\bar{r}^q_j\lambda^{6,q}_{t,j})$$
$$-\lambda^5_{t,j} \leq \epsilon_{t,j} \leq \lambda^5_{t,j}, \quad j \in \{1, ..., t-1\}, t \in N,$$
$$-\lambda^{6,q}_{t,j} \leq \mu^q_{t,j} \leq \lambda^{6,q}_{t,j}, \quad j \in \{1, ..., t-1\}, , t \in N, q \in Q$$

Next, constraint (29):

$$0 \geq -I^q_{t,0} + h^q \sum_{i=1}^{t}(-x^q_{i,0} - s^q_{i,0}) + h^q \sum_{i=1}^{t}\sum_{j=1}^{i-1}(r^q_i - x^q_{i,j}r^q_j - s^q_{i,j}r^q_j) - \sum_{i=1}^{t-1} I^q_{t,i}r^q_i,$$

By change of summation:

$$0 \geq -I^q_{t,0} + h^q \sum_{i=1}^{t}(-x^q_{i,0} - s^q_{i,0}) + \sum_{j=1}^{t-1}[(\sum_{i=j+1}^{t} (-h^q x^q_{i,j} - h^q s^q_{i,j})) + h^q - I^q_{t,j}]r^q_j + h^q r^q_t,$$

Using the following change of variable:

$$\nu^q_{t,j} = (\sum_{i=j+1}^{t} (-h^q x^q_{i,j} - h^q s^q_{i,j})) + h^q - I^q_{t,j}$$

we will have:

$$0 \geq -I^q_{t,0} + h^q \sum_{i=1}^{t}(-x^q_{i,0} - s^q_{i,0}) + \sum_{j=1}^{t-1} \nu^q_{t,j}r^q_j + h^q r^q_t,$$

then the constraint is equivalent to:

$$
\begin{aligned}
&0 \geq -I^q_{t,0} + h^q \sum_{i=1}^{t}(-x^q_{i,0} - s^q_{i,0}) + \sum_{j=1}^{t-1}(\nu^q_{t,j}\bar{r}^q_j + \beta\bar{r}^q_j\lambda^{7,q}_{t,j}) + h^q(\bar{r}^q_t + \beta\bar{r}^q_t), \\
&-\lambda^{7,q}_{t,j} \leq \nu^q_{t,j} \leq \lambda^{7,q}_{t,j}, \quad j \in \{1, ..., t-1\}, t \in N, q \in Q
\end{aligned}
$$

Next, constraint (30):

$$
\sum_{i=1}^{t}(-x^q_{i,0} - s^q_{i,0}) + \sum_{i=1}^{t}\sum_{j=1}^{i-1}(r^q_i - s^q_{i,j}r^q_j - x^q_{i,j}r^q_j) \geq 0
$$

By change of summation and writing the constraint as a less than or equal form we have:

$$
\sum_{i=1}^{t}(x^q_{i,0} + s^q_{i,0}) + \sum_{j=1}^{t-1}\sum_{i=j+1}^{t}(s^q_{i,j}r^q_j + x^q_{i,j}r^q_j - r^q_j) - r^q_t \leq 0
$$

Then, using the following change of variable:

$$
\tau^q_{t,j} = \sum_{i=j+1}^{t}(x^q_{i,j} + s^q_{i,j} - 1)
$$

we will have:

$$
\sum_{i=1}^{t}(x^q_{i,0} + s^q_{i,0}) + \sum_{j=1}^{t-1}(\tau^q_{t,j}r^q_j) - r^q_t \leq 0,
$$

then the constraint is equivalent to:

$$
\begin{aligned}
&\sum_{i=1}^{t}(x^q_{i,0} + s^q_{i,0}) + \sum_{j=1}^{t-1}(\tau^q_{t,j}\bar{r}^q_j + \beta\bar{r}^q_j\lambda^{8,q}_j) - (\bar{r}^q_t - \bar{r}^q_t\beta) \leq 0, \\
&-\lambda^{8,q}_{t,j} \leq \tau^q_{t,j} \leq \lambda^{8,q}_{t,j}, \quad j \in \{1, ..., t-1\}, t \in N, q \in Q
\end{aligned}
$$

Next, constraint (31):

$$
(x^m_{t,0} + \sum_{q=1}^{Q} x^q_{t,0}) + \sum_{i=1}^{t-1} x^m_{t,i}d_i + \sum_{i=1}^{t-1}\sum_{q=1}^{Q} x^q_{t,i}r^q_i - M_t y_t \leq 0,
$$

using the same approach we have the constraint as:

$$(x^m_{t,0} + \sum_{q=1}^{Q} x^q_{t,0}) + \sum_{i=1}^{t-1}(x^m_{t,i}\bar{d}_i + \alpha\bar{d}_i\lambda^9_i) + \sum_{i=1}^{t-1}\sum_{q=1}^{Q}(x^q_{t,i}\bar{r}^q_i + \beta\bar{r}^q_i\lambda^{10,q}_i) - M_t y_t \le 0,$$
$$-\lambda^9_{t,i} \le x^m_{t,i} \le \lambda^9_{t,i}, \quad j \in \{1, ..., t-1\}, t \in N$$
$$-\lambda^{10,q}_{t,i} \le x^q_{t,i} \le \lambda^{10,q}_{t,i}, \quad i \in \{1, ..., t-1\}, q \in Q, t \in N$$

Next, constraint (32):

$$x^m_{t,0} + \sum_{i=1}^{t-1}(x^m_{t,i}\bar{d}_i + \alpha\bar{d}_i\lambda^9_{t,i}) \ge 0$$
$$-\lambda^9_{t,i} \le x^m_{t,i} \le \lambda^9_{t,i}$$

Next, constraint (33):

$$x^q_{t,0} + \sum_{i=1}^{t-1}(x^q_{t,i}\bar{r}^q_i + \beta\bar{r}^q_i\lambda^{10,q}_{t,i}) \ge 0$$
$$-\lambda^{10,q}_{t,i} \le x^q_{t,i} \le \lambda^{10,q}_{t,i}$$

Next, constraint (34):

$$s^q_{t,0} + \sum_{i=1}^{t-1}(s^q_{t,i}\bar{r}^q_i + \beta\bar{r}^q_i\lambda^{11,q}_{t,i}) \ge 0$$
$$-\lambda^{11,q}_{t,i} \le s^q_{t,i} \le \lambda^{11,q}_{t,i}$$

Next, constraint (35):

$$I^q_{t,0} + \sum_{i=1}^{t-1}(I^q_{t,i}\bar{r}^q_i + \beta\bar{r}^q_i\lambda^{12,q}_{t,i}) \ge 0$$
$$-\lambda^{12,q}_{t,i} \le I^q_{t,i} \le \lambda^{12,q}_{t,i}$$

All the generated constraints together will result in the AARC model. This detailed process shows how much attention is required in order to ensure that the correct modelling is achieved in such a complex multi-stage robust formulation.

4.4 A Dynamic Problem with Stochastic Demand Timing

Stochastic demand timing is a novel and interesting concept that can be observed in a range of practical settings. In particular, this happens when a client company sends orders for a product to a supplier company, when the client company's product inventory level is empty. The order to the supplier is fixed, typically related to the inventory capacity of the customer. Hence, the supplier company knows very well the quantity that will be either picked up by or delivered to a customer, but is not able to know exactly when, although an interval of several days is known. This is particularly noticeable in operational or tactical production and inventory planning over several weeks with periods of a day, where demand and order quantities are well established, and is a typical context in process industries, which satisfy the demands of other industries. For example, this case is observed for non-mixable cement products that can be stored (see, e.g., Christiansen et al. (2011)) or calcium carbonate slurry products (see, e.g., Dauzère-Pérès et al. (2007)), where it is known that a vessel, a train, or a truck will arrive in an interval of several days to be completely filled.

Hence, order management is an interesting context, where stochastic demand timing is relevant. When a company has a list of potential customers' orders, predicted from historical data, with known quantities and time windows in which they should occur with their corresponding probabilities, solving the problem studied in this paper will provide the most efficient plan to answer these orders. Significant potential losses due to future orders can thus be estimated, and necessary actions to avoid these losses can be taken.

Let us consider the single-item uncapacitated dynamic lot sizing problem with a planning horizon of T periods in the classical deterministic sense, as follows:

$$\min \sum_{t=1}^{T} f_t y_t + \sum_{t=1}^{T} h_t s_t + \sum_{t=1}^{T} c_t x_t \tag{37}$$

$$\text{s.t. } x_t + s_{t-1} - s_t = D_t \qquad t = 1,\ldots,T \tag{38}$$

$$x_t \leq M_t y_t \qquad t = 1,\ldots,T \tag{39}$$

$$y_t \in \{0,1\}; x_t \geq 0; s_t \geq 0 \qquad t = 1,\ldots,T \tag{40}$$

For any period t, variables x_t and s_t represent production and inventory quantities, respectively, and binary y_t variables indicate whether a production setup takes place or not. The objective (37) is to find a minimum cost production plan, where the total cost consists of fixed setup costs f_t (charged only if production is strictly positive, i.e., $y_t = 1$), per unit inventory holding costs h_t, and per unit production costs c_t, respectively, for all periods in the horizon. We also assume all cost parameters to be strictly positive, i.e., no "free lunch".The flow balance constraints (38) ensure on-time satisfaction of demand D_t, whereas the relationship between production and setup variables is set by (39), where M_t is an upper bound on x_t, e.g., $M_t = \sum_{\ell=t}^{T} D_\ell$. Finally, the integrality and non-negativity constraints are provided by (40). Let us recall that this problem has a complexity of $O(T \log T)$, see, e.g., Wagelmans et al. (1992).

In addition to the deterministic demands $D_t, \forall t \in [1,T]$, that need to be satisfied on time, we simultaneously consider stochastic demand timing as follows. Let $[l_i, u_i] \subset [1,T]$ be an interval, indexed by i, where it is certain that a demand of d^i will fully occur, i.e., at once, in one period, with a probability of $p_t^i \geq 0$ for each period $t \in [l_i, u_i]$ and such that $\sum_{t=l_i}^{u_i} p_t^i = 1$. Note that $p_t^i = 0$ for $t \leq l_i - 1$ and $t \geq u_i + 1$. Let $\mathcal{I}$ be the set of such intervals with stochastic demand timing in the planning horizon and, for ease of notation, let $|\mathcal{I}| = n$.

In this paper, we make the following realistic assumptions:

- No backlog is allowed for deterministic demands and, accordingly, no backlog is allowed for any stochastic demand quantity d^i after period u_i. Note that, however, stochastic demand quantity d^i may be satisfied with inventory carried from before l_i, while backlogging is allowed within the interval $[l_i, u_i]$ with a variable backlog cost b_t. In Section **??**, the more general case where the variable backlog cost b_t^i also depends on d^i is discussed.

- Partial delivery of any stochastic demand quantity d^i is not allowed, i.e., d^i products must be delivered to the customer in one and only one period. Hence, each stochastic demand timing can be seen as a separate order, and the backlog cost is counted until d^i is fully satisfied. Note that the problem is easy to solve if partial delivery is allowed, as one can simply solve in that case a classical lot sizing problem with demand $p_t^i d^i$ in period t.

- As it is usually the case and w.l.o.g., backlog is more costly than inventory, i.e., $b_t > h_t$ $\forall t$.

For any period $t \in [l_i, u_i]$, the expected stochastic demand quantity to satisfy is $p_t^i d^i$. As this stochastic demand quantity cannot be produced after u_i, we note that, for any $t \leq u_i$ and per unit produced, the expected inventory is $\sum_{l=t+1}^{u_i} p_l^i$ (if one unit of product has already been produced) and the expected backlog is $\sum_{l=l_i}^{t-1} p_l^i$ (if one unit of product has not been produced yet). Hence, the expected holding and backlog cost for producing one unit of product to satisfy d^i in period t is denoted by $EC_i(t)$, which can be defined as follows for any $t \leq u_i$:

$$EC_i(t) = \sum_{l=t}^{u_i} h_l \sum_{k=l+1}^{u_i} p_k^i + \sum_{l=l_i}^{t-1} b_l \sum_{k=l_i}^{l} p_k^i \tag{41}$$

Note that the first and second terms of (41) correspond to the expected holding and backlogging costs, respectively. Also, note that the first term is equal to 0 for $t = u_i$, and the second term is equal to 0 for $t \leq l_i$. Next, we present a numerical example to illustrate the problem.

Consider a problem with five periods and two stochastic demand timing intervals, i.e., $T = 5$, $n = 2$. For the sake of simplicity, let the cost parameters be time independent, and let $h_t = 1.5$, $b_t = 6$, $f_t = 25$ and $c_t = 8$, $t = 1, \ldots, 5$. The remaining parameter values are given as follows:

t	1	2	3	4	5	
D_t	4	0	10	6	9	
p_t^1	0.45	0.35	0.2	0	0	$d^1 = 7$, $[l_1, u_1] = [1, 3]$
p_t^2	0	0	0.3	0.7	0	$d^2 = 5$, $[l_2, u_2] = [3, 4]$

We first note that in period 5, at most 9 units will be produced, i.e., the deterministic demand of period 5, and no stochastic demand. On the other hand, in the first three periods, d^1 and/or d^2 can be produced, while in period 4, d^2 can be produced, in addition to any deterministic demand that is produced. To illustrate (41), we provide the following detailed calculations for the cases of producing in period t when $l_i < t < u_i$, $t < l_i$ and $t = u_i$:
$EC_1(2) = h_2 p_3^1 + b_1 p_1^1 = 1.5 \times 0.2 + 6 \times 0.45 = 3$
$EC_2(1) = h_1(p_3^2 + p_4^2) + h_2(p_3^2 + p_4^2) + h_3 p_4^2 = 1.5 \times 1 + 1.5 \times 1 + 1.5 \times 0.7 = 4.05$
$EC_1(3) = b_1 p_1^1 + b_2(p_1^1 + p_2^1) = 6 \times 0.45 + 6 \times 0.8 = 7.5$
Recall that these are unit costs for expected holding and backlogging costs. For example, producing one unit of d^1 in period 2 will incur an expected cost of 3, in addition to the unit production cost of 8 and fixed cost of 25. □

The inventory variable s_t is a stochastic variable since d^i is stochastic, and thus modeling our problem by extending the model (37)-(40) is not trivial. Hence, as it is common in lot sizing, we propose to formalize our problem with the variables in $[0,1]$ z_{lt}, the fraction of the deterministic demand D_t produced in period $l \leq t$, and z_l^i, the fraction of the stochastic demand quantity d^i produced in period $l \leq u_i$. In order to illustrate the development of our model, we first reformulate the deterministic model (37)-(40) using the z_{lt} variables, which are linked to the original production variables as follows:

$$x_l = \sum_{t=l}^{T} z_{lt} D_t, \quad l = 1, \ldots, T. \tag{42}$$

Then, the deterministic model becomes:

$$\min \sum_{t=1}^{T} f_t y_t + \sum_{t=1}^{T} \sum_{l=1}^{t} (c_l + \sum_{k=l}^{t-1} h_k) z_{lt} D_t \tag{43}$$

$$\text{s.t.} \sum_{l=1}^{t} z_{lt} = 1 \qquad t = 1, \ldots, T \tag{44}$$

$$\sum_{t=l}^{T} z_{lt} D_t \leq M_l y_l \qquad l = 1, \ldots, T \tag{45}$$

$$y_t \in \{0, 1\} \qquad t = 1, \ldots, T \tag{46}$$

$$0 \leq z_{lt} \leq 1 \qquad t = 1, \ldots, T; \ l = 1, \ldots, t \tag{47}$$

We remark that the objective (37) is rewritten as (43) using (42) and the fact that inventory variables are no longer explicitly used. Constraints (44) ensure that the deterministic demands are satisfied in the horizon, and constraints (45) correspond to constraints (39) using (42).

Then, using z_l^i associated with the stochastic demand quantities, we next state the relationship between the original production variables and the new variables in a similar fashion to (42):

$$x_l = \sum_{t=l}^{T} z_{lt} D_t + \sum_{i \in \mathcal{I};\ l \leq u_i} z_l^i d^i, \quad l = 1, \ldots, T. \tag{48}$$

Our problem can then be modeled as follows:

$$\min \sum_{t=1}^{T} f_t y_t + \sum_{t=1}^{T} \sum_{l=1}^{t} (c_l + \sum_{k=l}^{t-1} h_k) z_{lt} D_t + \sum_{i \in \mathcal{I}} \sum_{l=1}^{u_i} (c_l + EC_i(l)) z_l^i d^i \tag{49}$$

$$\text{s.t. } (44), (47)$$

$$\sum_{l=1}^{u_i} z_l^i = 1 \qquad i \in \mathcal{I} \tag{50}$$

$$\sum_{t=l}^{T} z_{lt} D_t + \sum_{i \in \mathcal{I};\ l \leq u_i} z_l^i d^i \leq M_l y_l \qquad l = 1, \ldots, T \tag{51}$$

$$y_t \in \{0, 1\} \qquad t = 1, \ldots, T \tag{52}$$

$$0 \leq z_l^i \leq 1 \qquad i \in \mathcal{I}; \ l = 1, \ldots, u_i \tag{53}$$

In a similar fashion to (43), the objective (37) is rewritten as (49) using (48) and (41). Constraints (50) ensure that the stochastic demand quantities are produced in the horizon similar to constraints (44) for deterministic demands. Constraints (51) correspond to constraints (39) using (48).

Next, we remark the following result. $\arg\min EC_i(t) \in [l_i, u_i]$.

Proof. First, note that the first term of (41) is strictly decreasing over $[1, u_i]$ since $h_t > 0\ \forall t$, while the second term of (41) is strictly increasing over $[l_i, u_i]$ since $b_t > 0\ \forall t$. To prove that the minimum of $EC_i(t)$ is attained in $[l_i, u_i]$, it is sufficient to observe that the second term of (41) is 0 for $t \leq l_i$ while the first term of (41) attains its lowest value over $[1, l_i]$ at $t = l_i$. □

In the remainder of the paper, and for the sake of simplicity, we use the notation t_i^* to indicate the period where the minimum of $EC_i(t)$ is attained, i.e. $t_i^* = \arg\min EC_i(t)$. In case of multiple periods attaining this minimum, we assume that t_i^* indicates the earliest such period. Finally, we note that the problem can be rewritten with only stochastic demand quantities by considering that $p_t^i = 1$ and $l_i = t = u_i$ for D_t.

4.5 General Case of Stochastic Demand Timing

First, we investigate the general case of stochastic demand timing, in order to propose a general purpose dynamic programming algorithm. As we will discuss later, this algorithm will be improved from a computational complexity perspective when more restricted but realistic special cases are considered.

When one considers multiple intervals with stochastic demand timing, one can observe that such intervals may also have overlaps. Less obvious is a case when there is no particular order between such overlapping intervals, and therefore, we next define an essential property, in order to differentiate different cases of overlapping intervals.

Let d^i and d^j be two demands with stochastic timing. If $\sum_{l=1}^{t} p_l^i \geq \sum_{l=1}^{t} p_l^j\ \forall t \in [l_j, u_i]$, then we say that d^i **dominates** d^j.

Consider a problem with five periods and three stochastic demand timing intervals, i.e., $T = 5, n = 3$. Assume we are given the following data for these intervals:

t	1	2	3	4	5	
p_t^1	0.1	0.5	0.4	0	0	$[l_1, u_1] = [1, 3]$
p_t^2	0	0.3	0.2	0.5	0	$[l_2, u_2] = [2, 4]$
p_t^3	0	0	0.6	0.2	0.2	$[l_3, u_3] = [3, 5]$

Demand d^1 dominates d^2 since $0.1 + 0.5 \geq 0.3$ and $0.1 + 0.5 + 0.4 \geq 0.3 + 0.2$ both hold. On the other hand, neither d^2 nor d^3 dominate the other, since $0.3 + 0.2 \leq 0.6$ holds while $0.3 + 0.2 + 0.5 \geq 0.6 + 0.2$ is true. □

In this section, we look into the general case with multiple intervals of stochastic demand timing, where we do not have any dominance relationship between the overlapping intervals.

Let us also introduce the following definition, where we assume that $EC_i(t) = +\infty$ if $t \geq u_i + 1$. Let σ_i denote the sequence of length T for demand d^i in which periods are ranked in non-decreasing order of the production and expected unit holding and backlog cost $c_t + EC_i(t)$. More precisely, $\forall k = 2, \ldots, T$, either i) $c_{\sigma_i(k)} + EC_i(\sigma_i(k)) > c_{\sigma_i(k-1)} + EC_i(\sigma_i(k-1))$ or ii) both $c_{\sigma_i(k)} + EC_i(\sigma_i(k)) = c_{\sigma_i(k-1)} + EC_i(\sigma_i(k-1))$ and $\sigma_i(k) > \sigma_i(k-1)$ hold.

Using the first interval (i.e., $i = 1$) from Example 4.5, suppose that $c_1 + EC_1(1) = 12$, $c_2 + EC_1(2) = 11$ and $c_3 + EC_1(3) = 15$ (note this is simply $+\infty$ for periods 4 and 5). Then, by a slight abuse of notation, our ordering vector is $\sigma_1 = (2, 1, 3, 4, 5)$. □

Then, we propose the following result.

Theorem 1. *For two demands with stochastic timing d^i and d^j, if $\sigma_i = \sigma_j$, then there is an optimal solution in which d^i and d^j are produced in the same period.*

Proof. We know that there is an optimal solution in which d^i is produced in a single period t' and d^j is produced in a single period t''. If $\sigma_i = \sigma_j$ and $t'' \neq t'$ then, by definition of σ_i, the solution is only optimal if $c_{t'} + EC_i(t') = c_{t''} + EC_i(t'')$, otherwise the solution could be strictly improved by producing both demands d^i and d^j in t' if $c_{t'} + EC_i(t') < c_{t''} + EC_i(t'')$ or in t'' if $c_{t'} + EC_i(t') > c_{t''} + EC_i(t'')$. Finally, because $c_{t'} + EC_i(t') = c_{t''} + EC_i(t'')$, it is possible to change the solution and keep the same total cost by producing both demands d^i and d^j in t' or in t''. □

Theorem 1 implies that, for two stochastic demand timings such that $\sigma_i = \sigma_j$ and $u_i < u_j$, there is an optimal solution in which d_j is not produced between $u_i + 1$ and u_j. Note also that there are $O(T!)$ possible different sequences of periods in σ_i.

4.5.1 Dynamic Program for the General Case

Let $(sd^1, \cdots, sd^n)$ be a vector of binary parameters, where sd^i is defined for each stochastic demand timing interval $i \in \mathcal{I}$ in the same fashion as in Section **??**. Then, for the general dynamic program, we define $G(t, (sd^1, \cdots, sd^n))$, which indicates the value of the optimal solution for the horizon $[1, t-1]$ and the specific vector $(sd^1, \cdots, sd^n)$.

Note that a vector $(sd^1, \cdots, sd^n)$ is classified as *valid* at period t (or equivalently, $G(t, (sd^1, \cdots, sd^n))$ is valid) if:

- $sd^i = 0$ for all $i \in \mathcal{I}$ such that $t \leq l_i$,
- $sd^i = 0$ or $sd^i = 1$ for all $i \in \mathcal{I}$ such that $t \in [l_i + 1, u_i]$, and
- $sd^i = 1$ for all $i \in \mathcal{I}$ such that $t \geq u_i + 1$.

By definition, $G(1, (sd^1, \cdots, sd^n)) = 0$ holds, where $sd^i = 0$, $\forall i \in \mathcal{I}$. Let $\mathcal{SD}(t)$ denote the set of valid vectors at period t. For each vector $(sd^1, \cdots, sd^n) \in \mathcal{SD}(t)$, the recursion for $G(t, (sd^1, \cdots, sd^n))$ is formally defined as follows:

$$G(t, (sd^1, \cdots, sd^n)) = \min_{\substack{t' \leq t-1, \\ (sd'^1, \cdots, sd'^n) \in \mathcal{SD}(t')}} \Bigg(G(t', (sd'^1, \cdots, sd'^n)) + f_{t'} + \sum_{k=t'}^{t-1} c_{t'k} D_k + \sum_{\substack{i \in \mathcal{I}: \\ sd^i - sd'^i = 1}} d^i (c_{t'} + EC_i(t')) \Bigg) \quad (54)$$

The optimal cost for the full problem is given by $G(T + 1, (sd^1, \cdots, sd^n))$, where $sd^i = 1$, $\forall i \in \mathcal{I}$. We remark that, when $n = 1$, i.e., there is a single interval, it is easy to observe that this general dynamic program exactly maps to the one described in Section **??**: $G(t, sd^i)$ is reduced to a single stochastic demand timing while the validity arguments for sd^i remain (though now for a single interval), and the cost of producing d^i is only applied when sd^i value is changed from 0 to 1 in the new time period.

The complexity of the dynamic program is $O(T \max_{t\in[1,T]} |\mathcal{SD}(t)|)$. The value of $\max_{t\in[1,T]} |\mathcal{SD}(t)|$ is discussed in Lemma 2.

Lemma 2. *In the worst case,* $\max_{t\in[1,T]} |\mathcal{SD}(t)| \sim O(\min\{2^n, T!\})$

Proof. The worst case can be reached in two different ways:

1. If there exists $t \in [1,T]$ such that $t \in [l_i + 1, u_i]$, $\forall i \in \mathcal{I}$, i.e., all n intervals intersect with each other at least in one period. This leads to $O(2^n)$ combinations.

2. Following Theorem 1, it is possible to combine demands with the same sequence σ_i in the same indicator sd^i in the dynamic programming algorithm. This leads to a maximum of $O(T!)$ combinations. This is essentially a preprocessing stage to the algorithm.

□

Therefore, the time complexity of the algorithm may be exponential in n and in T. However, as stochastic demand intervals will be short in most practical settings (no more than 4 or 5 periods), a small number of intervals should be overlapping in any period t, leading to small sets $\mathcal{SD}(t)$. If at most k intervals are overlapping in any period t, then the complexity of the dynamic program is $O(Tk)$. Moreover, as we will see in Sections 4.5.2 and 4.5.3 for practical general cases, as well as in Section **??** for some relevant special cases, this time complexity can be effectively reduced to polynomial.

4.5.2 Time Independent Production Costs and Time Independent Ratio between Inventory and Backlog Costs

An interesting case in practice appears when the ratio between the unit inventory and backlog costs in each period is time independent, i.e., $h_t = \alpha_t h$ and $b_t = \alpha_t b$ with $\alpha_t > 0$ $\forall t$ (or, equivalently, $h_t/b_t = h/b$, $\forall t$). Moreover, we assume time independent production costs, i.e., $c_t = c$, $\forall t \in [1,T]$. Although this case is more restricted than the general case that does not specify cost functions or other key parameters of the problem, it is very common in practice, where hard to quantify backlog costs are often defined in terms of inventory holding costs. Moreover, its limitations are minimal, as there is no specification on how the actual cost levels would vary from one period to another, and time independent production costs are a common setting in the lot sizing literature. Because production costs are time independent, they can be ignored in the remainder of this section. Finally, as discussed in this section, this case can be solved in polynomial time. A more special case worth remarking is when the inventory and backlog costs are time independent, i.e., $\alpha_t = 1$ $\forall t$.

First, we note the step change from t to $t+1$:

$$\Delta(t) = EC_i(t+1) - EC_i(t) = \left(\sum_{l=t+1}^{u_i} h_l \sum_{k=l+1}^{u_i} p_k^i + \sum_{l=l_i}^{t} b_l \sum_{k=l_i}^{l} p_k^i \right)$$

$$-\left(\sum_{l=t}^{u_i} h_l \sum_{k=l+1}^{u_i} p_k^i + \sum_{l=l_i}^{t-1} b_l \sum_{k=l_i}^{l} p_k^i\right) = b_t \sum_{k=l_i}^{t} p_k^i - h_t \sum_{k=t+1}^{u_i} p_k^i$$

Because $\sum_{k=t+1}^{u_i} p_k^i = 1 - \sum_{k=l_i}^{t} p_k^i$, the expression above can be rewritten:

$$\Delta(t) = (h_t + b_t) \sum_{k=l_i}^{t} p_k^i - h_t \tag{55}$$

Note that (55) can also be used to show Proposition 4.4, since $\Delta(t) = -h_t < 0$ when $t \leq l_i - 1$. For the case of time independent ratio, we can rewrite the expression (55) as follows:

$$\Delta(t) = \alpha_t \Big((h + b) \sum_{k=l_i}^{t} p_k^i - h\Big) \tag{56}$$

Theorem 3. *If the ratio between the inventory and backlog costs is time independent, i.e., $h_t = \alpha_t h$ and $b_t = \alpha_t b$ $\forall t$, then $EC_i(t)$ is strictly decreasing until $t = t_i^*$ and strictly non-decreasing after $t = t_i^*$. Moreover, if inventory and backlog costs are time independent, i.e., $\alpha_t = 1$ $\forall t$, then $EC_i(t)$ is convex.*

Proof. We first observe that, in (56), $\sum_{k=l_i}^{t} p_k^i$ is strictly increasing with t when $l_i \leq t \leq u_i$ (while being 0 when $t \leq l_i - 1$, as noted earlier). Since $h + b > 0$, the value of $\Delta(t)$, starting from $-\alpha_t h < 0$ at $t = l_i - 1$, will also be strictly increasing. Hence, either (i) $t = t_i^* \leq u_i - 1$ holds due to the first observation of $(h+b) \sum_{k=l_i}^{t} p_k^i \geq h$ at t, or (ii) $t_i^* = u_i$ holds if $EC_i(u_i) - EC_i(u_i - 1) < 0$. In case (i), note that $(h + b) \sum_{k=l_i}^{t} p_k^i = h$ is possible, and hence the function $EC_i(t)$ is strictly non-decreasing (rather than strictly increasing). This concludes the proof of the first claim. Next, consider the case of $\alpha_t = 1$. Note that we can further simplify (56) by eliminating α_t. Then, we have:

$$EC_i(t + 1) = EC_i(t) + (h + b) \sum_{k=l_i}^{t} p_k^i - h$$

$$EC_i(t + 2) = EC_i(t) + (h + b) \sum_{k=l_i}^{t} p_k^i - h + (h + b) \sum_{k=l_i}^{t+1} p_k^i - h$$

where the second equation is simply the definition of $\Delta(t+1)$ with $EC_i(t+1)$ substituted using the first equation. Since $\sum_{k=l_i}^{t} p_k^i \leq \sum_{k=l_i}^{t+1} p_k^i$, it is possible to observe that $EC_i(t+2) + EC_i(t) \geq 2EC_i(t + 1)$. This concludes the convexity of $EC_i(t)$. □

The case of a convex $EC_i(t)$ function can be associated to the practical setting where, as one moves further away from $t = t_i^*$, not only the expected cost increases, but also the rate of the cost increases.

In line with the previous literature, we next define a regeneration interval $[t_1, t_2]$ as an interval of periods such that production takes place in periods t_1 and t_2 while no production occurs in periods t, $t_1 < t < t_2$. Then, we have the following result.

Given a regeneration interval $[t_1, t_2]$, let $\mathcal{I}_{t_1,t_2} = \{i \in \mathcal{I} : t_1 \leq t_i^* \leq t_2\}$. If the ratio between the inventory and backlog costs is time independent, and production costs are time independent, then in an optimal solution involving regeneration interval $[t_1, t_2]$, for every $i \in \mathcal{I}_{t_1,t_2}$, d^i will be produced either at t_1 or t_2.

Proof. First, note that the production of d^i for any $i \in \mathcal{I}_{i_1,i_2}$ cannot take place in a period $t < t_1$ (or $t > t_2$), since $EC_i(t) \geq EC_i(t_1)$ (or $EC_i(t) \geq EC_i(t_2)$, respectively) due to Theorem 3 and the fact that production costs are time independent. Since production of d^i for any $i \in \mathcal{I}$ takes place in a single period in an optimal solution due to Theorem **??**, and since, by definition, there is no production in any period t such that $t_1 < t < t_2$, d^i will be produced either in t_1 if $EC_i(t_1) \leq EC_i(t_2)$, or in t_2 otherwise. □

Next, we discuss how to use this result to define a dynamic program of polynomial complexity particularly due to the significantly reduced number of linkages between states. First, we note that the number of valid states is reduced, since now a state is valid only if $sd^i = 0$ for all $i \in \mathcal{I}$ such that $t \leq t_i^*$ (rather than $t \leq l_i$). Next, in order to account for the regeneration intervals, we replace $\mathcal{SD}(t')$ with $\mathcal{SD}(t', t)$ in the recursion (54) of the dynamic program, where we define any valid $\mathcal{SD}(t', t)$ as follows:

- If $t' \leq t_i^* \leq t - 1$ and $EC_i(t') \leq EC_i(t)$, then $sd^i = 1$ must hold at t,
- If $t' \leq t_i^* \leq t - 1$ and $EC_i(t') > EC_i(t)$, then $sd^i = 0$ must hold at t,
- If $t' \geq t_i^* + 1$, then $sd^i = 1$ must hold at t.

Note that the first case means that d^i must be produced at t' (since it is cheaper at t') whereas the second case means that d^i will be not produced at t'. In the third case, if $sd^i = 0$ holds at t', then d^i must be produced at t' since producing at t will be more expensive (whereas if $sd^i = 1$ holds at t', it means production of d^i is already completed earlier.)

With this transformation of valid states as well as interactions between them, we first note that, given an interval $i \in \mathcal{I}$ with stochastic demand timing, the optimal decision regarding a regeneration interval $[t_1, t_2]$ is trivial, unless $t_1 \leq t_i^* \leq t_2 - 1$ holds. Note that there are $O(T^2)$ nontrivial regeneration intervals satisfying $t_1 \leq t_i^* \leq t_2 - 1$, and for each of these regeneration intervals, we can pre-compute the set of valid vectors $\mathcal{SD}(t_1, t_2)$ as shown above, i.e., by calculating whether it is cheaper to produce d^i at the start or the end of the regeneration interval. With n intervals in total, this would result in at most $O(nT^2)$ computational effort.

Corollary 1. *In the case of time independent production costs and time independent ratio between inventory and backlog costs, the dynamic program has a worst case complexity of* $O(nT^2)$.

4.5.3 Time Independent Production Costs and Convex Probability Distributions

We next consider the case where the probability distribution for any stochastic demand timing is convex between l_i and u_j. Then, it is straightforward to observe that $EC_i(t)$ is convex, in the same fashion as in Theorem 3 when $\alpha_t = 1$ $\forall t$. Therefore, Proposition 4.5.2 holds in this case as well, and the worst case complexity of the dynamic program is $O(nT^2)$, as given in Corollary 1.

4.5.4 Final Remarks on Stochastic Timing

Many research avenues are worthwhile investigating from this novel stochastic setting. First, although we believe it is $\mathcal{NP}$-hard, the complexity of the general problem with backlog costs that are independent of the quantity of stochastic demand remains an open question to study. Second, the capacitated case with multiple products could be solved using a Lagrangian heuristic, such as the ones proposed in Trigeiro et al. (1989) and Brahimi et al. (2006), by relaxing the capacity constraints and solving the resulting single-item problems with the dynamic programs proposed in this paper. Another interesting extension of our work is to consider the case where $\sum_{t=l_i}^{u_i} p_t^i < 1$, i.e., there is a probability that demand d^i may not occur at all. In this case, the total demand on the planning horizon also becomes uncertain. This implies that some production quantity aimed at satisfying d^i might end up remaining in the inventory and thus could be used to satisfy other demands in the planning horizon. A last related research perspective would be to analyze the case with lost sales, where answering a demand too late would also result in products remaining in the inventory.

References

Attila, Ö. N., Agra, A., Akartunalı, K. & Arulselvan, A. (2017), A decomposition algorithm for robust lot sizing problem with remanufacturing option, *in* 'International Conference on Computational Science and Its Applications', Springer, pp. 684–695.

Ben-Tal, A., Goryashko, A., Guslitzer, E. & Nemirovski, A. (2004), 'Adjustable robust solutions of uncertain linear programs', *Mathematical programming* **99**(2), 351–376.

Bienstock, D. & Özbay, N. (2008), 'Computing robust basestock levels', *Discrete Optimization* **5**(2), 389–414.

Brahimi, N., Dauzère-Pérès, S. & Najid., N. (2006), 'Capacitated multi-item lot-sizing problems with time windows', *Operations Research* **54**(5), 951–967.

Christiansen, M., Fagerholt, K., Flatberg, T., Haugen, Ø., Kloster, O. & Lund, E. H. (2011), 'Maritime inventory routing with multiple products: A case study from the cement industry', *European Journal of Operational Research* **208**(1), 86–94.

Dauzère-Pérès, S., Nordli, A., Olstad, A., Haugen, K., Koester, U., Per Olav, M., Teistklub, G. & Reistad, A. (2007), 'Omya hustadmarmor optimizes its supply chain for delivering calcium carbonate slurry to european paper manufacturers', *Interfaces* **37**(1), 39–51.

Denizel, M., Ferguson, M. et al. (2009), 'Multiperiod remanufacturing planning with uncertain quality of inputs', *IEEE transactions on engineering management* **57**(3), 394–404.

See, C.-T. & Sim, M. (2010), 'Robust approximation to multiperiod inventory management', *Operations research* **58**(3), 583–594.

Tao, Z. & Zhou, S. X. (2014), 'Approximation balancing policies for inventory systems with remanufacturing', *Mathematics of Operations Research* **39**(4), 1179–1197.

Trigeiro, W. W., Thomas, L. J. & McClain, J. O. (1989), 'Capacitated lot sizing with setup times', *Management Science* **35**(3), 353–366.

Wagelmans, A., van Hoesel, S. & Kolen, A. (1992), 'Economic lot sizing: An $O(n \log n)$ algorithm that runs in linear time in the Wagner-Whitin case', *Operations Research* **40**(1), S145–S156.

Zhou, S. X., Tao, Z. & Chao, X. (2011), 'Optimal control of inventory systems with multiple types of remanufacturable products', *Manufacturing & Service Operations Management* **13**(1), 20–34.