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Exact Transient Solution of Maxwell Equations

by Michael Grinfeld and Pavel Grinfeld

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Exact Transient Solution of Maxwell Equations

Michael Grinfeld
Weapons and Materials Research Directorate,
DEVCOM Army Research Laboratory

Pavel Grinfeld
Drexel University

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14. ABSTRACT

In this report, we analyze a boundary value problem for the Maxwell equations, describing a transient process of establishing electrostatic equilibrium in an isotropic conducting plate. The solution is intended to be used for verification purposes when using the full magnetohydrodynamics code. To that end, various exact solutions of the Maxwell equations are required, and we establish one of those. The solution is representative when dealing with pulse loading problems. It clearly shows the importance of the displacement current concepts for analysis of transient processes in conductors.

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Contents

List of Figures

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1. Introduction

Consider a conductor with an initially uniform distribution of mobile charges (for instance, electrons). The uniform distribution of charges in a bounded conductor creates a nonzero electric field inside the conductor and thus generates electric currents. For the simplest models of a charged liquid, the current will continue until all the charges redistribute in such a way that the ultimate configuration of the charges will meet the following conditions: 1) there are no charges inside the conductor, 2) all charges are concentrated on the conductor's boundary, and 3) the final field inside the conductor vanishes. The transient processes include not only migration of charges, but also sophisticated evolutions of electric and magnetic fields. These processes can be analyzed based on the Maxwell model of electromagnetism. However, the system is, basically, too difficult for analytical treatment. Even when such a treatment is possible, the formal solutions become so complicated that they are even less transparent than the original equations. Therefore, the solutions should still be visualized by means of different asymptotic techniques and graphical tools. Fortunately, nowadays, computers make this procedure much easier. Moreover, different intermediate analytical treatment of the underlying mathematical models can be avoided almost completely. Still, a precise mathematical analysis remains unavoidable for several reasons, such as 1) qualitative analysis of the processes, 2) establishing general features of the solutions, 3) establishing asymptotic results, 4) verification and validation (V&V) of computer codes, and so on.

When dealing with V&V, researchers rely on the exact solutions of the underlying mathematical models in the simplest cases. Those cases should be relevant to the practical goals. When those goals change, the relevant simplest cases should be changed also.

In this report, we consider a 1-D nonstationary problem for the system of Maxwell equations permitting an exact solution. Namely, we consider an isotropic conducting plate, schematically shown in Fig. 1.

We use the Gauss system of units. Also, we neglect the permittivity and permeability of the conductor.

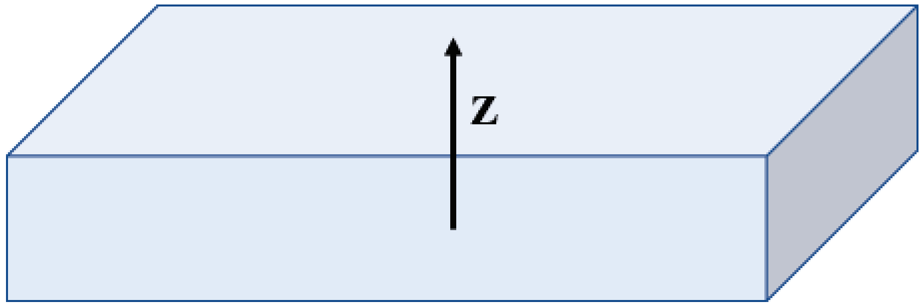

Fig. 1 Unbounded isotropic conducting plate

2. Formulation of the Problem

Let $Q(z,t)$ be the volumetric charge distribution of mobile electric charges. Let $I^i(z,t)$ be the electric current. Also, let E^i and H^i be the electric and magnetic fields.

For describing evolution of the fields, we postulate the Maxwell system

$$\nabla_i E^i = 4\pi Q\,, \tag{1}$$

$$\nabla_i H^i = 0\,, \tag{2}$$

$$\frac{\partial H^i}{\partial t} = -cz^{ijk}\nabla_j E_k\,, \tag{3}$$

and

$$\frac{\partial E^i}{\partial t} + 4\pi I^i = cz^{ijk}\nabla_j H_k\,. \tag{4}$$

The bulk master system, Eqs. 1–4, should be supplied with the charge conservation equation

$$\frac{\partial Q(z,t)}{\partial t} + \nabla_i I^i = 0\,. \tag{5}$$

We postulate the following simplest form of Ohm's law:

$$I^i = SE^i\,, \tag{6}$$

where S is an electric conductivity.

Inserting Eq. 6 for I^i in Eq. 5, we get the equation

$$\frac{\partial Q(z,t)}{\partial t} + S\nabla_i E^i = 0\,. \tag{7}$$

Now, eliminating $\nabla_i E^i$ between Eqs. 1 and 7, we get equations closed with respect to the charge density $Q(z,t)$

$$\frac{\partial Q(z,t)}{\partial t} + 4\pi S Q = 0 \,. \tag{8}$$

Given the initial distribution of charges $Q^\circ(z) \equiv Q(z,0)$, we can find distribution of charges $Q(z,t)$ for any $t > 0$. Indeed, multiplying Eq. 8 by $\exp(4\pi St)$, we can rewrite it as

$$\frac{\partial}{\partial t}(Q \exp(4\pi St)) = 0 \,. \tag{9}$$

Integrating Eq. 9 over interval $[0,t]$, we get

$$Q(z,t)e^{4\pi St} = Q(z,0) \tag{10}$$

and then

$$Q(z,t) = Q^\circ(z)e^{-4\pi St} \,. \tag{11}$$

(Compare this analysis with Landau and Lifshitz [1984].)

For the simplest model of electronic liquid, it is assumed infinitely compressible and sometimes called the "dust" model. Because of this feature, there appear boundary layers of charges with the density $\tau(\xi,t)$ per unit area of the boundary (where ξ^1, ξ^2 are the Gaussian coordinates at the boundary surfaces). Across the boundary surface, the electric field E^i experiences finite jump, satisfying the classical relationship

$$\left[E^i\right]_-^+ N_i = -4\pi\tau \,. \tag{12}$$

where N_i are components of the unit normal to the boundary.

Also, when the current $I^i(z,t)$ reaches the external boundaries and it still does not vanish, then the local charge density changes according to the charge conservation equation

$$\frac{\partial \tau(\xi,t)}{\partial t} = N_i I^i \,. \tag{13}$$

In Eq. 13, we consider the simplest situation, ignoring the surface flux of the 2D electric charges. In the following calculations of Section 3, we assume that the surface charge density vanishes at $t = 0$. Also, we notice that the surface charge density can both grow or decrease due to the bulk current.

3. One-Dimensional Solution of the Problem for the Plate Geometry

Let us apply Eq. 11 for the plate geometry, assuming that the initial charge distribution $Q^{\circ}(z) \equiv Q(z,0)$ depends only on the vertical coordinate Z. We look for the solution of the Maxwell system of equations and boundary that also depend on the single spatial coordinate Z. Moreover, we are looking for a solution with an identically vanishing magnetic field, i.e., $H^{i}(z,t) \equiv 0$. Next, we assume that the electric field has the only nonvanishing Z component for which we use the notation $E(Z,t)$. For the sake of simplicity, we consider the symmetric initial distribution $Q^{\circ}(Z) \equiv Q^{\circ}(-Z)$. Then, the same symmetry distribution will be true for any $t > 0$. Thus, we can limit ourselves considering the interval $[0,H]$ instead of $[-H,H]$, and use the boundary condition

$$E(0,t) = 0. \qquad (14)$$

Equation 11 implies

$$Q(Z,t) = Q^{\circ}(Z)e^{-4\pi St}. \qquad (15)$$

The bulk Eq. 1 implies

$$\frac{dE}{dZ} = 4\pi Q(Z,t). \qquad (16)$$

With the help of Eq. 15, we can rewrite Eq. 16 as follows:

$$\frac{\partial E(Z,t)}{\partial Z} = 4\pi Q^{\circ}(Z)e^{-4\pi St}. \qquad (17)$$

Integrating Eq. 17 over interval $(0,Z)$ and using boundary condition Eq. 14, we get

$$E(Z,t) = 4\pi e^{-4\pi St}\int_{0}^{Z} d\zeta Q^{\circ}(\zeta), \qquad (18)$$

for $Z < H$.

Outside of the plate, Eq. 16 should be replaced with Eq. 18 for the field $E(Z,t)$. We get

$$\frac{\partial E}{\partial Z} = 0 \quad . \tag{19}$$

for $Z > H$.

Equation 19 entails

$$E(Z,t) = F(t), \quad \text{for} \quad Z > H \quad . \tag{20}$$

Combining the Ohm's law Eq. 6 with relationship Eq. 18, we get

$$I(Z,t) = SE(Z,t) = 4\pi S e^{-4\pi S t} \int_0^Z d\zeta Q^\circ(\zeta), \;\; Z < H \quad . \tag{21}$$

Now, let us substitute $I(Z,t)$ from Eq. 21 into Eq. 13. In the 1D case, we get

$$\frac{d\tau}{dt} = I(H,t) = 4\pi S e^{-4\pi S t} \int_0^H d\zeta Q^\circ(\zeta) \quad . \tag{22}$$

Integrating Eq. 22 over the interval $[0,t]$ and using the initial condition $\tau(0) = 0$, we get

$$\tau(t) = (1 - e^{-4\pi S t}) \int_0^H d\zeta Q^\circ(\zeta) \quad . \tag{23}$$

Now, we turn to the jump condition, Eq. 12, of electric field, which can be rewritten as

$$E(H-0,t) - E(H+0,t) = -4\pi\tau(t) \quad . \tag{24}$$

In Eq. 24, $E(H-0,t)$ designates the limit value of the field $E(Z,t)$, when Z approaches H from inside the plate, whereas $E(H+0,t)$ designates the limit value of the field $E(Z,t)$ when Z approaches H from outside the plate. According to Eqs. 18 and 20, these limit values are equal to

$$\begin{aligned} E(H-0,t) &= 4\pi e^{-4\pi S t} \int_0^H d\zeta Q^\circ(\zeta), \\ E(H+0,t) &= F(t) \end{aligned} \quad . \tag{25}$$

Inserting the relationships of Eq. 25 into Eq. 24 and using Eq. 23 for the surface charge density, we arrive at the relationship

$$4\pi e^{-4\pi S t} \int_0^H d\zeta Q^\circ(\zeta) - F(t) = -4\pi(1 - e^{-4\pi S t}) \int_0^H d\zeta Q^\circ(\zeta) \quad . \tag{26}$$

Equation 26 implies

$$F(t) = 4\pi\int_0^H d\zeta Q^{\circ}(\zeta) = const\,. \tag{27}$$

4. Suggested Transient Solution from the Standpoint of the Maxwell Model

Having found the fields $E(z,t), I(z,t), \tau(t)$, can we claim that we found the exact solution to our problem in the framework of the Maxwell model of electromagnetism? Not yet. For such a claim, we have to verify that all the equations of the Maxwell model of electromagnetism, including the boundary conditions and initial data, are satisfied. It does not matter whether we used some of those equations explicitly or not. In particular, in our analysis we have not used the famous Eq. 4, containing the displacement current $\partial E^i / \partial t$. However, until we verify the validity of this equation, we cannot claim that our relations comprise the exact solution of the Maxwell model. Let us dwell on this equation since there are many models and applications that ignore the Maxwellian displacement current. Moreover, many outstanding thinkers, including Helmholtz and Kelvin, did not recognized the importance and correctness of this Maxwell invention. And nobody recognized Maxwell's model during his lifetime.

Our solution includes the relationship $H^i(z,t) = 0$, which still has the status of assumption, which also has to be verified in concert with solutions for $E(z,t), I(z,t)$ and $\tau(t)$. Let us begin with Eq. 1. Inserting Eqs. 14 and 18 into Eq. 1, we indeed see that Eq. 1 is satisfied. Then Eq. 2 is satisfied if $H^i(z,t) = 0$.

We leave to the readers the simple check that the rotation of our electric field vanishes, and this automatically implies the validity of Eq. 3 for our solution. Also, we leave to the readers the simple checks of validity for our solution of Eqs. 5, 6, 12, and 13. Of course, these checks are elementary since we established our solution explicitly using these equations.

The remainder of Eq. 4 in our 1D case with a vanishing electric field inside the plate reads as

$$\frac{\partial E}{\partial t} + 4\pi I = 0\ \cdot \tag{28}$$

In essence, we have not used this equation, but we still have to verify its validity for our solution. This can be done by substituting the formulae of Eqs. 18 and 21. Verifying Eq. 28 outside the plate, in view of Eq. 27, is straightforward.

To summarize, our simple solution is indeed the exact solution of the Maxwell model of electromagnetism.

5. The Role of the Displacement Current

Coming back to Eq. 28, we notice that the first term $\partial E / \partial t$ is the Maxwell's displacement current, whereas the second $-4\pi I$ is the conductive current. We see that these two components are exactly equal to each other. Depending on the substances, the bulk conductivity S can vary in the range of about 20 orders of magnitude. Then, both components of the current in Eq. 28 can change by 20 orders of magnitude. However, these two components of current are exactly equal to each other. Therefore, one cannot claim that in metals the conductivity current dominates over the displacement current. Depending on the particular problem, that statement can be true or false. For instance, for our exact solution the conductivity current is exactly equal to the displacement current regardless of the magnitude of conductivity.

So far, we analyzed the role of displacement current from a quantitative point of view. Now, let us analyze its role from a qualitative point of view. What if we just ignore the displacement current? The implications of such an action would be completely destructive. From a physical point of view, we would destroy the now classical electromagnetic theory of light. Our exact solution demonstrates that ignoring the displacement current is also destructive far from the theory of electromagnetic waves. For instance, ignoring the displacement current in Eq. 28 forces us to conclude that the convective current should also vanish. In other words, our elementary problem would not be solvable at all. This fact should be taken into account in magnetic hydrodynamics, in which neglecting the displacement current is the starting point of modeling (Landau and Lifshitz, 1984; Davidson, 2016).

From a purely mathematical point of view, neglecting the displacement current dramatically changes the global structure of the problem. The system of hyperbolic type becomes a problem of parabolic type with qualitatively different properties of solutions.

6. Conclusion

We analyzed a boundary value problem for the Maxwell equations, describing a transient process of establishing electrostatic equilibrium in an isotropic conducting plate. The solution is intended to be used for verification purposes when using the full magnetohydrodynamics code. To those purposes, various exact solutions of the Maxwell equations are required, and we established one of those. The solution is representative when dealing with pulse loading problems. It clearly shows the importance of the displacement current concepts for analyzing the transient processes in conductors.

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