A SOLUTION OF THE WAVE SOUND EQUATION IN SHALLOW WATER FOR REAL SPEED PROFILES AND SOLID BOTTOM UNDER SEDIMENT

by

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ABSTRACT

The wave equation has been solved by using normal mode theory. The medium is assumed to be horizontally stratified. A closed form solution has been found in the case of GANS - PEDERSEN types of density and sound-speed profiles. For real profiles of any given shape a numerical solution is employed that makes use of the VOLTERRA integral equation.

1 - INTRODUCTION

The use of normal modes theory for the calculation of shallow water sound propagation has first been developed by PEKERIS (Ref. 1, 2, 3) in 1948. His model was oversimplified as the sound speed was assumed to be constant in the water and the sea bottom was taken as an homogeneous fluid halfspace. A number of attempts have been made since to account for the variation of sound speed with depth : we can mention for instance a computer program worked out by NEWMAN and INGENITO (Ref. 4) for a two fluid model with the speed of sound varying with depth in the first fluid. This program makes use of the finite differences technique.

We develop herein a more realistic model based on a mode formulation exposed chapter 2 which is valid for horizontally stratified media (physical characteristics depending on depth only). This assumption enables the initial propagation equation to be transformed into a HELMHOLTZ type equation.

In our development the medium is taken as follows :

Sea water bounded by the atmosphere (plane surface) and infinite elastic bottom with or without an intermediate fluid layer corresponding to the sediment.

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In the water and the sediment the sound speed may vary with depth in any given manner. In the elastic bottom, both compressional and shear velocities are constant .

The density may vary with depth in the sediment and it stays constant in the water and the elastic bottom (although it could vary in the former if wanted).

The sound field dealt with is that created by an omnidirectional monochromatic point source.

The formulation developed in chapter 2 will be applied in chapter 3 to analytic velocity and density profiles of the GANS-PEDERSEN type. In chapter 4, it will be applied to realistic profiles of any shape given by a discrete number of data points.

2 - FORMULATION OF THE PROBLEM.

2.1 Sound propagation equation.

Let $\Psi(\vec{r},t)$ be the velocity potential.
By definition, the acoustic pressure $p(\vec{r},t)$ and the displacement velocity $\vec{v}(\vec{r},t)$ of a fluid element are given by :

$$(1)\quad p(\vec{r},t) = \rho \frac{\partial \Psi(\vec{r},t)}{\partial t} \; ; \; \vec{v}(\vec{r},t) = - \vec{\text{grad}}\, \Psi(\vec{r},t)$$

Taking into account the equations of motion and mass conservation and the state equation relating acoustic pressure to density variation, the potential $\Psi(\vec{r},t)$ obeys the following equation (first order approximation).

$$(2)\quad \Delta \Psi - \frac{1}{C^2(z)} \frac{\partial^2 \Psi}{\partial t^2} - \frac{1}{\rho(z)} \frac{d\rho}{dz} \frac{\partial \Psi}{\partial z} = - 4\pi\delta(\vec{r} - \vec{r}_e)\, e^{j\omega t}$$

where $C(z)$ is the speed of sound in the fluids (water or sediment) and $\rho(z)$ the fluid density. The second member of equation (2) represents the source term :

$\vec{r}$ and $\vec{r}_e$ are the vectors joining the origin of the referential to the point of observation and to the source respectively. Owing to the symmetry of revolution, one can make use of only two cylindrical coordinates r and z.

Therefore, $\psi(\vec{r},t)$ takes the form:

$$(3) \quad \Psi(\vec{r},t) = \Phi'(r,z)\, e^{j\omega t}$$

Equation 2 can be solved by using the HANKEL Transform of $\Phi'(r,z)$.

The wanted fonction $\Phi'(r,z)$ is a solution of the equation :

$$(4) \qquad \Phi'(r,z) = \frac{-1}{J\pi} \int_{-j\infty}^{+j\infty} \Phi(z,s)\, K_0(sr)\, s\, ds$$

where $K_0(sr)$ is the zero order second kind modified BESSEL function.

The $\Phi(z,s)$ function is a solution of the equation :

$$(5) \quad \frac{d^2\Phi}{dz^2} + \left[\frac{\omega^2}{C^2(z)} + S^2\right]\Phi - \frac{1}{\rho(z)}\frac{d\rho}{dz}\frac{d\Phi}{dz} = = 2\pi\delta(z - z_e)$$

where r is the horizontal range between source and observation point and z is the depth below sea level.

The s variable is a parameter that comes in when applying the HANKEL transform and that physically corresponds to the horizontal component of the wave vector.

In order to obtain the velocity potential, it is therefore necessary to first solve equation (5) and obtain the $\Phi(z,s)$ function, and then calculate integral (4).

The second order differential equation (5) obeys certain boundary conditions at the various interface levels : air-water, water-sediment, sediment-rock, which makes it a STURM-LIOUVILLE problem.

Note : The K_0 function is found here instead of the usual J_0 function because of the choice we made in writing $+ s^2$ in equation 5 (instead of $- s^2$). Changing s into $- js$ would lead to the forms mostly encountered in the literature. In the same way, there will be a change from real to imaginary and vice-versa when speaking of poles, integration contours, etc ....

2.2 Formulation of the boundary conditions.

Crossing (or boundary) conditions arise at each change of medium :

- Air-water interface ($z = 0$ plane) : acoustic pressure is null
- Water-sediment interface ($z = z_1$ plane) : pressure is continuous and the fluids undergo the same vertical motions on each side of the boundary
- Sediment-rock interface ($z = z_2$ plane) : the T_{zz} term of the stress tensor that acts on the elastic medium at $z = z_2+0$ must balance the pressure acting on the other side (at $z = z_2 - 0$). Furthermore, the boundary must undergo the same displacement as seen from each side. These two conditions lead to the following homogeneous condition :

$$(6) \quad \frac{1}{\rho_1(z_2)} \frac{1}{\Phi(z_2)} \frac{\partial \Phi}{\partial z}\bigg|_{z_2-0} = \frac{1}{\rho_2} \frac{1}{T_{zz}} \frac{\partial T_{zz}}{\partial z}\bigg|_{z_2+0} = K(s)$$

where the function Φ is a solution of (5) and where ρ_1 and ρ_2 are the specific masses in the sediment and the rock respectively. It can be demonstrated that the term K(s) may be expressed as a function of the compressional and shear velocities in the rock C_L and C_T, the variable s being the same as the one introduced in equation (4).

One would find :

$$(7)\quad K(s) = -\frac{j}{4\rho_2}\,\frac{\omega^4}{C_T^4}\,\frac{a}{\left[s^2 + \frac{\omega^2}{2C_T^2}\right]^2 - abs^2}$$

where

$$a = \left[\frac{\omega^2}{C_L^2} + s^2\right]^{1/2} \quad ; \quad b = \left[\frac{\omega^2}{C_T^2} + s^2\right]^{1/2}$$

The fact that the acoustic field must vanish at infinite distance leads the determinations to be taken for a and b.

- Source level plane (z = z_e) : the source condition can be transformed into a boundary condition at the source horizontal plane as follows :
  Continuity of pressure and opposite direction of the fluid motion on each side of the plane, leading :

$$(8)\quad \left.\frac{\partial\Phi}{\partial z}\right|_{z_{e-0}} - \left.\frac{\partial\Phi}{\partial z}\right|_{z_{e+0}} = -2$$

2.3 Solution of equation (5)

Let ϕ (z,s) be a solution of equation (5) that meets the previously exposed boundary conditions at z = 0, viz :

$$(9)\quad \phi(0,s) = 0 \;;\; \frac{1}{\rho(0)}\,\frac{\partial\phi(0,s)}{\partial z} = 1$$

Let $\psi\ (z,s)$ be a solution of equation (5) that meets the boundary conditions at $z = z_2$, viz :

$$\psi\ (z,s) = \psi_1\ (z,s) + K\ (s)\ \psi_2\ (z,s)$$

with :

$$(10) \qquad \psi_1\ (z_2,s) = 1 \quad ; \quad \psi_2\ (z_2,s) = 0$$

$$\frac{1}{\rho(z_2)} \left.\frac{\partial\psi_2}{\partial z}\right|_{z=z_2} = 1 \quad ; \quad \frac{1}{\rho(z_2)} \left.\frac{\partial\psi_1}{\partial z}\right|_{z=z_2} = 0$$

If these two functions $\phi\ (z,s)$ and $\psi\ (z,s)$ are linearly independant, the solution of equation (5) that obeys the three sets of conditions : surface, bottom and source, is given by the GREEN'S function :

$$(11) \qquad \Phi(z,z_e,s) = \begin{cases} -2\,\dfrac{\psi(z_e,s)\phi(z,s)}{\rho(0)\psi(0,s)} & 0 < z < z_e < z_2 \\[2ex] -2\,\dfrac{\phi(z_e,s)\psi(0,s)}{\rho(0)\psi(0,s)} & 0 < z_e < z < z_2 \end{cases}$$

It must be emphasized that solution (11) to equation (5) is not analytically known. We have just expressed the conditions to be fulfilled by the functions that are solutions of the problem described. These conditions are necessary and sufficient for all functions ϕ and ψ that meet conditions (9) and (10) respectively to give rise to a function Φ , solution to the problem.

2.4 Solution of equation (4) - Choice of the integration contour in the complex s plane.

The Φ (z, z_e, s) function just described is a complex function that has a certain number of poles s_n (complex, imaginary or real). These poles are the zeros of Ψ (0,s).(Here the STURM-LIOUVILLE problem is not an hermitian one).

From the expression for K(s) and the determinations chosen for a and b,the integration has to be made in the Re (s) $\geqslant$ 0 plane.

The cuts at $\pm j\omega/C_L$ and $\pm j\omega/C_T$ have been chosen so that only two of them are in the halfspace $R_e(s) > 0$ and they are parallel to the s real axis. The poles s_n of function Φ are simple and located in the fourth quadrant of the complex plane and so are their symetricals with respect to the origin, located in the second quadrant .

The chosen integration contour C is illustrated in figure 1.

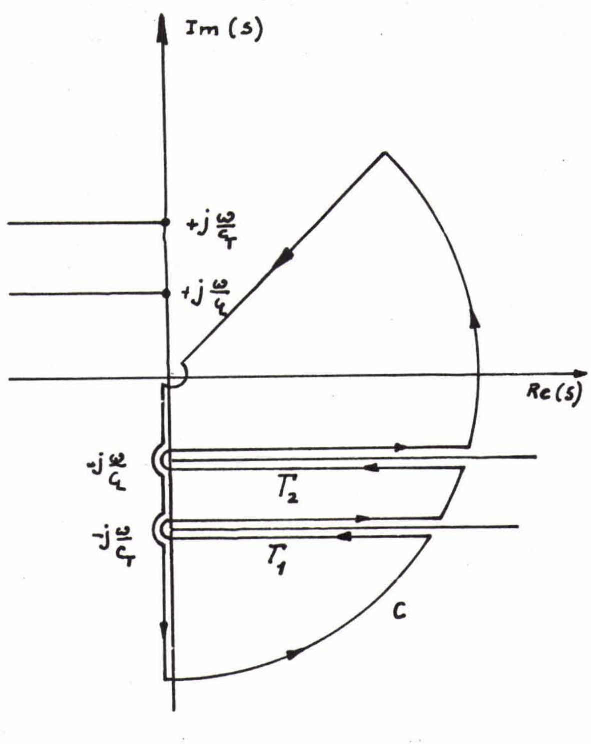

FIG. 1

Integration is made along the 1 + j straight path so that the second quadrant poles do not influence the Φ function. (This is of consideration in the case of a numerical integration of (4)).

With this integration contour, equation (4) may be written :

$$(12) \qquad \phi'(r,z) = \frac{1}{j\pi}\left[\int_{\Gamma_1} + \int_{\Gamma_2}\right] - 4\sum_n R_n K_0(s_n r)$$

where the R_n are the residues, given by :

$$(13) \qquad R_n = s_n \frac{\phi(z \text{ or } z_e, s_n)\psi(z_e \text{ or } z, s_n)}{\rho(0)\partial\psi(0,s_n)/\partial s}$$

Expression (12) consists of two terms :

The first term corresponds to the branch-line integrals calculated along cuts Γ_1 and Γ_2. Physically they represent waves that propagate along the sediment-rock interface at speeds C_L and C_T with amplitudes decreasing approximately as $1/r^2$ (ref.5).

The second term is a summation of discrete values associated with the residues which correspond to the roots of the dispersion equation $\psi\ (0,s) = 0$. Each term of this summation constitutes a propagation mode, i.e. a wave travelling with a horizontal wave vector given by s_n. The amplitudes of these waves decrease as $1/\sqrt{r}$, so that at ranges large compared to the water depth the branch-line integrals contribution becomes negligible.

In order to verify the above mathematical development, one may use it to solve the PEKERIS model. In this two layer model, the sea water is a fluid with constant sound speed C_1 and constant density ρ_1, bounded by planes at $z = 0$ and $z = z_2$ and countaining both the source (at z_e) and the receiver (at z). The sea bottom is taken as a fluid half-space with constant velocity C_2 and constant density ρ_2, extending from $z = z_2$ to z = infinity.

The $\phi\ (z,s)$ and $\psi\ (z,s)$ functions are then given by :

$$(14) \qquad \phi\ (z,s) = \rho_1 \frac{\sin\alpha_1 z}{\alpha_1}$$

$$(15) \qquad \psi(z,s) = \cos\alpha_1(z_2 - z) - \frac{\rho_1 K(s)}{\alpha_1} \sin\alpha_1(z_2 - z)$$

with :

$$\alpha_1 = \left[\frac{\omega^2}{C_1^2} + s\right]^{1/2}$$

Moreover, in this case of a fluid bottom, the expression for K(s) in (7) becomes :

$$K(s) = -\,j\,\frac{\alpha_2}{\rho_2} \qquad \text{With :} \qquad \alpha_2 = \left[\frac{\omega^2}{C_2^2} + s^2\right]^{1/2}$$

In the case $z < z_e$, function $\Phi\ (z, z_e, s)$ transforms into :

$$(16) \qquad \Phi(z,z_e s) = \frac{\sin\alpha_1 z}{\alpha_1} \times$$

$$\frac{\alpha_1 \cos \alpha_1(z_2 - z_e) + j\,\frac{\rho_1}{\rho_2}\,\alpha_2 \sin \alpha_1(z_2 - z_e)}{\alpha_1 \cos \alpha_1 z_2 + j\,\frac{\rho_1}{\rho_2}\,\alpha_2 \sin\alpha_1 z_2}$$

which is identical to the formula obtained by PEKERIS. Similar values for the residues R_n would also be found by applying equation (13).

3 - CLOSED FORM SOLUTIONS FOR THE GREEN'S FUNCTION

A large amount of work has been devoted, especially in the U.S.A., to study classes of C(z) function that would lead to closed form solutions for Φ (and $\partial\psi/\partial s$ that enters into the residue calculation).

In addition to C(z) = constant (PEKERIS) we can quote the GANS-PEDERSEN profile (Ref. 5 - 6), the parabolic profile (Ref. 7), the EPSTEIN profile (Ref. 8).

Among these various models we choose to program the GANS-PEDERSEN profile both as being of interest to solve a few practical cases we had to deal with, and in order to have a method for checking the program using the generalized numerical method exposed herunder. However, an improvement was brought to Pedersen's model as a density varying with depth could be introduced by the use of the exponential class of functions.

The adopted GANS-PEDERSEN modelling corresponded to the following description :

a) Sea water ($0 < z < z_1$)
   Constant density $\rho = \rho_0$
   Sound speed C(z) varying as :

$$C^2(z) = C_0^3/(C_0 - 2\gamma_0 z)$$

   where C_0 and γ_0 are constants.

b) In the sediment ($z_1 < z < z_2$) :

Density : $\rho(z) = \rho_1 \exp(\rho'_1 (z - z_2)/\rho_1)$

where ρ_1 and ρ'_1/ρ_1 are constants.

Sound speed : $C^2(z) = C_1^3/(C_1 - 2\gamma_1(z-z_1))$

where C_1 and γ_1 are constants.

Under these conditions, the ϕ and ψ functions that constitute Φ are given by the following expression :

(17)
$$\phi = -\frac{\pi}{3^{5/6}} \frac{C_0\rho_0}{(\gamma_0\omega^2)^{1/3}} (\zeta\zeta_0)^{1/3} \times$$

$$\left[J_{-1/3}(\zeta_0)J_{1/3}(\zeta) - J_{1/3}(\zeta_0)J_{-1/3}(\zeta)\right]$$

with :

$$\zeta = \frac{C_0^3}{3\gamma_0\omega^2}\left(\frac{\omega^2}{C^2(z)} + s^2\right)^{3/2} \quad ; \quad \zeta_0 = \frac{C_0^3}{3\gamma_0\omega^2}\left(\frac{\omega^2}{C_0^2} + s^2\right)^{3/2}$$

The ψ function may be written as :

(18) $\psi = Y + \rho_0 k_1(s)\, X$

with
$$X = -\frac{\pi}{3^{5/6}} \frac{C_0}{(\gamma_0\omega^2)^{1/3}} (\zeta\zeta_1)^{1/3} \times$$

$$\left[J_{-1/3}(\zeta_1)J_{1/3}(\zeta) - J_{1/3}(\zeta_1)J_{-1/3}(\zeta)\right]$$

$$Y = - \frac{\pi}{3^{1/2}} (\zeta\zeta_1{}^2)^{1/3} \times$$

$$\left[J_{-1/3}(\zeta) J_{-2/3}(\zeta_1) + J_{1/3}(\zeta) J_{2/3}(\zeta_1) \right]$$

$$k_1(s) = \frac{1}{\rho_1} \exp \left[\frac{\rho'_1}{\rho_1} (z_2 - z_1) \right] \times$$

$$\left\{ \frac{\rho'_1}{2\rho_1} + \frac{\left[\rho_1 K(s) - \frac{\rho'_1}{2\rho_1}\right] \frac{dX_1(\xi)}{dz} + \frac{dY_1(\xi_1)}{dz}}{\left[\rho_1 K(s) - \frac{\rho'_1}{2\rho_1}\right] X_1(\xi_1) + Y_1(\xi_1)} \right\}$$

With :
$$X_1 = - \frac{\pi}{3^{1/2}} \frac{C_1}{(3\gamma_1\omega^2)^{1/3}} (\xi\xi_2)^{1/3} \times$$

$$\left[J_{-1/3}(\xi_2) J_{1/3}(\xi) - J_{1/3}(\xi_2)\, J_{-1/3}(\xi) \right]$$

$$Y_1 = - \frac{\pi}{3^{1/2}} (\xi\xi_2^2)^{1/3} \times$$

$$\left[J_{-1/3}(\xi) J_{-2/3}(\xi_2) + J_{1/3}(\xi) J_{2/3}(\xi_2) \right]$$

$$\frac{dX_1}{dz} = \frac{\pi}{3^{1/2}} (\xi^2 \xi_2)^{1/3} \quad x$$

$$\left[J_{-1/3}(\xi_2) J_{-2/3}(\xi) + J_{1/3}(\xi_2) J_{2/3}(\xi) \right]$$

$$\frac{dY_1}{dz} = \frac{\pi}{3^{1/2}} \frac{(3\gamma_1 \omega^2)^{1/3}}{C_1} (\xi_2 \xi)^{2/3} \quad x$$

$$\left[J_{-2/3}(\xi_2) J_{2/3}(\xi) - J_{2/3}(\xi_2) J_{-2/3}(\xi) \right]$$

where the $J \pm 1/3$ and $J \pm 2/3$ are fractional order first kind BESSEL functions of the complex variable ξ or ξ_2.

The variables ζ_1, ξ and ξ_2 in the above expressions are given by :

$$\zeta_1 = \frac{C_1^{\,3}}{3\gamma_1\omega^2}\left(\frac{\omega^2}{C_1^{\,2}} + s^2\right)^{3/2}$$

$$\xi = \frac{C_1^{\,3}}{3\gamma_1\omega^2}\left[\frac{\omega^2}{C(z)^2} + s^2 - \left(\frac{\rho'_1}{2\rho_1}\right)^2\right]^{2/3}$$

$$\xi_2 = \frac{C_1^{\,3}}{3\gamma_1\omega^2}\left[\frac{\omega^2}{C(z_2)^2} + s^2 - \left(\frac{\rho'_1}{2\rho_1}\right)^2\right]^{2/3}$$

The calculation of the residues corresponding to the s_n poles of ψ (0,s) requires the knowledge of $\partial\psi / \partial s$. This derivative can be calculated in closed form without difficulty but this leads to a lengthy formula that will not be developed here.

4 - NUMERICAL FORM OF THE GREEN'S FUNCTION : (propagation in an underwater medium of any given characteristics).

The classes of function for C(z) and ρ (z) which permit to obtain closed form solutions to the wave propagation problem are too limited to account for all possible laws of variations encountered in practice. One solution is to divide the medium into layers in which sound velocity and density vary differently by a proper choice of the parameters ρ_i, ρ'_i / ρ_i, C_i , γ_i. Arrived at that degree of complexity one may as well envisage a completely numerical solution allowing any sound speed and density profile to be used. This method has been developed, programmed and will be described hereafter.

Let us rewrite equation (5) in a slightly different way by the system of two first order equations :

$$(19)\qquad \begin{aligned} &\rho(z)\frac{dU}{dz} + \left(\frac{\omega^2}{C^2(z)} + s^2\right) P = 0 \\ &\rho(z)U - \frac{dP}{dz} = 0 \end{aligned}$$

It can be shown that ϕ represents the solution P(z) with the bounding conditions :

$$P\,(0) = 0$$
$$U\,(0) = 1$$

and that ψ represents the solution P(z) with the conditions :

$$P\,(z_2) = 1$$
$$U\,(z_2) = K\,(s)$$

Let us now divide the medium into N horizontally stratified layers, the nth one being limited between $z = z_n$ and $z = z_{n+1}$

In each layer, it is possible to define a mean sound speed C_n and a mean density ρ_n. By adding similar terms on each side of equations (19), these equations can be written as follows :

$$(20)\qquad \begin{aligned} &\rho_n\frac{dU}{dz} + \left(\frac{\omega^2}{C_n^2} + s^2\right) P = \left(\frac{\omega^2}{C_n^2} - \frac{\omega^2}{C^2}\right)P + (\rho_n - \rho)\frac{dU}{dz} \\ &\rho_n U - \frac{dP}{dz} = (\rho_n - \rho)U \end{aligned}$$

The advantage of this form is that in each layer $(z_n - z_{n+1})$, the first member of the two equations (19) that were variable are now constant in (20). It becomes therefore possible to employ LAGRANGE's method of constants variation to solve equation (20).

This leads, after some arithmetic, to the following system :

$$(21)\quad P(z_{n+1}) = P(z_n)\cos\alpha_n(z_{n+1} - z_n) + \rho_n U(z_n)\frac{\sin\alpha_n(z_{n+1} - z_n)}{\alpha_n} + \int_{z_n}^{z_{n+1}} \left\{ \left\{ \left[\frac{\omega^2}{C_n^2} - \frac{\omega^2}{C^2(\zeta)} \right] P(\zeta) + \left[\rho_n - \rho(\zeta) \right] \frac{dU(\zeta)}{dz} \right\} \frac{\sin\alpha_n(z_{n+1} - \zeta)}{\alpha_n} - \left[\rho_n - \rho(\zeta) \right] U(\zeta)\cos\alpha_n(z_{n+1} - \zeta) \right\} d\zeta$$

$$(22)\quad \rho_n U(z_{n+1}) = - P(z_n)\alpha_n \sin\alpha_n(z_{n+1} - z_n) + \rho_n U(z_n)\cos\alpha_n(z_{n+1} - z_n) + \int_{z_n}^{z_{n+1}} \left\{ \left\{ \left[\frac{\omega^2}{C_n^2} - \frac{\omega^2}{C^2(\zeta)} \right] P(\zeta) + \left[\rho_n - \rho(\zeta) \right] \frac{dU(\zeta)}{dz} \right\} \cos\alpha_n(z_{n+1} - \zeta) + \left[\rho_n - \rho(\zeta) \right] U(\zeta)\alpha_n \sin\alpha_n(z_{n+1} - \zeta) \right\} d\zeta$$

where : $\alpha_n = \left(\frac{\omega^2}{C_n^2} + s^2\right)^{1/2}$

The functions ϕ and ψ are now solutions of a VOLTERRA type integral equation. If $U(z_n)$ and $P(z_n)$ are known, then $U(z_{n+1})$ and $P(z_{n+1})$ can be calculated, and thus from one layer to the next until U(z) and P(z). The values of U and P being known at the boundaries from the boundary conditions, these values serve to initialize the recurrent process.

Expressions (21) and (22) cannot however be programmed on a computer as such, but if the layers' thickness is adjusted so that C(z) and $\rho(z)$ do not vary too much around C_n and ρ_n then a TAYLOR expansion under the integral signs of (21) and (22) leads to an analytical formulation of the integrals.

The following expressions are reached after some arithmetic :

$$(23)\quad P(z_{n+1}) = P(z_n)\left\{\cos\alpha_n(z_{n+1} - z_n) + \sum_0^4 a_i J_{in} - \frac{1}{\rho(z_n)}\sum_1^3 b'_i K_{in}\right\} + U(z_n)\left\{\rho_n \frac{\sin\alpha_n(z_{n+1} - z_n)}{\alpha_n} + \rho(z_n)\sum_1^3 b_i J_{in} - \sum_0^4 a'_i K_{in}\right\}$$

$$(24)\quad \rho_n U(z_{n+1}) = P(z_n)\left\{ - \alpha_n \sin\alpha_n(z_{n+1} - z_n) + \sum_0^4 a_i K_{in} + \frac{\alpha_n^2}{\rho(z_n)}\sum_1^3 b'_i J_{in}\right\} + U(z_n)\left\{\rho_n \cos\alpha_n(z_{n+1} - z_n) + \rho(z_n)\sum_1^3 b_i K_{in} + \alpha_n^2 \sum_0^4 a'_i J_{in}\right\}$$

These expressions can be easily programmed as the coefficients a_i, b_i, a'_i, b'_i, J_{in} and K_{in} are analytical expressions that present no difficulty for computation. They are not presented here for the sake of simplification.

In order to calculate the residues corresponding to the poles s_n of ψ (0,s) it is necessary to get the $\partial\psi/\partial s$ values. This is done as follows :

Equating $\partial\psi / \partial s = 2 sx$ leads :

$$\frac{\partial P}{\partial s^2} \equiv \frac{\partial \psi}{\partial s^2} = x \quad \text{and} \quad \frac{\partial U}{\partial s^2} = \frac{1}{\rho}\frac{d}{dz}\frac{\partial\psi}{\partial s^2} = y$$

Derivating system (19) with respect to s^2 leads :

$$(25)\quad \begin{cases} \rho \dfrac{d}{dz}\dfrac{dU}{ds^2} + P + \left(\dfrac{\omega^2}{C^2(z)} + s^2\right)\dfrac{dP}{ds^2} = 0 \\[2ex] \dfrac{d}{dz}\left(\dfrac{dP}{ds^2}\right) - \rho\left(\dfrac{dU}{ds^2}\right) = 0 \end{cases}$$

with this system of equations meeting the boundary conditions :

$$x\ (z_2) = 0$$

$$y\ (z_2) = \partial K / \partial s^2$$

The analogy with the equations for U and P is obvious and the same recurrent methods can be used to get $x(z_{n+1})$ and $y(z_{n+1})$ as a function of $x\ (z_n)$ and $y\ (z_n)$.

5. COMPUTER PROGRAM

From the models developed in chapters 3 and 4, different programs were written down for the calculation of propagation losses versus horizontal range at a fixed frequency.

At the present time three main programs are available, namely (from the simplest to the most elaborate) :

1) Program "PEKTO" (from the names of PEKERIS and TOLSTOY) : Constant sound speed and density in the water ; Solid bottom without sediments.

2) Program "BESSEL" (because of the formulation in terms of $J \pm 1/3$ etc.): Sound speed and density varying as exposed in chapter 3 ; elastic bottom under the sediments.

3) Program "VOLTERRA" : Sound speed and density profiles of any given shape (as exposed in chapter 4); elastic bottom under the sediments.

The "VOLTERRA" program can obviously handle the computations corresponding to the profiles dealt with by the "BESSEL" and "PEKTO" programs but at greater cost. This was done however to check the accuracy of the "VOLTERRA"program.

These three programs have specific domains of application for which they are optimized.
Optimization was in all cases taken care of. For instance, the VOLTERRA program had to be divided into two parts : one program permitting to calculate both the branch-line integrals and the residue series of equation (12) and one program dealing only with the residues, i.e. the modes.

The reader will find hereunder a few details concerning the program structures.

In "PEKTO" and "BESSEL" the sound speed and density profiles are of course given by their analytic expressions while in VOLTERRA they are given by a set of data points. In this case the $C(z)$ and $\rho(z)$ values for any desired z during the computation are obtained from a subroutine that makes use of the natural cubic spline interpolation method.

In "BESSEL" and "VOLTERRA" the possibility to account for an absorption coefficient in the sediment (and water if desired) was introduced by taking a complex value for the sound speed. In that case the real part of the sound speed is introduced as previously and the imaginary part is a constant depending on the frequency and the medium characteristics.

All three programs use the same technique to search for the poles and perform the numerical integration if this is the case. They only differ in the way the GREEN's function ϕ and the $\frac{\partial\psi}{\partial s}$ derivative (used in the residues calculation) are evaluated.

- Calculation of the poles location in the complex plane and of the corresponding residues.

It was mentioned in paragraph 2.4 that the poles s_n were complex and located in the fourth gradiant. They can also be imaginary and on the half axis $Im(s) < 0$.

The program performs the computation of the $\phi(z\ s)$ function with s varying along Im (s) by steps Δs. At the same time it detects the minima of its denominator $\psi(o,s)$. The corresponding value of s can be regarded as the first approximate value of the imaginary part of the pole s_n. From this value the program searches for the exact location of s_n in the complex plane by the use of a NEWTON's method and then branches to the subroutine that computes the corresponding value of $\partial\psi/\partial s$, calls back to ϕ and ψ components of Φ and gives the residue as in equation (13). This procedure is carried out as long as s vary along $Im(s) < 0$ until a pre-established value s_{end} for which poles can no more exist.

- Numerical integration

Integral (4) is not directly computed in the program as this would present difficulties caused by too strong fluctuations of the GREEN's function along the imaginary axis. On the contrary the branch-line integrals of equation (12) are easier to calculate by an indirect method.

It can be demonstrated that these branch-line integrals may be expressed by the integral

$$(26) \quad \int_{-j\infty}^{(1+j)\infty} (\Phi - \Phi')\ K_0(sr)\ ds$$

where Φ' is given by

$$\Phi' = 2s \sum_n \frac{R_n}{s^2 - s_n^2}$$

s_n and R_n being the poles and corresponding residues.

It is possible to obtain an analytical approached value of (26) by :

$$(27) \qquad \frac{1}{2r} \sum_{p=0}^{N} \left[\frac{\Delta\Phi(s_p)}{s_p} + \frac{\Delta\Phi(s_{p+1})}{s_{p+1}} \right] \left[s_p K_1(s_p r) - s_{p+1} K_1 (s_{p+1} r) \right]$$

where K_1 is the first order modified BESSEL function and where $\Delta\Phi = \phi - \Phi'$.

The programming of equation (27) is easy. The achieved calculation accuracy is a function of the number N of values that have been computed for Φ along the two half lines "$s = 1 + j$" and $Im(s) < 0$ of the integration contour.

In order to obtain the total sound field as given by (4), one **just** needs to add to the previous results(branch-lines integrals) the sum of the residues (eq.12).

6. CONCLUSION

The various programs presented above have been written down on a CDC 6600 computer. The required computation time is a function of the given source frequency F.
This is evident since the number n of poles that determine the number of modes to be added is roughly given by $n = 2\ FH/C$ where C is the mean sound velocity in the water and H the water depth.

This computation time may hence reach large values. In spite of this it was found that the use of all programs offered numerous advantages beyond the mere aspect of sound field calculation in a given situation. For example they can be used to study the influence of various sea floor parameters on the sound propagation (sediments, density and layering, compressional and shear velocities etc) and the influence of variations in the sound velocity and density profiles (by the use of the "VOLTERRA" program in particular).

The "BESSEL" program can also be employed for the study of various sound channels (ref.9).

Finally it must pointed out that the complete sound field calculation may usefully help in the study of shadow zones and caustics where the geometrical optics approximation is no more valid.

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