AFRL-AFOSR-VA-TR-2022-0038

Univalent Type Theorems: Models, Equalities, and Coherence

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09/02/2021
Final Technical Report

Air Force Research Laboratory
Air Force Office of Scientific Research
Arlington, Virginia 22203
Air Force Materiel Command

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. REPORT DATE *(DD-MM-YYYY)*	**2. REPORT TYPE**	**3. DATES COVERED** *(From - To)*
02-09-2021	Final	01 Sep 2017 - 31 Jan 2021

4. TITLE AND SUBTITLE
Univalent Type Theorems: Models, Equalities, and Coherence

5a. CONTRACT NUMBER

5b. GRANT NUMBER
FA9550-17-1-0290

5c. PROGRAM ELEMENT NUMBER
61102F

6. AUTHOR(S)
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5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
THE UNIVERSITY OF LEEDS
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WOODHOUSE LA,
GB

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
AF Office of Scientific Research
875 N. Randolph St. Room 3112
Arlington, VA 22203

10. SPONSOR/MONITOR'S ACRONYM(S)
AFRL/AFOSR RTA2

11. SPONSOR/MONITOR'S REPORT NUMBER(S)
AFRL-AFOSR-VA-TR-2022-0038

12. DISTRIBUTION/AVAILABILITY STATEMENT
A Distribution Unlimited: PB Public Release

13. SUPPLEMENTARY NOTES

14. ABSTRACT
During the performance period of the grant, the PI, the PRDAs Karol Szumilo and Sina Hazratpour and the PI's PhD students developed research along three connected lines of enquiry: algebraic models of homotopy type theory, constructive models of univalent foundations, and connections between homotopy type theory and higher categories. Their research led to several key results, including a partial solution to the open problem of defining a constructive version of the simplicial model of univalent foundations. These results have already been used by researchers in the area and have been presented in several papers and preprints. The work has also opened up new research directions, such as a higher-categorical version of exact completions.

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			**17. LIMITATION OF ABSTRACT**	**18. NUMBER OF PAGES**	**19a. NAME OF RESPONSIBLE PERSON**
a. REPORT	**b. ABSTRACT**	**c. THIS PAGE**			TRISTAN NGUYEN
U	U	U	UU	4	**19b. TELEPHONE NUMBER** *(Include area code)* 426-7796

FA9550-17-1-0290
Univalent Type Theories: Models, Equalities, and Coherence
Final report - 01 September 2017 to 31 January 2021

Nicola Gambino

April 28th, 2021

1 Research

During the performance period of the grant, the PI, the PRDAs Karol Szumiło and Sina Hazratpour and the PI's PhD students developed research along three connected lines of enquiry: algebraic models of homotopy type theory, constructive models of univalent foundations, and connections between homotopy type theory and higher categories.

1.1 Algebraic models of homotopy type theory

The work of the PI and Sattler in [7] showed how the models of homotopy type theory (HoTT) defined by Coquand and his collaborators using uniform fibrations can be understood in clear mathematical terms using the theory of so-called *algebraic weak factorisation systems*. These are enhancements of ordinary weak factorisation systems, in which the usual lifting properties are replaced by lifting structures, providing explicitly defined diagonal fillers for lifting problems.

The work carried out led to two key results. The first, obtained in collaboration with the PhD student Marco Federico Larrea, showed that algebraic weak factorisation systems lead indeed to genuine, strict, models of Type Theory. The key result is obtained in two steps: the first is to show that uniform fibrations give rise to a comprehension category; the second is to show that this comprehension category satisfies the assumptions necessary to be able to perform the right adjoint splitting, which guarantees that all the rules governing substitutions in type theory are valid. The 45-page paper with these results has been accepted for publication in the *Journal of Symbolic Logic*.

Building on this work, the PI, the PDRA Sina Hazratpour and Steve Awodey developed a general method for relating algebraic weak factorisation systems and syntactic approaches to models of HoTT, as described in the work of Orton and Pitts. This involved developing a version of Kripke-Joyal forcing semantics for the internal type theory of a presheaf category. This provides an efficient method to relate precisely diagrammatic reasoning used in some of the literature on models of Homotopy Type Theory with the logical reasoning using the so-called internal languages used in some other literature. This is likely to have a very beneficial impact on the subject, as it will make possible to relate precisely notions and arguments that are at the moment only intuitively related. The PDRA met with Steve Awodey at a conference in December 2019 to discuss ideas and steady progress has been made since then, leading up to an extensive set of notes. We are currently working on completing a paper on this topic.

1.2 Towards a constructive model of Univalent Foundations

The second line of research focused on solving one of the key open problems in the area of Homotopy Type Theory, namely finding a constructive counterpart of the simplicial model of Univalent Foundations defined by Voevodsky, building on Simon Henry's proofs of the existence of the classical Kan–Quillen model structure on the category of simplicial sets.

This led to two sub-projects. The first, in collaboration with the PDRA Karol Szumiø and Christian Sattler, is to provide new proofs of the existence of the classical Kan–Quillen model structure on the category of simplicial sets in constructive logic. Simplicial sets provide a standard interpretation of classical topology. The Kan–Quillen model structure is a formal framework that makes this interpretation precise. It is also

a natural tool to be used for the purpose of interpreting of Homotopy Type Theory (HoTT). A major obstruction to that goal is that all the known constructions of the model structure rely on classical logic which is incompatible with the constructive character of HoTT. Inspired by the breakthough result of Simon Henry, we have provided two fully constructive proofs of its existence: one based on standard ideas of simplicial homotopy theory and one resulting from modern approaches originating in HoTT itself. The arguments were written up in a preprint that was posted to arXiv [8] and submitted for publication. Following a positive referee report, we are currently revising the paper.

The second sub-goal is to expand on the existence of the model structure and obtain the results necessary to model Univalent Foundations. Partial success was obtained by the PI with Simon Henry, leaving only a coherence issues to be addressed. This was written up in [6] and submitted for publication.

1.3 Homotopy Type Theory and Higher Categories

The new proofs of the existence of the constructive Kan-Quillen model structure led us to explore the possibility of generalising the result by replacing the category of sets by a more general category, so as to obtain model structure on categories of simplicial objects.

This was explored in a joint project by the PI, the PDRA Karol Szumiło, Simon Henry and Christian Sattler. We succeeded to show a very general result, asserting that categories of simplicial objects in a countably lextensive category admit a counterpart of the Kan-Quillen model structure. Here, a lextensive category is a category with finite limits and countable coproducts, which suitably interact with each other. This work required extensive use of ideas and methods of enriched category theory and enriched homotopical algebra. Key technical advances were made during a visit of Simon Henry and Christian Sattler to the University of Leeds in March 2020. A 67-page preprint on this has been completed [9] and submitted for publication.

One of the most interesting aspects of this work is that it suggests the possibility of developing a higher categorical version of the theory of exact completions, and use that in turn to define ∞-categorical counterparts of realisability toposes. This, in turn, can lead to new models of Homotopy Type Theory with previously unseen combination of logical features, in analogy with (but unavoidably more subtly than) the 1-categorical setting. Further interest in this work stems from the fact that it can inform the definition of the notion of an elementary higher topos, by providing examples on which to test the notion.

Another direction of research, led by the PDRA Karol Szumiłło, concerns internal languages of higher categories, in collaboration with Chris Kapulkin. This concerns the relationship between HoTT and higher category theory. Instead of considering one particular model of HoTT, we consider all possible models formalised as suitable higher categories. The goal is to prove that statements derivable in HoTT are exactly the ones that hold in all models. We have previously obtained a result of this type for a weak form of HoTT which includes only identity types (a minimum required to model homotopy theory in type theory). During the research period we were working on extending our approach to richer theories, including dependent products. Considerable progress during my visit at University of Western Ontario in August 2019. but some techinical challenges still need solution.

2 PhD supervision

During the performance period of the grant, the PI supervised the following PhD students, who completed their theses (completion date in brackets):

- Cesare Gallozzi, *Homotopy Type-Theoretic Interpretations of Constructive Set Theories* (2018), Gallozzi developed a new version of the type-theoretic interpretation of constructive set theory, originally due to Peter Aczel, using ideas of homotopy type theory. In particular, he obtained a refined analysis of the role played by the encoding of logic in validating various set-theoretic axioms. For this, he developed further work by the Peter Aczel and the PI on logic-enriched type theories. These results have appeared in a paper published in the journal *Mathematical Structures in Computer Science* [2].

- Marco Larrea, *Models of dependent type theory from algebraic weak factorisation systems* (2018). This work was discussed above and led to the publication of a joint paper with the PI [1]

- Jakob Vidmar, *Polynomial functors and W-types for groupoids* (2019). The PI and Vidmar have worked on extending the well-known groupoid model of type theory by Hofmann and Streicher to show the

validity of the rules for W-types. This work involved delicate calculations and further development of the theory of polynomial functors previously obtained by the PI and Joachim Kock. A paper is in planned.

- Raffael Stenzel, *On univalence, Rezk completeness and presentable quasi-categories* (2019). Stenzel worked on the problem of making precise the tantalising connection between Voevodsky's notion of univalence, introduced in the context of type theory, and Rezk's notion of completeness, introduced as part of his approach to ∞-categories using complete Segal spaces. Stenzel also worked on the very difficult problem of relating ∞-toposes and models of Homotopy Type Theory, obtaining significant partial results. Some papers have been written [4, 5].

Building on the activity generated by the grant and the PhD students, further students have been recruited (start date in brackets):

- Gabriele Lobbia (2018), working on topics in 3-dimensional category theory. In collaboration with him, the PI obtained a solution to an open question posed by Steve Lack, showing that pseudomonads in a Gray-category can be organised in a Gray-category. The results of this work have been published in the journal *Theory and Applications of Categories* [1].
- Matteo Spadetto (2020), working on Homotopy Type Theory, with a focus on weakenings of the rules of Martin-Löf type theory to match the idea of homotopy-invariance.
- Jack Romo (2020), jointly supervised with João Faria-Martins, working on relating ∞-dimensional and low-dimensional approaches to the cobordism hypothesis in Topological Quantum Field Theory.
- Luca Mesiti (2020), working on 2-dimensional category theory, with focus on proposed definitions of Grothendieck and elementary 2-toposes and a view towards applications to semantics of Homotopy Type Theory.
- Andrew Slattery (2020), working on 2-dimensional category theory, with a focus on the theory of 2-monads, pseudo-commutativity, with a view towards the development of 2-categorical commutative algebra and operad theory.

The group has now critical mass to sustain a weekly research meeting, promoting collaboration, interaction between students and dissemination of knowledge.

3 Conferences and seminars

The PI and the PDRAs gave several lectures at international conferences and seminars at research groups. The main ones are:

- The PI gave an invited seminar on the results in [1] at the Homotopy Type Theory Electronic Seminar in January 2019.
- The PI gave an invited lecture on [6] at the Mathematical Logic and Constructivity conference in Stockholm (August 2019).
- The PI gave an invited lecture on [3] at the inaugural meeting of the Italian Category Theory Seminar (June 2020) and a similar lecture at the the Coimbra Seminar on Algebra and Topology (October 2020).
- Karol Szumiło gave an invited lecture at the 2019 Homotopy Type Theory International Conference (Pittsburgh, PA, August 2019), where he reported on the progress of the project on internal languages of higher categories.He also gave similar talks at Yorkshire and Midlands Category Theory Seminar (YaMCATS, Sheffield, November 2018) and in seminars at University of Leeds (December 2018) and University of Cambridge (February 2019).
- Karol Szumiło presented the results of [8] at the 2019 Category Theory International Conference (Edinburgh, July 2019).
- Sina Hazratpour gave invited seminars on the collaborative work with Awodey and the PI at the Yorkshire and Midlands Category Theory Seminar and the Logic Seminar of the University of Amsterdam.

4 Other activities

In cooperation with Nima Rasekh, the PI and Karol Szumiło have organised Summer School on Higher Topos Theory and Univalent Foundations, which took place from June 24th to June 28th 2019 at University of Leeds. The school was addressed mainly to graduate students and its purpose was to introduce the participants to the two topics mentioned in the name and the relationship between them.

These topics originate from two, originally quite distant, areas of mathematics and are being unified by ongoing research, including our own. The event attracted participants with background in both areas and we believe it had a positive impact on interactions between them.

The school consisted of two lecture series and a number of talks, delivered by invited guests including some of the top experts in both areas: Peter LeFanu Lumsdaine, Charles Rezk, Benedikt Ahrens, André Joyal, Emily Riehl and Paige North. We have received very positive feedback from the participants.

The PI also served on the Scientific Committee of the International Conference on Homotopy Type Theory, held in Pittsburgh in August 2019, and is currently serving on the scientific committee of the International Conference in Category Theory 2021.

The PI also serves as an Associate Editor of the journal Mathematical Structures in Computer Science and on the editorial board of Applied Categorical Structures.

Publications

[1] Nicola Gambino and Marco Federico Larrea, *Models of Martin-Löf type theory from algebraic weak factorisation systems*, Journal of Symbolic Logic (To appear), available at `arXiv:1906.01491`.

[2] Cesare Gallozzi, *Homotopy type-theoretic interpretations of constructive set theories* (2019), available at `https://doi.org/10.1017/S0960129519000148`.

[3] Nicola Gambino and Gabriele Lobbia, *On the formal theory of pseudomonads and pseudodistributive laws*, Theory and Applications of Categories **37** (2021), no. 2, 14-56.

[4] Raffael Stenzel, *Univalence and completeness of Segal objects* (2019), available at `arXiv:1911.06640`.

[5] ———, *On the correspondence of object classifiers and Tarski universes* (2019), available at `arXiv:1911.01895`.

[6] Nicola Gambino and Simon Henry, *Towards a constructive simplicial model of Univalent Foundations* (2019), available at `https://arxiv.org/abs/1905.06281`.

[7] Nicola Gambino and Christian Sattler, *The Frobenius condition, right properness, and uniform fibrations,* Journal of Pure and Applied Algebra **221** (2021), no. 12, 3027-3068.

[8] Nicola Gambino, Christian Sattler, and Karol Szumiło, *The constructive Kan-Quillen model structure: two new proofs* (2019), available at `https://arxiv.org/abs/1907.05394`.

[9] Nicola Gambino, Christian Sattler, Karol Szumiło, and Simon Henry, *The effective model structure and ∞-groupoid objects* (2021), available at `https://arxiv.org/abs/2102.06146`.