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# CERTIFIED PROGRAMMING WITH DEPENDENT TYPES

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# Final report on the grant CERTIFIED PROGRAMMING WITH DEPENDENT TYPES (FA9550-16-1-0029)

Thorsten Altenkirch

08/14/2021

I will present the work done in temporal order repeating the contents of the annual reports plus a report on the last year of the project and then I will present a summary and conclusions.

## 1 Work in 2016

The project has made an excellent start with already two publications in international conferences and one accepted and further publications currently in preparation. This was made possible by having been able to hire two excellent research assistants (Ambrus Kaposi, January to March 2016) and Paolo Capriotti (April - December 2016). Both were also able to finish their PhDs in 2016 on topics which are closely related to the project [19, 15] A number of invited presentations on the subject of the grant were given by members of the team.

In the first year we concentrated on the following aspects of the project:

**Normalisation by evaluation (NBE) for dependent types**
: This follows from our previous work on using a feature of Homotopy Type Theory, namely Higher Inductive Types, to develop an intrinsically typed syntax of type theory. In our work on NBE we show that within this framework the method of normalisation by evaluation can be extended to dependent types. A journal paper on this result is currently in preparation. The algorithm with its correctness proof has been implemented using the Agda system. The paper [9] has appeared in the international conference on *Foundations of Computation and Deduction (FSCD)* which took place in Porto, Portugal.

**Extending Homotopy Type Theory with Strict Equality**
: This addresses a central problem in Homotopy Type Theory namely the existence of coherence problems which seem to be impossible to solve: an

example is the problem of defining Semisimplicial Types internally - this has been identified as an open problem in HoTT. In our work we follow up on an unpublished proposal by Voevodsky which is called Homotopy type System. Unlike Voevodsky we present a two-level system which doesn't require an extensional equality. This paper [5] has been published in the proceedings of *Computer Science Logic (CSL)* which took place in Marseille, France, this year. A followup journal publication is currently under preparation.

**Partiality as an effect** We revisited an open problem in Type Theory, namely how to represent possibly non-terminating programs in a satisfying manner. Previous attempts had some serious shortcomings in particular it wasn't possible that the construction constitutes a monad without further assumptions (here the countable axiom of choice). By using Higher Inductive Types we were able to define the partiality monad without using additional assumptions. This paper [7] has been accepted for *Foundations of Software Science and Computation Structures (FOSSACS)* which will take place in Uppsala, Sweden in 2017.

We have given a number of invited presentations which are related to the project (TA = Thorsten Altenkirch, AK = Ambrus Kaposi, PC = Paolo Capriotti) :

**TA** *What is a category? (in univalent type theory)*
CHoCoLa meeting, ENS Lyon, France, February 2016.

**AK** *Type Theory in Type Theory using Quotient Inductive Types*
Talk at the Theoretical computer science seminar, University of Birmingham, UK, February 2016.

**TA** *Why does Homotopy Type Theory matter?*
HoTT workshop at Fields Institute, Toronto, Canada, May 2016.

**AK** *Normalisation by Evaluation for Dependent Types*
22nd International Conference on Types for Proofs and Programs, TYPES 2016, Novi Sad, Serbia, May 2016

**PC** *Higher Categories in Homotopy Type Theory*
22nd International Conference on Types for Proofs and Programs, TYPES 2016, Novi Sad, Serbia, May 2016

**TA** *Partiality, revisited*
22nd International Conference on Types for Proofs and Programs, TYPES 2016, Novi Sad, Serbia, May 2016

**TA** *Why cubical type theory?*
The 5th International Congress on Mathematical Software, Berlin, Germany, July 2016.

**TA** *Naive Type Theory*
FOMUS - Foundations of Mathematics: Univalent Foundations and Set Theory, Bielefeld, Germany, July 2016.

**PC** $(\infty, 1)$*-Categories in Homotopy Type Theory*
YaMCATS meeting, Sheffield, UK, 11 July 2016

**PC** *Presheaf Models of Two-Level Type Theory*
Workshop on Categorical Logic and Univalent Foundations, Leeds, UK, July 2016

**TA** *Homotopy Type Theory as a Foundation of Mathematics ?*
Workshop on Categorical Logic and Univalent Foundations, Leeds, UK, July 2016

**TA** *A constructive justification of Homotopy Type Theory*
Homotopy Type Theory in Logic, Metaphysics and Philosophy of Physics, University of Bristol, UK, September 2016

**PC** *Semi-Segal types in Homotopy Type Theory*
Category Theory Seminar, Cambridge, UK, October 2016

# 2 Work in 2017

The work on the project continued successfully with the following highlights:

- The paper on the partiality monad has now appeared in print [6] giving an important application of quotient inductive types.
- As a followup to last year's conference paper on normalisation [9], a journal version with complete proofs has now been published [10].
- We have also published a paper on our view of cubical type theory: [11] based on earlier work.
- The work on 2-level type theory has been written up as a journal paper [13], this has been submitted but is not yet accepted.
- As an important step Capriotti and Krauss have presented a definition of higher categories in a 2-level type theory. This will be presented at POPL 2018: [16].
- A paper on the semantics of quotient inductive-inductive types has been accepted for FOSSACS 2018, providing foundations to the constructions we have carried out using them. [3].

In the summer 2017 we were able to host Ambrus Kaposi (University of Budapest) as a visitor for several weeks. As one can see this visit has been quote productive. The collaboration with Kaposi and his team is continuing.

We have given a number of invited courses and invited presentations which are related to the project (TA = Thorsten Altenkirch, PC = Paolo Capriotti) :

**TA** Computer Science ∩ Mathematics (Type Theory) - Computerphile (youtube), January 2017, over 140.000 views.

**TA** *Introduction to Homotopy Type Theory*, 22nd Estonian Winter School in Computer Science (EWSCS), Palmse, Estonia, January 2017. Invited course.

**TA** *Naïve Type Theory*, Midland Graduate School 2017 (MGS 2017) Leicester, UK, April 2017. Contributed course.

**TA** *Monadic containers and universes* 23rd International Conference on Types for Proofs and Programs, TYPES 2017, Budapest, Hungary, June 2017. Contributed talk.

**PC** *Notions of type formers* 23rd International Conference on Types for Proofs and Programs, TYPES 2017, Budapest, Hungary, June 2017. Contributed talk.

**TA** *From setoid hell to homotopy heaven?* 23rd International Conference on Types for Proofs and Programs, TYPES 2017, Budapest, Hungary, June 2017. Contributed talk.

**TA** *Introduction to homotopy type theory*, EU Types summer school, Ohrid, Macedonia, July 2017. Invited course.

**TA** *NaïveType Theory (tutorial)* Workshop on Homotopy Type Theory/ Univalent Foundations, Oxford, UK, September 2017. Invited tutorial.

**TA** *Naïve Type Theory (invited talk*), 7th South of England Regional Programming Language Seminar, Warwick, UK, September 2017

**TA** *Propositions as Types* - Computerphile (YouTube), September 2017, over 50.000 views

**TA** *Homotopy Type Theory: Vladimir Voevodsky* - Computerphile (YouTube), October 2017, over 40.000 views

**TA** *A Taste of Homotopy Type Theory*, Kolloqium über reine Mathematik, Hamburg, Germany, December 2017. Invited talk.

**PC** *Weak enrichment and cofibrant operads*, Midland Graduate School Christmas seminar, Nottingham, December 2017. Invited talk.

# 3 Work 2018 - June 2019

Recently we have focussed on Higher Inductive Types which are a central ingredient in Homotopy Type Theory (HoTT). While these are interesting to represent concepts from Homotopy Theory they are also extremely useful for other formalisation tasks in computer science. In [21] we present a partial solution to the problem to show that the free group of a set is a set. In [4] we

present a categorial semantics for a specific case of Higher Inductive Types, namely Quotient Inductive-Inductive Types (QIITs). An alternative, more syntactic investigation of QIITs is given in [20]. One issue is now to relate these two views of QIITs and also expand this to the case of truly higher inductive types. An important ingredient here is the presentation of higher categories in HoTT, which has been begun in [17].

We have given a number of invited courses and invited presentations which are related to the project

**TA** *The joy of QIITs* AMS Special Session on Homotopy Type Theory, San Diego, USA, January 2018. Invited talk.

**TA** *An introduction to (Homotopy) Type Theory* FMV Foundations in Mathematics, Munich, Germany, April 2018. Invited talk.

**TA** Towards higher models and syntax of type theory, Types, Homotopy Type Theory, and Verification, workshop, Bonn, June 2018.

**TA** *The partiality monad*, An Intersection of Neighbourhoods, Achim Jung Fest, Birmingham, September 2018

**TA** Why Type Theory matters, Lambda Days 2019, Krakow, February 2019

**TA** Naturality for Free - The category interpretation of directed type theory, Third Symposium on Compositional Structures, Oxford, March 2019

# 4 Work from July 2019 until June 2021

The development of 2-level type theory presents an important basis for many constructions that reason generically about higher categories. Based on our earlier work there is now a journal version available on arXiv [14] which has been submitted to a journal.

We also continued our work on Higher Inductive Types. As a test case for coherence theorems we have presented a constructions of the integers which interacts well with higher types [12]. Capriotti and Sattler have given a new generic constructions of an induction principle for generic HIITs [18]. In our recent still ongoing work we investigate the use of containers to model QIITs, first results have been presented at the annual types conference [8].

On the other hand many constructions in Type Theory do not rely on higher types but take place in a set-level type theory. To cover this important case but also to gain experience how to tackle higher types we have developed setoid type theory [2]. Setoid type theory can be translated into vanilla Martin-Löf Type Theory with some small additions (strict prop with general transports). In [1] we have shown that this can be extended to a non-univalent universe of sets.

The PI has given a number of talks related to the work on the grant and has also participated in an online panel discussion:

**TA** *On inductive types*, Workshop on Foundations and Applications of Univalent Mathematics, Herrsching, Germany, November 2019. Invited talk.

**TA** *The power of Pi*, Lambda Days 2020, Krakow, Poland, February 2020. Invited talk.

**TA** *Coinduction in Agda*, seminar at the University of Canterbury (online), May 2020, Invited talk.

**TA** *The Integers in Homotopy Type Theory*, Seminar at Tallinn University (online), December 2020. Invited Talk

**TA** Invited participant in panel discussion associated to Lambda Days (February 2021) Lambda Days Meetup "Dependent Types - salvation or plague?"

**TA** *Containers and Inductive Types*, talk at the workshop on Polynomial Functors at the Topos Institute Sydney, Australia (online), March 2021, invited talk.

# 5 Conclusions and summary

The project has substantially contributed to the development of Type Theory as an effective tool to formalize complex mathematical arguments in a concise and natural way and use this technology for the verification of software and hardware systems. It also delivers a novel way to underpin mathematical constructions by using structural reasoning instead of set-theoretic reasoning which is less modular and depends heavily on the choice of representations.

During the project we have published a substantial number of papers in high visibility publications, see the list of references below. Many of the publications have been frequently cited, for example our paper on QIITs [3] from 2018 has 50 citations, our paper on the partiality monad [7] has 39 citations and our first paper on 2-level type theory [5] has 42 citations, all numbers according to google scholar in September 2021. These are excellent numbers in this area.

These papers are also representative for the main progress that has been achieved during the project:

**Development of 2-level type theory** [5, 13]

We have been able to make precise a suggestion by Voevodsky which enables the formal development of many constructions that appear to be impossible in pure HoTT. This has been experimentally implemented i cubical agda and this will be the basis of the development of higher structures and categories in HoTT.

**The Theory of HIITs and QIITs** [6, 3, 20, 17, 12, **?**]

HIITs and QIITs are an important ingredient of many constructions in Type Theory. We have contributed substantially to understanding them in a generic way instead as a collection of examples.

**Type Theory in Type Theory** [9, 11, 2, 1]

> One of the applications of QIIts is the development of type theory in type theory as every good programmer wants to write a compiler in their own language. This approach captures many domain-specific languages using dependent types and it provides an important alternative to the traditional way of treating type theories by introducing untyped terms and a typing relation separately.

There are many loose ends in all these developments which we and others are investigating further. In the moment we are revisiting our old work on an alternative presentation of cubical type theory [11] which would lead to a model-independent characterisation and implementation of HoTT.

Apart from contributing to these results the projects has benefitted form the expertise of the RAs involved but also has helped their academic development: Paolo Capriotti, Ambrus Kaposi, Andrea Vezzosi and Christian Sattler. We have continued to collaborate with them after their time on the project and we have also worked with other people, most prominently Nicolai Kraus.

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# References

[1] Thorsten Altenkirch, Simon Boulier, Ambrus Kaposi, Christian Sattler, and Filippo Sestini. Constructing a universe for the setoid model. In *FoSSaCS*, pages 1–21, 2021.

[2] Thorsten Altenkirch, Simon Boulier, Ambrus Kaposi, and Nicolas Tabareau. Setoid type theory—a syntactic translation. In *International Conference on Mathematics of Program Construction*, pages 155–196. Springer, 2019.

[3] Thorsten Altenkirch, Paolo Capriotti, Gabe Dijkstra, Nicolai Kraus, and Fredrik Nordvall Forsberg. Quotient inductive-inductive types. 2017. To appear in FoSSaCS 2018.

[4] Thorsten Altenkirch, Paolo Capriotti, Gabe Dijkstra, Nicolai Kraus, and Fredrik Nordvall Forsberg. Quotient inductive-inductive types. In *Foundations of Software Science and Computation Structures - 21st International Conference, FOSSACS 2018, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2018, Thessaloniki, Greece, April 14-20, 2018, Proceedings*, pages 293–310, 2018.

[5] Thorsten Altenkirch, Nils Anders Danielsson, and Nicolai Kraus. Partiality, revisited - the partiality monad as a quotient inductive-inductive type. In *Foundations of Software Science and Computation Structures - 20th*

*International Conference, FOSSACS 2017, Uppsala, Sweden, April 22-29, 2017, Proceedings*, pages 534–549, 2017.

[6] Thorsten Altenkirch, Nils Anders Danielsson, and Nicolai Kraus. Partiality, revisited - the partiality monad as a quotient inductive-inductive type. In *Foundations of Software Science and Computation Structures - 20th International Conference, FOSSACS 2017, Uppsala, Sweden, April 22-29, 2017, Proceedings*, pages 534–549, 2017.

[7] Thorsten Altenkirch, Nils-Anders Danielsson, and Nicolai Kraus. Partiality, revisited: The partiality monad as a quotient inductive-inductive type. In *Foundations of Software Science and Computation Structures*, 2017. To appear.

[8] Thorsten Altenkirch and Ambrus Kaposi. A container model of type theory. *EUTYPES-TYPES 2021-Abstracts*.

[9] Thorsten Altenkirch and Ambrus Kaposi. Normalisation by evaluation for dependent types. In *1st International Conference on Formal Structures for Computation and Deduction, FSCD 2016, June 22-26, 2016, Porto, Portugal*, pages 6:1–6:16, 2016.

[10] Thorsten Altenkirch and Ambrus Kaposi. Normalisation by evaluation for type theory, in type theory. *Logical Methods in Computer Science*, 13(4), 2017.

[11] Thorsten Altenkirch and Ambrus Kaposi. Towards a cubical type theory without an interval. *Leibniz International Proceedings in Informatics*, 2017.

[12] Thorsten Altenkirch and Luis Scoccola. The integers as a higher inductive type. In *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 67–73, 2020.

[13] Danil Annenkov, Paolo Capriotti, and Nicolai Kraus. Two-level type theory and applications. 2017.

[14] Danil Annenkov, Paolo Capriotti, Nicolai Kraus, and Christian Sattler. Two-level type theory and applications. *arXiv preprint arXiv:1705.03307*, 2017.

[15] Paolo Capriotti. *Models of Type Theory with Strict Equality*. PhD thesis, University of Nottingham, 2016.

[16] Paolo Capriotti and Nicolai Kraus. Univalent higher categories via complete semi-Segal types. 2017. To appear in Proceedings of the ACM on Programming Languages (POPL).

[17] Paolo Capriotti and Nicolai Kraus. Univalent higher categories via complete semi-segal types. *PACMPL*, 2(POPL):44:1–44:29, 2018.

[18] Paolo Capriotti and Christian Sattler. Higher categories of algebras for higher inductive definitions. *EUTYPES-TYPES 2020-Abstracts*.

[19] Ambrus Kaposi. *Type theory in a type theory with quotient inductive types*. PhD thesis, University of Nottingham, 2016.

[20] Ambrus Kaposi, András Kovács, and Thorsten Altenkirch. Constructing quotient inductive-inductive types. *PACMPL*, 3(POPL):2:1–2:24, 2019.

[21] Nicolai Kraus and Thorsten Altenkirch. Free higher groups in homotopy type theory. *CoRR*, abs/1805.02069, 2018.