CLASSICAL AND GENERALIZED SOLUTIONS OF FRACTIONAL STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. For stochastic evolution equations with fractional derivatives, classical solutions exist when the order of the time derivative of the unknown function is not too small compared to the order of the time derivative of the noise; otherwise, there can be a generalized solution in suitable weighted chaos spaces. Presence of fractional derivatives in time leads to various modifications of the stochastic parabolicity condition. Interesting new effects appear when the order of the time derivative in the noise term is less than or equal to one-half.

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1. Introduction

Given a $\beta \in (0,1)$, and a smooth function $f = f(t)$, $t > 0$, the two most popular definitions of the derivative of order β are Riemann-Liouville

$$D_t^{\beta} f(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-s)^{-\beta} f(s)\, ds$$

and Caputo

$$\tilde{\partial}_t^{\beta} f(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} f'(s)\big)\, ds;$$

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t}\, dt. \tag{1.1}$$

The Riemann-Liouville derivative can be considered a true extension of the usual derivative to fractional orders. For example, a function does not have to be continuously differentiable to have Riemann-Liouville derivatives of order $\beta < 1$ [20]. On the other hand, the Caputo derivative is more convenient in initial-value problems, with no need for fractional-order initial conditions [19, Section 2.4.1].

The Kochubei extension of the Caputo derivative,

$$\partial_t^{\beta} f(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-s)^{-\beta} \big(f(s) - f(0+)\big)\, ds,$$

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$f(0+) = \lim_{t\to 0, t>0} f(t)$, seems to achieve the right balance between mathematical utility and physical relevance [8] and has been recently used in the study of large classes of stochastic partial differential equations [4, 11].

Let $w = w(t)$, $t \geq 0$, be a standard Brownian motion on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. The objective of this paper is to address fundamental questions about existence and regularity of solution for equations of the type

$$\partial_t^{\beta} X(t) = aX(t) + \partial_t^{\gamma} \int_0^t \big(\sigma X(s) + g(s)\big)\, dw(s),\ t > 0,\quad a, b \in \mathbb{R}. \tag{1.2}$$

With a suitable choice of a and σ, (1.2) covers the time-fractional versions of the Ornstein-Uhlenbeck process and the geometric Brownian motion, as well as certain evolution equations in function spaces. The emphasis is on derivation and analysis of explicit formulas for the solution, as opposed to the development of general theory.

Given the vast literature on the subject of fractional differential equations, Section 2 provides the necessary background, to make the presentation reasonably self-contained. Section 3 investigates the equation with $a = \sigma = 0$, corresponding to fractional derivatives or integrals of the Brownian motion, followed by the time fractional Ornstein-Uhlenbeck process ($\sigma = 0$) in Section 4 and the geometric Brownian motion in Section 5. Along the way we understand the origins of the condition $\beta - \gamma > -1/2$ at a more basic level than in [4, 11] and discover that the fractional in time Ornstein-Uhlenbeck process can exhibit the full range of sub-diffusive behaviors, including super-slow logarithmic. Section 6 investigates an SPDE version of (1.2) by replacing the numbers a, σ with fractional powers of the Laplace operator. Then the results of the previous sections lead to several versions of the stochastic parabolicity condition.

Throughout the paper, $\mathcal{C}(G)$ denotes the space of real-valued continuous functions on G and $\mathcal{C}_{loc}(G)$ is the space of functions that are continuous on every compact sub-set of G; $\Gamma = \Gamma(z)$, is the Gamma function, defined for all complex z except for the poles at $0, -1, -2, \ldots$ and, for z in the right half-plane, having the representation (1.1). Most of other notations, such as $\mathcal{L}[\cdot]$ and $\mathcal{E}$ for the Laplace transform and its domain, and $E_{\beta,\rho}$ for the two-parameter Mittag-Leffler function, are introduced in Section 2.

2. Background

2.1. **Fractional Derivatives and Integrals.** In this section we do not indicate the time variable as a subscript in the notations of the derivatives: for $\beta \in (0, 1)$,

$$D^{\beta} f(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-s)^{-\beta} f(s)\, ds, \tag{2.1}$$

$$\partial^{\beta} f(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-s)^{-\beta} \big(f(s) - f(0)\big)\, ds. \tag{2.2}$$

We also introduce the corresponding fractional integrals: for $p > 0$,

$$I^p f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s)\, ds, \tag{2.3}$$

$$J^p f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \big(f(s) - f(0+)\big)\, ds, \tag{2.4}$$

where $f(0+) = \lim_{t\to 0} f(t)$. In what follows, with all functions defined only for $t > 0$, we write $f(0)$ instead of $f(0+)$ By convention, $I^0 f(t) = f(t),\ J^0 f = f(t) - f(0)$. In particular, for the constant function $f(t) = 1$, $t \geq 0$,

$$I^p[1](t) = \frac{t^p}{\Gamma(p+1)}, \quad J^p[1](t) = 0, \quad D^\beta[1](t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}, \quad \partial^\beta[1](t) = 0. \tag{2.5}$$

Formulas (2.1)–(2.4) imply

$$D^\beta f(t) = \frac{d}{dt} I^{1-\beta} f(t), \tag{2.6}$$

$$\partial^\beta f(t) = \frac{d}{dt} J^{1-\beta} f(t). \tag{2.7}$$

Proposition 2.1. *For all $p, q > 0$,*

$$I^p(I^q f) = I^{p+q} f,\ \ J^p(J^q f) = J^{p+q} f. \tag{2.8}$$

Proof. For I,

$$\begin{aligned} \Gamma(p)\Gamma(q) I^p(I^q f)(t) &= \int_0^t \int_0^s (t-s)^{p-1}(s-r)^{q-1} f(r)\, dr\, ds \\ &= \int_0^t \left(\int_r^t (t-s)^{p-1}(s-r)^{q-1}\, ds \right) f(r)\, dr \\ &= \mathrm{B}(p,q) \int_0^t (t-r)^{p+q-1} f(r) dr = \Gamma(p)\Gamma(q) I^{p+q} f(t). \end{aligned}$$

The same computation works for J after noticing that if $p > 0$ and $f(t)$ is bounded near 0, then

$$\lim_{t\to 0+} |J^p f(t)| \leq C \lim_{t\to 0+} t^p = 0. \tag{2.9}$$

□

Proposition 2.2. *If $f \in \mathcal{C}_{loc}\big([0,+\infty)\big)$, then*

$$\partial^\beta I^\beta f(t) = D^\beta I^\beta f(t) = f(t), \qquad \partial^\beta J^\beta f(t) = f(t) - f(0). \tag{2.10}$$

Proof. Using (2.6), (2.7), (2.8) and keeping in mind that, similar to (2.9), $I^\beta f(0) = 0$, the result follows from the fundamental theorem of calculus:

$$\partial^\beta I^\beta f(t) = D^\beta I^\beta f(t) = \frac{d}{dt}\big(I^{1-\beta} I^\beta f\big)(t) \frac{d}{dt}\big(I f\big)(t) = \frac{d}{dt} \int_0^t f(s) ds = f(t).$$

Similarly,

$$\partial^\beta J^\beta f(t) = \frac{d}{dt}\big(J^{1-\beta} J^\beta f\big)(t) = \frac{d}{dt}\big(J f\big)(t) = \frac{d}{dt} \int_0^t \big(f(s) - f(0)\big) ds = f(t) - f(0).$$

□

Proposition 2.3. *If*

$$f(t) = f(0) + \int_0^t f'(s)\,ds, \tag{2.11}$$

then

$$J^p f = I^{1+p} f', \tag{2.12}$$

$$\partial^\beta f = I^{1-\beta} f'. \tag{2.13}$$

Proof. For (2.12), integrate by parts:

$$p\Gamma(p) J^p f(t) = \int_0^t \left(-\frac{\partial}{\partial s}(t-s)^p\right)\left(\int_0^s f'(r)\,dr\right) ds$$

$$= (t-s)^p \left(\int_0^s f'(r)\,dr\right)\Bigg|_{s=0}^{s=t} + \int_0^t (t-s)^p f'(s)\,ds = \Gamma(1+p) I^{1+p} f'(t),$$

and remember that $p\Gamma(p) = \Gamma(1+p)$.

For (2.13), differentiate (2.12) taking $p = 1-\beta$. □

Corollary 2.4. *If* (2.11) *holds, then*

$$I^\beta \partial^\beta f(t) = f(t) - f(0). \tag{2.14}$$

Proof. By (2.8) and (2.13),

$$I^\beta \partial^\beta f(t) = I^\beta I^{1-\beta} f'(t) = I f'(t) = \int_0^t f'(s) ds = f(t) - f(0).$$

□

2.2. **The Laplace Transform.** Recall that

$$f = f(t) \mapsto \mathcal{L}[f](\lambda) = \int_0^{+\infty} f(t) e^{-\lambda t}\,dt \tag{2.15}$$

is a one-to-one mapping defined on

$$\mathcal{E} = \Big\{ f \in L_{1,loc}((0,+\infty)) : \sup_{t>0} |f(t)| e^{ct} < \infty \ \text{ for some } \ c \in \mathbb{R} \Big\}. \tag{2.16}$$

We will use the following properties of the Laplace transform:

$$\mathcal{L}[If](\lambda) = \lambda^{-1} \mathcal{L}[f](\lambda); \tag{2.17}$$

$$f' \in \mathcal{E} \ \Rightarrow \ \mathcal{L}[f'](\lambda) = \lambda \mathcal{L}[f] - f(0); \tag{2.18}$$

$$h(t) = \int_0^t f(t-s) g(s)\,ds \ \Rightarrow \ \mathcal{L}[h](\lambda) = \mathcal{L}[f](\lambda)\,\mathcal{L}[g](\lambda); \tag{2.19}$$

$$f(t) = \frac{t^{\gamma-1}}{\Gamma(\gamma)}, \ \gamma > 0 \ \Rightarrow \ \mathcal{L}[f](\lambda) = \lambda^\gamma. \tag{2.20}$$

We now establish fractional versions of (2.17) and (2.18).

Proposition 2.5. *If* $f \in \mathcal{E} \cap \mathcal{C}_{loc}\big([0,+\infty)\big)$ *and* $\beta \in (0,1)$, *then*

$$\mathcal{L}[I^p f](\lambda) = \lambda^{-p}\mathcal{L}[f](\lambda), \tag{2.21}$$

$$\mathcal{L}[D^\beta f](\lambda) = \lambda^{\beta}\mathcal{L}[f](\lambda), \tag{2.22}$$

$$\mathcal{L}[J^p f](\lambda) = \lambda^{-p}\mathcal{L}[f](\lambda) - \lambda^{-p-1}f(0), \tag{2.23}$$

$$\mathcal{L}[\partial^\beta f](\lambda) = \lambda^{\beta}\mathcal{L}[f](\lambda) - \lambda^{\beta-1}f(0). \tag{2.24}$$

Proof. Equality (2.21) is an immediate consequence of (2.19), and then (2.22) follows from (2.18) and (2.6). To establish (2.23), we write

$$\begin{aligned}
\Gamma(p)\mathcal{L}[J^p f](\lambda) &= \int_0^{+\infty}\int_0^t (t-s)^{p-1}\big(f(s)-f(0)\big)e^{-\lambda t}\,dt \\
&= \int_0^{+\infty}\left(\int_s^{+\infty}(t-s)^{p-1}e^{-\lambda t}\,dt\right)\big(f(s)-f(0)\big)\,ds \\
&= \int_0^{+\infty}\left(\int_0^{+\infty}u^{p-1}e^{-\lambda u}\,dt\right)e^{-\lambda s}\big(f(s)-f(0)\big)\,ds \\
&= \Gamma(p)\Big(\lambda^p\mathcal{L}[f](\lambda) - \lambda^{p-1}f(0)\Big).
\end{aligned}$$

Then (2.7) and (2.18) imply (2.24). □

Next, we compute the Laplace transform of the standard Brownian motion. Define

$$\hat{w}(\lambda) = \int_0^{+\infty} e^{-\lambda t}\,dw(t); \tag{2.25}$$

for every $\lambda > 0$, the random variable $\hat{w}(\lambda)$ is Gaussian with mean zero and variance $1/(2\lambda)$. Then the stochastic Fubini theorem shows that

$$\mathcal{L}[w](\lambda) = \frac{\hat{w}(\lambda)}{\lambda}. \tag{2.26}$$

2.3. **The two parameter Mittag-Leffler function.** The function is defined by the power series

$$E_{\beta,\rho}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta k+\rho)}, \tag{2.27}$$

and, in a sense, is the fractional version of the exponential function. If $\beta > 0$, then, with the convention $1/\Gamma(-n) = 0$, $n = 0,1,2,\ldots$, the series on the right-hand side of (2.27) converges for all z and ρ. The particular case $\rho = 1$ is

$$E_{\beta,1}(z) := E_\beta(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta k+1)}. \tag{2.28}$$

Note that $E_0 = 1/(1-z),\ |z| \le 1$, $E_1(z) = e^z$, and $E_2(z) = \cosh(\sqrt{z})$.

Proposition 2.6 (The fractional Gronwall-Bellman inequality)**.** *If*

$$y(t) \le A(t) + B\int_0^t (t-s)^{\beta-1}y(s)\,ds,$$

where $y(t)\geq 0$, $A(t)\geq 0$ is non-decreasing, $B>0$, $\beta>0$, then

$$y(t)\leq A(t)E_{\beta}\big(B\Gamma(\beta)t^{\beta}\big).$$

Proof. See [23, Corollary 2]. □

In general, $E_{\beta,\rho}$ cannot be expressed using elementary functions, but, for many purposes, the following results suffice.

Proposition 2.7. *Let $\beta\in(0,1)$ and $\rho>0$.*

(1) *There exist numbers $C_1, C_2>0$ such that, for all $t>0$,*

$$|E_{\beta,\rho}(t)|\leq C_1(1+t)^{(1-\rho)/\beta}e^{t^{1/\beta}}+\frac{C_2}{1+t}. \tag{2.29}$$

(2) *There exists a number C so that, for all $t>0$,*

$$|E_{\beta,p}(-t)|\leq \frac{C}{1+t}. \tag{2.30}$$

(3) *Moreover, if $\rho\in(0,1)$, then*

$$\lim_{t\to+\infty} tE_{\beta,\rho}(-t)=\frac{1}{\Gamma(\rho-\beta)},\ \ \beta\neq\rho; \tag{2.31}$$

$$\lim_{t\to+\infty} t^2E_{\beta,\beta}(-t)=-\frac{1}{\Gamma(-\beta)}. \tag{2.32}$$

Proof. See [19, Theorem 1.5], [19, Theorem 1.6], and [19, Theorem 1.4], respectively. □

Next, define

$$y_{\beta,\rho}(t)=t^{\rho-1}E_{\beta,\rho}(at^{\beta}).$$

Proposition 2.8. *For every $a\in\mathbb{R}$, the family of functions $y_{\beta,\rho}$, $\beta\in(0,1)$, $\rho>0$, has the following properties:*

$$\mathcal{L}[y_{\beta,\rho}](\lambda)=\frac{\lambda^{\beta-\rho}}{\lambda^{\beta}-a}; \tag{2.33}$$

$$\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}y_{\beta,\rho}(s)\,ds=y_{\beta,\rho+\gamma}(t),\ \ \gamma>0; \tag{2.34}$$

$$D^{\gamma}y_{\beta,\rho}(t)=y_{\beta,\rho-\gamma}(t),\ \ \gamma\in(0,1),\ \ \rho>\gamma. \tag{2.35}$$

Proof. By (2.29), $y_{\beta,\rho}\in\mathcal{E}$ for all $\beta\in(0,1)$, $\rho>0$ and $a\in\mathbb{R}$. Then

$$\mathcal{L}[y_{\beta,\rho}](\lambda)=\sum_{k\geq 0}\int_0^{+\infty}\frac{a^k t^{k\beta+\rho-1}}{\Gamma(k\beta+\rho)}\,e^{-\lambda t}\,dt=\lambda^{-\rho}\sum_{k\geq 0}(a\lambda^{-\beta})^k$$
$$=\frac{\lambda^{-\rho}}{1-a\lambda^{-\beta}}=\frac{\lambda^{\beta-\rho}}{\lambda^{\beta}-a},$$

proving (2.33). Then, with (2.20) in mind, (2.34) and (2.35) follow from (2.21) and (2.22), respectively.

□

2.4. **Time fractional linear deterministic equations.** Consider the equation

$$\partial^{\beta} y(t) = ay(t) + f(t),\ t > 0,\ y(0) = y_0, \tag{2.36}$$

with $\beta \in (0,1),\ a \in \mathbb{R},\ f \in \mathcal{E} \cap \mathcal{C}_{loc}\big([0,+\infty)\big)$.

Definition 2.9. *A function $y \in \mathcal{C}_{loc}\big([0,+\infty)\big)$ is called a classical solution of* (2.36) *if*

$$J^{1-\beta} y(t) = \int_0^t \big(ay(s) + f(s)\big)\, ds,\ t > 0. \tag{2.37}$$

The following result is the analogue of [19, Example 4.3], where the Riemann-Liouville derivative is considered.

Theorem 2.10. *The unique solution of* (2.36) *in $\mathcal{E}$ is*

$$y(t) = y_0 E_{\beta}(at^{\beta}) + \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(a(t-s)^{\beta}\big) f(s)\, ds. \tag{2.38}$$

Proof. Take the Laplace transform on both sides of (2.37) and use (2.23):

$$\lambda^{\beta-1}\mathcal{L}[y](\lambda) - \lambda^{\beta-2} y_0 = \lambda^{-1}\Big(a\mathcal{L}[y](\lambda) + \mathcal{L}[f](\lambda)\Big),$$

or

$$\mathcal{L}[y](\lambda) = \frac{\lambda^{\beta-1}}{\lambda^{\beta}-a}\, y_0 + \frac{\mathcal{L}[f](\lambda)}{\lambda^{\beta}-a}.$$

The conclusion of the theorem now follows from (2.19), (2.33), and uniqueness of the Laplace transform on $\mathcal{E}$. □

2.5. **Chaos Expansion and Generalized Processes.** Below is a summary of the construction of the weighted chaos spaces; for details, see [12, 13, 16].

Introduce the following objects:

$\{\mathfrak{m}_k = \mathfrak{m}_k(t),\ t \in [0,T]\}$, an orthonormal basis in $L_2((0,T))$,

$$\mathcal{J} = \left\{ \boldsymbol{\alpha} = (\alpha_k,\ k \geq 1) : \alpha_k \in \{0,1,2,\ldots\},\ |\boldsymbol{\alpha}| := \sum_k \alpha_k < \infty \right\},$$

$$\xi_{\boldsymbol{\alpha}} = \prod_k \left(\frac{\mathrm{H}_{\alpha_k}(\xi_k)}{\sqrt{\alpha_k!}} \right),\ \mathrm{H}_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2},\ \xi_k = \int_0^T \mathfrak{m}_k(t)\, dw(t).$$

If $\eta \in L_2^W(\Omega)$, that is, a square-integrable functional of $w(t),\ t \in [0,T]$, then, by the Cameron-Martin theorem [3],

$$\eta = \sum_{\boldsymbol{\alpha} \in \mathcal{J}} \mathbb{E}(\eta \xi_{\boldsymbol{\alpha}})\, \xi_{\boldsymbol{\alpha}},\quad \mathbb{E}\eta^2 = \sum_{\boldsymbol{\alpha} \in \mathcal{J}} \Big|\mathbb{E}(\eta \xi_{\boldsymbol{\alpha}})\Big|^2;$$

see also [15, Theorem 5.1.12]. For example,

$$w(t) = \sum_{k\geq 1} \xi_k \, I\mathfrak{m}_k(t) = \sum_{k\geq 1} \xi_k \left(\int_0^t \mathfrak{m}_k(s)\, ds \right). \tag{2.39}$$

Let $\mathfrak{q} = \{q_k,\ k \geq 1\}$ be a sequence of positive numbers. We write

$$\mathfrak{q}^{\boldsymbol{\alpha}} := \prod_{k\geq 1} q_k^{\alpha_k}.$$

Definition 2.11. *Let $\mathfrak{q} = \{q_k,\ k \geq 1\}$ be a sequence such that $0 < q_k < 1$ for all k. The space $L_{2,\mathfrak{q}}\big((0,T)\big)$ is the closure of $L_2^W\big(\Omega; L_2(0,T)\big)$ with respect to the norm*

$$\|X\|_{2,\mathfrak{q}} = \left(\sum_{\boldsymbol{\alpha}\in\mathcal{J}} \big\|\mathbb{E}(X\xi_{\boldsymbol{\alpha}})\big\|^2_{L_2((0,T))} \right)^{1/2}.$$

An element of $L_{2,\mathfrak{q}}\big((0,T)\big)$ is represented by an expression of the form

$$X(t) = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} x_{\boldsymbol{\alpha}}(t)\xi_{\boldsymbol{\alpha}}$$

with non-random $x_{\boldsymbol{\alpha}} \in L_2((0,T))$ satisfying

$$\sum_{\boldsymbol{\alpha}\in\mathcal{J}} \mathfrak{q}^{\boldsymbol{\alpha}} \, \|x_{\boldsymbol{\alpha}}\|^2_{L_2((0,T))} < \infty,$$

and is called a `q-generalized process`.

For example, the white noise process

$$\dot{w}(t) = \sum_{k\geq 1} \xi_k \mathfrak{m}_k(t)$$

is a $\mathfrak{q}$-generalized process for every $\mathfrak{q}$ satisfying

$$\sum_{k\geq 1} q_k < \infty. \tag{2.40}$$

3. Fractional Derivatives of the Brownian Motion

Proposition 3.1. *If $\gamma \in (0,1)$, then*

$$\int_0^t (t-s)^{-\gamma} w(s)\, ds = \frac{1}{1-\gamma} \int_0^t (t-s)^{1-\gamma}\, dw(s). \tag{3.1}$$

Proof. Integrate by parts on the right-hand side of (3.1). □

Corollary 3.2. *Given $\gamma \in (0,1)$, define*

$$W_\gamma(t) = \frac{1}{(1-\gamma)\Gamma(1-\gamma)} \int_0^t (t-s)^{1-\gamma}\, dw(s),\ t > 0. \tag{3.2}$$

Then

$$J^{1-\gamma} w(t) = W_\gamma(t) = I^{1-\gamma} w(t), \tag{3.3}$$

and

$$\partial^{\gamma} w(t) = \frac{d}{dt} W_{\gamma}(t). \tag{3.4}$$

Proof. Equality (3.3) follows from (3.1), keeping in mind that $w(0) = 0$. After that, (2.7) implies (3.4). □

Definition 3.3. *A process $V = V(t)$, $t \in [0, T]$, is called a* `Gaussian Volterra process` *with kernel $K = K(t, s)$ if there exists a non-random function $K = K(t, s)$ such that $K(t, s) = 0, s > t$, $K \in L_2((0, T)^2)$, and*

$$\mathbb{P}\Big(V(t) = \int_0^t K(t, s)\, dw(s),\ t \in [0, T]\Big) = 1.$$

Proposition 3.4. *If*

$$\gamma \in (0, 1/2), \tag{3.5}$$

then $\partial_t^{\gamma} w$ is a Gaussian Volterra process with representation

$$\partial_t^{\gamma} w(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma}\, dw(s). \tag{3.6}$$

Proof. Similar to (2.9),

$$\lim_{t \to 0+} \int_0^t (t-s)^{-\gamma} w(s)\, ds = 0$$

with probability one. Therefore, it is enough to show that

$$\int_0^t \left(\int_0^s (s-r)^{-\gamma}\, dw(r) \right) ds = \frac{1}{1-\gamma} \int_0^t (t-r)^{1-\gamma}\, dw(r),$$

which follows by the stochastic Fubini theorem; condition (3.5) is necessary for the application of the stochastic Fubini theorem. □

Next, for $\beta, \gamma \in (0, 1)$, consider the equation

$$\partial_t^{\beta} X(t) = \partial_t^{\gamma} w(t),\ t > 0,\ X(0) = X_0, \tag{3.7}$$

Using (2.7), equation (3.7) becomes

$$\frac{d}{dt} J^{1-\beta} X(t) = \frac{d}{dt} J^{1-\gamma} w(t). \tag{3.8}$$

Together with (2.9), (3.8) implies that equation (3.7) should be interpreted as the integral equation

$$J^{1-\beta} X(t) = J^{1-\gamma} w(t). \tag{3.9}$$

Definition 3.5. *A classical solution of* (3.7) *on $[0, T]$ is a continuous process $X = X(t)$ such that*

$$\mathbb{P}\Big(J^{1-\beta} X(t) = J^{1-\gamma} w(t),\ t \in [0, T]\Big) = 1.$$

Theorem 3.6. *If*

$$\beta - \gamma > -\frac{1}{2}, \tag{3.10}$$

then

$$X(t) = X_0 + \frac{1}{\Gamma(1+\beta-\gamma)} \int_0^t (t-s)^{\beta-\gamma} dw(s) \tag{3.11}$$

is the unique classical solution of (3.7).

Proof. Apply $\partial_t^{1-\beta}$ to both sides of (3.9) and use (2.10), (2.8), and (2.7):

$$\begin{aligned} X(t) - X_0 &= \partial_t^{1-\beta} J^{1-\gamma} w(t) = \frac{d}{dt} J^{\beta} J^{1-\gamma} w(t) = \frac{d}{dt} J^{1+\beta-\gamma} w(t) \\ &= \frac{d}{dt} J^{1-(\gamma-\beta)} w(t); \end{aligned} \tag{3.12}$$

note that $1+\beta-\gamma > 0$ for all $\beta, \gamma \in (0,1)$. If $\gamma - \beta > 0$, then

$$\frac{d}{dt} J^{1-(\gamma-\beta)} w(t) = \partial_t^{\gamma-\beta} w(t), \tag{3.13}$$

and, under condition (3.10), equality (3.11) follows by Proposition 3.4.

If $\gamma - \beta \le 0$, then the function $t \mapsto J^{1-(\gamma-\beta)} w(t)$ is continuously differentiable in t:

$$J^{1-(\gamma-\beta)} w(t) = \frac{1}{\Gamma(1+|\gamma-\beta|)} \int_0^t (t-s)^{|\gamma-\beta|} w(s) ds$$

so that

$$\frac{d}{dt} J^{1-(\gamma-\beta)} w(t) = \frac{1}{|\gamma-\beta|\Gamma(1+|\gamma-\beta|)} \int_0^t (t-s)^{|\gamma-\beta|-1} w(s) ds$$

and (3.11) follows after integration by parts. □

We now make the following observations;

- In an ordinary differential equation, $\beta = \gamma = 1$, so that (3.10) holds.
- For every $t > 0$, $X(t)$ is a Gaussian random variable with variance

  $$\sigma^2(t) \propto \int_0^t s^{2(\beta-\gamma)} ds \propto t^{2(\beta-\gamma)+1};$$

  $f \propto g$ means f is proportional to g. The solution of (3.7) can thus exhibit the anomalous diffusion behavior $\sigma^2(t) \propto t^{\alpha}$ for all $\alpha \in (0,3)$ [6, Section 1.2]; regular diffusion $\sigma^2(t) \propto t$ corresponds to $\beta = \gamma$.
- The last equality in (3.12) suggests that the solution of (3.7) can be written as

  $$X(t) - X(0) = \partial_t^{\gamma-\beta} w(t), \tag{3.14}$$

  which makes perfect sense, but requires a justification. In particular, the proof of Theorem 3.6 shows that (3.10) is necessary for the right-hand side of (3.14) to define a continuous process, which in this case is a Gaussian Volterra process.

- Taking the Laplace transform on both sides of (3.7), with (2.26) in mind, results in
$$\lambda^{\beta}\mathcal{L}[X](\lambda) - \lambda^{\beta-1}X_0 = \lambda^{\gamma-1}\hat{w}(\lambda) \tag{3.15}$$
or
$$\mathcal{L}[X](\lambda) = \lambda^{-1}X_0 + \lambda^{\gamma-\beta-1}\hat{w}(\lambda), \tag{3.16}$$
which is consistent with (3.11).
- Condition (3.10) is standard in the study of fractional stochastic evolution equations [4, 11].

If condition (3.10) fails, so that $\gamma-\beta \geq 1/2$, then the solution of (3.7), as defined by (3.9) or (3.16), is a generalized process, best described using weighted chaos spaces.

Theorem 3.7. *If*
$$\mathfrak{m}_1(t) = \frac{1}{\sqrt{T}},\ \mathfrak{m}_k(t) = \sqrt{\frac{2}{T}}\cos\left(\frac{\pi t(k-1)}{T}\right),\ k\geq 2, \tag{3.17}$$
and
$$\gamma-\beta \geq \frac{1}{2}, \tag{3.18}$$
then (3.9) *defines a* $\mathfrak{q}$*-generalized process for every* $\mathfrak{q}$ *satisfying*
$$\sum_k k^{2(\gamma-\beta-1)}q_k < \infty. \tag{3.19}$$

Proof. Using (2.39), equalities (3.9) and (3.12) lead to
$$X(t) = X_0 + \sum_k \xi_k I^{1-(\gamma-\beta)}\mathfrak{m}_k(t). \tag{3.20}$$
By direct computation,
$$\left|\int_0^t (t-s)^{-(\gamma-\beta)}\cos(ks)\,ds\right| \leq Ck^{\gamma-\beta-1}, \tag{3.21}$$
cf. [1, Example 6.6.1], and then (3.19) follows. □

Note that

- As a quick consistency check, the extreme case $\beta=0, \gamma=1$ in (3.7) corresponds to $X=\dot{w}$, and then (3.19) becomes (2.40).
- The key step in the proof of Theorem 3.7 is asymptotic analysis, as $k\to\infty$, of
$$\int_0^t (t-s)^{-\kappa}\mathfrak{m}_k(s)ds,\ \kappa\in(0,1),$$
which is made possible by assumption (3.17). The $k^{-1+\kappa}$-asymptotic holds for other trigonometric basis, and, while it might not hold in general, the main constructions related to the chaos expansion, including the definition of the $\mathfrak{q}$-generalized processes, are intrinsic and do not depend on the choice of the bases, either in $L_2((0,T))$ or in $L_2^W(\Omega)$; see [9, 12].
- Equality (3.20) also holds under (3.10).

Remark 3.8. *With obvious modifications, the results of this section extend to equations of the type*

$$\partial_t^\beta X(t) = \partial_t^\gamma \int_0^t g(s)\,dw(s),$$

where $g \in L_2((0,T))$. *Under condition* (3.10), *the function* g *can be random as long as* g *is* $\mathcal{F}_t$*-adapted and*

$$\mathbb{E}\int_0^T g^2(t)\,dt < \infty.$$

4. Time Fractional Ornstein-Uhlenbeck Process

4.1. **Derivation of the equation.** Consider the harmonic oscillator

$$m\ddot{x}(t) + c^2 x(t) = F(t), \tag{4.1}$$

with a slight twist that the restoring force $-c^2 x$ does not depend on the mass m.

The force F has two components, damping F_d and external F_e:

$$F(t) = F_d(t) + F_e(t).$$

Traditional damping is

$$F_d(t) = -c_d \dot{x}(t),\ c_d > 0.$$

Instead, we assume that F_d has memory:

$$F_d(t) = -\int_0^t f_d(t-s)\dot{x}(s)\,ds. \tag{4.2}$$

A possible choice of the memory kernel in (4.2) is

$$f_d(t) = At^{-\beta},\ A > 0,\ \beta \in (0,1),$$

which corresponds to a continuous time random walk model with a heavy-tailed jump time distribution; cf. [17, Section 2.4].

If we also assume that the external force is the fractional derivative of the standard Bronwian motion,

$$F_e = \partial_t^\gamma w(t),\ \gamma \in (0,1),$$

then, with (2.13) in mind, equation (4.1) becomes

$$m\ddot{x}(t) + b\partial_t^\beta x(t) + c^2 x(t) = \partial_t^\gamma w(t). \tag{4.3}$$

Now take the Laplace transform on both sides of (4.3), with (2.26) in mind. Assuming zero initial conditions, the result is

$$\left(m\lambda^2 + b\lambda^\beta + c^2\right)\mathcal{L}[x](\lambda) = \lambda^{\gamma-1}\hat{w}(\lambda) \tag{4.4}$$

or

$$\mathcal{L}[x](\lambda) = \frac{\lambda^{\gamma-1}\hat{w}(\lambda)}{m\lambda^2 + b\lambda^\beta + c^2}. \tag{4.5}$$

Finally, we pass to the limit $m \to 0$ in (4.5); this procedure is known as the Smoluchowski-Kramers approximation [5]. With X denoting the corresponding limit of x, the result is

$$\mathcal{L}[X](\lambda) = \frac{\lambda^{\gamma-1}\hat{w}(\lambda)}{b\lambda^{\beta} + c^2}, \tag{4.6}$$

which, back in the time domain, and with re-scaled constants, becomes the equation describing the *time fractional Ornstein-Uhlenbeck process*:

$$\partial_t^{\beta} X(t) = -aX(t) + \partial_t^{\gamma} w(t), \ a > 0. \tag{4.7}$$

4.2. **Solution and its long-time behavior.** Similar to Definition 3.5, we say that a continuous process $X = X(t)$ is a classical solution of (4.7) on $[0, T]$ if

$$\mathbb{P}\left(J^{1-\beta} X(t) = -a \int_0^t X(s)\, ds + J^{1-\gamma} w(t), \ t \in [0, T] \right) = 1.$$

Theorem 4.1. *If* (3.10) *holds, then, for every $a \in \mathbb{R}$, $X_0 \in \mathbb{R}$, and $T > 0$, equation* (4.7) *has a unique solution in the class $\mathcal{E}$ and*

$$X(t) = X_0 E_{\beta}(-at^{\beta}) + \int_0^t (t-s)^{\beta-\gamma} E_{\beta,\beta-\gamma+1}\big(-a(t-s)^{\beta}\big)\, dw(s). \tag{4.8}$$

Proof. Take the Laplace transform on both sides of (4.7):

$$\mathcal{L}[X](\lambda) = \frac{\lambda^{\beta-1}}{\lambda^{\beta} + a} X_0 + \frac{\lambda^{\gamma-1}}{\lambda^{\beta} + a} \hat{w}(\lambda). \tag{4.9}$$

If $\beta - \gamma > -1/2$, that is, (3.10) holds, then inverting (4.9) yields (4.8). □

Equality (4.8) implies that, for every $t > 0$, $X(t)$ is a Gaussian random variable with mean

$$\mu(t) = X_0 E_{\beta}(-at^{\beta})$$

and variance

$$\sigma^2(t) = \int_0^t s^{2(\beta-\gamma)} E^2_{\beta,\beta-\gamma+1}(-as^{\beta})\, ds. \tag{4.10}$$

By (2.30),

$$\lim_{t\to+\infty} \mu(t) = 0.$$

To study $\sigma^2(t)$, we use (2.31) with $\rho = \beta - \gamma + 1$:

$$\lim_{t\to+\infty} tE_{\beta,\beta-\gamma+1}(-t) = \frac{1}{\Gamma(1-\gamma)}. \tag{4.11}$$

Note that (3.10) is necessary and sufficient for the convergence of (4.10) at zero. On the other hand, (4.11) implies that, depending on the values of γ, the integral in (4.10) can either converge or diverge at infinity:

$$s^{2(\beta-\gamma)} E^2_{\beta,\beta-\gamma+1}(-as^{\beta}) \sim s^{-2\gamma}, \ s \to +\infty. \tag{4.12}$$

Using (4.12), as well as the arguments similar to the proof of Theorem 3.7, we get the following characterization of the solution of (4.7) for all values of $\beta, \gamma \in (0, 1]$.

Theorem 4.2. (1) *If $\beta-\gamma\leq -1/2$, then X is a $\mathfrak{q}$-generalized process for every $\mathfrak{q}$ satisfying* (3.19)*;*

(2) *If $\beta-\gamma>-1/2$, then X is a Gaussian Volterra process* (4.8) *and*
- *If $\gamma>1/2$, then, as $t\to+\infty$, $X(t)$ converges in distribution to a Gaussian random variable with mean zero and variance*
$$\sigma^2_{\infty}(a,\beta,\gamma)=\int_0^{+\infty} s^{2(\beta-\gamma)}E^2_{\beta,\beta-\gamma+1}(-as^{\beta})\,ds; \tag{4.13}$$
- *If $\gamma=1/2$, then, as $t\to\infty$, $X(t)$ is a Gaussian random variable with mean of order $t^{-\beta}$ and variance of order $\ln t$;*
- *If $\gamma<1/2$, then, as $t\to\infty$, $X(t)$ is a Gaussian random variable with mean of order $t^{-\beta}$ and variance of order $t^{1-2\gamma}$.*

In particular, for $\beta\in(0,1]$ and $\gamma\in(0,1/2)$, the long-time behavior of X corresponds to that of a sub-diffusion; see, for example, [22] or [18, Section 6]. For $\gamma=1/2$, the result is an ultra-slow, or Sinai-type, diffusion [21].

Remark 4.3. *Different physical considerations lead to alternative forms of the time-fractional Ornstein-Uhlenbeck process: see, for example* [10] *and references therein.*

5. Time Fractional Geometric Brownian Motion

Similar to the geometric Brownian motion
$$dx(t)=ax(t)dt+\sigma x(t)dw(t),$$
which is
$$x(t)=x(0)\exp\left(\left(a-\frac{\sigma^2}{2}\right)t+\sigma w(t)\right), \tag{5.1}$$
define the time fractional geometric Brownian motion as the solution of the equation
$$\partial_t^{\beta}X(t)=aX(t)+\sigma\partial_t^{\gamma}\int_0^t X(s)\,dw(s),\ t>0,\quad \beta,\gamma\in(0,1), \tag{5.2}$$
with non-random initial condition $X(0)=X_0$.

By Theorem 3.6, if $\gamma-\beta<1/2$, then (5.2) is equivalent to the integral equation
$$X(t)=X_0+\frac{a}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}X(s)\,ds+\frac{\sigma}{\Gamma(1+\beta-\gamma)}\int_0^t(t-s)^{\beta-\gamma}X(s)\,dw(s); \tag{5.3}$$
using (4.8), we get a different, but equivalent, equation
$$X(t)=X_0E_{\beta}(at^{\beta})+\sigma\int_0^t(t-s)^{\beta-\gamma}E_{\beta,\beta-\gamma+1}\big(a(t-s)^{\beta}\big)X(s)\,dw(s). \tag{5.4}$$
Accordingly, we define a classical solution of (5.2) on $[0,T]$ as a continuous process $X=X(t)$ such that, for all $t\in[0,T]$, $\mathbb{E}X^2(t)<\infty$, $X(t)$ is $\mathcal{F}_t$-measurable, and (5.3) holds with probability one.

Because a closed-form expression of the type (5.1) is currently not available for $X(t)$, we will study (5.2) using chaos expansion.

To simplify the notations, let

$$\Phi(t) = t^{\beta-\gamma} E_{\beta,\beta-\gamma+1}(at^{\beta}). \tag{5.5}$$

Theorem 5.1. *Under condition* (3.10), *equation* (5.2) *has a unique classical solution for every $T>0$ and $X_0, a, \sigma \in \mathbb{R}$, with chaos expansion*

$$X(t) = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} X_{\boldsymbol{\alpha}}(t)\xi_{\boldsymbol{\alpha}},$$

where

$$\begin{split}\sum_{|\boldsymbol{\alpha}|=n} X_{\boldsymbol{\alpha}}(t)\xi_{\boldsymbol{\alpha}} = X_0\sigma^n \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} \\ \Phi(t-s_n)\Phi(s_n-s_{n-1})\ldots\Phi(s_2-s_1)E_{\beta}(as_1^{\beta})\, dw(s_1)\ldots dw(s_n),\end{split} \tag{5.6}$$

and

$$\begin{split}\mathbb{E}X^2(t) = X_0^2\Big(E_{\beta}^2(at^{\beta}) + \sum_{n=1}^{\infty}\sigma^{2n}\int_0^t\int_0^{s_n}\cdots\int_0^{s_2} \\ \Phi^2(t-s_n)\Phi^2(s_n-s_{n-1})\ldots\Phi^2(s_2-s_1)E^2_{\beta}(as_1^{\beta})\, ds_1\ldots ds_n\Big).\end{split} \tag{5.7}$$

Proof. Existence and uniqueness follow from (5.3) by the standard fixed point argument. To derive (5.6), we use the general result about chaos expansion for linear evolution equations [12, Section 6]. In particular, the functions $X_{\boldsymbol{\alpha}} = X_{\boldsymbol{\alpha}}(t)$, $\boldsymbol{\alpha}\in\mathcal{J}$, satisfy a system of equations, known as the `propagator`:

$$\begin{split}|\boldsymbol{\alpha}| = 0:\ \partial_t^{\beta} X_{(\mathbf{0})} = aX_{(\mathbf{0})},\ X_{(\mathbf{0})}(0) = X_0; \\ |\boldsymbol{\alpha}| > 0:\ \partial_t^{\beta} X_{\boldsymbol{\alpha}}(t) = aX_{\boldsymbol{\alpha}}(t) + \sigma\sum_{k\geq 1}\sqrt{\alpha_k}\, I^{1-\gamma}(X_{\boldsymbol{\alpha}-\boldsymbol{\epsilon}(k)}\,\mathfrak{m}_k)(t),\ X_{\boldsymbol{\alpha}}(0) = 0,\end{split} \tag{5.8}$$

where $\boldsymbol{\epsilon}(k)$ is the multi-index with $|\boldsymbol{\epsilon}(k)| = 1$ and the only non-zero element in position k. By Theorem 2.10,

$$\begin{split}X_{(\mathbf{0})}(t) = X_0 E_{\beta}(at^{\beta}), \\ X_{\boldsymbol{\alpha}}(t) = \sigma\sum_{k\geq1}\sqrt{\alpha_k}\int_0^t (t-s)^{\beta-1}E_{\beta,\beta}\big(a(t-s)^{\beta}\big)I^{1-\gamma}(X_{\boldsymbol{\alpha}-\boldsymbol{\epsilon}(k)}\,\mathfrak{m}_k)(s)\, ds, \\ |\boldsymbol{\alpha}| > 0.\end{split} \tag{5.9}$$

Changing the order of integration and using (2.34),

$$\begin{split}\int_0^t (t-s)^{\beta-1}E_{\beta,\beta}\big(a(t-s)^{\beta}\big)I^{1-\gamma}(X_{\boldsymbol{\alpha}-\boldsymbol{\epsilon}(k)}\,\mathfrak{m}_k)(s)\, ds \\ = \int_0^t (t-s)^{\beta-\gamma}E_{\beta,1+\beta-\gamma}\big(a(t-s)^{\beta}\big)X_{\boldsymbol{\alpha}-\boldsymbol{\epsilon}(k)}(s)\mathfrak{m}_k(s)\, ds,\end{split}$$

and then, iterating the result,

$$\begin{split}X_{\boldsymbol{\alpha}}(t) = \frac{\sigma^n}{\sqrt{\boldsymbol{\alpha}!}}\sum_{\pi\in\mathcal{P}^n}\int_0^t\int_0^{s_n}\cdots\int_0^{s_2}\Phi(t-s_n) \\ \times\Phi(s_n-s_{n-1})\ldots\Phi(s_2-s_1)X_{(\mathbf{0})}(s_1)\mathfrak{m}_{i_{\pi(n)}}(s_n)\ldots\mathfrak{m}_{i_{\pi(1)}}\, ds_1\ldots ds_n,\end{split} \tag{5.10}$$

where $\mathcal{P}^n$ is the permutations group of $\{1,\dots,n\}$ and $\{i_1,\dots,i_n\}$ is the characteristic set of $\boldsymbol{\alpha}$; cf. [12, Corollary 6.6]. After that, (5.6) follows from the connection between the Hermite polynomials and the iterated Itô integrals [7, Theorem 3.1]. Then (5.7) follows from (5.6) by Itô isometry.

It remains to show that the right-hand side of (5.7) is finite. To this end, we use (2.29) to write

$$\Phi^2(t)\le Ct^{r-1},\ r=2(\beta-\gamma)+1>0,$$

so that

$$\int_0^t\int_0^{s_n}\dots\int_0^{s_2}\Phi^2(t-s_n)\Phi^2(s_n-s_{n-1})\dots\Phi^2(s_2-s_1)\\ \times E_\beta^2(as_1^\beta)\,ds_1\dots ds_n\Big)\le\frac{C^n(T)}{\Gamma(nr+1)},$$

and convergence follows by the Stirling formula. □

Note that condition (3.10) is necessary and sufficient for the convergence of the integrals in (5.7), and once again we see that, without (3.10), no classical solution of (5.2) can exist.

On the other hand, each $X_{\boldsymbol{\alpha}}$ is well-defined by (5.9) for all $\beta,\gamma\in(0,1]$. Accordingly, we call the resulting formal sum $\sum_{\boldsymbol{\alpha}\in\mathcal{J}}X_{\boldsymbol{\alpha}}(t)\xi_{\boldsymbol{\alpha}}$ the `chaos solution` of (5.2). By construction, this solution exists and is unique.

Theorem 5.2. *If $\beta-\gamma\le-1/2$ and $\{\mathfrak{m}_k,\ k\ge1\}$ are given by* (3.17)*, then the chaos solution of* (5.2) *is a $\mathfrak{q}$-generalized process for every $\mathfrak{q}$ satisfying* (3.19).

Proof. The objective is to show that (3.19) implies

$$\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\mathfrak{q}^{\boldsymbol{\alpha}}|X_{\boldsymbol{\alpha}}(t)|^2<\infty,\ t>0, \tag{5.11}$$

and analysis of the proof of Theorem 5.1 shows that (5.11) will follow from

$$\sum_{k\ge1}q_k\left(\int_0^T\Phi(T-s)\,\mathfrak{m}_k(s)\,ds\right)^2<\infty. \tag{5.12}$$

To prove (5.12), define the operator Q on $L_2((0,T))$ by

$$Q\mathfrak{m}_k=\sqrt{q_k}\,\mathfrak{m}_k,\ k\ge1.$$

Then the operator Q is symmetric on $L_2((0,T))$,

$$Qf(t)=\sum_{k\ge1}\sqrt{q_k}\left(\int_0^T f(s)\mathfrak{m}_k(s)\,ds\right)\mathfrak{m}_k(t),$$

$$\sum_{k\ge1}q_k\left(\int_0^T\Phi(T-s)\,\mathfrak{m}_k(s)\,ds\right)^2=\int_0^T\big(Q\Phi(t)\big)^2\,dt,$$

and (5.12) follows from (3.21). □

6. Stochastic Fractional Parabolicity Conditions

Consider the stochastic equation

$$du(t,x) = bu_{xx}(t,x)dt + \Big(\varrho u_{xx}(t,x) + \sigma u_x(t,x) + cu(t,x)\Big)\,dw(t),\ t>0,\ x\in\mathbb{R}, \quad (6.1)$$

with real numbers b, ϱ, σ, c as parameters. It is well known that

- Equation (6.1) is well-posed in $L_2(\mathbb{R})$ if and only if $\varrho = 0$ and $2b - \sigma^2 \geq 0$; see, for example, [15, Section 2.3.1].
- Equation (6.1) is well-posed in a suitable chaos space if $b > 0$; cf. [14].

On other hand, a perturbation-type argument [4] shows that the following fractional version of (6.1),

$$\partial_t^\beta u(t,x) = au_{xx}(t,x) + \partial_t^\gamma \int_0^t \Big(\varrho u_{xx}(s,x) + \sigma u_x(s,x) + cu(s,x)\Big)\,dw(s) \qquad (6.2)$$

is well-posed in $L_2(\mathbb{R})$ if $\beta \in (0,1)$, $|\varrho|$ is sufficiently close to zero, and $0 < \gamma < 1/2$. Note that if $\gamma < 1/2$, then (3.10) holds for all $\beta \in (0,1)$.

The objective of this section is to establish more general sufficient conditions for well-posedness of (6.2) and similar equations.

Fix the numbers $b > 0,\ \sigma \in \mathbb{R},\ \beta,\gamma \in (0,1]$, $\alpha,\nu \in (0,2]$, and let

$$\Lambda = (-\boldsymbol{\Delta})^{1/2}$$

be the fractional Laplacian defined in the Fourier domain by

$$\frac{1}{(2\pi)^{\mathrm{d}/2}}\int_{\mathbb{R}^{\mathrm{d}}} e^{-ixy}(\Lambda f)(x)\,dx = \frac{|y|}{(2\pi)^{\mathrm{d}/2}}\int_{\mathbb{R}^{\mathrm{d}}} e^{-ixy} f(x)\,dx.$$

Consider the equation

$$\partial_t^\beta u(t,x) + b\Lambda^\alpha(t,x) = \sigma\,\partial_t^\gamma \int_0^t \Lambda^\nu u(s,x)\,dw(s),\ t>0,\ x\in\mathbb{R}^{\mathrm{d}}, \qquad (6.3)$$

with non-random initial condition $u(0,\cdot) \in L_2(\mathbb{R}^{\mathrm{d}})$.

Definition 6.1. *An $\mathcal{F}_t$-adapted process $u \in L_2\Big(\Omega; \mathcal{C}\big([0,T]; L_2(\mathbb{R}^{\mathrm{d}})\big)\Big)$ is called a solution of (6.3) if, for every $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^{\mathrm{d}})$,*

$$\mathbb{P}\Bigg(J^{1-\beta}\big(u,\varphi\big)_{L_2(\mathbb{R}^{\mathrm{d}})}(t) + b\int_0^t \big(u,\Lambda^\alpha\varphi\big)_{L_2(\mathbb{R}^{\mathrm{d}})}(s)\,ds = \sigma J^{1-\gamma}\left(\int_0^{\cdot} \big(u,\Lambda^\nu\varphi\big)_{L_2(\mathbb{R}^{\mathrm{d}})}(s)\,dw(s)\right)(t),\ t\in[0,T]\Bigg) = 1.$$

Equation (6.3) is called well-posed in $L_2(\mathbb{R}^{\mathrm{d}})$ if, for every initial condition $u(0,\cdot) \in \mathcal{C}_0^\infty(\mathbb{R}^{\mathrm{d}})$, there exists a unique solution u and

$$\mathbb{E}\|u\|^2_{L_2(\mathbb{R}^{\mathrm{d}})}(t) \leq C\|u(0,\cdot)\|^2_{L_2(\mathbb{R}^{\mathrm{d}})},$$

with C independent of the initial condition.

The L_2-isometry of the Fourier transform implies that, in terms of well-posedness in L_2, equation (6.3) with $\alpha = 2$ and $\nu = 1, 2$, is equivalent to (6.2); this equivalence might no longer hold for well-posedness in L_p, $p > 2$, see [2] when $\beta = \gamma = 1,\ \alpha = 2,\ \nu = 1$.

In the rest of the section we show that, under (3.10),

(1) For $\gamma \in (0, 1/2)$, equation (6.3) is well-posed in $L_2(\mathbb{R}^d)$ if and only if $\alpha \geq \nu$;
(2) For $\gamma = 1/2$, equation (6.3) is well-posed in $L_2(\mathbb{R}^d)$ if and only if $\alpha > \nu$;
(3) For $\gamma = (1/2) + \beta\varepsilon$, $\varepsilon \in (0, 1)$, equation (6.3) is well-posed in $L_2(\mathbb{R}^d)$ if $\alpha > \nu/(1-\varepsilon)$, and can be well-posed when $\alpha = \nu/(1-\varepsilon)$ under additional conditions on b and σ.

Theorem 6.2. *Assume that* (3.10) *holds. Then* (6.3) *is well-posed in* $L_2(\mathbb{R}^d)$ *in each of the following cases:*

- $\gamma \in (0, 1/2)$ *and* $\alpha \geq \nu$*;*
- $\gamma = 1/2$ *and* $\alpha > \nu$*;*
- $\gamma \in (1/2, 1]$ *and*

$$\alpha > \frac{\nu}{1 - \frac{\gamma - (1/2)}{\beta}}. \tag{6.4}$$

Proof. Denote by $U = U(t, y)$ the Fourier transform of u in the space variable. Then, by Fourier isometry, (6.3) is well-posed in $L_2(\mathbb{R}^d)$ if and only if

$$\mathbb{E}|U(t, y)|^2 \leq C|U(0, y)|^2,\ t > 0, \tag{6.5}$$

for some C independent of y. Accordingly, throughout the proof, C denotes a positive number independent of y.

Equation (6.3) in Fourier domain is

$$\partial_t^\beta U(t, y) = -b|y|^\alpha\, U(t, y) + \sigma|y|^\nu\, \partial_t^\gamma \int_0^t U(s, y)\, dw(s). \tag{6.6}$$

Notice that, for each $y \in \mathbb{R}^d$, equation (6.6) is of the same type as (5.2). Then (5.4) implies

$$\begin{aligned} \mathbb{E}|U(t, y)|^2 = {} & |U(0, y)|^2 E_\beta^2\big(-b|y|^\alpha t^\beta\big) \\ & + \sigma^2 |y|^{2\nu} \int_0^t (t-s)^{2(\beta-\gamma)} E^2_{\beta,\beta-\gamma+1}\big(-b|y|^\alpha (t-s)^\beta\big)\, \mathbb{E}|U(s, y)|^2\, ds. \end{aligned} \tag{6.7}$$

If $|y| \leq 1$, then (2.30) immediately implies

$$\mathbb{E}|U(t, y)|^2 \leq C\Big(|U(0, y)|^2 + \int_0^t (t-s)^{2(\beta-\gamma)}\, \mathbb{E}|U(s, y)|^2\, ds\Big),$$

and (6.5) follows by Proposition 2.6.

If $|y|>1$, and $\gamma\in(0,1/2)$, then we use (2.30) to write

$$E^2_{\beta}\big(-b|y|^{\alpha}t^{\beta}\big)\le C,$$
$$E^2_{\beta,\beta-\gamma+1}\big(-b|y|^{\alpha}(t-s)^{\beta}\big)\le \frac{C}{b^2|y|^{2\alpha}(t-s)^{2\beta}}, \tag{6.8}$$

and then (6.7) becomes

$$\mathbb{E}|U(t,y)|^2\le C\Big(|U(0,y)|^2+\int_0^t (t-s)^{-2\gamma}\,\mathbb{E}|U(s,y)|^2\,ds\Big), \tag{6.9}$$

so that (6.5) again follows by Proposition 2.6.

If $\gamma\ge 1/2$, then the integral on the right-hand side of (6.9) diverges. Accordingly, we replace (6.8) with

$$E^2_{\beta,\beta-\gamma+1}\big(-b|y|^{\alpha}(t-s)^{\beta}\big)\le \frac{C}{\big(a^2|y|^{2\alpha}(t-s)^{2\beta}\big)^{1-\varepsilon}},$$

taking $\varepsilon>0$ if $\gamma=1/2$ and $\varepsilon\beta>\gamma-1/2$ if $\gamma\in(1/2,1]$. Note that (3.10) is equivalent to $1-\varepsilon>0$. Then, instead of (6.9), we get

$$\mathbb{E}|U(t,y)|^2\le C\Big(|U(0,y)|^2+\int_0^t (t-s)^{2(\varepsilon\beta-\gamma)}\,\mathbb{E}|U(s,y)|^2\,ds\Big),$$

as long as $\alpha(1-\varepsilon)\ge\nu$, and conclude the proof by applying Proposition 2.6. □

Note that

(1) If $\gamma\ge(\beta+1)/2$, then (6.4) becomes $\alpha>2\nu$. For equation (6.2), this means $\varrho=\sigma=0$, which is consistent with [4].
(2) The results of [14] suggest that (6.3) is unlikely to have a $\mathfrak{q}$-generalized chaos solution when $\nu>\alpha$.

If $\gamma\in(0,1/2)$, then condition $\alpha\ge\nu$ is also necessary: (2.31) shows that (6.5) is not possible for large $|y|$ when $\alpha<\nu$. If $\gamma=\beta=1$, then (6.4) becomes $\alpha>2\nu$. On the other hand, similar to (6.1), equation (6.3) is well-posed in $L_2(\mathbb{R}^{\mathrm{d}})$ if $\gamma=\beta=1$, $\alpha=2,\ \nu=1$, and

$$2b\ge\sigma^2. \tag{6.10}$$

This observation suggests that, more generally, (6.3) could be well-posed if $\gamma-1/2=\beta\varepsilon,\ \varepsilon\in(0,1)$, and $\alpha=\nu/(1-\varepsilon)$, under an additional condition of the type (6.10). Dimensional analysis implies that the condition should be of the form $b\ge C(\beta,\gamma)|\sigma|^{1/(1-\varepsilon)}$. We conclude this section by establishing an upper bound for $C(\beta,\gamma)$, as well as addressing a similar question when $\gamma=1/2$.

Theorem 6.3. *Assume that* (3.10) *holds. Then equation* (6.3) *is*

- *NOT well-posed in $L_2(\mathbb{R}^{\mathrm{d}})$ if $\gamma=1/2$ and $\alpha=\nu$;*
- *well-posed in $L_2(\mathbb{R}^{\mathrm{d}})$ if $\gamma=(1/2)+\varepsilon\beta$, $\varepsilon\in(0,1)$,*

$$\alpha=\frac{\nu}{1-\varepsilon},\ \ b\ge\big(\sigma^2_{\infty}(1,\beta,\gamma)\big)^{1/(2-2\varepsilon)}\,|\sigma|^{1/(1-\varepsilon)}, \tag{6.11}$$

with $\sigma^2_{\infty}(a,\beta,\gamma)$ defined in (4.13).

Proof. Similar to the proof of the previous theorem, we need to study equality (6.7) for $|y|>1$, so we fix y with $|y|$ sufficiently large and define

$$V(t)=\mathbb{E}|U(t,y)|^2-|U(0,y)|^2E^2_{\beta}(-b|y|^{\alpha}t^{\beta}).$$

Then V is non-decreasing in t and satisfies

$$\begin{aligned} V(t)&=\sigma^2|y|^{2\nu}\int_0^t (t-s)^{2(\beta-\gamma)}E^2_{\beta,\beta-\gamma+1}\big(-b|y|^{\alpha}(t-s)^{\beta}\big)V(s)\,ds\\ &\quad+|U(0,y)|^2\ \sigma^2|y|^{2\nu}\int_0^t s^{2(\beta-\gamma)}E^2_{\beta,\beta-\gamma+1}\big(-b|y|^{\alpha}s^{\beta}\big)E^2_{\beta}\big(-b|y|^{\alpha}(t-s)^{\beta}\big)\,ds. \end{aligned} \tag{6.12}$$

If $\gamma=1/2$, we re-write (6.12) as

$$V(t)\geq |U(0,y)|^2\sigma^2|y|^{2\nu}\int_0^t s^{2\beta-1}E^2_{\beta,\beta-\gamma+1}\big(-b|y|^{\alpha}s^{\beta}\big)E^2_{\beta}\big(-b|y|^{\alpha}(t-s)^{\beta}\big)\,ds.$$

Changing the variables $T=(b|y|^{\alpha})^{1/\beta}t$, $\tau=(b|y|^{\alpha})^{1/\beta}s$, and keeping in mind that $\nu=\alpha$,

$$V(t)\geq |U(0,y)|^2\sigma^2 b^{-2}\int_0^T \tau^{2\beta-1}E^2_{\beta,\beta-\gamma+1}\big(-\tau^{\beta}\big)E^2_{\beta}\big((T-\tau)^{\beta}\big)\,d\tau.$$

Because $\lim_{|y|\to\infty}T=+\infty$ and, by (2.31), the last integral diverges at infinity, we conclude that (6.5) cannot hold.

Next, consider the case $\gamma\in(1/2,1]$ under the assumptions (6.11). The same computations as in the case $\gamma=1/2$ show that now the second integral on the right-hand side of (6.12) is uniformly bonded in $|y|$. To analyze the first integral, write

$$\varepsilon=\frac{\gamma-(1/2)}{\beta},\ \ \varpi=|y|^{-\varepsilon\alpha/\beta},$$

so that $\nu=\alpha(1-\varepsilon)$, and

$$\begin{aligned} &\sigma^2|y|^{2\nu}\int_0^t (t-s)^{2(\beta-\gamma)}E^2_{\beta,\beta-\gamma+1}\big(-b|y|^{\alpha}(t-s)^{\beta}\big)V(s)\,ds\\ &=\sigma^2|y|^{2\nu}\left(\int_0^{t-\varpi}+\int_{t-\varpi}^t\right)(t-s)^{2(\beta-\gamma)}E^2_{\beta,\beta-\gamma+1}\big(-b|y|^{\alpha}(t-s)^{\beta}\big)V(s)\,ds. \end{aligned} \tag{6.13}$$

We use (6.8) and $t-s\geq\varpi$ to bound the first integral on the right-hand side of (6.13) by

$$\frac{C\sigma^2|y|^{2\nu}}{\big(b|y|^{\alpha}\varpi^{\beta}\big)^2}\int_0^{t-\varpi}(t-s)^{2(\beta-\gamma)}V(s)\,ds\leq C\int_0^t (t-s)^{2(\beta-\gamma)}V(s)\,ds,$$

which, by (3.10), allows an application of Proposition 2.6. For the second integral, we use monotonicity of V to get an upper bound

$$\begin{aligned} V(t)\,\sigma^2|y|^{2\nu}&\int_{t-\varpi}^t (t-s)^{2(\beta-\gamma)}E^2_{\beta,\beta-\gamma+1}\big(-b|y|^{\alpha}(t-s)^{\beta}\big)\,ds\\ &=V(t)\,\sigma^2|y|^{2\nu}\int_0^{\varpi} s^{2(\beta-\gamma)}E^2_{\beta,\beta-\gamma+1}\big(-b|y|^{\alpha}s^{\beta}\big)\,ds, \end{aligned}$$

which, after the change of variable $\tau = (b|y|^{\alpha})^{1/\beta}s$ becomes

$$V(t)\,\frac{\sigma^2}{b^{2(1-\varepsilon)}}\int_0^{b^{1/\beta}|y|^{\nu/\beta}} \tau^{2(\beta-\gamma)}E^2_{\beta,\beta-\gamma+1}\big(-\tau^{\beta}\big)\,d\tau < V(t)\,\frac{\sigma^2}{b^{2(1-\varepsilon)}}\,\sigma^2_{\infty}(1,\beta,\gamma).$$

By assumption,

$$\frac{\sigma^2}{b^{2(1-\varepsilon)}}\,\sigma^2_{\infty}(1,\beta,\gamma)\le 1,$$

and then (6.5) follows from (6.12). □

As a final comment, note that, while the proof of Theorem 6.3 suggests that (6.11) might not be sharp, we do get the optimal bound (6.10) when $\alpha = 2$, $\nu = 1$, $\beta = \gamma = 1$, and $\varepsilon = 1/2$, because, with $E_{1,1}(t) = e^t$,

$$\sigma^2_{\infty}(1,1,1) = \int_0^{\infty} e^{-2t}\,dt = \frac{1}{2}.$$

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