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ALGORITHM FOR OPTIMAL FLIGHT VEHICLE DESIGN

by

Ye. V. Tarasov

FOREIGN TECHNOLOGY DIVISION

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ALGORITHM FOR OPTIMAL FLIGHT VEHICLE DESIGN

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ANNOTATION

This book discusses algorithmic methods of planning of flight vehicles using electronic digital computers.

One such method of optimal planning is developed and investigated, including a presentation of the variational problem (strict mathematical definition of the required conditions of optimization) considering analysis of the flight vehicle as a structure which receives various loads in flight, and as an object of control, plus an algorithm for its solution with a mathematical foundation for the algorithm itself. A multistage flight vehicle is used as an example to show the peculiarities of this algorithm when independent maneuver of the vehicle is possible after separation of initial stages or when the external and internal problems are analyzed together during interplanetary flights. In the algorithm, the criterion for improvement of the vehicle depends on the flight time, kinematic parameters of the vehicle at the end of the flight, its launch weight and the payload which it transports.

Combining the theory of the variational problem and of the computer algorithm for its solution provides us with a logically complete algorithm for optimal planning of a vehicle. This allows us, with properly selected power and aerodynamic systems of the vehicle, to determine its optimal planning parameters and flight characteristics.

As an example, appendices present algorithms for optimal planning of various vehicles corresponding to the effectiveness criterion selected, and several results of computer calculations.

The book is designed for scientific workers and engineers in the rocket and aviation industries.

Four tables; 66 figures; 46 bibliographic references.

FOREWORD

During the process of designing of a flight vehicle[1], the designer does not limit himself to development and analysis of any single plan. The planning process involves a number of compromise solutions, resulting in the creation of an effective (optimal) plan for the flight vehicle. There are many different criteria, characterizing the degree of perfection of a flight vehicle in some way. However, a numerical evaluation of the main purpose of planning can be expressed only by some single criterion. We will refer to this criterion as the criterion of effectiveness. The optimal plan for a vehicle must be considered that which corresponds to the highest (or lowest) value of the effectiveness criterion, while all remaining criteria should fall within certain limits.

During the rough development stage, the criteria most commonly used to express the evaluation of a vehicle are the total flight time, value of one of the kinematic parameters at the end of the flight (for example, final velocity or final flight range, etc.), the launch weight and payload, etc. If one of these criteria is accepted as the criterion of effectiveness, the values of the other criteria are either fixed or limited within certain ranges on the basis of the tactical and technical assignment.

In this work, by "optimal flight vehicle" we will mean a vehicle, the plan of which corresponds to the highest value of the criterion of effectiveness characteristic for the initial stage of rough planning.

The increasing interest in problems of optimal planning of vehicles has resulted from the demands of practice. Various trends and even "schools" have appeared, developing various methods of optimization. At

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[1]In this work, the term "flight vehicle" refers to a pilotless vehicle which is used only once.

the present time, in addition to the precise mathematical methods of optimization, approximate methods have also been developed and are being used: direct methods of variational calculus methods of mathematical programming and random search methods.

It is too early as yet to acknowledge superiority of any given method of planning an optimal flight vehicle. Each method has its own positive and negative aspects, and the great variety of methods allows us to penetrate more deeply into the essence of the problem and to find methods to solve it.

The variational methods of planning an optimal vehicle were born of the demands of practice and are now themselves influencing practice and the development of the mathematical theory of optimization.

The works of the pioneers of these methods, A. A. Kosmodem'yanskiy, A. M. Letov, D. Ye. Okhotsimskiy, I. V. Ostoslavskiy, S. V. Rumyantsev, B. I. Rabinovich, T. M. Eneyev, P. Chikal, A. Miyele, J. Leytman and others head a list of works which has grown tremendously[1], which have placed the mathematical theory of variational calculous in the service of the practice of flight vehicle planning.

The introduction of mathematical methods of optimization to the process of flight vehicle planning is possible only when electronic digital computers are used. This requires algorithmization of the methods of optimal planning of flight vehicles. Algorithmization facilitates the introduction of methods of optimal vehicle planning to practice, making them more usable.

In this work, we have studied and developed one algorithm for optimal flight vehicle planning. We present the solution of two problems faced by designers in optimal planning.

The solution of the first problem involves the strict mathematical definition of the necessary conditions for optimization of a flight vehicle. Development of the necessary conditions is based on the apparatus of variational calculus. However, the practice of planning of flight vehicles has set forth variational problems which have not been fully formulated in the mathematical theory of variational calculus.

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[1] A very incomplete conception of this list can be obtained from the bibliography in works [12, 23, 24, 29, 30].

The specifics of the problem of optimal planning of a flight vehicle required the analysis of the theoretical principles of the solution of the variational problem stated (see appendix). The most important works here are those of the School of L. S. Pontryagin [28], the works of G. A. Bliss [7] and M. R. Khestens [36]. This portion of the investigation was also influenced by the works of V. F. Krotov [20, 21] and V. A. Troitskiy [32, 33, 34].

The proper mathematical expression of the variational problem of optimal planning of a flight vehicle is determined by its physical content, i. e. by the extent to which the actual conditions of operation of the vehicle as a design and as a control object are properly represented. Therefore, § 1 of Chapter I is essentially dedicated to the methodological principles of the statement of the variational problem of optimal control.

It has historically developed that most publications dedicated to the application of methods of the mathematical theory of variational calculus [12, 23, 24, 30] have been written on problems of the flight dynamics of vehicles, analyzing only problems of optimization of the control functions. At first, this approach to the solution of the problem could have been justified to some extent, although the assumption that the planned parameters of the vehicle and the power unit were fixed in a number of cases rendered the problem of optimization senseless. However, any process of planning is difficult to imagine without discovering and studying the internal relationships between the problem of selection of a control program and the problem of selection of vehicle and power plant parameters.

The problem of optimal planning, methodologically properly formulated, allows the necessary conditions for optimization of a flight vehicle to be studied completely. This is performed in §2 and 3 of Chapter I.

The mathematical theory of variational methods of optimal planning of a flight vehicle might remain an independent end in itself unless it is analyzed together with the computational problem. The problem is that the solution of the variational problem of an optimal flight vehicle is generally reduced to a multipoint boundary problem, the solution of which requires its own mathematical apparatus. The multipoint boundary problem hinders the introduction of variational optimal planning methods to practice.

Therefore, the second problem was reduced to determination of an algorithm for solution of the multipoint boundary problem suitable for computer realization, and its mathematical justification.

The mathematical theory of the algorithm is presented in § 7 of the appendix. It was developed using [42] and [43].

The mathematical theory of the algorithm of the multipoint boundary problem finds its concrete application in the development of the algorithm of the variational method of optimal planning of a flight vehicle and power plant. The specific content of this algorithm, given in § 4 of Chapter I and § 3 of Chapters II and III, is presented considering its performance by universal digital computers. § 4 of Chapter I presents flow charts of the algorithm and its individual component parts (blocks). This algorithm will be referred to in this work as the computer algorithm for the variational method of optimal planning of a flight vehicle and power plant. Thus, the algorithm for the variational method of optimal planning of the flight vehicle synthesizes the theory of the variational problem of optimal planning of a flight vehicle and its computer algorithm and thereby achieves logical completeness.

The mathematical theory of the variational method of optimal planning of the object is presented in the appendix. It is included in order to make the book easier to read.

The algorithm for optimal planning of a multistage flight vehicle in many cases has its own peculiarities. For example, the algorithm for the variational method of optimization of the principal plan parameters and control of the flight modes of a multistage flight vehicle considering the possibility of independent maneuver of the stages does not fall within the framework of the general problem analyzed in Chapter I. The specifics of this problem are not so much found in the multistage nature of the vehicle as in the possibility of independent maneuver of stages after their separation. Chapter II is dedicated to the study of this problem.

The planning of spacecraft for trips to the planets of the solar system has its own peculiarities and complexities. Therefore, investigation of the variational method and algorithm for optimization of the principal plan parameters and flight control of a multistage spacecraft are discussed in a separate chapter -- Chapter III.

In this chapter, in the investigation of the problem of optimization of the principal plan parameters and modes of movement for interplanetary flights, the external and internal problems are analyzed in combination. As a result of the solution of the problem, a "single" optimal reference phase trajectory can be found. This is one of the main distinctions between this investigation and works published earlier in the scientific and technical literature.

The first and second chapters have appendices, in which the variational method and optimal planning algorithm of various flight vehicles are presented as examples.

Unfortunately, the terminology used in works on the optimization of flight vehicles has not yet been crystallized; therefore, each new term is explained as it is introduced.

All comments on the contents of this book should be sent to: Moscow, K-51, Petrovka, 24, Mashinostroyeniye Press.

CHAPTER I. VARIATIONAL METHOD OF OPTIMAL PLANNING OF A FLIGHT VEHICLE AND POWER PLANT

§ 1. Statement of the Variational Problem of Optimal Planning

A flight vehicle and power plant can be characterized by the plan parameters, design loads and design stresses, phase coordinates and control functions.

The plan parameters refer to the parameters determining the weight characteristics, form and dimensions of the flight vehicle and its design elements. They also include the thrust of the power plant, specific loads on design elements, design fuel reserve, etc.

The design loads express the calculated cases of loading of the flight vehicle and power plant.

The design stresses are the stresses in load bearing elements of the design, above which operation of the structure is not planned.

The phase coordinates refer to the parameters determining the position of the vehicle in space, its velocity, trajectory angle and weight at a given moment in time, etc.

The control functions refer to the parameters of regulation of the power plant and the control functions of the flight vehicle.

Generally speaking, the evaluation of the plan as a whole should be performed considering the influence of the aerodynamic and power design plans of the flight vehicle, the physical and chemical properties of the fuel and the physical and mechanical characteristics of the materials; economic-production and operational factors (cost of fuel and materials used, technological workability of design, reliability, availability of domestic raw materials and industrial base, etc.). However, this global approach to evaluation of a flight vehicle plan presently involves great difficulties, which become practically insurmountable when it is attempted to place them within a single computer algorithm. These difficulties result from the different levels of knowledge of these characteristics, the inability of contemporary mathematical methods to consider the entire combination of requirements for the planning problem and the shortcomings of available computers.

In the present work, the development of the method of optimal planning of a flight vehicle and power plant is performed on the assumption that the type of power plant has been fixed, the type of aerodynamic and strength design of the flight vehicle have been selected and the characteristics of the fuel and material have been determined.

This chapter is directed toward solution of the problem when the flight vehicle and power plant as objects of investigation are characterized only by the plan parameters, design loads and design stresses, the phase coordinates and control functions. The problem of selection of an effective combination of these parameters can be solved by quantitative evaluation of the effectiveness of the flight vehicle, which can be performed using the effectiveness criterion. In this work, the criterion of effectiveness, represented by I, is taken in general form as a function dependent on the value of the vector of phase coordinates at the final point $x_k = (V_k, \theta_k, H_k, L_k, t_k)$, the payload G_{pl} and the launch weight G_0:

$$I = I(x_k, G_{pl}, G_0).$$

This somewhat generalized description of the effectiveness criterion allows us to avoid particular evaluations of the planning goal. However, in solving a concrete problem of optimal planning of a flight vehicle, the criterion of effectiveness should be fixed in explicit form. For example, the payload or launch weight, one of the most capacious criteria for the improvement of a flight vehicle, is frequently used as the criterion of effectiveness.

In this chapter, we will study the problem of determining the neccessary conditions, the set of which may be sufficient for determination of the plan parameters, design loads, design stresses, phase coordinates and control functions as functions of time leading to the highest value of the criterion of effectiveness. These values will also be referred to as optimal in the following.

In order to solve the problem of optimization of parameters, we must expose the relationships between them and place certain limitations on their ranges of change.

Suppose Π is the v

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Here
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where the component ele
l_Φ^* -- the fuselage
l_{kp}^* -- the full win
l_c^* -- the full spa
$\bar{C}^*$ -- the relative
section para
vehicle;
η^* -- the reductio
fuselage;
S_Φ^* -- the area of
S_{kp}^* -- the area of
S_{em}^* -- the area of
F_a^* -- the area of
F_{kp}^* -- the area of
T_p -- the vector o
eters of the
the fuel tan
the power pl

[1] In Chapter I, sca
able V(t), θ(t), H(t),
ω(t) will be marked by

Suppose Π is the vector of a plan parameter. Then, by definition

$$\Pi = (\Gamma,\ T_p,\ Q_\text{т}^\text{р},\ P_{max},\ G_0,\ G_{\text{т.н}},\ G_\text{к},\ G_{0J},\ \mu_{pl}^*,\ \mu_\text{к},\ \mu_{\text{т.н}},\ a_0,\ b_{0I},\ J_0)^1.$$

Here

Γ is the "geometric" vector, determining the dimensions and form of the flight vehicle as a whole, the power plant and other units, and their relative dimensions, i. e.

$$\Gamma = \left(l_\Phi^*,\ l_{\text{кр}}^*,\ l_\text{с}^*,\ \bar{c}^*,\ \eta^*,\ \chi^*,\ S_\Phi^*,\ S_{\text{кр}}^*,\ S_{em}^*,\ F_a^*,\ \frac{S_{em}^*}{S_{\text{кр}}^*},\ \frac{F_a^*}{F_{\text{кр}}^*},\ \ldots\right),$$

where the component elements of the vector are:

l_Φ^* -- the fuselage length;

l_{kp}^* -- the full wing span;

l_c^* -- the full span of the stabilizer;

$\bar{C}^*$ -- the relative thickness of the profile measured in the cross section parallel to the plane of symmetry of the flight vehicle;

η^* -- the reduction in the wings with the portion covered by the fuselage;

S_Φ^* -- the area of characteristic cross section of the fuselage;

S_{kp}^* -- the area of the wings;

S_{em}^* -- the area of the empennage;

F_a^* -- the area of the output cross section of the motor nozzle;

F_{kp}^* -- the area of the critical nozzle cross section;

T_p -- the vector of the design, characteristic thermodynamic parameters of the power plant, influencing the size and weight of the fuel tanks, motor, turbine-pump unit and other units of the power plant, i. e.

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[1] In Chapter I, scalar quantities, except for g_0 and g, phase variable V(t), θ(t), H(t), L(t), s(t) and the control functions α(t) and ω(t) will be marked by an asterisk.

$$T_p = (T_3^*, p_k^*, p_a^*, p_H^*, p_b^*, \ldots),$$

where the component elements of the vector are:

p_k^* -- the pressure in the liquid-fuel rocket motor combustion chamber;

p_a^* -- the pressure in the output cross section of the motor nozzle;

p_b^* -- the pressure in the fuel tanks;

p_H^* -- the pressure at the output of the turbine pump unit;

T_3^* -- the gas temperature before the turbine;

Q_t^p -- the vector of specific heat flux, determining the specific heat fluxes to the characteristic points on the surface of the flight vehicle;

P_{max} --the vector of the maximum thrust, determining the maximum thrusts, corrected to the corresponding conditions, of the power plants of the stages of the flight vehicle;

J_0 -- the vector of specific thrust corresponding to P_{max}^* and determining the specific thrusts of the power plants of the stages corrected to the fixed conditions;

μ_k -- the vector of the relative final weight of the stage or the vector of the relative final weight determining the relative final weights of the stages, i. e.

$$\mu_k = (\mu_{k1}^*, \ldots, \mu_{kN}^*);$$

$\mu_{t \cdot o}$ --the vector of relative design reserve of working fluid, determining the relative design reserves of working fluid of each stage, i. e.

$$\mu_{T.o} = (\mu_{T.o}^{(1)*}, \ldots, \mu_{T.o}^{(n)*});$$

G_k -- the vector of the final weight of a stage or the vector of the final weight determining the final weights of the stages, i. e.

$$G_k = (G_{k1}^*, \ldots, G_{kN}^*);$$

G_0 -- the vector of the initial weight of a stage, or the vector of initial weight determining the initial weights of the stages, i. e.

$$G_0 = (G_{01}^*, \ldots, G_{0N}^*);$$

G_{0J} -- the vector of the payload (cargo) of the stage or the vector of the payload (cargo) determining the payloads of the stages, i. e.

$$G_{0J}=(G^{*}_{011},\ldots,G^{*}_{0N});$$

$G_{T \cdot o}$ -- the vector of design working fluid reserve, determining the design reserve of working fluid for each stage, i. e.

$$G_{T.o}=(G^{(1)*}_{T.o},\ldots,G^{(n)*}_{T.o});$$

a_0 -- the thrust/weight ratio vector, determining the thrust/weight ratio of the stages, i. e. $a_0 = (a^{*}_{01},\ldots, a^{*}_{0n})$;

b_{0i} -- the vector of the specific midship section, characterizing the specific loads from the initial weight of the stage on the component parts of the vehicle, i. e.

$$b_{0i}=\left(\frac{S^{(i)*}_{kp}}{G^{*}_{0J}},\ \frac{S^{(i)*}_{\phi}}{G^{*}_{0J}},\ \ldots\right).$$

It is important to keep in mind that the vector components noted must be mutually independent.

For a proper understanding of the geometric vector or plan parameter vector, we must note the strength vector δ. The strength vector is a vector, the component elements of which are the parameters characterizing the strength properties of the load-bearing structural elements related to their dimensions and form. For example, its components might be the thickness of the skin over the fuselage δ^{*}_{ϕ}, the wing δ^{*}_{kp}, the thickness of the fuel tank shells δ^{*}_{0} and δ^{*}_{f}, etc., i. e. $\delta = (\delta^{*}_{\phi}, \delta^{*}_{kp}, \delta^{*}_{0}, \delta^{*}_{f},\ldots)$.

The range of change of the plan parameter vector is limited. The limitation may result from conditions of arrangement of the parts of the flight vehicle, the level of technology, technological difficulties, etc. Thus, the selection of the vector of a plan parameter is related to the following condition of limitation of the plan parameter vector:

$$\Pi_{min}<\Pi<\Pi_{max}.$$

Furthermore, the following relationship obtains:

$$G_0=G_k+G^{*}_{T.o}\ \text{or}\ \mu_k+k_T\mu_{T.o}=1,$$ [1]

---

[1]Here and in the following, a product should be considered a tensor product.

where

$$k_{\text{т}} = k_{\text{т}}(Q_{\text{т}}^{p}, T_{p}, \Gamma, a_0, G_0, J_0) = (k_{\text{т}}^{(s)*})$$

(s is the number of the fuel sector) is a vector function considering the difference between the design fuel reserve and the operational reserve due to losses to evaporation and the requirement for a guaranteed fuel reserve, the fuel reserve required to fill the fuel lines, etc.

Due to this condition and the definition of the thrust/weight ratio vector, the vector of the plan parameter can be represented as follows:

$$\Pi = (\Gamma, T_{p}, Q_{\text{т}}^{p}, G_0, \mu_{p1}^{*}, \mu_{\text{к}}, a_0, b_{0l}, J_0),$$

although in this case we must keep in mind the relationships

$$\mu_{\text{к}} + k_{\text{т}}\mu_{\text{т.о}} - 1 = 0, \quad G_0 a_0 - P_{max} = 0,$$
$$G_0\mu_{\text{т.о}} - G_{\text{т.о}} = 0, \quad G_0 b_{0l} - \Gamma = 0,$$

which will be referred to as equations of plan parameter vector relationships.

In designing a flight vehicle for strength, the problem arises of selecting design loading cases, i. e. conditions under which the vehicle is most heavily loaded. If the vehicle is sufficiently strong in the design cases, we can be sure that it will be strong in all other cases of operation. The design loads are determined by the design loading cases, which generally occur among the following groups of loads:

a) loads on the vehicle in flight;
b) loads on the vehicle when being transported by a carrier (air to air and air to ground vehicles, etc.);
c) loads on the vehicle when launched;
d) loads on the vehicle during operations on the ground.

Generally speaking, for the class of vehicles which we are considering (pilotless single-operation vehicles), the design loading cases are primarily determined by the loads on the vehicle in flight. In this work, we evaluate the design loading cases of a flight vehicle which can appear only in flight. Furthermore, for greater definition we analyze only program-controlled apparatus, in which the maneuver loads are not random functions.

In the process of preliminary planning of a vehicle, establishment of the design loading cases and particularly determination of the parameters of the design loads is one of the most complex, cumbersome and important tasks. The responsibility of this task results from the fact that the loads fixed must provide a satisfactory but not excessive level of structural strength. This requires, in addition to a knowledge of all loads acting on the load-bearing elements of the apparatus at each point in its trajectory, a precise conception of the operating conditions of the load-bearing elements which influence their strength and provision of the ability of the structure to accept these loads without rupture. The difficulty of determining the design loading cases of a flight veh .le in flight are particularly increased if the phase trajectory is not known in advance, and even more so if a phase trajectory of the vehicle is such that the technical assignment is performed with the optimal value of the effectiveness criterion. Since determination of the optimal phase trajectory must be closely related to an evaluation of the load-bearing capacity of the structural elements of the vehicle, in this work we will use the design load vector N^p as the quantity to be varied, indicating the design cases of loading.

Two methods of construction of the system of design cases are encountered in practice: the method of the dominating loading and the method of the arbitrary loading.

The essence of the method of the dominating loading is that the stress rate resulting from the influence of a combination of loadings at those moments in time when one of the component loadings reaches its greatest value is studied. However, this method does not fully consider the specifics of the operating conditions of the structure. The approximate nature of this method also results from the fact that in the general case (with combined loadings), the maximum equivalent stress, taken as the criterion of danger of the complexly stressed state of the load-bearing element actually has no influence.

True, in this case in order to eliminate a possible misunderstanding in designation of design loading cases and particularly confusion in the interpretation of sufficient strength, the design cases of loading are fixed by the "strength norms," composed on the basis of the experience of planning, building, testing and operation of flight vehicles, accumulated constantly and systematized on the state-wide scale.

The "strength norms" establish the obligatory level of strength of a flight vehicle. Generally speaking, this is done by stating the maximum value of operational load; the maximum permissible value of negative acceleration; the maximum permissible velocity head and the safety factor, considering the number of times by which the design load should exceed the operating loads.

The essence of the method of arbitrary loads, which does not have these defects, is that the design load used in this case is the load at the moment in the flight when a certain arbitrary load reaches its maximum. The value of this load is determined considering the effective combined influence of forces and the effect of heating of structures.

Regardless of the method of construction of the design cases of loading, in this work the design loads will be represented by the design load vector N^p. It is related to the operating vector of design loads by the following relationship:

$$N^p = f_{k.b} N^e .$$

In calculations for the strength and weight of structures in a flight vehicle, it is insufficient to know the vector of a plan parameter and the vector of the design loads; the distribution of external loads, temperatures and specific or summary heat fluxes must also be fixed. It is assumed in this work that the vector of specific heat flux and the design load vector are fixed on the basis of known distributions of external loads, temperatures and specific heat fluxes. They are determined on the basis of the vector of design loads, using the strength norms. Knowledge of the vector of the plan parameter, the vector of design loads, the distribution of external loads, temperatures and specific heat fluxes with a fixed rule of change of permissible stresses (allowing the strength vector to be determined) makes it possible, considering the empirical coefficients, to calculate the weight of the flight vehicle and its component parts. The weight equation for the flight vehicle can be written in general as follows:

$$G_x^* = (1 + k_0^*) G_{p1}^* + \sum_{j=1}^{m} \gamma_j^* (1 + k_1^* + \dots + k_l^*) \times \int_0^{l_j} \frac{P_j^*(N^p, \Gamma, T_p, Q_z^p, G_0, a_0, h_{kr}, J_0, \sigma_{zj}^*)}{\sigma_{ej}^*} dx^*,$$

where $j = 1, \dots, m$ is the number of the load-bearing element of the flight vehicle;

P_j^* is the equivalent power function acting on the jth element;

σ_{ej}^* is the equivalent design stress of the jth element;

γ_j^* is the specific weight of the material of the jth element;

$k_0^*, k_1^*, \dots, k_l^*$ are empirical coefficients, produced statistically and considering the weight of the units, non-load-bearing parts, structural and technological peculiarities.

The appearance of empirical coefficients results from the fact that the share of the weight of nonload-bearing elements and the share of the weight reflecting the structural and technological peculiarities of the apparatus in question are not expressed in functional form using the plan parameter vector, the design load vector and the distribution of permissible stresses, but rather can be expressed only statistically.

Since the distribution of the external load, heat fluxes and temperatures is generally a function of x and since the inertial loads depend primarily on the weight of the structures (which is being determined), it is not difficult to see that any attempt to take the integrals in the weight equation by ordinary methods will be extremely difficult or quite impossible [40]. In one particular case, determination of these integrals in closed form can be performed only with a large number of simplifying assumptions. The most important step in the development of reliable methods for solution of the weight equation is the creation of a system allowing integration by simple methods considering the influence of all the most important variable quantities.

During the solution of other problems stated in this work, we will not analyze the various possible methods for solution of the weight equation considering the most important variables, but will simply assume that these solutions are available.

Suppose, using known methods, a solution to the weight equation is produced in the form:

$$\mu^{*}_{p1} = f^{*}_{\mu p}(N^{p}, \Gamma, T_{p}, Q^{p}_{t}, G_{0}, \mu_{k}, a_{0}, b_{0l}, J_{0}, \sigma^{p}, k),$$

or

$$\mu^{*}_{p1} = f^{*}_{\mu p}(N^{p}, R, Q^{p}_{t}, \sigma^{p}, G_{0}, \mu_{k}, a_{0}, b_{0l}, J_{0}, k),$$

where $R = (\Gamma, T_p)$ is the design parameter vector;

σ^p is the design stress vector, determining the design stress on the structure in its characteristic cross sections;[1]

k is the vector of the empirical coefficients.

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[1]Vector σ^p is similar to the vector of permissible stress. It should not be confused with the actual equivalent stresses arising in structural elements in flight.

This dependence will be referred to as the plan equation[1].

It should be kept in mind here that the design stress vector has an upper limit. Therefore, we should introduce the condition of limitation of the design stress vector, expressed in the form

$$\sigma^p \leq \sigma_{max},$$

where σ_{max} is the vector of maximum permissible stress.

This type of plan equation is rather broadly used for analysis of the effectiveness of various types of flight vehicles and various power plants, the difference between these equations for vehicles of a given class being not so much in the structure and form of the independent variables as in the number of terms included in the formula, the degree of their detailed development and the clarification of the empirical coefficients.

By the same method, we can produce

$$G_{0j} = G_0 f^j_{np}(N^p, R, Q^p_T, \sigma^p, G_0, \mu_k, a_0, b_{0l}, J_0, k).$$

Vectors N^p, σ^p, Q^p_T and R allow us to calculate the strength vector δ. It is determined from the strength vector equation

$$\delta = f_{0\delta}(N^p, \sigma^p, R, Q^p_T),$$

where $f_{0\delta} = (f^{1*}_{0\delta}, \dots, f^{m*}_{0\delta})$ is the vector function of the strengths of the load-bearing elements of the structure, the number of which is assumed equal to m.

The interval of change of strength vector δ is limited:

$$\delta_{min} \leq \delta \leq \delta_{max}.$$

The lower limit δ_{min} is determined by the design cases of loading of the vehicle before flight (§§ b-d, see loading groups) or by the production conditions. The upper limit δ_{max} is dictated by conditions of arrangement and placement of the load. Sometimes these conditions

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[1]In the technical literature, it is sometimes called the weight equation or the equation of existence of the flight vehicle [8].

lead to limitation of the corrected strength vector δ:

$$\bar{\delta} \leqslant \bar{\delta}_{max}.$$

Generally speaking, the design load vector or the design stress vector can be replaced by the strength vector. Then the variable quantities will be δ and σ^p or N^p. This replacement does not introduce any significant changes to the course of solution of the problem or influence its solution, due to the presence of the strength vector equation.

The placement of the operational reserve of working medium places certain requirements on the dimensions of the flight vehicle and must be related to the thermodynamic characteristics of the power plant and fuel compartment. Therefore, this interrelationship must be considered, which we will express as the following condition for placement of the working medium:

$$f_m(R, \delta, G_{т.о}, k_т, k_R) = 0,$$

or considering the equations relating the plan parameter vector

$$f_m(R, \zeta_i^p, a_0, G_0, \mu_к, J_0, k_R) = 0,$$

where k_R is the vector of the coefficients considering the free volume in the tanks, the volume of tank equipment, the free volume between tanks, etc;

$f_m = (f_m^{s^*})$ is the vector function.

If the mutual relationships between the plan parameters, design loads, strength characteristics and design stresses in the characteristic cross sections are limited to the relationships produced, it will be impossible to estimate the effectiveness of the flight vehicle to any extent, its ability to perform a required maneuver and correspond to the fixed TA.

Full determination of the physical and functional relationships requires determination of the interaction of the phase coordinates of the control function, plan parameters, effective loads and stresses.

The phase coordinates determining the phase state of the vehicle include the coordinates of the apparatus in the corresponding systems of coordinates, velocity and weight, required specific heat fluxes to structural elements and their temperature at the fixed moment in time.

They change with time. In the following, we will investigate the movement of the vehicle only in the plane of a great circle. The components of vector x, characterizing the phase state of the vehicle, will be V, θ, H, L, s, μ and Q_T, T_W. The phase variables $V(t)$, $\theta(t)$, $H(t)$, $L(t)$, $s(t)$, $Q_T(t)$, $T_W(t)$ will be assumed continuous and piecewise-differentiable throughout the entire attainable phase space.

In the following, the investigation of optimization of the parameters and control of a flight vehicle and power plant is performed without considering the dynamic characteristics of the control organs, i. e. only the static characteristics of the motor and balancing aerodynamic characteristics of the flight vehicle are analyzed. This removes the limitations on the inertial nature of the control systems and allows non-inertial changes in the control function. Therefore, we will use the balancing angle of attack α, the angle between the velocity vector and thrust vector ω and the motor adjustment parameters as the control functions.

In this chapter, we study the motor as an object of control with several degrees of freedom. The control functions used are the motor control parameters, with which the thrust and working medium flow rate per second are changed. Therefore, the number of control functions or the number of motor control parameters is equal to the number of its degrees of freedom. This approach to the motor as a control object allows the application of the investigations performed in this chapter to concrete types of power plants, for example jet engines and solid and liquid fueled rocket motors. The motor control parameters will be divided into two types: parameters choking the flow rate of working medium per second d_s^* ($s = 1,\ldots, n$) and parameters directly controlling the thrust of the motor r_p^* ($p = 1,\ldots, R$).

The parameters choking the working medium flow rate d_s^* will refer to control parameters which influence changes in the thrust by changing the flow rate of working medium per second. Parameters directly regulating thrust r_p^* are parameters which, when changed, do not lead to a change in the flow rate of working medium per second, but rather influence only the thrust. For example, they include the area of the critical or output cross section of the nozzle, the input area of the diffusor, etc., changes in which within certain limits lead to no change in the fuel flow rate.

Of course, this division of control parameters is quite arbitrary. It depends on the regulators used and the conditions of stable operation of the power plant when regulated. It must be noted that one of the requirements placed on control parameters is that they be autonomous, i. e. independent of each other.

Furthermore, we will consider that with unchanged external conditions, the thrust and flow rate of working medium per second are convex (concave) functions of d_s^* and r_p^* with positive derivatives from these to the maximum values of d_s^* and r_p^*. All control parameters can be reduced to this sort of dependence. The control parameters d_s^* and r_p^* are related only with the static characteristics of the power plant.

Suppose d is the vector of choking parameters or simply the choking vector, while r is the vector of direct thrust control parameters or the thrust vector. Then the control parameters of the flight vehicle and power plant will be d, r, α and ω. They can be fixed in the class of piecewise-continuous functions. Piecewise-continuous change in the control functions corresponds to the assumption that non-inertial control is permissible. However, this assumption is possible only if there is a finite number of moments in time at which the control functions undergo first order discontinuities. Furthermore, it is permitted if in the realization of the piecewise-continuous control, the left and right points of the discontinuity can be connected by modes of optimal pickup with slight overcontrol (with low overshoot factor) and if the time of this transient (almost aperiodic) process can be ignored in comparison to the remaining flight control time. Otherwise, we must look at the dynamic characteristics of the control organs and consider the influence of the overshoot factor on the plan parameters and loads. We will return to this fact in our investigation of the necessary conditions determining the optimal values of the control functions.

The area of change of the control functions is limited by the conditions

$$\alpha_{min} \leqslant \alpha \leqslant \alpha_{max},$$
$$\omega_{min} \leqslant \omega \leqslant \omega_{max},$$
$$d_{min} \leqslant d \leqslant d_{max}, \; r_{min} \leqslant r \leqslant r_{max}.$$

The limiting (maximal or minimal) values of the control functions depend on the design parameters, velocity and altitude of flight. They can be interrelated, and it is possible that the instantaneous value of one control function influences the limiting values of another. For example, in a flight vehicle with a supersonic ramjet engine, the limiting angle of attack is related to the stable operation of the diffusor, the maximum throat opening of which is limited by the temperature in the combustion chamber. Furthermore, this sort of interrelationship and this sort of influence may appear differently and may depend on the geometry of the vehicle, the velocity, flying altitude, etc. Limitation of the control functions and possible interaction of their limiting values are determined to a great extent by the conditions not allowing unstable operation of the power plant (surging of the compressor, diffusor, pulsating combustion chamber operation, etc.), flow separation over the lifting surface, etc. Therefore, the selection of a limiting (maximal or minimal) control vector is related to the limiting control condition, which in the general case can be represented by the following vector function:

$$\zeta = (\zeta_1^{*}, \ldots, \zeta_l^{*}) = \zeta(u_m, \alpha, d, r, \omega, V, H, R) = 0,$$

where l is the number of the control functions;

u_m is the vector of limiting values of the control functions:

$$u_m = (\alpha_{max}, \alpha_{min}, d_{max}, d_{min}, r_{max}, r_{min}, \omega_{max}, \omega_{min}).$$

Then the condition of permissible control will be the inequality $\zeta \neq 0$, represented in the following as

$$\zeta > 0.$$

The effective loads and effective stresses, acting through the control function, in the final analysis "organize" the required trajectory, over which the flight vehicle performs its assigned maneuver. However, in the process of flight the changes in these quantities are limited by the vector of design loads and the vector of design stresses. The vector function of effective loads N, determining the effective loads acting on the vehicle in flight, can be represented in the general case as the vector function

$$N = (N_1^{*}, \ldots, N_m^{*}) = f_N(n_x^{*}, n_y^{*}, \alpha, \omega, d, r, \mu^{*}, -\mu_{\tau}, T_w, Q_{\tau}, V, H, R, G_0, \delta)$$

or

$$N = f_N(n_x^{*}, n_y^{*}, \alpha, \omega, d, r, \mu^{*}, T_w, Q_{\tau}, V, H, R, G_0, \delta, G_{0J}),$$

where m is the number of design load parameters.

The flight should always be organized so that the condition of limitation of the effective load vector is fulfilled:

$$N^{e} \geqslant N \quad \text{or} \quad N^{e} \geqslant f_N.$$

During the flight, various surface and mass forces may act on the flight vehicle, including a variable temperature field. Although each effective load may not reach the design values, combinations are possible which exceed the design stresses defined considering heating of the structure.

Let σ be the vector function of effective stresses, determining the stresses in the load-bearing elements of the structure arising in flight (effective stresses). It can be functionally represented as follows in the general case:

$$\sigma = \sigma(N, T_w, Q_{\tau}, G_0, R, \delta).$$

then, a maneuver of the flight vehicle can be performed if the condition of limitation of the effective stress vector is not disrupted, as written considering changes in the design stress as functions of the temperature of the structure in the form

$$\sigma^p \geqslant f_\sigma(\dot{\sigma}, T_w)^{1)}$$

or

$$\sigma^p \geqslant f_\sigma(\dot{\alpha}, \omega, d, r, n_x^*, n_y^*, \mu, V, H, T_w, Q_{\tau}, G_0, R, \delta, G_{\omega}).$$

In evaluating the change in the effective stress vector, one should assume that all possible measures have been taken in the structure (use of compensators, multilayered panels, heat insulation, etc.) to decrease the harmful influence of aerodynamic heating on the elements of the flight vehicle.

During flight, the interaction of aerodynamic, inertial and elastic forces resulting from the finite rigidity of the structure of the flight vehicle leads to deformations. Therefore, the phenomena related to aerothermal elasticity can lead to undesirable processes in flight under certain conditions -- flutter, divergences, reversal of control surfaces, etc. -- which are very dangerous for the structure of the flight vehicle. These dangerous phenomena arise at the critical velocity head, the value of which depends on the geometric parameters of the flight vehicle, its arrangement, the temperature of the skin and is related through $c_y^{\alpha^*}$ to the velocity and flight altitude [5, 6]. However,

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[1] Depending on the statement of the problem, only this or the preceding condition of limitation should be used; however, this cannot always reflect the actual complex stressed state of the structure.

if certain requirements (rigidity norms) are followed during planning, the dangers can be avoided. On this basis, the conditions for elimination of dangerous aerothermal elasticity phenomena, which have been called the conditions of limitation of structural rigidity, can be represented in the general case as follows [6, 13]:

$$f_r(R, Q_\tau, T_w, V, H, \lambda, N^p) \leqslant 0.$$

Here $f_r = (f_r^{(1)*}, \ldots, f_r^{(k)*})$ is a vector function, where k is equal to the number of dangerous aerothermal elasticity phenomena.

In connection with the changes in external conditions and loadings, inertial forces and aerodynamic heating of the structural elements during the flight, various changes in the nature of thermodynamic parameters of the power plant are possible, some of which in many cases may result in unstable power plant operation (cavitation in the fuel pump, unstable operation due to excessive expansion in a nonadjustable nozzle, etc.). Therefore, conditions must be created which allow unstable operation of the power plant in flight to be avoided. We will represent these conditions as limitations on the vector function

$$f_{OW}(d, r, \alpha, \omega, \mu^*, V, H, T_w, Q_\tau, G_0, R) \leqslant 0$$

and will refer to them as the limiting conditions for stable operation of the power plant.

A flight vehicle is maneuvered under certain predetermined conditions with respect to the phase coordinates at the beginning and end of the flight. Fixation of the relationships between phase coordinates (or simply fixation of the phase coordinates) at the beginning and end of the flight will be represented as the following boundary conditions:

$$\left.\begin{aligned} &\varphi^*_{0\varrho}(V(t_0), \theta(t_0), H(t_0), L(t_0), s(t_0), t_0)=0, \\ &\qquad (\varrho=1, \ldots, \leqslant 6). \\ &\varphi^*_{\kappa\sigma}(V(t_\kappa), \theta(t_\kappa), H(t_\kappa), L(t_\kappa), s(t_\kappa), t_\kappa)=0, \\ &\qquad (\sigma=1, \ldots, \leqslant 6). \end{aligned}\right\} \quad (1.1.1)$$

Furthermore, for a multistage flight vehicle, the following dependences must be observed at the beginning and end of operation of each stage:

$$\left.\begin{aligned} &\mu_0 - 1 = 0, \\ &\varphi_\mu = \mu_\kappa^{(i)} - \mu_\kappa = 0, \\ &\varphi_Q = Q_{\tau\kappa}^{(i)} - Q_\tau^p = 0, \end{aligned}\right\} \quad (1.1.1)$$

where $\mu_0 = (\mu^*(t^{(i-1)})$ is the vector of the relative initial weight of each stage, or the vector of the relative initial weight;

$\mu_k^{(i)} = (\mu^*(t^i))$ is the vector function of the relative final weight of each stage;

$Q_{Tk}^{(i)}$ is the vector of required specific heat fluxes to the structural elements of a stage.

The first equation determines the relative initial weight of each stage. Fulfillment of the second condition indicates the end of the active sector of each stage. For the last stage (in the case of a multistage apparatus) when a passive sector at the end can be allowed, the second condition reflects the limitation on the phase variable $\mu^*(t)$. Since the function $\psi_\mu^{(n)*}$ is identical to zero, when the phase trajectory reaches the boundary of the phase variable $\mu^*(t)$, its derivative with respect to time is also equal to zero:

$$\frac{d\psi_\mu^{(n)*}}{dt} = 0$$

or, since

$$\psi_\mu^{(n)*\prime} = -\frac{\dot{a}_{0n}}{\dot{I}_{0n}} f^* = \mu_{Ts}^{(n)*}(t^{(n)}),$$

$$\mu_{Ts}^{(n)*} = p^{(n)*} = 0.$$

This last condition should be added to (1.1.1).

Therefore, when the condition $\psi_\mu^{(n)*} = 0$ is reached in the last stage, the optimal phase trajectory passes along this boundary to the final point, and in this sector of the phase trajectory

$$p^{(n)*} = 0 \quad \text{and} \quad \mu_{Ts}^{(n)*} = 0.$$

Thus, we have looked at all possible relationships between the plan parameters, design loads, strength characteristics and design stresses, considering various limitations, and have shown their interactions with the control functions and phase coordinates; the boundary conditions for maneuver of the flight vehicle have been considered. However, these relationships are insufficient -- there are no dependences revealing the change in the phase coordinates with time resulting from the influence of the control functions and considering the influence of the plan parameters on the aerodynamic forces and choke characteristics of the power plant. This requires that we use the differential equations of motion of the center of mass of the flight vehicle and the kinematic couples. We shall make the following assumptions (system of coordinates and diagram of forces shown on Figure 1.1):

-- the flight vehicle is looked upon as a material point;
-- the movement of the flight vehicle occurs in a vertical plane;
-- the movement of the flight vehicle occurs without slipping, i. e. the velocity vector is located in the plane of symmetry;
-- during the entire period of movement, the flight vehicle is balanced as to moments;
-- there are no turbulent disturbances of the atmosphere and its parameters in this sense are stable;
-- the earth is spherical and its rotation about its axis is ignored.

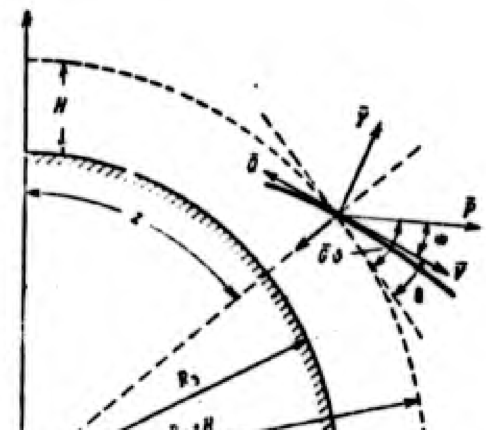

Figure 1.1. System of Coordinates and Diagram of Forces Acting on Flight Vehicle.

We write the equations of motion

$$\varphi_1^{\bullet} \equiv V' = \frac{g_0}{\mu^{\bullet}}(a_{0i}^{\bullet} p^{\bullet} \cos\omega - Q^{\bullet}) - g \sin\theta, \qquad (1.1.2)$$

$$\varphi_2^{\bullet} \equiv \theta' = \frac{g_0}{\mu^{\bullet} V}(a_{0i}^{\bullet} p^{\bullet} \sin\omega + Y^{\bullet}) - \frac{g\cos\theta}{V} + \frac{V}{R_з + H}\cos\theta, \qquad (1.1.3)$$

(1.1)

the kinematic couples

$$\varphi_3^* = H' = V \sin\theta, \qquad (1.1.4)$$

$$\varphi_4^* = L' = V \frac{R_3}{R_3 + H} \cos\theta, \qquad (1.1.5)$$

$$\varphi_5^* = s' = V \quad \text{or} \quad V_{cp} = \frac{1}{t_\kappa} \int_{t_0}^{t_\kappa} V\, dt. \qquad (1.1.6)$$

Here

$$Q^* = Q^*(V, H, b_{0i}, R, G_0, \alpha),$$
$$Y^* = Y^*(V, H, b_{0i}, R, G_0, \alpha),$$
$$g = g(H).$$

The introduction of equation (1.1.6) results primarily from the possible requirement for production of a fixed average velocity during maneuvering of the flight vehicle. This requirement is sometimes placed on certain types of flight vehicles to assure "controlability."

The choking characteristics of each type of power plant have their own peculiarities. However, it is characteristic that these characteristics depend on d, r and R. Furthermore, for some types of motors (jets), these characteristics are related to velocity V and altitude H of flight, while for others (liquid fueled and solid fueled rockets) they are influenced only by altitude. Therefore, generalizing the possible dependences, relative thrust and relative expenditure of working medium per second, we can represent them as the following functional dependence:

$$p^* = p^*(d, r, V, H, R, G_0, a_0, J_0, b_{0i}),$$
$$\mu^*_{rs} = \frac{\dot{a}_{0i}}{\dot{J}_{0i}} f^*(d, V, H, R, G_0, a_0, J_0, b_{0i}).$$

From which, since $\mu^*_s = -d\mu^*/dt$, we have

$$\varphi_6^* = \frac{d\mu^*}{dt} = -\frac{\dot{a}_{0i}}{\dot{J}_{0i}} f^*(d, V, H, R, G_0, a_0, J_0, b_{0i}). \qquad 1.1.7)\ (1.I)$$

It was noted above that the "behavior" of the structure of a flight vehicle must be evaluated not only on the basis of the actual surface and mass forces, but also on the basis of the temperature of its elements and the heat fluxes acting on them. This is particularly important at high flight velocities. The principal forms of transmission of heat to flight vehicle structural elements are convection radiation and conduction. The transmission of heat to the skin of a flight vehicle occurs due to convection and radiation. Usually, the physical processes related to transmission of heat to the structure are described in this manner. However, analytic representation of the processes accompanying heat transmission is very difficult, which reinforces the instability of the phenomena still more. In order to make the suggested method of solution more flexible and allow it to be applied regardless of the selected method of calculation of the specific heat fluxes and temperatures, let us represent their changes with time by the most general functional dependence

$$\frac{dQ_\tau}{dt^*}=f_q(\alpha_\tau, T_w, V; H, R, \delta),$$

$$\frac{dT_w}{dt^*}=f_w(\alpha_\tau, T_w, V, H, R, \delta),$$

which is typical at the present time. Here $Q_\tau = (Q_\tau^{(1)*},\ldots, Q_\tau^{(Q)*})$, $T_w = (T_w^{(1)*},\ldots, T_w^{(Q)*})$ are the vectors of actual specific heat flux and actual temperature of the structural elements, $f_q = (f_q^{(1)*},\ldots, f_q^{(Q)*})$ and $f_w = (f_w^{(1)*},\ldots, f_w^{(Q)*})$ are the corresponding vector functions. Analysis of calculation of structural temperatures and heat fluxes to the structural elements shows that the greatest volume of computational work is that involved in determining the heat conduction coefficient α_τ from the surrounding medium (boundary layer) to the surface of the structure with unstable heating. Therefore, a number of methods have been suggested for the calculation of α_τ, based on the semiempirical theory of turbulence, and a number of purely empirical formulas produced on the basis of experiments. Generalizing them, we can represent α_τ by the following functional dependence:

$$\alpha_\tau=\alpha_\tau(\alpha, V, H, T_w, R).$$

We then have

$$\left.\begin{aligned} \varphi_7 &\equiv \frac{dQ_\tau}{dt}=f_q(\alpha, V, H, T_w, \delta), && (1.1.8)\\ \varphi_8 &\equiv \frac{dT_w}{dt}=f_w(\alpha, V, H, T_w, R, \delta). && (1.1.9) \end{aligned}\right\} \quad (1.I)$$

In the functions f_q and f_w, we do not find the blackness coefficient, Stephan-Boltzman constant, heat conductivity coefficient or other physical-mechanical characteristics of the structural materials, since they are known parameters. The relationships produced are correct under the condition that the measures allowing reduction in heat flux to the structures do not require a change in the weight of the flight vehicle as a result of carrying away of the mass of heat insulating materials or cooling agents. In this case, equation (1.1.7) is correct as well. Since the influence of aerodynamic heating can appear only at rather high M numbers of flight, equations (1.1.8) and (1.1.9) should be used only after the flight vehicle has reached a certain velocity.

As the flight vehicle moves, as we have noted, we should not disrupt:

-- the condition of permissible control

$$\dot{\zeta} \geqslant 0, \qquad (1.1.10)$$

-- the condition of limitation of the actual (effective) load vector

$$N^p \geqslant f_{rf} N, \quad \text{or} \quad N^p \geqslant f_{sf} f_N, \qquad (1.1.11)$$

the condition of limitation of the effective stress vector

$$\sigma^p \geqslant f_\sigma \sigma,[1] \qquad (1.1.12)$$

-- the condition of limitation of the stable operation of the power plant

$$f_{o.y} \leqslant 0, \qquad (1.1.13)$$

-- the condition of limitation of structural rigidity $f_r \leqslant 0$.

Since vector function f_r does not depend on the control functions, the conditions of limitation of structural rigidity essentially reflect the conditions of limitation on changes in phase coordinates. We

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[1]Generally speaking, one of the conditions (1.1.11) or (1.1.12) can be replaced by the condition of limitation of the vector of effective strength $\delta \geq f_{kb}\delta^{\pi}$, where δ^{π} is the vector of effective strength.

introduce for analysis the vector function z_r, equal to

$$z_r = \left(\sum_{j=1}^{n} \frac{\partial f_r^{(1)*}}{\partial x_j^*} \varphi_j^*, \ldots, \sum_{j=1}^{n} \frac{\partial f_r^{(k)*}}{\partial x_j^*} \varphi_j^* \right),$$

where

$$x_j = \{V, \ldots, Q_\tau, T_w\},$$
$$\varphi_j = \{\varphi_1^*, \ldots, \varphi_q, \varphi_w\}.$$

If the structure "works" at the rigidity boundary, then

$$\phi_r = f_r = 0,$$

and therefore

$$z_r = 0.$$

In the case where $f_r \leq 0$ and $z_r \neq 0$. In the following, we will assume

$$z_r \leqslant 0, \quad (1.1.14a)$$

where the equality occurs when

$$\phi_r = 0. \quad (1.1.14b)$$

Equation (1.1.14b) for the phase trajectory sectors, one of which lies within the closed area of change of f_r^*, the other being at the rigidity boundary, acts as a sort of boundary condition.

The limiting conditions (1.1.10)-(1.1.14) should be formulated so that with the corresponding values of phase coordinates and plan parameter vector, the number of simultaneously occurring boundaries (equalities) will not exceed the number of control functions. Otherwise, the "extra" limiting conditions must be looked upon as conditions on the ends.

Furthermore, the plan parameters, strength characteristics or strength parameters, design loads and design stresses in the characteristic cross sections are interrelated by the following relationships:

the plan equation

$$\xi_1^{0*} \equiv \mu^*_{pl} - f^*_{np} = 0, \quad \xi_1^j \equiv G_{0j} - G_0 f^j_{np} = 0, \qquad (1.1.15)(1.1)$$

the equations relating the plan parameter vector

$$\left.\begin{array}{l} \xi_2^* \equiv G_0^* \mu^*_{pl} - G^*_{pl} = 0, \\ \xi_3 \equiv G_0 a_0 - P_{max} = 0, \\ \xi_4 \equiv G_0 b_{0l} - \Gamma = 0, \end{array}\right\} \qquad (1.1.16)(1.1)$$

the condition of placement of the working medium

$$\xi_5 \equiv f_m = 0. \qquad (1.1.17)(1.1)$$

the strength vector equation

$$\xi_6 \equiv \delta - f_{0\delta} = 0, \qquad (1.1.18)(1.1)$$

and their selection should not disrupt:

the plan parameter vector limiting condition

$$\Pi_{min} \leqslant \Pi \leqslant \Pi_{max}, \qquad (1.1.19)$$

the design stress vector limiting condition

$$\sigma_{max} \geqslant \sigma^p, \qquad (1.1.20)$$

the strength vector limiting condition

$$\delta_{min} \leqslant \delta \leqslant \delta_{max}, \qquad (1.1.21)$$

the adjusted strength vector limiting condition

$$\bar{\delta}_{max} \Gamma \geqslant \delta. \qquad (1.1.22)$$

In addition to (1.1.16), there are other equations of the relationship of the plan parameter vector

$$k_T \mu_{T.e} = 1 - \mu_k,$$
$$G_{T.0} = G_0 \mu_{T.0},$$

which need not be analyzed in the problem as relationships, since μ_{to} and G_{to} are not included in the remaining equations.

We can now formulate the problem of optimal planning of a flight vehicle and motor installation. It consists of the following: among the permissible values of the phase state vector x and control functions d, r, α and ω, plan parameter vector Π, design load vector N^P, design stress vector σ^P and strength vector δ, satisfying boundary conditions (1.1.1) and, possibly, (1.1.14b), coupling equations (1.I) and limitations (1.1.10)-(1.1.14a) and (1.1.19)-(1.1.22), find values for which the criterion of effectiveness of the flight vehicle

$$I^* = I(V_k, H_k, L_k, \theta_k, s_k, t_k, \mu^*_{p/})$$

reaches the precise upper boundary

$$I^* = \sup.$$

This problem is a variational problem, and therefore its solution must be sought on the basis of the mathematical apparatus of variational calculus. This is done by using the equations (see appendix)

$$\left.\begin{aligned} \varphi_9 &\equiv \zeta - \nu_u \nu_u = 0, \\ \varphi_{10} &\equiv D^P - D - \nu\nu = 0, \end{aligned}\right\} \qquad (1.1.23)\,(1.I)$$

$$\left.\begin{aligned} \varphi_{0\pi} &\equiv (\Pi_{max} - \Pi)(\Pi - \Pi_{min}) - \omega_\pi \omega_\pi = 0, \\ \varphi_{0\sigma} &\equiv (\sigma_{max} - \sigma^P) - \omega_\sigma \omega_\sigma = 0, \\ \varphi_{0\delta} &\equiv (\delta_{max} - \delta)(\delta - \delta_{min}) - \omega_\delta \omega_\delta = 0, \\ \varphi_{0\Gamma} &\equiv (\bar{\delta}_{max}\Gamma - \delta) - \omega_\Gamma \omega_\Gamma = 0 \end{aligned}\right\} \qquad (1.1.24) \quad (1.I)$$

to make a transition from the closed to the open area of permissible changes of control functions, design load vector, design stress vector and plan parameter vector, strength vector and relative strength vector.

Here $D^P = (N^P, \sigma^P, 0, 0_\gamma)$ is the design parameter vector;
$D = (f_{k \cdot b}N, fc, z_r, f_{o \cdot y})$ is the vector function of the required parameter;
$\nu_u(t)$, $\nu(t)$ are the vector functions of the conditional controls, which are arbitrary functions of time;
ω_π, ω_σ, ω_δ, ω_f are the vectors of conditional parameters which are independent of time.

In connection with this, the variational problem can be formulated as follows.

In the class of phase variables

$$V(t),\ \theta(t),\ H(t),\ L(t),\ s(t),\ \mu(t)^*,\ T_w(t),\ Q_T(t) \qquad (1.1.25)$$

control functions

$$d(t),\ r(t),\ \alpha(t),\ \omega(t), \qquad (1.1.26)$$

vector parameters

$$\Pi,\ N^p,\ \sigma^p,\ \delta, \qquad (1.1.27)$$

conditional control functions and parameters

$$\nu_w(t),\ \nu(t),\ \omega_n,\ \omega_e,\ \omega_k,\ \omega_r, \qquad (1.1.28)$$

permissible in the interval $t_0 \leq t \leq t_k$, satisfying the relationships (1.I) and boundary conditions (1.1.1) and possibly (1.1.14b), find phase variables (1.1.25), control functions (1.1.26) and vectors (1.1.27) for which the expression

$$I^*(V_k,\ \theta_k,\ H_k,\ L_k,\ s_k,\ t_k,\ \mu^*_{pl}) \qquad (1.1.29)$$

reaches its maximal value.

In the following, control u(t) = (α(t), ω(t), d(t), r(t)), phase trajectory x(t) = (V(t), θ(t), H(t), L(t), s(t), $\mu^*(t)$, $Q_T(t)$, $T_W(t)$) and vector parameters Π, N^P, σ^P and δ, satisfying the solution of this variational problem will be referred to as optimal.

The mathematical model of this variational problem, the functional of which depends not only on the phase coordinates, but also on the parameters, considering the relationships and limitations outlined, has not been fully discussed in the mathematical literature. The scientific and technical literature has contained analyses only of certain partial solutions of this problem [16, 29, 30]. Proof of the necessary and sufficient conditions of the maximum of the functional of this variational problem of optimal planning of a flight vehicle and the algorithm for its solution are presented in the appendix.

It can therefore be stated (see appendix) that the conditional extreme of functional (1.1.29) is reached on the same curves on which the unconditional extreme of the expression

$$\Phi^* = I^* + e_1^m \gamma_1^m + e_1^l \gamma_1^l + e_2^* \gamma_2^* + e_3 \gamma_3 +$$

$$+ e_4 \gamma_4 + e_5 \gamma_5 + e_6 \gamma_6 + e_7 \gamma_{0m} + e_8 \gamma_{0s} + e_9 \gamma_{0k} + e_{10} \gamma_{0r} + e_0 \psi_0 +$$

$$+ e_k \psi_k + e_q \psi_q + e_\mu \psi_\mu + \int_{t_0}^{t_k} F^* dt^{1)}, \qquad (1.1.30)$$

is realized, where e_0, e_q, e_k, e_μ ..., e_{10} are the vectors of constants of the Lagrange factors;

$$F^* = (x' - \varphi)\lambda(t); \quad \varphi = (\varphi_1^*, \ldots, \varphi_{10});$$

$$\psi_0 = (\psi_{01}^*, \ldots, \psi_{03}^*); \quad \psi_k = (\psi_{k1}^*, \ldots, \psi_{k6}^*);$$

$\lambda = (\lambda_1(t), \ldots, \lambda_k(t), \lambda_q(t), \lambda_\mu(t))$ is the vector function of the variable Lagrange factors.

§ 2. Necessary Condition for Optimization (Stability Condition)

The first necessary condition for the maximum of functional (1.1.30) is the condition of stability. As will be seen below, it includes the conditi of optimal cont:ol of the flight vehicle and the power plant, the condition of discontinuity at the moment of a sudden change of control and in the case of arrival at the rigidity boundary, the condition of discontinuity at the moment of separation of stages for multistage flight vehicles and the condition of optimization of parameters. Furthermore, the condition of transversality follows from it.

The stability condition is contained i. the equality of the first variation of functional (1.1.30) to zero (see appendix):

$$d\Phi^* = dI^* + e(d\gamma + d\psi) - \left[\left(F^* - x' \frac{\partial F^*}{\partial x'}\right) dt + \lambda\, dx\right]_{t_0} +$$

$$+ \int_{t_0}^{t_1^*} \left(\frac{\partial F^*}{\partial x} - \frac{d}{dt} \frac{\partial F^*}{\partial x'}\right) \Delta x\, dt + \int^{t_1^*} \frac{\partial F^*}{\partial u} \Delta u\, dt +$$

---

[1] Here and throughout the following, the product of vectors should be looked upon as a scalar product.

$$+\left(\int_{t_0}^{t_1^*}\frac{\partial F^*}{\partial \Pi}\,dt\right)\Delta\Pi+\left(\int_{t_0}^{t_1^*}\frac{\partial F^*}{\partial N^p}\,dt\right)\Delta N^p+\left(\int_{t_0}^{t_1^*}\frac{\partial F^*}{\partial \sigma^p}\,dt\right)\Delta\sigma^p+$$

$$+\left(\int_{t_0}^{t_1^*}\frac{\partial F^*}{\partial \delta}\,dt\right)\Delta\delta+\left[\left(F^*-x'\frac{\partial F^*}{\partial x'}\right)dt+\lambda\,dx\right]_{t_{1-}^*}-$$

$$-\left[\left(\frac{\partial F^*}{\partial x}-x'\frac{\partial F^*}{\partial x'}\right)dt+\lambda\,dx\right]_{t_1^*}+$$

$$+\int_{t_1^*}^{t^{(1)}}\left(\frac{\partial F^*}{\partial x}-\frac{d}{dt}\frac{\partial F^*}{\partial x'}\right)\Delta x\,dt+\int_{t_1^*}^{t^{(1)}}\frac{\partial F^*}{\partial u}\Delta u\,dt+$$

$$+\left(\int_{t_1^*}^{t^{(1)}}\frac{\partial F^*}{\partial \Pi}\,dt\right)\Delta\Pi+\left(\int_{t_1^*}^{t^{(1)}}\frac{\partial F^*}{\partial N^p}\,dt\right)\Delta N^p+$$

$$+\left(\int_{t_1^*}^{t^{(1)}}\frac{\partial F^*}{\partial \sigma^p}\,dt\right)\Delta\sigma^p+\left(\int_{t_1^*}^{t^{(1)}}\frac{\partial F^*}{\partial \delta}\,dt\right)\Delta\delta+$$

$$+\left[\left(F^*-x'\frac{\partial F^*}{\partial x'}\right)dt+\bar{\lambda}\,d\bar{x}+\lambda_6\,d\mu\right]_{t^{(1)}_{-}}-$$

$$-\left[\left(F^*-x'\frac{\partial F^*}{\partial x'}\right)dt+\bar{\lambda}\,d\bar{x}+\lambda_6\,d\mu\right]_{t^{(1)}_{+}}+$$

$$+\int_{t^{(1)}}^{t^{(1)}_{\Psi}}\left(\frac{\partial F^*}{\partial x}-\frac{d}{dt}\frac{\partial F^*}{\partial x'}\right)\Delta x\,dt+\int_{t^{(1)}}^{t^{(1)}_{\Psi}}\frac{\partial F^*}{\partial u}\Delta u\,dt+$$

$$+\left(\int_{t^{(1)}}^{t^{(1)}_{\Psi}}\frac{\partial F^*}{\partial \Pi}\,dt\right)\Delta\Pi+\left(\int_{t^{(1)}}^{t^{(1)}_{\Psi}}\frac{\partial F^*}{\partial N^p}\,dt\right)\Delta N^p+$$

$$+\left(\int_{t^{(1)}}^{t_{\Psi}}\frac{\partial F^*}{\partial \sigma^p}\,dt\right)\Delta\sigma^p+\left(\int_{t^{(1)}}^{t^{(1)}_{\Psi}}\frac{\partial F^*}{\partial \delta}\,dt\right)\Delta\delta+$$

$$
+\left[\left(F^{*}-x'\frac{\partial F^{*}}{\partial x'}\right)dt+\lambda\,dx\right]_{t^{(1)}_{r-}}-\left[\left(F^{*}-x'\frac{\partial F^{*}}{\partial x'}\right)dt+\lambda\,dx\right]_{t^{(1)}_{r+}}+
$$

$$
+\int\limits_{t^{(1)}_{r}}^{t^{(2)}_{r}}\left(\frac{\partial F^{*}}{\partial x}-\frac{d}{dt}\frac{\partial F^{*}}{\partial x'}\right)\Delta x\,dt+\int\limits_{t^{(1)}_{r}}^{t^{(2)}_{r}}\frac{\partial F^{*}}{\partial u}\Delta u\,dt+
$$

$$
+\left(\int\limits_{t^{(1)}_{r}}^{t^{(2)}_{r}}\frac{\partial F^{*}}{\partial \Pi}\,dt\right)\Delta\Pi+\left(\int\limits_{t^{(1)}_{r}}^{t^{(2)}_{r}}\frac{\partial F^{*}}{\partial N^{p}}\,dt\right)\Delta N^{p}+
$$

$$
+\left(\int\limits_{t^{(1)}_{r}}^{t^{(2)}_{r}}\frac{\partial F^{*}}{\partial \sigma^{p}}\,dt\right)\Delta\sigma^{p}+\left(\int\limits_{t^{(1)}_{r}}^{t^{(2)}_{r}}\frac{\partial F^{*}}{\partial \delta}\,dt\right)\Delta\delta+
$$

$$
+\left[\left(F^{*}-x'\frac{\partial F^{*}}{\partial x'}\right)dt+\lambda\,dx\right]_{t^{(2)}_{r-}}-\left[\left(F^{*}-x'\frac{\partial F^{*}}{\partial x'}\right)dt+\lambda\,dx\right]_{t^{(2)}_{r+}}+
$$

$$
+\int\limits_{t^{(2)}_{r}}^{t_{k}}\left(\frac{\partial F^{*}}{\partial x}-\frac{d}{dt}\frac{\partial F^{*}}{\partial x'}\right)\Delta x\,dt+\int\limits_{t^{(2)}_{r}}^{t_{k}}\frac{\partial F^{*}}{\partial u}\Delta u\,dt+
$$

$$
+\left(\int\limits_{t^{(2)}_{r}}^{t_{k}}\frac{\partial F^{*}}{\partial \Pi}\,dt\right)\Delta\Pi+\left(\int\limits_{t^{(2)}_{r}}^{t_{k}}\frac{\partial F^{*}}{\partial N^{p}}\,dt\right)\Delta N^{p}+
$$

$$
+\left(\int\limits_{t^{(2)}_{r}}^{t_{k}}\frac{\partial F^{*}}{\partial \sigma^{p}}\,dt\right)\Delta\sigma^{p}+\left(\int\limits_{t^{(2)}_{r}}^{t_{k}}\frac{\partial F^{*}}{\partial \delta}\,dt\right)\Delta\delta+
\qquad (1.2.1)
$$

$$
+\left[\left(F^{*}-x'\frac{\partial F^{*}}{\partial x'}\right)dt+\lambda\,dx\right]_{t_{k}}=0,
$$

where

$$
e=(e_{1},\;.\;.\;.\;.\;,e_{10},e_{q},e_{p},e_{r},e_{0},e_{k});
$$

$$
\zeta=(\zeta_{1},\;.\;.\;.\;.\;,\zeta_{6},\zeta_{0m},\zeta_{0l},\zeta_{0n},\zeta_{0r});
$$

$$
\varphi=(\varphi_{0},\varphi_{x},\varphi_{r},\varphi_{\varphi});
$$

$$
\bar{x}=(V,\theta,H,L,s,T_{w},Q_{s});
$$

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C^*_0 i

$$\bar{\lambda}=(\lambda_1^*, \lambda_2^*, \lambda_3^*, \lambda_4^*, \lambda_5^*, \lambda_q, \lambda_w);$$

$$u=(\alpha, \omega, d, r, v_w, \nu);$$

$$\frac{\partial F^*}{\partial \Pi}=\left(\frac{\partial F^*}{\partial R}, \frac{\partial F^*}{\partial a_0}, \frac{\partial F^*}{\partial b_{0i}}, \frac{\partial F^*}{\partial G_0}, \frac{\partial F^*}{\partial l_0}\right);$$

t_j^* is the moment in time at which the control function undergoes a first order discontinuity;

$t^{(i)}$ is the moment of separation of the ith stage.

For simplicity in recording the stability condition, we have assumed that a first order discontinuity in the control functions occurs once and that the flight vehicle has two stages.

Placing the corresponding requirements on the Lagrange coefficient λ and applying the main lemma of variational calculus to the integrals with free variation Δu (see appendix), we produce the Euler-Lagrange equation

$$\left.\begin{aligned} &\frac{\partial F^*}{\partial x}-\frac{d}{dt}\frac{\partial F^*}{\partial x'}=0, \\ &\frac{\partial F^*}{\partial u}=0. \end{aligned}\right\} \qquad (1.2.2)$$

Since F^* in its explicit form is independent of time, system (1.2.2) has the first integral

.2.1)

$$F^*-x'\frac{\partial F^*}{\partial x'}=C_0^* \quad \text{or} \quad H^*=-C_0^*. \qquad (1.2.3)$$

where

$$\begin{aligned} H^*=\lambda_1^*\left[\frac{g_0}{\mu^*}(a_{0i}^* p^* \cos\omega-Q^*)-g\sin\theta\right]+\lambda_2^*\left[\frac{g_0}{\mu^* V}(a_{0i}^* p^* \sin\omega+Y^*)-\right.\\ \left.-\frac{g\cos\theta}{V}+\frac{V\cos\theta}{R_3+H}\right]+\lambda_3^* V\sin\theta+\lambda_4^*\frac{VR_3}{R_3+H}\cos\theta+\\ +\lambda_5^* V-\lambda_6^*\frac{\dot{a}_{0i}}{f_{0i}^*}f^*+\lambda_q f_q+\lambda_w f_w; \end{aligned} \qquad (1.2.4)$$

C_0^* is a constant quantity.

Equation (1.2.3) can replace any of the differential equations in (1.2.2). where

According to system (1.I), equation (1.2.2) can be represented in explicit form as follows:

$$\left.\begin{aligned}
&\lambda_1^{\bullet\prime} = -\lambda_1^\bullet \frac{\partial\dot\varphi_1}{\partial V} - \lambda_2 \frac{\partial\dot\varphi_2}{\partial V} - \lambda_3 \frac{\partial\dot\varphi_3}{\partial V} - \lambda_4 \frac{\partial\dot\varphi_4}{\partial V} - \\
&\quad - \lambda_5 \frac{\partial\dot\varphi_5}{\partial V} - \lambda_6^\bullet \frac{\partial\dot\varphi_6}{\partial V} - \frac{\partial\varphi_7}{\partial V}\lambda_q - \frac{\partial\varphi_8}{\partial V}\lambda_w - \\
&\quad - \frac{\partial\varphi_9}{\partial V}\lambda_s - \frac{\partial\varphi_{10}}{\partial V}\lambda_D, && (1.2.5)\\
&\lambda_2^{\bullet\prime} = -\lambda_1^\bullet \frac{\partial\dot\varphi_1}{\partial\theta} - \lambda_2^\bullet \frac{\partial\dot\varphi_2}{\partial\theta} - \lambda_3^\bullet \frac{\partial\dot\varphi_3}{\partial\theta} - \lambda_4^\bullet \frac{\partial\dot\varphi_4}{\partial\theta} - \\
&\quad - \frac{\partial\varphi_{10}}{\partial\theta}\lambda_D, && (1.2.6)\\
&\lambda_3^{\bullet\prime} = -\lambda_1^\bullet \frac{\partial\dot\varphi_1}{\partial H} - \lambda_2^\bullet \frac{\partial\dot\varphi_2}{\partial H} - \lambda_4^\bullet \frac{\partial\dot\varphi_4}{\partial H} - \lambda_6^\bullet \frac{\partial\dot\varphi_6}{\partial H} - \\
&\quad - \frac{\partial\varphi_7}{\partial H}\lambda_q^\bullet - \frac{\partial\varphi_8}{\partial H}\lambda_w - \frac{\partial\varphi_9}{\partial H}\lambda_s - \frac{\partial\varphi_{10}}{\partial H}\lambda_D, && (1.2.7)\\
&\lambda_4^{\bullet\prime} = 0 \quad \text{or} \quad \lambda_4^\bullet = \text{const}, && (1.2.8)\\
&\lambda_5^{\bullet\prime} = 0 \quad \text{or} \quad \lambda_5^\bullet = \text{const}, && (1.2.9)\\
&\lambda_6^{\bullet\prime} = -\lambda_1^\bullet \frac{\partial\dot\varphi_1}{\partial\mu^\bullet} - \lambda_2^\bullet \frac{\partial\dot\varphi_2}{\partial\mu^\bullet} - \frac{\partial\varphi_{10}}{\partial\mu^\bullet}\lambda_D, && (1.2.10)\\
&\lambda_q' = -\frac{\partial\varphi_{10}}{\partial Q_q}\lambda_D, && (1.2.11)\\
&\lambda_w' = -\frac{\partial\varphi_7}{\partial T_w}\lambda_q - \frac{\partial\varphi_8}{\partial T_w}\lambda_w - \frac{\partial\varphi_{10}}{\partial T_w}\lambda_D, && (1.2.12)\\
&\lambda_1^\bullet \frac{\partial\dot\varphi_1}{\partial d} + \lambda_2^\bullet \frac{\partial\dot\varphi_2}{\partial d} + \lambda_6^\bullet \frac{\partial\dot\varphi_6}{\partial d} + \frac{\partial\varphi_9}{\partial d}\lambda_s + \frac{\partial\varphi_{10}}{\partial d}\lambda_D = 0, && (1.2.13)\\
&\lambda_1^\bullet \frac{\partial\dot\varphi_1}{\partial r} + \lambda_2^\bullet \frac{\partial\dot\varphi_2}{\partial r} + \frac{\partial\varphi_9}{\partial r}\lambda_s + \frac{\partial\varphi_{10}}{\partial r}\lambda_D = 0, && (1.2.14)\\
&\lambda_1^\bullet \frac{\partial\dot\varphi_1}{\partial\alpha} + \lambda_2^\bullet \frac{\partial\dot\varphi_2}{\partial\alpha} + \frac{\partial\varphi_7}{\partial\alpha}\lambda_q + \frac{\partial\varphi_8}{\partial\alpha}\lambda_w + \frac{\partial\varphi_9}{\partial\alpha}\lambda_s + \\
&\quad + \frac{\partial\varphi_{10}}{\partial\alpha}\lambda_D = 0, && (1.2.15)\\
&\lambda_1^\bullet \frac{\partial\dot\varphi_1}{\partial\omega} + \lambda_2^\bullet \frac{\partial\dot\varphi_2}{\partial\omega} + \frac{\partial\varphi_9}{\partial\omega}\lambda_s + \frac{\partial\varphi_{10}}{\partial\omega}\lambda_D = 0, && (1.2.16)\\
&\lambda_s \nu_s = 0, \quad \lambda_D \nu = 0, && (1.2.17)
\end{aligned}\right\} \quad (1.\text{II})$$

where

$$\frac{\partial\varphi_1^*}{\partial V}=\frac{g_0}{\mu^*}\left(a_{0i}^*\frac{\partial p^*}{\partial V}\cos\omega-\frac{\partial Q^*}{\partial V}\right),$$

$$\frac{\partial\varphi_2^*}{\partial V}=\left\{-\frac{1}{V}\left[\frac{g_0}{\mu V}(a_{0i}^*p^*\sin\omega+Y^*)-\frac{g\cos\theta}{V}-\right.\right.$$

$$\left.\left.-\frac{V\cos\theta}{R_3+H}\right]+\frac{g_0}{\mu V}\left(a_{0i}^*\frac{\partial p^*}{\partial V}\sin\omega+\frac{\partial Y^*}{\partial V}\right)\right\},$$

$$\frac{\partial\varphi_3^*}{\partial V}=\sin\theta,\quad \frac{\partial\varphi_4^*}{\partial V}=\frac{R_3}{R_3+H}\cos\theta,$$

$$\frac{\partial\varphi_5^*}{\partial V}=1,\quad \frac{\partial\varphi_6^*}{\partial V}=-\frac{a_{0i}^*}{I_0^*}\frac{\partial f^*}{\partial V},$$

$$\frac{\partial\varphi_7}{\partial V}=\frac{\partial f_q}{\partial V},\quad \frac{\partial\varphi_8}{\partial V}=\frac{\partial f_w}{\partial V},\quad \frac{\partial\varphi_9}{\partial V}=\frac{\partial\zeta}{\partial V},\quad \frac{\partial\varphi_{10}}{\partial V}=-\frac{\partial D}{\partial V};$$

$$\frac{\partial\varphi_1^*}{\partial\theta}=-g\cos\theta,\quad \frac{\partial\varphi_2^*}{\partial\theta}=\left(\frac{g}{V}-\frac{V}{R_3+H}\right)\sin\theta,$$

$$\frac{\partial\varphi_3^*}{\partial V}=V\cos\theta,\quad \frac{\partial\varphi_4^*}{\partial\theta}=-\frac{R_3V}{R_3+H}\sin\theta,\quad \frac{\partial\varphi_{10}}{\partial\theta}=-\frac{\partial D}{\partial\theta},$$

$$\frac{\partial\varphi_1^*}{\partial H}=\frac{g_0}{\mu^*}\left(a_{0i}^*\frac{\partial p^*}{\partial H}\cos\omega-\frac{\partial Q^*}{\partial H}\right)-\frac{\partial g}{\partial H}\sin\theta,$$

$$\frac{\partial\varphi_2^*}{\partial H}=\frac{g_0}{\mu V}\left(a_{0i}^*\frac{\partial p^*}{\partial H}\sin\omega+\frac{\partial Y^*}{\partial H}\right)-\frac{\partial g}{\partial H}\frac{\cos\theta}{V}-\frac{V\cos\theta}{(R_3+H)^2},$$

$$\frac{\partial\varphi_4^*}{\partial H}=-\frac{VR_3}{(R_3+H)^2}\cos\theta,\quad \frac{\partial\varphi_6^*}{\partial H}=-\frac{a_{0i}^*}{I_0^*}\frac{\partial f^*}{\partial H},$$

$$\frac{\partial\varphi_7}{\partial H}=\frac{\partial f_q}{\partial H},\quad \frac{\partial\varphi_8}{\partial H}=\frac{\partial f_w}{\partial H},\quad \frac{\partial\varphi_9}{\partial H}=\frac{\partial\zeta}{\partial H},\quad \frac{\partial\varphi_{10}}{\partial H}=-\frac{\partial D}{\partial H};$$

$$\frac{\partial\varphi_1^*}{\partial\mu^*}=-\frac{g_0^*}{\mu^{*2}}(a_{0i}^*p^*\cos\omega-Q^*),$$

$$\frac{\partial\varphi_2^*}{\partial\mu^*}=-\frac{g_0^*}{V\mu^{*2}}(a_{0i}^*p^*\sin\omega+Y^*),$$

$$\frac{\partial\varphi_{10}}{\partial\mu^*}=-\frac{\partial D}{\partial\mu^*};$$

$$\frac{\partial\varphi_{10}}{\partial Q_{ч}}=-\frac{\partial D}{\partial Q_{ч}};$$

$$\frac{\partial\varphi_7}{\partial T_w}=\frac{\partial f_q}{\partial T_w},\quad \frac{\partial\varphi_8}{\partial T_w}=\frac{\partial f_w}{\partial T_w},\quad \frac{\partial\varphi_{10}}{\partial T_w}=-\frac{\partial D}{\partial T_w};$$

$$\frac{\partial\dot{\varphi}_1}{\partial d}=\frac{\dot{g}_0\dot{a}_{0l}}{\mu^\bullet}\frac{\partial p^\bullet}{\partial d}\cos\omega,\quad \frac{\partial\dot{\varphi}_2}{\partial d}=\frac{\dot{g}_0\dot{a}_{0l}}{\mu V}\frac{\partial p^\bullet}{\partial d}\sin\omega,$$

$$\frac{\partial\dot{\varphi}_6}{\partial d}=-\frac{\dot{a}_{0l}}{\dot{J}_{0l}}\frac{\partial f^\bullet}{\partial d},\quad \frac{\partial\varphi_9}{\partial d}=\frac{\partial\mathcal{K}}{\partial d},\quad \frac{\partial\varphi_{10}}{\partial d}=-\frac{\partial D}{\partial d};$$

$$\frac{\partial\dot{\varphi}_1}{\partial r}=\frac{\dot{g}_0\dot{a}_{0l}}{\mu^\bullet}\frac{\partial p^\bullet}{\partial r}\cos\omega,\quad \frac{\partial\dot{\varphi}_2}{\partial r}=\frac{\dot{g}_0\dot{a}_{0l}}{\mu V}\frac{\partial p^\bullet}{\partial r}\sin\omega,$$

$$\frac{\partial\varphi_9}{\partial r}=\frac{\partial\mathcal{K}}{\partial r},\quad \frac{\partial\varphi_{10}}{\partial r}=-\frac{\partial D}{\partial r};$$

$$\frac{\partial\dot{\varphi}_1}{\partial\alpha}=-\frac{\dot{g}_0}{\mu^\bullet}\frac{\partial Q^\bullet}{\partial\alpha},\quad \frac{\partial\dot{\varphi}_2}{\partial\alpha}=\frac{\dot{g}_0}{\mu V}\frac{\partial Y^\bullet}{\partial\alpha},\quad \frac{\partial\varphi_7}{\partial\alpha}=\frac{\partial f_q}{\partial\alpha},$$

$$\frac{\partial\varphi_8}{\partial\alpha}=\frac{\partial f_w}{\partial\alpha},\quad \frac{\partial\varphi_9}{\partial\alpha}=\frac{\partial\mathcal{K}}{\partial\alpha};\quad \frac{\partial\varphi_{10}}{\partial\alpha}=-\frac{\partial D}{\partial\alpha};$$

$$\frac{\partial\dot{\varphi}_1}{\partial\omega}=-\frac{\dot{g}_0\dot{a}_{0l}p^\bullet}{\mu}\sin\omega,\quad \frac{\partial\varphi_2}{\partial\omega}=\frac{\dot{g}_0\dot{a}_{0l}}{\mu V}\cos\omega,\quad \frac{\partial\varphi_9}{\partial\omega}=\frac{\partial\mathcal{K}}{\partial\omega},$$

$$\frac{\partial\varphi_{10}}{\partial\omega}=-\frac{\partial D}{\partial\omega}.$$

The control programmed defined on the basis of the solutions of the Euler-Lagrange equations (1.II) and the coupling equation (1.I) will be called the optimal control program, and the corresponding phase trajectory will be called the optimal phase trajectory. The Euler-Lagrange equations (1.2.13)-(1.2.17), related to the presence of control functions in the coupling equations should be interpreted as the optimal control conditions.

The solutions of equations (1.2.17) separate the areas of possible optimal controls. They can be as follows:

$$\lambda_u=0,\ \nu_u\neq 0,\ \lambda_D=0,\ \nu\neq 0, \qquad (1.2.18)$$

$$\lambda_u\neq 0,\ \nu_u=0,\ \lambda_D=0,\ \nu=0, \qquad (1.2.19)$$

$$\lambda_u=0,\ \nu_u=0,\ \lambda_D=0,\ \nu\neq 0, \qquad (1.2.20)$$

$$\lambda_D\neq 0,\ \nu=0,\ \lambda_u=0,\ \nu_u\neq 0, \qquad (1.2.21)$$

$$\lambda_D=0,\ \nu=0,\ \lambda_u=0,\ \nu_u\neq 0 \qquad (1.2.22)$$

or $\lambda_u=0,\ \nu_u=0.$

Solutions (1.2.18)-(1.2.20) determine a control not involving the limiting value of the vector of the required parameter D, while solutions (1.2.21), (1.2.22) define a control involving the limiting value of D. In connection with this, let us introduce a number of concepts of optimal control.

Optimal control will refer to stable control, if it is determined from the Euler-Lagrange equations (1.2.13)-(1.2.16) according to solutions (1.2.18). Thus, the control function found falls within its limiting values and is not limited by the limiting value of vector D, while the equations appear as follows for the corresponding control function:

$$\frac{g_0 \dot{a}_{0i}}{\mu^*}\left(\lambda_1^* \cos\omega + \lambda_2^* \frac{\sin\omega}{V}\right)\frac{\partial p^*}{\partial d} - \lambda_6^* \frac{\dot{a}_{0i}}{J_{0i}^*}\frac{\partial f^*}{\partial d} = 0, \tag{1.2.23}$$

$$\frac{\partial p^*}{\partial r} = 0, \tag{1.2.24}$$

$$\frac{g_0}{\mu^*}\left(\lambda_1^* \frac{\partial Q^*}{\partial \alpha} - \frac{\lambda_2^*}{V}\frac{\partial Y^*}{\partial \alpha}\right) - \frac{\partial f_\varphi}{\partial \alpha}\lambda_\varphi - \frac{\partial f_\omega}{\partial \alpha}\lambda_\omega = 0, \tag{1.2.25}$$

$$\lambda_1 \sin\omega - \lambda_2 \frac{\cos\omega}{V} = 0. \tag{1.2.26}$$

Here we have assumed $\lambda_1^* \cos\omega + \lambda_2^* \frac{\sin\omega}{V} \neq 0$, since otherwise we come to degeneration of the variational problem.

The control determined from the Euler-Lagrange equations (1.2.13)-(1.2.16) considering (1.2.19) is referred to as the optimal limiting control. In this case, the control function takes on one of its limiting values according to the condition $\zeta = 0$, while equations (1.2.13)-(1.2.16) become

$$\frac{\partial \mathcal{K}}{\partial d}\lambda_u + \frac{g_0 \dot{a}_{0i}}{\mu^*}\left(\lambda_1 \cos\omega + \lambda_2 \frac{\sin\omega}{V}\right)\frac{\partial p^*}{\partial d} - \lambda_6 \frac{\dot{a}_{0i}}{J_{0i}^*}\frac{\partial f^*}{\partial d} = 0, \tag{1.2.27}$$

$$\frac{\partial \mathcal{K}}{\partial r}\lambda_u + \frac{g_0 \dot{a}_{0i}}{\mu^*}\left(\lambda_1^* \cos\omega + \lambda_2 \frac{\sin\omega}{V}\right)\frac{\partial p^*}{\partial r} = 0, \tag{1.2.28}$$

$$-\frac{\partial \mathcal{K}}{\partial \alpha}\lambda_u + \frac{g_0}{\mu^*}\left(\lambda_1^* \frac{\partial Q^*}{\partial \alpha} - \frac{\lambda_2^*}{V}\frac{\partial Y^*}{\partial \alpha}\right) - \frac{\partial f_\varphi}{\partial \alpha}\lambda_\varphi - \frac{\partial f_\omega}{\partial \alpha}\lambda_\omega = 0, \tag{1.2.29}$$

$$-\frac{\partial \mathcal{K}}{\partial \omega}\lambda_u + \frac{g_0 \dot{a}_{0i} p^*}{\mu^*}\left(\lambda_1^* \sin\omega - \lambda_2^* \frac{\cos\omega}{V}\right) = 0. \tag{1.2.30}$$

Equations (1.2.20) characterize the conditions of possible switching (transition) from the stable to optimal limiting control or back.

The control defined from the Euler-Lagrange equations (1.2.13)-(1.2.16) considering (1.2.21) is called the optimal coupled control. In this case, the control function takes on values resulting from the boundary of the required parameter vector according to equation $D^p - D = 0$. The Euler-Lagrange equations (1.2.13)-(1.2.16) become

$$\frac{g_0 \dot{a}_{0i}}{\mu^*}\left(\lambda_1^* \cos\omega + \lambda_2^* \frac{\sin\omega}{V}\right)\frac{\partial p^*}{\partial d} - \lambda_6^* \frac{\dot{a}_{0i}}{\dot{f}_{0i}} \frac{\partial f^*}{\partial d} - \frac{\partial D}{\partial d}\lambda_D = 0, \qquad (1.2.31)$$

$$\frac{g_0 \dot{a}_{0i}}{\mu^*}\left(\lambda_1^* \cos\omega + \lambda_2^* \frac{\sin\omega}{V}\right)\frac{\partial p^*}{\partial r} - \frac{\partial D}{\partial r}\lambda_D = 0, \qquad (1.2.32)$$

$$\frac{g_0}{\mu^*}\left(\lambda_1^* \frac{\partial \bar{Q}^*}{\partial \alpha} - \frac{\lambda_2^*}{V}\frac{\partial Y^*}{\partial \alpha}\right) - \lambda_q \frac{\partial f_q}{\partial \alpha} - \frac{\partial f_w}{\partial \alpha}\lambda_w + \frac{\partial D}{\partial \alpha}\lambda_D = 0, \qquad (1.2.33)$$

$$\frac{g_0 \dot{a}_{0i} P^*}{\mu^*}\left(\lambda_1^* \sin\omega - \lambda_2^* \frac{\cos\omega}{V}\right) + \frac{\partial D}{\partial \omega}\lambda_D = 0. \qquad (1.2.34)$$

Equations (1.2.22) characterize the conditions of possible transition from optimal coupled control to stable control or to optimal limiting control.

Thus, the optimal operating modes of the power plant are possible with maximum and minimum modes corresponding to equations (1.2.27) and (1.2.24) or (1.2.28), with a choked mode corresponding to equations (1.2.23) and (1.2.24) or (1.2.28). If the maximum, minimum or choked mode of the power plant determined by equations (1.2.23) and (1.2.24) or (1.2.28) leads to disruption of the limiting conditions of the vector of the required parameter D [conditions (1.1.11)-(1.1.14)], the operating mode of the power plant will be determined by the boundary of the required parameter vector. In this last case, the thrust is generally less than its maximum value. Therefore, the thrust of the power plant on the optimal phase trajectory of the flight vehicle is a piecewise-continuous, piecewise-smooth function and in the general case, possibly, consists of the following different sectors: the maximum thrust mode, the choked and "coupled" (choked) modes, the minimum thrust mode.

Optimal control α includes its limiting values and stable control, defined by equation (1.2.25). If this control α causes disruption of the limiting condition of the vector of the required parameter D, control α is determined from the condition of location of vector D at the boundary where α has values less than the limiting values. Thus, control function α on the optimal phase trajectory may be piecewise-continuous, piecewise-smooth and in the general case, possibly, consists of sectors of the limiting control, sectors of stable and "coupled" control.

The optimal control ω includes its limiting values and stable control, defined by equation (1.2.26). In the case when this control disrupts the conditions of limitation of required parameter vector D, control ω will be coupled to the boundary value of D, at which ω generally takes on values within its limits. Therefore, control function ω on the optimal phase trajectory of the flight vehicle may be piecewise-continuous, piecewise-smooth and in the general case, possibly, consists of sectors of limiting control, sectors of stable and "coupled" control.

With certain simplifying assumptions, we can produce certain quantitative relationships between values of α and ω with stable control. Suppose the required maneuver of the flight vehicle is such that the slight aerodynamic heating of the vehicle can be ignored. Then, by simultaneous solution of the Euler-Lagrange equations (1.2.25), (1.2.26), we find

$$\mathrm{tg}\,\omega=\frac{\frac{\partial Q_i^*}{\partial \alpha}}{\frac{\partial Y^*}{\partial \alpha}}, \tag{1.2.35}$$

since $\frac{\partial Q^*}{\partial \alpha} = \frac{\partial Q_i}{\partial \alpha}$[1]. At supersonic flight velocities in the area of linear dependence of aerodynamic lift on α, we have $\frac{\partial c_y^*}{\partial \alpha} = c_y^{\alpha *}$ and $\frac{\partial c_{xi}^*}{\partial \alpha} = 2c_y^{\alpha *}\alpha$, which, with low values of ω, leads to the equation

$$\omega \approx 2\alpha. \tag{1.2.36}$$

Consequently, during stable control α and ω, while the angle of attack is low, angle ω is equal to twice the angle of attack. We can note that approximately the same relationship between α and ω is retained with larger values of α. For example, with values of angle

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[1] Although at supersonic flight velocities the drag resulting from the lift consists of wave and inductive drag, it is methodologically simpler not to separate the two [22]. Therefore, both here and below inductive drag will be taken to mean all drag resulting from lift.

of attack near the critical values $\partial Y^*/\partial\alpha=0$, and therefore according to (1.2.35) $\omega = \pi/2$. However, according to experimental data [44] for wings of various geometries and M numbers between 1.3 and 2.5, α_{kp} = 40-45°. Thus, with stable control, relationship (1.2.36) is approximately retained between α and ω up to the maximum angles of attack (up to α_{kp}). Furthermore, taking the angle between the direction of thrust and the chord of the wing or approximately the axis of the flight vehicle as γ^*, considering (1.2.36) we find

$$\gamma^* \approx \alpha,$$

that is with stable control the angle of attack is equal to the angle of deflection of the thrust force from the axis of the vehicle. These considerations indicate a number of conclusions. Whereas in many cases the maximum angle of attack of a flight vehicle can be assumed close to α_{kp}, the limiting angles γ^* are frequently low. In this connection, we can state that the limiting control ω, following stable control α and ω, occurs earlier than limiting control α. This conclusion always retains its force where $\alpha_{max} \geq \omega_{max}$ and $|\alpha_{min}| \geq |\omega_{min}|$. Then the optimal control of a flight vehicle, in addition to the optimal coupled control, can in the general case consist of sectors of limiting control α and ω, sectors of limiting control ω and stable control α and ω.

We note that when the dependence of Q^* on α is ignored, the Euler-Lagrange equations (1.2.25), (1.2.26) become

$$\lambda_3 \equiv 0 \text{ and } \omega \equiv 0.$$

In this connection, it can be assumed that if the aerodynamic lift is fully "sufficient" for the performance of a required flight vehicle maneuver, with the optimal control the use of thrust to create additional lift should be contemplated only with a decrease in inductive aerodynamic drag by decreasing the required α.

The Euler-Lagrange equations do not exhaust the condition of stability. It also includes the condition of discontinuity of Lagrange coefficients λ at the moment of a sudden change in the control function t_i^*. Since a non-inertial change in the control function does not cause a disruption of continuity of the phase coordinates, in order to observe equation (1.2.1) we will have

$$\left[F^{\bullet}-x'\frac{\partial F^{\bullet}}{\partial x'}\right]_{t^{\bullet}_{i-}}=\left[F^{\bullet}-x'\frac{\partial F^{\bullet}}{\partial x'}\right]_{t^{\bullet}_{i+}}.$$

or

$$\left.\begin{array}{l}[C_0^{\bullet}]_{t^{\bullet}_{i-}}=[C_0^{\bullet}]_{t^{\bullet}_{i+}},\\ [\lambda]_{t^{\bullet}_{i-}}=[\lambda]_{t^{\bullet}_{i+}}\ \text{or}\ [\lambda_1^{\bullet}]_-=[\lambda_1^{\bullet}]_+,\ldots,[\lambda_6^{\bullet}]_-=[\lambda_6^{\bullet}]_+,\\ [\lambda_{\varphi}]_-=[\lambda_{\varphi}]_+,\ [\lambda_{\omega}]_-=[\lambda_{\omega}]_+.\end{array}\right\}\quad(1.2.37)$$

The discontinuity conditions (1.2.37) are analogues of the Erdman-Weierstrass conditions [7]. They require that the Lagrange factor γ and constant C_0^* be continuous at all control break points.

In studying the conditions of discontinuity of the Lagrange coefficients at moments of stage separation, let us agree on a number of assumptions which do not lead to disruption of the accuracy of the solutions determined by the assumptions made earlier. As the main assumption, we assume instantaneous separation of the "spent" stage, corresponding to the condition

$$t_-^{(l)}=t_+^{(l)},$$

and we will arbitrarily assume that separation of the stage introduces no perturbation to the motion of the vehicle and does not change its orientation. This allows us to consider the phase coordinates V, θ, H, L, s, Q_T and T_W to be continuous at moments of stage separation. Then, for fulfillment of the condition of stability we produce

$$\left.\begin{array}{l}\left[F-x'\dfrac{\partial F}{\partial x'}\right]_{t_-^{(l)}}=\left[F-x'\dfrac{\partial F}{\partial x'}\right]_{t_+^{(l)}}\ \text{or}\ [C_0]_{t_-^{(l)}}=[C_0]_{t_+^{(l)}},\\ [\lambda]_{t_-^{(l)}}=[\lambda]_{t_+^{(l)}}\ \text{or}\ [\lambda_1^{\bullet}]_{t_-^{(l)}}=[\lambda_1^{\bullet}]_{t_+^{(l)}},\ldots,\\ [\lambda_6^{\bullet}]_{t_-^{(l)}}=[\lambda_6^{\bullet}]_{t_+^{(l)}},\ [\lambda_{\varphi}]_{t_-^{(l)}}=[\lambda_{\varphi}]_{t_+^{(l)}},\ [\lambda_{\omega}]_{t_-^{(l)}}=[\lambda_{\omega}]_{t_+^{(l)}}.\end{array}\right\}\quad(1.2.38)$$

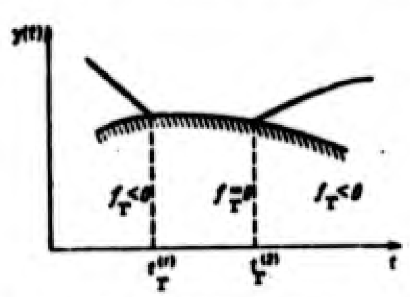

Figure 1.2. Diagram of Entry of Optimal Phase Trajectory to Boundary of Rigidity Limitation and Departure from It.

These equations indicate that at points of separation of the stages, continuity of the Lagrange coefficients $\lambda_1^*, \ldots, \lambda_5^*, \lambda_q, \lambda_\omega$ and constant first integral C_0^* should be retained. The Lagrange coefficient λ_6^* may undergo first order discontinuities at points of stage separation, its value to the right of the point of discontinuity being determined by (1.2.3).

Thus, the possible first order discontinuities in the control functions and vectors $\bar{n}$, N^p, σ^p and $\bar{s}$ should not disrupt the continuity of the Lagrange coefficients $\lambda_1^*, \ldots, \lambda_5^*, \lambda_q, \lambda_\omega$ and the constant first integral C_0^*.

In the case of "operation" of the structure at the boundary of the rigidity limitation at the points of entry $t_r^{(1)}$ and departure $t_r^{(2)}$, where the phase trajectory moves from the area determined by inequality $\psi_r < 0$ to the boundary determined by the equation $\psi_r = 0$ and back (Figure 1.2), in order to satisfy the stability condition, the following condition of discontinuity of Lagrange coefficients at the rigidity boundary should occur according to (1.2.1):

$$\left[F^* - x' \frac{\partial F^*}{\partial x'}\right]_{t_{r-}^{(1)}} = \left[F^* - x' \frac{\partial F^*}{\partial x'}\right]_{t_{r+}^{(1)}}$$

or

$$[C_0^*]_{t_{r-}^{(1)}} = [C_0^*]_{t_{r+}^{(1)}}, \tag{1.2.39}$$

$$\lambda(t_{r-}^{(1)}) - \lambda(t_{r+}^{(1)}) - \frac{\partial \psi_r}{\partial x} e_r = 0, \tag{1.2.40}$$

$$[C_0^*]_{t_{r-}^{(2)}} = [C_0^*]_{t_{r+}^{(2)}}, \quad \lambda(t_{r-}^{(2)}) = \lambda(t_{r+}^{(2)}), \tag{1.2.41}$$

where

$$\frac{\partial \psi_r}{\partial x} = \left(\frac{\partial \psi_r^{(1)}}{\partial V}, \ldots, \frac{\partial \psi_r^{(1)}}{\partial T_m}\right).$$

Thus, upon arriving at the rigidity limitation boundary at the points of entry $t_r^{(1)}$, the integration constant retains its value, while the Lagrange coefficient λ may generally be discontinuous, keeping in mind (1.2.40) however, upon departure from the rigidity limitation boundary at the points of departure $t_r^{(2)}$ (points where the phase trajectory goes over from the boundary determined by the equation $\psi_r = 0$, to the internal area fixed by the inequality $\psi_r < 0$), the Lagrange coefficients $\lambda(t_r^{(s)})$ remain continuous, and the integration constant C_0^* retains its value.

Further, due to the equality of the first variation of functional (1.1.30) to 0, according to (1.2.1) the following conditions obtain:

$$C_0^* dt_0 + \lambda_{10}^* dV_0 + \lambda_{20}^* d\vartheta_0 + \lambda_{30}^* dH_0 + \lambda_4^* dL_0 + \lambda_5^* ds_0 + \lambda_{q0} dQ_{\tau 0} +$$
$$+ \lambda_{w0} dT_{w0} - \sum_{\varrho=1}^{6} e_{0\varrho} d\psi_{0\varrho} = 0$$

or, since $Q_{\tau 0} = 0$ and T_{w0} is fixed in advance, we produce

$$C_0^* dt_0 + \lambda_{10}^* dV_0 + \lambda_{20}^* d\vartheta_0 + \lambda_{30}^* dH_0 + \lambda_4^* dL_0 +$$
$$+ \lambda_5^* ds_0 - \sum_{\varrho=1}^{6} e_{0\varrho}^* d\psi_{0\varrho}^* = 0, \qquad (1.2.42)$$

$$C_0^* dt_k + \lambda_{1k}^* dV_k + \lambda_{2k}^* d\vartheta_k + \lambda_{3k}^* dH_k + \lambda_4^* dL_k +$$
$$+ \lambda_5^* ds_k + \sum_{\varrho=1}^{6} e_{k\varrho}^* d\psi_{k\varrho}^* + \sum_{j=1}^{6} \frac{\partial I^*}{\partial x_{jk}^*} dx_{jk}^* = 0, \qquad (1.2.43)$$

$$\left.\begin{array}{l} \lambda_{6k} + e_p = 0, \\ \lambda_q + e_q = 0, \\ \lambda_{wk} = 0, \end{array}\right\} \qquad (1.2.44)$$

where

$$x_{jk} = V_k;\ \vartheta_k;\ H_k;\ L_k;\ s_k;\ t_k;$$
$$\lambda_{6k} = (\lambda_6^*(t_-^{(1)}),\ \lambda_6^*(t_-^{(2)}), \dots,\ \lambda_6^*(t_k)).$$

Placing the corresponding requirements on the Lagrange coefficients $e_{0\varrho}^*$ and $e_{k\varrho}^*$ and setting the coefficients with free variations of the phase coordinates equal to zero, we can produce from equations (1.2.42) and (1.2.43) the conditions of transversality (see appendix).

One of the peculiarities of the solution of the problem formulated is that the parameters of the vehicle are not fixed. They must be selected so that the conditions of their optimality are satisfied. The conditions of optimality of flight vehicle parameters follow from the condition of stability (1.2.1), if the Lagrange coefficients $e_1^*, \dots, e_{10}^*$, e_q, e_p are determined in the proper manner with variation of some of the parameters, while the coefficients with the variations of other parameters are set equal to zero. We will then have

$$\frac{\partial I^*}{\partial \dot{\varphi}_{pl}} + c_1^{0*}\frac{\partial f_1^{0*}}{\partial \dot{\varphi}_{pl}} + c^*_2\frac{\partial f^*_2}{\partial \dot{\varphi}_{pl}} + \frac{\partial \varphi_m}{\partial \dot{\varphi}_{pl}} c_7 = 0, \tag{1.2.45}$$

$$c^*_2\frac{\partial f^*_2}{\partial G_{pl}} + \frac{\partial \varphi_m}{\partial G_{pl}} c_7 = 0, \tag{1.2.46}$$

$$c_1^{0*}\frac{\partial f_1^{0*}}{\partial G_0} + \frac{\partial f_1^{f}}{\partial G_0} c_1^{f} + \frac{\partial f^*_2}{\partial G_0} c_2^* + \frac{\partial \varphi_3}{\partial G_0} c_3 + \frac{\partial \varphi_4}{\partial G_0} c_4 + \frac{\partial \varphi_5}{\partial G_0} c_5 +$$

$$+ \frac{\partial \varphi_7}{\partial G_0} c_7 + \int_{t_0}^{t_k} \frac{\partial F}{\partial G_0} dt = 0, \tag{1.2.47a}$$

$$c_1^{0*}\frac{\partial f_1^{0*}}{\partial G_{0J}} + \frac{\partial f_1^{f}}{\partial G_{0J}} c_1^{f} + \frac{\partial \varphi_4}{\partial G_{0J}} c_4 + \frac{\partial \varphi_5}{\partial G_{0J}} c_5 + \frac{\partial \varphi_7}{\partial G_{0J}} c_7 + \int_{t_0}^{t_k} \frac{\partial F}{\partial G_{0J}} dt = 0, \tag{1.2.47b}$$

$$\frac{\partial \varphi_3}{\partial P_{max}} c_3 + \frac{\partial \varphi_7}{\partial P_{max}} c_7 = 0, \tag{1.2.48}$$

$$c_1^{0}\frac{\partial f_1^{0}}{\partial a_0} + \frac{\partial f_1^{f}}{\partial a_0} c_1^{f} + \frac{\partial \varphi_3}{\partial a_0} c_3 + \frac{\partial \varphi_5}{\partial a_0} c_5 + \frac{\partial \varphi_7}{\partial a_0} c_7 + \int_{t_0}^{t_k} \frac{\partial F}{\partial a_0} dt = 0, \tag{1.2.49}$$

$$c_1^{0}\frac{\partial f_1^{0}}{\partial b_w} + \frac{\partial f_1^{f}}{\partial b_w} c_1^{f} + \frac{\partial \varphi_4}{\partial b_w} c_4 + \frac{\partial \varphi_7}{\partial b_w} c_7 + \int_{t_0}^{t_k} \frac{\partial F}{\partial b_w} dt = 0, \tag{1.2.50}$$

$$c_1^{0}\frac{\partial f_1^{0}}{\partial p_n} + \frac{\partial f_1^{f}}{\partial p_n} c_1^{f} + \frac{\partial \varphi_5}{\partial p_n} c_5 + \frac{\partial \varphi_7}{\partial p_n} c_7 - c_p = 0, \tag{1.2.51}$$

$$c_1^{0}\frac{\partial f_1^{0}}{\partial l_0} + \frac{\partial f_1^{f}}{\partial l_0} c_1^{f} + \frac{\partial \varphi_5}{\partial l_0} c_5 + \frac{\partial \varphi_7}{\partial l_0} c_7 + \int_{t_0}^{t_k} \frac{\partial F}{\partial l_0} dt = 0, \tag{1.2.52}$$

$$c_1^{0}\frac{\partial f_1^{0}}{\partial N^p} + \frac{\partial f_1^{f}}{\partial N^p} c_1^{f} + \frac{\partial \varphi_6}{\partial N^p} c_6 + \int_{t_0}^{t_k} \lambda_p dt = 0, \tag{1.2.53}$$

$$c_1^{0}\frac{\partial f_1^{0}}{\partial \Gamma} + \frac{\partial f_1^{f}}{\partial \Gamma} c_1^{f} + \frac{\partial \varphi_4}{\partial \Gamma} c_4 + \frac{\partial \varphi_5}{\partial \Gamma} c_5 + \frac{\partial \varphi_6}{\partial \Gamma} c_6 + \frac{\partial \varphi_7}{\partial \Gamma} c_7 +$$

$$+ \frac{\partial \varphi_{10}}{\partial \Gamma} c_{10} + \int_{t_0}^{t_k} \frac{\partial F}{\partial \Gamma} dt = 0, \tag{1.2.54}$$

$$e_1^{0*}\frac{\partial f_1^{0*}}{\partial T_p}+\frac{\partial f_1^J}{\partial T_p}e_1^J+\frac{\partial \beta_5}{\partial T_p}e_5+\frac{\partial \beta_6}{\partial T_p}e_6+\frac{\partial \beta_7}{\partial T_p}e_7+\int_{t_0}^{t_k}\frac{\partial F}{\partial T_p}dt=0, \tag{1.2.55}$$

$$e_1^{0*}\frac{\partial f_1^{0*}}{\partial Q_i^p}+\frac{\partial f_1^J}{\partial Q_i^p}e_1^J+\frac{\partial \beta_5}{\partial Q_i^p}e_5+\frac{\partial \beta_6}{\partial Q_i^p}e_6-e_q=0. \tag{1.2.56}$$

$$e_1^{0*}\frac{\partial f_1^{0*}}{\partial \sigma^p}+\frac{\partial f_1^J}{\partial \sigma^p}e_1^J+\frac{\partial \beta_6}{\partial \sigma^p}e_6+\frac{\partial \beta_{0\sigma}}{\partial \sigma^p}e_8+\int_{t_0}^{t_k}\lambda_D dt=0. \tag{1.2.57}$$

$$\frac{\partial f_m}{\partial \delta}e_6+\frac{\partial \beta_6}{\partial \delta}e_6+\frac{\partial \beta_{0\delta}}{\partial \delta}e_9+\frac{\partial \beta_{0r}}{\partial \delta}e_{10}+\int_{t_0}^{t_k}\frac{\partial F}{\partial \delta}dt=0, \tag{1.2.58}$$

$$e_7 w_a=0,\ e_8 w_\sigma=0,\ e_9 w_\delta=0,\ e_{10} w_r=0. \tag{1.2.59}$$

Excluding the Lagrange coefficients $e_1, \ldots, e_6$, e_μ and e_q from equations (1.2.44)-(1.2.59), we produce the condition of optimality of plan parameters μ_{p1}, G_{p1}, G_0, P_{max}, μ_k, a_0, b_{0i}, J_0, Γ, T_p, Q_i^p and the design load vector N^p, the design stress vector σ^p and the strength vector δ. The solutions of equations (1.2.59) allow us to determine whether the optimal value of the parameter is equal to its limiting value or lies within the area.

Analyzing equations (1.2.45)-(1.2.58), we can draw certain conclusions. In the general case in equation (1.2.53), the algebraic sum

$$\left(e_1^{0*}\frac{\partial f^{0*}_{\pi p}}{\partial N^p}+\frac{\partial f^J_{\pi p}}{\partial N^p}e_1^J+\frac{\partial f_m}{\partial N^p}e_6\right)$$

or at least one of its terms cannot be equal to zero. This indicates the obligatory inclusion in the optimal control of modes of coupled control resulting from the presence of limitations on the effective load vectors[1]. Similarly, in equation (1.2.57) the integral is generally not equal to zero, since the algebraic sum included in the equation or at least one of its terms in the general case cannot be equal to zero. Consequently, the optimal control must include modes of coupled control resulting from the presence of limitations on the effective load vector[1]. For example, suppose G_{p1} is in the open area. Then, according to (1.2.59) and (1.2.45) and (1.2.46), we produce $e_2 = e_7 = 0$ and $e_1^{0*} = -\partial I^*/\partial \mu_{p1} \neq 0$. Furthermore, the dependence of $f^*_{\pi p}$ on N^p and σ^p is always such that $\partial f^*_{\pi p}/\partial N^p \neq 0$ and $\partial f^*_{\pi p}/\partial \sigma^p \neq 0$. Therefore, if even e_1^J, e_6, e_8 are equal to zero, the expressions $\frac{\partial I^*}{\partial \mu_{p1}}\ \frac{\partial f^*_{\pi p}}{\partial N^p}$ and $\frac{\partial I^*}{\partial \mu_{p1}}\ \frac{\partial f^*_{\pi p}}{\partial \sigma^p}$ are always not equal to zero,

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[1] See footnote on page 19.

which allows us to affirm that the integral in equations (1.2.53) and (1.2.57) is in the general case not equal to zero.

Thus, preliminary analysis of equations (1.2.45)-(1.2.59) means that in the general case the optimal phase trajectory of a flight vehicle always includes a sector with coupled control, resulting from the presence of limitations on the vector of effective stresses (Figure 1.3).

It must be noted in this connection that the statement and solution of the problem of performance of a required maneuver with maximum value of effectiveness criterion I, produced by optimization of the control functions and plan parameter vector alone, generally leads to determination of an extreme and plan parameter vector for which the vehicle will either be unnecessarily heavy or its structure will be damaged upon performance of certain maneuvers due to excessive loading.

The Euler-Lagrange equations (1.II), conditions of transversality (1.2.42)-(1.2.43) and discontinuity conditions (1.2.37)-(1.2.41), together with equations (1.2.45)-(1.2.59) determine the condition of stability of the functional (effectiveness criterion) I*.

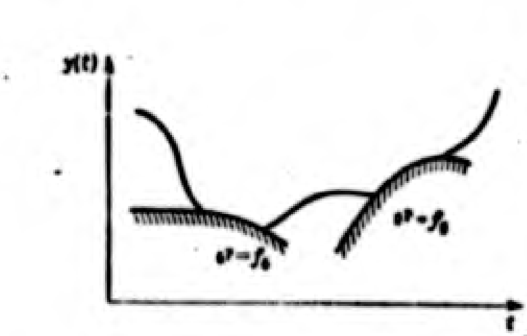

Figure 1.3. Schematic Representation of Optimal Phase Trajectory.

Satisfaction of the condition of stability and boundary conditions (1.1.1) in the solution of the system of coupling equations (1.I) considering the limitations introduced allows us to find the optimal program of the control functions $\alpha(t)$, $d(t)$, $r(t)$, $\omega(t)$, the optimal phase trajectory $V(t)$, $\theta(t)$, $H(t)$, $L(t)$, $s(t)$, $T_w(t)$, $\nu^*(t)$, $Q_T(t)$ and the optimal values of the plan parameter vector Π, the vector of design loads N^p, the vector of design stresses σ^p and the strength vector δ, for which the criterion of effectiveness I* reaches its maximal value.

§ 3. Optimal
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§ 3. Optimal Control of a Flight Vehicle and Power Plant (Necessary Weierstrass Condition, Maximum Principle and Control Condition)

The Euler-Lagrange equations relating to the first necessary condition allow us to produce the optimal control and thereby the optimal phase trajectory, which may be a minimizing, maximizing or saddle curve. Therefore, additional necessary conditions are required, allowing more confident maximization of functional (1.1.29). These additional necessary conditions are the necessary Weierstrass condition and the Clebsch condition which follows from it for the open area of change of control functions and the maximum principle (for our case the minimum principle) for the closed area of change (see appendix). It is quite probable that investigation of the Weierstrass condition or the minimum principle will produce additional information concerning the condition of formation of the limiting and stable control $\alpha(t)$, $\omega(t)$, $d(t)$ and $r(t)$ on the optimal phase trajectories of the flight vehicle, leading to the maximum value of the functional, even before computation. Since the Clebsch condition follows from the necessary Weierstrass condition, which is "stronger," it will not be analyzed here.

The necessary Weierstrass condition or the minimum principle can give additional information on the conditions of formation and existence of the limiting and stable controls only. Therefore, investigation of the necessary Weierstrass condition or the minimum principle at the point in question in the interval $[p_0, t_k]$ will be performed on the assumption that the corresponding control function at this point is not related to the required parameter vector by a limitation. We note in advance that at those points where the control is restricted by conditions of limitation of the required parameter vector, i. e. is determined from the equation $D^p - D = 0$, on the basis of the maximum principle we can find only equations similar to the Euler-Lagrange equations with coupled control of the corresponding function.

The Weierstrass condition in the case of maximum value of functional (1.1.29) has the form

$$E^{*} = H - \tilde{H} < 0,$$

where

$$H = \lambda\varphi,\ \tilde{H} = \lambda\tilde{\varphi},$$
$$\lambda = (\lambda_1^{*}, \ldots, \lambda_q, \lambda_\omega),$$
$$\varphi = (\varphi_1^{*}, \ldots, \varphi_n),$$
$$\tilde{\varphi} = (\tilde{\varphi}_1^{*}, \ldots, \tilde{\varphi}_n),$$

$\tilde{\phi}$ corresponds to values of ϕ with permissible control functions.

By permissible control functions, we mean values of the functions within the framework of the limiting values for which the coupling equations (1.I) are satisfied, while a permissible phase trajectory is a phase trajectory which is realized with a permissible change in the control function and satisfies the boundary conditions.

Due to the continuity between angular points, the existence of which results from separation of stages, functions $V(t)$, $\theta(t)$, $H(t)$, $L(t)$, $s(t)$, $\mu(t)$, $Q_T(t)$ and $T_W(t)$, the conditions

$$V=\tilde{V},\ \theta=\tilde{\theta},\ H=\tilde{H},\ L=\tilde{L},\ s=\tilde{s},\ \mu^*=\tilde{\mu}^*,\ Q_T=\tilde{Q}_T,\ T_w=\tilde{T}_w$$

are observed.

Thus, function E can be reduced to the form $E^* = H^* - \tilde{H}^*$, thereby producing

$$H^* \leq \tilde{H}^*, \tag{1.3.1}$$

where

$$H^* = \frac{g_0}{\mu^*} a_\omega^* p^* \left(\lambda_1^* \cos \alpha + \lambda_2^* \frac{\sin \alpha}{V}\right) - \frac{g^*}{\mu^*}\left(\lambda_1^* \dot{Q}_i - \lambda_2^* \frac{\gamma^*}{V}\right) -$$
$$- \lambda_6^* \frac{\dot{a}_\omega}{f_\omega} f^* + \lambda_e f_e + \lambda_w f_w.$$

In order to expand the area of existence of possible optimal controls, we will perform our investigation of the Weierstrass condition under the assumption of the existence among the permissible controls of a zero control, i. e. a control for which the corresponding control function is equal to zero.

Condition (1.3.1) should be fulfilled with any permissible control, i. e. H^* with optimal control is always less than or equal to H^* with any permissible non-optimal control, including the zero control. Therefore, the condition

$$H^* \leq H^0, \tag{1.3.2}$$

occurs, where H^0 is the value of H with the zero control.

Thus, condition (1.3.2) follows from the Weierstrass condition (1.3.1) and, like it, is local. In the following, condition (1.3.2) will be called the control condition or the condition of permissible non-zero control. It shows that with any non-zero control the value of H* should be less than or equal to the value of H^0. If the control condition (1.3.2) throughout the range of permissible non-zero values of the control function is not fulfilled, the optimal control satisfying the Weierstrass condition (1.3.1) must be assumed to be the zero control.

The control condition is a weak condition in comparison to the Weierstrass condition. Its fulfillment does not indicate satisfaction of the Weierstrass condition, but only shows the possibility of its fulfillment by a non-zero control. Therefore, the control condition does not allow us to judge the optimal control, but in many cases helps us very simply to determine the area of possible optimal control.

In order to determine the optimal control, we must use the mathematical theory of the maximum principle [28], a generalization of the necessary Weierstrass condition to cover the case when the optimal control being determined is related to limitations based on inequalities. The maximum principle (since the maximum value of the functional is determined) consists in that with the optimal control determined among the permissible values of control functions d, r, α and ω, function H* reaches the precise lower boundary at each moment in time:

$$m^* = \inf \{H^*(\lambda_i^*, x_i^*, \omega, \alpha, d, r, \Pi, N_P, \mathscr{F})\},$$

$$\begin{aligned} &\omega_{min} \leqslant \omega \leqslant \omega_{max}, \\ &\alpha_{min} \leqslant \alpha \leqslant \alpha_{max}, \\ &d_{min} \leqslant d \leqslant d_{max}, \\ &r_{min} \leqslant r \leqslant r_{max}. \end{aligned} \qquad (1.3.3)$$

Condition (1.3.3) allows us to determine the optimal control. In this sense, condition (1.3.3) is the condition of optimal control. It does not contradict condition (1.3.2). The control condition, being "weaker," facilitates and in many cases greatly simplifies determination of the optimal control from the maximum principle. Therefore, investigation of the optimal control according to (1.3.3) should be performed on the basis of the control condition (1.3.2).

If the control function is determined unambiguously at each moment in time from (1.3.3), the optimal mode of motion within the fixed boundary conditions is unique and the phase trajectories satisfy the systems (1.I) and (1.II).

It has been demonstrated in [11, 20], that the condition of optimal control (1.3.3) does not always have a unique solution. If the non-uniqueness of solution (1.3.3) is retained in sector (τ_1, τ_2) of the interval $t_0 \leq t \leq t_k$, a slipping mode occurs. Then, generally speaking, there is no control with which it is possible to realize motion along the phase trajectories of the equations of motion of the system (1.1). Therefore, there is no maximum of the functional in the class of permissible control functions. However, there is almost always a sequence of phase trajectories of equations of motion which converges to a certain sequence of limiting curves, satisfying the fixed boundary conditions, while the values of the functional approach their upper boundary. Thus, determination of the maximum of the functional is essentially reduced to determination of this sequence. Determining it, we can realize the optimal mode as accurately as desired while remaining in the class of permissible control functions. The general theory of slipping optimal modes and the method of their calculation were first developed in detail in [11, 20].

Approximation of the slipping mode even with low accuracy leads to a control, the realization of which requires the use of regulators with high over-control factors, requiring that the problem be stated once more considering the dynamic characteristics and the dependence of the over-control factor on the plan parameters and design load parameters. If this is inconvenient for any reason, we can go over to approximate solution of the slipping optimal mode, the method of which is given in [20].

Therefore, the problem of investigation of (1.3.3) is determination of the optimal controls using the control condition (1.3.2) and location of any possible area of non-uniqueness of the solutions of (1.3.3.), leading to slipping modes.

The control functions α, ω, d and r are autonomous, i. e. are independent of each other. Due to this, condition (1.3.3) considering (1.3.2) can be analyzed separately for each control function.

For the thrust vector, the control condition has the form

$$H^* = \frac{g_0 \dot{a}_{0i} p^*}{\mu^*}\left(\lambda_1^* \cos\omega + \lambda_2^* \frac{\sin\omega}{V}\right) - \lambda_6^* \frac{\dot{a}_{0i}}{\dot{I}_{0i}} f^* \leqslant 0.$$

Furthermore, for the control function ω

$$H_\omega^* = p^*\left(\lambda_1^* \cos\omega + \lambda_2^* \frac{\sin\omega}{V}\right).$$

If $\lambda_6^* \leq 0$, then, taking inertial flight as the zero control ω, control condition (1.3.2) and the condition of optimal control can be represented as

$$H_\omega^* \leqslant 0 \text{ or } p^*\left(\lambda_1^* \cos\omega + \lambda_2^* \frac{\sin\omega}{V}\right) \leqslant 0, \tag{1.3.4}$$

$$m_\omega^* = \inf_{\omega_{min} < \omega < \omega_{max}} H_\omega^*. \tag{1.3.5}$$

We must note here that the zero control ω -- inertial flight -- is a singular control ω or a singular solution of (1.3.5), appearing in connection with the possibility of cutoff of the power plant.

In order to determine the conditions under which (1.3.5) is fulfilled and clarify the ambiguity of control ω, let us study the curves of change of H^*_ω with respect to ω with arbitrary fixed values of λ_1^* and λ_2^* and V. Thus, we have

$$\frac{\partial H^*_\omega}{\partial\omega}=p^*\left(-\lambda_1^*\sin\omega+\lambda_2^*\frac{\cos\omega}{V}\right). \qquad (1.3.6)$$

$$\frac{\partial^2 H^*_\omega}{\partial\omega^2}=-p^*\left(\lambda_1^*\cos\omega+\lambda_2^*\frac{\sin\omega}{V}\right). \qquad (1.3.7)$$

From (1.3.7), according to (1.3.4), we produce

$$\frac{\partial^2 H^*_\omega}{\partial\omega^2}\geqslant 0.$$

Thus, the dependence of H^*_ω on ω at each moment in time is expressed by a curve which bulges downward (Figure 1.4). Therefore, (1.3.5) where $p^* \neq 0$ can be fulfilled both with stable control and with limiting control ω but, significantly, at each moment in time only by one of these. Consequently, at each moment in time there is an unambiguous control with respect to ω. Let us now study the area of values of λ_1^* and λ_2^* for which stable or limiting control ω ($-\pi/2 < \omega < \pi/2$) is possible with $p^* \neq 0$. For this, we analyze (1.3.6), keeping in mind (1.3.4). Conditions (1.3.4) can be fulfilled with the following values of λ_1^* and λ_2^*:

$$\lambda_1^*\leqslant 0,\ \lambda_2^*\leqslant 0,$$

$$\lambda_1^*\leqslant 0,\ \lambda_2^*\geqslant 0, \qquad (1.3.8)$$

$$\lambda_1^*\geqslant 0,\ \lambda_2^*\sin\omega\leqslant 0 \qquad (1.3.9)$$

or $\lambda_1^*\geqslant 0,\ \lambda_2^*<0$ where $\omega>0$, (1.3.10)

$\lambda_1^*\geqslant 0,\ \lambda_2^*>0$ where $\omega<0$. (1.3.11)

Among the conditions (1.3.8)-(1.3.11), we shall study values of ω with which a precise lower boundary of H^*_ω is possible. To do this, we analyze cases of change in the sign of the derivative $\partial H^*_\omega/\partial\omega$. If (1.3.8) and (1.3.9) occur, the precise lower boundary of H^*_ω can be reached with

$$\frac{\partial H^*_\omega}{\partial \omega} = -\lambda^*_1 \sin \omega + \lambda^*_2 \frac{\cos \omega}{v} = 0, \qquad (1.3.12)$$

which corresponds to stable control with respect to ω. In order to produce equation (1.3.12), expression $\partial H^*_\omega/\partial\omega$ must change its sign from minus to plus upon movement from ω_{min} to ω_{max}. Therefore, in the area of positive values of ω (1.3.12) is possible only with (1.3.8), while in the area of negative values of ω it is possible only with (1.3.9) (see Figure 1.4). However, (1.3.8) and (1.3.9) also permit retention of the sign of $\partial H^*_\omega/\partial\omega$, leading to fulfillment of the condition of optimal control (1.3.5) with limiting control. Thus, (1.3.8) retains only $\partial H^*_\omega/\partial\omega < 0$, while (1.3.9) retains $\partial H^*_\omega/\partial\omega > 0$, correspondingly leading to $\omega = \omega_{max}$, $\omega = \omega_{min}$. Thus, in the area of values

$$\lambda^*_1 < 0, \quad \lambda^*_2 = -|\lambda^*_2| \, \mathrm{sign}(\sin \omega) \qquad (1.3.13)$$

the condition of optimal control (1.3.5) has an unambiguous solution, corresponding to the stable or limiting control with respect to ω, permitting transition from stable to limiting control and back.

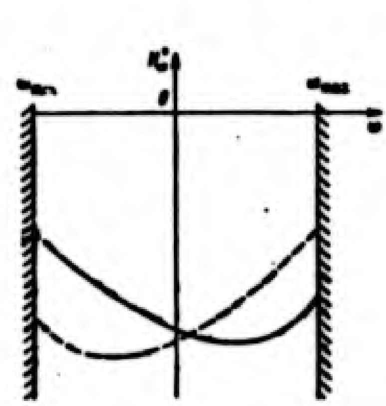

Figure 1.4. Change in H^*_ω as a Function of Angle ω Under the Condition $\lambda_1 < 0$:
———— $\lambda_2 < 0$, $0 < \omega < \omega_{max}$,
-------- $\lambda_2 > 0$, $\omega_{min} < \omega < 0$

Under conditions (1.3.10) and (1.3.11), we produce from (1.3.6) respectively

$$\left.\begin{array}{l}\frac{\partial H_{\omega}^{*}}{\partial \omega}<0, \\ \frac{\partial H_{\omega}^{*}}{\partial \omega}>0.\end{array}\right\} \quad (1.3.14)$$

Therefore, the precise lower boundary of H_{ω}^{*} is reached at $\omega = \omega_{min}$ and $\omega = \omega_{max}$ respectively (Figure 1.5). Thus, in the area of values

$$\begin{gathered}\lambda_1^{*} \geqslant 0, \\ \lambda_2^{*}=-|\dot{\lambda}_2^{*}| \operatorname{sign}(\sin \omega),\end{gathered} \quad (1.3.15)$$

where

$$\operatorname{sign}(\sin \omega)\begin{cases}=1 & \text{where } \omega=\omega_{max}, \\ =-1 & \text{where } \omega=\omega_{min},\end{cases}$$

the precise lower boundary of H_{ω}^{*} is reached only with limiting control.

Keeping in mind (1.3.13) and (1.3.15), regardless of the sign of λ_1^{*}, we produce

$$\lambda_2^{*} \sin \omega \leqslant 0.$$

Then, according to (1.3.6) and (1.3.7) where $\lambda_1^{*} = 0$ and $\lambda_2^{*} \neq 0$, the condition of optimal control (1.3.5) is fulfilled only with limiting control ω_{max} or ω_{min}, when $\lambda_2^{*} < 0$ and $\lambda_2^{*} > 0$ respectively. Consequently, where $\lambda_2^{*} \neq 0$ in area (1.3.15) of the values of λ_1^{*} and λ_2^{*}, the condition of optimal control (1.3.5) has an unambiguous solution, corresponding to one of the limiting controls ω. Non- transition from one limiting control ω to the other in case (1.3.15) is achieved at the moment

$$\lambda_1^{*}=0 \text{ and } \lambda_2^{*}=0.$$

If the last condition is retained with respect to time, i. e.

$$\lambda_1^{*} \equiv 0 \text{ and } \lambda_2^{*} \equiv 0,$$

we arrive at an infinite sequence of limiting controls ω. The identities decrease the order of the system of Euler-Lagrange equations (1.11) by two units and lead to degeneration $k = 2$ of the variational problem. However, in many cases they cause all the Lagrange coefficients to be identical to zero, leading to full degeneration of the variational problem.

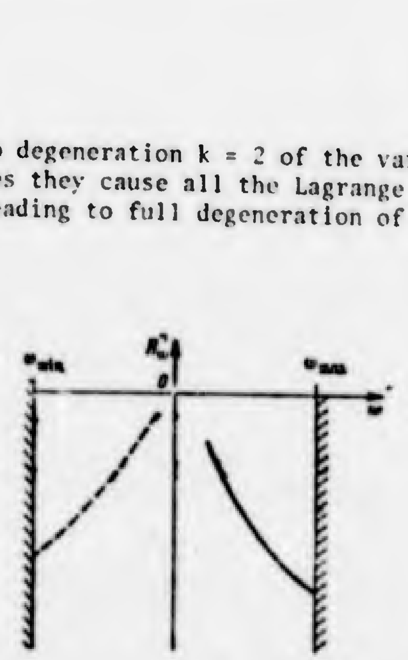

Figure 1.5. Change in H^*_ω as a Function of angle ω where $\lambda_1 \geq 0$:
— $\lambda_2 \leq 0$, $\omega = \omega_{max}$,
--- $\lambda_2 \geq 0$, $\omega = \omega_{min}$

Let us now determine the area of values of λ^*_1 and λ^*_2 satisfying the control condition (1.3.4), with which optimal control of angle ω is possible if it is limited by the interval $1/2\pi < \omega < 3/2\pi$. These values of ω are fulfilled for descending flight vehicles, when maneuvers involve deceleration due to the force of gravity. Then ω_{max} and ω_{min} fall within the limits

$$\pi < \omega_{max} < \frac{3}{2}\pi, \quad \frac{1}{2}\pi < \omega_{min} < \pi.$$

Let us introduce the new angles $\bar{\omega}$, reducing the first values of ω to the first positive and negative quadrant by substituting $\omega = \pi + \bar{\omega}$. Then the control condition (1.3.4) is corrected to the form

$$-(\lambda^*_1 \cos\bar{\omega} + \lambda^*_2 \sin\bar{\omega}) < 0.$$

Therefore, all preceding conclusions produced for values $-\pi/2 < \omega < \pi/2$ are retained also for $\bar{\omega}$ $(-\pi/2 < \bar{\omega} < \pi/2)$ when the signs of λ^*_1 and λ^*_2 are reversed. Thus, in the area of values

$$\lambda^*_1 \geqslant 0, \ \lambda^*_2 = |\lambda^*_2| \operatorname{sign}(\sin\bar{\omega})$$

the optimal control condition (1.3.5) has an unambiguous solution, corresponding to the stable or limiting control ω, while in the area

$$\lambda_1^* \leqslant 0,\ \lambda_2^* = |\lambda_2^*| \operatorname{sign}(\sin \bar{\omega})$$

it has an unambiguous solution with the limiting control.

Representing

$$\lambda_\omega^* = -\frac{\partial H_\omega^*}{\partial \omega},$$

on the basis of the results produced, we can formulate the following conclusions: the maximum principle with limiting and stable control ω will be satisfied if in addition to the inequality

$$p^*\left(\lambda_1^* \cos\omega + \lambda_2^* \frac{\sin\omega}{V}\right) < 0$$

one of the following conditions

$$\left.\begin{aligned} &\lambda_\omega^* = 0 \text{ where } \omega_{min} \leqslant \omega \leqslant \omega_{max}, \\ &\lambda_\omega^* < 0 \text{ where } \omega = \omega_{max}, \\ &\lambda_\omega^* > 0 \text{ where } \omega = \omega_{min}, \end{aligned}\right\} \qquad (1.3.16)$$

is fulfilled, and where $\lambda_1^* < 0$ the optimal control may be either the limiting or the stable control, while where $\lambda_1^* \geq 0$ it can only be the limiting control (where $-\pi/2 < \omega < \pi/2$).

This conclusion is correct for the condition $\lambda_6^* \leq 0$. If $\lambda_6^* > 0$, the inequality

$$\lambda_1^* \cos\omega + \lambda_2^* \frac{\sin\omega}{V} > 0,$$

is possible, and does not disrupt the control condition of the thrust vector where $p^* \neq 0$. However, it occurs only where $\lambda_1^* \geq 0$, since otherwise in the area of $\omega = 0$ it will be disrupted or transformed to inequality (1.3.4). In order to explain the condition of formation of this inequality, let us expand the area of change of ω slightly, assuming

$$-\frac{\pi}{2} \leqslant \omega \leqslant \frac{\pi}{2}.$$

Then where $\lambda_2^* < 0$ or where $\lambda_2^* > 0$ in the area of $\omega = \pi/2$ or $\omega = -\pi/2$ respectively, this inequality is disrupted and the condition (1.3.4) occurs. Thus, if it is possible to satisfy the condition of optimal control (1.3.5) with this inequality, it is only possible where $\omega_{min} > -\pi/2$ and $\omega_{max} < \pi/2$. Furthermore, since in this case

$$\frac{\partial^2 H_\omega^*}{\partial \omega^2} = -p^* \left(\lambda_1^* \cos\omega + \lambda_2^* \frac{\sin\omega}{V}\right) < 0,$$

the optimal control condition (1.3.5) corresponds to limiting control. Here when the condition

$$m_\omega^* = (H_\omega^*)_{\omega_{max}} = (H_\omega^*)_{\omega_{min}}$$

arises and is retained over a certain time sector, it is possible in principle to have a slipping mode as the optimal control ω.

Consequently, where $\lambda_6^* > 0$ and $\lambda_1^* > 0$ in the limited area of change of $\omega(\omega_{min} > -\pi/2,\ \omega_{max} < \pi/2)$, the inequality $\lambda_1^* \cos\omega + \lambda_2^* \sin\omega/V > 0$ is possible, satisfying the thrust vector control condition where $p^* \neq 0$. In this case, the optimal control condition is satisfied only by limiting control, which when the equation $(H_\omega^*)_{\omega_{max}} = (H_\omega^*)_{\omega_{min}}$ is retained over a certain time sector can in principle be transformed to a slipping mode of ω. However, it must be noted that with a sufficiently "broad" limiting area $\omega(\omega_{max} \leq \pi/2$ and $\omega_{min} \geq -\pi/2)$, this case does not occur. Therefore, in the following (unless specifically stated otherwise) we will use only control condition (1.3.4) for qualitative evaluation of optimal control in our investigation of the condition of optimization of the other control functions α, r and d.

Let us now present a certain geometric interpretation of the results just produced. For simplicity, we shall use the function $\sin\omega$ in place of ω for the control function. Then the Euler-Lagrange equation (1.2.30) becomes

$$\lambda_\omega^* = \lambda_1^* \frac{\partial \cos\omega}{\partial \sin\omega} + \frac{\lambda_2^*}{V},$$

where

$$\lambda_\omega^* = -\lambda_\omega \frac{\partial K}{\partial \sin\omega} \frac{p^*}{g_0 \dot{G} V^*}.$$

In the interval $0 \leq \cos\omega \leq 1$, the dependence $\cos\omega = \phi(\sin\omega)$ is such that the tangent is always located above the curve (Figure 1.6). Therefore, where $\omega \neq 0$, the following condition occurs

$$\left[(\cos\omega - \cos\bar{\omega}) - (\sin\omega - \sin\bar{\omega}) \frac{\partial \cos\omega}{\partial \sin\omega}\right] > 0,$$

where $\underline{\omega}$ corresponds to the value of sin ω at the point of contact;
$\bar{\omega}$ corresponds to the value of sin ω at an arbitrary point.

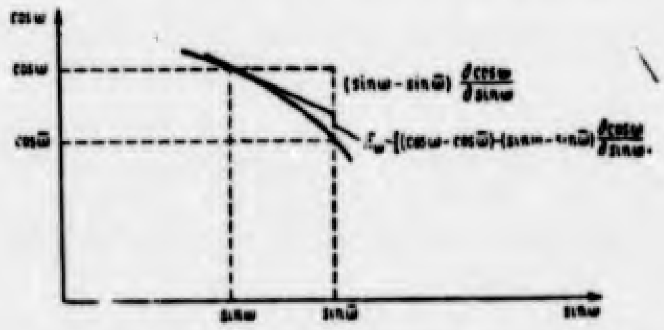

Figure 1.6. Geometric Interpretation of the Necessary Weierstrass Condition for Angle ω

The Weierstrass condition (1.3.1) relative to control function ω becomes

$$E_{\omega}^{*}=\left[\lambda_{1}^{*}(\cos\omega-\cos\bar{\omega})+\frac{\lambda_{2}^{*}}{V}(\sin\omega-\sin\bar{\omega})\right]<0.$$

Considering the Euler-Lagrange equation, this inequality can be corrected to the form

$$E_{\omega}^{*}=\lambda_{1}^{*}\left[(\cos\omega-\cos\bar{\omega})-(\sin\omega-\sin\bar{\omega})\frac{\partial\cos\omega}{\partial\sin\omega}+\frac{\lambda_{\omega}^{*}}{\lambda_{1}^{*}}\times\right.$$
$$\left.\times(\sin\omega-\sin\bar{\omega})\right]<0.$$

From this, we see that stable control for which $\lambda_{\omega}^{*}=0$ is possible only where $\lambda_{1}^{*}<0$. At the same time, the necessary Weierstrass condition is fulfilled with the limiting control if where

$$\lambda_{1}^{*}<0 \quad \text{or} \quad \lambda_{1}^{*}>0$$
$$-\lambda_{\omega}^{*}(\sin\omega-\sin\bar{\omega})\geqslant 0.$$

This last condition is easy to reduce to inequalities (1.3.16). Based on these results, we note that with stable control ω we always have

$$\lambda_{1}^{*}\cos\omega+\lambda_{2}^{*}\frac{\sin\omega}{V}<0.$$

Thus, with stable control ω, the necessary Weierstrass condition corresponds to the control condition (1.3.4).

For control function α we have

$$H_\alpha^* = -\frac{g_0}{\mu^*}\left(\lambda_1^* Q_l^* - \lambda_2^* \frac{Y^*}{V}\right) + \lambda_q f_q + \lambda_w f_w.$$

Then the control condition and optimal control condition are produced in the form

$$H_\alpha^* \leqslant 0 \quad \text{or} \quad -\frac{g_0}{\mu^*}\left(\lambda_1^* Q_l^* - \lambda_2^* \frac{Y^*}{V}\right) + \lambda_q (f_q - f_q^0) + \lambda_w (f_w - f_w^0) \leqslant 0, \tag{1.3.17}$$

$$m_\alpha^* = \inf_{\alpha_{min} < \alpha < \alpha_{max}} H_\alpha^*, \tag{1.3.18}$$

where f_q^0, f_w^0 are the functions f_q and f_w where $\alpha = 0$.

Investigation of possible solutions of (1.3.18) with various fixed λ_1, λ_2 λ_q λ_w, V and H, μ^* should be performed considering (1.3.17). The dependence of f_q and f_w on the angle of attack α is very difficult to determine, and it appears differently under the influence of various factors. In connection with this and in order to simplify study of (1.3.17) and (1.3.18), we will assume a very weak dependence of f_q and f_w on α, such that $f_q \approx f_q^0$, $f_w = f_w^0$. Therefore, (1.3.17) and (1.3.18) are reduced to the form

$$-\lambda_1^* Q_l^* + \lambda_2^* \frac{Y^*}{V} \leqslant 0, \tag{1.3.19}$$

$$m_\alpha^* = \inf_{\alpha_{min} < \alpha < \alpha_{max}} \left(-\lambda_1^* Q_l^* + \lambda_2^* \frac{Y^*}{V}\right). \tag{1.3.20}$$

We can formally arrive at these same conditions when the coupling equations (1.1.8) and (1.1.9), considering the influence of aerodynamic heating on the "operation" of the flight vehicle structure, are omitted. However, essentially the assumption of weak dependence of f_q and f_w on α and ignoring the coupling equations (1.1.8) and (1.1.9) represent a different approach to evaluation of maneuvers of the flight vehicle. In the first case, a certain approximation is used in determining indicators of aerodynamic heating T_w and Q_T, while in the second case the influence of aerodynamic heating on the load-bearing work of the structure of the flight vehicle is ignored, which is possible in principle only with relatively low values of T_w and Q_T.

According to the definition of H^*_α, we will have

$$\frac{\partial H^*_\alpha}{\partial \alpha}=\left(-\lambda_1^* \frac{\partial Q_i^*}{\partial Y^*}+\lambda_2^* \frac{1}{V}\right) \frac{\partial Y^*}{\partial \alpha}=\frac{1}{Y^*}\left[\left(-\lambda_1 Q_i^*+\lambda_2 \frac{Y^*}{V}\right)\times \times \frac{\partial Y^*}{\partial \alpha}+\lambda_1^* Q_i^{2*} \frac{\partial K_i^*}{\partial \alpha}\right], \tag{1.3.21}$$

$$\frac{\partial^2 H_\alpha}{\partial \alpha^2}=-\lambda_1^* \frac{\partial^2 Q_i^*}{\partial Y^{2*}}\left(\frac{\partial Y^*}{\partial \alpha}\right)^2+\left(-\lambda_1 \frac{\partial Q_i^*}{\partial Y^*}+\lambda_2 \frac{1}{V}\right) \frac{\partial^2 Y^*}{\partial \alpha^2},$$

$$\frac{\partial^2 H_\alpha^*}{\partial \alpha^2}=-\lambda_1 \frac{\partial^2 Q_i^*}{\partial \alpha^2}+\frac{\lambda_2}{V} \frac{\partial^2 Y^*}{\partial \alpha^2}. \tag{1.3.22}$$

The aerodynamic characteristics of the flight vehicle are such that in the general case where $\alpha < \alpha_{kp}$, we have [22]

$$\frac{\partial Y^*}{\partial \alpha}>0, \quad \frac{\partial^2 Q_i^*}{\partial Y^{2*}}>0 \quad \text{or} \quad \frac{\partial^2 Q_i^*}{\partial \alpha^2}>0. \tag{1.3.23}$$

Furthermore, based on the experience of planning flight vehicles [10, 22, 26], we can assume that α_{max} and α_{min} will always be in the range of values of α for which it is possible to produce the linear dependence of Y^* on α with great accuracy, particularly where $M > 1$. Then, we can practically always assume

$$\frac{\partial^2 Y^*}{\partial \alpha^2}\approx 0. \tag{1.3.24}$$

The error caused by using the linear dependence of Y^* on α and therefore by condition (1.3.24) in an investigation of the optimal control condition (1.3.20) can be significant for the estimation of the sign of $\partial^2 H^*_\alpha/\partial\alpha^2$, but only where $\lambda^*_2/V \gg \lambda^*_1$, since always $\partial^2 Qi/\partial\alpha^2 > \partial^2 Y/\partial\alpha^2$. However, if λ^*_2/B is greater than λ^*_1, this is possible only with limiting control α. For example, suppose $(\alpha_{\pi p}) \leq 30\text{-}40°$. In the case of stable control, the Euler-Lagrange equation (1.2.25) indicates that $2\alpha = \lambda^*_2/V\,\lambda^*_1$, i. e. at the beginning of the range of limiting values of α, the ratio $\frac{\lambda^*_2/V}{\lambda^*_1}$ cannot be greater than one.

Thus, with a linear dependence of Y^* on α, the sign of the derivative $\partial^2 H^*_\alpha/\partial\alpha^2$ is determined by the sign of the Lagrange coefficient λ^*_1. Then we produce

$$\frac{\partial^2 H^*_\alpha}{\partial \alpha^2} \geqslant 0 \quad \text{where} \quad \lambda_1^* \leqslant 0, \qquad (1.3.25)$$

$$\frac{\partial^2 H^*_\alpha}{\partial \alpha^2} \leqslant 0 \quad \text{where} \quad \lambda_1 > 0, \qquad (1.3.26)$$

while the derivative $\partial^2 H^*_\alpha/\partial\alpha^2$ can be equal to zero only where $\lambda_1 = 0$. In the case of (1.3.25), the curve expressing the dependence of H^*_α on α bulges downward, while in the case of (1.3.26) it bulges upward. Therefore, fulfillment of the optimal control condition (1.3.20) is possible at each moment in time with (1.3.25) with fully defined control: stable or some limiting (see Figure 1.7), while with (1.3.26) it is possible with limiting control (see Figure 1.8). In the latter case, the proof of unambiguity of the solution requires additional analysis.

Let us investigate the area of values of λ_1^* and λ_2^* for which stable or limiting control is possible, keeping in mind

$$\left.\begin{aligned} &\frac{\partial Q_i^*}{\partial \alpha} > 0 \ \text{or} \ \frac{\partial Q_i^*}{\partial Y^*} > 0 \quad \text{where } 0 < \alpha < \alpha_{max}, \\ &\frac{\partial Q_i^*}{\partial \alpha} < 0 \ \text{or} \ \frac{\partial Q_i^*}{\partial Y^*} < 0 \quad \text{where } \alpha_{min} \leqslant \alpha < 0, \\ &Q_i^* = 0, \ \frac{\partial Q_i^*}{\partial \alpha} = 0 \ \text{where } \alpha = 0. \end{aligned}\right\} \qquad (1.3.27)$$

According to the control condition (1.3.19), the following inequalities should be observed:

$$\lambda_1^* < 0 \ \text{and} \ \lambda_2^* Y^* < 0$$

or

$$\lambda_1^* < 0, \ \lambda_2^* < 0, \ 0 < \alpha, \qquad (1.3.28)$$

$$\lambda_1^* < 0, \ \lambda_2^* > 0, \ \alpha < 0, \qquad (1.3.29)$$

$$\lambda_1^* > 0, \ \lambda_2^* < 0, \qquad (1.3.30)$$

$$\lambda_1^* > 0, \ \lambda_2^* > 0. \qquad (1.3.31)$$

Keeping in mind (1.3.21) and (1.3.27), we note that cases (1.3.28) and (1.3.29) permit solution of the optimal control condition (1.3.20) with the equality

$$\frac{\partial H^*_\alpha}{\partial \alpha} = -\lambda_1^* \frac{\partial Q_i^*}{\partial \alpha} + \frac{\lambda_2^*}{V} \frac{\partial Y^*}{\partial \alpha} = 0,$$

which corresponds to stable control α; in the area of positive values of λ_2^*, $\alpha < 0$, while in the area of negative values $\alpha > 0$. Therefore, the transition through $\alpha = 0$ with stable control is possible only with $\lambda^*_2 = 0$.

Furthermore, (1.3.28) and (1.3.29) do not exclude the possibility of retention of constant sign by $\partial H^*_\alpha / \partial\alpha$. Thus, $\partial H^*_\alpha/\partial\alpha < 0$ can occur only with (1.3.28), while $\partial H^*_\alpha \partial\alpha > 0$ can occur only with (1.3.29). In these cases, the condition of optimal control (1.3.20) is satisfied with $\alpha = \alpha_{max}$ and $\alpha = \alpha_{min}$ respectively. The equation

$$\inf_{\alpha=\alpha_{min}} H^*_\alpha = \inf_{\alpha=\alpha_{max}} H^*_\alpha$$

in connection with (1.3.28) and (1.3.29), is possible only where $\lambda_2 = 0$, and therefore, according to the control condition, we produce $\lambda_1 = 0$. Consequently, the non-uniqueness of the solution of the optimal control condition (1.3.20), resulting from the non-inertial transition from one limiting control to the other, is allowed only with equations $\lambda_1 = 0$ and $\lambda_2 = 0$, which, if retained in time produce $\lambda_1 \equiv 0$ and $\lambda_2 \equiv 0$, leading to degeneration k = 2 of the variational problem and frequently even to its full degeneration. In the case $\lambda_1 = 0$, the optimal control condition (1.3.20) is satisfied where $\lambda_2^* < 0$ over $\alpha = \alpha_{max}$, while where $\lambda_2^* > 0$ it is satisfied over $\alpha = \alpha_{min}$.

Thus, in the area

$$\lambda_1^* < 0, \qquad \lambda_2^* = -|\lambda_2^*| \operatorname{sign} \alpha \tag{1.3.32}$$

the solutions of the optimal control condition (1.3.20) are unambiguous and correspond to the stable or one of the limiting controls (Figure 1.7). In conclusion we note that since in the area of non-linear dependence of Y* on angle α with large values but where $\alpha < \alpha_{kp}$ we find

$$\frac{\partial^2 Y^*}{\partial\alpha^2} < 0 \quad \text{where } \alpha > 0,$$
$$\frac{\partial^2 Y^*}{\partial\alpha^2} > 0 \quad \text{where } \alpha < 0,$$

the conclusions produced earlier concerning satisfaction of the optimal control condition α are fully retained with (1.3.32).

In the case both of linear and of non-linear dependence of Y on α, with large values we retain the condition

$$\frac{\partial K_i^*}{\partial\alpha} < 0.$$

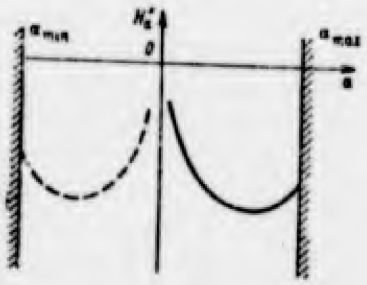

Figure 1.7. Change in H_α^* as a Function of Angle of Attack α with the Condition $\lambda_1 < 0$:

—— $\lambda_2 \leq 0$, $0 \leq \alpha \leq \alpha_{max}$,
. $\lambda_2 \geq 0$, $\alpha_{min} \leq \alpha \leq 0$

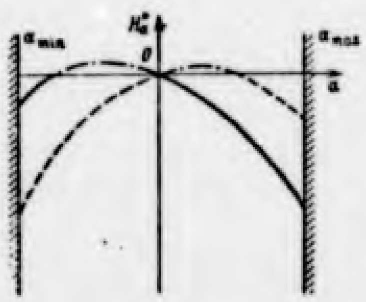

Figure 1.8. Change in H_α^* as a Function of Angle of Attack α with the Condition $\lambda_1 \geq 0$:

—— where $\lambda_2 \leq 0$ $\alpha = \alpha_{max}$,
--- where $\lambda_2 \geq 0$ $\alpha = \alpha_{min}$

Therefore with (1.3.30) and (1.3.31), according to (1.3.21) we can write

$$\left.\begin{aligned} \frac{\partial H_\alpha^*}{\partial \alpha} &< 0 \quad \text{where } \alpha > 0, \\ \frac{\partial H_\alpha^*}{\partial \alpha} &> 0 \quad \text{where } \alpha < 0. \end{aligned}\right\} \qquad (1.3.33)$$

Thus, where $\lambda_1^* \geq 0$, the optimal control condition (1.3.20) is satisfied only with limiting values of α (Figure 1.8). We represent

$$\lambda_\alpha^* = \frac{\partial H_\alpha^*}{\partial \alpha}.$$

We then produce the following conclusions: the principle of the minimum with limiting and stable control α will be satisfied if in addition to the inequality

$$H_\alpha^* < 0 \quad \text{where } \alpha \neq 0$$

one of the following conditions is also fulfilled:

$$\left.\begin{aligned} \lambda_\alpha^* &= 0 \quad \text{where } \alpha_{min} \leq \alpha \leq \alpha_{max}, \\ \lambda_\alpha^* &< 0 \quad \text{where } \alpha = \alpha_{max}, \\ \lambda_\alpha^* &> 0 \quad \text{where } \alpha = \alpha_{min}. \end{aligned}\right\} \qquad (1.3.34)$$

where with $\lambda_1^* < 0$ the optimal control α can be either the limiting or the stable control, while where $\lambda_1^* \geq 0$ it can only be the limiting control.

These conclusions can be given a definite physical explanation. Suppose in place of α, the control function is Y.

Then the Euler-Lagrange equation (1.2.29) becomes

$$\lambda_4^* = -\lambda_1^* \frac{\partial Q_i^*}{\partial Y^*} + \frac{\lambda_3^*}{V}, \qquad (1.3.35)$$

where

$$\lambda_4^* = -\lambda_8 \frac{\partial \kappa}{\partial Y^*} \frac{\mu^*}{g_0}.$$

The characteristic dependence $Q_i^* = \phi^*(Y^*)$ is shown on Figure 1.9. It is such that the curve $\phi^*(Y^*)$ is always above the tangent. Therefore, we have

$$(\bar{Q}_i^* - Q_i^*) - (\bar{Y}^* - Y^*) \frac{\partial Q_i^*}{\partial Y^*} > 0,$$

where Q_i^* is the value of Q_i^* at the point of contact, corresponding to Y^*;

$\bar{Q}_i^*$ is the value of Q_i^* with arbitrary $\bar{Y}^* \neq Y^*$.

The Weierstrass condition (1.3.1) relative to the control function Y^* can be written as follows:

$$E_4^* = \left[\lambda_1^*(\bar{Q}_i^* - Q_i^*) - \frac{\lambda_3^*}{V} (\bar{Y}^* - Y^*)\right] \leqslant 0.$$

Using (1.3.35), let us convert it to the form

$$E_4^* = \lambda_1^* \left[(\bar{Q}_i^* - Q_i^*) - (\bar{Y}^* - Y^*) \frac{\partial Q_i^*}{\partial Y^*} - \frac{\lambda_4^*}{\lambda_1^*} (\bar{Y}^* - Y^*)\right] \leqslant 0.$$

We can see that the stable control, when $\lambda_\alpha^* = 0$, is possible only with $\lambda_1^* < 0$. Limiting control will correspond to the Weierstrass condition if where $\lambda_1^* < 0$

$$\lambda_4^*(\bar{Y}^* - Y^*) > 0,$$

where $\lambda_1^* > 0$

$$\lambda_4^*(Y^* - \bar{Y}^*) \leqslant 0.$$

These conditions can be easily reduced to inequalities (1.3.34).

Thus, the possibility of stable control α where $\lambda_1^* < 0$ and its absence where $\lambda_1^* > 0$ is determined by the process of interaction of the aerodynamic lift and its inductive drag in the form of dependence $Q^* = \phi^*(Y^*)$, given on Figure 1.9.

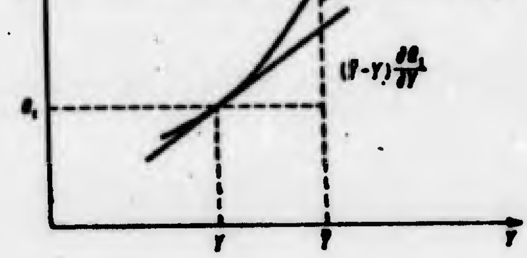

Figure 1.9. Characteristic Dependence of Aerodynamic Inductive Drag Q_i on Lift Y: V = const, H = const

Inequalities (1.3.30) and (1.3.31) do not exclude non-uniqueness of the solution of the optimal control condition (1.3.20). It occurs where

$$\inf_{\alpha=\alpha_{max}} H_\alpha^* = \inf_{\alpha=\alpha_{min}} H_\alpha^*,$$

from which

$$\frac{Q_{i\,max}^* - Q_{i\,min}^*}{Y_{max}^* + |Y^*|_{min}} = -\frac{\lambda_3^*}{V\lambda_1^*}. \qquad (1.3.36)$$

Then equation (1.3.36) can be fulfilled under the following conditions:

$$Y_{max}^* < |Y_{min}^*|, \ \lambda_3^* < 0, \qquad (1.3.37)$$

$$Y_{max}^* > |Y^*|_{min}, \ \lambda_3^* > 0, \qquad (1.3.38)$$

$$Q_{i\,max}^* = Q_{i\,min}^*, \ \lambda_3^* = 0, \qquad (1.3.39)$$

Equation (1.3.36) is the condition of transition from one limiting control to the other. If it is retained over some interval (τ_1, τ_2), a slipping mode arises, consisting of an infinitely great sequence of limiting control α_{max} and α_{min}. With this control, the movement of the flight vehicle occurs with the maximum inductive drag, which may result from the requirement of maximum controlled deceleration. Therefore, slipping mode α will be interpreted as one form of maximum controlled deceleration. Equation (1.3.36) decreases the order of system (1.II) by one in the slipping mode.

Furthermore, we must note that where $0 \leq \alpha \leq \alpha_{max}$ with (1.3.30), (1.3.31), according to (1.3.21) and control condition (1.3.19), the solutions of (1.3.20) are the limiting control $\alpha = \alpha_{max}$ and the zero control. Transition from the limiting control to zero control is determined by the equation

$$-\lambda_1^* Q_{i\,max}^* + \lambda_2^* \frac{Y_{max}^*}{V} = 0.$$

Thus, where $\lambda_1^* > 0$ when the following inequalities are fulfilled

$$\left.\begin{array}{l} Y_{max}^* \geqslant |Y_{min}^*|,\ \lambda_2^* < 0, \\ Y_{max}^* \leqslant |Y_{min}^*|,\ \lambda_2^* > 0, \\ Q_{i\,max}^* = Q_{i\,min}^*,\ \lambda_2^* \neq 0 \end{array}\right\} \qquad (1.3.40)$$

the solution of (1.3.20) is always unambiguous and corresponds to one of the limiting controls, while where conditions (1.3.37)-(1.3.39) are fulfilled, the solution may not be unique. We should discuss particularly the conditions of formation and retention of the slipping optimal mode. The existence of a slipping mode is related to retention over a fixed time interval (τ_1, τ_2) of the condition

$$H_*^{ck} = -\lambda_1^* Q_{i\,max}^* + \lambda_2^* \frac{Y_{max}^*}{V} = -\lambda_1^* Q_{i\,min}^* + \lambda_2^* \frac{Y_{min}^*}{V}.$$

Therefore, the slipping mode will correspond at each moment in time to the optimal control α when the following condition is fulfilled:

$$m_*^* = H_*^{ck}.$$

The slipping modes of control α (and other control functions) introduce a number of difficulties to the solution of variational problems. The possibility of their formation creates particular difficulties of a computational nature. Although there are certain approximate methods for solution of variational problems with slipping modes [11, 20], it must be noted that there are as yet no sufficiently general methods for solution of this problem.

It is very important here to be able to solve the variational problem within the framework of one computation algorithm with and without slipping modes. Another difficulty is the difficulty of technical achievement. Of course, slipping modes cannot be realized. In this connection, if the solution of the variational problem is related to the existence of a slipping mode, we must either return anew to the initial conditions of the problem to introduce a limitation resulting from the inertial nature of the control system, or go over to approximate methods of calculation of the slipping mode.

Thus, retention of unambiguity of the solution of (1.3.20) with respect to time leads either to the necessity of introduction of new conditions in the solution of the variational problem, or to approximate calculation methods.

All solutions found in the investigation of the optimal control condition for α (1.3.20), have been produced with no relation to control ω. When there are control functions ω, they must not contradict the solutions for the optimal control condition for ω (1.3.5) in their physical significance. Therefore, the investigation of optimal control α and ω will be performed considering their mutual influence.

We have

$$H^{*}_{\alpha,\omega}=\lambda_1 \frac{g_0}{\mu^*}(a^*_{0i}p^*\cos\omega-Q^*_i)+\lambda^*_2\frac{g_0}{\mu^* V}(a^*_{0i}p^*\sin\omega+Y^*).$$

The condition of optimal control of α and ω simultaneously is

$$m^*=\inf_{\substack{\alpha_{min}<\alpha<\alpha_{max}\\ \omega_{min}<\omega<\omega_{max}}} H^{*}_{\alpha,\omega}. \tag{1.3.41}$$

It can be fulfilled when the control conditions are realized

$$\lambda^*_1\frac{g_0}{\mu^*}(a^*_{0i}p^*\cos\omega-Q^*_i)+\lambda^*_2\frac{g_0}{\mu^* V}(a^*_{0i}p^*\sin\omega+Y^*)\leqslant 0 \tag{1.3.42}$$

as well as the control conditions relative to α where $p^* \geq 0$

$$-\lambda^*_1 Q^*_i+\lambda^*_2\frac{Y^*}{V}\leqslant 0. \tag{1.3.43}$$

These two conditions lead to yet another condition:

$$p^*\left(\lambda^*_1\cos\omega+\lambda^*_2\frac{\sin\omega}{V}\right)\leqslant 0. \tag{1.3.44}$$

Suppose $|\omega| \leq \pi/2$. Then the value of expression $(a^*_{0i}p^* \cos\omega - Q^*_i)$ is independent of the sign of ω and α, and is determined only by their absolute values. The value of expression $(a^*_{0i}\,p^* \sin\omega + Y^*)$ depends both on the absolute values of α and ω and on their signs. Therefore, with any defined values of α and ω, the value of the expression $\left(\frac{a_{0i}^* p^* \sin\omega}{Y^*}+1\right)$ or $\left(1+\frac{Y^*}{a_{0i}^* p^* \sin\omega}\right)$ becomes greatest when the signs correspond.

Optimal control condition (1.3.41) will be represented in the form

$$m^* = \inf_{\substack{\alpha_{min}<\alpha<\alpha_{max} \\ \omega_{min}<\omega<\omega_{max}}} \left[\lambda_1^* \frac{g_0}{\mu^*}\left(a_{0i}^* p^* \cos\omega - Q_i^*\right) + \lambda_2^* Y^* \frac{g_0}{\mu V}\left(1+\frac{a_{0i}^* p^* \sin\omega}{Y^*}\right)\right],$$

or

$$m^* = \inf_{\substack{\alpha_{min}<\alpha<\alpha_{max} \\ \omega_{min}<\omega<\omega_{max}}} \left[\lambda_1^* \frac{g_0}{\mu^*}\left(a_{0i}^* p^* \cos\omega - Q_i^*\right) + \lambda_2^* \sin\omega \frac{g_0 a_{0i}^* p^*}{\mu^* V}\left(1+\frac{Y^*}{a_{0i}^* p^* \sin\omega}\right)\right].$$

These conditions of optimal control can be determined if according to (1.3.43) and (1.3.44), we have

$$\lambda_2^* Y^* \leqslant 0 \quad \text{where } \lambda_1^* \leqslant 0.$$

$$\lambda_2^* \sin\omega \leqslant 0 \quad \text{where } \lambda_1^* \geqslant 0.$$

Based on these inequalities, we note that according to (1.3.42) the condition of simultaneous optimal control α and ω can be fulfilled if $Y^*/\sin \geq 0$ or $\sin\omega/Y^* > 0$, i. e. if the directions of the thrust vector and aerodynamic lift vector correspond. Thus, regardless of the sign of the Lagrange coefficient λ_1^*, the condition of simultaneous optimal control α and ω can be fulfilled when the inequality

$$\lambda_2^* Y^* \leqslant 0$$

is fulfilled, and the directions of the thrust vector and lift vector correspond. In this connection, in the active sector of an optimal phase trajectory of a flight vehicle, slipping mode α cannot be encountered. If it is possible, it will occur during the passive sector, when condition (1.3.44) is not fulfilled. Therefore, in the active sector of an optimal phase trajectory, the optimal control condition for α and ω (1.3.41) is satisfied where $\lambda_1^* < 0$, both with stable control α and ω and with limiting control α and ω, while where $\lambda_1^* > 0$ it will be satisfied only with limiting control α and ω.

For control function r we will have

$$H_r^* = \frac{g_0}{\mu^*} a_{0i}^* p^* \left(\lambda_1^* \cos\omega + \lambda_2^* \frac{\sin\omega}{V}\right).$$

Assuming the possibility of the zero control for the force of gravity, when p* = 0, the control condition and optimal control condition for function r can be represented in the form

$$p^* \left(\lambda_1^* \cos\omega + \lambda_2^* \frac{\sin\omega}{V}\right) < 0, \tag{1.3.45}$$

$$m^* = \inf_{r_{min} < r < r_{max}} p^* \left(\lambda_1^* \cos\omega + \lambda_2^* \frac{\sin\omega}{V}\right). \tag{1.3.46}$$

Usually, the permissible values of r_{min} and r_{max} are such that when they occur, p* > 0. Then the control condition (1.3.45) for r becomes similar to control condition (1.3.4) for ω. In this connection, the optimal control condition (1.3.46) will be fulfilled with those values of r for which $p^* = (p^*_{max})_r$. Therefore, only in the case of an upward-bulging dependence p = Φ(r) and the given V, H and d is stable control r possible with $r_{opt} \leq r_{max}$ (Figure 1.10), the stable control defined by the condition

$$\frac{\partial p^*}{\partial r} = 0. \tag{1.3.47}$$

In other cases, the optimal control r will be its value with the given V, H and d for which p* reaches its maximum. In the following, it is this value of r which we will consider the maximum r_{max} (Figure 1.11). Thus, optimal control r is possible either with its stable value r_{opt}, determined from (1.3.47), or with its maximum value r_{max}. However, this conclusion is correct only until stable operation of the power plant is disrupted. Otherwise, control r will be related to the condition of stable operation of the power plant.

For control function d we produce

$$H_d^* = \frac{g_0}{\mu^*} a_{0i}^* p^* \left(\lambda_1^* \cos\omega + \lambda_2^* \frac{\sin\omega}{V}\right) - \lambda_6^* \frac{\dot{a}_{0i}}{f_{0i}^*} f^*.$$

The minimum value of d can be various. For a more precise physical representation of the role of optimal control d, we assume d_{min} such that

$p^* = 0$. Then the control condition for d can be represented in the form

$$\frac{g_0}{\mu^*} a^*_{0i} p^* \left(\lambda^*_1 \cos\omega + \lambda^*_2 \frac{\sin\omega}{V}\right) - \lambda^*_6 \left(\frac{\dot{a}_{0i}}{J^*_{0i}} - \mu^*_{ts\,min}\right) < 0.$$

The presence of $u^*_{ts\ min} \neq 0$ where $p^* = 0$ results from the existence of thermodynamic losses in every actual power plant, which must be covered by the expenditure of a certain quantity of fuel. However, for simplification of our investigations we can assume $u^*_{ts\ min} = 0$. This assumption is quite correct where $p^* > 0$. Therefore, control condition d can be represented as

$$H^*_p = \frac{g_0}{\mu} a^*_{0i} p^* \left(\lambda^*_1 \cos\omega + \lambda^*_2 \frac{\sin\omega}{V}\right) - \lambda^*_6 \frac{\dot{a}_{0i}}{J^*_{0i}} f^* < 0. \quad (1.3.48)$$

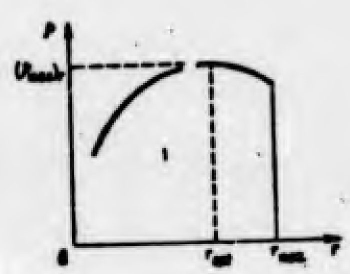

Figure 1.10. Determination of Stable Control r: V = const, H = const, d = const, $r_{opt} < r_{max}$, $\partial^2 p/\partial r^2 < 0$

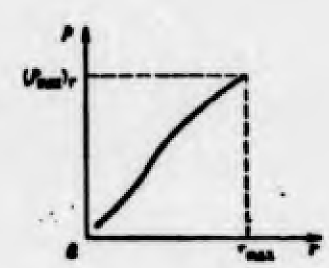

Figure 1.11. Determination of Optimal Limiting Control: V = const, H = const, d = const, $r_{opt} = r_{max}$

The optimal control condition for d is written as follows:

$$m^* = \inf_{d_{min} < d < d_{max}} \left[\frac{g_0}{\mu^*} a^*_{0i} p^* \left(\lambda^*_1 \cos\omega + \lambda^*_2 \frac{\sin\omega}{V}\right) - \lambda^*_6 \frac{\dot{a}_{0i}}{J^*_{0i}} f^*\right]. \quad (1.3.49)$$

Since the choking parameters d^*_s are independent, control condition (1.3.48) and optimal control condition (1.3.49) can be analyzed for each choking parameter separately, considering the other parameters constant. Then in place of choking parameter d we can use p^* as the control function, expressing function f^* as the following formula:

$$f^* = f^* (p^*, d^*_i, V, H, R, O_0, a_0, J_0, b_{0i}).$$

where d_l^* are the regulation parameters except for d_s^* $(l = 1, \ldots, n - 1)$.

Therefore, the optimal control condition (1.3.49) for d_s^* is now represented as:

$$m = \inf_{<p<(p_{max})d_s} \left[\frac{g_0}{\mu^*} a_{0l}^* p^* \left(\lambda_1^* \cos\omega + \lambda_2^* \frac{\sin\omega}{V} \right) - \lambda_6^* \frac{a_{0l}^*}{J_{0l}^*} f^* \right]. \tag{1.3.50}$$

We should note the identical nature of the optimal control conditions (1.3.49) and (1.3.50). However, (1.3.50) can be used to produce the solution more clearly. Investigation of the solutions of the optimal control condition (1.3.50) will be performed considering control condition (1.3.48) and optimal control ω and α, considering r^*, d_l^* $(l = 1, \ldots, n - 1)$, V, H, μ^*, ω, λ_1^*, λ_2^* and λ_6^* constant. Keeping in mind the control condition for ω (1.3.4), we find that control condition (1.3.48) will be fulfilled where

$$\lambda_6^* \geqslant 0, \tag{1.3.51}$$

$$\lambda_6^* < 0. \tag{1.3.52}$$

We have

$$\frac{\partial H_p^*}{\partial p^*} = \frac{g_0}{\mu^*} a_{0l}^* \left(\lambda_1^* \cos\omega + \lambda_2^* \frac{\sin\omega}{V} \right) - \lambda_6^* \frac{a_{0l}^* \partial f^*}{J_{0l}^* \partial p^*} \tag{1.3.53}$$

or, since

$$\frac{\partial f^*}{\partial p^*} = \frac{\partial}{\partial p^*} \left(\frac{p^*}{p_{sp}^*} \right) = \frac{1}{p_{sp}^*} \left(1 - \frac{p^*}{p_{sp}^*} \frac{\partial p_{sp}^*}{\partial p^*} \right),$$

then

$$\frac{\partial H_p^*}{\partial p^*} = \frac{1}{p^*} \left[\frac{g_0}{\mu^*} a_{0l}^* p^* \left(\lambda_1^* \cos\omega + \lambda_2^* \frac{\sin\omega}{V} \right) - \lambda_6^* \frac{a_{0l}^*}{J_{0l}} f^* + \right.$$
$$\left. + \lambda_6^* \frac{a_{0l}^*}{J_{0l}^*} \frac{p^{*2}}{p_{sp}^{*2}} \frac{\partial p_{ya}^*}{\partial p^*} \right]. \tag{1.3.54}$$

$$\frac{\partial^2 H_p^*}{\partial p^{*2}} = -\lambda_6^* \frac{a_{0l}^* \partial^2 f^*}{J_{0l}^* \partial p^{*2}} \quad \text{or} \quad \frac{\partial^2 H_p^*}{\partial p^{*2}} = \lambda_6^* \frac{\partial^2 p^*}{\partial \mu_{ya}^{*2}} \Big/ \left(\frac{\partial p^*}{\partial \mu_{ya}^*} \right)^3 \tag{1.3.55}$$

On the basis of (1.3.51) and (1.3.52), let us determine the possible solutions for the optimal control condition (1.3.50). In the case of

(1.3.51), with control condition ω (1.3.4), according to (1.3.53) we have $\partial H_p^*/\partial p^* < 0$. Therefore, the solution of (1.3.50) or (1.3.49) is only control $p^* = (p^*_{max})_{d_s}$, referred to in the following as the maximum thrust mode and represented simply by p^*_{max} rather than $(p^*_{max})_{d_s}$.

Thus, in order to satisfy (1.3.49) or (1.3.50) at the extreme points, where the Lagrange coefficient λ_6^* is greater than or equal to zero, control function p^* should take on its maximum value. From this it also follows that if under control condition ω (1.3.4) the choking modes of operation of the power plant occur, they are possible only in the case $\lambda_6^* < 0$. Let us now investigate the solution of the optimal control condition (1.3.50) with (1.3.52).

Here first of all we must note the possibility of production of two equivalent solutions. Thus, at a certain moment in time with $p^* \neq 0$, the equation

$$\frac{g_0}{\mu^*}\left(\lambda_1^* \cos\omega + \lambda_2^* \frac{\sin\omega}{V}\right) - \lambda_6^* \frac{1}{J_{0t}^* P_{sp}^*} = 0, \tag{1.3.56}$$

is possible, which with the choking mode, according to (1.3.54), characterizes the cruising thrust mode ($p^* = p^*_{kp}$), for which $p^*_{sp} = p^*_{sp\ max}$. If at a subsequent moment in time control condition (1.3.48) is disrupted, then after satisfying (1.3.56), two controls are possible in principle: flight by inertia (zero control) or the slipping mode reflecting the pulsating thrust mode $p^* = p^*_{max}$ or $p^* = p^*_{kp}$ and $p^* = 0$. They satisfy condition (1.3.50) identically and correspond to $H_p^* = 0$. In this sense, the two solutions are equivalent. In the first case, equation (1.3.56) is the condition for transition from mode $p^* = p^*_{max}$ (or $p^* = p^*_{kp}$) to inertial flight, while in the second case it is the condition for transition to the slipping mode while retaining (1.3.56) with respect to time. From the point of view of simplicity of performance, the zero control (inertial flight) is more acceptable than the difficult slipping mode. However, after satisfying equation (1.3.56), preference should be given to the control for which the effectiveness criterion is higher.

Since we are interested only in modes up to $p = p_{max}$, no matter what the choking characteristic, throughout this range of p^* with increasing μ^*_{ts}, thrust p^* increases. Therefore, where $\lambda_6 < 0$, according to (1.3.55) we produce

$$\frac{\partial^2 H_p^*}{\partial p^{*2}} \approx -\frac{\partial^2 p^*}{\partial P_{Ts}^{*2}}, \qquad (1.3.57)$$

which indicates the opposite nature of the signs of $\frac{\partial^2 H_p^*}{\partial p^{*2}}$ and $\frac{\partial^2 p^*}{\partial P_{Ts}^{*2}}$.

The solution of (1.3.50), characterizing the optimal modes of operation of the power plant, will be determined after preliminarily dividing the possible choking characteristics into two standard groups. This allows us to order and generalize the results of the solution of (1.3.50) depending on the group of the choking characteristic.

The first group includes choking characteristics for which the value of the maximum possible thrust p^*_{max} practically corresponds with the value of the cruising thrust p^*_{kp}, corresponding with the given V and H to the maximum p^*_{sp} (Figure 1.12-1.14). For choking characteristics of the second group, the value of p^*_{max} is significantly greater than that of p^*_{kp} (Figure 1.15).

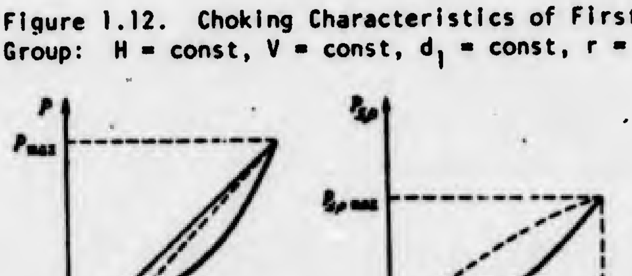

Figure 1.12. Choking Characteristics of First Group: H = const, V = const, d_1 = const, r = const

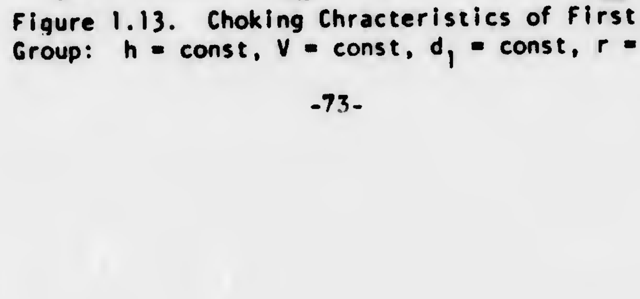

Figure 1.13. Choking Chracteristics of First Group: h = const, V = const, d_1 = const, r = const

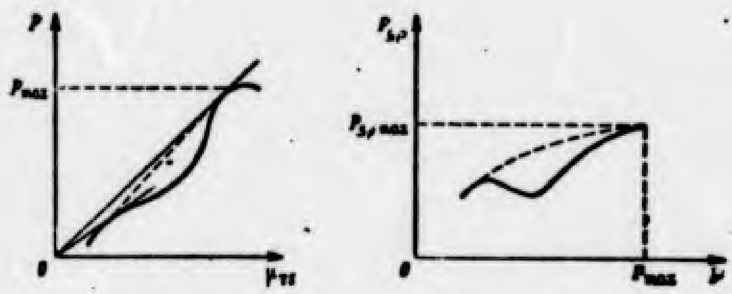

Figure 1.14. Choking Characteristics of First Group: H = const, V = const, d_1 = const, r = const

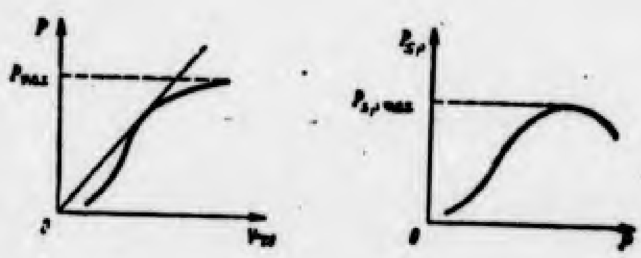

Figure 1.15. Choking Characteristics of Second Group: H = const, V = const, d_1 = const, r = const

Let us analyze the solution of (1.3.50) for the first group of choking characteristics. In this group, the choking characteristics in the range $0 \le p^* \le p^*_{max}$ can have varying curvature (see Figure 1.13 and 1.14), leading to certain difficulties in the investigation of (1.3.50). In order to eliminate these difficulties, we must turn to the "method of improvement" of choking characteristics. It consists in replacing the arbitrary form of the choking characteristic with a characteristic having a constant curvature and a value of thrust for each value of fuel flow rate per second ν^*_{ts} and constants r^*, d^*_1, V and H, equal to or greater than the true value. Thus, the "method of improvement" allows us to give the choking characteristic a constant curvature by increasing the specific thrust at certain points on the characteristic with constant fuel flow rate per second. It then becomes obvious that if with the improved choking characteristic the choking modes of operation of the power plant do not correspond to the condition of optimal control (1.3.50), they do not correspond to it with arbitrary form of the choking characteristic.

In order to produce information from condition (1.3.50) on the optimal operating modes of the power plant, let us "improve" the arbitrary form of the choking characteristic, replacing the characteristic while retaining the values of $p^*_{kp} = p^*_{max}$ by characteristics which bulge upward by increasing p^*_{sp} (see Figures 1.13 and 1.14). In this connection we see that for all choking characteristics in the first group within the limits of $0 \leq p \leq p_{kp}$, $\partial p^*_{sp}/\partial p^* > 0$. This can be immediately seen, since the tangent of the angle connecting the coordinate origin to the point of the choking characteristic is equal to the specific thrust p^*_{sp} corresponding to this point.

Since $\lambda^*_6\,(\partial p^*_{sp}/\partial p^*) < 0$ and in view of control condition (1.3.48), within the range $p^* > 0$ according to (1.3.54) we will have $\partial H^*_p/\partial p^* < 0$ (Figure 1.16). Therefore, for the first group of choking characteristics, the solution of (1.3.50) corresponds only to the maximum thrust mode. If in the maximum thrust mode condition (1.3.48) is disrupted, the optimal control will be the zero control ($p^* = 0$ and $\mu^*_{ts} = 0$). The condition of transition from the maximum thrust mode to inertial flight (zero control) then becomes equation (1.3.56).

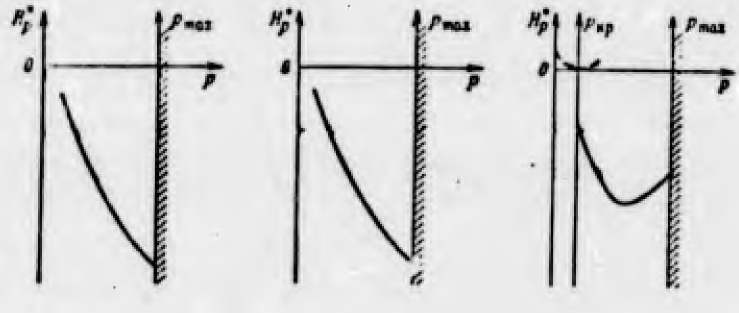

Figure 1.16. Dependence of Function H^*_p on Relative Thrust p (left, where $\lambda_6 \geq 0$; others, where $\lambda_6 < 0$.

Among the choking characteristics of the first group, the linear choking characteristic corresponding to the ray (1.1.17) occupies a special position. However, the linear choking characteristic is not realized in practice. Due to various thermodynamic losses (change in heat liberation coefficient, index of polytropy, etc.) for a given altitude H and control function r*, the specific thrust drops with decreasing flow rate of the fuel (slightly with high degrees of choking, considerably with lower degrees of choking)[1]. Therefore, it can be fully approximated by an upward bulging choking characteristic for which $p^*_{max} = p^*_{kp}$ and which, with high degrees of choking, is quite similar to the ray connecting the coordinate origin with the point p^*_{max}. Thus, for power plants in which the choking characteristics are in the first group, the optimal control is the maximum thrust mode or inertial flight, the transition between which being performed by equation (1.3.56).

Let us study the conditions of optimal control (1.3.50) for choking characteristics in the second group (see Figure 1.1.15). Here, as in the preceding case, we apply the "method of improvement" to the portion of the choking characteristic with the value $p^* < p^*_{kp}$, allowing us to represent it as an upward bulging curve by increasing the corresponding points of p^*_{sp}. Then, according to (1.3.48), where $p^* \geq 0$, it follows from (1.3.54) that in the interval $0 \leq p^* \leq p^*_{kp}$, $\partial H^*_p/\partial p^* < 0$ and $\partial H^*_p/\partial p^* > 0$ are possible only where $p^* \geq p^*_{kp}$. Therefore, the optimal control of the power plant corresponding to (1.3.50) occurs where $p^*_{kp} \leq p^* \leq p^*_{max}$ or with the zero control ($p^* = 0$, $\mu^*_{ts} = 0$).

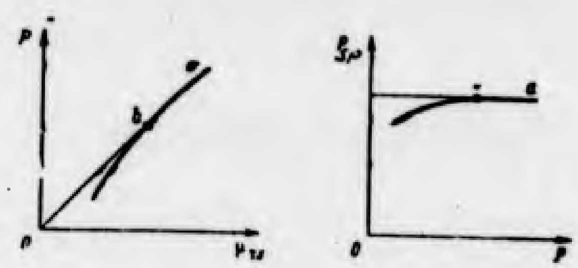

Figure 1.17. Actual Linear Choking Characteristic: H = const, r = const

If where $p^* > p^*_{kp}$ the choking characteristic is upward convex

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[1] The degree of choking is the ratio of the instantaneous thrust value p^* to the maximum thrust value p^*_{max}.

(Figure 1.15) throughout, then since (see Figure 1.16)

$$\frac{\partial^2 H_p^*}{\partial p^{*2}} = -\lambda^*_6 \frac{\partial^2 f^*}{\partial p^{*2}} > 0$$

solution of (1.3.50) with

$$\frac{\partial H_p^*}{\partial p^*} = \frac{g_0 a_{0l}^*}{\mu^*}\left(\lambda_1^* \cos\omega + \lambda_2^* \frac{\sin\omega}{V}\right) - \lambda_6^* \frac{a_{0l}^* \partial f^*}{j_{0l}^* \partial p^*} = 0$$

is possible.

At point $p^* = p^*_{kp}$ we have

$$\frac{g_0}{\mu^*}\left(\lambda_1^* \cos\omega + \lambda_2^* \frac{\sin\omega}{V}\right) - \lambda_6^* \frac{1}{j_{0l}^* p_{sp}^*} = 0,$$

which leads to $H_p^* = 0$. If subsequently condition (1.3.48) is disrupted, after $p^* = p_{kp}$ two controls are possible: the zero control (inertial flight) and the slipping mode, reflecting the pulsating thrust mode $p^* = p_{kp}$ and $p^* = 0$, $\psi^*_{ts} = 0$. They correspond identically to $H_p^* = 0$. Therefore, preference must be given to the control for which the criterion of effectiveness is higher.

Thus, for choking characteristics in a second group, bulging upward where $p^* \geq p^*_{kp}$, the solution of (1.3.50) is unambiguous at each moment in time and corresponds where $\partial H_p^*/\partial p^* > 0$ to the maximum thrust mode, where $p^* \geq p^*_{kp}$ and $\partial H_p^*/\partial p^* = 0$ -- to the choking mode and where (1.3.48) is disrupted -- to the zero control ($\psi^*_{ts} = 0$, $p^* = 0$)[1]. The conditions of transition from the maximum thrust mode to the choking mode and from the choking mode to inertial flight are the following equations, respectively

$$\left.\begin{aligned} &\frac{\partial H_p^*}{\partial p} = \frac{g_0 a_{0l}^*}{\mu^*}\left(\lambda_1^* \cos\omega + \lambda_2^* \frac{\sin\omega}{V}\right) - \lambda_6^* \frac{a_{0l}^*}{j_{0l}^*}\left(\frac{\partial f^*}{\partial p^*}\right)_{p_{max}} = 0, \\ &\frac{g_0 a_{0l}^*}{\mu^*}\left(\lambda_1^* \cos\omega + \lambda_2^* \frac{\sin\omega}{V}\right) - \lambda_6^* \frac{a_{0l}^*}{j_{0l}^* p_{sp\,max}^*} = 0. \end{aligned}\right\} \quad (1.3.58)$$

Among the choking modes, the mode with $p^* = p^*_{kp}$, corresponding to the value of maximum specific thrust, occupies a particular position.

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[1] Similar results for one particular problem were produced in [21].

It is the most economical mode. Therefore, in contrast to the other choking modes, it is usually referred to as the cruising mode. Then, the remaining choking modes with optimal control p* can be interpreted as transitional modes from the maximum thrust mode to the cruising mode, or as transitional modes from lower thrust to the maximum thrust. This is the "physical" sense of the choking modes of operation of the power plant with optimal flight modes of the flight vehicle. For certain power plants, they are not used during the flight, although it is operationally possible to achieve the operation of the power plant at thrust values less than the cruising mode. However, as the solutions of (1.3.50) show, these operating modes of the power plant do not correspond to optimal control p*. In this sense, the cruising mode is the minimum thrust mode.

The conclusions which we have produced concerning the conditions of optimal control p* can be extended to each parameter of regulation d_i^* ($i = 1, \ldots, n-1$).

Let us now give a geometric interpretation to our conclusions. First of all, it is characteristic for the "improved" choking characteristics of the first group where $p^* \leq p^*_{max}$ (Figure 1.18) that

$$\bar{E}^{\bullet}_{p_{\kappa p}} = \left[(\dot{p}^{\bullet}_{\kappa p} - \bar{p}^{\bullet}) - (\dot{\mu}^{\bullet}_{\tau s \kappa p} - \bar{\dot{\mu}}^{\bullet}_{\tau s}) \frac{\partial \dot{p}^{\bullet}_{\kappa p}}{\partial \dot{\mu}^{\bullet}_{\tau s}} \right] \geqslant 0 \qquad (1.3.59)$$

and

$$\bar{E}^{\bullet}_{p_0} = \left(p^{\bullet} - \mu^{\bullet}_{\tau s} \frac{\partial p^{\bullet}}{\partial \dot{\mu}^{\bullet}_{\tau s}} \right) < 0 \qquad (1.3.60)$$

with permissible control $\bar{p}^* = 0$, while for downward-bulging choking characteristics, the following condition is always maintained (Figure 1.19):

$$\bar{E}^{\bullet}_{p} = \left[(\dot{\bar{p}}^{\bullet} - p^{\bullet}) - (\bar{\dot{\mu}}^{\bullet}_{\tau s} - \dot{\mu}^{\bullet}_{\tau s}) \frac{\partial p^{\bullet}}{\partial \dot{\mu}^{\bullet}_{\tau s}} \right] > 0. \qquad (1.3.61)$$

For choking characteristics of the second group where $p^*_{kp} \leq p^* \leq p^*_{max}$, if they are upward-convex, we have (Figure 1.20)

$$\bar{E}^{\bullet}_{p} = \left[(\bar{p}^{\bullet} - p^{\bullet}) - (\bar{\dot{\mu}}^{\bullet}_{\tau s} - \dot{\mu}^{\bullet}_{\tau s}) \frac{\partial p^{\bullet}}{\partial \dot{\mu}^{\bullet}_{\tau s}} \right] < 0, \qquad (1.3.62)$$

where as if they bulge downward, we have (1.3.61).

The necessary Weierstrass condition for the control function p* (or d_s^*) becomes

$$E_f^{\circ}=\left[\frac{g_0}{\mu^{\circ}}\, a^{\circ}_{0i}\left(\lambda_1^{\circ}\cos\omega+\lambda_2^{\circ}\frac{\sin\omega}{V}\right)(p^{\circ}-\bar{p}^{\circ})-\lambda_6^{\circ}\frac{a^{\circ}_{0i}}{J^{\circ}_{0i}}(f^{\circ}-\bar{f}^{\circ})\right]\leqslant 0. \qquad (1.3.63)$$

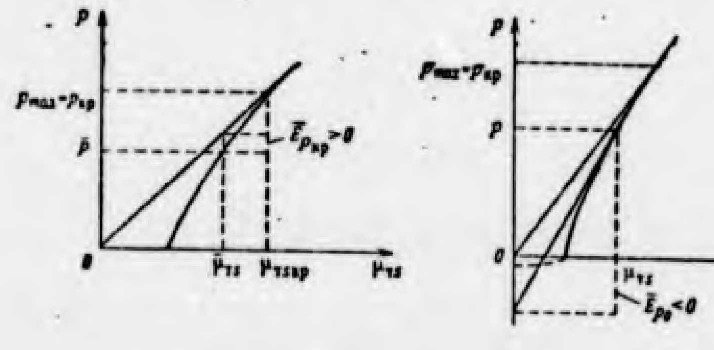

Figure 1.18. Geometric Interpretation of Necessary Weierstrass Condition for Control Function p and Upward-Convex Choking Characteristics of First Group

In the case of the choking mode of operation of the power plant, the Euler-Lagrange equation (1.2.23) is produced in the form

$$\frac{g_0 a^{\circ}_{0i}}{\mu^{\circ}}\left(\lambda_1^{\circ}\cos\omega+\lambda_2^{\circ}\frac{\sin\omega}{V}\right)-\lambda_6^{\circ}\frac{a^{\circ}_{0i}}{J^{\circ}_{0i}}\frac{\partial f^{\circ}}{\partial p^{\circ}}=0. \qquad (1.3.64)$$

Then with the choking mode of operation of the power plant, the necessary Weierstrass condition (1.3.63) becomes

$$-\left(\lambda_1^{\circ}\cos\omega+\lambda_2^{\circ}\frac{\sin\omega}{V}\right)\left[(\bar{p}^{\circ}-p^{\circ})-(\bar{\mu}^{\circ}_{TS}-\mu^{\circ}_{TS})\frac{\partial p^{\circ}}{\partial\mu^{\circ}_{TS}}\right]\leqslant 0.$$

Keeping in mind the control condition for ω (1.3.4), we have

$$(\bar{p}^{\circ}-p^{\circ})-(\bar{\mu}^{\circ}_{TS}-\mu^{\circ}_{TS})\frac{\partial p^{\circ}}{\partial\mu^{\circ}_{TS}}\leqslant 0. \qquad (1.3.65)$$

Figure 1.19. Geometric Interpretation of Necessary Weierstrass Condition for Control Function p and Downward-Convex Choking Characteristic

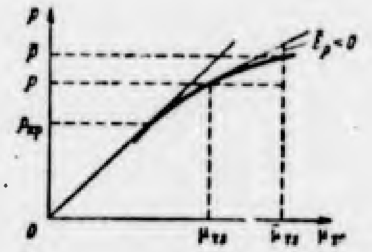

Figure 1.20. Geometric Interpretation of Necessary Weierstrass Condition for Control Function p and Upward-Convex Choking Characteristic of Second Group

This inequality reflects the necessary Weierstrass condition for the control function p with the choking mode of the power plant. Now, knowing the geometric properties of the choking characteristics expressed by equations (1.3.59)-(1.3.62), we can speak of the possibility of choking modes as optimal controls. For example, for the choking characteristics of the first group according to (1.3.59)-(1.3.61), with no value of p^* in the range $0 < p^* < p^*_{max} = p^*_{kp}$ is the Weierstrass necessary condition fulfilled, while for choking character tics of the second group according to (1.3.61)-(1.3.62), choking modes will satisfy (1.3.65) only in the case of choking characteristics which bulge upward and at $p^*_{kp} \leq p^* \leq p^*_{max}$. Thus, on the basis of the geometric properties of the choking characteristics, reflecting their physical properties, the same results are produced concerning optimal control of the power plant as in the investigation of the control condition (1.3.48) and optimal control condition (1.3.50).

It was demonstrated earlier that in many cases of limiting control ω where $\lambda^*_6 > 0$, it is possible in principle for the following inequality to exist without disrupting the control condition for p^* (1.3.48):

$$\lambda_1^* \cos\omega + \lambda_2^* \frac{\sin\omega}{V} > 0.$$

Then, based on the values of derivatives $\partial H^*_p/\partial p^*$ and $\partial^2 H_p/\partial p^{*2}$, according to (1.3.54) and (1.3.55) or keeping in mind (1.3.65), we can show that the power plant with choking characteristics of the first group bulging upward should operate either at the maximum thrust mode or at $p^* = 0$, while for characteristics which bulge downward it can operate in a choked mode. For power plants with choking characteristics of the second group according to (1.3.54), a choked mode is also possible with $0 < p^* < p^*_{kp}$ if where $p^* < p^*_{kp}$ the characteristic bulges downward.

Thus, with the limiting control for ω and $\lambda_1^* > 0$, $\lambda_6^* > 0$, for power plants with choking characteristics which bulge downward where $p^* < p^*_{kp}$, a choked mode is possible as the optimal control p. This choking of thrust, obviously, results from the "requirement" of a certain curvature of the trajectory.

Up to this point, we have studied the necessary Weierstrass condition considering the deflecting of the thrust vector from the axis of the vehicle, resulting from the presence of control function ω. Suppose the thrust vector is always directed along the longitudinal axis of the apparatus and the control functions are α, d and r. Then

$$H^* = \lambda_1^* \frac{g_0 \dot{a}_{0i} p^* \cos\alpha - \dot{Q}_i}{\mu^*} + \lambda_2^* \frac{g_0 \dot{a}_{0i} p^* \sin\alpha + Y^*}{\mu^* V} - \lambda_6^* \frac{\dot{c}_{0i}}{\dot{J}_{0i}} f^*. \quad (1.3.66)$$

It follows from (1.3.66) that

$$H_\alpha^* = \lambda_1^* \frac{\dot{a}_{0i} p^* \cos\alpha - \dot{Q}_i}{\mu^*} + \lambda_2^* \frac{\dot{a}_{0i} p^* \sin\alpha + Y^*}{\mu^* V}.$$

The optimal control condition α becomes

$$m = \inf_{\alpha_{min} < \alpha < \alpha_{max}} \left(\lambda_1^* \frac{\dot{a}_{0i} p^* \cos\alpha - \dot{Q}_i}{\mu^*} + \lambda_2^* \frac{\dot{a}_{0i} p^* \sin\alpha + Y^*}{\mu^* V} \right). \quad (1.3.67)$$

In the investigation of this condition, we must keep in mind the following zero controls:

$$\alpha = 0 \text{ and } p^* = 0, \quad (1.3.68)$$

$$\alpha \neq 0, \; p^* = 0, \quad (1.3.69)$$

which are singular points for expression (1.3.67). Zero controls (1.3.68) and (1.3.69) allow us to write the following control conditions where $\lambda_6^* \le 0$:

$$\lambda_1^* \frac{\dot{a}_{0i} p^* \cos\alpha - \dot{Q}_i}{\mu^*} + \lambda_2^* \frac{\dot{a}_{0i} p^* \sin\alpha + Y^*}{\mu^* V} < 0, \quad (1.3.70)$$

$$-\lambda_1^* \frac{\dot{Q}_i}{\mu^*} + \lambda_2^* \frac{Y^*}{\mu^* V} < 0. \quad (1.3.71)$$

If $p^* \geq 0$ and $|\alpha| \geq 0$, then it follows from the control condition that

$$p^*\left(\lambda_1^* \cos\alpha + \lambda_2^* \frac{\sin\alpha}{V}\right) \leqslant 0. \tag{1.3.72}$$

Inequality (1.3.72) can be analyzed in place of (1.3.70) as the control condition. Thus, where $p^* = 0$ the optimal control condition (1.3.67) will be satisfied by the solutions produced earlier in our analysis of control condition (1.3.19) and optimal control condition (1.3.20). Therefore, where $p^* = 0$ and $\lambda_1^* < 0$, optimal control condition (1.3.67) is satisfied both with stable and with limiting control α, while where $p^* = 0$ and $\lambda_1^* \geq 0$ it is satisfied only with the limiting control. If $p^* \neq 0$, control conditions (1.3.71) and (1.3.72) are fulfilled if

$$\lambda_1^* \leqq 0 \quad \text{and} \quad \lambda_2^* \sin\alpha \leqslant 0. \tag{1.3.73}$$

Under these conditions and with $p^* \neq 0$ in the range $|\alpha| < \pi/2$, the slipping mode cannot be produced on the optimal phase trajectory. Thus, in the active sectors of the optimal phase trajectory where $|\alpha| < \pi/2$, the condition of optimal control (1.3.67) cannot be satisfied by the slipping control mode.

On the basis of the expression for H_α^*, we find

$$\frac{\partial H_\alpha^*}{\partial\alpha} = -\lambda_1^* \frac{a_{0i}^* p^* \sin\alpha + \frac{\partial Q_i^*}{\partial\alpha}}{\mu^*} + \lambda_2^* \frac{a_{0i}^* p^* \cos\alpha + \frac{\partial Y^*}{\partial\alpha}}{\mu^* V}. \tag{1.3.74}$$

$$\frac{\partial^2 H_\alpha^*}{\partial\alpha^2} = -\frac{a_{0i}^* p}{\mu^*}\left(\lambda_1^* \cos\alpha + \lambda_2^* \frac{\sin\alpha}{V}\right) - \lambda_1^* \frac{\frac{\partial^2 Q_i^*}{\partial\alpha^2}}{\mu^*} + \lambda_2^* \frac{\frac{\partial^2 Y^*}{\partial\alpha^2}}{\mu^* V}. \tag{1.3.75}$$

In practice, over a very broad range of M numbers beyond the limits of the linear dependence of Y^* on α, according to [22], we can assume that

$$\frac{\partial^2 Y^*}{\partial\alpha^2} \leqslant 0 \text{ where } \alpha > 0 \text{ and } \frac{\partial^2 Y^*}{\partial\alpha^2} > 0 \text{ where } \alpha < 0.$$

Then where $p^* \neq 0$ and $\lambda_1^* \leq 0$, according (1.3.73)

$$\frac{\partial^2 H_\alpha^*}{\partial\alpha^2} > 0,$$

which leads to fulfillment of optimal control condition (1.3.67) with stable or limiting control α, and produces

$$\lambda_{\alpha}^{*}=\frac{\partial H_{\alpha}^{*}}{\partial\alpha}=0 \text{ where } \alpha_{min}\leqslant\alpha\leqslant\alpha_{max},$$

$$\lambda_{\alpha}<0 \text{ where } \alpha=\alpha_{max},$$

$$\lambda_{\alpha}>0 \text{ where } \alpha=\alpha_{min}.$$

If where $p^{*}\neq 0$, $\lambda_1^{*}>0$, then according to (1.3.74), optimal control condition (1.3.67) is satisfied only by limiting control α.

Consequently, when $|\alpha|<\pi/2$ where $\lambda_1^{*}\leq 0$, optimal control condition (1.3.67) in the passive and active sectors of the optimal phase trajectory is satisfied both by the stable and by the limiting control α, while where $\lambda_1^{*}\geq 0$, optimal control condition (1.3.67) is satisfied only by limiting control α.

For control functions r and d we have

$$H_{r}^{*}=-\frac{g_0}{\mu^{*}}a_{0i}^{*}p^{*}\left(\lambda_1^{*}\cos\alpha+\lambda_2^{*}\frac{\sin\alpha}{V}\right),$$

$$H_{d}^{*}=-\frac{g_0}{\mu^{*}}a_{0i}^{*}p^{*}\left(\lambda_1^{*}\cos\alpha+\lambda_2^{*}\frac{\sin\alpha}{V}\right)-\lambda_6^{*}\frac{\dot{a}_{0i}^{*}}{J_{0i}^{*}}f^{*}.$$

Therefore, the optimal control conditions for these functions are respectively

$$m^{*}=\inf_{r_{min}<r<r_{max}} H_{r}^{*}, \quad (1.3.76)$$

$$m^{*}=\inf_{d_{min}<d<d_{max}} H_{d}^{*}. \quad (1.3.77)$$

It is not difficult to show that where $p^{*}>0$, in connection with the control condition for α (1.3.72), the optimal control conditions for r and d are satisfied with optimal control α in the same solutions for which the optimal control conditions for r and d (1.3.46) and (1.3.49) are satisfied with optimal control ω. Therefore, optimal control conditions (1.3.76) and (1.3.77) for r and d will not be studied here.

Thus, the conclusions concerning optimal controls r and d are not related to the presence of control functions for ω or α.

Let us now analyze the control condition and optimal control condition for functions p^* and α, of a flight vehicle without lifting surfaces with its thrust always directed along the axis of the body. In this type of flight vehicle, the aerodynamic lift is created by deflection of the axis of the body from the velocity vector, and in the case $p^* = 0$ the vehicle moves along a ballistic trajectory. In this case, ignoring inductive drag of the body, we produce

$$H^* = \lambda_1^* \frac{\dot{a}_{0i} p^* \cos\alpha}{\mu^*} + \lambda_2^* \frac{\dot{a}_{0i} p^* \sin\alpha + Y^*}{\mu V} - \lambda_6^* \frac{\dot{a}_{0i}}{\dot{J}_{0i}} f^*$$

or, ignoring the terms with $\sin^2\alpha$ in Y^*, according to [22] we produce an expression for H^* in the form

$$H^* = \lambda_1^* \frac{\dot{a}_{0i} p^* \cos\alpha}{\mu^*} + \lambda_2^* \frac{(\dot{a}_{0i} p^* + Y^*) \sin\alpha}{\mu^* V} - \lambda_6^* \frac{\dot{a}_{0i}}{\dot{J}_{0i}} f^*, \tag{1.3.78}$$

where

$$\tilde{Y}^* = \frac{S_0^{(I)*}}{G_{0i}^*} \frac{\rho V^2}{2} c_y^{\alpha*}.$$

Then the control condition and optimal control condition for p^* and α can be written as

$$\lambda_1^* \frac{\dot{a}_{0i} p^* \cos\alpha}{\mu^*} + \lambda_2^* \frac{(\dot{a}_{0i} p^* + \tilde{Y}^*) \sin\alpha}{\mu^* V} - \lambda_6^* \frac{\dot{a}_{0i}}{\dot{J}_{0i}} \leqslant 0, \tag{1.3.79}$$

$$-\lambda_1^* \frac{\dot{a}_{0i} p^* (1 - \cos\alpha)}{\mu^*} + \lambda_2^* \frac{(\dot{a}_{0i} p^* + \tilde{Y}^*) \sin\alpha}{\mu^* V} \leqslant 0, \tag{1.3.80}$$

$$m^* = \inf_{0<p<p_{max}} \left(\lambda_1^* \frac{\dot{a}_{0i} p^* \cos\alpha}{\mu^*} + \lambda_2^* \frac{\dot{a}_{0i} p^* \sin\alpha}{\mu^* V} - \lambda_6^* \frac{\dot{a}_{0i}}{J_{0i}} f^* \right), \tag{1.3.81}$$

$$m^* = \inf_{\alpha_{min}<\alpha<\alpha_{max}} \left[\lambda_1^* \frac{\dot{a}_{0i} p^* \cos\alpha}{\mu^*} + \lambda_2^* \frac{(\dot{a}_{0i} p^* + \tilde{Y}^*) \sin\alpha}{\mu^* V} \right]. \tag{1.3.82}$$

If the inequality

$$\lambda_1^* \frac{\dot{a}_{0i} p^* \cos\alpha}{\mu^*} + \lambda_2^* \frac{(\dot{a}_{0i} p^* + \tilde{Y}^*) \sin\alpha}{\mu^* V} \geqslant 0,$$

occurs, then control conditions (1.3.79) and (1.3.80) for nonzero control are satisfied with

$$\lambda_1^* > 0 \quad \text{and} \quad \lambda_6^* > 0,$$

while the optimal control for α, in view of the fact that $\partial^2 H_\alpha^* / \partial\alpha^2 < 0$, is the limiting control, which in principle, when the equality $H^*_{\alpha\,max} = H^*_{\alpha\,min}$ is retained, can go over to a slipping mode.

If the inequality

$$\lambda_1^* a_{01}^* p^* \cos\alpha + \lambda_2^* (a_{01}^* p^* + \bar{Y}^*) \frac{\sin\alpha}{V} < 0,$$

is correct, then where $\partial^2 H_\alpha^* / \partial\alpha^2 > 0$, optimal control condition (1.3.82) can be satisfied either with stable or with limiting control, the latter always occuring where $\lambda_1^* \geq 0$, if $\lambda_2^* \sin\alpha < 0$.

Thus, in the case $\lambda_1^* < 0$, the optimal control condition for α is satisfied either by stable or by limiting control, while where $\lambda_0^* \geq 0$, it is satisfied only by limiting control.

Due to the formal adequacy of the control conditions and optimal control conditions for α (1.3.80) and (1.3.82) for the control conditions and optimal control conditions for ω, the conclusions of optimal control for r and d will be similar to those produced above.

Up to the present, the construction of optimal controls for α ω, d and r on the basis of the condition of optimal control and the control condition has been performed within an open area of values of the vector of the required parameter. However, if one of the controls is coupled due to the location of the required parameter vector at the boundary, investigation of the necessary Weierstrass condition and the maximum principle relative to this function, expressed by conditions (1.3.1) and (1.3.3), should be performed considering the equation

$$D^p - D = 0.$$

In this case, turning to the Lagrange coefficient λ_D, on the basis of the condition of optimal control for a coupled control function, we produce equations similar to the Euler-Lagrange equations (1.2.31)-(1.2.34) with coupled behavior of the corresponding control function.

§ 4. Computer Algorithm for Variational Method of Optimal Planning

In the preceding sections, we formulated the problem of the variational method of optimal planning of a flight vehicle and power plant. Its solution has produced the first order necessary conditions: the conditions of stability (Euler-Lagrange equations, condition of optimality of parameters, condition of transversality and discontinuity condition) and the Weierstrass condition or the maximum principle. The production of these results has required that we overcome a number of difficulties arising due to the necessity of working out the theoretical principles of the solution of the problem as stated. In this sense, the preceding sections can be looked upon as a finished work, leading to the solution of a complex variational problem.

However, the concrete expression of the solution of this problem consists of the numerical results, allowing us to understand and evaluate the solution in the form of numerical values of optimal controls and parameters, the optimal phase trajectory of a flight vehicle.

In this connection, another problem arises: the problem of organizing a computational procedure using a universal electronic digital computer, allowing us to carry the solution produced through to numerical results.

The problem of organization of the computational procedure using the universal digital computer is primarily a problem of development of an algorithm. The difficulty of this problem lies in the fact that the algorithm developed must allow us to use a universal digital computer to solve an extremely broad range of problems following from the variational problem of optimal planning formulated earlier (§ 1). Then the peculiarities of any given program will be determined by the properties of the concrete programming task, not by the algorithm itself.

Since the solution of the variational problem of optimal planning of a flight vehicle and power plant has led only to the definition of first order necessary conditions, it is desirable to develop an algorithm for a computational procedure such that it includes definition and satisfaction (simultaneously with completion of computer calculations) of the second order necessary conditions, which would allow us to consider the solution sufficient as well. All of this taken together allows us to affirm that the development of an algorithm for the computational procedure is another, independent work, no less difficult than the preceding work, requiring its own special investigation.

True, it will be more correct, on the basis of the essence of the planning process and the requirements which follow from it, to analyze the variational problem of optimal planning of the flight vehicle and power plant and the problem of its computational procedure (development of the algorithm and actual computation) together. This unity generates the problem of the variational method of optimal planning of the flight vehicle and power plant. Its solution will lead to the logical completion of the variational method of optimal planning of a flight vehicle and power plant using universal electronic digital computers.

In this sense, the solution of the variational problem of optimization of parameters, control of the flight vehicle and power plant should be looked upon as the first stage, and development of the algorithm for its computation as the second and final stage of the investigation of the variational method of optimal planning of the flight vehicle.

Let us now go over to the second stage in our investigation -- presentation of the algorithm for the computational procedure, the theoretical bases of which are presented in the appendix.

First of all, for compactness of our notations and computations, we must introduce a number of new symbols, the usage of which will allow us, based on equation system (1.I)-(1.II) and the conditions of optimality of the parameters (1.2.45)-(1.2.58), to write the new system as

$$\left.\begin{aligned} &x'=f(x,u,\varrho,a),\\ &\eta'=\chi(x,u,\varrho,a),\\ &a'=0,\\ &\xi(x,u,\varrho,a)=0, \end{aligned}\right\} \qquad (1.4.1)$$

where

$$x=(V,\ldots,T_{w},\lambda_1^{*},\ldots,\lambda_q,\lambda_w),$$

$$\varrho=(\lambda_x,\lambda_D),$$

$$u=(\alpha,\omega,\dot{d},r,\nu_x,\nu),$$

$$a=(\Pi,N^p,\sigma^p,\delta),$$

$$f=\left(\varphi_1,\ldots,\varphi_8,\frac{\partial H}{\partial V},\ldots,\frac{\partial H}{\partial T_w}\right),$$

$$\chi=\left(\frac{\partial F^{*}}{\partial \bar{G}_0},\ \frac{\partial F^{*}}{\partial a_0},\ \frac{\partial F^{*}}{\partial b_N},\ \frac{\partial F^{*}}{\partial \mu_x},\ \frac{\partial F^{*}}{\partial \Gamma},\ \frac{\partial F^{*}}{\partial T^p},\ \frac{\partial F^{*}}{\partial D^p},\ \frac{\partial F^{*}}{\partial Q_\tau^p},\ \frac{\partial F^{*}}{\partial \delta}\right),$$

$$\xi=\left(\varphi_9,\varphi_{10},\frac{\partial F^{*}}{\partial \alpha},\ldots,\frac{\partial F^{*}}{\partial \nu_x},\ \frac{\partial F^{*}}{\partial \nu}\right).$$

We make use here of the fact that, by using the conditions of optimality of parameters, we can write the Euler-Lagrange equations related to the parameters Π, N^p, σ^p and δ (see appendix) as follows:

$$\left.\begin{aligned}
&\eta_1' = -\lambda_1^{\bullet}\frac{\partial\varphi_1^{\bullet}}{\partial a_0} - \lambda_2^{\bullet}\frac{\partial\varphi_2^{\bullet}}{\partial a_0} - \lambda_6^{\bullet}\frac{\partial\varphi_6^{\bullet}}{\partial a_0} - \frac{\partial\varphi_{10}}{\partial a_0}\lambda_D,\\
&\eta_2' = -\lambda_1^{\bullet}\frac{\partial\varphi_1^{\bullet}}{\partial b_{0l}} - \lambda_2^{\bullet}\frac{\partial\varphi_2^{\bullet}}{\partial b_{0l}} - \lambda_6^{\bullet}\frac{\partial\varphi_6^{\bullet}}{\partial b_{0l}} - \frac{\partial\varphi_{10}}{\partial b_{0l}}\mu_D,\\
&\eta_3' = -\lambda_1^{\bullet}\frac{\partial\varphi_1^{\bullet}}{\partial G_0} - \lambda_1^{\bullet}\frac{\partial\varphi_2^{\bullet}}{\partial G_0} - \lambda_6^{\bullet}\frac{\partial\varphi_6^{\bullet}}{\partial G_0} - \frac{\partial\varphi_{10}}{\partial G_0}\lambda_D,\\
&\eta_4' = -\lambda_1^{\bullet}\frac{\partial\varphi_1^{\bullet}}{\partial G_{0J}} - \lambda_2^{\bullet}\frac{\partial\varphi_2^{\bullet}}{\partial G_{0J}} - \lambda_6^{\bullet}\frac{\partial\varphi_6^{\bullet}}{\partial G_{0J}} - \frac{\partial\varphi_{10}}{\partial G_{0J}}\lambda_D,\\
&\eta_5' = -\lambda_1^{\bullet}\frac{\partial\varphi_1^{\bullet}}{\partial\Gamma} - \lambda_2^{\bullet}\frac{\partial\varphi_2^{\bullet}}{\partial\Gamma} - \lambda_6^{\bullet}\frac{\partial\varphi_6^{\bullet}}{\partial\Gamma} - \frac{\partial\varphi_7}{\partial\Gamma}\lambda_q - \frac{\partial\varphi_8}{\partial\Gamma}\lambda_w\\
&\quad - \frac{\partial\varphi_9}{\partial\Gamma}\lambda_u - \frac{\partial\varphi_{10}}{\partial\Gamma}\lambda_D,\\
&\eta_6' = -\lambda_1^{\bullet}\frac{\partial\varphi_1^{\bullet}}{\partial T_p} - \lambda_2^{\bullet}\frac{\partial\varphi_2^{\bullet}}{\partial J_0} - \lambda_6^{\bullet}\frac{\partial\varphi_6^{\bullet}}{\partial J_0} - \frac{\partial\varphi_{10}}{\partial J_0}\lambda_D,\\
&\eta_7' = -\lambda_1^{\bullet}\frac{\partial\varphi_1^{\bullet}}{\partial T_p} - \lambda_2^{\bullet}\frac{\partial\varphi_2^{\bullet}}{\partial T_p} - \lambda_6^{\bullet}\frac{\partial\varphi_6^{\bullet}}{\partial T_p} - \frac{\partial\varphi_7}{\partial T_p}\lambda_q -\\
&\quad - \frac{\partial\varphi_8}{\partial T_p}\lambda_w - \frac{\partial\varphi_{10}}{\partial T_p}\lambda_D,\\
&\eta_8' = -\lambda_D,\ \eta_9' = -\lambda_D,\\
&\eta_{10}' = -\frac{\partial\varphi_7}{\partial\delta}\lambda_q - \frac{\partial\varphi_8}{\partial\delta}\lambda_w - \frac{\partial\varphi_{10}}{\partial\delta}\lambda_D
\end{aligned}\right\} \quad (1.II)$$

Therefore, when the algorithm is used in a computer, equation system (1.II) produced earlier should be supplemented by this sytem of equations. This system simplifies calculation of the conditions of optimality of the parameters, converting them to purely boundary conditions, since now the integrals in (1.2.45)-(1.2.58) can be replaced with the difference in the boundary values of η.

In the following, curve $x(t) = (V(t), \ldots, \lambda_W(t)$, produced by solution of system (1.4.1), will be referred to as the conditional phase trajectory in contrast to the phase trajectory $x(t) = (V(t), \ldots, T_W(t))$, while x_i^* (t) will be referred to as the conditional phase variables.

Furthermore, boundary conditions (1.1.1), equations (1.1.15)-(1.1.18) and the transversality conditions produced from (1.2.42)-(1.2.43), the conditions of optimality of parameters (1.2.45)-(1.2.58), after exclusion of the constant Lagrange coefficients e from this, will be written as follows:

$$R_0(x(t_0))=(R_0^{p*}(x(t_0)))=0\ (p=1,\ldots,\tau),$$

$$R_\kappa(t_\kappa,\ x(t_\kappa),\ x(t^{(i)}),\ \eta(t_\kappa),\ \eta(t_-^{(i)}),\ a)= \tag{1.4.2}$$

$$=(R_\kappa^{m*}(t_\kappa,\ x(t_\kappa),\ x(t^{(i)}),\ \eta(t_\kappa),\ \eta(t_-^{(i)}),\ a))=0 \tag{1.4.3}$$

$$(m=1,\ldots,n-\tau),\ \ \eta(t_+^{(i)})=(\eta(t_0),\ \eta(t_+^{(1)}),\ldots,\eta(t_+^{(\kappa-1)}))=0. \tag{1.4.4}$$

Equation system (1.4.1) contains r parameters and l variables x(t), where r + l = n. The last equation of this system unambiguously defines the dependence of u and ρ on x(t) and a. Therefore, its solution requires n boundary conditions. However, at the initial point $t = t_0$ there are only, in addition to (1.4.4), $\tau < n$ conditions, while the m additional conditions required are fixed at certain intermediate points in the interval $(t_0, t_k]$.

Thus, definition of the optimal parameters and controls, the optimal phase trajectory involves the solution of a multipoint boundary problem, in which the system of differential equations and boundary conditions are generally nonlinear.

In this connection, we can formulate the basic requirement for the computer algorithm for the optimal parameters and controls.

First of all, it must include a "strategy" for solution of equation system (1.4.1) with which, in addition to fulfillment of the necessary Weierstrass condition or the condition for optimal control and the conditions of limitation on the vector of the required parameter along the conditional phase trajectory x(t), the boundary conditions (1.4.2)-(1.4.4) are satisfied. Furthermore, the solution of this problem by universal computer must be time-limited.

What is the essence of the algorithm and whatare the initial positions in the construction of its mathematical model?

The essence of the algorithm is reduction of the multipoint boundary problem to a Cauchy problem by fixation of the parameters and the missing initial conditions in the corresponding manner and production on this basis of a series of solutions in which, generally speaking, the boundary conditions (1.4.3) may not be fulfilled. However, the logic of the algorithm leads to adjustment of the fixed parameters and missing boundary conditions in an arbitrary manner so that finally a solution appears which satisfies boundary conditions (1.4.3).

The "strategy" of the algorithm includes the solution of three "tactical" problems, the presentation of which can be made more clear if we introduce for convenience a number of new symbols. Let $\bar{x}(t_0) = \bar{x}_0 = (x_0^{(1)*}, x_0^{(2)*}, \ldots, x_0^{(\tau)*})$ be the vector of conditional variables, fixed at the initial point, or the vector of fixed initial coordinates, and $x(t_0) = x_0 = (x_0^{(\tau+1)*}, \ldots, x_0^{(1)*})$ be the vector of conditional phase variables missing at the initial point or the vector of desired initial coordinates [1].

We further represent

$$p=(p_1^*,\ldots,p_m^*)=((R_k^{(\tau+1)})_2^*,\ldots,(R_k^{(n)})^{2*}), \tag{1.4.5}$$

$$p_\Sigma^*=\sum_{j=1}^{m} p_j^*. \tag{1.4.6}$$

where p_j^* is a functional;

p_Σ^* is the summary functional.

In these symbols, the strategy of the algorithm can be formulated as follows: determine vectors x_0 and a for which $p = 0$.

Let us now assume that the vectors $a^{(0)}$ and $x_0^{(0)}$ have been fixed for which, generally speaking, the boundary conditions of (1.4.3) are not satisfied and $p = p^{(0)}$. Now, in order to perform our accepted strategy at point $(x_0^{(0)}, a^{(0)}, p^{(0)})$, we must organize search for the direction of movement which, at least in a small area around this point, leads to the new point $(x_0^{(1)}, a^{(1)}, p^{(1)})$, where $p^{(1)} = p_{min}^{(0)}$. This direction at point $(x_0^{(0)}, a^{(0)}, p^{(0)})$ will be referred to as the direction of swiftest descent. Further, if we do not use special measures, at point $(x_0^{(1)}, a^{(1)}, p^{(1)})$, we must organize search for the direction of swiftest descent again, in order to arrive in small area

---

[1] Here for simplification, the initial conditions of (1.4.2) are expressed as fixed phase coordinates $x_0^{(p)*}$ $(p = 1, \ldots, \tau)$, while $1 < n$.

around this point at the new point $(x_0^{(2)}, a^{(2)}, p^{(2)})$, where $p^{(2)} = p_{min}^{(1)}$. Thus, we can finally find the vectors $a^{(s)}$ and $x_0^{(s)}$ corresponding to $p_{min}^{(s-1)} = 0$. In this connection, the first tactical problem is organization at these defined points of search for the direction of swiftest descent. Construction of the mathematical model of the algorithm basically involves solution of this problem. However, if we move to the point $(x_0^{(s)}, a^{(s)}, p^{(s)})$ (where $p^{(s)} = 0$) along a path, each ϵ-step of which involves search for the direction of swiftest descent in order to minimize functional p, the "cost" (in time) of the path may be relatively high.

Therefore, the problem arises of the expediency of using the direction of swiftest descent which has been found, determination of which usually requires considerable computation, for movement in considerably larger steps, as far as possible. This procedure can have a positive effect, resulting in the achievement of a deeper descent in each direction with a smaller volume of computation. Essentially, we are speaking of the determination of the local minimum with the direction of steepest descent determined at a given point. This is the sense of the second tactical problem, the solution of which should lead in the ideal case to determination of the maximum step $\Delta x_0^{(1)}$ and $\Delta a^{(1)}$ from the initial point $(x_0^{(1)}, a^{(1)}, p^{(1)})$ to the point with p_{min} for the direction of descent in question.

In the process of solution of these two tactical problems, a certain amount of information appears on the structure of the function $p(x_0, a)$. It is quite natural to desire in some way to use this information for the organization of further descent with lower expenditures. This is the third tactical problem, the problem of self-teaching of the machine for organization of further rapid descent.

The course of performance of the computational procedure algorithm for the multipoint boundary problem by universal computer can be followed on the flow chart shown on Figure 1.21.

After this general characterization of the algorithm, let us go over to a detailed description.

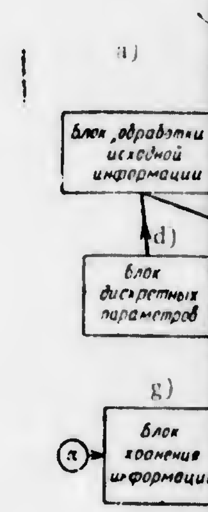

Figure 1.2
of Optima
Trajectory

KEY: a,
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Mathematical Model

If we give a geom
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described by system (1
model of the algorithm
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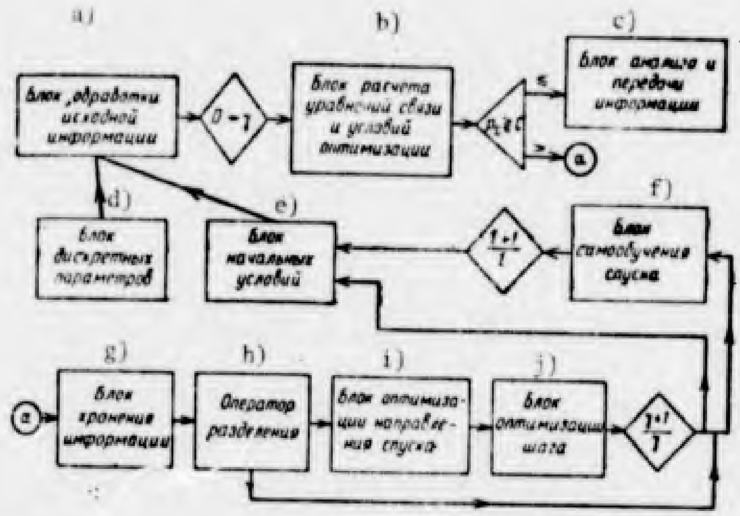


Figure 1.21. Flow Chart of Algorithm for Computation of Optimal Controls, Parameters and Optimal Phase Trajectory.

KEY: a, Initial Formation Processing Unit; b, Unit Calculating Coupling Equations and Optimization Conditions; c, Information Analysis and Transmission Unit; d, Discreet Parameters Unit; e, Initial Conditions Unit; f, Descent Self-Teaching Unit; g, Information Storage Unit; h, Separation Operator; i, Descent Direction Optimization Unit; j, Step Optimization Unit

Mathematical Model of the Algorithm

If we give a geometric interpretation to the task before the algorith, its solution must be directed toward]ocation in an m-dimensional space of the point of intersection of the hypersurface $p_j^*(x_0, a) = 0$ $(j = 1, \ldots, m)$ considering events in the sector $[1_0, t_k]$ described by system (1.4.1). Howev ., in constructing the mathematical model of the algorithm, due to its complexity, we must give preference to analytical methods. Then the multipoint boundary problem is reduced to the solution of the following variational problem:

among permissible values of x_0 and a, defined by the inequalities

$$x_{0\,min} \leqslant x_0 \leqslant x_{0\,max},$$
$$a_{min} \leqslant a \leqslant a_{max},$$

considering the presence of equations (1.4.1), more properly (1.I)-(1.II), in variations and determination of $\bar{x}_0$, find values for which

$$p_i^* = \inf,$$

where

$$p_j^* \leqslant \Delta_j (j = 1, \ldots, i-1, i+1, \ldots, m),$$

where

$$\Delta_j > 0.$$

Omitting the intermediate calculations, given in the appendix, let us write the necessary conditions for the minimum of p_i^*, expressed by the following system of conjugate equations:

$$\left.\begin{aligned}
y_1^{*\prime} &= -\frac{\partial f}{\partial V} y - \frac{\partial \eta}{\partial V} z_\eta - \frac{\partial \xi}{\partial V} y_n, \\
y_2^{*\prime} &= -\frac{\partial f}{\partial \theta} y - \frac{\partial \eta}{\partial \theta} z_\eta - \frac{\partial \xi}{\partial \theta} y_n, \\
y_3^{*\prime} &= -\frac{\partial f}{\partial H} y - \frac{\partial \eta}{\partial H} z_\eta - \frac{\partial \xi}{\partial H} y_n, \\
y_4^{*\prime} &= 0, \quad y_5^{\prime} = 0, \\
y_6^{*\prime} &= -\frac{\partial f}{\partial \mu} y - \frac{\partial \eta}{\partial \mu} z_\eta - \frac{\partial \xi}{\partial \mu} y_n, \\
y_7^{*\prime} &= -\frac{\partial f}{\partial Q_\eta} y - \frac{\partial \eta}{\partial Q_\eta} z_\eta - \frac{\partial \xi}{\partial Q_\eta} y_n, \\
y_8^{*\prime} &= -\frac{\partial f}{\partial T_w} y - \frac{\partial \eta}{\partial T_w} z_\eta - \frac{\partial \xi}{\partial T_w} y_n, \\
y_{11}^{*\prime} &= -\frac{\partial f}{\partial \lambda_1} y - \frac{\partial \eta}{\partial \lambda_1} z_\eta - \frac{\partial \xi}{\partial \lambda_1} y_n, \\
y_{12}^{*\prime} &= -\frac{\partial f}{\partial \lambda_2} y - \frac{\partial \eta}{\partial \lambda_2} z_\eta - \frac{\partial \xi}{\partial \lambda_2} y_n, \\
y_{13}^{*\prime} &= -\frac{\partial f}{\partial \lambda_3} y - \frac{\partial \eta}{\partial \lambda_3} z_\eta - \frac{\partial \xi}{\partial \lambda_3} y_n, \\
y_{14}^{*\prime} &= -\frac{\partial f}{\partial \lambda_4} y - \frac{\partial \eta}{\partial \lambda_4} z_\eta - \frac{\partial \xi}{\partial \lambda_4} y_n, \\
y_{15}^{*\prime} &= -\frac{\partial f}{\partial \lambda_5} y - \frac{\partial \eta}{\partial \lambda_5} z_\eta - \frac{\partial \xi}{\partial \lambda_5} y_n, \\
y_{16}^{*\prime} &= -\frac{\partial f}{\partial \lambda_6} y - \frac{\partial \eta}{\partial \lambda_6} z_\eta - \frac{\partial \xi}{\partial \lambda_6} y_n, \\
y_{17}^{\prime} &= -\frac{\partial f}{\partial \lambda_q} y - \frac{\partial \eta}{\partial \lambda_q} z_\eta - \frac{\partial \xi}{\partial \lambda_q} y_n.
\end{aligned}\right\} \qquad (1.\text{III})$$

$$
\left.
\begin{aligned}
&y'_{18} = -\frac{\partial f}{\partial \lambda_w} y - \frac{\partial \chi}{\partial \lambda_w} z_\eta - \frac{\partial \xi}{\partial \lambda_w} y_s, \\
&z'_1 = -\frac{\partial f}{\partial \eta_1} y - \frac{\partial \chi}{\partial \eta_1} z_\eta - \frac{\partial \xi}{\partial \eta_1} y_s, \\
&\text{and similarly} \\
&z'_2 = 0,\ z'_3 = 0,\ z'_4 = 0,\ z'_5 = 0, \\
&z'_6 = 0,\ z'_7 = 0,\ z'_8 = 0,\ z'_9 = 0,\ z'_{10} = 0 \\
&z'_{11} = -\frac{\partial f}{\partial a_0} y - \frac{\partial \chi}{\partial a_0} z_\eta - \frac{\partial \xi}{\partial a_0} y_s, \\
&z'_{12} = -\frac{\partial f}{\partial b_{0l}} y - \frac{\partial \chi}{\partial b_{0l}} z_\eta - \frac{\partial \xi}{\partial b_{0l}} y_s, \\
&z'_{13} = -\frac{\partial f}{\partial G_0} y - \frac{\partial \chi}{\partial G_0} z_\eta - \frac{\partial \xi}{\partial G_0} y_s, \\
&z'_{14} = -\frac{\partial f}{\partial G_{0J}} y - \frac{\partial \chi}{\partial G_{0J}} z_\eta - \frac{\partial \xi}{\partial G_{0J}} y_s, \\
&z'_{15} = -\frac{\partial f}{\partial \Gamma} y - \frac{\partial \chi}{\partial \Gamma} z_\eta - \frac{\partial \xi}{\partial \Gamma} y_s, \\
&z'_{16} = -\frac{\partial f}{\partial I_0} y - \frac{\partial \chi}{\partial I_0} z_\eta - \frac{\partial \xi}{\partial I_0} y_s, \\
&z'_{17} = -\frac{\partial f}{\partial T_p} y - \frac{\partial \chi}{\partial T_p} z_\eta - \frac{\partial \xi}{\partial T_p} y_s, \\
&z'_{18} = -\frac{\partial f}{\partial N^p} y - \frac{\partial \chi}{\partial N^p} z_\eta - \frac{\partial \xi}{\partial N^p} y_s, \\
&z'_{19} = -\frac{\partial f}{\partial \sigma^p} y - \frac{\partial \chi}{\partial \sigma^p} z_\eta - \frac{\partial \xi}{\partial \sigma^p} y_s, \\
&z'_{20} = -\frac{\partial f}{\partial \delta} y - \frac{\partial \chi}{\partial \delta} z_\eta - \frac{\partial \xi}{\partial \delta} y_s, \\
&z'_{21} = -\frac{\partial f}{\partial Q^p_\eta} - \frac{\partial \chi}{\partial Q^p_\eta} z_\eta - \frac{\partial \xi}{\partial Q^p_\eta} y_s, \\
&z'_{22} = -\frac{\partial f}{\partial \mu_\kappa} y - \frac{\partial \chi}{\partial \mu_\kappa} z_\eta - \frac{\partial \xi}{\partial \mu_\kappa} y_s = 0, \\
&z'_{23} = -y \frac{\partial f}{\partial G_{\mu l}} - z_\eta \frac{\partial \chi}{\partial G_{\mu l}} - \\
&\quad - y_s \frac{\partial \xi}{\partial G_{r,r}} = 0, \\
&\frac{\partial f}{\partial d} y + \frac{\partial \chi}{\partial d} z_\eta + \frac{\partial \xi}{\partial d} y_s = 0, \\
&\frac{\partial f}{\partial r} y + \frac{\partial \chi}{\partial r} z_\eta + \frac{\partial \xi}{\partial r} y_s = 0,
\end{aligned}
\right\} \quad (1.III)
$$

$$
\left.
\begin{aligned}
& y\frac{\partial f}{\partial \alpha}+z_n\frac{\partial \chi}{\partial \alpha}+y_n\frac{\partial \xi}{\partial \alpha}=0,\\
& y\frac{\partial f}{\partial \omega}+z_n\frac{\partial \chi}{\partial \omega}+y_n\frac{\partial \xi}{\partial \omega}=0,\\
& \frac{\partial f}{\partial \lambda_n}y+\frac{\partial \chi}{\partial \lambda_n}z_n+\frac{\partial \xi}{\partial \lambda_n}y_n=0,\\
& \frac{\partial f}{\partial \lambda_D}y+\frac{\partial \chi}{\partial \lambda_D}z_n+\frac{\partial \xi}{\partial \lambda_D}y_n=0,\\
& \frac{\partial f}{\partial \nu_n}y+\frac{\partial \chi}{\partial \nu_n}z_n+\frac{\partial \xi}{\partial \nu_n}y_n=0 \quad \text{or}\\
& 2y_9\nu_n-y_{23}\lambda_n=0,\\
& \frac{\partial f}{\partial \nu}y+\frac{\partial \chi}{\partial \nu}z_n+\frac{\partial \xi}{\partial \nu}y_n=0 \quad \text{or}\\
& 2y_{10}\nu-y_{24}\lambda_D=0.
\end{aligned}
\right\} \qquad (1.\text{III})
$$

Here

$y = (y_1^*, y_2^*, y_3^*, \ldots, y_8^*, y_1^* \ \ldots \ y_{18}^*)$ is the vector of conjugate coefficients corresponding to the coupling equations (1.1.2)-(1.1.9) and the Euler-Lagrange equations (1.2.5)-(1.2.12);

$y_n = (y_9^*, y_{10}^*, y_{19}^*, \ldots, y_{24}^*)$ is the vector of conjugate coefficients corresponding to coupling equations (1.1.23) and Euler-Lagrange equations (1.2.13)-(1.2.17);

$z_n = (z^*, z_2^*, \ldots, z_{10}^*)$ is the vector of conjugate coefficients corresponding to the Euler-Lagrange equations presented above, coupled with optimization of parameters.

Keeping in mind the Euler-Lagrange equations (1.2.17) and the latter two equations, we produce

$$y_{23}\nu_n=0, \quad y_{24}\nu=0.$$

Equation system (1.III) is conjugate to equation system (1.I)-(1.II) in variations. In this system, the coefficients with

$$y=(y_1^\bullet, y_2^\bullet, \ldots, y_{24}^\bullet) \quad \text{and} \quad z=(z_1^\bullet, \ldots, z_{22}^\bullet)$$

are determined in the process of integration of equation system (1.I)-(1.II). Therefore, in order to determine the direction of swiftest descent at point t_0, equation system (1.III) must be integrated simultaneously with system (1.I)-(1.II) from points t_k and $t^{(i)}$ to point t_0. The conditions at points t_k and $t^{(i)}$ for integration of equation system (1.I)-1.II) from t_k and $t^{(i)}$ to t_0 are determined in the preceding integration from t_0 to $t^{(i)}$ and t_k, while the conditions for integration of the conjugate system (1.III) are determined after determination of functionals p_j^* and their partial derivatives with respect to the conditional phase variables and the parameters at points t_k and $t^{(i)}$ respectively. However, generally speaking, the values of $t^{(i)}$ and t_k are unknown, while the intermediate and final points are fixed by relationships (1.4.3). Therefore, we can select the corresponding functions from (1.4.3) as the functions ψ_i^{0*}, determining the intermediate and final points. Experience has shown that selection of an arbitrary function from (1.4.3) can complicate the computational procedure. The following can be recommended as almost universal stop functions:

$$\psi_i^{0*} = \mu^*(t^{(i)}) - \mu_{\kappa j}^* = 0 \quad (i = 1, \ldots, n-1). \tag{1.4.7}$$

If point t_k is not fixed in explicit form in the boundary conditions (1.4.3), the final point frequently must be determined through the stop function

$$\psi_\kappa^{0*} = \mu^*(t_\kappa) - \mu_{\kappa N}^* = 0^{1)}. \tag{1.4.8}$$

Then in order to determine the direction of swiftest descent at point t_0, equation system (1.III) must be integrated together with system (1.I)-(1.II) from points $\psi_k^{0*} = 0$ and $\psi_i^{0*} = 0$ to point t_0. The conditions at points $\psi_k^{0*} = 0$ and $\psi_i^{0*} = 0$ for performance of integration of the conjugate system (1.III) are determined as follows (see appendix):

$$\left.\begin{aligned} & y_\kappa = \left.\frac{\partial p}{\partial x}\right|_{\psi^0}, \quad y_{6\kappa} = -\left.\frac{p'}{\mu'}\right|_{\psi^0}, \\ & z_\kappa = \left.\frac{\partial p}{\partial a}\right|_{\psi^0}, \quad z_{m\kappa} = \left.\frac{\partial p}{\partial \mu_\kappa} + \frac{p'}{\mu'}\right|_{\psi^0}. \end{aligned}\right\} \tag{1.4.9}$$

---

[1] It is assumed here that the active sector is analyzed at the end point. Otherwise, a function which can be satisfied in practically any situation must be selected as ψ_k^{0*}.

Here

$$p=(p_1,\ldots,p_m),\ \psi^0=(\psi_1^{0*},\ldots,\psi_\kappa^{0*}),$$

$$y_\kappa=({}_1y_{1\kappa},\ldots,{}_1y_{5\kappa},{}_1y_{7\kappa},{}_1y_{8\kappa},{}_1y_{11\kappa},\ldots,{}_1y_{18\kappa},\ldots,{}_my_{18\kappa}),$$

$$z_\kappa=({}_1z_{1\kappa},\ldots,{}_1z_{21\kappa},{}_1\dot{z}_{23\kappa},\ldots,{}_mz_{23\kappa})$$

$$y_{6\kappa}=({}_1y_{6\kappa},\ldots{}_my_{6\kappa})$$

$$z_{22\kappa}=({}_1z_{22\kappa},\ldots,{}_mz_{22\kappa}).$$

In the process of integration of system (1.I)-(1.II), the phase variables undergo first order discontinuity at points $\psi_i^{0*}=0$, while μ changes suddenly. Therefore, at these points, the conjugate coefficient $y_6^*(t)$ changes suddenly Representing the value of y(t) to the right of $\psi_i^{0*}=0$ by $y(t_-^{(i)})$ (we should consider that in this case integration is performed from t_k or, at least, from $\psi_i^{0*}=0$ to t_0), we produce

$$\begin{aligned}{}_sy_6^*(t_+^{(i)})=\frac{1}{\mu'(t_+^{(i)})}\ [{}_sy_6^*(t_-^{(i)})\mu'(t_-^{(i)})+{}_sy(t_-^{(i)})(x'(t_-^{(i)})-\\ -x'(t_+^{(i)}))+{}_sz(t_-^{(i)})(\eta'(t_-^{(i)})-\eta'(t_+^{(i)}))]\ (s=1,\ldots,m).\end{aligned} \tag{1.4.10}$$

Knowledge of the initial conditions at points $\psi_k^{0*}=0$ and $\psi_i^{0*}=0$, determined by relationships (1.4.9), and the discontinuity conditions $y_6^*(t^{(i)})$ according to (1.4.10) allows us to integrate system (1.I)-(1.III) from $\psi_k^{0*}=0$ or $\psi_i^{0*}=0$ to t_0 and thereby to find $y(t_0)$ and $z(t_0)$. Furthermore, due to the property of the conjugate system of equations (1.III) and equation system (1.I)-(1.II) in variations (see appendix), the following system of linear equations occurs:

$$\left.\begin{aligned}\Delta p_1^*&={}_1y(t_0)\delta x_0+{}_1z(t_0)\delta a,\\ &\ldots\ldots\ldots\ldots\\ \Delta p_s^*&={}_sy(t_0)\delta x_0+{}_sz(t_0)\delta a,\\ &\ldots\ldots\ldots\ldots\\ \Delta p_m^*&={}_my(t_0)\delta x_0+{}_mz(t_0)\delta a.\end{aligned}\right\} \tag{1.4.11}$$

Here δx_0 is the variation of the vector of conditional phase variables missing at the initial point;

δa is the variation of the vector of parameters;

$$y(t_0)=({}_1y_1^{\bullet}(t_0),\ldots,{}_1y_{18}^{\bullet}(t_0),\ldots,{}_my_{18}^{\bullet}(t_0)),$$

$$z(t_0)=({}_1z_{11}^{\bullet}(t_0),\ldots,{}_1z_{21}^{\bullet}(t_0),{}_1z_{23}^{\bullet}(t_0),\ldots{}_mz_{21}^{\bullet}(t_0),{}_mz_{23}^{\bullet}(t_0)),$$

$${}_sz_{22}(t_0)=\Big\{{}_sz_{22}^{\bullet}(t_-^{(l)})-\frac{1}{\mu'^{\bullet}(t_+^{(l)})}\big[{}_sy_6^{\bullet}(t_-^{(l)})\mu'^{\bullet}(t_-^{(l)})+{}_sy(t_-^{(l)})(x'(t_-^{(l)})-$$

$$-x'(t_+^{(l)}))+{}_sz(t_-^{(l)})(\eta'(t_-^{(l)})-\eta'(t_+^{(l)}))\big]\Big\}\ (l=1,\ldots n).$$

Now, solving the system of linear equations (1.4.11), we can determine δx_0 and δa, allowing us to find the direction of swiftest descent at $(x_0^{(1)}, a^{(1)}, p^{(1)})$.

Thus, by integrating system (1.I)-(1.II) with properly selected x_0 and a, we can find at points $\psi_k^{0*} = 0$ and $\psi_I^{0*} = 0$ the values of discrepancies R_k and the functionals of the multipoint boundary problem p, making it possible on the basis of the equations of initial conditions of the conjugate system (1.4.9) and the conditions of discontinuity of the conjugate coefficients (1.4.10) to integrate system (1.I)-(1.III) from $\psi_k^{0*} = 0$ and $\psi_i^{0*} = 0$ to t_0 and determine $y(t_0)$ and $z(t_0)$; after this, turning to the system of linear equations (1.4.11), we can determine the direction of swiftest descent at the point $(x_0^{(1)}, a^{(1)}, p^{(1)})$. This is the essence of the model of the algorithm for calculation of the direction of swiftest descent.

The flow chart of the algorithm for the computational procedure of the multipoint boundary problem shown on Figure 1.21 shows the solution of system (1.I)-(1.II) with the properly selected x_0 and a from t_0 to $\psi_i^{0*} = 0$ and $\psi_k^{0*} = 0$, determination of discrepancies R_k and functionals p in the unit for calculation of coupling equations and optimization conditions and the unit for processing of results of integration, while calculation of the initial conditions of the conjugate system, integration of system (1.I)-(1.III) from points $\psi_K^{0*} = 0$ and $\psi_i^{0*} = 0$ to t_0 and solution of linear equation system (1.4.11), showing the direction of swiftest descent, is performed by the unit for optimization of the direction of descent (BONS).

A flow chart for the unit for calculation of coupling equations and optimization conditions and the unit for optimization of the direction of descent is shown on Figure 1.22. With continuous transition from the first unit to the second (represented by the character

α on the figure), these two units should be interpreted as a single unit of rapid descent (BBS). Therefore, the diagram shown on Figure 1.22 can be looked upon as a flow chart for the rapid descent unit.

In the process of completion of the computational procedure for the multipoint boundary problem, the requirement arises of estimating the deviation of the criterion of effectiveness I from its maximum value. This estimate can be performed (see appendix) if the conditional phase trajectory produced in a given iteration lies near the extreme passing through the neighborhood of the fixed boundary conditions, while the values of parameters a are near optimal.

In this case we have

$$dI^* = {}_jy(t_0)\delta x_0 + {}_jz(t_0)\delta a.$$

Here the values of the conjugate coefficients ${}_jy(t_0)$ and ${}_jz(t_0)$ are determined by integration of the system of conjugate equations (1.III) along the corresponding conditional phase trajectory, with the initial conditions

$$\left.\begin{aligned} {}_jy_\kappa &= \frac{\partial I^*}{\partial x}\Big|_{\psi_\kappa^{0*}}, & {}_j\dot{y}_{6\kappa} &= -\frac{I'^*}{\mu'^*}\Big|_{\psi_\kappa^{0*}}, \\ {}_jz_\kappa &= \frac{\partial I^*}{\partial a}\Big|_{\psi_\kappa^{0*}}, & {}_j\dot{z}_{22\kappa} &= \frac{I'^*}{\mu'^*}\Big|_{\psi_\kappa^{0*}}. \end{aligned}\right|$$

Thus, using the values of δx_0 and δa found by solving the system of linear equations (1.4.11), we produce the value of dI^*.

Step Optimization Algorithm

In the ideal case the optimization algorithm should lead to determination of the step $\Delta x_0^{(\gamma)}$ and $\Delta a^{(\gamma)}$ in the calculated direction of swiftest descent from point $(x_0^{(\gamma)}, a^{(\gamma)}, p^{(\gamma)})$ which would lead to $p_j^{(\gamma+1)} < p_j^{(\gamma)}$ and $p_j^{(\gamma+1)} - p_j^{(\gamma)} = \sup$. This step will be called the maximum step.

However, it is not as yet possible to construct a strict mathematical model of the algorithm for calculation of the maximum step. An important role here is played by the difficulties in considering the nonlinearities of the equations of system (1.I)-(1.II) and the boundary conditions (1.4.3) with respect to the phase coordinates, controls and parameters. Furthermore, it must be kept in mind that not only is the dependence of p_j^* on x_0 and a unknown, but it is also related to events in the sector $[t_0, t_k]$ described by system (1.I)-(1.II). Therefore, the algorithm for step optimization can be constructed only for search for the maximum values of $\Delta x_0^{(\gamma)}$ and $\Delta a^{(\gamma)}$.

Generally speaking, several algorithms can be constructed for search of the value of a step near the maximum, but preference must be given to algorithms with:

1) simple machine search logic;
2) a cost (in the sense of volume of main memory and machine time) of search not higher than the search cost of swiftest descent;
3) great possibility for determination of the maximum value of a step or a step near the maximum in a limited amount of machine time;
4) search, if only $(x_0^{(\gamma)}, a^{(\gamma)}, p^{(\gamma)})$ is not a local minimum, basically leading to the condition $p_\Sigma^{(\gamma + 1)} < p_\Sigma^{(\gamma)}$.

Analysis of a number of step optimization algorithms has indicated preference for the two algorithms described below, corresponding to the requirements formulated. The realization of these algorithms by computer requires the construction of the step optimization unit (BOSH).

The first algorithm is the algorithm for step optimization, in which the summary functional p_Σ^* is represented as a function of one independent variable ε^*. The step optimization unit realizing this algorithm in the computer will be referred to as the unit for single-parameter step optimization (BOOSH).

In the second algorithm, the summary functional p_Σ^* is represented as a multidimensional function of independent variables ε_j^* $(j = 1, \ldots, m)$. The step optimization unit realizing this algorithm in the computer will be referred to as the unit for multiparameter step optimization (VMOSH).

Unit for Single Parameter Step Optimization (BOOSH)

The primary specific feature of the algorithm of this unit is reduction of the summary functional p_Σ^* to a function of one dimensionless independent variable ε^*. Introduction of the dimensionless independent variable ε^* allows us to avoid a number of difficulties related to step metrics. Generally speaking, the concept of the step in minimization of a function of several independent variables (each a function of p_Σ, due to the differing physical nature, is not as simple as it might seem at first glance.

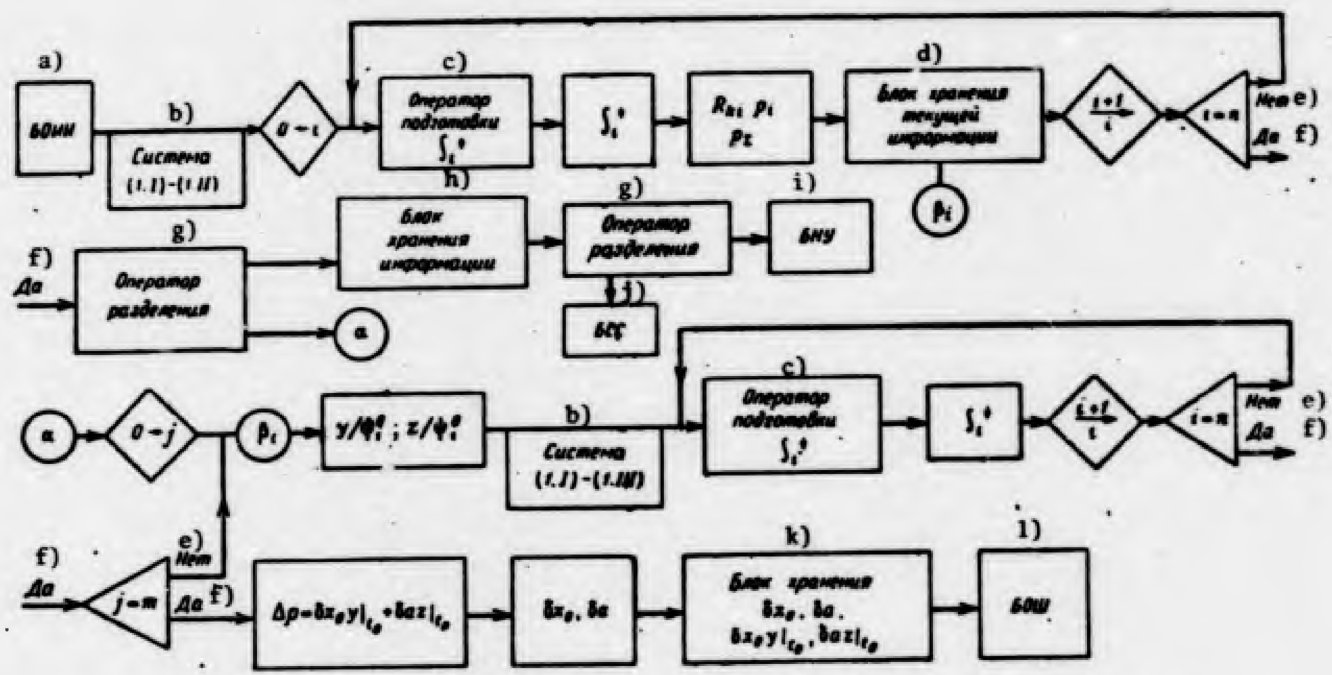


Figure 1.22. Flow Chart of Unit for Calculation of Coupling Equations and Optimization Conditions and Unit for Optimization of Direction of Descent;

KEY: a, BOII; b, System; c, Operator for Preparation of; d, Unit for Storage of Current Information; e, No; f, Yes; g, Separation Operator; h, Information Storage Unit; i, BNU; j, BSS; k, Unit for Storage of; l, BOSH.

The value of a step in the direction of swiftest descent is to be calculated as a share of the length of the limiting step, equal to the difference between the corresponding values of $x^*_{\kappa 0}$ and a^*_p in the given iteration and their limiting values.

The current, jth step for each quantity is

$$\left.\begin{aligned} \Delta x^{(j)*}_{\kappa 0} &= s^*_\kappa \Delta x^{np*}_{\kappa 0\,max} \varepsilon^* \quad (\kappa = 1, \ldots, r < m), \\ \Delta a^{(j)*}_p &= s^*_{ap} \Delta a^{np*}_{p\,max} \varepsilon^* \quad (p = 1, \ldots, d;\; d + r = m). \end{aligned}\right\} \qquad (1.4.12)$$

Here ε^* is a coefficient determining the current step as a fraction of the limiting permissible step in the direction of swiftest descent, where $0 \leq \alpha \leq 0$;

$\Delta x^{\pi p*}_{\kappa 0\ max}$, $\Delta a^{\pi p*}_{p\ max}$ are the limiting permissible steps, equal to

$$\left.\begin{aligned} &\Delta x^{np*}_{\kappa 0\,max} = (x^{np*}_{\kappa 0\,max} - x^{(j)*}_{\kappa 0}) \text{ where } \delta x^*_{\kappa 0} \geqslant 0 \\ &\text{or} \\ &\Delta x^{np*}_{\kappa 0\,max} = (x^{np*}_{\kappa 0\,min} - x^{(j)*}_{\kappa 0}) \text{ where } \delta x^*_{\kappa 0} < 0, \\ &\Delta a^{np*}_{p\,max} = (a^{np*}_{p\,max} - a^{(j)*}_p) \text{ where } \delta a^*_p \geqslant 0\,. \\ &\text{or} \\ &\Delta a^{np*}_{p\,max} = (a^{np*}_{p\,min} - a^{(j)*}_p) \text{ where } \delta a^*_p < 0; \end{aligned}\right\} \qquad (1.4.13)$$

s^*_κ, s^*_{ap} are the scaling factors, generating $\delta x^*_{\kappa 0}$ and δa^*_p in the internal area of the limiting steps and having the following form:

$$s^*_\kappa = \frac{1}{\tilde{\Delta}^*_{max}} \frac{\delta x^*_{\kappa 0}}{\Delta x^{np*}_{\kappa 0\,max}},$$

$$s^*_{ap} = \frac{1}{\tilde{\Delta}^*_{max}} \frac{\delta a^*_p}{\Delta a^{np*}_{p\,max}},$$

where $\tilde{\Delta}^*_{max}$ is the maximum value of the ratios

$$\frac{\delta x^*_{\kappa 0}}{\Delta x^{np*}_{\kappa 0\,max}} \text{ and } \frac{\delta a^*_p}{\Delta a^{np*}_{p\,max}}.$$

Functional $p^{(j)}$ is a vector function with the components $p_1^{(j)*}, \ldots, p_m^{(j)*}$. Therefore, in order to simplify calculations, the values of δx_0 and δa produced from the solution of the system of linear equations (1.4.11) may be used only for determination of the direction of swiftest descent, rather than looking upon them as a criterion of the measure of the step.

This approach to estimation of the step measure in the direction of swiftest descent in order to determine its optimal value has a number of advantages over other methods. First of all, it allows us to probe in the direction of swiftest descent the entire path from the fixed values of $x_0^{(\gamma)}$ and $a^{(\gamma)}$ to the limiting values; secondly, it is located in the ε^*-area of point $(p^{(\gamma)}, x_0^{(\gamma)}, a^{(\gamma)})$; third, it allows us to reveal a difficult situation (location at the point of the local minimum) and either move as far as possible from the dangerous point or shift control to a different unit in the algorithm.

Generally speaking, search for the optimal step can be performed among the jth steps calculated according to (1.4.12), comparing the values of $p_\Sigma^{(j)*}$. However, this approach to determination of Δx and Δa absolutely fails to consider the multidimensionality of $p^{(j)} = (p_1^{(j)*}, \ldots, p_m^{(j)*})$. Therefore, it may hinder the search

$$p_\Sigma^{*\inf} = \inf_{\substack{x_{0\,\min}^{\text{np}} < x_0 < x_{0\,\max}^{\text{rp}},\\ a_{\min}^{\text{np}} < a < a_{\max}^{\text{rp}}}} \{p_\Sigma^{(j)*}\},$$

since the change in components $x_0^{(j)}$ and $a^{(j)}$ influences the change in components $p^{(j)}$ differently. Thus, the change of any component of element $x^{(j)}$ or $a^{(j)}$ at a given step will influence the components of vector $p^{(j)}$ differently: some will decrease, others will increase, and depending on this the value of $p_\Sigma^{(j)*}$ may either decrease or remain as before, or even increase. In this connection, it is very important, in determining the step to consider the multidimensionality of $p^{(j)}$ in some way. This requires that each component element of vectors Δx_0 and Δa have some individual influence on the change in each component of vector $p^{(j)}$. This requirement cannot as yet be fully satisfied. However, in the algorithm which we suggest the value of each component of vectors $\Delta x_0^{(j)}$ and $\Delta a^{(j)}$ is organized to a certain extent in consideration of its individual influence on the component of vector $p^{(j)}$.

Calculation of the value of each component element of vectors $\Delta x_0^{(j)}$ and $\Delta a^{(j)}$ considering its individual influence on the components of vector $p^{(j)}$ is the second specific feature of this step optimization algorithm. This specific feature is based on the following positions.

Let us assume that there is only one functional and that at point t_0 one phase variable x_0 or one parameter a is missing. Suppose they are p_1^* and x_{10}^* or a_1^*.

Then, according to (1.4.11), we produce

$$\Delta p_1^* = {}_1y_1^* \delta x_{10}^* \quad \text{or} \quad \Delta p_1^* = {}_1z_1^* \delta a_1^*.$$

From this, since $\Delta p_1^* < 0$, we have

$${}_1y_1^* \delta x_{10}^* < 0 \quad \text{or} \quad {}_1z_1^* \delta a_1^* < 0.$$

Thus, if we consider only the individual influence of the step of any missing conditional phase variable $x_{\kappa 0}^{(j)}$ or any parameter $a_p^{(j)*}$ ($\kappa + p = 1, \ldots, m$) on any functional p_s^* ($s = 1, \ldots, m$), it will manifest itself positively, i. e. decrease p_s^* with a corresponding change in $x_{\kappa 0}^{(j)*}$ or $a_p^{(j)*}$ in the direction determined from the condition

$$\left.\begin{array}{l} {}_sy_\kappa^*(t_0)\delta x_{\kappa 0}^* < 0, \\ {}_sz_p^*(t_0)\delta a_p^* < 0. \end{array}\right\} \qquad (1.4.14)$$

However, the direction of swiftest descent found by solution of linear equation system (1.4.11) does not always correspond to the sign of $\delta x^*_{\kappa 0}$ or δa_p^* determined according to conditions (1.4.14). Obviously, in case the step does not correspond to the conditional phase variable $\Delta x_{\kappa 0}^*$ or parameter Δa_p^*, the step should not be made at "full power," defined by the value of ε^* according to (1.4.12). In this case, the scale of the limiting permissible step selected from (1.4.13) should be decreased. Therefore, the limiting permissible step should be determined after each iteration using the following relationships:

$$\Delta \dot{x}_{\kappa 0}^{np*} = (x_{\kappa 0\,max}^{p*} - x_{\kappa 0}^{(j)*})\, e_\kappa^{(j)*} \quad \text{where} \quad \delta x_{\kappa 0}^* \geq \dot{0}$$

or

$$\Delta x_{\kappa 0}^{\pi p*} = (x_{\kappa 0\,\min}^{\pi p*} - x_{\kappa 0}^{(\gamma)*})\, e_{\kappa}^{(\gamma)} \quad \text{where} \quad \delta x_{\kappa 0}^{*} < 0.$$

$$\Delta a_{p}^{\pi p*} = (a_{p\,\max}^{\pi p*} - a_{p}^{(\gamma)*})\, e_{p}^{(\gamma)*} \quad \text{where} \quad \delta a_{p}^{*} \geq 0$$

or

$$\Delta a_{p}^{\pi p*} = (a_{p\,\min}^{\pi p*} - a_{p}^{(\gamma)*})\, e_{p}^{(\gamma)*} \quad \text{where} \quad \delta a_{p}^{*} < 0.$$

The coefficients e_{κ}^{*}, e_{p}^{*} allow us to consider the individual influence on the components of $p^{(j)}$ in calculating the value of each component element of vector step $\Delta x^{(j)}$ and $\Delta a^{(j)}$.

Based on the preceding discussions, we can construct the following formulas

$$\left.\begin{aligned} e_{\kappa}^{(\gamma)*} &= \frac{1}{p_{\Sigma}^{(\gamma)*}} \sum_{s=1}^{m} p_{s}^{(\gamma)*} \operatorname{sign}\left[1 - \operatorname{sign}\left({}_{s}y_{\kappa}^{*}(t_0)\,\delta x_{\kappa 0}^{*}\right]\right), \\ e_{p}^{(\gamma)*} &= \frac{1}{p_{\Sigma}^{(\gamma)*}} \sum_{s=1}^{m} p_{s}^{(\gamma)*} \operatorname{sign}\left[1 - \operatorname{sign}\left({}_{s}z_{p}^{*}(t_0)\,\delta a_{p}^{*}\right)\right]. \end{aligned}\right\} \quad (1.4.15)$$

Calculation of $e_{\kappa}^{(\gamma)*}$ and $e_{p}^{(\gamma)*}$ is planned to be performed in the "e" unit (B-"e").

However, it is hardly always expedient to organize search for the maximum step $\Delta x_{0\,\max}$ and $\Delta a_{\max}$ in the area of the limiting step $\Delta x^{\pi p}$ and $\Delta a^{\pi p}$. For example, with low values of $p^{(\gamma)}$, it is obviously incorrect to move far from the point $(x^{(\gamma)}, a^{(\gamma)}, p^{(\gamma)})$. Furthermore, for some components of x_0 and a it is sometimes difficult to imagine their limiting values in advance (for example, the initial values of the Lagrange coefficients). Therefore, it would be desirable to change the area of search with the corresponding change in the value of $p^{(\gamma)}$. Performance of this task must be assigned to the computer, in order that it will occur automatically. This is achieved by organizing a search area unit (BOP), in which calculation of the instantaneous limiting values $x_{0\pi p}^{(\gamma+1)}$ and $a_{\pi p}^{(\gamma+1)}$ is performed using the following relationships:

$$x_{0\max}^{(\gamma+1)} = x^{(\gamma)}(1 + k_x p_{\Sigma}^{(\gamma)} \operatorname{sign} x^{(\gamma)})$$

or

$$x_{0\min}^{(\gamma+1)} = x^{(\gamma)}(1 - \bar{k}_x p_\Sigma^{(\gamma)} \operatorname{sign} x^{(\gamma)}),$$

$$a_{\max}^{(\gamma+1)} = a^{(\gamma)}(1 + k_a p_\Sigma^{(\gamma)} \operatorname{sign} a^{(\gamma)})$$

or

$$a_{\min}^{(\gamma+1)} = a^{(\gamma)}(1 - \bar{k}_a p_\Sigma^{(\gamma)} \operatorname{sign} a^{(\gamma)}),$$

where k_x, $\bar{k}_x$, k_a, $\bar{k}_a$ are fixed coefficients.

The BOP allows us to "hold" a reduction which has been achieved, narrowing the area of search and decreasing the functional, and gives the search the property of positive feedback (Figure 1.23), practically forbidding search at the point of optimal values of $x_0^{(\gamma)}$ and $a^{(\gamma)}$ (at the point of equilibrium) and expanding the potential capabilities for search by a large step in the case of a considerable mismatch (large value of functional p_Σ^*).

The presence of feedback, deforming the area of search for the maximum step depending on the values of the functionals is the third specific feature of the algorithm for single-parameter optimization of the step.

However, the output parameters of the BOP, $x_{0\pi p}^{(\gamma+1)}$ and $a_{\pi p}^{(\gamma+1)}$ should not go beyond the limiting values of $x_0^{\pi p}$ and $a^{\pi p}$. Therefore, comparison of the output parameters of the BOP with their limiting values must be performed. This is performed in the comparison operator, which outputs the final judgments concerning $x_{0\pi p}^t$ and $a_{\pi p}^t$ for the instantaneous search.

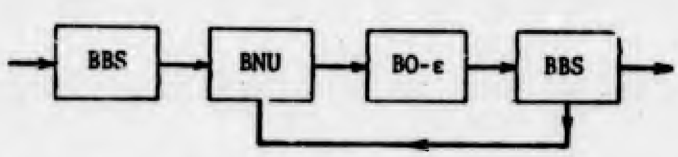


Figure 1.23. Diagram of Feedback for Determination of Search Area.

Then the instantaneous value of the step of the conditional phase variable $x_{\kappa 0}$ and parameter a_p in the instantaneous value unit (BTZ) is calculated as follows:

$$\Delta x_{\kappa 0}^{(j)*} = (x_{\kappa 0\,max}^{\tau *} - x_{\kappa 0}^{(\gamma)*})\, e_{\kappa}^{(\gamma)*} \varepsilon^{*} \quad \text{where} \quad \delta x_{\kappa 0}^{*} \geqslant 0$$

or

$$\left.\begin{aligned} \Delta x_{\kappa 0}^{(j)*} &= (x_{\kappa 0\,min}^{\tau *} - x_{\kappa 0}^{(\gamma)*})\, e_{\kappa}^{(\gamma)*} \varepsilon^{*} \quad &\text{where} \quad \delta x_{\kappa 0}^{*} < 0, \\ \Delta a_{p}^{(j)*} &= (a_{p\,max}^{\tau *} - a_{p}^{(\gamma)*})\, e_{p}^{(\gamma)*} \varepsilon^{*} \quad &\text{where} \quad \delta a_{p}^{*} \geqslant 0 \end{aligned}\right\} \qquad (1.4.16)$$

or

$$\Delta a_{p}^{(j)*} = (a_{p\,min}^{\tau *} - a_{p}^{(\gamma)*})\, e_{p}^{(\gamma)*} \varepsilon^{*} \quad \text{where} \quad \delta a_{p}^{*} < 0,$$

where $\varepsilon^{*} = \varepsilon_{j}^{*}$ or $\varepsilon^{*} = \varepsilon^{(j)*}$,

$$\varepsilon_{j}^{*} = \varepsilon_{0}^{*} \beta_{1}^{(j)*}, \quad \varepsilon^{(j)*} = \varepsilon_{0}^{*} \beta_{2}^{(j)*}, \qquad (1.4.17)$$

j is the step number;

ε_0^*, $\beta_1^{(j)*}$, $\beta_2^{(j)*}$ are the fixed coefficients, where $\beta_1^{(j)*} < 1$, $\beta_2^{(j)*} > 1$;

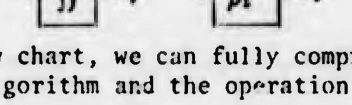
$x_{\kappa 0\,max}^{\tau *}$, $x_{\kappa 0\,min}^{\tau *}$ are the instantaneous maximum and minimum values of $x_{\kappa 0}^{*}$ at the input to the BTZ;

$a_{p\,max}^{\tau *}$, $a_{p\,min}^{\tau *}$ are the instantaneous maximum and minimum values of a_p^* at the input to the BTZ.

The instantaneous value of the conditional phase variable $x_{\kappa 0}$ and parameter a_p^* in the BTZ is

$$\left.\begin{aligned} x_{\kappa 0}^{(j)*} &= x_{\kappa 0}^{(\gamma)*} + \Delta x_{\kappa 0}^{(j)*}, \\ a_{p}^{(j)*} &= a_{p}^{(\gamma)*} + \Delta a_{p}^{(j)*}. \end{aligned}\right\} \qquad (1.4.18)$$

A flow chart of the single-parameter step optimization unit is shown on Figure 1.24. In it, calculation of $e_{\kappa}^{(\gamma)*}$, $e_{p}^{(\gamma)*}$, ε^* and, therefore, $\Delta x_{\kappa 0}^{(j)*}$, $\Delta a_{p}^{(j)*}$, $x_{\kappa 0}^{(j)*}$, $a_{p}^{(j)*}$ is performed according to relationships (1.4.15)-(1.4.18) in the initial conditions unit (BNU), while the unit for calculation of coupling equations and optimization conditions (BRUSIUO) in the unit for calculation of $p_{\Sigma}^{(j)*}$ are represented by the following two symbols



Following this flow chart, we can fully comprehend the logic of the step optimization algorithm and the operation of the unit. It is as follows.

After leaving a given iteration (rapid descent unit), $e_{\kappa}^{(\gamma)*}$, $e_{p}^{(\gamma)*}$ and the instantaneous limiting steps $\Delta x_{0\pi p}^{t}$ and $\Delta a_{\pi p}^{t}$ are calculated in the BNU as functions of the sign of $\delta x_{\kappa 0}^{*}$ and δa_{p}^{*}. Then, using the fixed coefficient ε_{0}^{*}, we determine $\Delta x_{\kappa}^{(0)*}$, $\Delta a_{p}^{(0)*}$ and $x_{\kappa 0}^{(0)*}$, $a_{p}^{(0)*}$. After this, we go over to the BRUSIUO and calculate $p_{s}^{(0)*}$ and $p_{\sigma}^{(0)*}$. The comparison unit compares $p_{\Sigma}^{(0)*}$ with $p_{\Sigma}^{(\gamma)*}$. If $p_{\Sigma}^{(\gamma)*} < p_{\Sigma}^{(0)*}$, the computer repeats this cycle of operations until $p_{\Sigma}^{(j)*} < p_{\Sigma}^{(\gamma)*}$, calculating in the BNU at each jth step $\varepsilon_{j}^{*} = \varepsilon_{0}^{*}\beta_{1}^{(j)*}$ and $x_{\kappa 0}^{(j)*}$, $a_{p}^{(j)*}$. If the result is achieved, the problem arises of either going back to repetition of the operation to find a p_{Σ}^{*} even less than $p_{\Sigma}^{(j)*}$, or accepting the result produced and going over to a new iteration or, using the information produced concerning function $p_{\Sigma}^{*}(\varepsilon)$, constructing it by an approximate method and using this approximate construction of $p_{\Sigma}^{*}(\varepsilon)$ to determing coefficient $\varepsilon^{\inf *}$ in the interval $0 \le \varepsilon^{*} \le \varepsilon_{j-1}^{*}$, corresponding to $p_{\varepsilon}^{\inf *}$.

In the first case, each time the favorable situation ($p_{\Sigma}^{(\tau + 1)*} < p_{\Sigma}^{(\gamma)*}$) arises, the temptation will always arise to make one more step, while with the unfavorable situation ($p_{\Sigma}^{(\gamma + 1)} > p_{\Sigma}^{(\gamma)*}$), doubt can arise as to the correctness of the step selected. Therefore, in the first case in order to produce satisfactory results we must either plan in advance a large number of steps, requiring a definite expenditure of machine time, or suffer from doubts as to the unused capabilities. In the second case, as in the first, the question arises as to whether the results produced is achieved at too high a price. In the third case, the results of preceding steps are used for approximate construction of the function $p_{\Sigma}^{*}(\varepsilon)$. With a successful approximation, it helps to judge the true nature of the dependence $p_{\Sigma}^{*}(\varepsilon)$ with high probability and without great expense, thereby allowing us to determine $\varepsilon^{\inf *}$ with $p_{\Sigma}^{\inf *}$. The advantages of the third case are obvious: it does not require advance planning of a large number of unnecessary steps, as is required in the first case, but an increase in the number of steps in comparison with the second method of only one or two significantly increases the reliability of the selection of $p_{\Sigma}^{\inf *}$, near the true value. True, the great advantages of the third method depend to a great extent on the method of construction of the approximate version of $p_{\Sigma}^{*}(\varepsilon)$.

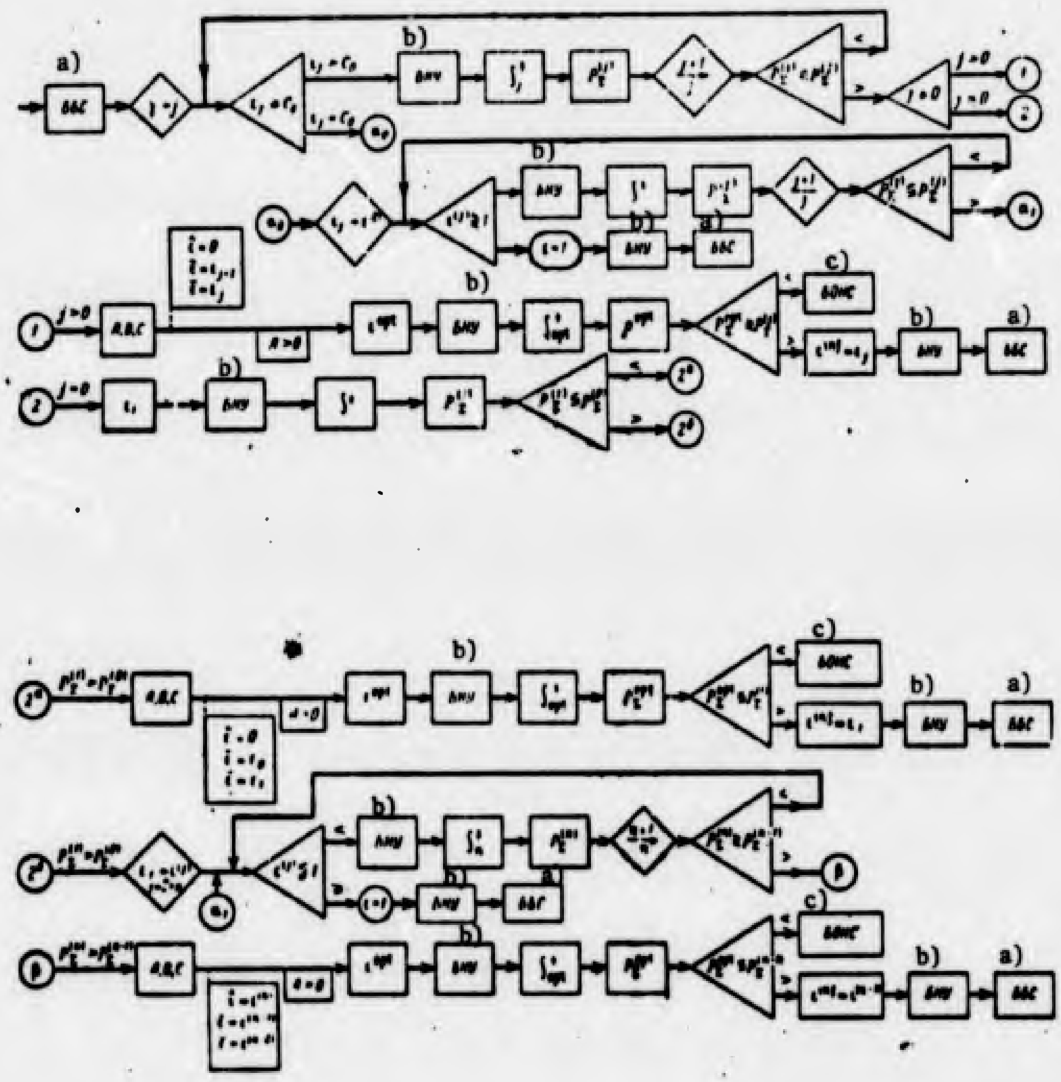


Figure 1.24. Structural Diagram of Step Optimization Unit

KEY: a, BBS; b, BNU; c, BONS

The construction of an approximate $p^*_\Sigma(\varepsilon)$ can be performed using an interpolation polynomial, using the points $(p_\Sigma^{(j)*}, \varepsilon^*_j)$ as points of interpolation. Generally speaking, the interpolation polynomial can be of any order of up to (j + 1). Of course, increasing the order of the polynomial increases the accuracy of the interpolation formula. However, an unnecessary increase in accuracy does not produce any noticeable effect and may complicate the logic of construction of the polynomial, cause unjustified expenditures and complexities in seeking out the point $(\varepsilon^{min*}, p_\Sigma^{min*})$, resulting in loss of the possibility of using this portion of the program for similar cases. Therefore, among the interpolation polynomials, preference should be given to the quadratic polynomial

$$p^*_\Sigma = A^*\varepsilon^{*2} + B^*\varepsilon^* + C^*. \tag{1.4.19}$$

The quadratic polynomial has a simple logical structure for computers, in many cases can give good coincidence with the true curve $p^*_\Sigma(\varepsilon)$, allows us to organize search for the point $(\varepsilon^{min*}, p_\Sigma^{min*})$ rather simply and rapidly. Furthermore, the section in which the algorithm for construction of the quadratic polynomial and search for this point are performed can be used in similar situations in other portions of the algorithm for solution of the multipoint boundary problem.

The coefficients A*, B* and C* of the quadratic polynomial (1.4.19) are calculated in the section →[A,B,C]← using the following relationships:

$$\left.\begin{aligned} A^* &= \frac{\hat{p}^*_\Sigma}{(\hat{\varepsilon}^* - \tilde{\varepsilon}^*)(\hat{\varepsilon}^* - \bar{\varepsilon}^*)} - \frac{\tilde{p}^*_\Sigma}{(\hat{\varepsilon}^* - \tilde{\varepsilon}^*)(\tilde{\varepsilon}^* - \bar{\varepsilon}^*)} + \\ &\quad + \frac{\bar{p}^*_\Sigma}{(\hat{\varepsilon}^* - \bar{\varepsilon}^*)(\tilde{\varepsilon}^* - \bar{\varepsilon}^*)}, \\ B^* &= \frac{\tilde{p}^*_\Sigma - \hat{p}^*_\Sigma}{\tilde{\varepsilon}^* - \hat{\varepsilon}^*} - A^*(\hat{\varepsilon}^* + \tilde{\varepsilon}^*), \\ C^* &= \frac{\hat{p}^*_\Sigma\tilde{\varepsilon}^* - \tilde{p}^*_\Sigma\hat{\varepsilon}^*}{\tilde{\varepsilon}^* - \hat{\varepsilon}^*} + A^*\hat{\varepsilon}^*\tilde{\varepsilon}^*. \end{aligned}\right\} \tag{1.4.20}$$

In the case in question, we have

$$\begin{aligned} \hat{p}^*_\Sigma &= p_\Sigma^{(1)*}, & \hat{\varepsilon}^* &= 0, \\ \tilde{p}^*_\Sigma &= p_\Sigma^{(j-1)*}, & \tilde{\varepsilon}^* &= \varepsilon^*_{j-1}, \\ \bar{p}^*_\Sigma &= p_\Sigma^{(j)*}, & \bar{\varepsilon}^* &= \varepsilon^*_j. \end{aligned}$$

Since $p_\Sigma^{(j)*} < p_\Sigma^{(\gamma)*} < p_\Sigma^{(j-1)*}$, then $A^* > 0$ and $B^* < 0$.

Therefore, we produce

$$\varepsilon^{min^*} = \varepsilon^{opt^*} = -\frac{B^*}{2A^*}. \qquad (1.4.21)$$

After calculation of ε^{opt^*} in the section $\to\boxed{\varepsilon^{opt}}\leftarrow$, a transfer is made from ε^{opt^*} to BNU and further to $\to\boxed{f^{\dagger}\ opt}\to$ and $\to\boxed{P_\Sigma^{opt}}\to$. Then the condition $p_\Sigma^{opt} \gtrless p_\Sigma^{(j)}$ is made. This test is necessary, since poor interpolation of the true function $p_\Sigma^*(\varepsilon)$ is quite possible using the quadratic polynomial (1.4.19). If $p_\Sigma^{opt^*} > p_\Sigma^{(j)}$, the true dependence $p_\Sigma^*(\varepsilon)$ is of complex structure, not suitable for interpolation using the simple polynomial. In this situation, a complex algorithm is required to construct good interpolation and perform search for $(\varepsilon^{min^*}, p_\Sigma^{min^*})$. Since its introduction is unjustified due to the low probability of these situations, we assume $\varepsilon^{inf^*} = \varepsilon_j^*$ and return to the BNU for subsequent transfer to the BBS. If $p_\Sigma^{opt^*} < p_\Sigma^{(j)*}$, the interpolation has been organized satisfactorily, and we can immediately transfer to the BONS.

True, a difficult situation may arise in which the cycle is completed with $\varepsilon_j^* = \varepsilon_0^*$ minimal in the sense of ε_j^*, while still $p_\Sigma^{(\gamma)*} < p_\Sigma^{(j)*}$. This situation sometimes arises when the point $(x_0^{(\gamma)*}, a^{(\gamma)*}, p_\Sigma^{(\gamma)*})$ is located near the local minimum. In this case, we must go to the operations indicated on line (α_1). One specific feature of this portion of the step optimization section is the search for $p_\Sigma^* < p_\Sigma^{(\gamma)*}$, on the first side of ε_0, i. e. where $\varepsilon = \varepsilon^{(j)}$. Here the search is completed either with $p_\Sigma^{(\gamma)*} > p_\Sigma^{(j)*}$ with a transfer to line (α_1), or when we reach $\varepsilon^{(j)*} = 1$ by an exit to the BBS with $p_\Sigma^{(j)*} > p_\Sigma^{(\gamma)*}$. The exit to the BBS with $p_\Sigma^{(j)*} > p_\Sigma^{(\gamma)*}$ and $\varepsilon^{(j)*} = 1$ is forced. However, some consolation can be found with these poor results in the fact of movement to the maximum distance from the difficult point.

If where $\varepsilon^* = \varepsilon_0^*$ we find $p_\Sigma^{(\gamma)*} > p_\Sigma^{(0)*}$, we should go over to the sequence of operations indicated on the flow chart by (2). Its logic is clear. Here the comparison $p_\Sigma^{(1)*} \gtrless p_\Sigma^{(0)*}$ allows us to select one path of two. Where $p_{\Sigma}^{(1)*} < p_{\Sigma}^{(0)*}$, using the interpolation of $p_{\Sigma}(\varepsilon)$ indicated above, we determine the point $(\varepsilon^{inf*}, p_{\Sigma}^{inf*})$ (line (2a)). However, if $p_{\Sigma}^{(1)*} > p_{\Sigma}^{(0)*}$, further search for the least p_{Σ}^{*} with $\varepsilon^{*} = \varepsilon^{(j)*} > \varepsilon_0^{*}$ is performed. This search is organized according to the portion of the flow chart represented by (2b) . With each new $\varepsilon^{(j)*}$, the step in the direction of swiftest descent increases, until $p_{\Sigma}^{(n)*} > p_{\Sigma}^{(n-1)*}$, which is tested in the comparison unit $p_{\Sigma}^{(n)*} \gtrless p_{\Sigma}^{(n-1)*}$ (see line (2b) , Figure 1.24). Then a transfer is made to the operations connected on the diagram by line (β) . Point $(\varepsilon^{inf*}, p_{\Sigma}^{inf*})$ is determined similarly to the preceding method after interpolation of function $p_{\Sigma}(\varepsilon)$ by quadratic polynomial (1.3.19). If always $p_{\Sigma}^{(n)*} < p_{\Sigma}^{(n-1)*}$, after reaching $\varepsilon^{*} = \varepsilon^{(j)*} = 1$, a transfer is made to the BNU and further to the BBS.

The flow chart of the single-parameter step optimization section, particularly the portion producing optimization of "ε^{*}" (BO-"ε") demonstrates the fourth specific feature of its algorithm. It consists in the construction by performance of a number of experiments of an approximate dependence $p_{\Sigma}^{*}(\varepsilon)$, which allows rather inexpensive and frequently quite accurate prediction of the determination of point $(\varepsilon^{inf*}, p_{\Sigma}^{inf*})$ in the calculated direction of swiftest descent.

On the flow chart for BOOSH, there is some uncertainty in the selection of the first step defined by the value of ε_0^{*}, and in the selection of coefficients $\beta_1^{(j)*}$ and $\beta_2^{(j)*}$. However, the values of ε_0^{*}, $\beta_1^{(j)*}$ and $\beta_2^{(j)*}$ determine the strategy for the search for ε^{inf*}.

In solving the problem of locating ε^{inf*}, we must attempt to select, from all permissible strategies for search, that strategy which leads to achievement of the goal in the best manner. We are concerned here with determination of the coefficient ε^{inf*} giving us p_{Σ}^{inf*} by an efficient means, i. e. optimization of the very method of optimization of ε^{*}.

In order to select an efficient strategy for search, we must introduce a certain measure which takes on a definite value in analysis of the strategies before beginning of calculation of p_{Σ}^{*} as a function of ε^{*}. It is hardly possible to perform an objective estimation of the search strategy on the basis of the minimum value of p_{Σ}^{*}: this estimation would depend more on luck.

Generally speaking, the effectiveness of the search strategy S can be represented as a dependence of the quantity of information J concerning the structure of the minimizing function p_Σ^* and the possible location of the area of its minimum and as a dependence of the number of experiments m, on the basis of which the information was collected. Then, the criterion of effectiveness can be expressed as follows:

$$h = \max_J \{S(m, J)\}.$$

The criterion of effectiveness h reaches its optimal value h* when the following equation is satisfied:

$$h^* = \min_m \{h(m)\} = \min_m \max_J \{S(m, J)\}.$$

Actually, the optimal search strategy must be considered that which provides the maximum information concerning the position of the minimum point in the minimum number of experiments. This minimum estimation of the search strategy is quite careful, but corresponds to the requirements of the search problem.

Unfortunately, this optimal strategy does not exist for all functions. For unimodal functions -- functions having one peak (or one dip) -- we can select a similar objective criterion of effectiveness of the search strategy, using the quantitative estimate.

Suppose $p_\Sigma^*(\varepsilon)$ is a unimodal function and $0 \leq \varepsilon^* \leq 1$. We note that the conditions of continuity and differentiability are not applied to the function, which is quite important. Let us select a search strategy for the minimum if p_Σ^* for which the value of ε_k^* is defined as the function Φ_k^* of the preceding values of arguments ε_j^* and the values found for $p_\Sigma^*(\varepsilon_j)$. Then with fixed ε_1^* we can write

$$\varepsilon_k^* = \varphi_k^*(\varepsilon_1, \ldots, \varepsilon_{k-1}, p_\Sigma^{(1)}(\varepsilon_1), \ldots, p_\Sigma^{(k-1)}(\varepsilon_{k-1}))$$
$$(k = 2, \ldots, n).$$

Furthermore, let us assume that knowledge of ε_k^* allows us in some way to determine the lower bound s and the upper bound r of the interval containing the minimum value of p_Σ^*, i. e.

$$s = s(\varepsilon_k, p_\Sigma^{(k)}) \quad (k = 2, \ldots, n), \quad r = r(\varepsilon_k, p_\Sigma^{(k)}).$$

Strategy S is expressed as follows:

$$S=\{\varepsilon_1, \tilde{\varepsilon}_1, \dots, \tilde{\varepsilon}_n, s, r\}.$$

If we introduce the length of interval $l = r - s$, a function of S_n, then optimal strategy S_n^* in the sense defined earlier should have the following property:

$$L_n^*(S^*)=\min_{n} \max_{S_n} \{L_n(S_n)\},$$

or in other words

$$\sup L_n(S_n^*) \leqslant \inf \sup L_n(S_n)+\Delta^{*}.[1]$$

This optimal strategy S_n^*, for which the worst result (in the sense of the length of interval L_n) is better with arbitrarily small tolerance $\Delta^* > 0$ than the worst results of other strategies, can be determined for all functions $p_\Sigma^*(\varepsilon)$ belonging to the class of unimodal functions.

Since the value of L_n is intentionally selected for the worst case, there is no undesirable dependence on the results of tests, so that $\sup L_n(S_n)$ can be looked upon as an a priori criterion of effectiveness of the strategies, the minimum of which determines the optimal strategy.

The Fibonacci method [43, 46], must be looked upon as one of the most effective sequential strategies of search for the minimum of a unimodal function having the optimal value of the criterion of effectiveness which we have introduced.

An important point in the use of the Fibonacci method is proper selection of the number of experiments. In planning search, one should select the minimum shift Δ for which the results of two experiments can be differentiated. The condition of differentiability determines the limiting number of tests, after which a further increase is useless. Suppose the maximum number of experiments for the Fibonacci method is m. We can then produce

$$F_{m+1}^* < \frac{l}{\Delta} < F_{m+2}^*,$$

[1] Determination of strategy S_n^* and proof of its optimality in this sense are presented in [43].

where F^*_{m+1}, F^*_{m+2} is the Fibonacci number for (m + 1) and (m + 2).

If we take the value of Δ in the limit $0.03 \leq \varepsilon < 0.05$, we must consider the value of m = 6 to be effective, since according to the inequality there is no reason to perform more than six experiments.

Where m = 6, it would be possible in principle to produce twelve minimum search strategies $p^*_\Sigma S^*_6 = (S^{(1)*}_6, \ldots, S^{(12)*}_6)$, which could be used to perform search by the Fibonacci method. All of these strategies are equally probable in the general case.

Of course, the search algorithm can be realized by only one of these, which actually leads to the minimum value of the unimodal function. If it were possible to perform search p^{inf*}_Σ using each strategy $S^{(1)*}_6, \ldots, S^{(12)*}_6$ individually, the probable values of ε^{inf*} would be equal to those shown in Table 1.1 [1].

Table 1.1

S^*_6	1	2	3	4	5	6	7	8	9	10	11	12
ε	0,077	0,231	0,231	0,3846	0,3845	0,538	0,538	0,692	0,3846	0.769	0,769	0,923

Judging from the data of Table 1.1, we note that the search results repeat for various strategies. Therefore, according to the Fibonacci method where m = 6, we can arrive at the following values of ε*, equal to 0.077; 0.231, 0.385, 0.538, 0.692, 0.769 and 0.923, near one of which in the interval $\varepsilon^* - \Delta^* \leq \varepsilon^* \leq \varepsilon^* + \Delta^*$ we find ε^{inf*}.

This result is possible only for unimodal functions $p^*_\Sigma(\varepsilon)$. However, the true dependence of p^*_Σ on ε* is unknown. In this connection, we must investigate the entire interval of ε* from zero to one, but with the least possible number of experiments. Therefore, as the reference

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[1] In order to simplify calculations, Δ* was assumed equal to zero.

values of ε^* we must take those which were produced earlier (see Table 1.1) for various strategies of search $S_6^{(j)}$ using the Fibonacci method. Then in the case of unimodal function $p_\Sigma^*(\varepsilon)$, it will be possible to produce the value of p_Σ^* relatively rapidly and at least less than with the remaining values of ε^*; otherwise, the BOOSH algorithm will find the least values of p_Σ^* in the interval of ε^* from zero to one, based on the reference values of ε^*. This approach to the search for $\varepsilon^{\inf *}$ requires, regardless of the form of function $p_\Sigma^*(\varepsilon)$, performance of a number of tests which is less than or equal to the number of tests of the guaranteed most effective sequential search strategy -- the Fibonacci method.

The reference values of ε^* produced allow us to select values of ε_0^* and $\beta_1^{(j)*}$, $\beta_2^{(j)*}$. It should be kept in mind here that the most probable determination of $p_\Sigma^{\inf *}$ is where $0.1 < \varepsilon^* < 0.5$. Therefore, we can accept $\varepsilon_0^* = 0.231$ or $\varepsilon^* = 0.385$, and take the corresponding $\beta_1^{(j)*}$ and $\beta_2^{(j)*}$ from Table 1.2.

Table 1.2

ε_0 \ $\beta_i^{(j)}$	$\beta_1^{(1)}$	$\beta_1^{(2)}$	$\beta_2^{(1)}$	$\beta_2^{(2)}$	$\beta_2^{(4)}$	$\beta_2^{(4)}$	$\beta_2^{(5)}$	$\beta_2^{(6)}$
0,385	0,6	0,2	1,4	1,8	2	2,4	2,6	—
0,231	0,33	—	1,66	2,23	3	3,3	4	4,3

Multiparametric Step Optimization Section (BMOSH)

The single-parameter step optimization unit (BOOSH) performs search for $\Delta x_{\kappa 0}^*$ and Δa_p^* for which p_Σ^* reaches values closest to inf by representing p_Σ^* as a function depending only on one independent variable ε^*. This organization of the step optimization unit greatly simplifies search in the required direction of values of p_Σ^* less than the initial $p_\Sigma^{(\gamma)*}$. However, by its nature p_Σ^* is a multivariate function. Therefore, any BOOSH strategy is limited in its possibilities.

If we now assume that the direction of search of $p_{\Sigma}^{inf*} = \inf p_{\Sigma}^{(j)*}$ has been correctly determined, the possibilities of the algorithm of the section for multiparametric step optimization in which the summary functional p_{Σ}^{*} is represented as a multidimensional function, can be practically unlimited. The result of operation of this section would depend to a great extent on the strategy for search of the maximum step value on the basis of each component of the vectors x_0 and a. However, it must be noted that it is practically impossible to find a measure of effectiveness of a search strategy using many variables for the minimum of a function which does not depend on luck to a great extent. Therefore, it is as yet difficult to find an objective method of comparing strategies for multiparametric step optimization and we cannot determine a strategy for multiparametric optimization which is optimal in any sense. Figuratively speaking, whereas we must write "unidimensional limitation" across the BOOSH, the BMOSH suffers from "excessive dimensionality." All of this, plus the great cumbersomeness of the BMOSH, limits the capabilities of this section and requires that its usage be approached carefully. True, there is some hope that in the near future the "excessive dimensionalities" will be less oppressive for experimental programmers, since investigations in the solution of the problem of multidimensional search are being conducted on a broad scale.

Let us analyze one possible BMOSH algorithm. Its idea is based on a suggestion by S. S. Lavrov for the use of baricentric coordinates for the production of simple formulas and effective methods for the solution of a number of computational problems for functions of many variables [1].

One advantage of the BMOSH algorithm presented below is its interrelationship with the BOOSH algorithm presented earlier. Construction of the search area section (BOP) remains as before; in the instantaneous values section (BTZ), the instantaneous value of the step of the arbitrary phase variables $x_{\kappa 0}^{*}$ and a_{p}^{*} is calculated as follows:

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[1] S. S. Lavrov, use of baricentric coordinates for the solution of certain computational problems, *Zhurnal Vychislitel'noy Matematiki I Matematicheskoy Fiziki* Vol. 4, No. 5, 1964.

$$
\left.
\begin{array}{ll}
 & \Delta x_{\varkappa 0}^{(j)*}=(x_{\varkappa 0\max}^{r*}-x_{\varkappa 0}^{(j)*})\varepsilon_{\varkappa}^{*} \text{ if } \delta x_{\varkappa 0}^{*}\geqslant 0, \\
\text{or} & (\varkappa=1,\ldots,l), \\
 & \Delta x_{\varkappa 0}^{(j)*}=(x_{\varkappa 0\min}^{r*}-x_{\varkappa 0}^{(j)*})\varepsilon_{\varkappa}^{*} \text{ if } \delta x_{\varkappa 0}^{*}<0, \\
 & \Delta a_{p}^{*}=(a_{p\max}^{r*}-a_{p}^{(j)*})\varepsilon_{p}^{*} \text{ if } \delta a_{p}^{*}\geqslant 0 \\
\text{or} & (p=l+1,\ldots,d+l=m), \\
 & \Delta a_{p}^{*}=(a_{p\min}^{r*}-a_{p}^{(j)*})\varepsilon_{p}^{*} \text{ if } \delta a_{p}^{*}<0.
\end{array}
\right\} \quad (1.4.23)
$$

Thus, the instantaneous value of the step for each component of the vectors x_0 and a is determined not only by the limiting step of this component, but also by the corresponding value of the coefficient ε_i^* (i = 1,..., m). In this connection, the summary functional will depend on the m independent variables ε_j^*, each of which changes between 0 and 1 or 0 and ε_i^{max*}.

The BMOSH algorithm should provide for determination of values of ε_i^* for which the value of p_Σ^* is close to or equal to p_Σ^{inf*}. The difficulty in construction of this algorithm is related not only to the multidimensional nature of p_Σ^*, but also to the absence of any information on the structure of function $p_\Sigma^*(\varepsilon_i)$. Data can be produced on multidimensional function $p_\Sigma^*(\varepsilon)$ only by experiment, i. e. by calculating p_Σ^* with various values of vector $\varepsilon = (\varepsilon_1^*,\ldots, \varepsilon_m^*)$.

Let us assume that in the direction of swiftest descent which has been determined it is actually possible to find values of ε_j^* (j = 1,..., m) for which p_Σ^* reaches its minimum value p_Σ^{inf}. Then, if we have sufficiently properly constructed our approximation (in some sense) of the multidimensional function $p_\Sigma^*(\varepsilon_j)$, we can find, determining the minimum of this approximating function, the value of ε_j^* for which $p_\Sigma^* = p_\Sigma^{min*}$.

Therefore, the BMOSH algorithm must on the one hand provide for construction of the polynomial F^* (ε_j), rather precisely approximating the summary functional p_Σ^* as a multidimensional function of ε_j^* and, on the other hand, must include a simple method of determining the global extreme of the multidimensional polynomial $F^*(\varepsilon_j)$. Frequently these requirements are rather incompatible, so that sometimes they must be purchased at the price of the accuracy of the approximation.

If we use baricentric coordinates, we can rather simply construct an interpolation multidimensional kth power polynomial (where $k \leq 3$) for the multidimensional function $p^*_\Sigma(\varepsilon_j)$. One of the specific features of the construction of the interpolation polynomial using baricentric variables is that the problem of construction of the interpolation polynomial $F^*(\varepsilon_j)$ of power k for function $p^*_\Sigma(\varepsilon_j)$ is equivalent to the problem of interpolation of this function using a homogeneous polynomial $F^*(\mu_0, \ldots, \mu_m)$ of the same power as the baricentric variables $\mu_0, \ldots, \mu_m$ (see footnote on page 116).

We select arbitrarily the m + 1 base points $p^{(j)*}_\Sigma$ $(j = 0, \ldots, m)$, located at a common position, so that the determinant

$$D=\begin{vmatrix} 1 & 1 & \ldots & 1 \\ \varepsilon^*_{10} & \varepsilon^*_{11} & \ldots & \varepsilon^*_{1m} \\ \cdot & \cdot & \cdot & \cdot \\ \varepsilon^*_{m0} & \varepsilon^*_{m1} & \ldots & \varepsilon^*_{mm} \end{vmatrix}$$

is not equal to zero, where $\varepsilon^*_{1j}, \ldots, \varepsilon^*_{mj}$ are the values of ε^* at point $p^{(j)*}_\Sigma$, looked upon as cartesian coordinates in an m-dimensional euclidean space. Let us now place arbitrary mass μ^*_j at each point, so that the summary mass is equal to unity

$$\sum_{j=0}^{m} \mu^*_j = 1. \tag{1.4.24}$$

Then, the center of gravity of this system of masses p^*_Σ will have the cartesian coordinates

$$\varepsilon^*_j = \mu^*_0 \varepsilon^*_{j0} + \mu^*_1 \varepsilon^*_{j1} + \ldots + \mu^*_m \varepsilon^*_{jm} \quad (j = 1, 2, \ldots, m). \tag{1.4.25}$$

The quantities $\mu^*_0, \mu^*_1, \ldots, \mu^*_m$ are called baricentric coordinates of point p^*_Σ. For arbitrary point p^*_Σ with coordinates $\varepsilon^*_1, \ldots, \varepsilon^*_m$, the baricentric coordinates $\mu_0^*, \ldots, \mu_m^*$ are determined from the system of equations (1.4.24) and (1.4.25) unambiguously, since determinant D of this system is not equal to zero.

Let us now analyze the problem of construction of a second power interpolation polynomial for $p_\Sigma^*(\varepsilon_j)$ with a special selection of interpolation points. Let us divide the one mass into two equal parts and distribute them between the base points. The homogeneous polynomial $F^*(\mu_0, \ldots, \mu_m)$ can be written in the form

$$F^*(\mu_0, \ldots, \mu_m) = \sum_{j=0}^{m} \frac{1}{2} a_{jj}\mu_j^2 + \sum_{j,k=0}^{m} a_{jk}\mu_j\mu_k. \quad (1.4.26)$$

Equating the values of polynomial F^* and functions p_Σ^* at the points of interpolation, we produce

$$a_{jk}^* = 4p_{\Sigma k}^{(j)*} - (p_{\Sigma j}^{(j)*} + p_{\Sigma k}^{(j)*}) \quad (j, k = 0, 1, \ldots, m;\ j < k), \quad (1.4.27)$$

where

$p_{\Sigma j}^{(j)*} = p_\Sigma^*(\varepsilon_{1j}^*, \ldots, \varepsilon_{mj}^*)$ $(j = 0, 1, \ldots, m)$ are the base values of p_Σ,

$p_{\Sigma k}^{(j)*} = p_\Sigma^*\left(\frac{\varepsilon_{1j}^* + \varepsilon_{1k}^*}{2}, \ldots, \frac{\varepsilon_{mj}^* + \varepsilon_{mk}^*}{2}\right)$ $(j, k = 0, 1, \ldots, m)$ are the values of p_Σ^* at the mid points of all sectors connecting the base points in pairs.

Let us find the global minimum of polynomial F^*, having in mind the limited area of change of vector ε^*. The existence of limitations on the area of change of the components of vector ε^*, defined by inequalities $0 \le \varepsilon_j^* \le \varepsilon_j^{max*}$, will be considered by introducing the following equations:

$$\varphi_j^* = \left(\varepsilon_j^{max*} - \sum_{i=0}^{m} \mu_i^*\varepsilon_{ji}^*\right) \cdot \sum_{i=0}^{m} \mu_i^*\varepsilon_{ji}^* - v_j^{*2} = 0 \quad (j = 1, \ldots, m). \quad (1.4.28)$$

Thus, the global minimum F^* should be defined considering relationships (1.4.24) and (1.4.28), leading to the necessity of composing the new function

$$Q^* = F^* + \lambda^*\left(\sum_{j=0}^{m} \mu_j^* - 1\right) + \sum_{j=1}^{m} \varphi_j^*\lambda_{\varepsilon j}^* = 0.$$

Setting the partial derivatives of function Q^* equal to zero with respect to the independent variables $\mu_0^*, \ldots, \mu_m^*$ and ν_j^*, we produce

$$\frac{\partial Q^*}{\partial \mu_i^*} = \sum_{k=0}^{m} a_{ik}^* \mu_k^* + \lambda^* + \sum_{j=1}^{m} \lambda_{0j}^* \frac{\partial \varphi_j^*}{\partial \mu_i^*} = 0 \quad (i = 0, \ldots, m),$$

$$\lambda_{0j}^* \nu_j^* = 0 \qquad (j = 1, \ldots, m).$$

Keeping in mind the relationships (1.4.24), (1.4.27) and (1.4.28), we can reduce them to

$$\left.\begin{array}{l} \sum_{k=0}^{m} 4 p_{\Sigma k}^{(i)*} \mu_k^* + \mu^* + \sum_{j=1}^{m} \lambda_j^* = p_{\Sigma i}^{(i)*} \quad (i = 0, 1, \ldots, m), \\ \sum_{k=0}^{m} \mu_k^* = 1, \\ \lambda_j^* = 0 \quad (j = 1, \ldots, m), \qquad (1.4.30) \end{array}\right\} \qquad (1.4.29)$$

where

$$\mu^* = \lambda^* - \sum_{k=0}^{m} p_{\Sigma k}^{(k)*} \mu_k^*, \quad \lambda_j^* = \lambda_{j0}^* \frac{\partial \varphi_j^*}{\partial \mu_{0i}^*},$$

where, if as a result of solution of (1.4.29) we produce

$$\sum_{i=0}^{m} \mu_i^* \varepsilon_{pi}^* < 0 \quad \text{and (or)} \left(\varepsilon_l^{\max *} - \sum_{i=0}^{m} \mu_i^* \varepsilon_{li}^* \right) < 0,$$

system (1.4.29) must be converted by replacing equations (1.4.30) with the following equations:

$$\left.\begin{array}{ll} & \sum_{i=0}^{m} \mu_i^* \varepsilon_{pi}^* = 0 \quad (p = 1, \ldots, m) \\ \text{and (or)} & \\ & \sum_{i=0}^{m} \mu_i^* \varepsilon_{li}^* = \varepsilon_l^{\max *} \quad (l = 1, \ldots, m; \text{ where } l \neq p), \\ & \lambda_s^* = 0 \quad (s = 1, \ldots, m; \text{ where } l \neq p \neq s). \end{array}\right\} \qquad (1.4.31)$$

Thus, in order to determine vector ε^* for which the summary functional p_Σ^* possibly reaches its minimum, after calculation of the base points we must first solve the system of linear equations (1.4.29) and find the values of μ_i^* $(i = 0, 1, \ldots, m)$. If with these values of μ_i^* we produce $\phi_j^* < 0$, we must solve the system of linear equations (1.4.29) and (1.4.31) without equations (1.4.30), determining the new values of μ_j^*. In this last case, the minimum value of p_Σ^* is reached with certain limiting values of ε_j^*. After calculating μ_i^{opt*} for which the interpolation polynomial $F^{opt*}(\mu_0^{opt*}, \ldots, \mu_m^{opt*})$ with the limitations used reaches its minimum, we return to equation system (1.4.25) and determine ε_j^{opt*} $(j = 1, \ldots, m)$. This allows us to determine the values of vectors x_0^{opt*} and a^{opt*}, possibly corresponding to p_Σ^{opt*}.

Comparing p_Σ^{opt*} with F^{opt*}, we can judge the results of interpolation. When

$$|p_\Sigma^{opt*} - F^{opt*}| < c^{opt*} \qquad (1.4.32)$$

we will consider the interpolation satisfactory, i. e.

$$\varepsilon_j^{inf*} = \varepsilon_j^{opt*} \quad \text{and} \quad p_\Sigma^{inf*} = p_\Sigma^{opt*}$$

and therefore the task of the BMOSH is performed.

If inequality (1.4.32) is disrupted, then if there is at least one ε_j^{opt*} within the limit $0 < \varepsilon_j^{opt*} < \varepsilon_j^{max*}$, and with a value of p_Σ^{opt*} less than the maximum value of the summary functional at the base point $p_{\Sigma b}^{max*}$, then we go over once more to construction of the interpolation polynomial F and determination of its extreme, replacing the previous system of base points ε_{ij}^* $(j = 1, \ldots, m;\ i = 0, 1, \ldots, m)$ with a new system ε_{ji}^* $(j = 1, \ldots, m;\ i = 0, 1, \ldots, k-1, k+1, \ldots, m)$ and ε_j^{opt*}, in which the point ε_{jk}^{max*} with the value of $p_{\Sigma b}^{max*}$ is replaced by the point ε_j^{opt*} $(j = 1, \ldots m)$ with p_Σ^{opt}. This process can be considered cyclical, with fixed maximum number of cycles r. If after r cycles the inequality is still not satisfied, we take as the solution of ε_j^{inf*} the point ε_j^{min*} corresponding to the least value of the summary functional at the base points $p_{\Sigma b}^{min*}$.

In the case of disruption of inequality (1.4.32) and movement of all ε_j^{opt*} to their limiting values or with $p_\Sigma^{opt*} \geq p_{\Sigma b}^{max*}$, we accept as the solution of ε_j^{inf*} the point ε_j^{min*} corresponding to $p_{\Sigma b}^{min*}$.

In all case [illegible] related to disruption of inequality (1.4.32), a rare case is possible for which $\varepsilon_j^{min*} = \varepsilon_j^*$ ($j = 1, \ldots, m$), i. e. the base point with the minimum value of the functional corresponds to the initial point, found in the preceding iteration. Then we must accept $\varepsilon_j^{min*} = \varepsilon_j^{min*}$ as the solution.

Furthermore, in the process of performance of this algorithm, the situation may arise when system (1.4.29) or (1.4.29) and (1.4.30) will have no unique solution. The way out of this position can be found by an attempt to replace the old system of base points with a new system: $\bar{\varepsilon}_{j1}^* = \varepsilon_j^{min*}$, $\bar{\varepsilon}_{j(i+1)}^* = \varepsilon_{ji}^*$ ($j = i = 1, \ldots, m$).

These situations may be repeated with the maximum number of cycles not exceeding m. This position can be escaped by transferring each time from the preceding to a new system of base points

$$\bar{\varepsilon}_{j1}^* = \frac{1}{s} \varepsilon_j^{min*}, \quad \bar{\varepsilon}_{j(i+1)} = \varepsilon_{ji}^* \; (j, i = 1, \ldots, m),$$

where $s = 1, \ldots, m$ is the ordinal number of the cycle.

A flow chart of the section for multiparametric optimization is shown on Figure 1.25.

a)

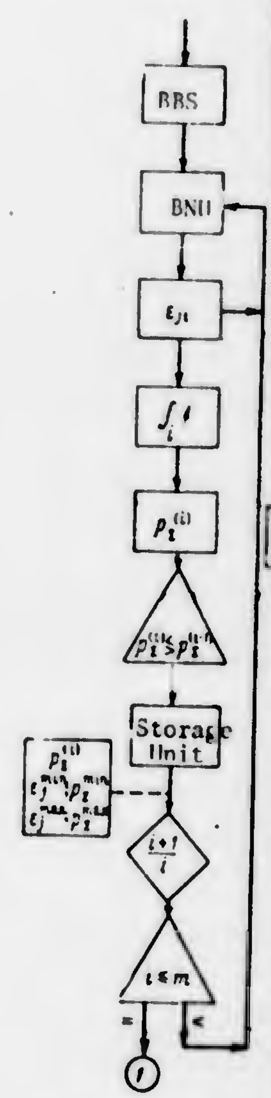

Figure 1.25. Fl
Optimization Sec

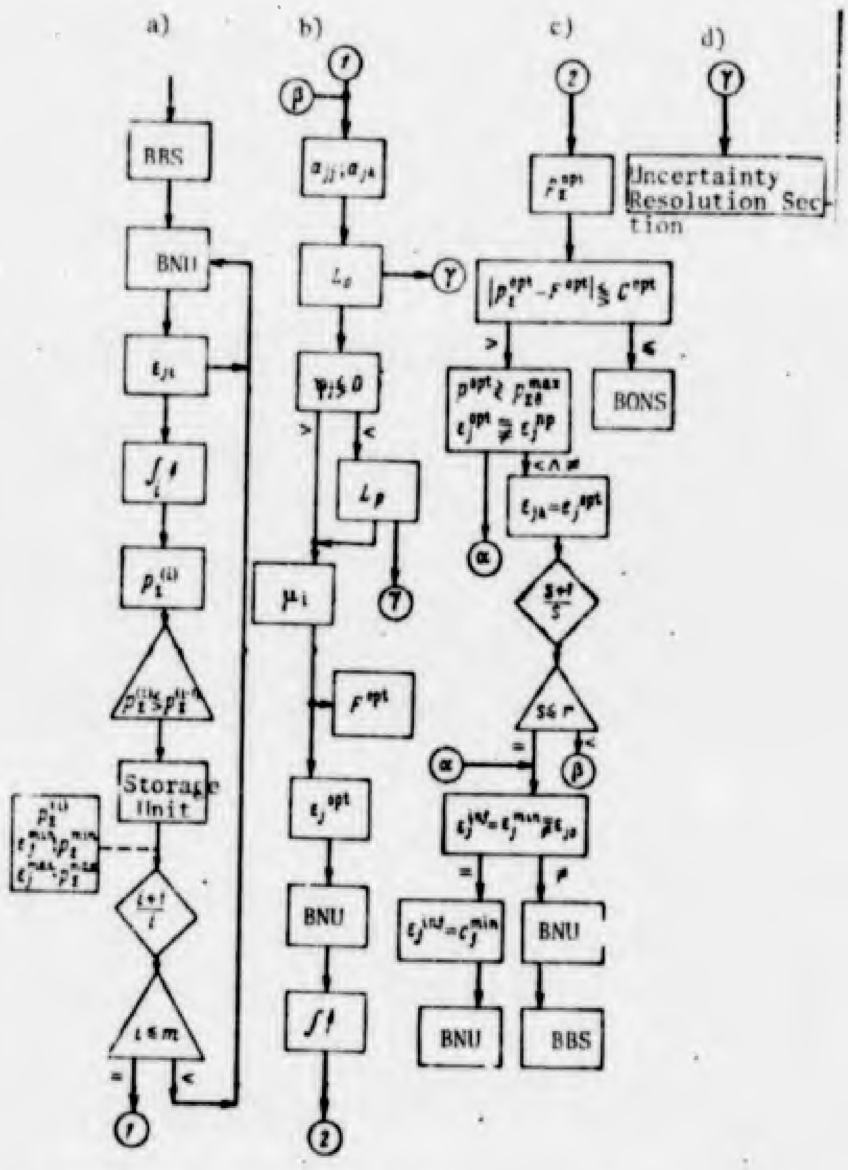

Figure 1.25. Flow Chart of Multiparametric Step Optimization Section.

APPENDIX TO CHAPTER 1

Variational Method of Optimal Planning of Single-Stage, Liquid-Fueled LRBM

This appendix is primarily designed as an illustration of the general theory presented in this chapter. However, it is also of independent interest due to the specifics of the problem studied.

As an example, we shall analyze the variational method of optimal planning of a single stage long range ballistic missile (LRBM) with a liquid-fueled motor, capable of delivering a known payload G_{pl} over a fixed range L_k with minimum launch weight G_0, i. e. the problem of optimizing the plan parameter vector Π, the vector of the parameter of design loads N^P, the vector of design stresses σ^P, the strength vector δ and the controls $p(t)$ and $\alpha(t)$.

Figure 1.26. Diagram of Missile: 1, power plant; 2, tail section; 3, fuel tank; 4, fuel section; 5, oxidizer tank; 6, warhead.

The solution of this problem consists of several stages. First, we select the structural plan of the LRBM and the power plant; in our case, we will consider this stage to have been completed. We then select the component elements of the vectors Π, N^P, σ^P and δ, the number of which is necessary for performance of the preliminary planning of the LRBM, then formulate the variational problem.

Suppose the structural plan of the missile is a "tank over tank" arrangement (Figure 1.26). Then the missile will consist of the following main parts: warhead, fuel section, motor section and tail section. The warhead contains the payload. The fuel section consists of the two tanks containing the main fuel components: fuel and oxidizer. The space between tanks is used to contain the control system apparatus. The power plant is placed in the tail sector.

Furthermore, let us assume that planning is performed considering the following requirements: the nose portion must separate from the main body of the rocket in flight; the fuel tanks are load-bearing structures with separating ends; the fuel feed system is a turbine pump arrangement; the motor contains four combustion chambers.

Considering the component parts of the missile, the structural formula for its launch weight is represented as

$$G_0 = G_{T\Sigma} + G_{np} + G'_{to} + G_{bt} + G_{xo} + G_{pp} + G_{ay} + G_{ue}.$$

Here

G_{np} is the weight of the nose portion;

G'_{to} is the weight of the tanks of main fuel components and related structural elements;

G_{bt} is the weight of the blow tank;

G_{xo} is the weight of the tail section;

G_{pp} is the weight of the power plant;

G_{ay} is the weight of the control apparatus;

G_{ue} is the weight of the unnamed elements.

The total weight of fuel placed in the tanks is

$$G_{T\Sigma} = G^{p}_{T\Sigma} + G_T^{(c)} + G_T^{(3)} + G_T^{(H)} + G_T^{(n)}.$$

The summary design fuel weight $G^{p}_{T\Sigma}$ is directly expended in accelerating the missile. The pre-launch fuel reserve $G_T^{(c)}$ is burned when the motor is started before liftoff of the missile from the launch platform. The portion of the fuel referred to as the fuel fill $G_T^{(3)}$ remains in the fuel lines after the motor is turned off. This portion is included in the final weight of the missile and in the weight of the power plant. Furthermore, there is always a guaranteed fuel reserve $G_T^{(g)}$, required in case of a deviation from the planned flight mode. It is also a part of the final weight of the missile and is included in the weight of the fuel section. The weight of the air cushion in the fuel tanks is evaluated by the weight of the gas blow $G_T^{(H)}$. It is also included in the final weight of the missile and considered part of the fuel section weight.

Thus, the structural formula of the final (passive) weight of the missile G_k can be expressed as follows:

$$G_{к}=G_{пр}+G_{э.у}+G'_{ωh}+G_{bt}+G_{т}^{(н)}+G_{т}^{(г)}+G_{х.о}+G_{ρρ}+G_{т}^{(з)}+G_{ue}.$$

then

$$G_0=G_{к}+G^{р}_{тз}+G_{т}^{(с)}.$$

From this, in relative quantities we produce

$$\mu_{пс}=\mu_{к}-\left[\frac{G_{э.у}}{G_{00}}\,\frac{G_0-G_{пр}}{G_0}+\frac{G_{ue}}{G_{п0}}\cdot\frac{G_0-G_{пр}}{G_0}+\right.$$
$$\left.+\frac{G'_{т.о}+G_{bt}+G_{т}^{(г)}+G_{т}^{(н)}}{G^{р}_{тз}}\,\frac{G^{р}_{тзр}}{G_0}+\frac{G_{х.о}}{G_0}+\frac{G_{пр}+G_{т}^{(з)}}{P^{п}_{max}}\,\frac{P^{п}_{max}}{G_0}\right], \tag{1}$$
$$\mu^{р}_{тз}=1-\mu_{к}-\overline{G}_{т}^{(с)},$$

or

$$\mu_{пр}=\frac{1}{1-\mu_{э.у}-\mu_{н.э}}\{\mu_{к}-[\mu_{э.у}+\mu_{ue}+a_{т.о}(1-\mu_{к}-\overline{G}_{т}^{(с)})+$$
$$+(\gamma_{ρρ}\overline{G}_{т}^{(з)})a_0+\mu_{х.о}]\}. \tag{2}$$

This equation (2) is the plan equation for a single stage liquid fuel missile.

In order to reduce this plan equation to "working" form, we must reveal the dependence of μ_{ue}, a_{tc}, γ_{pp}, μ_{xo}, $\overline{G}_t^{(c)}$, $\overline{G}_t^{(3)}$ on G_0, Π, N^P, σ^P and δ.

Ordinarily, $G_t^{(g)}$, $G_t^{(c)}$ and $G_t^{(3)}$ are estimated as fractions of the fuel flow rate per second G_{ts}, i. e.

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[1] For example, see [4, 18].

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[1] For example,

$$G_{\text{T}}^{(r)} = k_{\text{T}}^{(r)} G_{\text{T}s}, \quad G_{\text{T}}^{(c)} = k_{\text{T}}^{(c)} G_{\text{T}s}, \quad G_{\text{T}}^{(э)} = k_{\text{T}}^{(э)} G_{\text{T}s}.$$

In this connection, we have

$$\left.\begin{aligned} &\bar{G}_{\text{T}}^{(c)} = \frac{G_{\text{T}}^{(c)}}{G_0} = k_{\text{T}}^{(c)} \frac{a_0}{P_{sp}^{n}}, \qquad \bar{G}_{\text{T}}^{(э)} = \frac{G_{\text{T}}^{(э)}}{P^{n}} = \frac{k_{\text{T}}^{(э)}}{P_{sp}^{n}}, \\ &\bar{G}_{\text{T}}^{(r)} = \frac{G_{\text{T}}^{(r)}}{G_{\text{T}\Sigma}^{p}} = k_{\text{T}}^{(r)} \frac{a_0}{P_{sp}^{n} \mu_{\text{T}\Sigma}^{p}} \text{ or } \quad \bar{G}_{\text{T}}^{(r)} = \frac{k_{\text{T}}^{(r)}}{(1-\mu_{\text{k}}) \frac{P_{sp}^{n}}{a_0} - k_{\text{T}}^{(c)}}. \end{aligned}\right\} \quad (3)$$

The analytic dependence of γ_{pp} and a_{to} on the parameters of the fuel supply and motor system is related to the selection of arrangements of fuel tank and power plant systems, with the selection of the motor supply pumping system.

One of the important problems before a designer in the creation of a liquid fueled motor is the problem of selection of a motor system. His solution will to a great extent determine the future capabilities of the power plant. Planning is then reduced to determination of values of the power plant parameters which allow the best usage of the power and weight characteristics of the motor system selected. Determination of the analytic dependence of the weight characteristics γ_{pp} and power characteristics P and P_{sp} of the power plant on its parameters will be performed assuming the possibility of realization of highly economical power plant systems in which there are no losses of specific thrust related to movement of the working medium by the turbine. Closed system motors are of this type. Although there are two large groups of such systems [1], we will discuss only the system in which the turbine is supplied with working fluid from the main components, with the fluid fed beyond the turbine into the main combustion chamber, where the working medium participates in the combustion process together with the liquid components. A concrete system with precombustion of the generator gas was selected, according to which one of the components (oxidizer) moves from the pump to the gas generator completely, while the other component (fuel) is fed in a quantity providing the required gas temperature before the turbine of the turbine-pump unit. After the turbine, the working fluid is sent to the combustion chamber, to which the remaining portion of one of the components (fuel) is sent in a quantity so as to provide combustion at the selected component weight ratio.

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[1] For example, see [17].

In this connection, the principal units determining the weight of the power plant are the turbine-pump unit, gas generator, combustion chamber and nozzle, frame and fastening elements and accessory equipment.

Let us study the weight characteristics of the turbine-pump unit. The component elements of this unit are the pump and turbine. The pump includes the following main elements: input portion of pump (weight and dimensions determined primarily by flow rate of component through pump); vane wheel body with spiral casing and vane wheel plus shaft (dimensions and weight of these elements determined primarily by required pressure beyond pump and rotating speed); pump diffusor (dimensions and weight depending on pressure beyond pump and flow rate of component);

The dimensions and weight of the turbine unit are determined primarily by the temperature of the working medium at the input and the rotating speed of the pump. The temperature of the working fluid at the input is limited by the strength of the turbine blades and is generally fixed in advance. Therefore, the weight of the turbine unit is determined basically by the rotating speed of the pump. Thus, the weight of the turbine-pump unit is determined by the summary flow rate of fuel per second G_{ts}, the pressure in the combusion chamber p_k and the rotating speed of the turbine rotor end. It can be expressed as the following dependence:[1]

$$G_{\text{ТРЦ}} = G_{\text{ТРЦ}}(k_j^{\text{ТРЦ}}, G_{\text{тs}}, p_{\text{к}}, n, \gamma_{\text{т}}).$$

We then produce

$$\gamma_{\text{ТРЦ}} = \gamma_{\text{ТРЦ}}(k_j^{\text{ТРЦ}}, G_0, a_0, P^{\text{п}}_{sf}, p_{\text{к}}, n, \gamma_{\text{т}}).$$

On the basis of statistics and analysis of newly planned TPU for engines operating with precombustion, we can find the values of the coefficients k_j^{TPU} $(j = 1, \ldots, m)$.

The weight of the combusion chamber G_{cc} consists of the weight of the chamber head G_{cc}^{g} and the cylindrical portion (the combustion chamber is assumed cylindrical in form) with the input to the nozzle G_{cc}^{0}.

[1] For example, see [39].

During operation of the motor, the individual elements of the combution chamber are subjected to various loads. The internal and external shells of the chamber carry loads which are particularly varied in their nature. They are subjected to the effects of radial and axial hydrostatic pressures, thermal loads arising due to heating of the internal shell, dynamic and vibration loads, etc. Therefore, in designing liquid fuel motors for strength, the most difficult and least developed task is design of the shells and head of the chamber, due to the difficulty of detailed determination of the stresses arising at these points during operation. Since precise consideration of the loads is quite difficult, chamber shell design is usually performed somewhat approximately. The thickness of the chamber head depends on the pressure in the combustion chamber p_k.

In this connection, let us represent the weight of the combustion chamber as follows:

$$G_{cc} = G_{cc}\,(k_i^{cc},\ G_{\tau s},\ p_{\kappa},\ q_{cc},\ T_{cc},\ \tau_{cc},\ R_{cc}).$$

The values of the relative flow stress q_{cc}, time spent by the fuel in the chamber τ_{cc} depend primarily only on the components. Therefore, in planning a combustion chamber for known components, the parameters q_{cc} and τ_{cc} can be considered constant and fixed. Going over to the relative weight, we find

$$\gamma_{cc} = \gamma_{cc}\,(k_i^{cc},\ G_0,\ a_0,\ P_{sp}^{\circ},\ p_{\kappa},\ q_{cc},\ R_{cc},\ T_{cc},\ \tau_{cc}).$$

The possible values of static coefficients k_i^{cc} (i = 1,..., 1) are found on the basis of planning experience.

Using the assumptions made in estimating the weight of the combustion chamber, the weight of the gas generator G_{gg} can be represented as

$$G_{gg} = G_{gg}(k_i^{gg},\ G_{\tau s},\ p_{gg},\ q_{gg},\ R_{gg},\ T_{gg},\ \tau_{gg}).$$

Depending on the type of gas generator we will have:
an oxidative gas generator

$$G_{\Delta\varphi s} = \frac{(1 + k_{gg})\,k_{cc}}{(1 + k_{\varphi n})\,k_{gg}}\,G_{\tau s},$$

a reducing generator

Then for the oxidative generator

$$\gamma = \gamma\ (k_i^{??},\ G_0,\ a,\ P_{sp}^{\pi},\ p_{??},\ k_{??},\ k_{??},\ q_{??},\ T_{??},\ P_{??},\ \tau_{??}).$$

Let us now determine the parameters which determine the weight of the output portion of the motor nozzle G_c and its analytic dependence on them. We will assume the nozzle to be conical with fixed half aperture angle α_c. Generally speaking, the aperture angle of the output (supercritical) portion of the nozzle influences its weight significantly, as well as the nature of the process in the nozzle and the specific thrust of the engine. However, the factors determining the optimal value of the aperture angle of the expanding portion of the nozzle are quite contradictory and can be considered only experimentally. Therefore, α_c is assumed fixed from the experience of planning of nozzles.

The weight of the nozzle can be expressed as

$$G_c = \frac{\gamma_M^c \delta_c F_{kp}}{\sin \alpha_c} (f_a - 1).$$

From this we produce[1]

$$\gamma_c = k^c \frac{2\gamma_M^c 3/k\ D_{kp}}{\sigma_B^c \sin 2\alpha_c P_{sp}^{\pi}} (f_a - 1).$$

The weight of the frame and fastening depend only on the values of the maximum thrust and are expressed as[2]

$$G_p = k_1^p + k_2^p P_{max}^{\pi}.$$

All lines and valves of the motor installation are usually evaluated on the basis of the flow rate of fuel per second. It is

$$G_{ap} = k^{ap} G_{T s}.$$

Here k_1^p, k_2^p and k^{ap} are static coefficients.

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[1] For example, see [3].
[2] For example, see [4].

Then

$$\gamma_p = k_1^p \frac{1}{a_0 G_0} + k_2^p,$$

$$\gamma_{ap} = k^{as} \frac{1}{P_{sp}^{n}}.$$

Since

$$\gamma_{en} = \gamma_{TPU} + \gamma_{\varphi_1} + \gamma_{cc} + \gamma_{\tau} + \gamma_p + \gamma_{ap},$$

(4)

the specific weight of the motor γ_m with a fixed weight relationship of the main components is a function of the initial weight G_0, thrust to weight ratio a_0, pressure in the combustion chamber p_k, pressure at the output of the nozzle p_a and rotating speed of the turbine-pump unit n, i. e. we have

$$\gamma_{en} = \gamma_{be}(G_0, a_0, p_k, p_a, n). \tag{5}$$

The dependence just produced, of course, does not absolutely precisely determine the value of the specific weight of the planned motor and its accessory units, which depends to a certain extent on the quality of the design and the creative abilities of the designer, but physically quite properly reflects the change in specific weight (weight) of the motor as a function of the parameters used and can allow its estimation with a certain degree of accuracy.

Furthermore, the power and economy characteristics of a power plant depend on the values of p_k and p_a, since the specific thrust in a vacuum P^{π}_{sp} and the relative maximum thrust $p_h = P_H/P^{\pi}_{max}$ are respectively

$$P^{n}_{sp} = \varphi_c \varphi_p \varphi_{\omega a} \frac{2\beta_{th} k}{\sqrt{k^2 - 1}} \left(\frac{2}{k+1}\right)^{\frac{1}{k-1}} \times$$

$$\times \sqrt{1 - \left(\frac{p_a}{p_k}\right)^{\frac{k-1}{k}}} \left[1 + \frac{k-1}{2k} \frac{\left(\frac{p_a}{p_k}\right)^{\frac{k-1}{k}}}{1 - \left(\frac{p_a}{p_k}\right)^{\frac{k-1}{k}}}\right]. \tag{6}$$

$$p_h = \frac{P_H}{P^{\pi}_{max}} = 1 - \frac{f_a g}{p_k P^{\pi}_{y\pi}} \frac{p_{H0}}{p_k} \Delta_H, \tag{7}$$

where $\Delta_H = p_H/p_{H0}$, p_H and p_{H0} are the relative pressure of the atmosphere, pressure of the atmosphere at altitude H and H = 0. The analysis of these dependences indicates that with a certain increase in p_k the power and economy characteristics of the power plant improve.

However, transition to high p_k results in a near proportional increase in the thermal fluxes through the walls of the motor. They can be reduced, but this always results in losses in specific thrust, which may even result in a decrease in the effective specific thrust with high pressure in the combustion chamber.

The influence of p_a on the power and economy characteristics is more complex. Its estimation is influenced by the effect of the height of the nozzle. True, generally speaking, for ballistic missiles of the types we are studying a decrease in p_a has a favorable influence on the power and economy characteristics of the power plant. Consequently, the change in p_k, p_a, n and a_0 G_0 (or P^{π}_{max}) influences the power, economy and weight characteristics of the power plant.

The use of closed power plant systems (motors with precombustion) also allows effective utilization of high combustion chamber pressures p_k. Therefore, the planning of high pressure pumps with high efficiency and satisfactory weight characteristics allows us to achieve high p_k and consequently to create highly economical motors. Possibly, the planning of high speed turbine-pump units will be expedient, allowing the creation of pump units with low weight and small size. However, high speed pumps have low cavitation properties. They require higher pressure of components at the intake. High pressures at the intake to the pump unit may worsen the weight characteristics of the tank section. Thus, a change in the rotating speed of the pump unit n and pressure in the combustion chamber p_k influences not only the characteristics of the motor, but also the weight characteristics of the tank section. Therefore, the selection of n and p_k must be made in consideration of their influence on the weight characteristics of the tank section. A change in p_k is limited by the upper limit $p_{k\,max}$. It is determined by the difficulties of creating high pressure pumps and by thermal losses in the combustion chamber and nozzle. Usually, this limit is established experimentally and on the basis of preliminary analysis of power plants. Experience and preliminary analysis also allow us to estimate the upper limit for the change in rotating speed of the pump unit n_{max}.

Determinating of the pump unit rotating speed n, pressure in the combustion chamber p_k and fuel flow rate per second G_{ts} determines the pressure at the input to the oxidizer pump p^o_{BC} and the fuel pump p^g_{BC}. In order to assure operation of the pumps without cavitation, as they are planned the condition $p^o_{BC} \geq p^o_{BC\ min}$ and $p^g_{BC} \geq p^g_{BC\ min}$ must be maintained. However, any "excess" increase in pressure at the input to the pump may lead to worsening of the weight characteristics of the tank section. Therefore, the pressure at the input to the pumps should be selected from the condition $p^o_{BC} = p^o_{BC\ min}$ and $p^g_{BC} = p^g_{BC\ min}$. The pressures at the outputs of the pumps for the main components, determined from the condition of operation of the pumps without cavitation will be[1]

$$\left.\begin{aligned} p^{o}_{\text{вс}\,\min} &= p^{o}_{s} + m^{o}_{s} n^{4/3} G^{2/3}_{\text{т.о}} = p^{o}_{s} + m^{o}_{s} n^{4/3} \left(\frac{a_{0} G_{0}}{P^{\text{п}}_{G_t}}\right)^{2/3}, \\ p^{\text{г}}_{\text{вс}\,\min} &= p^{\text{г}}_{s} + m^{\text{г}}_{s} n^{4/3} G^{2/3}_{\text{т.г}} = p^{\text{г}}_{s} + m^{\text{г}}_{s} n^{4/3} \left(\frac{a_{0} i_{0}}{P^{\text{п}}_{t_f}}\right)^{2/3}, \end{aligned}\right\} \quad (8)$$

where

$$m^{o}_{s} = 10\gamma^{1/3}_{\text{т.о}} \left(\frac{1}{C_{\text{кр}}} \sqrt{\frac{k_{\text{ср}}}{1 + k_{\text{ср}}}}\right)^{4/3},$$

$$m^{\text{г}}_{s} = 10\gamma^{1/3}_{\text{тг}} \left(\frac{1}{C_{\text{кр}}} \sqrt{\frac{1}{1 + k_{\text{ср}}}}\right)^{4/3}.$$

For the pump units of liquid fueled motors, the critical cavitation factors C_{kp} are determined experimentally.

During flight of a missile, in order to assure operation of the pump unit without cavitation, the pressures at the inputs must correspond to the condition

$$p^{\text{г}}_{\text{вс}} \geqslant p^{\text{г}}_{\text{вс}\,\min}, \quad p^{o}_{\text{вс}} \geqslant p^{o}_{\text{вс}\,\min}. \quad (9)$$

Since the pressure at the input to the pumps (in the direction of increasing pressure) can be regulated only by blowing into the tanks of the main components, condition (9) in turn leads to a limitation on the

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[1] For example, see [25].

lower value of blowing pressure in the oxidizer tank p_{bo}^{min} and fuel tank p_{bg}^{min}. The creation of conditions under which the pumps will operate without cavitation is one of the purposes of blowing of the tanks. Therefore, during flight of a missile in order to retain operation of the pumps without cavitation throughout the entire trajectory, the following condition must be fulfilled:

$$\left.\begin{aligned} p_{b_g} &\geqslant p^{g}_{nc\,min} + \Delta p^{g} - n_{x1} h^{g}_{x} \gamma_{F_g} - \frac{\gamma_{f_g} C^{2}_{nx}}{2g_0}, \\ p_{b_o} &\geqslant p^{o}_{nc\,min} + \Delta p^{o} - n_{x1} h^{o}_{x} \gamma_{\tau\,o} - \frac{\gamma_{\tau,o} C^{2}_{nx}}{2g_0}, \end{aligned}\right\} \qquad (10)$$

where C_{BX} is the velocity of a component as the input to the pump.

In many cases, in order to decrease the values of p_{bo} and p_{bg} without worsening the operation of the pump and without reducing its rotating speed, the required minimum pressure at the input of the pump is achieved by using special devices before the pump, increasing the pressure before the intake to the main pump. These devices are usually liquid ejectors or supplementary pumps[1]. If we allow the possibility of the use of liquid ejector-type devices before the pumps in our power plant, we must estimate the weight of these devices. At the present time, sufficient experimental data have not been accumulated to produce acceptible statistical factors for estimation of the weight characteristics of pre-pump ejectors. However, we can assume that the weight of the ejectors depends on their characteristic parameters G_{ts}, $p^{0}_{BC\,min}$, $p^{g}_{BC\,min}$, π_{ej} and p_k, i. e.

$$G_{e_j} = G_{e_j}(G_{\tau s}, p_k, \pi^{o}_{e_j}, \pi^{g}_{e_j}, p^{o}_{nc\,min}, p^{g}_{nc\,min}).$$

Then

$$\gamma_{e_j} = \gamma_{e_j}(a_0 G_0, p^{n}_{s\rho}, p_k, \pi^{o}_{e_j}, \pi^{g}_{e_j}, p^{o}_{nc\,min}, p^{r}_{nc\,min}).$$

In this connection, the following conditions should be fulfilled of the input to the ejectors

$$p^{o}_{e_j} \geqslant \frac{p^{o}_{nc\,min}}{\pi^{o}_{e_j}}, \quad p^{g}_{e_j} \geqslant \frac{p^{g}_{nc\,min}}{\pi^{g}_{e_j}}.$$

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[1] For example, see [25, 39].

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and therefore, the following conditions should also be observed:

$$\left.\begin{aligned} p_{б.о} \geqslant \frac{p^{о}_{вс\ min}}{\pi^{о}_{e_j}} + \Delta p^{о} - n_{x1} h^{о}_{x} \gamma_{т.о}, \\ p_{б.г} \geqslant \frac{p^{г}_{вс\ min}}{\pi^{г}_{e_j}} + \Delta p^{г} - n_{x1} h^{г}_{x} \gamma_{т.г}. \end{aligned}\right\} \qquad (11)$$

The problems arising in the planning of a nozzle are related to the selection of pressure at the nozzle cross section p_a. For example, in selecting a planned motor it must be kept in mind that a decrease in p_a leads to an increase in the size and therefore weight of the nozzle, and the thrust of the engine at the surface and at low altitudes will be significantly decreased; increasing p_a leads to a decrease in the size of the nozzle and thereby decreases its weight, but the increase in thrust with increasing altitude is less, resulting in relatively low thrust at high altitude.

Furthermore, we must keep in mind that when the gases in the nozzle over-expand to a pressure below $p_a < p_{a\ min}$, the gas flow will separate from the walls of the nozzle due to the appearance of compression jumps, significantly decreasing the specific thrust of the motor and possibly even leading to unstable operation[1]. Preliminary analysis shows that p_a has an upper limit $p_{a\ max}$, the value of which is generally less than atmospheric pressure at sea level. Thus, the change in p_a must be limited to

$$p_{a\ min} \leqslant p_a \leqslant p_{a\ max}. \qquad (12)$$

Using experimental data and preliminary calculations[2], we can assume

$$p_{a\ min} = 0{,}29 p_{H_0},\quad p_{a\ max} = 0{,}8 p_{H_0}. \qquad (13)$$

Planning of the nozzle should be performed considering the fact that the summary area of the output cross section of the nozzle F_a

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[1] For example, see [3, 17].
[2] For example, see [3].

should not exceed the mid-ship section of the missile. Therefore, the selection of p_a and p_k (more accurately of the ratio p_a/p_k) is limited also by the following:

$$\leq 0{,}25S. \qquad (14)$$

Production of an analytic expression a_{to} allows us not only to perform an estimate of the weight of the fuel sector, but also to determine the input parameters characterizing its dimensions. The structural formula of the weight of the fuel sector G_{to} can be represented as follows:

$$G_{т.о}=G_{б.г}+G_{б.о}+G_{б}+G_{ар.б}+G_{к}+G_{т}^{(и)}+G_{т}^{(г)}.$$

We usually assume

$$G_{ар.б}=k^{ар.б}G_{тз}^{о},$$
$$G_{б}=k^{б}G_{об.б}.$$

Then

$$a_{ар.б}=k_{б}^{ар.б},\quad a=k^{б}a_{б}.$$

The relationship between the volumes of tanks of the main components and the quantity of fuel placed in the tanks is usually expressed in the form [1]

$$V_{б}^{о}=k_{V}^{о}\frac{G_{тз}^{о}}{\gamma_{т.о}},\quad V_{б}^{г}=k_{V}^{г}\frac{G_{тз}^{г}}{\gamma_{т.г}},$$

where k_V^o, k_V^g are empirical coefficients.

Generally speaking, the volume of the tanks of the main components is

$$V_{б}^{о}=\frac{\pi D^2}{4}l_{б}^{о}-\Delta V_{б}^{о},$$

$$V_{б}^{г}=\frac{\pi D^2}{4}l_{б}^{г}-\Delta V_{б}^{г},$$

[1] For example, see [4].

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where ΔV_t^o, ΔV_t^g represent the difference between the volume of the cylindrical bases and their actual volumes in the oxidizer and fuel tanks.

Suppose the base forms of the oxidizer and fuel tanks are planned identical and depend only on D, i. e.

$$\Delta V_t^o = \Delta V_t^g = 2\Delta V_b.$$

Then

$$V_t^o = \frac{\pi D^2}{4} l_t^o - 2\Delta V_b,$$

$$V_t^g = \frac{\pi D^2}{4} l_t^g - 2\Delta V_b.$$

Furthermore, using the preceding dependences, we can write

$$G_{\tau z}^o = k_z^o \frac{k_{gn}}{1 + k_{gn}} G_{\tau z}^p,$$

$$G_{\tau z}^g = k_z^g \frac{G_{\tau z}^p}{1 + k_{gn}}.$$

Here

$$k_z^o = 1 + \left(k_\tau^{(y)} + k_\tau^{(c)} + k_\tau^{(s)}\right) \frac{(1 + k_{gn}) a_0}{k_{cr} P_{sp}^n \left(1 - \mu_k - \bar{G}_\tau^{(c)}\right)},$$

$$k_z^g = 1 + \left(k_\tau^{(y)} + k_\tau^{(c)} + k_\tau^{(s)}\right) \frac{(1 + k_{gn}) a_0}{P_{sp}^n \left(1 - \mu_k - \bar{G}_\tau^{(c)}\right)}.$$

Inconnection with this, we have

$$V_b^o = k_p^o G_{\tau z}^p, \quad V_b^g = k_p^g G_{\tau z}^p,$$

where

$$k_p^o = k_V^o k_z^o \frac{k_{gn}}{(1 + k_{gn}) \gamma_{\tau o}},$$

$$k_p^g = k_V^g k_z^g \frac{1}{(1 + k_{gn}) \gamma_{\tau g}}.$$

Keeping in mind the relationships produced, we find

$$\frac{\pi D^2}{4}\left(l_t - \frac{16\Delta V_{b}}{\pi D^2}\right) = k_p (1 + \chi) G_{\tau z}^p$$

or

$$\frac{\pi D^3}{4G_0}\left(\bar{l}_b - \frac{16\Delta V_{bt}}{\pi D^3}\right) = k_p^g(1+\chi)(1-\mu_k-\bar{G}_t^{(c)}), \tag{15}$$

where

$$l_t^o = \left[\chi l_b + \frac{8\Delta V_{bt}}{\pi D^2}(1-\chi)\right]\frac{1}{1+\chi},$$

$$l_t^g = \left[l_b - \frac{8\Delta V_{bt}}{\pi D^2}(1-\chi)\right]\frac{1}{1+\chi},$$

where

$$\chi = \frac{k_p^o}{k_p^g}.$$

The weight of the shells of the tanks of the main components will be

$$G_b = G_{bo} + G_{bg} = \gamma_m^{bo}\delta_o\left[2k_{bt}^o\frac{\pi D^2}{4} + 2(k_{bt}^o-1)\pi D h_{bt}^o + \pi D l_b^o\right] + \gamma_m^{bg}\delta_g\left[2k_{bt}^g\frac{\pi D^2}{4} + 2(k_{bt}^g-1)\pi D h_{bt}^g + \pi D l_b^g\right].$$

Here and in the following is assumed that the thickness of the shell of the oxidizer tank δ_o and the thickness of the shell of the fuel tank δ_g are identical. Furthermore, from the condition of placement and collection of fuel we will consider the relative height of the base $\bar{h}_b^o = \bar{h}_b^o/D$ and $\bar{h}_b^g = \bar{h}_b^g/D$ fixed.

Then, keeping in mind the identical form of the bases, we produce

$$G_b = \frac{\pi D^2}{2}(\gamma_m^{bo}\delta_o + \gamma_m^{bg}\delta_g)\left[k_{bt} + 4(k_{bt}-1)\bar{h}_{bt}\right] + \pi D(\gamma_m^{bo}\delta_o l_b^o + \gamma_m^{bg}\delta_g l_b^g)$$

or, considering the values of l_t^o and l_t^g, we find

$$G_b = \frac{\pi D^2}{4}\left\{(\gamma_m^{bo}\delta_o + \gamma_m^{bg}\delta_g)\left[k_{bt} + 4(k_{bt}-1)\bar{h}_{bt}\right] + \frac{2}{1+\chi}\left[(\chi\gamma_m^{bo}\delta_o + \gamma_m^{bg}\delta_g)\bar{l}_b - \frac{8\Delta V_{bt}}{\pi D^3}(1-\chi)(\gamma_m^{bg}\delta_g - \gamma_m^{bo}\delta_o)\right]\right\}.$$

From this we have

$$a_{b}=\frac{\pi D^{2}}{2G_{0}(1-\mu_{\kappa}-\bar{G}_{\tau}^{(c)})}\Big\{(\gamma_{M}^{bo}\delta_{0}+\gamma_{M}^{bg}\delta_{g})C_{bt}+$$

$$+\frac{2}{1+\chi}\Big[(\chi\gamma_{M}^{bo}\delta_{0}+\gamma_{M}^{bg}\delta_{g})\bar{l}_{b}-\frac{8\Delta V_{bt}}{\pi D^{3}}(1-\chi)(\gamma_{M}^{bg}\delta_{g}-\gamma_{M}^{bo}\delta_{0})\Big]\Big\} \qquad (16)$$

or, considering (15)

$$a_{b}=\frac{2k_{p}^{g}(1+\chi)}{D\left(\bar{l}_{b}-\frac{16\Delta V_{bt}}{\pi D^{3}}\right)}\Big\{(\gamma_{M}^{bo}\delta_{0}+\gamma_{M}^{bg}\delta_{g})C_{bt}+$$

$$+\frac{2}{1+\chi}\Big[(\chi\gamma_{M}^{bo}\delta_{0}+\gamma_{M}^{bg}\delta_{g})\bar{l}_{b}-\frac{8\Delta V_{bt}}{\pi D^{3}}(1-\chi)(\gamma_{M}^{bg}\delta_{g}-\gamma_{M}^{bo}\delta_{0})\Big]\Big\}. \qquad (17)$$

where

$$C_{bt}=k_{bt}+4(k_{bt}-1)\bar{h}_{ct}$$

In order to compose the structural formula of the weight of the blow system, let us use the following tank blowing plan: the fuel tank is blown through a reducer and air pressure accumulator, while the oxidizer tank is blown by gas taken from the gas generator. Therefore, the weight of the blowing system will consist of the weight of the gas and the blowing cylinder for the fuel tank plus the weight of accessory equipment. We therefore produce

$$a_{bt}+a_{\tau}^{(н)}=c_{б}^{н}\tilde{V}_{б}^{н},$$

where

$$c_{б}^{н}=\left[k_{б}^{н}\frac{\gamma_{M}^{б.н}p_{0}^{н}f_{н.б}^{н}}{\sigma_{в.t}^{н}}\eta_{б}^{н}\left(1+\frac{R_{н}}{R_{н}-r_{н}}\right)+\frac{p_{0}^{н}}{RT_{0}^{н}}\right]. \qquad (18)$$

$$\tilde{V}_{б}^{н}=\frac{V_{б}^{t}}{G_{\tau г}^{p}}=\frac{k_{p}^{g}p_{б.g}^{(н)}}{(1-k)}\Big\{p_{0}^{н}-(p_{б.g}^{(н)}+\Delta p_{red})-k_{ct}\frac{273\chi(p_{0}^{н})^{3k}}{(T_{0}^{н})^{3}}\times$$

$$\times\Big[\frac{(p_{0}^{н})^{2-3k}-(p_{б.g}^{(н)}+\Delta p_{red})^{2-3k}}{2-3k}-\frac{p_{б.g}^{(н)}}{1-3k}[(p_{0}^{н})^{1-3k}-(p_{б.g}^{(н)}+\Delta p_{red})]\Big]\Big\}^{-1}. \qquad (19)$$

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[1] The volume of compressed gas (volume of the blow cylinder) $V_{t}^{(H)}$ is determined by transforming the state equations and energy equation, written in the form

At the end of the powered flight sector, the blow pressure $p_{to}^{(k)}$ and $p_{tg}^{(k)}$ are fixed so as to provide cavitation-free operation of the pumps. Therefore, we can assume

$$p^{(к)}_{б.г} = p^{н}_{к\,min} + \Delta p^{н} - \frac{a_0}{\mu_к} h^{н}_{з.к} \gamma_{т.г}$$

since

$$p^{(к)}_{б.г} = p^{пр}_{б.г}.$$

$$(20)$$

The values of pressure p_o^H and temperature T_o^H are usually fixed on the basis of planning experience.

Thus, we produce

$$a_{т.о} = c_{ср.б} + (1 + c_б)\mu_б + c_б^н \bar{V}_б^н + \frac{h_т^{(н)}}{\frac{1-\mu_к}{a_0} p^н_{ср} - h_т^{(с)}}, \qquad (21)$$

where

$$c_{ср.б} = k^{ср.б}, \quad c_б = k^б.$$

$$G^{bg}_{тg} T^{(к)}_{tg} = \int_{G^н_{gк}}^{G^н_{g0}} (T - \Delta T)\, dG,$$

where G_{tg}^{bg} is the weight of the blow gas in the fuel tank at the end of operation;

$G_{g0}^{(h)}$, $G_{gk}^{(h)}$ is the weight of compressed gas in the blow tank at the beginning and end of operation;

$T_{bg}^{(k)}$ is the temperature of the blow gas in the fuel tank at the beginning of operation;

T is the instantaneous gas temperature before the reducer.

In order to estimate the change in temperature Δt occuring upon transition of the gas through the choking device due to the Joule effect, we used an equation similar to that used by Linde [3]:

$$\Delta T = k_{zh} \left(p - p^{(к)}_{тg}\right) \left(\frac{273}{T}\right)^2.$$

The weight of the control apparatus G_{ay} consists of the weight of the control organs (steering wheel, steering machines, trimmers), the weight of the control devices (on board electric power supplies, timing mechanisms) and the weight of the instrument section itself. Generally speaking, G_{ay} can be estimated in preliminary planning only on the basis of statistical data.

It is usually assumed that

$$G_{a.y} = k_{a.y} G_{00}$$

or

$$\mu_{a.y} = k_{a.y}, \tag{22}$$

where K_{ay} is the statistical coefficient, depending on the fixed flight range.

The nose portion consists of the payload, shell, payload heat insulation, external heat-insulating cover, etc. The weight of the nose portion includes the weight of the devices used to retain it, separate it and stabilize it. If due to technical conditions the total weight of the nose portion G_{np} is not fixed in planning, but rather only the weight of the payload G_{pl}, usually the total weight of the nose portion is determined in preliminary planning from the relationship

$$G_{np} = k_{pl} G_{pl},$$

where k_{pl} is a static coefficient depending on the fixed flight range and design factors.

To achieve some simplification in the presentation of the solution of this problem, we will estimate the weight of the nose portion using this dependence. Then

$$\mu_{np} = k_{pl} \mu_{pl}. \tag{23}$$

In estimating the weight of the tail section of the missile G_{xo}, it should be kept in mind that the design case for the tail section is the launch [25], when the body receives the wind load and the weight of the missile. Furthermore, it must be kept in mind that in missiles having no stabilizing devices, the body of the tail section is usually made cylindrical or near cylindrical. Then the weight of the tail section can be assumed equal to

$$G_{x.o} = k^{x.o}\gamma_{M}^{x.o}\pi D\delta_{x.o} l_{x.o}.$$

The length of the body of the tail section can be estimated as follows:

$$l_{x.o} = l_{x.o}^{sup} + D_{kp}(\sqrt{f_a} - 1)\,\mathrm{ctg}\,\alpha_c,$$

where l_{xo}^{sup} is the portion of the length of the body of the tail section to the nozzle.

The calculation of δ_{xo} and l_{xo}^{sup} is very difficult. The values of these parameters can be more properly selected on the basis of planning experience using known prototypes. We then produce

$$\mu_{x.o} = c_{x.o}\left[1 + \frac{D_{kp}}{l_{x.o}^{sup}}(\sqrt{f_a} - 1)\,\mathrm{ctg}\,\alpha_c\right]\frac{D}{G_0}, \tag{24}$$

where

$$c_{x.o} = k^{x.o}\gamma_{M}^{x.o}\pi\delta_{x.o} l_{x.o}^{sup}. \tag{25}$$

The weight of the unnamed elements G_{ue} includes the weight of fairings, haches, guards, paint, etc. G_{ue} can only be estimated after design development of the units of the missile. Therefore, we will calculate G_{ue} on the basis of statistical data as a fraction of G_{00}:

$$G_{ue} = k_{ue}G_{00}.$$

Then

$$\mu_{ue} = k_{ue}.$$

Dependences (3, 5, 7, 19 and 21-24) allow us to write the plan equation for a single stage missile with liquid fueled engine in explicit form:

$$\mu_{pl} = \frac{1}{k\,(1 - \mu_{a.y} - \mu_{n.y})}\Big\{\mu_k - \mu_{a.y} - \mu_{ue} - \gamma_{ue}a_0 - \left(1 - \mu_k - k_T^{(c)}\frac{a_0}{P_{sp}^{n}}\right)\left\{\frac{k_T^{(g)}}{\frac{1-\mu_k}{a_0}P_{sp}^{n} - k_T^{(c)}} + c_{ap.\delta} + \right. \tag{26}$$

$$+(1+c_б'')\frac{2k_p^{\theta}(1+\chi)}{\left(\bar{l}_б-\frac{16\Delta V_{бс}}{\pi D^3}\right)D}\left[(\gamma_м^{б.о}\delta_o+\gamma_м^{б.г}\delta_г)c_{бс}+\right.$$

$$+\frac{2}{1+\chi}\left[(\chi\gamma_м^{б.о}\delta_0+\gamma_м^{б.г}\delta_g)\bar{l}_б-\frac{8\Delta V_{бг}}{\pi D^3}(1-\chi)\times\right.$$

$$\left.\left.\times(\gamma_м^{б.г}\delta_г-\gamma_м^{б.о}\delta_0)\right]\right]+c_б''\bar{V}_б''\Big\}-c_{х.п}\left[1-\frac{D_{кр}}{I_{х.п}^{пуст}}(1\ \bar{J}_a-1)\operatorname{ctg}\alpha_c\right]\frac{D}{G_0}\Big\}. \quad (26)$$

Thus, the plan equation in general form can be represented as:

$$\alpha_{pt}=f_{пр}(G_0,\ \mu_к,\ a_0,\ p_к,\ p_a,\ n,\ D,\ \delta_0,\ \delta_g,\ \bar{l}_б),$$

i. e. the plan equation includes the parameters G_0, μ_k, a_0, p_k, p_a, n, D, δ_0, δ_g and $\bar{I}_t$.

According to the terminology used in § 1, they can be classified as follows:

G_0, μ_k, a_0, p_k, p_a, n, D, $\bar{I}_t$ are component elements of the plan parameter vector Π;

δ_g, δ_0 are component elements of the strength vector δ.

According to the terminology of this same paragraph, we see that the vectors of the initial weight of the carrier, relative dry weight and thrust to weight ratio are degenerated to the scalars G_0, μ_k and a_0, while the component elements of the vector of design characteristics of thermodynamic parameters of the power plant are only p_k, p_a, n and the geometric vector D and $\bar{I}_t$. One specific feature of these parameters is that with fixed component constants $c_{ap\cdot t}$, $c_b, \ldots, c_{xo}$, they determine not only the weight but also the geometric characteristics of the units and of the missile as a whole. Therefore, fixation of G_0, μ_k, a_0, p_k, p_a, n, D, $\bar{I}_t$ and the component constants $c_{ap\cdot t}$, $c_b, \ldots, c_{xo}$ allows us to perform full preliminary planning of the missile once the design and power system have been fixed.

In our problem the power units determining the design cases of loading of the missile in flight are the tanks of the main components (fuel and oxidizer tanks). This results from the fact that the tanks of the main components are used as a load-bearing body, receiving the forces acting on the missile. For other elements -- the tail section, engine frame, etc. (except for the nose portion), the design loading cases generally are not related to the effects of loads on the missile in flight, which is reflected on the composition of the plan equation parameters. Possible design cases of loading of the nose portion, as planning experience has shown, can be determined only during the flight of the nose after separation in the passive sector. Due to the difficulties of calculation arising as a result of difficulties in precise calculation of the temperature fields of the shell of the nose portion, it is assumed that the nose withstands the loads acting on it, and its weight characteristics are estimated by the approximate method indicated earlier. Furthermore, this assumption allows us to simplify somewhat the presentation of the solution of our problem. This assumption introduces no errors in principle to the results of the investigation, while the quantitative error which does arise hardly influences the summary error of the plan equation, resulting from statistical and other coefficients.

Generally speaking, the fuel tanks should be designed for strength in several sections. Generally, the most heavily loaded sections of the tank are the sectors where the base connects to the cylindrical portion[1]. Therefore, we will use the cross sections A-A and B-B (Figure 1.27) at the junction points between the cylindrical shell and base as the design sections. Also, we will analyze tanks of equal thickness, determined by the thickness of the shell in the design cross sections δ_g and δ_o. Thus, in this example the variable parameter is strength vector δ, or more accurately its components δ_g and δ_o, not the design load vector N^p.

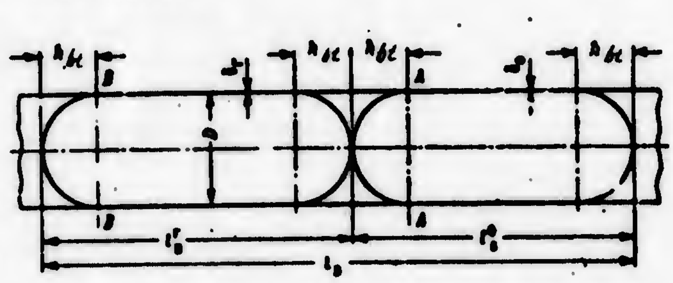

Figure 1.27. Schematic Diagram of Fuel and Oxidizer Tanks

We will calculate the tanks for the rupture loads. In connection with this, the design stress vector with fixed temperature will be a known quantity. Its components are

[1] For example, see [5, 35].

$$\sigma^{p}_{б.о} = \frac{\sigma^{б.о}_{вр}}{\eta_1^{б.о}}, \quad \sigma^{p}_{б.г} = \frac{\sigma^{б.г}_{вр}}{\eta_1^{б.г}}. \tag{27}$$

During the flight of a missile, the following conditions should be fulfilled

$$\left.\begin{array}{l} \delta_g \geqslant f^{г}_{5f}\delta^{п}_{г}, \\ \delta_o \geqslant f^{о}_{5f}\delta^{п}_{о}, \end{array}\right\} \tag{28}$$

where δ^{π}_{g}, δ^{π}_{o} are the actual thickness of the fuel and oxidizer tank skins respectively.

Fulfillment of these conditions with fixed loading of the missile is related to a certa n extent to the selection of the blow pressures for the fuel tank p_{tg} and oxidizer tank p_{to}, the changes in which are limited by the lower boundary determined from the condition of cavitation-free operation of the pump units

$$p_{б.г} = p^{min}_{б.г} = p^{г}_{вх\,min} + \Delta p^{г} - n_{x1} h^{г}_{x} \gamma_{т.г} - \frac{\gamma_{т.г} c^{2}_{вх}}{2g_0},$$

$$p_{б.о} = p^{min}_{б.о} = p^{о}_{вх\,min} + \Delta p^{о} - n_{x1} h^{о}_{x} \gamma_{т.о} - \frac{\gamma_{т.о} c^{2}_{вх}}{2g_0}.$$

or from the condition of stability of the envelope with the meridianal compressive stress

$$p_{б.г} = p^{кр}_{б.г}, \quad p_{б.о} = p^{кр}_{б.о}.$$

It should be kept in mind here that the actual thickness of the fuel and oxidizer tank shells at any point on the phase trajectory will be minimal if p_{tg}, p_{to} and δ_g, δ_o are determined for $p^{kp}_{t} \geq p^{min}_{t}$ from the conditions

$$\left.\begin{array}{l} \eta_1^{б.г}\sigma^{p}_{б.г} = \sigma^{б.г}_{вр} \text{ и } \eta_2^{б.г}\sigma_{mg} = \sigma^{б.г}_{кр}, \\ \eta_1^{б.о}\sigma^{p}_{б.о} = \sigma^{б.о}_{вр} \text{ и } \eta_2^{б.о}\sigma_{mo} = \sigma^{б.о}_{кр}, \end{array}\right\} \tag{29}$$

where σ^{e}_{mg}, σ^{e}_{mo} are the meridianal stresses experienced in the calculated cross section of the fuel (or oxidizer) tank.

Therefore, fulfillment of conditions (28) must be checked assuming

where $p_б^{кр} \geqslant p_б^{min}$ $p_б = p_б^{кр}$ $(p_б = \{p_{б.г}, p_{б.о}\},\ p_б^{кр} = \{p_{б.г}^{кр}, p_{б.о}^{кр}\})$

where $p_б^{min} > p_б^{кр}$ $p_б = p_б^{min}$ $(p_б^{min} = \{p_{б.г}^{min}, p_{б.о}^{min}\})$.

It is important to consider that when the stronger condition (28) is fulfilled with the values of p_t selected, the instantaneous value of blow pressure p_t can be selected at any level in the range $p_t^{max} \leq p_t \leq p_t^{kp}$ or $p_t^{kp} \geq p_t^{min}$, or $p_t^{max} \leq p_t \leq p_t^{min}$ where $p_t^{min} > p_t^{kp}$, and p_{to}^{max} and p_{tg}^{max} are determined from equations (27). This freedom in the selection of p_t allows us to determine the simplest realization of $p_t(t)$.

Conditions (28) can be replaced with the equivalent conditons

$$\left.\begin{aligned} \frac{\sigma_{m\,f}^{б.г}}{\eta_1^{б.г}} &= \sigma_{б.г}^{p} > \sigma_{кр\,г.}^{н} \\ \frac{\sigma_{m\,f}^{б.о}}{\eta_1^{б.о}} &= \sigma_{б.о}^{p} > \sigma_{кр\,о.}^{н} \end{aligned}\right\} \qquad (30)$$

Let us determine the stresses acting in the shell of a cylindrical tank loaded simultaneously with the excess summary pressure p^p, axial forces N and bending moment M_{bnd}, using the momentless theory of shells.

The meridianal stress is

$$\sigma_m = \frac{p_б^p D}{4\delta} - \frac{N}{\pi D\delta} \pm \frac{4M_{bnd}}{\pi D^2\delta}.^{1)}$$

The circular stress, determined from the Laplace equation is

$$\sigma_t = \frac{p^p D}{2\delta},$$

where

$$p^p = f_{Sf}(p_б - p_н + \gamma_т h' n_{x1}).$$

---

[1] In this and the following equations the indices representing fuel and oxidizer tanks are omitted for brevity.

In a shell loaded with excess pressure the circular stress is always positive ($\sigma_\tau > 0$). The meridianal stress, depending on the magnitude and sign of the bending moment and axial force, may be tensile or compressive.

Since the critical compressive stress σ_{kp} determining the condition of stability of the shell is

$$\sigma_{\kappa p} = k_\sigma \frac{E_t \delta}{D},$$

from the equation $\sigma_{kp} = \eta_2 \sigma_m^e = \eta_2 \sigma_m^\pi$ we produce

$$p_6^{\kappa p} = \frac{4}{\pi D^2}\left[\left(N \pm \frac{4M_{\text{бал}}}{D}\right) - k_\sigma \frac{\pi E_t}{\eta_2^6} \delta^2\right],$$

where k_σ is the experimental critical compressive stress coefficient.

If with the given loading condition of the missile $p_t^{kp} \geq p_t^{min}$, then, by using the hypothesis of greatest tangent stresses, we can determine the actual equivalent stress in the form

$$\sigma_{\text{экв}}^{\pi} = \sigma_\tau^{\pi} - \sigma_m^{\pi},$$

or

$$\sigma_{\text{экв}}^{\pi} = \frac{2}{\pi D \delta}\left[N \pm \frac{4M_{\text{бал}}}{D} + f_{\text{экв}}^6 \frac{\pi D^2}{4}(\gamma_\tau h^\pi n_{x1} - p_n) - k_\sigma \frac{f_{\text{экв}}^6 \pi E_t}{2\eta_2^6} \delta^2\right].$$

Then condition (30) becomes

$$\sigma_{\text{в}\tau}^{6.\varphi} - \frac{2\eta_1^{6.\varphi}}{\pi D \delta_\varphi}\left[N_\varphi \pm \frac{4M_{\text{бал}\,\varphi}}{D} + f_{\text{экв}}^{6.\varphi} \frac{\pi D^2}{4}(\gamma_{\tau.\varphi} h^\varphi n_{x1} - p_n) - k_\sigma \frac{f_{\text{экв}}^{6.\varphi} \pi E_t^{6.\varphi}}{2\eta_2^{6.\varphi}} \delta_\varphi^2\right] \geq 0,$$

$$\sigma_{\text{в}\tau}^{6.o} - \frac{2\eta_1^{6.o}}{\pi D \delta_o}\left[N_o \pm \frac{4M_{\text{бал}\,o}}{D} + f_{\text{экв}}^{6.o} \frac{\pi D^2}{4}(\gamma_{\tau.o} h^o n_{x1} - p_n) - k_\sigma \frac{f_{\text{экв}}^{6.o} \pi E_t^{6.o}}{2\eta_2^{6.o}} \delta_o^2\right] \geq 0$$

or

$$\dot{\sum_i} \geq N_{\text{пр}\,\varphi}$$

$$\dot{\sum_i} \geq N_{\text{пр}\,o}.$$

where $\Sigma_t = \frac{\pi D^2}{2} \frac{\sigma_{Bt}}{\eta_1}\left(1 + k_1 \frac{f_{rf}^{A} E_t^{t} \eta_1^{A}}{\sigma_{B}^{t} \eta_1^{B} D}\right)$ is the strength indicator of a tank considering the influence of temperature;

$\Sigma_t^{g}, \Sigma_t^{o}$ are the strength indicators of the fuel and oxidizer tanks;

$N_{\pi p} = N \pm \frac{4M_{bnd}}{D} + f_{vf}^{A} \frac{\pi D^2}{4}(\gamma_r h^r n_{x1} - p_0)$ is the actual equivalent load in the design cross section of the tanks;

$N_{\pi p g}, N_{\pi p o}$ is the actual equivalent load in the design cross section of the fuel and oxidizer tanks.

The tank strength indicators change on the phase trajectory of the missile only as a result of changing the strength characteristics σ_{Bt}^{p} and E_t^{t} of the material of the fuel tanks as a function of temperature. If the influence of temperature on σ_B^{t} and E^{t} is not considered, then $\Sigma^{t} = \Sigma$, and the strength indicator remains constant throughout the phase trajectory.

The actual design load of the fuel tanks $N_{\pi pg}$ and $N_{\pi po}$ changes over the trajectory and depends on the coefficients of axial and transverse loadings, the instantaneous weight of the missile or the instantaneous weight of the fuel, the flight altitude and velocity, etc. Let us expand the values of N, M_{bnd} and h^{T} for the cross sections A-A of the oxidizer tank and B-B of the fuel tank. In this connection we have

$$N_A = f_{rf}^{A} [n_{x1}(G_{p.o} + G'_{a.o}) + R_{o.A}],$$

$$h_A^{o} = \left[\frac{k_{ro} G_{T\Sigma}}{\gamma_{T.o}(1 + k_{ro})} + \Delta V_0^{o}\right]\frac{4}{\pi D^2} - h_{bd}^{o},$$

$$M_{bnd\,A} = f_{sf}^{A} (M_a^{A} + M_0^{A}),$$

$$N_B = f_{sf}^{B}\left[n_{x1}\left(G_{g.o} + G'_{a.g} + \frac{k_{ro} G_{T\Sigma}}{1 + k_{ro}} + C'_{a.o}\right) + R_{oB}\right],$$

$$h_B = \left[\frac{G_{T\Sigma}}{\gamma_{T.g}(1 + k_{ro})} + \Delta V_g^{0}\right]\frac{4}{\pi D^2} - h_{bg},$$

$$M_{bnd.B} = f_{sf}^{B}(M_a^{B} + M_0^{B}).$$

Thus, the functional dependence of $N_{\pi p}$ on the phase coordinates and parameters in generalized form looks as follows:

$$N_{\pi p} = N_{\pi p}(V, H, \mu, \alpha, p, G_0, a_0, D, \delta_r, \delta_0).$$

If under this loading condition of the missile $p_t^{min} > p_t^{kp}$, we must at first calculate the meridianal stress

$$\sigma_m = f_{sf}^{\delta} \frac{p_\delta^{min} D}{4\delta} - \frac{N}{\pi D \delta} \pm \frac{4 M_{bnd}}{\pi D \delta}.$$

In the case $\sigma_m < 0$, the equivalent stress, according to the hypothesis of maximum tangential stresses, is defined as:

$$\sigma_{eq} = \sigma_t - \sigma_m = \frac{N_{op}}{\pi D \delta} + f_{sf}^{\delta} \frac{D}{4\delta} (p_\delta^{min} + \gamma_t h' n_{x1} - p_a).$$

Then condition (30) becomes

$$\Phi_i^{f} > \overline{N}_{op\,f},$$
$$\Phi_i^{o} > \overline{N}_{op\,o},$$

where

$$\Phi_i = \frac{\pi D \delta}{\eta_1^{\delta}} \sigma_{ui}^{\delta} - f_{sf}^{\delta} \frac{\pi D^2}{4} \left(p_{tc\,min} + \Delta p^{i} - \frac{\gamma_t c_{oi}^2}{2 g_0} \right),$$

$$\overline{N}_{op} = N_{op} - f_{sf}^{\delta} \frac{\pi D^2}{4} (\gamma_t \overline{h}_x^{i} n_{x1} + p_a),$$

$$\overline{h}_x^{i} = h_x^{i} - h^{i}.$$

In the case $\sigma_m \geq 0$, the equivalent stress is equal to the maximum stress. Usually for the cross sections A-A and B-B, the maximum stress is σ_τ[1]. Then

$$\sigma_{eq} = \sigma_t = \frac{p^{\delta} D}{2\delta} = f_{sf}^{\delta} \frac{D}{2\delta} (p_\delta^{min} - p_a + \gamma_t h' n_{x1}).$$

Therefore, condition (30) is reduced to the form

$$\frac{2\delta}{\eta_1^{\delta f} D} \sigma_{ui}^{\delta f} + f_{sf}^{\delta f} (\gamma_{t.f} \overline{h}_x^{f} n_{x1} + p_a) - (p_{tc\,min}^{f} + \Delta p^{f}) - \frac{\gamma_{t.f} c_{os}^2}{2 g_0} > 0,$$

$$\frac{2\delta_o}{\eta_1^{\delta o} D} \sigma_{ui}^{\delta o} + f_{sf}^{\delta o} (\gamma_{t.o} \overline{h}_x^{o} n_{x1} + p_a) - (p_{tc\,min}^{o} + \Delta p^{o}) - \frac{\gamma_{t.o} c_{os}^2}{2 g_0} > 0$$

---

[1] For example, see [35].

or

$$\tilde{\Phi}_i^{\circ} \geq 0,$$

$$\tilde{\Phi}_i^{\circ} \geq 0,$$

where

$$\tilde{\Phi}_i = \frac{2\delta}{\eta_i^{\delta} D}\sigma_{a\,i}^{\delta} + f_{ji}^{\delta}(\gamma_i h_s' n_{si} + p_a) - (p_{ac\,min} + \Delta p') - \frac{\gamma_i c_{ai}^2}{2g_0}.$$

With these last relationships, our analysis of possible relationships between the vector of plan parameters, strength vector, design stress vector and various limiting conditions for a single-stage ballistic missile is basically completed. Summing up our analysis, let us present the relationships and limiting conditions sequentially as in § 1, using the terminology of this paragraph. Thus, we have the plan equation

$$\beta_1 = \mu_{\kappa i} - f_{np}(G_0, \mu_{\kappa}, a_0, p_{\kappa}, p_a, n, D, \bar{l}_6, \tau_n, \tau_0) = 0. \qquad (I)$$

Here the function $f_{\pi p}$ is equal to the right portion of equation (26);

the coupling equation of the plan parameter vector

$$\left.\begin{aligned} \beta_2 = \mu_{n.g} - \frac{G_{n.g}}{G_0} = 0, \quad \beta_3 = a_0 - \frac{P_{max}^n}{G_0} = 0, \\ \beta_4 = b_0 - \frac{\pi D^2}{4G_0} = 0; \end{aligned}\right\} \qquad (I)$$

the condition of arrangement of the working medium [see equation (15)]

$$\beta_5 = \frac{\pi D^3\left(i_6 - \frac{16\Delta V_{\partial t}}{\pi D^3}\right)}{4G_0 k_\gamma^{g}(1+\chi)\left(1 - \mu_{\kappa} - G_T^{(c)}\right)} - 1 = 0; \qquad (I)$$

the condition of limiting of the plan parameter vector

$$P_{max} \leq (P_{max})_{max},$$

$$p_{\kappa\,min} \leq p_{\kappa} \leq p_{\kappa\,max},$$

$$p_a \leq p_{a\,max},$$

$$F_a < 0{,}25S,$$

$$n \leq n_{max},$$

$$D \leq D_{max},$$

$$\bar{l}_6 \leq l_{6\,max};$$

the condition of limitation of the strength vector

$$\lambda_{\eta}^{min} \leqslant \lambda_{\eta},$$
$$\lambda_{0}^{min} \leqslant \lambda_{0};$$

The condition of limitation of the actual vector

where $p_t^{kp} \geq p_t^{min}$

$$\Sigma_t - N_{np}^{n} \geqslant 0;$$

where $p_t^{min} > p_t^{kp}$

$$\Phi_t - N_{np}^{p} \geqslant 0, \quad \text{if} \quad \sigma_m < 0,$$

or

$$\tilde{\Phi}_t \geqslant 0, \quad \text{if} \quad \sigma_m \geqslant 0;$$

the condition of limitation of stable operation of the power plant

$$0{,}29 p_{H_0} - p_a \leqslant 0,$$
$$p_{oc}^{o} \geqslant p_{oc\,min}^{o} \;\&\; p_{nc}^{o} \geqslant p_{nc\,min}^{o},$$
$$p_{b.o_1} \geqslant p_{b.o_1}^{min}, \quad p_{b.o} \geqslant p_{b.o}^{min}.$$

However, on the basis of the last two limiting conditions for stable operation of the power plant, as we can see from the preceding considerations, we have based the selection of pressures at the input to the pumps and constructed one of the conditions of limitation of the real strength vector. Therefore, in the following they will not be included in the number of limiting conditions. Furthermore, it should be noted that in many cases when it is not possible to maintain the condition of limitation of the actual strength vector throughout the active sector of the phase trajectory by controlling the angle of attack $\alpha(t)$ (for example, with a concrete assignment for a change in angle of attack $\alpha = \alpha$ (V, H); flight in airless space) they must be looked upon as final conditions.

The differential equations of motion of a missile in the phase space V, θ, H, L, T_w, μ considering the assumptions made in § 1, can be represented as follows:

$$
\left.\begin{aligned}
&\varphi_1 = V' = \frac{g_0}{\mu}(a_0 p\cos\alpha - b_0\overline{Q}) - g\sin\theta,\\
&\varphi_2 = \theta' = \frac{g_0}{\mu V}(a_0 p\sin\alpha + b_0\overline{Y}) - \frac{g\cos\theta}{V} + \frac{V\cos\theta}{R_0+H},\\
&\varphi_3 = H' = V\sin\theta,\\
&\varphi_4 = L' = V\frac{R_0}{R_0+H}\cos\theta,\\
&\varphi_5 = \mu' = -\frac{a_0}{P^0_{sp}}\frac{p}{p_{sp}},\\
&\varphi_6^0 = T'_{w\,0} = f_w^0(V, H, T_{w0}, \delta_0),\ \varphi_6^* = T'_{w*} = f_w^*(V, H, T_{w0}, \delta_0).
\end{aligned}\right| \qquad \text{(I)}
$$

Generally speaking, c_x and c_y depend on α, V, H, D, $\overline{I}$ and the geometric form of the nose portion. The dependence of c_x on α where $\alpha < 20°$ is very slight [1], and therefore will not be considered in the following. On the basis of theoretical and experimental investigations, we can represent the analytic dependence of c_x and c_y on these parameters, but it is usually cumbersome and contains certain inaccuracies. Therefore, in solving this problem it is more convenient to use the known dependence $c_x = c_x(V, H)$ and $c_y = c_y(\alpha, V, H)$ of some prototype, the geometric dimensions of which lie within the fixed range of variable parameters of the missile being planned. Of course, this leads to certain errors in the estimation of the aerodynamic coefficients c_x and c_y. However, since the drag is much less than the thrust, the errors arising will have little influence on the estimation of the flying characteristics of the missile. Furthermore, with a rather narrow range of change of variable geometric parameters, the deviation of the true values of aerodynamic coefficients from their values in the prototype may fall within the limits of accuracy of the theoretical calculations of the aerodynamic coefficients.

The control functions p(t) and α(t) have certain limitations, leading to the presence of the permissible control conditions, written as

$$0 < p < p_{max} = p_b,$$
$$\alpha_{min}(V, H) < \alpha < \alpha_{max}(V, H).$$

In connection with the presence of various limitations, let us introduce according to § 1 the conditional control functions and conditional parameters, allowing us to make a transition from the closed to the open area of permissible changes of the control functions, strength vector and plan parameter vector. We then produce

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[1] For example, see [22].

$$\left.\begin{aligned}\varphi_{9\alpha} &\equiv (\alpha-\alpha_{min})(\alpha_{max}-\alpha)-v_\alpha^2=0,\\ \varphi_{9p} &\equiv p(p_h-p)-v_p^2=0\end{aligned}\right\} \qquad \text{(I)}$$

where $p_t^{kp} \geq p_t^{min}$,

$$\varphi_{10}^{kp}=\left(\dot{\sum}_{\alpha t}^{g}-N_{np.g}\right)\left(\dot{\sum}_{\alpha t}^{o}-N_{np.o}\right)-v_{11}^2=0$$

where $p_t^{min} > {}_t^{kp}$,

$$\varphi_{10}^{-}\equiv(\Phi_t^g-\overline{N}_{np.g})(\Phi_t^o-\overline{N}_{np.o})-v_{12}^2=0, \quad \text{if} \quad \sigma_m<0,$$

$$\varphi_{10}^{+}\equiv\tilde{\Phi}_t^g\tilde{\Phi}_t^o-v_{13}^2=0, \qquad \text{if} \quad \sigma_m>0,$$

or in place of the last three conditions we introduce one condition

$$\left.\begin{aligned}
&\varphi_{10}\equiv X_\delta-v_\delta^2=0,\\
&\psi_{01}^{k}\equiv[(P_{max})_{max}-P_{max}]-w_1^2=0,\\
&\psi_{02}^{k}\equiv(p_k-p_{k\,min})(p_{k\,max}-p_k)-w_2^2=0,\\
&\psi_{03}^{k}\equiv(p_a-p_{a\,min})(p_{a\,max}-p_a)(0{,}25k_cS-F_a)-w_3^2=0,\\
&\psi_{04}^{k}\equiv(n_{max}-n)-w_4^2=0,\\
&\psi_{05}^{k}\equiv(D_{max}-D)-w_5^2=0,\\
&\psi_{06}^{k}\equiv(\bar{l}_{6\,max}-\bar{l}_6)-w_6^2=0,\\
&\psi_{07}^{k}\equiv(a_{0\,max}-a_0)(a_0-a_{0\,min})-w_7^2=0,\\
&\psi_{08}^{k}\equiv(b_{0\,max}-b_0)(b_0-b_{0\,min})-w_8^2=0,\\
&\psi_{06}^{r}\equiv(\delta_g-\delta_g^{min})-w_{6.g}^2=0,\\
&\psi_{06}^{o}\equiv(\delta_o-\delta_o^{min})-w_{6.o}^2=0.\\
&\text{Here}\\
&\alpha_{max}=\alpha_{max}(V,H),\ \alpha_{min}=\alpha_{min}(V,H),\\
&X_\delta=\left(\sum_t-N_{np}\right)(\Phi_t-\overline{N}_{np})\tilde{\Phi}_t.
\end{aligned}\right\} \qquad \text{(I)}$$

For a ballistic missile, the boundary conditions are fixed as follows:

initial point

$$\left.\begin{aligned}
&t_0=0,\ V_0=0,\ \theta_0=\frac{\pi}{2},\ H_0=0,\ L_0=0,\\
&T_{w\,g}(t_0)=T_{w\,g.0}^{giv},\quad T_{w0}(t_0)=T_{w\,o0}^{giv},\ \mu_0=1;
\end{aligned}\right.$$

final point

$$\left.\begin{aligned}
&L(t_k)=L_k^{giv},\ H(t_k)=H_k^{giv},\\
&\psi_\mu\equiv\mu(t_k)-\mu_k=0.
\end{aligned}\right\} \qquad (31)$$

This last equation leads to the presence of limiting conditions on phase variable μ:

$$\mu > \mu_к.$$

where from the moment of arrival at boundary $\psi_m = \mu(t_n) - \mu_k = 0$, the power plant should be turned off.

In this connection, the problem stated earlier can be reduced to a variational problem which is formulated as follows: in the interval $t_0 \leq t \leq t_k$, in the class of permissible phase variables

$$V(t),\ \theta(t),\ H(t),\ L(t),\ \mu(t),\ T_{wg}(t),\ T_{wo}(t), \tag{32}$$

control functions

$$\alpha(t),\ p(t), \tag{33}$$

and parameters

$$a_0,\ \mu_к,\ P_{max},\ b_0,\ p_к,\ p_e,\ n,\ D,\ T_б,\ \delta_g,\ \delta_o, \tag{34}$$

satisfying relationships (I) and boundary conditions (31), find phase variables (32), control functions (33) and parameters (34) for which G_0 reaches the minimum value with fixed G_{pl}.

In order to solve this variational problem, using the general solution of § 1 as a guidance, we compose the expression

$$\begin{aligned}\Phi = -G_0 &+ e_1\psi_1 + e_2\psi_2 + e_3\psi_3 + e_4\psi_4 + e_5\psi_5 + \\ &+ e_{01}\psi_{01} + e_{02}\psi_{02} + e_{03}\psi_{03} + e_{04}\psi_{04} + e_{05}\psi_{05} + \\ &+ e_{06}\psi_{06} + e_{07}\psi_{07} + e_{08}\psi_{08} + e_{\alpha}\psi_{\alpha} + e_{\alpha}\psi_{\alpha} + \\ &+ e_{p}\psi_{p} + \int_{t_0}^{t_k} F\,dt,\end{aligned} \tag{35}$$

where

$$\begin{aligned}F = (V' - \varphi_1)\lambda_1 &+ (\theta' - \varphi_2)\lambda_2 + (H' - \varphi_3)\lambda_3 + (L' - \varphi_4)\lambda_4 + \\ &+ (\mu' - \varphi_6)\lambda_6 + (T'_{wg} - \varphi_g^0)\lambda_{8g} + (T'_{wo} - \varphi_o^0)\lambda_{8o} - \\ &- \varphi_{9\alpha}\lambda_{9\alpha} - \varphi_{9p}\lambda_{9p} - \varphi_{11}\lambda_{10}.\end{aligned}$$

We can now write the first necessary conditions for the maximum of functional $(-G_0)$ or the minimum G_0 -- the stability condition.

The Euler-Lagrange equations will be as follows:

$$\begin{aligned}
&\dot{\lambda}_1 = -\lambda_1 \frac{\partial \varphi_1}{\partial V} - \lambda_2 \frac{\partial \varphi_2}{\partial V} - \lambda_3 \frac{\partial \varphi_3}{\partial V} - \lambda_4 \frac{\partial \varphi_4}{\partial V} - \lambda_{8g} \frac{\partial \varphi_{8g}}{\partial V} - \\
&\qquad - \lambda_{8o} \frac{\partial \varphi_{8o}}{\partial V} - \lambda_{9o} \frac{\partial \varphi_{9o}}{\partial V} - \lambda_{10} \left(\frac{\partial \varphi_{10}}{\partial V}\right)_{T_w},\\
&\dot{\lambda}_2 = -\lambda_1 \frac{\partial \varphi_1}{\partial \theta} - \lambda_2 \frac{\partial \varphi_2}{\partial \theta} - \lambda_3 \frac{\partial \varphi_3}{\partial \theta} - \lambda_4 \frac{\partial \varphi_4}{\partial \theta},\\
&\dot{\lambda}_3 = -\lambda_1 \left(\frac{\partial \varphi_1}{\partial H}\right)_{p_h} - \lambda_2 \left(\frac{\partial \varphi_2}{\partial H}\right)_{p_h} - \lambda_4 \frac{\partial \varphi_4}{\partial H} - \lambda_6 \frac{\partial \varphi_6}{\partial H} - \\
&\qquad - \lambda_{8g} \frac{\partial \varphi_8^g}{\partial H} - \lambda_{8o} \frac{\partial \varphi_8^o}{\partial H} - \lambda_{9o} \frac{\partial \varphi_{9o}}{\partial H} - \lambda_{9p} \frac{\partial \varphi_{9p}}{\partial H} - \lambda_{10} \left(\frac{\partial \varphi_{10}}{\partial H}\right)_{p_h, T_w},\\
&\dot{\lambda}_4 = 0 \quad \text{or} \quad \lambda_4 = \text{const},\\
&\dot{\lambda}_6 = -\lambda_1 \frac{\partial \varphi_1}{\partial \mu} - \lambda_2 \frac{\partial \varphi_2}{\partial \mu} - \lambda_{10} \frac{\partial \varphi_{10}}{\partial \mu}.
\end{aligned} \qquad (11)$$

$$\begin{aligned}
&\dot{\lambda}_{8g} = -\lambda_{8g} \frac{\partial \varphi_8^g}{\partial T_{wg}} - \lambda_{10} \frac{\partial \varphi_{10}}{\partial T_{wg}},\\
&\dot{\lambda}_{8o} = -\lambda_{8o} \frac{\partial \varphi_8^o}{\partial T_{wo}} - \lambda_{10} \frac{\partial \varphi_{10}}{\partial T_{wo}},\\
&\lambda_1 \frac{\partial \varphi_1}{\partial p} + \lambda_2 \frac{\partial \varphi_2}{\partial p} + \lambda_6 \frac{\partial \varphi_6}{\partial p} + \lambda_{9p} \frac{\partial \varphi_{9p}}{\partial p} + \lambda_{10} \frac{\partial \varphi_{10}}{\partial p} = 0,\\
&\lambda_1 \frac{\partial \varphi_1}{\partial \alpha} + \lambda_2 \frac{\partial \varphi_2}{\partial \alpha} + \lambda_{9o} \frac{\partial \varphi_{9o}}{\partial \alpha} + \lambda_{10} \frac{\partial \varphi_{10}}{\partial \alpha} = 0,\\
&\qquad \lambda_{9p} v_p = 0, \quad \lambda_{9o} v_o = 0.
\end{aligned}$$

Let us analyze the last four Euler-Lagrange equations, determining the optimal control of the thrust p(t) and angle of attack α(t). In § 1 in the investigation of the second necessary condition -- the Weierstrass condition -- considering the minimum principle and the control condition it was demonstrated that, generally speaking, for liquid fueled motors the maximum thrust mode $p = p_{max} = p_h$ and the minimum thrust mode $p = 0$ should be taken as the optimal control p(t).

Therefore, in the following we will use only the limiting control $p = p_h$ and $p = 0$ as the optimal, the presence of a passive sector being characterized by the condition

$$H_p = \frac{g_0 a_0}{\mu}\left(\lambda_1 \cos\alpha + \lambda_2 \frac{\sin\alpha}{V}\right) - \lambda_6 \frac{a_0}{P^n_{sp}} > 0. \qquad (36)$$

However, from the moment of arrival at boundary ı, when the condition $\nu = \nu_k$ arises, the optimal coupled control n is formed, which, regardless of condition (36), leads to $p = 0$.

The optimal control of angle of attack $\alpha(t)$ according to the Euler-Lagrange equations may be stable control, optimal limiting control or optimal coupled control.

They correspond to the Euler-Lagrange equations:
stable control

$$\lambda_1 \frac{\partial \varphi_1}{\partial \alpha} + \lambda_2 \frac{\partial \varphi_2}{\partial \alpha} = 0$$

or

$$\left.\begin{aligned} &\lambda_1 \frac{g v^2 \rho F}{\mu} \sin\alpha - \lambda_2 \frac{g_0}{\mu V}\left(a_1 p \cos\alpha + b_0 \frac{\partial \bar{F}}{\partial \alpha}\right) = 0, \\ &\lambda_{9} = 0, \quad \lambda_{10} = 0; \end{aligned}\right\} \qquad (37)$$

limiting control

$$\left.\begin{aligned} &\lambda_8 = -\lambda_1 \frac{g v^2 \rho F}{\mu} \sin\alpha - \lambda_2 \frac{g_0}{\mu V}\left(a_1 p \cos\alpha + b_0 \frac{\partial \bar{F}}{\partial \alpha}\right), \\ &\lambda_{10} = 0, \quad \nu_8 = 0; \end{aligned}\right\} \qquad (38)$$

coupled control

$$\left.\begin{aligned} &\lambda_6 = \left[\lambda_1 \frac{g v^2 \rho F}{\mu} \sin\alpha - \lambda_2 \frac{g_0}{\mu}\left(a_1 p \cos\alpha + b_0 \frac{\partial \bar{F}}{\partial \alpha}\right)\right] \frac{1}{\frac{\partial X_6}{\partial \alpha}}, \\ &\lambda_{9} = 0, \quad \nu_6 = 0, \end{aligned}\right\} \qquad (39)$$

where

$$\lambda_8 = \lambda_9 (\alpha_{max} + \alpha_{min} - 2\alpha).$$

With coupled control, corresponding to Euler-Lagrange equation (39), the angle of attack is determined from the equation

$$X_6 = 0. \qquad (40)$$

In this case we have

$$\delta_f^n = f_{sf} \delta_f \qquad \text{or} \qquad \delta_o^n = f_{sf} \delta_o,$$

i. e. the required thickness of the fuel or oxidizer tank shell reaches its limiting value and differs from the actual thickness only by f_{sf} times. Adjustment of the angle of attack by coupled control allows the design of the tanks for the main components to withstand all loads arising on the phase trajectory of the missile.

Since ϕ_j ($j = 1, \ldots, 10$) is not dependent in explicit form on t, system (II) has the first integral

$$\lambda_1\varphi_1 + \lambda_2\varphi_2 + \lambda_3\varphi_3 + \lambda_4\varphi_4 + \lambda_6\varphi_6 + \lambda_8^c\varphi_8^c + \lambda_8^o\varphi_8^o = -C_0, \tag{41}$$

where C_0 is the integration constant.

The design plan of the missile is such that after completion of the active sector the nose portion must be separated. Therefore, from the condition of stability we produce the condition of discontinuity of the Lagrange coefficients at the moment of separation of the nose portion (in the terminology of § 1, the condition of discontinuity at the moment of separation of a stage). Considering the assumptions made in § 1, they can be written as

$$\left.\begin{array}{c} [\lambda_1]_{t_k^-} = [\lambda_1]_{t_k^+}, \ldots, [\lambda_8^o]_{t_k^-} = [\lambda_8^o]_{t_k^+}, \\ [C_0]_{t_k^-} = [C_0]_{t_k^+}, \end{array}\right\} \tag{42}$$

$$\lambda_6(t_n^-) \ldots = -c_{..} \tag{43}$$

We must keep in mind here that in the sector $[t_n^+, t_k]$ the coupling equations do not include the phase coordinate ν and therefore $\lambda_6 \equiv 0$. Furthermore, the conclusions produced in the analysis of the conditions of discontinuity (1.2.37) allow us to affirm that the Lagrange coefficients $\lambda_1, \ldots, \lambda_4, \lambda_6, \lambda_8^c, \lambda_8^o$ and the integration constant C_0 are continuous with a sudden change in the control functions $p(t)$ and $\alpha(t)$.

Thus, in the sector $[t_0, t_k]$ the Lagrange coefficients $\lambda_1, \ldots, \lambda_8^c$, λ_8^o and the integration constant C_0 are continuous throughout, while the Lagrange coefficient λ_6 is continuous throughout the sector $[t_0, t_n^-]$.

Based on this conclusion of continuity of the integration constant C_0, we can write the following equality for the moment of separation of the nose portion of the missile, corresponding to the moment when the boundary $t = t_k$ is reached:

$$\left[\lambda_1\left(\frac{g_0 a_0 p\cos\alpha - k_0\tilde{Q}}{\mu} - g\sin\theta\right) + \lambda_2\left(\frac{g_0 a_0 p\sin\alpha + k_0\tilde{Y}}{\mu V} - \right.\right.$$

$$\left.\left. - \frac{g\cos\theta}{V} + \frac{V\cos\theta}{R_3 + H}\right) + \lambda_3 V\sin\theta + \lambda_4 V\frac{R_3}{R_3 + H}\cos\theta - \lambda_6\frac{a_0 p}{P^n_f p_{3f}}\right]_{t_n^-} =$$

$$= \left[-\lambda_1\left(\frac{S\tilde{Q}}{k_{k.n} G_{gl}} + g\sin\theta\right) + \lambda_2\left(\frac{S\tilde{Y}}{k_{k.n} G_{gl} V} - \frac{g\cos\theta}{V} + \right.\right.$$

$$\left.\left. + \frac{V\cos\theta}{R_3 + H}\right) + \lambda_3 V\sin\theta + \lambda_4 V\frac{R_3}{R_3 + H}\cos\theta\right]_{t_n^+}.$$

Then, keeping in mind the discontinuity condition (42), we find that this equation can be fulfilled only when

$$\frac{g_0 a_0 p}{\mu}\left(\lambda_1\cos\alpha + \lambda_2\frac{\sin\alpha}{V}\right) - \lambda_6\frac{a_0 p}{P^n_f p_{3f}}\bigg|_{t_n^-} = 0, \tag{44}$$

which corresponds to the condition of transition from the optimal limiting control $p = p_{max}$ to the optimal limiting control $p = 0$ (36).

Consequently, the transition to coupled control $p = 0$ upon reaching the boundary of phase variable μ at the boundary $\mu - \mu_k = 0$ corresponds to the moment of "optimal" cut-off of the power plant. Therefore, in the sector $[t_n^-, t_k]$, coupled control p will correspond to the optimal limiting control $p = 0$.

The conditions of optimality of the parameters become

$$e_1 + e_2 = 0,$$

$$-1 - e_1\frac{\partial f_{np}}{\partial G_0} + e_2\frac{G_{gt}}{G_0} + e_3\frac{P_{max}}{G_0^2} + e_4\frac{S}{G_0^2} + e_5\frac{\partial \bar{z}_5}{\partial G_0} + e_{03}\frac{\partial \bar{z}_{03}}{\partial G_0} = 0,$$

$$e_3\frac{1}{G_0} + e_{01} = 0,$$

$$-e_1\frac{\partial f_{np}}{\partial a_0} + e_3 + e_5\frac{\partial \bar{z}_5}{\partial a_0} + e_{03}\frac{\partial \bar{z}_{03}}{\partial a_0} + e_{07}(a_{max} + a_{0min} - 2a_0) -$$

$$-\int_{t_0}^{t_n}\left(\lambda_1\frac{\partial \varphi_1}{\partial a_0} + \lambda_2\frac{\partial \varphi_2}{\partial a_0} + \lambda_6\frac{\partial \varphi_6}{\partial a_0} + \lambda_{10}\frac{\partial \varphi_{10}}{\partial a_0}\right)dt = 0,$$

$$-e_1\frac{\partial f_{np}}{\partial \mu_k} + \lambda_6^n + e_5\frac{\partial \bar{z}_5}{\partial \mu_k} = 0,$$

$$-e_1 \frac{\partial f_{np}}{\partial p_\kappa} + e\,\frac{\partial \beta_5}{\partial p_\kappa} + e_{02}(p_{\kappa\,max} + p_{\kappa\,min} - 2p_\kappa) + e_{03}\frac{\partial \beta_{03}}{\partial p_\kappa} -$$

$$-\int_{t_0}^{t_n}\left(\lambda_6 \frac{\partial \varphi_6}{\partial p_\kappa} + \lambda_{9p}\frac{\partial \varphi_{9p}}{\partial p_\kappa} + \lambda_{10}\frac{\partial \varphi_{10}}{\partial p_\kappa}\right) dt = 0,$$

$$-e_1 \frac{\partial f_{np}}{\partial p_a} + e_5 \frac{\partial \beta_5}{\partial p_a} + e_{03}\frac{\partial \beta_{03}}{\partial p_a} - \int_{t_0}^{t_n}\left(\lambda_6 \frac{\partial \varphi_6}{\partial p_a} + \lambda_{9p}\frac{\partial \varphi_{9p}}{\partial p_a} + \lambda_{10}\frac{\partial \varphi_{10}}{\partial p_a}\right)dt = 0,$$

$$-e_1 \frac{\partial f_{np}}{\partial D} - e_4 \frac{\pi D}{2G_0} + e_5 \frac{\partial \beta_5}{\partial D} + e_{03}\frac{\partial \beta_{03}}{\partial D} - e_{05} -$$

$$-\int_{t_0}^{t_n} \lambda_{10}\frac{\partial \varphi_{10}}{\partial D}\,dt - \int_{t_0}^{t_n}\lambda_1 \frac{\partial \varphi_1}{\partial D}\,dt = 0,$$

$$-e_1 \frac{\partial f_{np}}{\partial l_6} + e_5 \frac{\partial \beta_5}{\partial l_6} - e_{06} - \int_{t_0}^{t_n}\frac{\partial \varphi_{10}}{\partial l_6}\,dt = 0,$$

$$e_4 + e_{08}(b_{0\,max} + b_{0\,min} - 2b_0) - \int_{t_0}^{t_n}\left(\lambda_1 \frac{\partial \varphi_1}{\partial b_0} + \lambda_2 \frac{\partial \varphi_2}{\partial b_0} + \lambda_{10}\frac{\partial \varphi_{10}}{\partial b_0}\right) dt = 0,$$

$$-e_1 \frac{\partial f_{np}}{\partial n} - e_{04} - \int_{t_0}^{t_n}\lambda_{10}\frac{\partial \varphi_{10}}{\partial n}\,dt = 0. \qquad (45)$$

$$-e_1 \frac{\partial f_{np}}{\partial \delta_r} + e_6^r - \int_{t_0}^{t_n}\left(\lambda_{8r}\frac{\partial \varphi_8^r}{\partial \delta_r} + \lambda_{10}\frac{\partial \varphi_{10}}{\partial \delta_r}\right)dt = 0, \qquad (46)$$

$$-e_1 \frac{\partial f_{np}}{\partial \delta_0} + e_6^n - \int_{t_0}^{t_n}\left(\lambda_{8o}\frac{\partial \varphi_8^n}{\partial \delta_0} + \lambda_{10}\frac{\partial \varphi_{10}}{\partial \delta_0}\right) dt = 0, \qquad (47)$$

$$\left.\begin{array}{llll} e_{01}w_1 = 0, & e_{02}w_2 = 0, & e_{03}w_3 = 0, & e_{04}w_4 = 0, \\ e_{05}w_5 = 0, & e_{06}w_6 = 0, & e_{07}w_7 = 0, & e_{08}w_8 = 0, \\ e_6^r w_{6.r} = 0, & e_6^n w_{6.o} = 0. & & \end{array}\right\} \qquad (48)$$

Analysis of equations (45)-(47) indicates a number of remarkable conclusions, which are possible considering that, generally speaking, $e_1 \neq 0$.

If we assume that throughout the phase trajectory the conditions of strong limitation of the required thickness vector are fulfilled, then since $\lambda_{10} = 0$, $\lambda_8 = \lambda_{8p} = 0$ and $\partial f_{np}/\partial \delta_r \neq 0$, $\partial f_{np}/\partial \delta_o \neq 0$,

$\partial f_{\pi p}/\partial m \neq 0$, the optimal values of the tank shell thickness for the fuel and oxidizer tanks will be δ_g^{min} and δ_o^{min}, while the optimal value of the rotating speed of the pump unit will be the limiting permissible speed on the basis of experience of planning of such pump units, n_{max}. this result is a result of the fact that with the assumptions made, the design cases of loading of the tanks of the main components are not related to flight of the missile. Therefore, if cases are possible when the optima values δ_g^{opt} and δ_o^{opt} are such that $\delta_g^{opt} > \delta_g^{min}$ and $\delta_o^{opt} > \delta_o^{min}$, there should always be a sector on the phase trajectory over which the optimal control α is coupled control, determined from equation (40). This result occurs in connection with the fact that the design cases of loading of the tanks of the main components arise in the powered sector of the trajectory of the rocket. Thus, at the extremes, in addition to stable control $\alpha(t)$, determined by the Euler-Lagrange equation (37), there should always be coupled control α, expressed by the conditions $\delta_g = f_{sf}^{tg}\delta_g^{\pi}$ and $\delta_o = f_{sf}^{to}\delta_p^{\pi}$ or equation (40). This conclusion corresponds with the conclusion drawn earlier in § 1.

Furthermore, if the situation $p_{tg}^{kp} > p_{tg}^{min}$ and $p_{to}^{kp} > p_{to}^{min}$ is always realized on the phase trajectory, the optimal values of δ_g and δ_o will be equal to the greatest minimum value of the real thickness of the shells of the fuel and oxidizer tanks expressed by conditions (29) and determined by satisfaction of equations (46) and (47). In this case the optimal value of n is n_{max}. However, this situation can arise primarily only with low values of $p_{BC\ min}^{g}$ and $p_{BC\ min}^{o}$. Thus, if we can produce values of $p_{BC\ min}^{g}$ and $p_{BC\ min}^{o}$, for which at the extreme where $\delta_g = f_{sf}^{tg}\delta_g^{\pi}$ and $\delta_o = f_{sf}^{to}\delta_o^{\pi}$, the conditions $p_{tg}^{kp} > p_{tg}^{min}$ and $p_{to}^{kp} > p_{to}^{min}$ always occur, the weight of the pump system and tank section and therefore the launch weight G_0 estimated as a function of the parameters δ_g, δ_o and n will be produced minimum for fulfillment of the maneuver. Therefore, there is reason to organize various measures and introduce the corresponding design improvements to the pump unit corresponding to a decrease in $p_{BC\ min}$. We can see from (8) that the minimum pressure of the input to the pumps, determined from the condition of cavitation-free operation of the pump unit, will always be greater for the oxidizer pump, since the volumetric flow rate of oxidizer is always greater than the volumetric flow rate of fuel, and $p_s^o > p_s^g$.

In this connection, the situation is possible on the phase trajectory when the condition $p_{tg}^{kp} > p_{tg}^{min}$ is created for the fuel tank, while for the oxidizer tank in some sectors $p_{to}^{kp} > p_{to}^{min}$, while in other sectors $p_{to}^{min} > p_{to}^{kp}$. Then, the optimal thickness δ_g will be equal to the greatest minimum value of the actual thickness of the shell of the fuel tank expressed by conditions (29), while the optimal thickness δ_0 will be greater than the greatest minimum value of the actual thickness of the oxidizer tank and $n < n_{max}$, since the increase in the rotating speed of the pump will be limited by the conditions of cavitation-free operation of the oxidizer pump.

Excluding the coefficients $e_{.}, \ldots, e_5$ from the conditions of optimality of the parameters, we produce

$$1+\frac{\eta_{1к} m_G}{(\eta_{1к}-1)\,m_a}+\frac{m_a M_G^{пр}+m_G M_a^{пр}}{m_a \eta_{1к}}=0,$$

$$1+\frac{(\eta_{2к}-\lambda_6^a)\,m_G}{(\eta_{2к}-1)\,m_{p_к}}+\frac{m_{p_к} M_G^{пр}+m_G M_{p_к}^{пр}}{m_{p_к}\eta_{2к}}=0,$$

$$1+\frac{\eta_{3к} m_G}{(\eta_{3к}-1)\,m_{p_г}}+\frac{m_{p_г} M_G^{пр}+m_G M_{p_г}^{пр}}{m_{p_г}\eta_{3к}}=0,$$

$$1+\frac{\eta_{4к} m_G}{(\eta_{4к}-1)\,m_{p_о}}+\frac{m_{p_о} M_G^{пр}+m_G M_{p_о}^{пр}}{m_{p_о}\eta_{4к}}=0,$$

$$1+\frac{\eta_{5к} m_G}{(\eta_{5к}-1)\,m_D}+\frac{m_D M_G^{пр}+m_G M_D^{пр}}{m_D \eta_{5к}}=0,$$

$$1+\frac{\eta_{6к} m_G}{(\eta_{6к}-1)\,m_н}+\frac{m_н M_G^{пр}+m_G M_н^{пр}}{m_н \eta_{6к}}=0,$$

$$1+\frac{\eta_{7к} m_G}{(\eta_{7к}-1)\,m_\xi^г}+\frac{m_\xi^г M_G^{пр}+m_G M_{\xi г}^{пр}}{m_\xi^г \eta_{7к}}=0,$$

$$1+\frac{\eta_{8к} m_G}{(\eta_{8к}-1)\,m_\xi^о}+\frac{m_\xi^о M_G^{пр}+m_G M_{\xi о}^{пр}}{m_\xi^о \eta_{8к}}=0.$$

Here

$$\eta_{1к}=\int_{t_0}^{t_к}\left[\frac{S}{G_0^2}\left(\lambda_1\frac{\partial\varphi_1}{\partial h_к}+\lambda_2\frac{\partial\varphi_2}{\partial h_к}\right)+\lambda_{10}\left(\frac{S}{G_0^2}\frac{\partial\varphi}{\partial h_к}-\frac{\partial\bar{G}}{\partial h_к}\left(\frac{\partial\varphi}{\partial G}\right)_{G_0}\right)\right]dt,$$

$$\eta_{1k} = -\int_{t_0}^{t_k} \left\{ \lambda_1 \frac{\partial q_1}{\partial a_0} + \lambda_2 \frac{\partial q_2}{\partial a_0} + \lambda_6 \frac{\partial q_6}{\partial a_0} + \lambda_{10} \left[\left(\frac{\partial q_{10}}{\partial a_0} \right)_{l_6} + \left(\frac{\partial q_{10}}{\partial l_6} \right)_{a_0} \frac{\partial \hat{l}_6}{\partial a_0} \right] \right\} dt,$$

$$\eta_{2k} = -\frac{\partial \hat{l}_6}{\partial \psi_k} \int_{t_0}^{t_k} \lambda_{10} \left(\frac{\partial q_{10}}{\partial l_6} \right)_{\psi_k} dt,$$

$$\eta_{3k} = -\int_{t_0}^{t_k} \left\{ \lambda_6 \frac{\partial q_6}{\partial p_k} + \lambda_{9p} \frac{\partial q_{9p}}{\partial p_k} + \lambda_{10} \left[\left(\frac{\partial q_{10}}{\partial p_k} \right)_{l_6} + \left(\frac{\partial q_{10}}{\partial l_6} \right)_{p_k} \frac{\partial \hat{l}_6}{\partial p_k} \right] \right\} dt,$$

$$\eta_{4k} = -\int_{t_0}^{t_k} \left\{ \lambda_6 \frac{\partial q_6}{\partial p_a} + \lambda_{9p} \frac{\partial q_{9p}}{\partial p_a} + \lambda_{10} \left[\left(\frac{\partial q_{10}}{\partial D} \right)_{l_6} + \left(\frac{\partial q_{10}}{\partial l_6} \right)_{p_a} \frac{\partial \hat{l}_6}{\partial p_a} \right] \right\} dt,$$

$$\eta_{5k} = -\int_{t_0}^{t_k} \left\{ \frac{nD}{2G_0} \left(\lambda_1 \frac{\partial q_1}{\partial b_0} + \lambda_2 \frac{\partial q_2}{\partial b_0} \right) + \lambda_{10} \left[\left(\frac{\partial q_{10}}{\partial D} \right)_{l_6} + \left(\frac{\partial q_{10}}{\partial l_6} \right)_D \frac{\partial \hat{l}_6}{\partial D} + \right. \right.$$

$$\left. \left. + \frac{nD}{2G_0} \frac{\partial q_{10}}{\partial b_0} \right] \right\} dt - \int_{t_0}^{t_k} \lambda_1 \frac{\partial q_1}{\partial D} dt,$$

$$\eta_{6k} = -\int_{t_0}^{t_k} \lambda_{10} \frac{\partial q_{10}}{\partial n} dt, \qquad \eta_{7k} = -\int_{t_0}^{t_k} \left(\lambda_{9r} \frac{\partial q_9^{p}}{\partial b_r} + \lambda_{10} \frac{\partial q_{10}}{\partial b_r} \right) dt,$$

$$\eta_{8k} = -\int_{t_0}^{t_k} \left(\lambda_{9n} \frac{\partial q_9^{n}}{\partial b_n} + \lambda_{10} \frac{\partial q_{10}}{\partial b_n} \right) dt;$$

$$m_G = \left(\frac{\partial f_{np}}{\partial G_0} \right)_{l_6} + \left(\frac{\partial f_{np}}{\partial l_6} \right)_{G_0} \frac{\partial \hat{l}_6}{\partial G_0} + \frac{G_{p.t}}{G_0^2},$$

$$m_a = \left(\frac{\partial f_{np}}{\partial a_0} \right)_{l_6} + \left(\frac{\partial f_{np}}{\partial l_6} \right)_{a_0} \frac{\partial \hat{l}_6}{\partial a_0},$$

$$m_{\psi_k} = \left(\frac{\partial f_{np}}{\partial \psi_k} \right)_{l_6} + \left(\frac{\partial f_{np}}{\partial l_6} \right)_{\psi_k} \frac{\partial \hat{l}_6}{\partial \psi_k},$$

$$m_{p_k} = \left(\frac{\partial f_{np}}{\partial p_k} \right)_{l_6} + \left(\frac{\partial f_{np}}{\partial l_6} \right)_{p_k} \frac{\partial \hat{l}_6}{\partial p_k},$$

$$m_{p_a} = \left(\frac{\partial f_{np}}{\partial p_a} \right)_{l_6} + \left(\frac{\partial f_{np}}{\partial l_6} \right)_{p_a} \frac{\partial \hat{l}_6}{\partial p_a},$$

$$m_D = \left(\frac{\partial f_{np}}{\partial D} \right)_{l_6} + \left(\frac{\partial f_{np}}{\partial l_6} \right)_D \frac{\partial \hat{l}_6}{\partial D}.$$

$$m_a = \frac{\partial f_{np}}{\partial a}, \quad m_\lambda^n = \frac{\partial f_{np}}{\partial \lambda_n}, \quad m_b^n = \frac{\partial f_{np}}{\partial b_0};$$

$$M_G^{np} = -c_{01}\frac{P_{max}}{G_0} + c_{03}\frac{\partial \vartheta_m}{\partial G_0} - c_{06}\frac{\partial \bar{l}_6}{\partial G_0} - c_{08}\frac{S}{G_0^2}(b_{0max} + b_{0min} - 2b_0),$$

$$M_a^{np} = -c_{01}G_0 + c_{03}\frac{\partial \vartheta_m}{\partial a_0} - c_{06}\frac{\partial \bar{l}_6}{\partial a_0} + c_{07}(a_{max} + a_{min} - 2a_0),$$

$$M_{p_k}^{np} = -c_{06}\frac{\partial \bar{l}_6}{\partial p_k}, \quad M_{p_a}^{np} = c_{03}\frac{\partial \vartheta_m}{\partial p_a} - c_{06}\frac{\partial \bar{l}_p}{\partial p_a},$$

$$M_{p_k}^{np} = c_{02}(p_{k\,max} + p_{k\,min} - 2p_k) + c_{03}\frac{\partial \vartheta_m}{\partial p_k} - c_{06}\frac{\partial \bar{l}_6}{\partial p_k},$$

$$M_D^{np} = c_{03}\frac{\partial \vartheta_m}{\partial D} - c_{05} - c_{06}\frac{\partial \bar{l}_6}{\partial D} + c_{08}\frac{\pi D}{2G_0}(b_{0max} + b_{0min} - 2b_0),$$

$$M_n^{np} = -c_{04}, \quad M_{\lambda_0}^{np} = -c_\lambda^0, \quad M_{b_0}^{np} = -c_b^0.$$

In order to produce the partial derivatives $\bar{I}_t$ with respect to the various parameters, we must use the relationship $\delta_5 = 0$.

Suppose the optimal values of the parameters are found within their area of limitation. Then we will have

$$1 + \frac{\eta_{1k} m_G}{(\eta_{2k} - 1) m_a} = 0, \tag{49}$$

$$1 - \frac{(\eta_{2k} - \lambda_6^n) m_a}{m_{p_k} \eta_{1k}} = 0 \tag{50}$$

or, considering (44) and the mode $p = p_{max} = p_h$, we produce

$$\left.\begin{aligned}
&1 - \frac{\left[\eta_{2k} - \frac{g_0 p_h P_{sp}^n}{\eta_k}\left(\lambda_1 \cos\alpha + \lambda_2 \frac{\sin\alpha}{V}\right)\right] m_a}{\eta_{1k} m_{p_k}} = 0,\\
&1 - \frac{\eta_{1k} m_a}{m_{p_k} \eta_{1k}} = 0, \qquad 1 - \frac{\eta_{1k} m_a}{m_{p_a} \eta_{1k}} = 0,\\
&1 - \frac{\eta_{1k} m_a}{m_D \eta_{1k}} = 0, \qquad 1 - \frac{\eta_{1k} m_a}{m_n \eta_{1k}} = 0,\\
&1 - \frac{\eta_{1k} m_a}{m_b^n \eta_{1k}} = 0, \qquad 1 - \frac{\eta_{1k} m_a}{m_\lambda \eta_{1k}} = 0.
\end{aligned}\right\} \tag{50}$$

Since at the end of the powered sector fixation of T_{wg}, T_{wo} and at the final point fixation of V_k, θ_k and t_k is free (i. e. these phase coordinates are not fixed), the condition of transversality is produced in the form

$$\lambda_{8g.n}=0, \quad \lambda_{8o.n}=0, \tag{51}$$

$$\lambda_{1k}=0, \quad \lambda_{3k}=0, \tag{52}$$

$$C_0=0 \tag{53}$$

or, considering the first integral in t_k, we find

$$\lambda_{3k}\sin\theta_k+\lambda_6\cos\theta_k=0. \tag{54}$$

It should be kept in mind that due to the continuity of C_0, throughout the sector $[t_0, t_k]$, condition (53)need be satisfied only in one point. Since the first integral can be replaced by any of the Euler-Lagrange differential equations, after performance of this replacement condition (53) will always be satisfied, since it is used at each point for solution of system (II). In the following the first integral will be replaced by the Euler-Lagrange equation, related to phase coordinate ν. Then, excluding the Lagrange coefficient λ_6 from the remaining Euler-Lagrange equations and optimization conditions of the parameters, we could consider condition (53) always fulfilled. However, at point t_n, due to the continuity condition C_0 (44), λ_6 is not included in the first integral. Therefore, λ_6^{π} cannot be excluded from the conditions of optimality of the parameters using the first integral and this requires use of the continuity condition C_0 (44), due to which it can be considered always fulfilled in the following. Then, condition (53) will also be always fulfilled if condition (54) is satisfied at t_k.

Thus, at points t_n and t_k we must satisfy the following sixteen conditions:

(49)-(52), (54) and $\nu(t) - \mu_k = 0$, $L(t_k) = L_k^{fix}$, $H(t_k) = H_k = 0$.

For the solution of system (I)-(II) at point t_0, in addition to condition (31), we must fix the eight parameters a_0, μ_k, D, p_k, p_a, n, δ_g, δ_o and the six Lagrange coefficients λ_{10}, λ_{20}, λ_{30}, λ_4, λ_{8g0}, λ_{8o0}, the values of which are unknown, and the two stop functions to determine points t_p and t_k. It is assumed here that the values of G_0 and $\overline{I}_t$ are determined from the equations $\beta_1 = 0$ and $\beta_5 = 0$. However, one of the coefficients $\lambda_{10}, \lambda_{20}, \lambda_{30}, \lambda_4, \lambda_{8g0}, \lambda_{8o0}$ can be excluded, for example λ_4 by dividing it into the equations of system (II), the conditions of optimality of parameters (49), (50) and the conditions of transversality (51), (52) and (54). Then equation (49) is converted to the condition

$$1+\frac{\eta_k m_0}{m_a\left(\eta_a - \frac{1}{\lambda_4}\right)}=0,$$

which in the following can always be considered fulfilled, while system (II), a portion of the conditions of optimality (50) and the condition of transversality (51), (52) and (54), due to their homogeneity relative to the Lagrange coefficients, remain unchanged.

Consequently, determination of the optimal values of parameters a_0, μ_k, D, p_k, p_a, $\overline{I}_t$, n, δ_g and δ_o, optimal equations $\alpha(t)$ and $\varphi(t)$ and thereby the optimal phase trajectory, on the basis of which the planned single-stage ballistic missile will be able to deliver the known payload G_{pl} over the fixed range L_k with minimum launch weight G_0 is reduced to solution of the multipoint (three point) boundary problem presented in general form in § 4 and in the appendix.

The multipoint (three point) boundary problem produced as a result of solution of the variational problem can be formulated as follows: determine parameters

$$a_0,\ \mu_k,\ D,\ p_k,\ p_a,\ n,\ \delta_g,\ \delta_o, \tag{55}$$

$$G_0,\ \overline{I}_t \tag{56}$$

and Lagrange coefficients

$$\lambda_{10},\ \lambda_{20},\ \lambda_{30},\ \lambda_{8g0},\ \lambda_{8o0} \tag{57}$$

for which the solution of the system

$$
\left.
\begin{aligned}
& V' = f_1, \quad \theta' = f_2, \quad H' = f_3, \quad L' = f_4, \quad \mu' = f_5, \\
& T'_{w\rho} = f_6, \quad T_{w0} = f_7, \quad 0 = f_8, \quad 0 = f_9, \\
& \lambda'_1 = f_{10} = -\lambda_1 \frac{\partial f_1}{\partial V} - \lambda_2 \frac{\partial f_2}{\partial V} - \lambda_3 \frac{\partial f_3}{\partial V} - \frac{\partial f_4}{\partial V} - \lambda_{8\rho} \frac{\partial f_6}{\partial V} - \\
& \qquad - \lambda_{80} \frac{\partial f_7}{\partial V} + \lambda_a \frac{\partial \alpha_{np}}{\partial V} - \lambda_6 \frac{\partial \varphi_6}{\partial V}, \\
& \lambda'_2 = f_{11} = -\lambda_1 \frac{\partial f_1}{\partial \theta} - \lambda_2 \frac{\partial f_2}{\partial \theta} - \lambda_3 \frac{\partial f_3}{\partial \theta} - \frac{\partial f_4}{\partial \theta}, \\
& \lambda'_3 = f_{12} = -\lambda_1 \frac{\partial f_1}{\partial H} - \lambda_2 \frac{\partial f_2}{\partial H} - \lambda_3 \frac{\partial H_4}{\partial H} - \lambda_{8\rho} \frac{\partial f_6}{\partial H} - \\
& \qquad - \lambda_{80} \frac{\partial f_7}{\partial H} + \lambda_a \frac{\partial \alpha_{np}}{\partial H} - \lambda_6 \frac{\partial \varphi_6}{\partial H}, \\
& \lambda'_{8\rho} = f_{13} = -\lambda_{8\rho} \frac{\partial f_6}{\partial T_{w\rho}} - \lambda_6 \frac{\partial \varphi_6}{\partial T_{w\rho}}, \\
& \lambda'_{80} = f_{14} = -\lambda_{80} \frac{\partial f_7}{\partial T_{w0}} - \lambda_6 \frac{\partial \varphi_6}{\partial T_{w0}}, \\
& 0 = f_{15} = -\lambda_1 \frac{\partial f_1}{\partial \alpha} - \lambda_2 \frac{\partial f_2}{\partial \alpha} - \lambda_6 \frac{\partial \varphi_6}{\partial \alpha} - \lambda_a,
\end{aligned}
\right\} \quad \text{(III)}
$$

$$
\left.
\begin{aligned}
& 0 = -\lambda_0 \nu_{\lambda}, \quad 0 = -\lambda_6 \nu_6, \\
& \eta'_1 = \zeta_1 = -\lambda_1 \frac{\partial f_1}{\partial a_0} - \lambda_2 \frac{\partial f_2}{\partial a_0} - \xi \frac{\partial f_5}{\partial a_0} - \lambda_6 \frac{\partial \varphi_6}{\partial a_0}, \\
& \eta'_2 = \zeta_2 = -\lambda_6 \frac{\partial \varphi_6}{\partial \varphi_a}, \\
& \eta'_3 = \zeta_3 = -\lambda_1 \frac{\partial f_1}{\partial p_a} - \lambda_2 \frac{\partial f_2}{\partial p_a} - \xi \frac{\partial f_5}{\partial p_a} - \lambda_6 \frac{\partial \varphi_6}{\partial p_a}, \\
& \eta'_4 = \zeta_4 = -\lambda_1 \frac{\partial f_1}{\partial p_a} - \lambda_2 \frac{\partial f_2}{\partial p_a} - \xi \frac{\partial f_5}{\partial p_a} - \lambda_6 \frac{\partial \varphi_6}{\partial p_a}, \\
& \eta'_5 = \zeta_5 = -\frac{\pi D}{2G_0} \left(\lambda_1 \frac{\partial f_1}{\partial b_0} + \lambda_2 \frac{\partial f_2}{\partial b_0} \right) - \\
& \qquad - \lambda_6 \left[\frac{\partial \varphi_6}{\partial D} + \frac{\pi D}{2G_0} \left(\frac{\partial \varphi_6}{\partial b_0} \right)_D \right] \Bigg|_{t_a}^{t_a} - \lambda_1 \frac{\partial \varphi_1}{\partial D} \Bigg|_{t_a}^{t_a}, \\
& \eta'_6 = \zeta_6 = -\lambda_6 \frac{\partial \varphi_6}{\partial n}, \quad \eta'_7 = \zeta_7 = -\lambda_6 \frac{\partial \varphi_6}{\partial \theta_\rho} - \lambda_{8\rho} \frac{\partial f_6}{\partial \theta_\rho}, \\
& \eta'_8 = \zeta_8 = -\lambda_6 \frac{\partial \varphi_6}{\partial \theta_0} - \lambda_{80} \frac{\partial \varphi}{\partial \theta_0}
\end{aligned}
\right\}
$$

leads to zero values of the functionals

$$p_1 = c^{(1)}\left[1 - \frac{(\eta_{2\kappa} - \xi_n) m_a}{\eta_{1\kappa} m_{\mu_\kappa}}\right]^2, \tag{58}$$

$$p_2 = c^{(2)}\left(1 - \frac{\eta_{3\kappa} m_a}{\eta_{1\kappa} m_{p_\kappa}}\right)^2, \quad p_3 = c^{(3)}\left(1 - \frac{\eta_{4\kappa} m_a}{\eta_{1\kappa} m_{p_a}}\right)^2,$$

$$p_4 = c^{(4)}\left(1 - \frac{\eta_{5\kappa} m_a}{\eta_{1\kappa} m_D}\right)^2, \quad p_5 = c^{(5)}\left(1 - \frac{\eta_{6\kappa} m_a}{\eta_{1\kappa} m_n}\right)^2,$$

$$p_6 = c^{(6)}\left(1 - \frac{\eta_{7\kappa} m_a}{\eta_{1\kappa} m_\xi^r}\right)^2, \quad p_7 = c^{(7)}\left(1 - \frac{\eta_{8\kappa} m_a}{\eta_{1\kappa} m_\xi^0}\right)^2,$$

$$p_8 = c^{(8)}\lambda_{cn}^2, \quad p_9 = c^{(9)}\lambda_{\kappa 0\, n}^2,$$

$$p_{10} = c^{(10)}\lambda_{1\kappa}^2, \quad p_{11} = c^{(11)}\lambda_{2\kappa}^2,$$

$$p_{12} = c^{(12)}(1 + \lambda_{3\kappa} \operatorname{tg} \theta_\kappa), \quad p_{13} = c^{(13)}\left(1 - \frac{L_\kappa}{L_\kappa^{\text{тр}}}\right)^2, \tag{59}$$

$$\beta_1 = 0, \quad \beta_6 = 0, \tag{60}$$

$$\varphi_1^0 = \mu(t) - \mu_\kappa = 0, \quad \varphi_2^0 = H(t_\kappa) = 0.$$

Here

$$\varphi_1 = f_1, \ \varphi_2 = f_2, \ \varphi_3 = f_3, \ \varphi_4 = f_4, \ f_5 = -\frac{a_0 \bar{p}}{P_{\xi_r}^{n}},$$

$$\varphi_{8r} = f_6, \quad \varphi_{80} = f_7, \quad f_{9_2} = f_8, \quad \varphi_{10} = f_9;$$

$$\frac{\partial \varphi_1}{\partial V} = -\frac{g_0 b_0}{\mu}\frac{\partial \bar{Q}}{\partial V}, \quad \frac{\partial \varphi_2}{\partial V} = \frac{g_0 b_0}{\mu V}\frac{\partial \bar{Y}}{\partial V} - \frac{1}{V}\left[\frac{g_0}{\mu V}(a_0 p_h \sin\alpha + b_0 Y) - \right.$$

$$\left. - \frac{g\cos\theta}{V} - \frac{V\cos\theta}{R_3 + H}\right],$$

$$\frac{\partial \varphi_3}{\partial V} = \sin\theta, \quad \frac{\partial \varphi_4}{\partial V} = \frac{R_3}{R_3 + H}\cos\theta;$$

$$\frac{\partial \varphi_1}{\partial \theta} = -g\cos\theta, \quad \frac{\partial \varphi_2}{\partial \theta} = \left(\frac{g}{V} - \frac{V}{R_3 + H}\right)\sin\theta;$$

$$\frac{\partial \varphi_3}{\partial \theta} = V\cos\theta, \quad \frac{\partial \varphi_4}{\partial \theta} = -\frac{R_3}{R_3 + H}\sin\theta;$$

$$\frac{\partial \varphi_1}{\partial H} = \frac{g_0}{\mu}\left(a_0 \bar{p}\frac{\partial p_h}{\partial H}\cos\alpha - b_0\frac{\partial \bar{Q}}{\partial H}\right) - \frac{\partial g}{\partial H}\sin\theta,$$

$$\frac{\partial \varphi_2}{\partial H} = \frac{g_0}{\mu V}\left(a_0 \bar{p}\frac{\partial p_h}{\partial H}\sin\alpha + b_0\frac{\partial \bar{Y}}{\partial H}\right) - \frac{\partial g}{\partial H}\frac{\cos\theta}{V} - \frac{V\cos\theta}{(R_3 + H)^2},$$

$$\frac{\partial\varphi_4}{\partial H}=-\frac{R_3}{(R_3+H)^2}V\cos\theta;$$

$$\frac{\partial\varphi_1}{\partial\mu}=-\frac{g_0}{\mu^2}\left(a_0\bar{p}p_h\cos\alpha-b_0\bar{Q}\right),$$

$$\frac{\partial\varphi_2}{\partial\mu}=-\frac{g_0}{\mu^2V}\left(a_0\bar{p}p_h\sin\alpha+b_0Y\right),$$

$$\frac{\partial\varphi_1}{\partial\alpha}=-\frac{g_0a_0\bar{p}p_h}{\mu}\sin\alpha,\quad \frac{\partial\varphi_2}{\partial\alpha}=\frac{g_0}{\mu V}\left(a_0\bar{p}p_h\cos\alpha+b_0\frac{\partial\bar{Y}}{\partial\alpha}\right);$$

$$\frac{\partial\varphi_1}{\partial a_0}=\frac{g_0\bar{p}p_h}{\mu}\cos\alpha,\quad \frac{\partial\varphi_2}{\partial a_0}=\frac{g_0\bar{p}p_h}{\mu V}\sin\alpha,\quad \frac{\partial f_5}{\partial a_0}=-\frac{\bar{p}}{P^n_{yf}};$$

$$\frac{\partial\varphi_1}{\partial p_k}=\frac{g_0\bar{p}}{\mu}\frac{\partial p_h}{\partial p_k}\cos\alpha,\quad \frac{\partial\varphi_2}{\partial p_k}=\frac{g_0\bar{p}}{\mu V}\frac{\partial p_h}{\partial p_k}\sin\alpha,$$

$$\frac{\partial f_5}{\partial p_k}=\frac{a_0\bar{p}}{(P^n_{yd})^2}\frac{\partial P^n_{yd}}{\partial p_k};$$

$$\frac{\partial\varphi_1}{\partial p_a}=\frac{g_0a_0\bar{p}}{\mu}\frac{\partial p_h}{\partial p_a}\cos\alpha,\quad \frac{\partial\varphi_2}{\partial p_a}=\frac{g_0a_0\bar{p}}{\mu V}\frac{\partial p_h}{\partial p_a}\sin\alpha,$$

$$\frac{\partial f_5}{\partial p_a}=\frac{a_0\bar{p}}{(P^n_{yd})^2}\frac{\partial P^n_{yd}}{\partial p_a};$$

$$\frac{\partial\varphi_1}{\partial b_0}=-\frac{g_0}{\mu}\bar{Q},\quad \frac{\partial\varphi_2}{\partial b_0}=\frac{g_0}{\mu V}Y,$$

$$\frac{\partial\varphi_1}{\partial D}\Bigg|_{t_h}^{t_k}=-\frac{\pi D}{2k_{p}G_{p}}\bar{Q};$$

$$\xi=\frac{P^n_{yp}}{a_0}\left(\lambda_1\varphi_1+\lambda_2\varphi_2+\lambda_3\varphi_3+\varphi_4\right);$$

$$\bar{p}=1 \text{ if } H_p\leqslant 0;$$

$$\bar{p}=0 \text{ if } H_p\geqslant 0;$$

$$\text{if } p_a^{\min}>p_c^{\text{кр}}$$

$$\sigma_m\geqslant 0$$

$$\frac{\partial\varphi_6}{\partial V}=\frac{\partial\Phi_t}{\partial V},\quad \frac{\partial\varphi_6}{\partial H}=\frac{\partial\Phi_t}{\partial H},\quad \frac{\partial\varphi_6}{\partial\mu}=\frac{\partial\Phi_t}{\partial\mu},\quad \frac{\partial\varphi_6}{\partial\alpha}=\frac{\partial\Phi_t}{\partial\alpha},$$

$$\frac{\partial\varphi_6}{\partial a_0}=\frac{\partial\Phi_t}{\partial a_0},\quad \frac{\partial\varphi_6}{\partial\mu_k}=\frac{\partial\Phi_t}{\partial\mu_k},\quad \frac{\partial\varphi_6}{\partial p_k}=\frac{\partial\Phi_t}{\partial p_k},\quad \frac{\partial\varphi_6}{\partial p_a}=\frac{\partial\Phi_t}{\partial p_a},$$

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that the c
$p = p_h$ and
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ing we wil
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$$\frac{\partial\varphi_6}{\partial D}=\left(\frac{\partial\tilde{\Phi}_t}{\partial D}\right)_{b_0},\quad \frac{\partial\varphi_6}{\partial b_0}=\left(\frac{\partial\tilde{\Phi}_t}{\partial b_0}\right)_D,\quad \frac{\partial\varphi_6}{\partial n}=\frac{\partial\tilde{\Phi}_t}{\partial n},$$

$$\frac{\partial\varphi_6}{\partial\delta_r}=\frac{\partial\tilde{\Phi}_t}{\partial\delta_r},\quad \frac{\partial\varphi_6}{\partial\delta_e}=\frac{\partial\Phi_t}{\partial\delta_e};$$

$$\vartheta_m<0$$

$$\frac{\partial\varphi_6}{\partial V}=\frac{\partial\Phi_t}{\partial V}-\frac{\partial\overline{N}_{np}}{\partial V},\quad \frac{\partial\varphi_6}{\partial H}=\frac{\partial\Phi_t}{\partial H}-\frac{\partial\overline{N}_{np}}{\partial H},\quad \frac{\partial\varphi_6}{\partial\mu}=-\frac{\partial\overline{N}_{np}}{\partial\mu},$$

$$\frac{\partial\varphi_6}{\partial a}=-\frac{\partial\overline{N}_{np}}{\partial a},\quad \frac{\partial\varphi_6}{\partial a_0}=-\frac{\partial\overline{N}_{np}}{\partial a_0},\quad \frac{\partial\varphi_6}{\partial\mu_\kappa}=-\frac{\partial\overline{N}_{np}}{\partial\mu_\kappa},$$

$$\frac{\partial\varphi_6}{\partial p_\kappa}=\frac{\partial\Phi_t}{\partial p_\kappa}-\frac{\partial\overline{N}_{np}}{\partial p_\kappa},\quad \frac{\partial\varphi_6}{\partial p_e}=\frac{\partial\Phi_t}{\partial p_e}-\frac{\partial\overline{N}_{np}}{\partial p_e},$$

$$\frac{\partial\varphi_6}{\partial D}=\left(\frac{\partial\Phi_t}{\partial D}-\frac{\partial\overline{N}_{np}}{\partial D}\right)_{b_0},\quad \frac{\partial\varphi_6}{\partial b_0}=-\left(\frac{\partial\overline{N}_{np}}{\partial b_0}\right)_D,$$

$$\frac{\partial\varphi_6}{\partial n}=\frac{\partial\Phi_t}{\partial n},\quad \frac{\partial\varphi_6}{\partial\delta_r}=\frac{\partial\Phi_t}{\partial\delta_r}-\frac{\partial\overline{N}_{np.e}}{\partial\delta_r},\quad \frac{\partial\varphi_6}{\partial\delta_e}=\frac{\partial\Phi_t}{\partial\delta_e}-\frac{\partial\overline{N}_{np.e}}{\partial\delta_e};$$

where $p_6^{\kappa p}\geqslant p_6^{min}$

$$\frac{\partial\varphi_6}{\partial V}=\frac{\partial\sum_t}{\partial V}-\frac{\partial N_{np}}{\partial V},\quad \frac{\partial\varphi_6}{\partial H}=\frac{\partial\sum_t}{\partial H}-\frac{\partial N_{np}}{\partial H},\quad \frac{\partial\varphi_6}{\partial\mu}=-\frac{\partial N_{np}}{\partial\mu},$$

$$\frac{\partial\varphi_6}{\partial a}=-\frac{\partial N_{np}}{\partial a},\quad \frac{\partial\varphi_6}{\partial D}=\left(\frac{\partial\sum_t}{\partial D}-\frac{\partial N_{np}}{\partial D}\right)_{b_0},\quad \frac{\partial\varphi_6}{\partial b_0}=-\frac{\partial N_{np}}{\partial b_0},$$

$$\frac{\partial\varphi_6}{\partial\delta_r}=\frac{\partial\sum_t}{\partial\delta_r}-\frac{\partial N_{np.e}}{\partial\delta_r},\quad \frac{\partial\varphi_6}{\partial\delta_e}=\frac{\partial\sum_t}{\partial\delta_e}-\frac{\partial N_{np.e}}{\partial\delta_e}.$$

System (III) is written considering the conclusion made earlier that the optimal modes of the power plant are the maximum thrust mode $p = p_h$ and the mode $p = 0$. It is further assumed that on the basis of the fixed parameters (55) using expression (59) is always possible to determine condition (56) unambiguously. Therefore, in the following we will consider condition (59) always fulfilled. As stop functions we select functions ψ_1^0 and ψ_k^0, in connection with which condition (60) can also be considered always fulfilled. Thus, we must, using the thirteen constants (55) and (57), while solving system (III) reduce the thirteen functionals (58) to 0.

In order to organize the computational procedure of this multipoint boundary problem, following the mathematical model of the computational algorithm presented in § 4, we compose the conjugate system in the form

$$
\left.
\begin{aligned}
\dot y_1^{(n)} &= -\sum_{i=1}^{15} y_i^{(n)} \frac{\partial f_i}{\partial V} - \sum_{j=1}^{8} z_j^{(n)} \frac{\partial \zeta_j}{\partial V} \quad (n=1, \ldots, 13),\\
\dot y_2^{(n)} &= -\sum_{i=1}^{15} y_i^{(n)} \frac{\partial f_i}{\partial \vartheta} - \sum_{j=1}^{8} z_j^{(n)} \frac{\partial \zeta_j}{\partial \vartheta},\\
\dot y_3^{(n)} &= -\sum_{i=1}^{15} y_i^{(n)} \frac{\partial f_i}{\partial H} - \sum_{j=1}^{8} z_j^{(n)} \frac{\partial \zeta_j}{\partial H},\\
\dot y_4^{(n)} &= 0 \quad \text{or} \quad y_4^{(n)} = \text{const},\\
\dot y_5^{(n)} &= -\sum_{i=1}^{15} y_i^{(n)} \frac{\partial f_i}{\partial \mu} - \sum_{j=1}^{8} z_j^{(n)} \frac{\partial \zeta_j}{\partial \mu},\\
\dot y_6^{(n)} &= -\sum_{i=1}^{15} y_i^{(n)} \frac{\partial f_i}{\partial T_{wg}} - \sum_{j=1}^{8} z_j^{(n)} \frac{\partial \zeta_j}{\partial T_{wg}},\\
\dot y_7^{(n)} &= -\sum_{i=1}^{15} y_i^{(n)} \frac{\partial f_i}{\partial T_{wo}} - \sum_{j=1}^{8} z_j^{(n)} \frac{\partial \zeta_j}{\partial T_{wo}},\\
\dot y_{10}^{(n)} &= -\sum_{m=10}^{15} y_m^{(n)} \frac{\partial f_m}{\partial \lambda_1} - \sum_{j=1}^{8} z_j^{(n)} \frac{\partial \zeta_j}{\partial \lambda_1},\\
\dot y_{11}^{(n)} &= -\sum_{m=10}^{15} y_m^{(n)} \frac{\partial f_m}{\partial \lambda_2} - \sum_{j=1}^{8} z_j^{(n)} \frac{\partial \zeta_j}{\partial \lambda_2},\\
\dot y_{12}^{(n)} &= -\sum_{m=10}^{15} y_m^{(n)} \frac{\partial f_m}{\partial \lambda_3} - \sum_{j=1}^{8} z_j^{(n)} \frac{\partial \zeta_j}{\partial \lambda_3},\\
\dot y_{13}^{(n)} &= -\sum_{m=10}^{15} y_m^{(n)} \frac{\partial f_m}{\partial \lambda_{8p}} - \sum_{j=1}^{8} z_j^{(n)} \frac{\partial \zeta_j}{\partial \lambda_{8p}},\\
\dot y_{14}^{(n)} &= -\sum_{m=10}^{15} y_m^{(n)} \frac{\partial f_m}{\partial \lambda_{8o}} - \sum_{j=1}^{8} z_j^{(n)} \frac{\partial \zeta_j}{\partial \lambda_{8o}},
\end{aligned}
\right\} \quad (IV)
$$

$$
\left.
\begin{aligned}
&\sum_{i=1}^{15} y_i^{(n)} \frac{\partial f_i}{\partial \alpha} + \sum_{j=1}^{8} z_j^{(n)} \frac{\partial \zeta_j}{\partial \alpha} = 0, \\
&\sum_{m=10}^{15} y_m^{(n)} \frac{\partial f}{\partial \varepsilon} + y_{16}^{(n)} \nu_\varepsilon = 0, \quad \lambda_\varepsilon y_{16}^{(n)} + 2 y_8^{(n)} \nu_\varepsilon = 0, \\
&\sum_{m=10}^{15} y_m^{(n)} \frac{\partial f_m}{\partial \lambda_6} + \sum_{j=1}^{8} z_j^{(n)} \frac{\partial \zeta_j}{\partial \lambda_6} - y_{17}^{(n)} \nu_6 = 0, \\
&\lambda_6 y_{17}^{(n)} + 2 \nu_6 y_9^{(n)} = 0, \\
\dot{z}_9^{(n)} &= -\sum_{i=1}^{15} y_i^{(n)} \frac{\partial f_i}{\partial G_0} - \sum_{j=1}^{8} z_j^{(n)} \frac{\partial \zeta_j}{\partial G_0}, \\
\dot{z}_{10}^{(n)} &= -\sum_{i=1}^{15} y_i^{(n)} \frac{\partial f_i}{\partial a_0} - \sum_{j=1}^{8} z_j^{(n)} \frac{\partial \zeta_j}{\partial a_0}, \\
\dot{z}_{11}^{(n)} &= -\sum_{i=1}^{15} y_i^{(n)} \frac{\partial f_i}{\partial \mu_\kappa} - \sum_{j=1}^{8} z_j^{(n)} \frac{\partial \zeta_j}{\partial \mu_\kappa}, \\
\dot{z}_{12}^{(n)} &= -\sum_{i=1}^{15} y_i^{(n)} \frac{\partial f_i}{\partial p_\kappa} - \sum_{j=1}^{8} z_j^{(n)} \frac{\partial \zeta_j}{\partial p_\kappa},
\end{aligned}
\right\} \qquad \text{(IV)}
$$

$$
\left.
\begin{aligned}
\dot{z}_{13}^{(n)} &= -\sum_{i=1}^{15} y_i^{(n)} \frac{\partial f_i}{\partial p_a} - \sum_{j=1}^{8} z_j^{(n)} \frac{\partial \zeta_j}{\partial p_a}, \\
\dot{z}_{14}^{(n)} &= -\sum_{i=1}^{15} y_i^{(n)} \frac{\partial f_i}{\partial D} - \sum_{j=1}^{8} z_j^{(n)} \frac{\partial \zeta_j}{\partial D}, \\
\dot{z}_{15}^{(n)} &= -\sum_{i=1}^{15} y_i^{(n)} \frac{\partial f_i}{\partial n} - \sum_{j=1}^{8} z_j^{(n)} \frac{\partial \zeta_j}{\partial n}, \\
\dot{z}_{16}^{(n)} &= -\sum_{i=1}^{15} y_i^{(n)} \frac{\partial f_i}{\partial \delta_r} - \sum_{j=1}^{8} z_j^{(n)} \frac{\partial \zeta_j}{\partial \delta_r}, \\
\dot{z}_{17}^{(n)} &= -\sum_{i=1}^{15} y_i^{(n)} \frac{\partial f_i}{\partial \delta_0} - \sum_{j=1}^{8} z_j^{(n)} \frac{\partial \zeta_j}{\partial \delta_0}, \\
&\dot{z}_l^{(n)} = 0 \quad \text{or} \quad z_l^{(n)} = \text{const} \quad (l = 1, \ldots, 8).
\end{aligned}
\right\}
$$

The boundary conditions for the conjugate system (IV) at point $\psi_1^0 = 0$ are

$$y_{1n}^{(1)} = \frac{\partial p_1}{\partial V},\ y_{2n}^{(1)} = \frac{\partial p_1}{\partial \theta},\ y_{3n}^{(1)} = \frac{\partial p_1}{\partial H},\ y_4^{(1)} = 0.$$

$$y_{5n}^{(1)} = \left(\frac{\partial p_1}{\partial \mu}\right)_{p_n} - \frac{\dot p_1}{\dot \mu},\ y_{6n}^{(1)} = \frac{\partial p_1}{\partial T_{wr}},\ y_{7n}^{(1)} = \frac{\partial p_1}{\partial T_{wn}},$$

$$y_{10n}^{(1)} = \frac{\partial p_1}{\partial \lambda_1},\ y_{11n}^{(1)} = \frac{\partial p_1}{\partial \lambda_2},\ y_{12n}^{(1)} = \frac{\partial p_1}{\partial \lambda_3},\ y_{13n}^{(1)} = \frac{\partial p_1}{\partial \vartheta_{0p}},$$

$$y_{14n}^{(1)} = \frac{\partial p_1}{\partial \vartheta_{0e}},\ z_1^{(1)} = \frac{\partial p_1}{\partial \eta_1},\ z_2^{(1)} = \frac{\partial p_1}{\partial \eta_2},$$

$$z_{3n}^{(1)} = \ldots = z_8^{(1)} = 0,\ z_{9n}^{(1)} = \frac{\partial p_1}{\partial G_0},\ z_{10n}^{(1)} = \frac{\partial p_1}{\partial a_0},$$

$$z_{11n}^{(1)} = \left(\frac{\partial p_1}{\partial p_k}\right)_{\mu} + \frac{\dot p_1}{\dot \mu},\ z_{12n}^{(1)} = \frac{\partial p_1}{\partial p_k},\ z_{13n}^{(1)} = \frac{\partial p_1}{\partial p_a},\ z_{14n}^{(1)} = \frac{\partial p_1}{\partial D},$$

$$z_{15n} = \frac{\partial p_1}{\partial n},\ z_{16n}^{(1)} = \frac{\partial p_1}{\partial \vartheta_p},\ z_{17n}^{(1)} = \frac{\partial p_1}{\partial \vartheta_e};$$

$$y_{1n}^{(2)} = 0,\ y_{2n}^{(2)} = 0,\ y_{3n}^{(2)} = 0,\ y_4^{(2)} = 0,\ y_{5n}^{(2)} = -\frac{\dot p_2}{\dot \mu},$$

$$y_{6n}^{(2)} = \ldots = y_{14n}^{(2)} = 0,\ z_1^{(2)} = \frac{\partial p_2}{\partial \eta_1},\ z_2^{(2)} = 0,\ z_3^{(2)} = \frac{\partial p_2}{\partial \eta_3},$$

$$z_4^{(2)} = \ldots = z_8^{(2)} = 0,\ z_{9n}^{(2)} = \frac{\partial p_2}{\partial G_0},\ z_{10n}^{(2)} = \frac{\partial p_2}{\partial a_0},$$

$$z_{11n}^{(2)} = \left(\frac{\partial p_2}{\partial p_k}\right)_{\mu} + \frac{\dot p_2}{\dot \mu},\ z_{12n}^{(2)} = \frac{\partial p_2}{\partial p_k},\ z_{13n}^{(2)} = \frac{\partial p_2}{\partial p_a},\ z_{14n}^{(2)} = \frac{\partial p_2}{\partial D},$$

$$z_{15n}^{(2)} = \frac{\partial p_2}{\partial n},\ z_{16n}^{(2)} = \frac{\partial p_2}{\partial \vartheta_p},\ z_{17n}^{(2)} = \frac{\partial p_2}{\partial \vartheta_e};$$

$$y_{1n}^{(3)} = \ldots = y_4^{(3)} = 0,\ y_{5n}^{(3)} = -\frac{\dot p_3}{\dot \mu},\ y_{6n}^{(3)} = y_{7n}^{(3)} = y_{10n}^{(3)} =$$

$$= \ldots = y_{14}^{(3)} = 0,\ z_1^{(3)} = \frac{\partial p_3}{\partial \eta_1},\ z_2^{(3)} = z_3^{(3)} = 0,$$

$$z_4^{(3)} = \frac{\partial p_3}{\partial \eta_4},\ z_5^{(3)} = \ldots = z_8^{(3)} = 0,\ z_{9n}^{(3)} = \frac{\partial p_3}{\partial G_0},$$

$$z_{10n}^{(3)} = \frac{\partial p_3}{\partial a_0},\ z_{11n}^{(3)} = \left(\frac{\partial p_3}{\partial p_k}\right)_{\mu} + \frac{\dot p_3}{\dot \mu},\ z_{12n}^{(3)} = \frac{\partial p_3}{\partial p_k},\ z_{13n}^{(3)} = \frac{\partial p_3}{\partial p_a},$$

$$z_{14n}^{(3)} = \frac{\partial p_3}{\partial D},\ z_{15n}^{(3)} = \frac{\partial p_3}{\partial n},\ z_{16n}^{(3)} = \frac{\partial p_3}{\partial \vartheta_p},\ z_{17n}^{(3)} = \frac{\partial p_3}{\partial \vartheta_e};$$

$$y_{1n}^{(4)} = \ldots = y_4^{(4)} = 0,\ y_{5n}^{(4)} = -\frac{\dot p_4}{\dot \mu},\ y_{6n}^{(4)} = y_{7n}^{(4)} = y_{10n}^{(4)} =$$

t point

$$= \ldots = y_{14n}^{(4)} = 0, \quad z_1^{(4)} = \frac{\partial p_4}{\partial \eta_1}, \quad z_2^{(4)} = \ldots = z_4^{(4)} = 0,$$

$$z_5^{(4)} = \frac{\partial p_4}{\partial \eta_5}, \quad z_6^{(4)} = 0, \quad z_7^{(4)} = 0, \quad z_8^{(4)} = 0, \quad z_{9n}^{(4)} = \frac{\partial p_4}{\partial G_0},$$

$$z_{10n}^{(4)} = \frac{\partial p_4}{\partial a_0}, \quad z_{11n}^{(4)} = \left(\frac{\partial p_4}{\partial v_{1n}}\right)_p + \frac{\dot{p}_4}{\mu}, \quad z_{12n}^{(4)} = \frac{\partial p_4}{\partial p_n}, \quad z_{13n}^{(4)} = \frac{\partial p_4}{\partial p_a},$$

$$z_{14n}^{(4)} = \frac{\partial p_4}{\partial D}, \quad z_{15n}^{(4)} = \frac{\partial p_4}{\partial n}, \quad z_{16n}^{(4)} = \frac{\partial p_4}{\partial \vartheta_r}, \quad z_{17n}^{(4)} = \frac{\partial p_4}{\partial \vartheta_0};$$

$$y_{1n}^{(5)} = 0, \quad y_{2n}^{(5)} = 0, \quad y_{3n}^{(5)} = 0, \quad y_4^{(5)} = 0, \quad y_{5n}^{(5)} = -\frac{\dot{p}_5}{\mu},$$

$$y_6^{(5)} = y_{7n}^{(5)} = y_{10n}^{(5)} = \ldots = y_{14n}^{(5)} = 0, \quad z_1^{(5)} = \ldots =$$

$$= z_5^{(5)} = 0, \quad z_6^{(5)} = \frac{\partial p_5}{\partial \eta_6}, \quad z_7^{(5)} = z_8^{(5)} = 0, \quad z_{9n}^{(5)} = \frac{\partial p_5}{\partial G_0},$$

$$z_{10n}^{(5)} = \frac{\partial p_5}{\partial a_0}, \quad z_{11n}^{(5)} = \frac{\partial p_5}{\partial v_{1n}} + \frac{\dot{p}_5}{\mu}, \quad z_{12n}^{(5)} = \frac{\partial p_5}{\partial p_n}, \quad z_{13n}^{(5)} = \frac{\partial p_5}{\partial p_a},$$

$$z_{14n}^{(5)} = \frac{\partial p_5}{\partial D}, \quad z_{15n}^{(5)} = \frac{\partial p_5}{\partial n}, \quad z_{16n}^{(5)} = \frac{\partial p_5}{\partial \vartheta_r}, \quad z_{17n}^{(5)} = \frac{\partial p_5}{\partial \vartheta_0};$$

$$y_{1n}^{(6)} = \ldots = y_4^{(6)} = 0, \quad y_{5n}^{(6)} = -\frac{\dot{p}_6}{\mu}, \quad y_{6n}^{(6)} = 0, \quad y_{7n}^{(6)} = 0,$$

$$y_{10n}^{(6)} = \ldots = y_{14n}^{(6)} = 0, \quad z_1^{(6)} = \ldots = z_6^{(6)} = 0,$$

$$z_7^{(6)} = \frac{\partial p_6}{\partial \eta_7}, \quad z_8^{(6)} = 0, \quad z_{9n}^{(6)} = \frac{\partial p_6}{\partial G_0}, \quad z_{10n}^{(6)} = \frac{\partial p_6}{\partial a_0}, \quad z_{11n}^{(6)} = \left(\frac{\partial p_6}{\partial v_{1n}}\right)_p +$$

$$+ \frac{\dot{p}_6}{\mu}, \quad z_{12n}^{(6)} = \frac{\partial p_6}{\partial p_n}, \quad z_{13n}^{(6)} = \frac{\partial p_6}{\partial p_a}, \quad z_{14n}^{(6)} = \frac{\partial p_6}{\partial D}, \quad z_{15n}^{(6)} = \frac{\partial p_6}{\partial n},$$

$$z_{16n}^{(6)} = \frac{\partial p_6}{\partial \vartheta_r}, \quad z_{17n}^{(6)} = \frac{\partial p_6}{\partial \vartheta_0};$$

$$y_{1n}^{(7)} = \ldots = y_4^{(7)} = 0, \quad y_{5n}^{(7)} = -\frac{\dot{p}_7}{\mu}, \quad y_{6n}^{(7)} = y_{7n}^{(7)} = y_{10n}^{(7)} =$$

$$= \ldots = y_{14n}^{(7)} = 0, \quad z_1^{(7)} = \ldots = z_7^{(7)} = 0,$$

$$z_8^{(7)} = \frac{\partial p_7}{\partial \eta_8}, \quad z_{9n}^{(7)} = \frac{\partial p_7}{\partial G_0}, \quad z_{10n}^{(7)} = \frac{\partial p_7}{\partial a_0}, \quad z_{11n}^{(7)} = \left(\frac{\partial p_7}{\partial v_{1n}}\right)_p +$$

$$+ \frac{\dot{p}_7}{\mu}, \quad z_{12n}^{(7)} = \frac{\partial p_7}{\partial p_n}, \quad z_{13n}^{(7)} = \frac{\partial p_7}{\partial p_a}, \quad z_{14n}^{(7)} = \frac{\partial p_7}{\partial D},$$

$$z_{15n}^{(7)}=\frac{\partial p_7}{\partial n},\ z_{16n}^{(7)}=\frac{\partial p_7}{\partial l_0},\ z_{17n}^{(7)}=\frac{\partial p_7}{\partial \beta_0};$$

$$y_{1n}^{(8)}=\ldots=y_{4}^{(8)}=0,\ y_{5n}^{(8)}=-\frac{\dot{p}_8}{\mu},\ y_{6n}^{(8)}=y_{7n}^{(8)}=y_{10n}^{(8)}=$$

$$=\ldots=y_{12n}^{(8)}=0,\ y_{13n}^{(8)}=\frac{\partial p_8}{\partial r_{8q}},\ y_{14n}^{(8)}=0,$$

$$z_{1}^{(8)}=\ldots=z_{10n}^{(8)}=0,\ z_{11n}^{(8)}=\frac{\dot{p}_8}{\mu},\ z_{12n}^{(8)}=\ldots=z_{17n}^{(8)}=0,$$

$$y_{1n}^{(9)}=\ldots=y_{4}^{(9)}=0,\ y_{5n}^{(9)}=-\frac{\dot{p}_9}{\mu},\ y_{6n}^{(9)}=y_{7n}^{(9)}=y_{10n}^{(9)}=$$

$$=\ldots=y_{13n}^{(9)}=0,\ y_{14n}^{(9)}=\frac{\partial p_9}{\partial r_{8n}},\ z_{1}^{(9)}=\ldots=z_{10n}^{(9)}=0,$$

$$z_{11n}^{(9)}=\frac{\dot{p}_9}{\mu},\quad z_{12n}^{(9)}=\ldots=z_{17n}^{(9)}=0.$$

The boundary conditions for the conjugate system (IV) at point $\dot{\psi}_к^0 = 0$ are as follows:

$$y_{1к}^{(10)}=y_{2к}^{(10)}=y_{4}^{(10)}=\ldots=y_{7к}^{(10)}=0,\ y_{3к}^{(10)}=-\frac{\dot{p}_{10}}{\dot{\psi}_к^0},$$

$$y_{10к}^{(10)}=\frac{\partial p_{10}}{\partial \lambda_1},\ y_{11к}^{(10)}=\ldots=y_{14к}^{(10)}=0,$$

$$z_{1}^{(10)}=\ldots=z_{17к}^{(10)}=0;$$

$$y_{1к}^{(11)}=y_{2к}^{(11)}=y_{4}^{(11)}=\ldots=y_{7к}^{(11)}=0,\ y_{3к}^{(11)}=-\frac{\dot{p}_{11}}{\dot{\psi}_к^0},$$

$$y_{10к}^{(11)}=0,\ y_{11к}^{(11)}=\frac{\partial p_{11}}{\partial \lambda_2},\ y_{12к}^{(11)}=y_{13к}^{(11)}=y_{14к}^{(11)}=0,\ z_{1}^{(11)}=$$

$$=\ldots=z_{17к}^{(11)}=0;$$

$$y_{1к}^{(12)}=0,\ y_{2к}^{(12)}=\frac{\partial p_{12}}{\partial \vartheta},\ y_{3к}^{(12)}=-\frac{\dot{p}_{12}}{\dot{\psi}_к^0},\ y_{4}^{(12)}=\ldots=$$

$$=y_{7к}^{(12)}=0,\ y_{10к}^{(12)}=0,\ y_{11к}^{(12)}=0,\ y_{12к}^{(12)}=\frac{\partial p_{12}}{\partial \lambda_3},$$

$$y_{13к}^{(12)}=0,\ y_{14к}^{(12)}=0;\ z_{1}^{(12)}=\ldots=z_{17к}^{(12)}=0;$$

$$y_{1к}^{(13)}=0,\ y_{2к}^{(13)}=0,\ y_{3к}^{(13)}=-\frac{\dot{p}_{13}}{\dot{\psi}_к^0},\ y_{4}^{(13)}=\frac{\partial p_{13}}{\partial L},$$

$$y_{5к}^{(13)}=\ldots=y_{7к}^{(13)}=0,\ y_{10к}^{(13)}=\ldots=y_{14к}^{(13)}=0,$$

$$z_{1}^{(13)}=\ldots=z_{17к}^{(13)}=0.$$

The discontinuity conditions of the conjugate coefficients at point $\psi_1^n = 0$ are such that all conjugate coefficients, with the exception of y_5, are continuous. Conjugate coefficient $y_5(t_n^+)$ is

$$y_5^{(s)}(t_n^+) = \left[y_1^{(s)}(t_n^-)(\dot{V}(t_n^+) - \dot{V}(t_n^-)) + y_2^{(s)}(t_n^-)(\dot{\theta}(t_n^+) - \dot{\theta}(t_n^-)) + \right.$$

$$\left. + y_{10}^{(s)}(t_n^-)(\dot{\lambda}_1(t_n^+) - \dot{\lambda}_1(t_n^-)) + y_{12}^{(s)}(t_n)(\dot{\lambda}_3(t_n^+) - \dot{\lambda}_3(t_n^-))\right] \frac{P_3^n}{a_0}.$$

$$(s = 10, \ldots, 13).$$

The system of linear equations allowing us to determine $\delta\lambda_{10}$, $\delta\lambda_{20}$, $\delta\lambda_{30}$, $\delta\lambda_{8r0}$, $\delta\lambda_{8g0}$, δa_0, $\delta\mu_k$, δp_k, δp_a, δD, δn, $\delta\delta_r$, $\delta\delta_0$, is reduced to the following form:

$$\Delta p_l = y_{100}^{(l)}\delta\lambda_{10} + y_{110}^{(l)}\delta\lambda_{20} + y_{120}^{(l)}\delta\lambda_{30} + y_{130}^{(l)}\delta\lambda_{8r0} + y_{140}^{(l)}\delta\lambda_{8g0} +$$

$$+ \left(z_{100}^{(l)} - \frac{z_{90}^{(l)}}{m_G}\frac{\partial f_{np}}{\partial a_0}\right)\delta a_0 + \left(z_{110}^{(l)} - \frac{z_{90}^{(l)}}{m_G}\frac{\partial f_{np}}{\partial \mu_k}\right)\delta\mu_k +$$

$$+ \left(z_{120}^{(l)} - \frac{z_{90}^{(l)}}{m_G}\frac{\partial f_{np}}{\partial p_k}\right)\delta p_k + \left(z_{130}^{(l)} - \frac{z_{90}^{(l)}}{m_G}\frac{\partial f_{np}}{\partial p_a}\right)\delta p_a +$$

$$+ \left(z_{140}^{(l)} - \frac{z_{90}^{(l)}}{m_G}\frac{\partial f_{np}}{\partial D}\right)\delta D + \left(z_{150}^{(l)} - \frac{z_{90}^{(l)}}{m_G}\frac{\partial f_{np}}{\partial n}\right)\delta n +$$

$$+ \left(z_{160}^{(l)} - \frac{z_{90}^{(l)}}{m_G}\frac{\partial f_{np}}{\partial \delta_r}\right)\delta\delta_r + \left(z_{170}^{(l)} - \frac{z_{90}^{(l)}}{m_G}\frac{\partial f_{np}}{\partial \delta_0}\right)\delta\delta_0$$

$$(l = 1, \ldots, 9).$$

$$\Delta p_s = y_{100}^{(s)}\delta\lambda_{10} + y_{110}^{(s)}\delta\lambda_{20} + y_{120}^{(s)}\delta\lambda_{30} + y_{130}^{(s)}\delta\lambda_{8r0} + y_{140}^{(s)}\delta\lambda_{8g0} +$$

$$+ \left(z_{100}^{(s)} - \frac{z_{90}^{(s)}}{m_G}\frac{\partial f_{np}}{\partial a_0}\right)\delta a_0 + \left\{z_{110}^{(s)} - \left[y_1^{(s)}(t_n^-)(\dot{V}(t_n^+) - \right.\right.$$

$$- \dot{V}(t_n^-)) + y_2^{(s)}(t_n^-)(\dot{\theta}(t_n^+) - \dot{\theta}(t_n^-)) + y_{10}^{(s)}(t_n^-)(\dot{\lambda}_1(t_n^+) -$$

$$\left.\left. - \dot{\lambda}_1(t_n^-)) + y_{12}^{(s)}(t_n^-)(\dot{\lambda}_3(t_n^+) - \dot{\lambda}_3(t_n^-))\right]\frac{P_3^n}{a_0} - \frac{z_{90}^{(s)}}{m_G}\frac{\partial f_{np}}{\partial \mu_k}\right\}\delta\mu_k +$$

$$+ \left(z_{120}^{(s)} - \frac{z_{90}^{(s)}}{m_G}\frac{\partial f_{np}}{\partial p_k}\right)\delta p_k + \left(z_{130}^{(s)} - \frac{z_{90}^{(s)}}{m_G}\frac{\partial f_{np}}{\partial p_a}\right)\delta p_a +$$

$$+ \left(z_{140}^{(s)} - \frac{z_{90}^{(s)}}{m_G}\frac{\partial f_{np}}{\partial D}\right)\delta D + \left(z_{150}^{(s)} - \frac{z_{90}^{(s)}}{m_G}\frac{\partial f_{np}}{\partial n}\right)\delta n +$$

$$+ \left(z_{160}^{(s)} - \frac{z_{90}^{(s)}}{m_G}\frac{\partial f_{np}}{\partial \delta_r}\right)\delta\delta_r + \left(z_{170}^{(s)} - \frac{z_{90}^{(s)}}{m_G}\frac{\partial f_{np}}{\partial \delta_0}\right)\delta\delta_0$$

$$(s = 10, \ldots, 13).$$

In order to estimate the deviations of G_0 from the optimal value near the extreme, we should additionally integrate system (IV) from $\psi_k^0 = 0$ to t_0 with the following initial conditions:

$$
{}_jy_{1k} = \ldots = {}_jy_{7k} = 0, \quad {}_jy_{10k} = \ldots = {}_jy_{14k} = 0,
$$
$$
{}_jz_1 = \ldots = {}_jz_8 = {}_jz_{10k} = \ldots = {}_jz_{17k} = 0, \quad {}_jz_{9k} = 1.
$$

We then produce

$$
\begin{aligned}
dG_0 = {} & {}_jy_{10\,0}\,\delta\lambda_{10} + {}_jy_{11\,0}\,\delta\lambda_{20} + {}_jy_{12\,0}\,\delta\lambda_{30} + {}_jy_{13\,0}\,\delta\lambda_{40} + \\
& + {}_jy_{14\,0}\,\delta\lambda_{50} + \left({}_jz_{10\,0} - \frac{{}_jz_{9\,0}}{m_0}\,\frac{\partial f_{np}}{\partial a_0}\right)\delta a_0 + \\
& + \Big\{ {}_jz_{11\,0} - \big[{}_jy_1(t_k^-)\big(\dot V(t_k^+) - \dot V(t_k^-)\big) + {}_jy_2(t_k^-)\big(\dot\theta(t_k^+) - \\
& - \dot\theta(t_k^-)\big) + {}_jy_{10}(t_k^-)\big(\dot\lambda_1(t_k^+) - \dot\lambda_1(t_k^-)\big) + {}_jy_{12}(t_k^-)\big(\dot\lambda_3(t_k^+) - \\
& - \dot\lambda_3(t_k^-)\big)\big]\frac{P^{\circ}_{3\mu}}{a_0} - \frac{{}_jz_{9\,0}}{m_0}\,\frac{\partial f_{np}}{\partial \psi_k}\Big\}\delta\psi_k + \\
& + \left({}_jz_{12\,0} - \frac{{}_jz_{9\,0}}{m_0}\,\frac{\partial f_{np}}{\partial p_k}\right)\delta p_k + \left({}_jz_{13\,0} - \frac{{}_jz_{9\,0}}{m_0}\,\frac{\partial f_{np}}{\partial p_a}\right)\delta p_a + \\
& + \left({}_jz_{14\,0} - \frac{{}_jz_{9\,0}}{m_0}\,\frac{\partial f_{np}}{\partial D}\right)\delta D + \left({}_jz_{15\,0} - \frac{{}_jz_{9\,0}}{m_0}\,\frac{\partial f_{np}}{\partial n}\right)\delta n + \\
& + \left({}_jz_{16\,0} - \frac{{}_jz_{9\,0}}{m_0}\,\frac{\partial f_{np}}{\partial \delta_g}\right)\delta\delta_g + \left({}_jz_{17\,0} - \frac{{}_jz_{9\,0}}{m_0}\,\frac{\partial f_{np}}{\partial \delta_o}\right)\delta\delta_o .
\end{aligned}
$$

This relationship allows us to estimate ΔG_0 near the defined solution.

This concludes basically the description of the mathematical model of the algorithm for the solution of the formulated multipoint (three point) boundary problem. It remains only, following the flow chart presented in § 4, to compose the computer program, the computer realization of which allows us to perform the necessary combination of calculations determining the minimum value of G_0 and the optimal values of the parameters a_0, ψ_k, P_{max}, D, l_b, p_k, p_a, n, δ_g, δ_o and optimal controls $p(t)$ and $\alpha(t)$, thereby making it possible to perform preliminary planning of the LPBM capable of delivering the payload over the fixed range with minimum launch weight.

If we assume the gravitational field to be homogeneous and do not consider aerodynamic forces, this variational problem can be solved completely. This idealization of the problem has certain advantages and disadvantages. However, one thing is definite: the solution produced can be used as a good "zero" solution of the multipoint boundary problem. Therefore, there is some reason to present the results of the solution of the thus idealized variational problem of optimization of plan parameters, design loads and controls of a single-stage liquid fueled missile, capable of delivering a certain payload to a fixed range with minimum launch weight. These results are presented for ranges L_k = 1000-2000 km on Figures 1.28-1.49. In the calculations, we used typical plan equation coefficients for the current stage of development.

As general conclusions, we can note that first of all, in the powered stage the optimal operating mode of the engine is only the maximum thrust mode, i. e. there are no intermediate passive sectors in (t_0, t_π); secondly, the optimal pitch control is that in which the pitch angle is constant.

Furthermore, it should be kept in mind that for these ranges the optimal values of thrust to weight ratio, pressures in the combustion chamber, rotating speeds of the pump unit and pitch angle are such that they can be averaged and assumed to be: a_0 = 2.2, p_k = 125 kg/cm^2, $n = 25 \cdot 10^3$ rpm, θ = 48.2°. This will result in a deviation of G_0 from the optimal value by not over 0.5%.

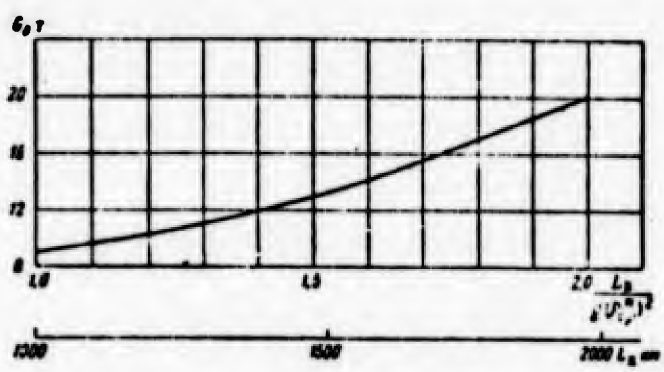

Figure 1.28. Optimal Launch Weight as a Function of Range and Relative Flight Range

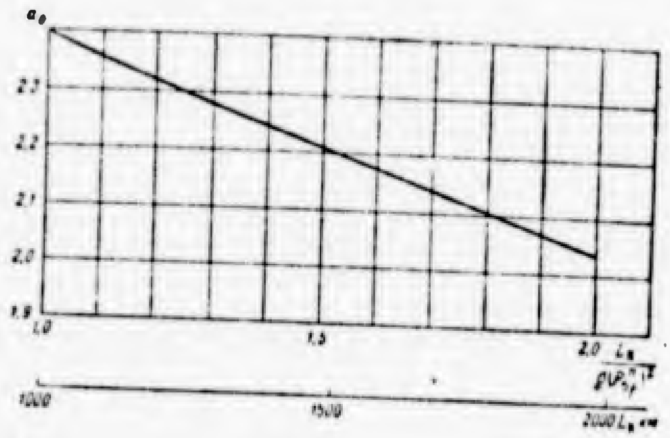

Figure 1.29. Optimal Thrust to Weight Ratio as a Function of Range and Relative Flight Range

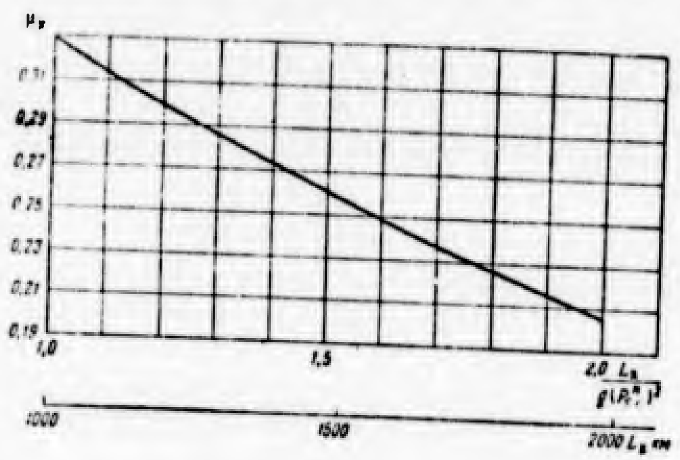

Figure 1.30. Optimal Final Weight as a Function of Range and Relative Flight Range

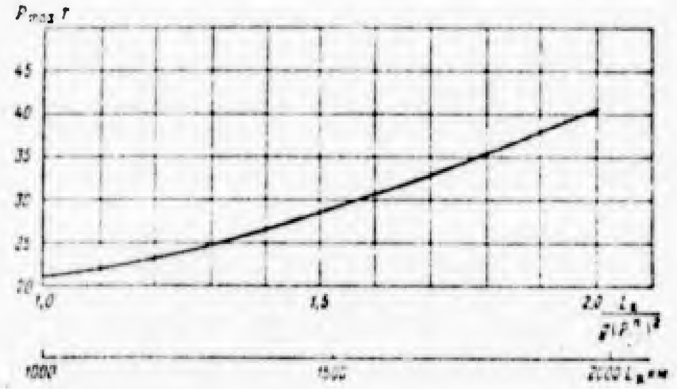

Figure 1.31. Optimal Thrust as a Function of Range and Relative Flight Range

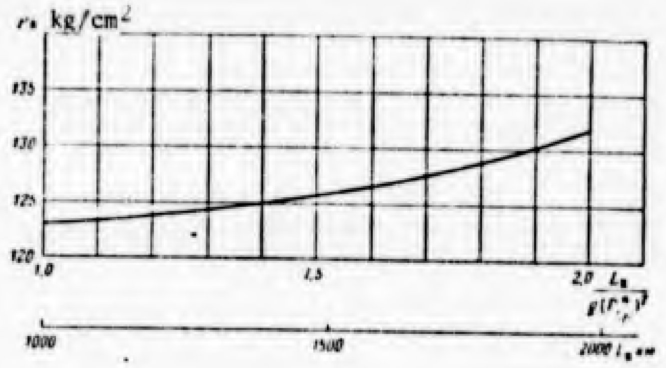

Figure 1.32. Optimal Pressure in Combustion Chamber as a Function of Range and Relative Flight Range

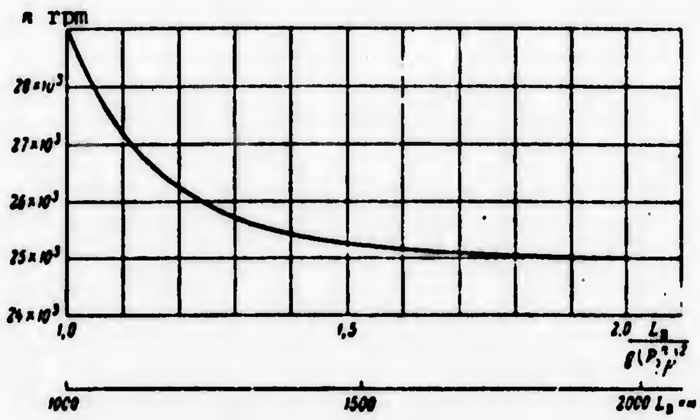

Figure 1.33. Optimal Rotating Speed of Pump Unit as a Function of Range and Relative Flight Range

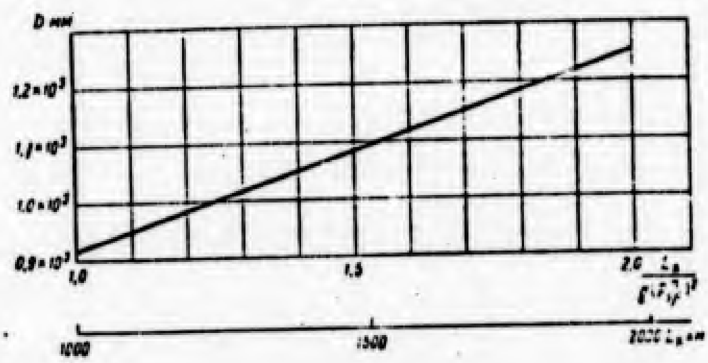

Figure 1.34. Optimal Diameter of Body as a Function of Range and Relative Flight Range

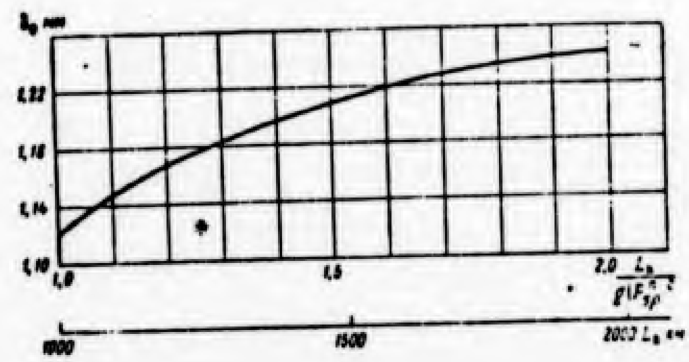

Figure 1.35. Optimal Thickness of Oxidizer Tank Shell as a Function of Range and Relative Flight Range

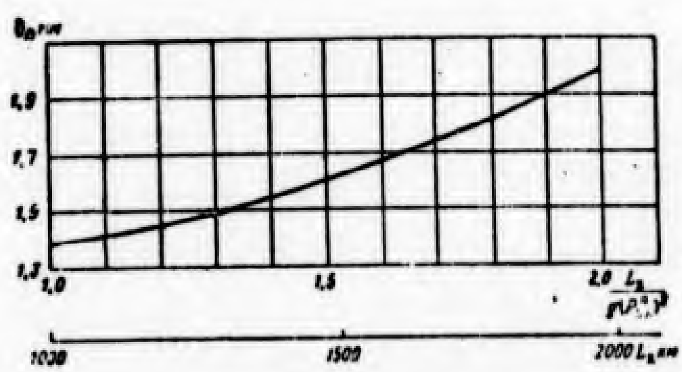

Figure 1.36. Optimal Thickness of Fuel Tank Shell as a Function of Range and Relative Flight Range

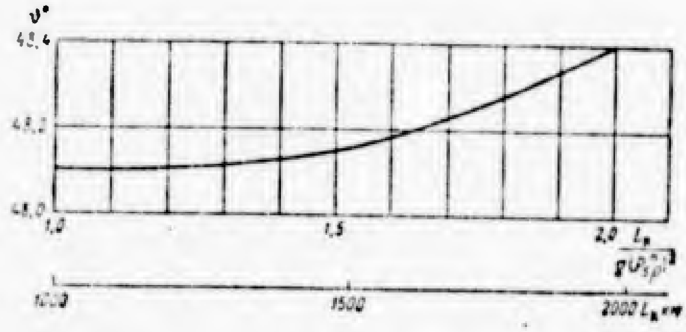

Figure 1.37. Optimal Pitch Angle as a Function of Range and Relative Flight Range

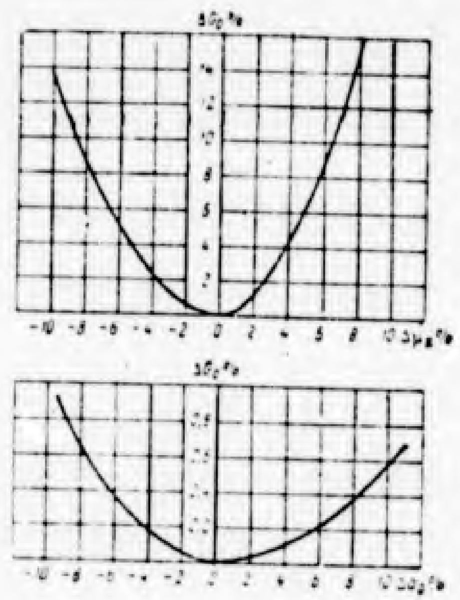

Figure 1.38. Deviation of Initial Weight From Optimal Value with Deviation of Parameters μ_k and a_0 from Optimal Values

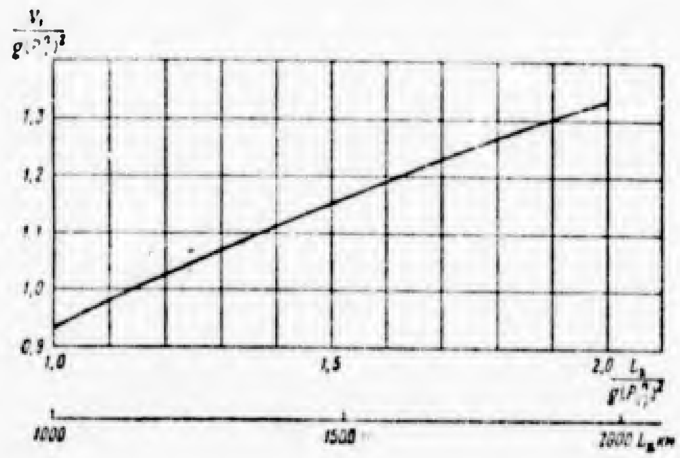

Figure 1.39. Optimal Relative Velocity at End of Powered Sector as a Function of Range and Relative Flight Range [$V_1 = V(t_n)$]

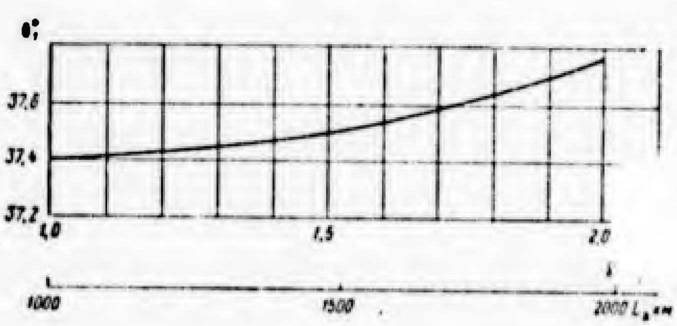

Figure 1.40. Optimal Final Power Angle as a Function of Range and Relative Flight Range

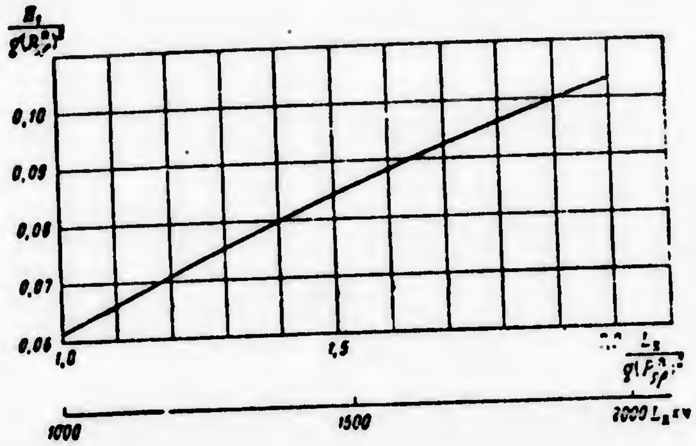

Figure 1.41. Optimal Relative Height of End of Powered Sector as a Function of Range and Relative Flight Range

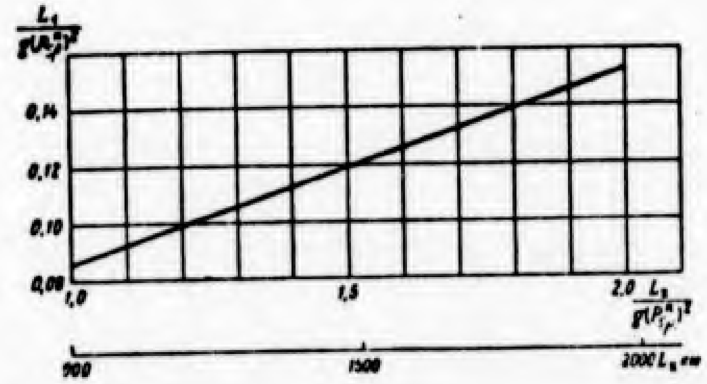

Figure 1.42. Optimal Relative Range at End of Powered Sector as a Function of Range and Relative Flight Range

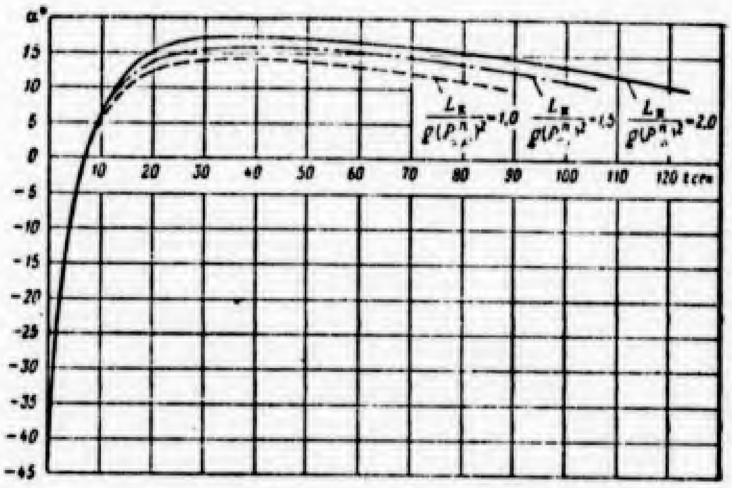

Figure 1.43. Optimal Angle of Attack as a Function of Time of Powered Sector

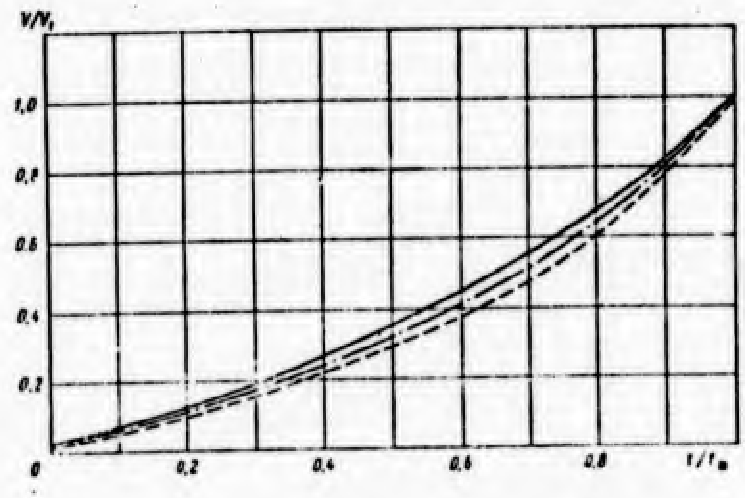

Figure 1.44. Optimal Relative Velocity in Active Sectors as a Function of Relative Time of Active Sector —— $\frac{L_к}{g(P^n_{..})^2}=2$, —·— $\frac{L_к}{g(P^n_{..})^2}=1.5$, – – – $\frac{L_к}{g(P^n_{..})^2}=1$

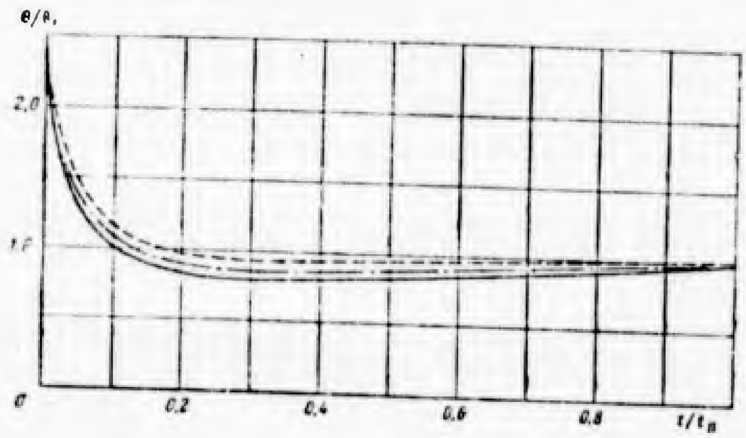

Figure 1.45. Optimal Relative Power Angle as a Function of Relative Time of Powered Sector:

$—\ \frac{L_к}{g(P^п_{уд})^2}=2;\ —\cdot—\ \frac{L_к}{g(P^п_{уд})^2}=1{,}5;\ ---\ \frac{L_к}{g(P^п_{уд})^2}=1$

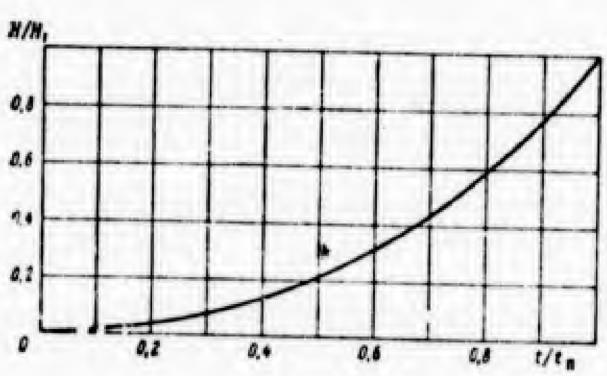

Figure 1.46. Optimal Relative Height of Powered Sector as a Function of Relative Time of Powered Sector:

$\frac{L_к}{g(P^п_{уд})^2}=1;\ 1{,}5;\ 2$

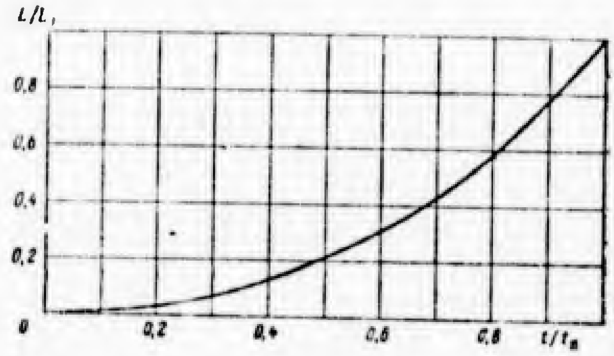

Figure 1.47. Optimal Relative Range of Powered Sector as a Function of Relative Time of Powered Sector:

$$\frac{L_k}{g\left(P^{n}_{sp}\right)^2}=1,\ 1{,}5,\ 2$$

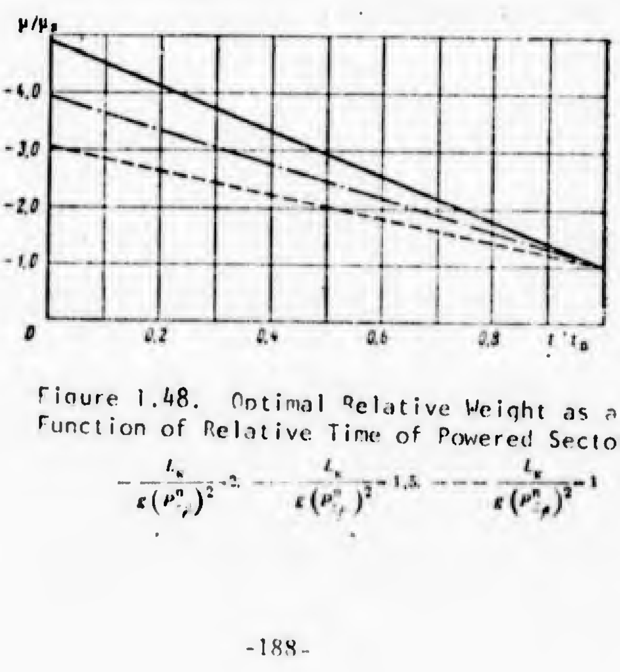

Figure 1.48. Optimal Relative Weight as a Function of Relative Time of Powered Sector:

$$— \frac{L_k}{g\left(P^{n}_{sp}\right)^2}=2;\quad -\cdot- \frac{L_k}{g\left(P^{n}_{sp}\right)^2}=1{,}5;\quad --- \frac{L_k}{g\left(P^{n}_{sp}\right)^2}=1$$

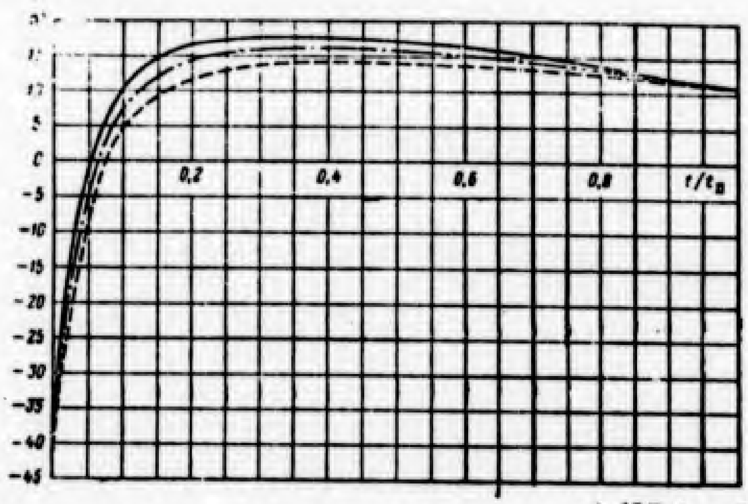

Figure 1.49. Optimal Angle of Attack as a Function of Relative Time of Powered Sector:

$$—\ \frac{L_k}{g(P^n_{3f})^2}=2\quad ---\ \frac{L_k}{g(P^n_{3f})^2}=1.5\quad - - -\ \frac{L_k}{g(P^n_{3f})^2}=1$$

Our attention is drawn by the fact that for various ranges L_k on the optimal phase trajectory, the curves

$$\frac{V}{V_n}=f_1\left(\frac{t}{t_n}\right),\ \frac{\theta}{\theta_n}=f_2\left(\frac{t}{t_n}\right),\ \frac{H}{H_n}=f_3\left(\frac{t}{t_n}\right)\ \text{and}\ \frac{L}{L_n}=f_4\left(\frac{t}{t_n}\right)$$

practically correspond (Figures 1.44-147).

The curves shown on Figure 1.38 allow us to estimate the deviation from the minimum value of launch weight of a missile as the parameters μ_k and a_0 deviate from their optimal values.

CHAPTER II. VARIATIONAL METHOD OF OPTIMIZATION OF MULTISTAGE FLIGHT VEHICLE CONSIDERING POSSIBILITY OF INDEPENDENT MANEUVER OF STAGES

§ 1. Statement of Problem

In the initial stage of planning of a flight vehicle and power plant, the problem arises of determining at least the main plan parameters and the possible modes of motion providing for fulfillment of the TA of the flight vehicle with the maximum value of the criterion of effectiveness I. Only the main plan parameters and modes of motion are used in this stage of planning, since establishment of the precise weight relationships and the dependences between parameters of a flight vehicle and power plant not yet in existence (and particularly their inscription in analytic form) is a problem which is very difficult in practice. Furthermore, in this stage of planning the necessity frequently arises of determining an efficient arrangement and aerodynamic plan for the flight vehicle, analysis of which is usually performed with somewhat lower requirements for accuracy of the various weight relationships and therefore can be performed considering their dependences only on the main plan parameters.

The essence of the matter is that the main plan parameters influence the entire process of planning in a certain manner.

In one way or another, the problem of determining the main plan parameters, modes of movement of the flight vehicle and operating modes of the power plant allowing them to be planned for performance of the TA with the maximum value of the criterion of effectiveness has practical sense. The necessity of solving this problem is encountered each time the initial stage of planning of a flight vehicle is undertaken. For a multistage flight vehicle, the solution of this problem is even more sensible, and is of particular interest.

In this chapter, we study the problem of determining the main plan parameters and modes of movement of a multistage (for simplicity we assume two stages) flight vehicle, capable of fulfilling a formulated TA with the maximum value of the criterion of effectiveness I or, more accurately, where I = sup, which in general form will be assumed equal to

$$I = I(V_k, \theta_k, H_k, L_k, t_k, G_0, G_{p.t}). \qquad (2.1.1)$$

It is assumed that the second stage, after separation from the "main" vehicle, can perform independent maneuver (return to the initial base, travel to a fixed destination, etc.).

The problem is analyzed of optimization of the main plan parameters and modes of movement of several objects moving along various trajectories in order to perform the TA set before them and preliminarily accelerated by a single object (booster), the parameters and modes of movement of which are also optimized. It is considered that the structural-power and aerodynamic plans of the flight vehicle, the type and characteristics of the power plant are fixed, the physical-chemical characteristics of the materials and fuels are known.

Both rocket and jet motors may be analyzed as power plants for the stages. The problem at hand has certain specific features. Therefore, solutions are possible which may fall outside the framework of the solution of the variational problem of Chapter I. Therefore, the necessity arises of special analysis of this problem, and there is reason for greater definition and concreteness to study its solution individually.

The main plan parameters of the multistage flight vehicle will be G_{0J} (J = I, II), where $G_0 = G_{0I}$, and G_{p1}, μ_{kJ}, a_{0i} (i = 1, 2), J_{0i}, b_{0i}, $\nu_k^{(H)}$.

The other plan parameters are considered fixed from the experience of planning or the technical and operational requirements.

The plan equation and other weight relationships can be represented in general form as follows:

$$\left.\begin{aligned} \varphi_1^{(0)} &\equiv G_{p.t} - G_0\mu_{p.t} = 0, \\ \varphi_1^{(1)} &\equiv G_{0II} - G_0\mu_{0I} = 0, \\ \varphi_2 &\equiv \mu_{0I} + \mu_{kI} - \mu_{k1} = 0, \end{aligned}\right\} \qquad (2.1.2)(2.I)$$

where

$$\mu_{p.t} = f_{\text{op}}(a_{0i}, b_{0i}, J_{0i}, \mu_{k\,J}, \mu_k^{(m)}, G_0, G_{0\,II}),$$
$$\mu_{0I} = f_{\text{op}}^{(1)}(a_{01}, b_{01}, J_{01}, \mu_{k\,I}, \mu_k^{(m)}, G_0),$$
$$\mu_{0\,II} = f_{\text{op}}^{(2)}(a_{02}, b_{02}, J_{02}, \mu_{k\,II}, G_0, G_{0\,II}).$$

The coupling equations of the plan parameters will be expressed also in the form

$$\left.\begin{aligned} \varphi_3^{(i)} &= a_{0i} - \frac{P_{\max}^{(i)}}{G_{0J}} = 0 \quad (i = 1, 2), \\ \varphi_4^{(i)} &= b_{0i} - \frac{S^{(i)}}{G_{0J}} = 0. \end{aligned}\right\} \qquad (2.1.3)(2.1)$$

The relationships (2.1.2) and (2.1.3) are correct within the permissible range of change of the main parameters. For this and a number of other reasons dictated by technical and operational conditions, conditions of arrangement, etc., we must consider the condition of limitation of plan parameters

$$\left.\begin{aligned} &G_{0J\min} \leqslant G_{0J} \leqslant G_{0J\max}, \\ &a_{0i\min} \leqslant a_{0i} \leqslant a_{0i\max}, \\ &(P_{\max i})_{\min} \leqslant P_{\max i} \leqslant (P_{\max i})_{\max}, \\ &J_{0i\min} \leqslant J_{0i} \leqslant J_{0i\max}, \\ &b_{0i\min} \leqslant b_{0i} \leqslant b_{0i\max}, \\ &S_{\min}^{(i)} \leqslant S^{(i)} \leqslant S_{\max}^{(i)}, \\ &G_{p.t\min} \leqslant G_{p.t} \leqslant G_{p.t\max}. \end{aligned}\right\} \qquad (2.1.4)$$

The beginning and end of flight of the vehicle are related by the boundary conditions which can be written in general form as:

$$\left.\begin{aligned} &t_0 = 0, \quad \mu_0 = 1, \quad \psi_{0\varrho}(V_0, \vartheta_0, H_0, L_0) = 0 \\ &\qquad\qquad (\varrho = 1, \ldots, m \leqslant 4), \\ &\psi_{k\sigma}(V_k, \vartheta_k, H_k, L_k, t_k) = 0 \\ &\qquad\qquad (\sigma = 1, \ldots, n \leqslant 5). \end{aligned}\right\} \qquad (2.1.5)$$

At the beginning and end of the 'operation' of each stage, the following equations must be fulfilled:

$$\left.\begin{aligned} &\mu_0(t_-^{(1)}) = 1, \\ &\varphi_\mu^{(i)} = \mu(t_-^{(i)}) - \mu_{k\,J} = 0. \end{aligned}\right\} \qquad (2.1.5)$$

These last equations also express the limitation on the phase variable μ in the form

$$\mu(t) \geqslant \mu_{\kappa I}.$$

If after achieving the equality $\mu(t^{(2)}) = \mu_{kII}$ further flight is possible in principle, then as was shown in Chapter I, §1, it should be performed with the power plant switched off.

After separation, the booster and second stage perform independent maneuvers, the purposes of which are different. Therefore, at the end of the independent flight of the booster, the boundary conditions which must be fulfilled are generally different from (2.1.5), and represented in the form

$$\tilde{\varphi}_{\kappa\pi}(\tilde{V}_{\kappa}, \tilde{\theta}_{\kappa}, \tilde{H}_{\kappa}, \tilde{L}_{\kappa}, \tilde{t}_{\kappa}) = 0$$
$$(\pi = 1, \ldots, l \leqslant 5). \qquad (2.1.6)$$

We have introduced the tilde here in order to distinguish the phase variables of the booster from the phase variables of the second stage after separation.

Furthermore, the following conditions should be observed:

$$\left.\begin{array}{l} \tilde{\varphi}_{\mu} = \mu(\tilde{t}^{(1)}) - \mu_{\kappa I} = 0, \\ \tilde{\varphi}_{\mu}^{(\kappa)} = \mu(\tilde{t}_{\kappa}) - \mu_{\kappa}^{(\kappa)} = 0. \end{array}\right\} \qquad (2.1.6)$$

This latter equation expresses the limitation on the phase variable μ in the form

$$\tilde{\mu}(t) \geqslant \mu_{\kappa}^{(\kappa)},$$

where, if after equation $\tilde{\mu}(\tilde{t}_k) \geq \mu_k^{(H)}$ is fulfilled, further flight is possible in principle, it should occur with the power plant turned off.

Conditions (2.1.6) follow from the requirements placed on the flight of the booster after separation. They can be formulated from the TA, dictated by the track of the flight vehicle or a number of other factors.

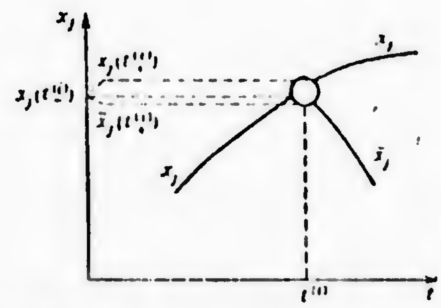

Figure 2.1. Arbitrary Representation of Branching of Phase Variable x_j and Absence of Double Discontinuity at Point $t^{(i)}$

In the following we will assume, as in Chapter I, § 2, that separation of the stages occurs instantaneously and without perturbations. This allows us to assume the phase variables V(t), θ(t), H(t) and L(t) at the moment of separation to be continuous, indicated by the conditions

$$\left.\begin{array}{ll}\varphi_V=V(t_-^{(1)})-V(t_+^{(1)})=0, & \tilde{\varphi}_V=V(t_-^{(1)})-\tilde{V}(t_+^{(1)})=0,\\ \varphi_\theta=\theta(t_-^{(1)})-\theta(t_+^{(1)})=0, & \tilde{\varphi}_\theta=\theta(t_-^{(1)})-\tilde{\theta}(t_+^{(1)})=0,\\ \varphi_H=H(t_-^{(1)})-H(t_+^{(1)})=0, & \tilde{\varphi}_H=H(t_-^{(1)})-\tilde{H}(t_+^{(1)})=0,\\ \varphi_L=L(t_-^{(1)})-L(t_+^{(1)})=0, & \tilde{\varphi}_L=L(t_-^{(1)})-\tilde{L}(t_+^{(1)})=0,\\ \varphi_t=t_-^{(1)}-t_+^{(1)}=0, & \tilde{\varphi}_t=t_-^{(1)}-\tilde{t}_+^{(1)}=0.\end{array}\right\} \quad (2.1.7)$$

Condition (2.1.7) reflects the absence at point $t^{(1)}$ of a first order double discontinuity -- the first discontinuity and the second discontinuity[1] -- to the right of which the phase variables are, respectively:

$$V(t_+^{(1)}),\ \ldots,\ L(t_+^{(1)}) \text{ и } \tilde{V}(t_+^{(1)}),\ \ldots,\ \tilde{L}(t_+^{(1)}) \text{ (fig. 2.1).}$$

Making the assumptions introduced in Chapter I, § 1, we write

[1] The names "first discontinuity" and "second discontinuity" are arbitrarily assigned.

the equations of motion

$$\varphi_1^{(i)} = V' = \frac{g_0}{\mu}(a_{0i} p \cos\alpha - b_{0i}\bar{Q}^{(i)}) - g \sin\theta,$$

$$(i = 1, 2),$$

$$\varphi_2^{(i)} = \theta' = \frac{g_0}{\mu V}(a_{0i} p \sin\alpha + b_{0i}\bar{Y}^{(i)}) - \frac{g\cos\theta}{V} + \frac{V\cos\theta}{R_3 + H},$$

kinematic couplings

$$\varphi_3^{(i)} = H' = V\sin\theta,$$

$$\varphi_4^{(i)} = L' = \frac{VR_3}{R_3 + H}\cos\theta, \qquad (2.1)$$

divergence characteristic equation

$$\varphi_5^{(i)} = \mu' = -\frac{a_{0i}}{J_{0i}} f^{(i)}(V, H, p, a_{0i}, J_{0i}).$$

and the coupling equations for the booster after separtion of the stage are written as:

equations of motion

$$\tilde{\varphi}_1 = \tilde{V}' = \frac{g_0}{\mu q_{k1}}(a_{01}\tilde{p}\cos\tilde{\alpha} - b_{01}\tilde{Q}) - \tilde{g}\sin\tilde{\theta},^{1}$$

$$\tilde{\varphi}_2 = \tilde{\theta}' = \frac{g_0}{\mu q_{k1}\tilde{V}}(a_{01}\tilde{p}\sin\tilde{\alpha} + b_{01}\tilde{Y}) - \frac{\tilde{g}\cos\tilde{\theta}}{\tilde{V}} + \frac{\tilde{V}\cos\tilde{\theta}}{R_3 + \tilde{H}},$$

kinematic couplings

$$\tilde{\varphi}_3 = \tilde{H}' = \tilde{V}\sin\tilde{\theta},$$

$$\tilde{\varphi}_4 = \tilde{L}' = \frac{\tilde{V}R_3}{R_3 + \tilde{H}}\cos\tilde{\theta}, \qquad (2.1)$$

divergence characteristic equation

$$\tilde{\varphi}_5 = \tilde{\mu}' = -\frac{a_{01}}{J_{01}} f^{(1)}.$$

Here

$$\bar{Q}^{(i)} = \frac{\rho V^2}{2} c_x^{(i)},\ \bar{Y}^{(i)} = \frac{\rho V^2}{2} c_y^{(i)},$$

$$\tilde{Q} = \frac{\tilde{\rho}\tilde{V}^2}{2}\tilde{c}_x,\ \tilde{Y} = \frac{\tilde{\rho}\tilde{V}^2}{2}\tilde{c}_y.$$

---

[1] Here for simplification we assume $P_{max}^{(I)} = P_{max}^{(II)}$ and $S^{(I)} = S^{(II)}$.

where

$$
\begin{aligned}
&c_x^{(i)} = c_{x0}^{(i)}(V, H) + c_{xi}^{(i)}(V, H, \alpha),\\
&c_y^{(i)} = c_y^{(i)}(V, H, \alpha),\\
&\tilde{c}_x = \tilde{c}_{x0}(\tilde{V}, \tilde{H}) + \tilde{c}_{xi}(\tilde{V}, \tilde{H}, \tilde{\alpha}),\\
&\tilde{c}_y = \tilde{c}_y(\tilde{V}, \tilde{H}, \tilde{\alpha}),\\
&f^{(1)} = f^{(1)}(\tilde{V}, \tilde{H}, \tilde{p}, J_{01}, a_{01}).
\end{aligned}
$$

The permissible change in control functions $\alpha(t)$, $p(t)$ and $\tilde{\alpha}(t)$, $\tilde{p}(t)$ lies within fixed limits expressed by the permissible control condition:

$$
\left.
\begin{aligned}
&\alpha_{min}^{(i)}(V, H) \leqslant \alpha(t) \leqslant \alpha_{max}^{(i)}(V, H),\\
&0 \leqslant p^{(i)}(t) \leqslant p_{max}^{(i)}(V, H),\\
&\tilde{\alpha}_{min}(\tilde{V}, \tilde{H}) \leqslant \tilde{\alpha}(\tilde{t}) \leqslant \tilde{\alpha}_{max}(\tilde{V}, \tilde{H}),\\
&0 \leqslant \tilde{p}(\tilde{t}) \leqslant \tilde{p}_{max}(\tilde{V}, \tilde{H}).
\end{aligned}
\right\} \tag{2.1.8}
$$

Generally speaking, a change in the phase variables may also be related to certain limitations. They arise due to the corresponding requirement: on the flight trajectories or are dictated by the TA. For simplification, we will assume that limitations on changes in the phase variables can exist only during the flight of the first and second stages. Suppose they are represented as follows:

$$f_\Phi^{(i)}(V, \theta, H) \geqslant 0. \tag{2.1.9}$$

We can now more concretely formulate the task formulated in this chapter.

It is reduced to determination of the plan parameters α_{0i}, J_{0i}, $P_{max}^{(i)}$, h_{0i}, $S^{(i)}$ and $_{kJ}$, $^{v}{}_{kI}$, $_{k}^{(H)}$, G_{0J}, G_{pI} and controls $\alpha(t)$ and $p(t)$ of the multistage flight vehicle, capable with limitations (2.1.8) and (2.1.9) of performing maneuvers within the framework of the boundary conditions ψ = 0 and ψ_{k0} = 0 such that the phase trajectory of the booster at the end of its flight satisfies the boundary conditions ψ_{kT} = 0, while the criterion of effectiveness I reaches its maximum value (more accurately, the precise upper boundary).

Controls $\alpha(t)$, $p(t)$ and $\tilde{\alpha}(t)$, $\tilde{p}(t)$, as in Chapter I, are in the class of piecewise continuous functions, while the phase variables $V(t)$, $\theta(t)$, $H(t)$, $L(t)$ and $\tilde{V}(t)$, $\tilde{\theta}(t)$, $\tilde{H}(t)$, $\tilde{L}(t)$ are in the class of piecewise-smooth functions. Phase variable $\nu(t)$ undergoes one first order discontinuity at point $p^{(1)}$, its value to the right of the discontinuity being fixed [see (2.1.5) and (2.1.6)].

Essentially, this is a variational problem. However, this problem is stated for the first time as a variational problem, so that its solution has not yet been found. Due to its specific peculiarity of branching at points $t^{(i)}$ of the phase trajectory into several trajectories it is somewhat different from the general variational problem given in Chapter I, and therefore should be individually analyzed.

Following the mathematical theory presented earlier (see appendix), let us go over from the closed to the open area of change of the main plan parameters and control functions $\alpha(t)$, $p(t)$ and $\tilde{\alpha}(t)$, $\tilde{p}(t)$. To do this, we introduce the following equations in system (2.I)

$$\left.\begin{aligned}
&\varphi_6^{(i)} = (\alpha - \alpha_{min}^{(i)})(\alpha_{max}^{(i)} - \alpha) - \nu_\alpha^{(i)2}(t) = 0,\\
&\varphi_7^{(i)} = p(p_{max}^{(i)} - p) - \nu_p^{(i)2}(t) = 0,\\
&\varphi_{01}^{(i)} = (G_{0i} - G_{0i\,min})(G_{0i\,max} - G_{0i}) - w_{1i}^2 = 0,\\
&\varphi_{02}^{(i)} = (a_{0i} - a_{0i\,min})(a_{0i\,max} - a_{0i}) - w_{2i}^2 = 0,\\
&\varphi_{03}^{(i)} = [P_{max\,i} - (P_{max\,i})_{min}]\,[(P_{max\,i})_{max} - P_{max\,i}] - w_{3i}^2 = 0,\\
&\varphi_{04}^{(i)} = (b_{0i} - b_{0i\,min})(b_{0i\,max} - b_{0i}) - w_{4i}^2 = 0,\\
&\varphi_{05}^{(i)} = (J_{0i} - J_{0i\,min})(J_{0i\,max} - J_{0i}) - w_{5i}^2 = 0,\\
&\varphi_{06}^{(i)} = (S^{(i)} - S_{min}^{(i)})(S_{max}^{(i)} - S^{(i)}) - w_{6i}^2 = 0,\\
&\varphi_{07} = (G_{p.t} - G_{p.t\,min})(G_{p.t\,max} - G_{p.t}) - w_7^2 = 0.
\end{aligned}\right\} \quad (2.I)$$

$$\left.\begin{aligned}
&\tilde{\varphi}_6 = (\tilde{\alpha} - \tilde{\alpha}_{min})(\tilde{\alpha}_{max} - \tilde{\alpha}) - \tilde{\nu}_\alpha^2(t) = 0,\\
&\tilde{\varphi}_7 = (\tilde{p}_{max} - \tilde{p})\,\tilde{p} - \tilde{\nu}_p^2(t) = 0,
\end{aligned}\right\} \quad (1.\tilde{I})$$

If the phase trajectory of the flight vehicle reaches the boundary

$$f_\varphi^{(i)} = 0, \qquad (2.1.10)$$

then this equation must act as the final conditions for the sector of the trajectory which lies within the area and is immediately adjacent to the sector on the boundary. Furthermore, at this time the control will be determined from the condition of equality of function $f_\varphi^{(i)}$ to zero, leading to the equation

$$\chi^{(l)} \equiv \frac{\partial f_{\phi}^{(l)}}{\partial V} \ddot{\varphi}_1^{(l)} + \frac{\partial f_{\phi}^{(l)}}{\partial \theta} \ddot{\varphi}_2^{(l)} + \frac{\partial f_{\phi}^{(l)}}{\partial H} \ddot{\varphi}_3^{(l)} = 0,$$

in which controls $\alpha(t)$ and $p(t)$ appear in explicit form. Then in system (2.I) we introduce the equation

$$\ddot{\varphi}_{\phi}^{(l)} \equiv \chi^{(l)} - f_{\phi}^{(l)} \nu(t) = 0, \qquad (2.I)$$

where $\nu(t)$ is an arbitrary function of time, $\nu \equiv 0$ where

$$f_{\phi}^{(l)} \equiv 0.$$

After all of these transformations, we can formulate the variational problem which follows from this problem. It consists in the following:

in the interval $t_0 \leq t \leq t_k$, in the class of permissible

phase variables

$$V(t),\ \theta(t),\ H(t),\ L(t),\ \mu(t), \qquad (2.1.11)$$

control functions

$$\alpha(t),\ p(t), \qquad (2.1.12)$$

plan parameters

$$G_{\alpha J},\ G_{pJ},\ a_{0l},\ P_{max\,l},\ J_{0l},\ b_{0l},\ S^{(l)},\ \mu_{\kappa J},\ \mu_{\kappa l}, \qquad (2.1.13)$$

arbitrary control functions and parameters

$$\nu_{\alpha}^{(l)}(t),\ \nu_{p}^{(l)}(t),\ \nu^{(l)}(t),\ w_{1l},\ w_{2l},\ w_{3l},\ w_{4l},\ w_{5l},\ w_{6l},\ w_{7}, \qquad (2.1.14)$$

satisfying couplings (1.I) and boundary conditions (2.1.5), find phase variables (2.1.11), control functions (2.1.12) and parameters (2.1.13) with which, in the interval $t^{(1)} \leq \tilde{t} < t_k$, in the class of permissible

phase variables

$$\tilde{V}(\tilde{t}),\ \tilde{\theta}(\tilde{t}),\ \tilde{H}(\tilde{t}),\ \tilde{L}(\tilde{t}),\ \tilde{\mu}(\tilde{t}), \qquad (2.1.15)$$

control functions

$$\tilde{\alpha}(\tilde{t}),\ \tilde{p}(\tilde{t}), \qquad (2.1.15)$$

plan parameters

$$\mu_{\kappa}^{(n)},\ a_{01},\ b_{01},\ J_{01},\ S^{(1)},\ P_{\max l}, \tag{2.1.15}$$

arbitrary control functions and parameters

$$\tilde{v}_{\nu}(t),\ \tilde{v}_{p}(t),\ w_{31},\ w_{41},\ w_{51},\ w_{61},$$

satisfying the boundary conditions(2.1.6) and couplings (1.1), the expression

$$I(V_{\kappa},\ \theta_{\kappa},\ H_{\kappa},\ L_{\kappa},\ t_{\kappa},\ G_{p\,l},\ G_0) \tag{2.1.16}$$

reaches its maximum value.

We compose the expression

$$\begin{aligned}\Phi = I &+ e_1^{(0)}\gamma_1^{(0)} + e_1^{(1)}\gamma_1^{(1)} + e_2\gamma_2 + \sum_{i=1}^{2} e_3^{(i)}\gamma_3^{(i)} + \\ &+ \sum_{i=1}^{2} e_4^{(i)}\gamma_4^{(i)} + \sum_{j=1}^{6}\sum_{i=1}^{2} e_{0j}^{(i)}\gamma_{0j}^{(i)} + e_{07}\gamma_{07} + \\ &+ \sum_{Q=1}^{m} e_Q\gamma_{0Q} + \sum_{s=1}^{n} e_{\kappa s}\gamma_{\kappa s} + \sum_{i=1}^{2} e_p^{(i)}\gamma_p^{(i)} + \\ &+ \sum_{s=1}^{l} \tilde{e}_{\kappa s}\tilde{\gamma}_{\kappa s} + \tilde{e}_p^{(n)}\tilde{\gamma}_p^{(n)} + \tilde{e}_p\tilde{\gamma}_p + e_V\gamma_V + \\ &+ \tilde{e}_V\tilde{\gamma}_V + e_\theta\gamma_\theta + \tilde{e}_\theta\tilde{\gamma}_\theta + e_H\gamma_H + \tilde{e}_H\tilde{\gamma}_H + \\ &+ e_L\gamma_L + \tilde{e}_L\tilde{\gamma}_L + e_t\gamma_t + \tilde{e}_t\tilde{\gamma}_t + \\ &+ \int_{t_0}^{t^{(1)}} F_1\,dt + \int_{t^{(1)}}^{\tilde{T}_\kappa} \tilde{F}\,dt + \int_{t^{(2)}}^{t_\kappa} F_2\,dt,\end{aligned} \tag{2.1.17}$$

where

$$F_i = \sum_{j=1}^{n}(\dot{x}_j - \varphi_j^{(i)})\lambda_j, \quad \tilde{F} = \sum_{j=1}^{7}(\dot{\tilde{x}}_j - \tilde{\varphi}_j)\tilde{\lambda}_j.$$

$\lambda_i^{(i)}$, $\tilde{\lambda}_j$ are the variable Lagrange coefficients,

$e_1^{(0)}, \ldots, e_t$ are the constant Lagrange coefficients,

$$x_j = V, \vartheta, H, L, \mu; \quad \tilde{x}_j = \tilde{V}, \tilde{\vartheta}, \tilde{H}, \tilde{L}, \tilde{\mu}.$$

Relationship (2.1.17) is always such that $\Phi = I$. According to the appendix, we can state that the conditional extreme of the criterion of effectiveness I corresponds with the unconditional extreme of functional Φ.

Let us now go over to determination of the unconditional maximum of functional Φ, determining its necessary conditions.

§ 2. Necessary Conditions for Optimization of Criterion of Effectiveness (Condition of Stability, Weierstrass Condition and Maximum Principle Considering Control Condition)

The necessary conditions for the maximum value of the criterion of effectiveness of a multistage flight vehicle will include the condition of stability, the Weierstrass condition and the maximum principle, which we will analyze considering the control condition.

From the condition of stability, i. e. $d\Phi = 0$, we produce: the Euler-Lagrange equations and the conditions of optimization of parameters, discontinuity conditions and branching conditions at points of stage separation and the conditions of transversality. Going over to their determination, we write

$$\begin{aligned}
d\Phi = {} & dI + e_1^{(0)} d\xi_1^{(0)} + e_1^{(1)} d\xi_1^{(1)} + e_2 d\xi_2 + \\
& + \sum_{i=1}^{2} e_3^{(i)} d\xi_3^{(i)} + \sum_{i=1}^{2} e_4^{(i)} d\xi_4^{(i)} + \sum_{j=1}^{6} \sum_{i=1}^{2} e_{0j}^{(i)} d\xi_{0j}^{(i)} + \\
& + e_{0\tau} d\xi_{0\tau} + \sum_{\varrho=1}^{m} e_\varrho d\xi_{0\varrho} + \sum_{\nu=1}^{n} e_{\kappa\nu} d\xi_{\kappa\nu} + \\
& + \sum_{i=1}^{2} e_\mu^{(i)} d\xi_\mu^{(i)} + \sum_{\varkappa=1}^{l} \tilde{e}_{\nu\varkappa} d\tilde{\xi}_{\nu\varkappa} + \tilde{e}_\mu^{(n)} d\tilde{\xi}_\mu^{(n)} + \\
& + \tilde{e}_\mu d\tilde{\xi}_\mu + e_V d\xi_V + \tilde{e}_V d\tilde{\xi}_V + \tilde{e}_\vartheta d\tilde{\xi}_\vartheta + e_\vartheta d\xi_\vartheta + \\
& + e_H d\xi_H + \tilde{e}_H d\tilde{\xi}_H + e_L d\xi_L + \tilde{e}_L d\tilde{\xi}_L + \\
& + e_t d\xi_t + \tilde{e}_t d\tilde{\xi}_t - \left[\left(F_1 - \sum_{j=1}^{5} x_j' \frac{\partial F_1}{\partial x_j'} \right) dt + \right.
\end{aligned}$$

$$
+\sum_{j=1}^{s}\frac{\partial F_1}{\partial x_j'}dx_j\Bigg]_{t_0}+\int_{t_0}^{t^{(1)}}\sum_{j=1}^{s}\left(\frac{\partial F_1}{\partial x_j}-\frac{d}{dt}\frac{\partial F_1}{\partial x_j'}\right)\Delta x_j\,dt+
$$

$$
+\int_{t_0}^{t^{(1)}}\sum_{m=1}^{s}\frac{\partial F_1}{\partial u_m}\Delta u_m\,dt+\int_{t_0}^{t^{(1)}}\sum_{l=1}^{6}\frac{\partial F_1}{\partial a_l^{(1)}}\Delta a_l^{(1)}\,dt+
$$

$$
+\int_{t^{(1)}}^{\tilde{T}_k}\sum_{j=1}^{s}\left(\frac{\partial\tilde{F}_1}{\partial\tilde{x}_j}-\frac{d}{dt}\frac{\partial\tilde{F}_1}{\partial x_j'}\right)\Delta\tilde{x}_j\,dt+\int_{t^{(1)}}^{\tilde{T}_k}\sum_{l=1}^{6}\frac{\partial\tilde{F}_1}{\partial a_l^{(1)}}\Delta a_l^{(1)}\,dt+
$$

$$
+\int_{t^{(1)}}^{\tilde{T}_k}\sum_{m=1}^{4}\frac{\partial\tilde{F}_1}{\partial\tilde{u}_m}\Delta u_m\,dt+
$$

$$
+\left[\left(F_1-\sum_{j=1}^{s}x_j'\frac{\partial F_1}{\partial x_j'}\right)dt+\sum_{j=1}^{s}\frac{\partial F_1}{\partial x_j'}dx_j\right]_{t_-^{(1)}}-
$$

$$
-\left[\left(F_2-\sum_{j=1}^{s}x_j'\frac{\partial F_2}{\partial x_j'}\right)dt+\sum_{j=1}^{s}\frac{\partial F_2}{\partial x_j'}dx_j\right]_{t_+^{(1)}}-
$$

$$
-\left[\left(\tilde{F}_1-\sum_{j=1}^{s}\tilde{x}_j'\frac{\partial\tilde{F}_1}{\partial\tilde{x}_j'}\right)d\tilde{t}+\sum_{j=1}^{s}\frac{\partial\tilde{F}_1}{\partial\tilde{x}_j'}d\tilde{x}_j\right]_{t_+^{(1)}}+\ldots+
$$

$$
+\left[\left(F_2-\sum_{j=1}^{s}x_j'\frac{\partial F_2}{\partial x_j'}\right)dt+\sum_{j=1}^{s}\frac{\partial F_2}{\partial x_j'}dx_j\right]_{t_{1-}^{*}}+e_\phi^{(2)}df_\phi^{(2)}-
$$

$$
-\left[\left(F_2-\sum_{j=1}^{s}x_j'\frac{\partial F_2}{\partial x_j'}\right)dt+\sum_{j=1}^{s}\frac{\partial F_2}{\partial x_j'}dx_j\right]_{t_{1+}^{*}}+\ldots+
$$

$$
+\left[\left(F_2-\sum_{j=1}^{s}x_j'\frac{\partial F_2}{\partial x_j'}\right)dt+\sum_{j=1}^{s}\frac{\partial F_2}{\partial x_j'}dx_j\right]_{t_k}+
$$

$$
+\left[\left(\tilde{F}_1-\sum_{j=1}^{s}\tilde{x}_j'\frac{\partial\tilde{F}_1}{\partial\tilde{x}_j'}\right)d\tilde{t}+\sum_{j=1}^{s}\frac{\partial\tilde{F}_1}{\partial x_j'}d\tilde{x}_j\right]_{\tilde{T}_k}=0, \qquad (2.2.1)
$$

where

It is

$f_\phi^{(2)}$ = 0 on

appearing a

on the phas

t_1^{cx}.

Accord

bility (2.2

be written

where

$$a_i^{(1)} = U_0,\ a_{01},\ \mu_{\kappa 1},\ \mu_{\kappa 1},\ J_{01},\ b_{01};\ u_m = u(t),\ p(t),\ v_a^i(t),\ v_p^i(t),\ v^i(t);$$
$$\tilde{u}_m = \tilde{u}(t),\ \tilde{p}(t),\ \tilde{v}_a(t),\ \tilde{v}_p(t).$$

It is assumed here that the phase trajectory reaches the boundary $f_\Phi^{(2)} = 0$ only once in the sector $(t^{(1)}, t_k)$ at point t_1^*, the boundary appearing as a result of the existence of the condition of limitation on the phase coordinates (2.1.9), then leaves the boundary at point t_1^{cx}.

According to the appendix, on the basis of the condition of stability (2.2.1), the Euler-Lagrange equations in the sector $[t_0, t_k]$ can be written as

$$\left.\begin{aligned}
\varphi_9^{(i)} &\equiv \lambda_1^{(i)\prime} = -\sum_{j=1}^{5} \lambda_j^{(i)} \frac{\partial \varphi_j^{(i)}}{\partial V} - \lambda_a^{(i)} \frac{\partial a_{np}^{(i)}}{\partial V} - \\
&\quad - \lambda_p^{(i)} \frac{\partial p_{max}^{(i)}}{\partial V} - \lambda_\Phi^{(i)} \frac{\partial \chi^{(i)}}{\partial V}, \\
\varphi_{10}^{(i)} &\equiv \lambda_2^{(i)\prime} = -\sum_{j=1}^{4} \lambda_j^{(i)} \frac{\partial \varphi_j^{(i)}}{\partial \theta} - \lambda_\Phi^{(i)} \frac{\partial \chi^{(i)}}{\partial \theta}, \\
\varphi_{11}^{(i)} &\equiv \lambda_3^{(i)\prime} = -\sum_{j=1}^{5} \lambda_j^{(i)} \frac{\partial \varphi_j^{(i)}}{\partial H} - \lambda_a^{(i)} \frac{\partial a_{np}^{(i)}}{\partial H} - \\
&\quad - \lambda_p^{(i)} \frac{\partial p_{max}^{(i)}}{\partial H} - \lambda_\Phi^{(i)} \frac{\partial \chi^{(i)}}{\partial H}, \\
\varphi_{12}^{(i)} &\equiv \lambda_4^{(i)\prime} = 0 \ \text{ or } \ \lambda_4^{(i)} = \text{const}, \\
\varphi_{13}^{(i)} &\equiv \lambda_a^{(i)} + \lambda_1^{(i)} \frac{\partial \varphi_1^{(i)}}{\partial a} + \lambda_2^{(i)} \frac{\partial \varphi_2^{(i)}}{\partial a} +
\end{aligned}\right\} \quad (2.II)$$

$$\left.\begin{aligned}
&+ \lambda_\Phi^{(i)} \frac{\partial \chi^{(i)}}{\partial a} = 0, && (2.2.2) \\
\varphi_{14}^{(i)} &\equiv \lambda_p^{(i)} + \lambda_1^{(i)} \frac{\partial \varphi_1^{(i)}}{\partial p} + \lambda_2^{(i)} \frac{\partial \varphi_2^{(i)}}{\partial p} + \\
&+ \lambda_3 \frac{\partial \varphi_5^{(i)}}{\partial p} + \lambda_\Phi^{(i)} \frac{\partial \chi^{(i)}}{\partial p} = 0, && (2.2.3) \\
\lambda_5^{(i)\prime} &= -\lambda_1^{(i)} \frac{\partial \varphi_1^{(i)}}{\partial \vartheta} - \lambda_2^{(i)} \frac{\partial \varphi_2^{(i)}}{\partial \vartheta} - \lambda_\Phi^{(i)} \frac{\partial \chi^{(i)}}{\partial \vartheta}, \\
&\lambda_a^{(i)} v_a^{(i)} = 0, \quad \lambda_p^{(i)} v_p^{(i)} = 0, \quad \lambda_\Phi^{(i)} f_\Phi^{(i)} = 0. && (2.2.4)
\end{aligned}\right\}$$

The Euler-Lagrange equations (2.2.2) and (2.2.4) allow us to determine the optimal values of control function $\alpha(t)$. With stable control, when $\alpha_{min}^{(i)} \le \alpha \le \alpha_{max}^{(i)}$, $\nu_{\alpha}^{(i)} \neq 0$, $\lambda_{\alpha}^{(i)} = 0$, $\lambda_{\phi}^{(i)} = 0$, and therefore

$$\varphi_{13}^{(i)} = -\lambda_1^{(i)}\left(a_{0i} p \sin\alpha + b_{0i} \frac{\partial \bar{Q}^{(i)}}{\partial\alpha}\right) + \frac{\lambda_2^{(i)}}{V}\left(a_{0i} p \cos\alpha + b_{0i} \frac{\partial \bar{Y}^{(i)}}{\partial\alpha}\right) = 0. \quad (2.2.5)$$

If $\alpha = \alpha_{np}$, then $\nu_{\alpha}^{(i)} \equiv 0$, $\lambda_{\alpha}^{(i)} \neq 0$, $\lambda_{\phi}^{(i)} \equiv 0$.

Then

$$\varphi_{13}^{(i)} = -\lambda_1^{(i)} \frac{g_0}{\mu}\left(a_{0i} p \sin\alpha_{np} + b_{0i} \frac{\partial \bar{Q}^{(i)}}{\partial\alpha}\right) + \lambda_2^{(i)} \frac{g_0}{\mu V}\left(a_{0i} p \cos\alpha_{np} + b_{0i} \frac{\partial \bar{Y}^{(i)}}{\partial\alpha}\right) + \lambda_{\alpha}^{(i)} = 0. \quad (2.2.6)$$

In the case of movement of the phase trajectory along the boundary formed by the conditions of limitation on the phase variables $f_{\phi}^{(i)} = 0$, the coupled control of the angle of attack $\alpha(t)$ will be determined from the equation $\chi^{(i)} = 0$, where $\nu_{\alpha}^{(i)} \equiv 0$, $\lambda_{\alpha}^{(i)} \equiv 0$, $\nu^{(i)} \equiv 0$, and we therefore produce

$$\varphi_{13}^{(i)} = -\lambda_1^{(i)} \frac{g_0}{\mu}\left(a_{0i} p \sin\alpha + b_{0i} \frac{\partial \bar{Q}^{(i)}}{\partial\alpha}\right) + \lambda_2^{(i)} \frac{g_0}{\mu V}\left(a_{0i} p \cos\alpha + b_{0i} \frac{\partial \bar{Y}}{\partial\alpha}\right) + \lambda_{\phi}^{(i)} \frac{\partial \chi^{(i)}}{\partial\alpha} = 0. \quad (2.2.7)$$

We must now determine permissibility of the stable control α as an optimal control where $\alpha_{min}^{(i)} \le \alpha \le \alpha_{max}^{(i)}$ by studying the Weierstrass condition or the maximum principle (minimum principle).

Let us represent

$$X^{(i)} = a_{0i} p \cos\alpha + b_{0i}\bar{Q}^{(i)}, \quad Z^{(i)} = a_{0i} p \sin\alpha + b_{0i}\bar{Y}^{(i)}$$

and accept as control function Z in place of α (here we arbitrarily assume control p to be fixed). In this case the Weierstrass condition becomes (see appendix)

$$\lambda_1^{(i)}(X^{(i)} - \tilde{X}^{(i)}) + \frac{\lambda_2^{(i)}}{V}(Z^{(i)} - \tilde{Z}^{(i)}) \le 0, \quad (2.2.8)$$

where

$\tilde{z}^{(i)}$ is the permissible value of control function $z^{(i)}$, where $\tilde{z}^{(i)} \neq z^{(i)}$;

$\tilde{X}^{(i)} = X^{(i)}(\tilde{z}^{(i)})$.

Then with the stable control, the Weierstrass condition (2.2.8) considering Euler-Lagrange equation (2.2.5), which, in connection with the replacement of α by Z, becomes

$$\lambda_1^{(i)} \frac{\partial X^{(i)}}{\partial Z^{(i)}} + \frac{\lambda_2^{(i)}}{V} = 0,$$

can be written as follows:

$$\lambda_1^{(i)} \left[(X^{(i)} - \tilde{X}^{(i)}) + \frac{\partial X^{(i)}}{\partial Z^{(i)}} (\tilde{Z}^{(i)} - Z^{(i)}) \right] \leqslant 0.$$

Since where $-\pi/2 < \alpha < \pi/2$, the tangent to the curve $X^{(i)} = X^{(i)} \times (z^{(i)})$ will always be located above it, we find

$$X^{(i)} - \tilde{X}^{(i)} + \frac{\partial X^{(i)}}{\partial Z^{(i)}} (\tilde{Z}^{(i)} - Z^{(i)}) > 0.$$

Thus, where $\lambda_1 < 0$, it is possible in principle for the optimal control α to be stable control satisfying the Weierstrass condition. Therefore, if the angle of attack determined unambiguously from the Euler-Lagrange equation (2.2.5) lies within the limits $\alpha_{min} \leq \alpha \leq \alpha_{max}$, the optimal control is stable, while otherwise it is limiting control, and the condition $f_\phi^{(i)} \geq 0$ should be satisfied.

The Euler-Lagrange equations (2.2.3)-(2.2.4) make it possible to determine the optimal value of control function p(t). However, in sectors where the condition of limitation on phase variables (2.1.9) is not in force, as the investigations in Chapter 1, § 3 have shown, the optimal operating modes of the rocket engines (liquid or solid fueled), satisfying the Weierstrass condition or the maximum principle, can only be the following modes: $p = 0_{max}$ and $p = 0$, while the optimal operating modes of jet engines (turbojet or supersonic ram jet) satisfying the Weierstrass condition or the maximum principle may be the maximum thrust mode, the minimum thrust mode or a choked mode, the presence of which depends on the type of choking characteristic of the power plant.

However, according to the control condition, flight with the engine operating will be optimal where

$$\frac{g_0 P_{0i}}{\mu}\left(\lambda_1^{(i)}\cos\alpha+\lambda_2^{(i)}\frac{\sin\alpha}{V}\right)-\lambda_5^{(i)}\frac{a_{0i}}{J_{0i}}f^{(i)}\leqslant 0. \qquad (2.2.9)$$

Of course, it is assumed here that the phase trajectory does not extend along the boundary $f_\phi^{(i)} = 0$. Otherwise, the control condition may be disrupted in order to assure fulfillment of the limiting condition on phase variables (2.1.9). If with optimal $p > 0$ we can assure $f_\phi^{(i)} = 0$ by the corresponding control α, then the optimal control α will be the coupled control, while $\alpha_{min} \leq \alpha \leq \alpha_{max}$. If, however, with optimal $p > 0$ it is impossible to provide $f_\phi^{(i)} \geq 0$ by some control α, then where $f_\phi^{(i)} = 0$ we must go over to coupled control p, determined by the equation $\chi^{(i)} = 0$.

From stability condition (2.2.1) we can also produce the Euler-Lagrange equations in sector $[\bar{t}^{(i)}, \bar{t}_k]$. They are produced in the form

$$\left.\begin{aligned}
&\tilde{\varphi}_8\equiv\dot{\tilde{\lambda}}_1=-\sum_{j=1}^{5}\tilde{\lambda}_j\frac{\partial\tilde{\varphi}_j}{\partial V}-\tilde{\lambda}_\alpha\frac{\partial\tilde{\alpha}_{\cdot p}}{\partial V}-\tilde{\lambda}_p\frac{\partial\tilde{p}_{max}}{\partial V},\\
&\tilde{\varphi}_9\equiv\dot{\tilde{\lambda}}_2=-\sum_{j=1}^{4}\tilde{\lambda}_j\frac{\partial\tilde{\varphi}_j}{\partial\theta},\\
&\tilde{\varphi}_{10}\equiv\dot{\tilde{\lambda}}_3=-\sum_{j=1}^{5}\tilde{\lambda}_j\frac{\partial\tilde{\varphi}_j}{\partial H}-\tilde{\lambda}_\alpha\frac{\partial\tilde{\alpha}_{\cdot p}}{\partial H}-\tilde{\lambda}_p\frac{\partial\tilde{p}_{max}}{\partial H},\\
&\tilde{\varphi}_{11}\equiv\dot{\tilde{\lambda}}_4=0 \text{ or } \quad \tilde{\lambda}_4=\text{const},\\
&\tilde{\varphi}_{12}\equiv\tilde{\lambda}_\alpha+\tilde{\lambda}_1\frac{\partial\tilde{\varphi}_1}{\partial\tilde{\alpha}}+\tilde{\lambda}_2\frac{\partial\tilde{\varphi}_2}{\partial\tilde{\alpha}}=0,\\
&\tilde{\varphi}_{13}\equiv\tilde{\lambda}_p+\tilde{\lambda}_1\frac{\partial\tilde{\varphi}_1}{\partial\tilde{p}}+\tilde{\lambda}_2\frac{\partial\tilde{\varphi}_2}{\partial\tilde{p}}+\tilde{\lambda}_5\frac{\partial\tilde{\varphi}_5}{\partial\tilde{p}}=0,\\
&\tilde{\varphi}_{14}\equiv\dot{\tilde{\lambda}}_5=-\tilde{\lambda}_1\frac{\partial\tilde{\varphi}_1}{\partial\tilde{\mu}}-\tilde{\lambda}_2\frac{\partial\tilde{\varphi}_2}{\partial\tilde{\mu}},\\
&\tilde{\lambda}_\alpha\tilde{v}_\alpha=0,\ \tilde{\lambda}_p\tilde{v}_p=0.
\end{aligned}\right\} \qquad (2.\bar{1}1)$$

Investigation of equations (2.Ĩ1), the Weierstrass condition and the maximum principle considering the control condition allows us to note that the optimal control conditions $\tilde{p}$ and $\tilde{\alpha}$ for the booster are similar to the conditions (2.2.5)-(2.2.9) determined earlier without coupled control.

Thus, where $\tilde{\lambda}_1 < 0$, the optimal control can possibly be stable control, satisfying the Weierstrass condition. Therefore, if the angle of attack determined from equation

$$-\tilde{\lambda}_1\left(a_{01}\tilde{p}\sin\tilde{\alpha}+b_{01}\frac{\partial\tilde{Q}}{\partial\tilde{\alpha}}\right)+\frac{\tilde{\lambda}_2}{\tilde{V}}\left(a_{01}\tilde{p}\cos\tilde{\alpha}+b_{01}\frac{\partial\tilde{Y}}{\partial\tilde{\alpha}}\right)=0,$$

lies within the limits $\tilde{\alpha}_{min} \leq \tilde{\alpha} \leq \tilde{\alpha}_{max}$, the optimal control is stable, while otherwise it is the limiting control which exists only with $\tilde{\lambda}_1 \geq 0$.

In connection with the independence of $\phi_j^{(i)}$ and $\tilde{\phi}_j$ in explicit form on t, systems (2.I1) and (2.Ĩ1) have first integrals which are respectively

$$F_i-\sum_{j=1}^{s} x_j^{(i)'}\frac{\partial F_i}{\partial x_j^{(i)'}}=C_0^{(i)}$$

$$\tilde{F}-\sum_{j=1}^{s} \tilde{x}_j'\frac{\partial \tilde{F}}{\partial \tilde{x}_j'}=\tilde{C}_0$$

or

$$\sum_{j=1}^{s} \lambda_j^{(i)}\varphi_j^{(i)}=-C_0^{(i)}, \qquad (2.2.10)$$

$$\sum_{j=1}^{s} \tilde{\lambda}_j\tilde{\varphi}_j=-\tilde{C}_0. \qquad (2.2.11)$$

where $C_0^{(i)}$, $\tilde{C}_0$ are the integration constants.

In order to satisfy the conditions of stability, the following conditions of optimality of the main plan parameters must be fulfilled:

$$\frac{\partial I}{\partial G_{p.l}} + e_1^{(0)} \frac{\partial \beta_1^{(0)}}{\partial G_{p.l}} + e_{07} \frac{\partial \beta_{07}}{\partial G_{p.l}} = 0,$$

$$\frac{\partial I}{\partial G_0} + e_1^{(0)} \frac{\partial \beta_1^{(0)}}{\partial G_0} + e_1^{(1)} \frac{\partial \beta_1^{(1)}}{\partial G_0} + e_2 \frac{\partial \beta_2}{\partial G_0} + e_3^{(1)} \frac{\partial \beta_3^{(1)}}{\partial G_0} +$$

$$+ e_4^{(1)} \frac{\partial \beta_4^{(1)}}{\partial G_0} + e_{01}^{(1)} \frac{\partial \beta_{01}^{(1)}}{\partial G_0} = 0,$$

$$e_1^{(0)} \frac{\partial \beta_1^{(0)}}{\partial a_{01}} + e_1^{(1)} \frac{\partial \beta_1^{(1)}}{\partial a_{01}} + e_2 \frac{\partial \beta_2}{\partial a_{01}} + e_3^{(1)} \frac{\partial \beta_3^{(1)}}{\partial a_{01}} + e_{02}^{(1)} \frac{\partial \beta_{02}^{(1)}}{\partial a_{01}} +$$

$$+ \int_{t_0}^{t^{(1)}} \frac{\partial F_1}{\partial a_{01}}\, dt + \int_{t^{(1)}}^{\widetilde{T}_k^{(1)}} \frac{\partial \widetilde{F}}{\partial a_{01}}\, dt = 0,$$

$$e_3^{(1)} \frac{\partial \beta_3^{(1)}}{\partial P_{max1}} + e_{03}^{(1)} \frac{\partial \beta_{03}^{(1)}}{\partial P_{max1}} = 0,$$

$$e_1^{(0)} \frac{\partial \beta_1^{(0)}}{\partial b_{01}} + e_1^{(1)} \frac{\partial \beta_1^{(1)}}{\partial b_{01}} + e_2 \frac{\partial \beta_2}{\partial b_{01}} + e_4^{(1)} \frac{\partial \beta_4^{(1)}}{\partial b_{01}} + e_{04}^{(1)} \frac{\partial \beta_{04}^{(1)}}{\partial b_{01}} +$$

$$+ \int_{t_0}^{t^{(1)}} \frac{\partial F_1}{\partial b_{01}}\, dt + \int_{t^{(1)}}^{\widetilde{T}_k} \frac{\partial \widetilde{F}}{\partial b_{01}}\, dt = 0,$$

$$e_1^{(0)} \frac{\partial \beta_1^{(0)}}{\partial J_{01}} + e_1^{(1)} \frac{\partial \beta_1^{(1)}}{\partial J_{01}} + e_2 \frac{\partial \beta_2}{\partial J_{01}} + e_{05}^{(1)} \frac{\partial \beta_{05}^{(1)}}{\partial J_{01}} +$$

$$+ \int_{t_0}^{t^{(1)}} \frac{\partial F_1}{\partial J_{01}}\, dt + \int_{t^{(1)}}^{\widetilde{T}_k^{(1)}} \frac{\partial \widetilde{F}}{\partial J_{01}}\, dt = 0,$$

$$e_4^{(1)} \frac{\partial \beta_4^{(1)}}{\partial S^{(1)}} + e_{06}^{(1)} \frac{\partial \beta_{06}^{(2)}}{\partial S^{(1)}} = 0,$$

$$e_1^{(0)} \frac{\partial \beta_1^{(0)}}{\partial \mu_{k1}} + e_1^{(1)} \frac{\partial \beta_1^{(1)}}{\partial \mu_{k1}} + e_2 \frac{\partial \beta_2}{\partial \mu_{k1}} - e_\mu^{(1)} = 0,$$

$$e_2 \frac{\partial \beta_2}{\partial \mu_{k1}} + \int_{t^{(1)}}^{\widetilde{T}_k} \frac{\partial \widetilde{F}}{\partial \mu_{k1}}\, dt = 0,$$

$$e_1^{(0)} \frac{\partial\varphi_1^{(0)}}{\partial\varphi_\kappa^{(n)}} + e_1^{(1)} \frac{\partial\varphi_1^{(1)}}{\partial\varphi_\kappa^{(n)}} + e_2 \frac{\partial\varphi_2}{\partial\varphi_\kappa^{(n)}} - \tilde{e}_\mu^{(n)} + \int_{\tilde{t}_\kappa^{(1)}}^{\tilde{t}_\kappa} \frac{\partial\tilde{F}}{\partial\varphi_\kappa^{(n)}} dt = 0,$$

$$e_1^{(0)} \frac{\partial\varphi_1^{(0)}}{\partial G_{011}} + e_1^{(1)} \frac{\partial\varphi_1^{(1)}}{\partial G_{011}} + e_3^{(2)} \frac{\partial\varphi_3^{(2)}}{\partial G_{011}} + e_4^{(2)} \frac{\partial\varphi_4^{(2)}}{\partial G_{011}} +$$

$$+ e_{01}^{(2)} \frac{\partial\varphi_{01}^{(2)}}{\partial G_{011}} = 0,$$

$$e_1^{(0)} \frac{\partial\varphi_1^{(0)}}{\partial a_{c2}} + e_3^{(2)} \frac{\partial\varphi_3^{(2)}}{\partial a_{c2}} + e_{02}^{(2)} \frac{\partial\varphi_{02}^{(2)}}{\partial a_{c2}} + \int_{t^{(1)}}^{t^{(2)}} \frac{\partial F_2}{\partial a_{c2}} dt = 0,$$

$$e_3^{(2)} \frac{\partial\varphi_3^{(2)}}{\partial P_{max2}} + e_{03}^{(2)} \frac{\partial\varphi_{03}^{(2)}}{\partial P_{max2}} = 0,$$

$$e_1^{(0)} \frac{\partial\varphi_1^{(0)}}{\partial b_{c2}} + e_4^{(2)} \frac{\partial\varphi_4^{(2)}}{\partial b_{c2}} + e_{04}^{(2)} \frac{\partial\varphi_{04}^{(2)}}{\partial b_{c2}} + \int_{t^{(1)}}^{t_\kappa} \frac{\partial F_2}{\partial b_{c2}} dt = 0,$$

$$e_1^{(0)} \frac{\partial\varphi_1^{(0)}}{\partial I_{c2}} + e_{05}^{(2)} \frac{\partial\varphi_{05}^{(2)}}{\partial I_{c2}} + \int_{t^{(1)}}^{t^{(2)}} \frac{\partial F_2}{\partial I_{c2}} dt = 0,$$

$$e_4^{(2)} \frac{\partial\varphi_4^{(2)}}{\partial S^{(2)}} + e_{06}^{(2)} \frac{\partial\varphi_{06}^{(2)}}{\partial S^{(2)}} = 0,$$

$$e_1^{(0)} \frac{\varphi_1^{(0)}}{\partial\mu_{\kappa 11}} - e_\mu^{(2)} + \int_{t^{(2)}}^{t_\kappa} \frac{\partial F_2}{\partial\mu_{\kappa 11}} = 0,$$

$$\left.\begin{array}{ll} e_{01}^{(1)} w_{11} = 0, & e_{01}^{(2)} w_{12} = 0, \\ e_{02}^{(1)} w_{21} = 0, & e_{02}^{(2)} w_{22} = 0, \\ e_{03}^{(1)} w_{31} = 0, & e_{03}^{(2)} w_{32} = 0, \\ e_{04}^{(1)} w_{41} = 0, & e_{04}^{(2)} w_{42} = 0, \\ e_{05}^{(1)} w_{51} = 0, & e_{05}^{(2)} w_{52} = 0, \\ e_{06}^{(1)} w_{61} = 0, & e_{06}^{(2)} w_{62} = 0, \\ \multicolumn{2}{c}{e_{07} w_7 = 0} \end{array}\right\} \qquad (2.2.12)$$

Excluding the constant Lagrange coefficients $e_1^{(0)}$, $e_1^{(1)}$, e_2, $e_3^{(i)}$ and $e_4^{(i)}$ from these equations, we produce the conditions of optimality of the main plan parameters in the following converted form:

$$\frac{m_{G_{p.d}}}{m_a^{(1)}} A_a^{(1)} - \frac{\partial I}{\partial G_{p.d}} + \frac{m_{G_{p.d}}}{m_a^{(1)}} B_a^{(1)} - B_{n.r} = 0, \qquad (2.2.13)$$

$$\frac{m_{G_0}}{m_a^{(1)}} A_a^{(1)} - \left(\frac{\partial I}{\partial G_0} + A_0\right) + \frac{m_{G_0}}{m_a^{(1)}} B_a^{(1)} - B_0 = 0, \qquad (2.2.14)$$

$$\left.\begin{aligned}
&1 + \frac{\left(A_\mu^{(1)} + B_\mu^{(1)}\right) m_a^{(1)}}{\left(A_a^{(1)} + B_a^{(1)}\right) m_\mu^{(1)}} = 0,\\
&1 - \frac{\left(A_j^{(1)} + B_j^{(1)}\right) m_a^{(1)}}{\left(A_a^{(1)} + B_a^{(1)}\right) m_j^{(1)}} = 0,\\
&\frac{\partial \beta_4^{(1)}}{\partial S^{(1)}} - \frac{\left(A_b^{(1)} + B_b^{(1)}\right) m_a^{(1)}}{\left(A_a^{(1)} + B_a^{(1)}\right) m_b^{(1)}} = 0,\\
&1 + \frac{\left(A_a^{(n)} + B_\mu^{(n)}\right) m_a^{(1)}}{\left(A_a^{(1)} + B_a^{(1)}\right) m_\mu^{(n)}} = 0,\\
&1 + \frac{A_\mu^{(2)} m_a^{(2)}}{\left(A_a^{(2)} + B_a^{(2)}\right) m_\mu^{(2)}} = 0,\\
&1 + \frac{A_\mu^{(2)} m_a^{(1)}}{\left(A_a^{(1)} + B_a^{(1)}\right) m_\mu^{(2)}} = 0,\\
&1 - \frac{\left(A_j^{(2)} + B_j^{(2)}\right) m_a^{(2)}}{\left(A_a^{(2)} + B_a^{(2)}\right) m_j^{(2)}} = 0,\\
&\frac{\partial \beta_4^{(2)}}{\partial S^{(2)}} - \frac{\left(A_b^{(2)} + B_b^{(2)}\right) m_a^{(2)}}{\left(A_n^{(2)} + B_a^{(2)}\right) m_b^{(2)}} = 0,
\end{aligned}\right\} \qquad (2.2.15)$$

where

$$m_{G_{p.d}} = \frac{\partial \beta_1^{(0)}}{\partial G_{p.d}},$$

$$m_{G_0} = \frac{\partial \beta_1^{(0)}}{\partial G_0} - \frac{\partial \beta_4^{(1)}}{\partial G_0} \frac{\partial \beta_1^{(0)}}{\partial b_{c1}} - \left(\frac{\partial \beta_1^{(1)}}{\partial G_0} - \frac{\partial \beta_4^{(1)}}{\partial G_0} \frac{\partial \beta_1^{(1)}}{\partial b_{c1}}\right) \times$$

$$\times\left(\frac{\partial\beta_1^{(0)}}{\partial G_{011}}-\frac{\partial\beta_4^{(2)}}{\partial G_{011}}\frac{\partial\beta_1^{(0)}}{\partial b_{02}}\right),$$

$$m_a^{(1)}=\frac{\partial\beta_1^{(0)}}{\partial a_{01}}-\frac{\partial\beta_1^{(1)}}{\partial a_{01}}\left(\frac{\partial\beta_1^{(0)}}{\partial G_{011}}-\frac{\partial\beta_4^{(2)}}{\partial G_{011}}\frac{\partial\beta_1^{(0)}}{\partial b_{02}}\right),$$

$$m_\mu^{(1)}=\frac{\partial\beta_1^{(0)}}{\partial\mu_{\kappa 1}}-\frac{\partial\beta_1^{(1)}}{\partial\mu_{\kappa 1}}\left(\frac{\partial\beta_1^{(0)}}{\partial G_{011}}-\frac{\partial\beta_4^{(2)}}{\partial G_{011}}\frac{\partial\beta_1^{(0)}}{\partial b_{02}}\right),$$

$$m_J^{(1)}=\frac{\partial\beta_1^{(0)}}{\partial J_{01}}-\frac{\partial\beta_1^{(1)}}{\partial J_{01}}\left(\frac{\partial\beta_1^{(0)}}{\partial G_{011}}-\frac{\partial\beta_4^{(2)}}{\partial G_{011}}\frac{\partial\beta_1^{(0)}}{\partial b_{02}}\right),$$

$$m_b^{(1)}=\frac{\partial\beta_1^{(0)}}{\partial b_{01}}-\frac{\partial\beta_1^{(1)}}{\partial b_{01}}\left(\frac{\partial\beta_1^{(0)}}{\partial G_{011}}-\frac{\partial\beta_4^{(2)}}{\partial G_{011}}\frac{\partial\beta_1^{(0)}}{\partial b_{02}}\right),$$

$$m_\mu^{(\kappa)}=\frac{\partial\beta_1^{(0)}}{\partial\mu_\kappa^{(\kappa)}}-\frac{\partial\beta_1^{(1)}}{\partial\mu_\kappa^{(\kappa)}}\left(\frac{\partial\beta_1^{(0)}}{\partial G_{011}}-\frac{\partial\beta_4^{(2)}}{\partial G_{011}}\frac{\partial\beta_1^{(0)}}{\partial b_{02}}\right),$$

$$m_a^{(2)}=\frac{\partial\beta_1^{(0)}}{\partial a_{02}},\quad m_\mu^{(2)}=\frac{\partial\beta_1^{(0)}}{\partial\mu_{\kappa 11}},$$

$$m_J^{(2)}=\frac{\partial\beta_1^{(0)}}{\partial J_{02}},\quad m_b^{(2)}=\frac{\partial\beta_1^{(0)}}{\partial b_{12}},$$

$$A_0=-(\eta_{0\kappa}^{(1)}+\tilde{\eta}_{0\kappa})\frac{\partial\beta_4^{(1)}}{\partial G_0}+\left(\frac{\partial\beta_1^{(1)}}{\partial G_0}-\frac{\partial\beta_4^{(1)}}{\partial G_0}\frac{\partial\beta_1^{(1)}}{\partial b_{01}}\right)A_{011}-$$

$$-\tilde{\eta}_{2\kappa}\left(\frac{\partial\beta_2}{\partial G_0}-\frac{\partial\beta_4^{(1)}}{\partial G_0}\frac{\partial\beta_2}{\partial b_{01}}\right),\quad A_{011}=\eta_{0\kappa}^{(2)}\frac{\partial\beta_4^{(2)}}{\partial G_{011}},$$

$$B_0=e_{01}^{(1)}\frac{\partial\beta_{01}^{(1)}}{\partial G_0}-e_{03}^{(1)}\frac{\partial\beta_{03}^{(1)}/\partial P_{max1}}{\partial\beta_3^{(1)}/\partial P_{max1}}\frac{\partial\beta_3^{(1)}}{\partial G_0}-e_{04}^{(1)}\frac{\partial\beta_4^{(1)}}{\partial G_0}\frac{\partial\beta_{04}^{(1)}}{\partial b_{01}}-$$

$$-\left(\frac{\partial\beta_1^{(0)}}{\partial G_0}-\frac{\partial\beta_4^{(1)}}{\partial G_0}\frac{\partial\beta_1^{(1)}}{\partial b_{01}}\right)B_{011},$$

$$B_{011}=e_{01}^{(2)}\frac{\partial\beta_{01}^{(2)}}{\partial G_{011}}-e_{03}^{(2)}\frac{\partial\beta_3^{(2)}}{\partial G_{011}}\frac{\partial\beta_{03}^{(2)}/\partial P_{max2}}{\partial\beta_3^{(2)}/\partial P_{max2}}-e_{04}^{(2)}\frac{\partial\beta_4^{(2)}}{\partial G_{011}}\frac{\partial\beta_{04}^{(2)}}{\partial b_{02}},$$

$$B_{a.J}=e_{07}\frac{\partial\beta_{07}}{\partial G_{pJ}},$$

$$A_a^{(1)}=\eta_{1\kappa}^{(1)}+\tilde{\eta}_{1\kappa}-\tilde{\eta}_{3\kappa}\frac{\partial\beta_2}{\partial a_{01}}+\frac{\partial\beta_1^{(1)}}{\partial a_{01}}A_{011},$$

$$B_a^{(1)} = e_{02}^{(1)} \frac{\partial \xi_{02}^{(1)}}{\partial a_{01}} - e_{03}^{(1)} \frac{\partial \xi_{03}^{(1)} / \partial P_{\max 1}}{\partial \xi_3^{(1)} / \partial P_{\max 1}} - \frac{\partial \xi_1^{(1)}}{\partial a_{01}} B_{011},$$

$$A_\mu^{(1)} = \frac{\partial \xi_2}{\partial \mu_{\kappa 1}} \tilde{\eta}_{3\kappa} - \frac{\partial \xi_1^{(1)}}{\partial \mu_{\kappa 1}} A_{011} - e_\mu^{(1)},$$

$$B_\mu^{(1)} = \frac{\partial \xi_1^{(1)}}{\partial \mu_{\kappa 1}} B_{011},$$

$$A_\mu^{(\kappa)} = \frac{\partial \xi_2}{\partial \mu_\kappa^{(\kappa)}} \tilde{\eta}_{3\kappa} - \frac{\partial \xi_1^{(1)}}{\partial \mu_{\kappa 1}} A_{011} + \tilde{e}_\mu^{(\kappa)} - \tilde{\eta}_{4\kappa},$$

$$B_\mu^{(\kappa)} = \frac{\partial \xi_1^{(1)}}{\partial \mu_\kappa^{(\kappa)}} B_{011},$$

$$A_J^{(1)} = \eta_{2\kappa}^{(1)} + \tilde{\eta}_{2\kappa} + \frac{\partial \xi_1^{(1)}}{\partial J_{01}} A_{011} - \frac{\partial \xi_2}{\partial J_{01}} \tilde{\eta}_{3\kappa},$$

$$B_J^{(1)} = e_{05}^{(1)} \frac{\partial \xi_{05}^{(1)}}{\partial J_{01}} - \frac{\partial \xi_1^{(1)}}{\partial J_{01}} B_{011},$$

$$A_b^{(1)} = \frac{\partial \xi_4^{(1)}}{\partial S^{(1)}} \left(\eta_{0\kappa}^{(1)} + \tilde{\eta}_{0\kappa} - \frac{\partial \xi_1^{(1)}}{\partial b_{01}} A_{011} - \frac{\partial \xi_2}{\partial b_{01}} \tilde{\eta}_{3\kappa} \right),$$

$$B_b^{(1)} = e_{04}^{(1)} \frac{\partial \xi_4^{(1)}}{\partial S^{(1)}} \frac{\partial \xi_{04}^{(1)}}{\partial b_{01}} - e_{06}^{(1)} \frac{\partial \xi_{06}^{(1)}}{\partial S^{(1)}} - \frac{\partial \xi_4^{(1)}}{\partial S^{(1)}} \frac{\partial \xi_1^{(1)}}{\partial b_{01}} B_{011},$$

$$A_a^{(2)} = \eta_{1\kappa}^{(2)}, \quad B_a^{(2)} = e_{02}^{(2)} \frac{\partial \xi_{02}^{(2)}}{\partial a_{02}} - e_{03} \frac{\partial \xi_{03}^{(2)} / \partial P_{\max}^{(2)}}{\partial \xi_3^{(2)} / \partial P_{\max}^{(2)}},$$

$$A_\mu^{(2)} = e_\mu^{(2)} - \eta_{3\kappa},$$

$$A_J^{(2)} = \eta_{2\kappa}^{(2)}, \quad B_J^{(2)} = e_{05}^{(2)} \frac{\partial \xi_{05}^{(2)}}{\partial J_{02}},$$

$$A_b^{(2)} = \frac{\partial \xi_4^{(2)}}{\partial S^{(2)}} \eta_{0\kappa}^{(\kappa)}, \quad B_b^{(2)} = e_{04}^{(2)} \frac{\partial \xi_4^{(2)}}{\partial S^{(2)}} \frac{\partial \xi_{04}^{(2)}}{\partial b_{02}} - e_{06}^{(2)} \frac{\partial \xi_{06}^{(2)}}{\partial S^{(2)}},$$

$$\eta_{0\kappa}^{(1)} = \int_{t_0}^{t^{(1)}} \frac{\partial F_1}{\partial b_{01}} dt, \quad \eta_{0\kappa}^{(\kappa)} = \int_{t^{(1)}}^{t_\kappa} \frac{\partial F_2}{\partial b_{02}} dt, \quad \tilde{\eta}_{0\kappa} = \int_{t^{(1)}}^{\tilde{t}_\kappa} \frac{\partial \tilde{F}}{\partial b_{01}} dt,$$

$$\eta_{1\kappa}^{(l)} = \int_{t^{(l-1)}}^{t^{(l)}} \frac{\partial F_l}{\partial a_{0l}} dt, \quad \tilde{\eta}_{1\kappa} = \int_{t^{(1)}}^{\tilde{t}_\kappa^{(1)}} \frac{\partial \tilde{F}}{\partial a_{01}} dt, \quad \eta_{2\kappa}^{(l)} = \int_{t^{(l-1)}}^{t^{(l)}} \frac{\partial F_l}{\partial J_{0l}} dt,$$

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$$\tilde{\eta}_{2\kappa} = \int_{\tilde{t}^{(1)}}^{\tilde{T}_{\kappa}^{(1)}} \frac{\partial \tilde{F}}{\partial J_{01}} dt, \quad \eta_{3\kappa} = \int_{t^{(2)}}^{t_{\kappa}} \frac{\partial F_2}{\partial \mu_{\kappa 1}} dt, \quad \tilde{\eta}_{3\kappa} = \int_{t^{(1)}}^{\tilde{T}_{\kappa}} \frac{\partial \tilde{F}}{\partial \mu_{\kappa 1}} dt,$$

$$\tilde{\eta}_{4\kappa} = \int_{\tilde{T}_{\kappa}^{(1)}}^{\tilde{T}_{\kappa}} \frac{\partial \tilde{F}}{\partial \mu_{\kappa}^{(\kappa)}} dt,$$

where

$$\begin{aligned} &\eta_{00}^{(l)} = 0, && \tilde{\eta}_{00} = 0, \\ &\eta_{10}^{(l)} = 0, && \tilde{\eta}_{10} = 0, \\ &\eta_{20}^{(l)} = 0, && \tilde{\eta}_{20} = 0, \\ &\eta_{30} = 0, && \tilde{\eta}_{30} = 0, && \tilde{\eta}_{40} = 0. \end{aligned}$$

Equations (2.2.12) allow us to determine whether the optimal values of the main plan parameters are located on the boundary of the area of limitation, when

$e_{0j}^{(j)} \neq 0$, $e_{07} = 0$ and $w_{ji} = 0$, $w_{07} = 0$ $(i = 1, 2;\ j = 1, \ldots, 6)$, or within the area, when $w_{ji} \neq 0$, $w_{07} \neq 0$ and $e_{0j}^{(i)} = 0$, $e_{07} = 0$.

Suppose the optimal values of the main plan parameters are located within the area of limitations. Then the conditions of optimality of parameters (2.2.13)-(2.2.15) become

$$\frac{m_{G_{p.l}}}{m_a^{(1)}} A_a^{(1)} - \frac{\partial I}{\partial G_{p.l}} = 0, \qquad (2.2.16)$$

$$\left.\begin{aligned} &\frac{m_{G_0}}{m_a^{(1)}} A_a^{(1)} - \left(A_0 + \frac{\partial I}{\partial G_0}\right) = 0, \\ &1 + \frac{A_{\mu}^{(1)} m_a^{(1)}}{A_a^{(1)} m_{\mu}^{(1)}} = 0, \quad 1 - \frac{A_j^{(1)} m_a^{(1)}}{A_a^{(1)} m_j^{(1)}} = 0, \\ &\frac{\partial \zeta_4^{(1)}}{\partial S^{(1)}} - \frac{A_b^{(1)} m_a^{(1)}}{A_a^{(1)} m_b^{(1)}} = 0, \quad 1 + \frac{A_{\mu}^{(\kappa)} m_a^{(1)}}{A_a^{(1)} m_{\mu}^{(\kappa)}} = 0, \end{aligned}\right\} \qquad (2.2.17)$$

$$\left.\begin{aligned} &1 + \frac{A_{\mu}^{(2)} m_a^{(2)}}{A_a^{(2)} m_{\mu}^{(2)}} = 0, \quad 1 + \frac{A_{\mu}^{(2)} m_a^{(1)}}{A_a^{(1)} m_{\mu}^{(2)}} = 0, \\ &1 - \frac{A_j^{(2)} m_a^{(2)}}{A_a^{(2)} m_j^{(2)}} = 0, \quad \frac{\partial \zeta_4^{(2)}}{\partial S^{(2)}} - \frac{A_b^{(2)} m_a^{(2)}}{A_a^{(2)} m_b^{(2)}} = 0. \end{aligned}\right\} \qquad (2.2.18)$$

The condition of stability (2.2.1) can be fulfilled if the so-called branching conditions are fulfilled, i. e. the conditions allowing us to determine the Lagrange coefficients at points $t^{(i)}$ to the right of the first discontinuity ($\lambda_j^{(i+1)}(t_+^{(i)})$) and the right of the second discontinuity ($\tilde{\lambda}_j(t_+^{(i)})$) of the phase variables

According to (2.2.1), we produce

$$
\begin{aligned}
&\lambda_1(t_-^{(1)})+e_V+\tilde{e}_V=0,\\
&\tilde{\lambda}_1(t_+^{(1)})+\tilde{e}_V=0, \quad \lambda_1(t_+^{(1)})+e_V=0;\\
&\lambda_2(t_-^{(1)})+e_\theta+\tilde{e}_\theta=0,\\
&\tilde{\lambda}_2(t_+^{(1)})+\tilde{e}_\theta=0, \quad \lambda_2(t_+^{(1)})+e_\theta=0;\\
&\lambda_3(t_-^{(1)})+e_H+\tilde{e}_H=0,\\
&\tilde{\lambda}_3(t_+^{(1)})+\tilde{e}_H=0, \quad \lambda_3(t_+^{(1)})+e_H=0;\\
&\lambda_4(t_-^{(1)})+e_L+\tilde{e}_L=0,\\
&\tilde{\lambda}_4(t_+^{(1)})+\tilde{e}_L=0, \quad \lambda_4(t_+^{(1)})+e_L=0;\\
&C_0(t_-^{(1)})+e_t+\tilde{e}_t=0,\\
&\tilde{C}_0(t_+^{(1)})+\tilde{e}_t=0, \quad C_0(t_+^{(1)})+e_t=0.
\end{aligned}
$$

From this, after excluding the Lagrange coefficients e_V, $\tilde{e}_V$, ..., e_t, $\tilde{e}_t$, we find the branching conditions

$$
\left.
\begin{aligned}
&\lambda_1(t_-^{(1)})-\lambda_1(t_+^{(1)})-\tilde{\lambda}_1(t_+^{(1)})=0,\\
&\lambda_2(t_-^{(1)})-\lambda_2(t_+^{(1)})-\tilde{\lambda}_2(t_+^{(1)})=0,\\
&\lambda_3(t_-^{(1)})-\lambda_3(t_+^{(1)})-\tilde{\lambda}_3(t_+^{(1)})=0,\\
&\lambda_4(t_-^{(1)})-\lambda_4(t_+^{(1)})-\tilde{\lambda}_4(t_+^{(1)})=0,\\
&C_0^{(1)}(t_-^{(1)})-C_0^{(2)}(t_+^{(1)})-\tilde{C}_0(t_+^{(1)})=0.
\end{aligned}
\right\} \qquad (2.2.19)
$$

The last equation of branching conditions (2.2.19) can, based on the expressions of the first integral (2.2.11), be reduced to the form

$$
C_0^{(2)}(t_+^{(1)})=C_0^{(1)}(t_-^{(1)})+\sum_{j=1}^{4}(\lambda_j(t_-^{(1)})-\lambda_j(t_-^{(1)}))\tilde{\xi}_j+ \\
+\tilde{\lambda}_5(t_-^{(1)})\tilde{\xi}_5\Big|_{t=t_+^{(1)}}. \qquad (2.2.20)
$$

Thereby, determination of $\lambda_j^{(i+1)}(t_+^{(1)})$ and $\tilde{\lambda}_5(t_+^{(1)})$ with known values of $C_0^{(1)}(t_-^{(1)})$, $\lambda_j(t_-^{(1)})$, $\xi_j\big|_{t = t_+^{(1)}}$ unambiguously determines $C_0^{(2)}(t_+^{(1)})$ and $\tilde{C}_0(t_+^{(1)})$.

Branching conditions (2.2.19) show that the Lagrange coefficients and constant of the first integral at point $t^{(1)}$ can undergo double first order discontinuity, and to the right of the first discontinuity they are $\lambda_1(t_+^{(1)}),\ldots,\lambda_4(t_+^{(1)})$, $C_0^{(2)}(t_+^{(1)})$ and to the right of the second discontinuity $\tilde{\lambda}_1(t_+^{(1)}),\ldots,\tilde{\lambda}_4(t_+^{(1)})$, $\tilde{C}_0(t_+^{(1)})$ (Figure 2.2).

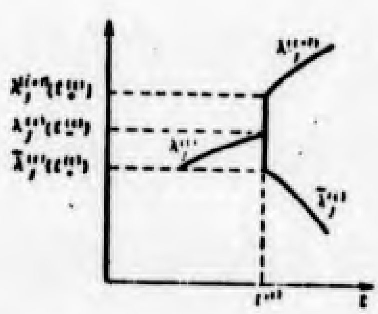

Figure 2.2. Arbitrary Representation of Double Discontinuity of Lagrange Coefficient λ_i at Point $t^{(i)}$

Thus, although the phase variables V(t), θ(t), H(t), and L(t) at point $t^{(1)}$ are continuous, the Lagrange coefficients $\lambda_1,\ldots,\lambda_4$ and C_0 undergo double first order discontinuity. This is one of the peculiarities of the branching condition.

The number of relationships in the branching condition (2.2.19) is insufficient for unambiguous determination of the values of Lagrange coefficients $\lambda_1,\ldots,\lambda_5$ and the constant of the first integral C_0 both to the right of the first discontinuity and to the right of the second discontinuity. Therefore, at point $t^{(1)}$ the Lagrange coefficients $\lambda_1,\ldots,\lambda_5$ either to the right of the first discontinuity or to the right of the second discontinuity may take on arbitrary values, independent of the values of the corresponding Lagrange coefficients and C_0 to the left of the discontinuity and the phase coordinates at point $t^{(1)}$. This is another peculiarity of the branching condition. It allows, as we will see below, satisfaction of the fixed boundary conditions for the booster and the conditions of transversality at points $t_k^{(1)}$ and $\tilde{t}_k$.

If the phase trajectory strikes the boundary $f_\phi^{(i)} = 0$, existing due to the existence of the limiting condition on phase coordinates (2.1.9), at points of entry and exit from the boundary in order to satisfy the stability condition, the following equations must be fulfilled:

$$\left.\begin{aligned} &\lambda_1^{(i)}(t_{i-}^*) - \lambda_1^{(i)}(t_{i+}^*) + e_\Phi^{(i)}\frac{\partial f_\Phi^{(i)}}{\partial V} = 0 \qquad (i = 1, \ldots, m), \\ &\lambda_2^{(i)}(t_{i-}^*) - \lambda_2^{(i)}(t_{i+}^*) + e_\Phi^{(i)}\frac{\partial f_\Phi^{(i)}}{\partial \theta} = 0, \\ &\lambda_3^{(i)}(t_{i-}^*) - \lambda_3^{(i)}(t_{i+}^*) + e_\Phi^{(i)}\frac{\partial f_\Phi^{(i)}}{\partial H} = 0, \end{aligned}\right\} \quad (2.2.21)$$

$$\lambda_4^{(i)}(t_{i-}^*) - \lambda_4^{(i)}(t_{i+}^*) = 0, \quad C_0^{(i)}(t_{i-}^*) - C_0^{(i)}(t_{i+}^*) = 0, \quad (2.2.22)$$

$$\lambda_5^{(i)}(t_{i-}^*) - \lambda_5^{(i)}(t_{i+}^*) = 0, \quad (2.2.23)$$

$$\left.\begin{aligned} &\lambda_1^{(i)}(t_{i-}^{cx}) - \lambda_1^{(i)}(t_{i+}^{cx}) = 0, \quad \lambda_2^{(i)}(t_{i-}^{cx}) - \lambda_2^{(i)}(t_{i+}^{cx}) = 0, \\ &\lambda_3^{(i)}(t_{i-}^{cx}) - \lambda_3^{(i)}(t_{i+}^{cx}) = 0, \quad \lambda_4^{(i)}(t_{i-}^{cx}) - \lambda_4^{(i)}(t_{i+}^{cx}) = 0, \\ &\lambda_5^{(i)}(t_{i-}^{cx}) - \lambda_5^{(i)}(t_{i+}^{cx}) = 0, \quad C_0^{(i)}(t_{i-}^{cx}) - C_0^{(i)}(t_{i+}^{cx}) = 0. \end{aligned}\right\} \quad (2.2.24)$$

After conversion of expressions (2.2.21), we find

$$\left.\begin{aligned} &\lambda_1^{(i)}(t_{i-}^*) - \lambda_1^{(i)}(t_{i+}^*) - (\lambda_2^{(i)}(t_{i-}^*) - \lambda_2^{(i)}(t_{i+}^*))\frac{\partial f_\Phi^{(i)}/\partial V}{\partial f^{(i)}/\partial \theta} = 0, \\ &\lambda_3^{(i)}(t_{i-}^*) - \lambda_3^{(i)}(t_{i+}^*) - (\lambda_2^{(i)}(t_{i-}^*) - \lambda_2^{(i)}(t_{i+}^*))\frac{\partial f_\Phi^{(i)}/\partial H}{\partial f_\Phi^{(i)}/\partial \theta} = 0. \end{aligned}\right\} \quad (2.2.25)$$

Consequently, upon transition through the entry point t_1^*, some Lagrange coefficients -- λ_1, λ_2, and λ_3 -- according to (2.2.25) may undergo first order discontinuity, while others -- λ_4, λ_5 and C_0 -- remain continuous according to (2.2.22) and (2.2.23). At points of departure from the boundary t_1^{cx} [points where the phase trajectory transfers from the boundary fixed by equation $f_\phi^{(i)} = 0$ to the internal area fixed by inequality (2.1.9)], the Lagrange coefficients $\lambda_1, \ldots, \lambda_5$ and C_0, according to (2.2.24), are continuous. Furthermore, it must be noted that, fixing the value of the Lagrange coefficient $\lambda_2(t^*_{1+})$ or $c^{(i)}_\varphi$ at the entry points t^*_1 to the right of the discontinuity (point t^*_1), we can determine the value of Lagrange coefficients $\lambda_1(t^*_{1+})$ and $\lambda_3(t^*_{1+})$ to the right of the discontinuity according to (2.2.25), and equations (2...22) and (2.2.25) should be fulfilled. Conditions (2.2.22)-(2.2.25) will be referred to as the conditions of discontinuity of the Lagrange coefficients at the points of entry and exit from the boundary of limitation on the phase variables.

We will not analyze the conditions of discontinuity of Lagrange coefficients and C_0 at points of discontinuity of control $\alpha(t)$ and $p(t)$, produced from the condition of stability (2.2.1), since they are similar to the conditions presented in Chapter I, § 2. These discontinuity conditions require continuity of the Lagrange coeffcients $\lambda_1, \ldots, \lambda_5$ and C_0 at the break points of control functions $\alpha(t)$ and $p(t)$.

The condition of stability (2.2.1) leads to another important condition -- the condition of transversality. Since the phase variables of the initial and final points of the phase trajectory of the flight vehicle and the final point of the phase trajectory of the booster are not interrelated, the conditions of transversality are as follows:

at point t_0

$$\lambda^{(1)}_{10}\,dV_0+\lambda^{(1)}_{20}\,d\vartheta_0+\lambda^{(1)}_{30}\,dH_0+\lambda^{(1)}_{4}\,dL_0-\sum_{\varrho=1}^{m} e_\varrho\, d\varphi_{0\varrho}=0; \quad (2.2.26)$$

at point t_k

$$\left(C^{(2)}_0+\frac{\partial I}{\partial t_k}\right)dt_k+\left(\lambda^{(k)}_{1k}+\frac{\partial I}{\partial V_k}\right)dV_k+\left(\lambda^{(k)}_{2k}+\frac{\partial I}{\partial \vartheta_k}\right)d\vartheta_k+$$

$$+\left(\lambda^{(k)}_{3k}+\frac{\partial I}{\partial H_k}\right)dH_k+\left(\lambda^{(k)}_{4}+\frac{\partial I}{\partial L_k}\right)dL_k+\sum_{s=1}^{m} e_{ks}\, d\varphi_{ks}=0; \quad (2.2.27)$$

at point $\tilde{t}_k$

$$\tilde{C}_0\, d\tilde{t}_k+\tilde{\lambda}_{1k}\, d\tilde{V}_k+\tilde{\lambda}_{2k}\, d\tilde{\vartheta}_k+\tilde{\lambda}_{3k}\, d\tilde{H}_k+\tilde{\lambda}_4\, d\tilde{L}_k+$$

$$+\sum_{s=1}^{l}\tilde{e}_{ks}\, d\tilde{\varphi}_{ks}=0; \quad (2.2.28)$$

at points $t^{(i)}$

$$e_{\rho}^{(i)} + \lambda_{5\kappa}^{(i)} = 0; \tag{2.2.29}$$

at point $\tilde{t}_k^{(1)}$

$$e_{\rho}^{(\kappa)} + \tilde{\lambda}_{5\kappa} = 0. \tag{2.2.30}$$

Selecting coefficients e_ρ ($\rho = 1, \ldots, m \leq 4$), $e_{k\sigma}$ ($\sigma = 1, \ldots, n \leq 5$) and $\tilde{e}_{k\pi}$ ($\pi = 1, \ldots, l \leq 5$) such that the expressions before the corresponding phase variable differentials in equations (2.2.26)-(2.2.28) are equal to zero, we produce

$$\left.\begin{aligned} &\lambda_{10}^{(1)} - e_{\rho}\frac{\partial \psi_{0\rho}}{\partial V_0} = 0, \quad \lambda_{20}^{(1)} - e_{\rho}\frac{\partial \psi_{0\rho}}{\partial \vartheta_0} = 0, \\ &\lambda_{30}^{(1)} - e_{\rho}\frac{\partial \psi_{0\rho}}{\partial H_0} = 0, \quad \lambda_{4}^{(1)} - e_{\rho}\frac{\partial \psi_{0\rho}}{\partial L_0} = 0, \end{aligned}\right\} \tag{2.2.31}$$

$$\left.\begin{aligned} &\lambda_{1\kappa}^{(\kappa)} + \frac{\partial I}{\partial V_\kappa} + e_{\kappa\sigma}\frac{\partial \psi_{\kappa\sigma}}{\partial V_\kappa} = 0, \quad \lambda_{2\kappa}^{(\kappa)} + \frac{\partial I}{\partial \vartheta_\kappa} + e_{\kappa\sigma}\frac{\partial \psi_{\kappa\sigma}}{\partial \vartheta_\kappa} = 0, \\ &\lambda_{3\kappa}^{(\kappa)} + \frac{\partial I}{\partial H_\kappa} + e_{\kappa\sigma}\frac{\partial \psi_{\kappa\sigma}}{\partial H_\kappa} = 0, \quad \lambda_{4}^{(2)} + \frac{\partial I}{\partial L_\kappa} + e_{\kappa\sigma}\frac{\partial \psi_{\kappa\sigma}}{\partial L_\kappa} = 0, \\ &C_0^{(2)} + \frac{\partial I}{\partial t_\kappa} + e_{\kappa\sigma}\frac{\partial \psi_{\kappa\sigma}}{\partial t_\kappa} = 0, \end{aligned}\right\} \tag{2.2.32}$$

$$\left.\begin{aligned} &\tilde{\lambda}_{1\kappa} + \tilde{e}_{\kappa\pi}\frac{\partial \tilde{\psi}_{\kappa\pi}}{\partial \tilde{V}_\kappa} = 0, \quad \tilde{\lambda}_{2\kappa} + \tilde{e}_{\kappa\pi}\frac{\partial \tilde{\psi}_{\kappa\pi}}{\partial \tilde{\vartheta}_\kappa} = 0, \\ &\tilde{\lambda}_{3\kappa} + \tilde{e}_{\kappa\pi}\frac{\partial \tilde{\psi}_{\kappa\pi}}{\partial \tilde{H}_\kappa} = 0, \quad \tilde{\lambda}_{4} + \tilde{e}_{\kappa\pi}\frac{\partial \tilde{\psi}_{\kappa\pi}}{\partial \tilde{L}_\kappa} = 0, \\ &\tilde{C}_0 + \tilde{e}_{\kappa\pi}\frac{\partial \psi_{\kappa\pi}}{\partial \tilde{t}_\kappa} = 0. \end{aligned}\right\} \tag{2.2.33}$$

Thus, we have found and analyzed all necessary first order conditions, the fulfillment of which allows us to determine the optimal values of the main plan parameters and the optimal mode of motion of the multistage flight vehicle, capable of maneuvering with boundary conditions fixed for it and for the booster with the maximum value of the criterion of effectiveness. This requires integration of equation systems (2.I), (2.Ĩ) and (2.II), (2.ĨI) so that the boundary conditions (2.1.5) and final conditions for the booster (2.1.6), as well as the conditions of optimality of the main plan parameters (2.2.12)-(2.2.15) and the conditions of transversality (2.2.31)-(2.2.33) are fulfilled, while the branching condition (2.2.19), discontinuity condition (2.2.22)-(2.2.25) and condition of discontinuity at points of discontinuity of the control functions must be fulfilled. This problem is a multipoint boundary problem, the algorithm of which was analyzed in detail in Chapter I, § 4 and the appendix. Therefore, our presentation of the construction of the computational procedure of this multipoint boundary problem will be performed considering the results produced in Chapter I, § 4.

§ 3. Computational Algorithm for Variational Method of Optimization of Multistage Flight Vehicle Considering Possibility of Independent Maneuver of Booster and Vehicle

Proper construction of the procedure for calculation of the optimal values of the main plan parameters and optimal control of the multistage flight vehicle is possible only after precise formalization of the multipoint boundary problem, to which the variational problem stated in § 1 of this chapter has been reduced. Approaching its formulation, we introduce a number of assumptions which simplify our presentation but do not change the essence of the problem.

Suppose at the initial point t_0 the concrete phase variable values are fixed, i. e. the following conditions obtain:

$$\psi_{01} \equiv V_0 - V_0^{\text{н}} = 0, \quad \psi_{02} \equiv \theta_0 - \theta_0^{\text{н}} = 0,$$
$$\psi_{03} \equiv H_0 - H_0^{\text{н}} = 0, \quad \psi_{04} \equiv L_0 - L_0^{\text{н}} = 0.$$

Then equation (2.2.31) becomes

$$\lambda_{10}^{(1)} - c_1 = 0, \quad \lambda_{20}^{(1)} - c_2 = 0, \quad \lambda_{30}^{(1)} - c_3 = 0, \quad \lambda_4^{(1)} - c_4 = 0,$$

and therefore the Lagrange coefficients $\lambda_{10}^{(1)}, \ldots, \lambda_{50}^{(1)}$ are unknown.

Since the first integral of (2.2.10) can be replaced by any of the differential equations of system (2.11), suppose it is replaced by the Euler-Lagrange equation coupled to the phase variable μ [seventh equation and system (2.11)]. In this connection, according to (2.2.10), we represent

$$\lambda_8^{(i)} = \xi^{(i)}, \tag{2.3.1}$$

where

$$\xi^{(i)} = \frac{J_{0i}}{a_{0i} f^{(i)}} - (\lambda_1^{(i)} \varphi_1^{(i)} + \lambda_2^{(i)} \varphi_2^{(i)} + \lambda_3^{(i)} \varphi_3^{(i)} + \lambda_4^{(i)} \varphi_4^{(i)} + C_0^{(i)}).$$

Using equation (2.3.1) to exclude Lagrange coefficient $\lambda_5^{(i)}$ from equation system (2.II), let us reduce the order of this system by unity. Now at point t_0 we will have the unknown Lagrange coefficients $\lambda_{10}^{(1)}, \ldots, \lambda_4^{(1)}$ and the constant of the first integral $C_0^{(1)}$.

Furthermore, the final conditions (2.1.5) and conditions of transversality (2.2.32) after exclusion of the coefficients $e_{k\upsilon}$ at point t_k and the expression of the first integral for this point, giving us only six relationships, can be expressed in general form as follows:

$$\varphi_\varkappa^*(\lambda_{1k}^{(\varkappa)}, \ldots, \lambda_4^{(2)}, C_0^{(2)}, V_\varkappa, \ldots, t_\varkappa, b_{0i}, \mu_{\varkappa j}, G_0, G_{p,l}) = 0 \quad (\varkappa = 1, \ldots, 5), \tag{2.3.2}$$

$$\psi_\varkappa^0 = 0, \tag{2.3.3}$$

where (2.3.3) represents one of the conditions (2.1.5) not included in (2.3.2) (for example, the condition with index 1).

The final conditions (2.1.6) and the conditions of transversality (2.2.33), after exclusion of coefficients $\tilde{e}_{k\pi}$ at point $\tilde{t}_k$ and the expression of the first integral for this point, making up only six relationships, can be represented in general form as follows:

$$\tilde{\varphi}_m(\tilde{\lambda}_{1\varkappa}, \ldots, \tilde{\lambda}_4, \tilde{c}_0, \tilde{V}_\varkappa, \ldots, \tilde{t}_\varkappa, b_{0i}, \mu_{\varkappa 1}, \mu_\varkappa^{(n)}) = 0 \quad (m = 1, \ldots, 5), \tag{2.3.4}$$

$$\tilde{\varphi}_\varkappa^0 = 0, \tag{2.3.5}$$

where (2.3.5) represents one of the conditions (2.1.6) not included in relationship (2.3.4) (for example,the condition with index 1).

After these transformations, we can now formulate the multipoint boundary problem. It consists of the following: determine at point t_0 -- $\lambda_{10}^{(1)}, \ldots, \lambda_4^{(1)}$ and $C_0^{(1)}$; at point $t^{(1)}$ -- $\lambda_{10}^{(2)}, \ldots, \lambda_{50}^{(2)}$ or $C_0^{(2)}$; the main plan parameters a_{0i}, b_{0i}, J_{0i}, $t_k^{(H)}$, μ_{kJ} and G_{pl} are such that as a result of integration of system (2.I)-(2.II) from t_0 to t_k and system (2.I)-(2.II) considering the branching condition (2.2.19) from $t^{(1)}$ to t_k, conditions (2.3.2)-(2.3.4) and the condition of optimality of the main plan parameters (2.2.13)-(2.2.15) are fulfilled, considering (2.2.12), using $\psi_\mu^{(1)}$ and (2.3.3), (2.3.5) as the stop functions. It is considered here that the remaining principal plan parameters G_{0J}, μ_{ki}, $S^{(i)}$ and $P_{max}^{(i)}$ can be determined using equations (2.1.2)-(2.1.3).

Thus, in order to satisfy the twenty conditions (2.2.13)-(2.2.15) and (2.3.2), (2.3.4), there are twenty unknown parameters $\lambda_{10}^{(1)}, \ldots, C_0^{(1)}$, $\lambda_{10}^{(2)}, \ldots, C_0^{(2)}$ and a_{0i}, J_{0i}, b_{0i}, μ_{kJ}, $\mu_k^{(II)}$, G_{pl}, allowing us to speak of the possibility of solving the multipoint boundary problem as stated.

In order to decrease the number of unknown parameters and thereby the number of boundary conditions, we can use the homogeneity of systems (2.II) and (2.II) relative to the Lagrange coefficients λ_j. For this, we must divide the equations of system (2.II) and system (2.II), the conditions of optimality of the main plan parameters (2.2.16)-(2.2.18) by any one of the Lagrange coefficients $\lambda_{10}^{(1)}, \ldots, \lambda_4^{(1)}$, for example by $|\lambda_{10}^{(1)}|$, assuming that the optimal values of the main plan parameters lie within the permissible area, and the converted transversality conditions in relationships (2.3.2), (2.3.4). Then condition (2.2.18) and the converted transversality conditions in relationships (2.3.4) will not be changed due to their homogeneity relative to Lagrange coefficients λ_j, while coefficient $\lambda_{10}^{(1)}$ will be excluded from condition (2.2.16) and the transformed transversality conditions in relationship (2.3.2) using equation (2.2.17). After these transformations, equation (2.2.16) becomes

$$\frac{m_{G_{p.t}}}{m_a^{(1)}} A_a^{(1)} - \left(\frac{m_{G_0}}{m_a^{(1)}} A_a^{(1)} - A_0 \right) \frac{\partial I / \partial G_{p.t}}{\partial I / \partial G_0} = 0, \tag{2.3.6}$$

while conditions (2.3.2) are rerresented as follows:

$$\varphi_k(\lambda_{1k}^{(k)}, \ldots, \lambda_1^{(2)}, A_a^{(2)}, m_a^{(2)}, C_0^{(2)}, V_k, \ldots, t_k, b_{0i}, \mu_{kJ}, G_0, G_{p.t}) = 0. \tag{2.3.7}$$

Thus, we have produced twelve unknown parameters $\lambda_{20}^{(1)}, \ldots, C_0^{(1)}$, $\lambda_{10}^{(2)}, \ldots, C_0^{(2)}$, a_{0i}, b_{0i}, J_{0i}, μ_{kJ}, $\mu_k^{(H)}$ and G_{pl}, using which we must satisfy the twelve boundary conditions (2.3.6), (2.2.18) and (2.3.4), (2.3.7), since condition (2.2.17) can now be considered always satisfied.

If these boundary conditions are represented as functionals

$$
\left.
\begin{aligned}
&p_1=c_1\left[-\frac{m_{O_{pf}}}{m_a^{(1)}}A_a^{(1)}-\left(\frac{m_{G_0}}{m_a^{(1)}}-A_0\right)\frac{\partial I/\partial G_{pf}}{\partial I/\partial G_0}\right]=0,\\
&p_2=c_2\left(1+\frac{A_p^{(2)}m_a^{(1)}}{A_a^{(1)}m_p^{(2)}}\right)^2,\\
&p_3=c_3\left(1+\frac{A_p^{(2)}m_a^{(2)}}{A_a^{(2)}m_a^{(2)}}\right)^2,\\
&p_4=c_4\left(1-\frac{A_J^{(2)}m_a^{(2)}}{A_a^{(2)}m_J^{(2)}}\right)^2,\\
&p_5=c_5\left(\frac{\partial\lambda_4^{(2)}}{\partial S^{(2)}}-\frac{A_b^{(2)}m_a^{(2)}}{A_a^{(2)}m_b^{(2)}}\right)^2\\
&p_l=c_l\varphi_l^2 \quad (l=6,\ldots,10),\\
&p_{11}=c_{11}\left(1+\frac{A_p^{(1)}m_a^{(1)}}{A_a^{(1)}m_p^{(1)}}\right)^2,\\
&p_{12}=c_{12}\left(1-\frac{A_J^{(1)}m_a^{(1)}}{A_a^{(1)}m_J^{(1)}}\right)^2\\
&p_{13}=c_{13}\left(\frac{\partial\lambda_4^{(1)}}{\partial S^{(1)}}-\frac{A_b^{(1)}m_a^{(1)}}{A_a^{(1)}m_b^{(1)}}\right)^2,\\
&p_{14}=c_{14}\left(1+\frac{A_p^{(n)}m_a^{(1)}}{A_a^{(1)}m_p^{(n)}}\right)^2,\\
&\tilde{p}_m=\tilde{c}_m\tilde{\varphi}_m^2 \quad (m=1,\ldots,5),
\end{aligned}
\right|
\tag{2.3.8}
$$

then the algorithm of the multipoint boundary problem constructed according to the appendix and Chapter I, § 4, allows us to determine the following main plan parameters:

$$a_{0l},\ \mu_{\kappa J},\ J_{0l}^{*},\ b_{0l},\ \mu_\kappa^{(n)},\ G_{p.l} \qquad (l=J=1,2) \tag{2.3.9}$$

and the following Lagrange coefficients

$$\lambda_{10}^{(2)},\ \lambda_{20}^{(l)},\ \lambda_{30}^{(l)},\ \lambda_4^{(l)},\ C_0^{(l)} \qquad (l=1,2). \tag{2.3.10}$$

with which over $[t_0, t_k]$ the solution to system

$$
\left.
\begin{aligned}
& V' = \varphi_1^{(l)}, \quad \theta' = \varphi_2^{(l)}, \quad H' = \varphi_3^{(l)}, \quad L' = \varphi_4^{(l)}, \\
& \mu' = \varphi_5^{(l)}, \quad 0 = \varphi_6^{(l)}, \quad 0 = \varphi_7^{(l)}, \quad 0 = \varphi_8^{(l)}, \\
& \lambda_1^{(l)\prime} = \varphi_9^{(l)}, \ \lambda_2^{(l)\prime} = \varphi_{10}^{(l)}, \ \lambda_3^{(l)\prime} = \varphi_{11}^{(l)}, \ 0 = \varphi_{12}^{(l)}, \\
& 0 = \varphi_{13}^{(l)}, \quad 0 = \varphi_{14}^{(l)}, \\
& \eta_0^{(l)\prime} = \zeta_0^{(l)} = -\sum_{j=1}^{8} \lambda_j^{(l)} \frac{\partial \varphi_j^{(l)}}{\partial b_{0l}}, \\
& \eta_1^{(l)\prime} = \zeta_1^{(l)} = -\sum_{j=1}^{8} \lambda_j^{(l)} \frac{\partial \varphi_j^{(l)}}{\partial a_{0l}}, \\
& \eta_2^{(l)\prime} = \zeta_2^{(l)} = -\sum_{j=1}^{8} \lambda_j^{(l)} \frac{\partial \varphi_j^{(l)}}{\partial J_{0l}}, \\
& \eta_3' = \zeta_3 = -\sum_{j=1}^{8} \lambda_j^{(n)} \frac{\partial \varphi_j}{\partial \mu_{k11}}
\end{aligned}
\right| \qquad (2.III)
$$

and over $[t^{(1)}, \tilde{t}_k]$ considering (2.2.19), the solution of the system

$$
\left.
\begin{aligned}
& \tilde{V}' = \tilde{\varphi}_1, \ \tilde{\theta}' = \tilde{\varphi}_2, \ \tilde{H}' = \tilde{\varphi}_3, \ \tilde{L}' = \tilde{\varphi}_4, \ \tilde{\mu}' = \tilde{\varphi}_5, \\
& 0 = \tilde{\varphi}_6, \quad 0 = \tilde{\varphi}_7, \\
& \tilde{\lambda}_1' = \tilde{\varphi}_8, \ \tilde{\lambda}_2' = \tilde{\varphi}_9, \ \tilde{\lambda}_3' = \tilde{\varphi}_{10}, \ \tilde{\lambda}_4' = \tilde{\varphi}_{11} = 0, \\
& 0 = \tilde{\varphi}_{12}, \quad 0 = \tilde{\varphi}_{13}, \ \tilde{\eta}_0' = \tilde{\zeta}_0 = -\sum_{j=1}^{7} \tilde{\lambda}_j \frac{\partial \tilde{\varphi}_j}{\partial b_{01}}, \\
& \tilde{\eta}_1' = \tilde{\zeta}_1 = -\sum_{j=1}^{7} \tilde{\lambda}_j \frac{\partial \tilde{\varphi}_j}{\partial a_{01}}, \\
& \tilde{\eta}_2' = \zeta_2 = -\sum_{j=1}^{7} \lambda_j \frac{\partial \tilde{\varphi}_j}{\partial J_{01}}, \\
& \tilde{\eta}_3' = \tilde{\zeta}_3 = -\sum_{j=1}^{7} \tilde{\lambda}_j \frac{\partial \tilde{\varphi}_j}{\partial \mu_{k1}}, \\
& \tilde{\eta}_4' = \tilde{\zeta}_4 = -\sum_{j=1}^{7} \tilde{\lambda}_j \frac{\partial \tilde{\varphi}_j}{\partial \mu_k^{(n)}}
\end{aligned}
\right| \qquad (2.\tilde{I}II)
$$

lead to zero values of functions (2.3.8).

For this, according to the mathematical model of the algorithm, we must compose the conjugate system in the form:

over $[t_k, t_0]$

$$
\left.
\begin{aligned}
&\dot{y}^{(i)}_{l,m} = -\sum_{j=1}^{14} y^{(i)}_{j,m} \frac{\partial \varphi^{(i)}_j}{\partial x_l} - \sum_{n=0}^{3} z^{(i)}_{n,m} \frac{\partial \zeta^{(i)}_n}{\partial x_l} \\
&(l = 1, \ldots, 5, 9, \ldots, 12, 18;\ m = 1, \ldots, 19;\ i = 1, 2;\\
&x_l = \{V, \theta, H, L, \mu, \lambda_1, \lambda_2, \lambda_3, \lambda_4, C_0\}),\\
&\sum_{j=1}^{14} y^{(i)}_{j,m} \frac{\partial \varphi^{(i)}_j}{\partial \alpha} + \sum_{n=0}^{3} z^{(i)}_{n,m} \frac{\partial \zeta^{(i)}_n}{\partial \alpha} = 0,\\
&\sum_{j=1}^{14} y^{(i)}_{j,m} \frac{\partial \varphi^{(i)}_j}{\partial \lambda_\theta} + y^{(i)}_{15,m} \nu^{(i)}_\theta = 0, \quad 2y^{(i)}_{6,m} \nu^{(i)}_\theta - y^{(i)}_{15,m} \lambda^{(i)}_\theta = 0,\\
&\sum_{j=1}^{14} y^{(i)}_{j,m} \frac{\partial \varphi^{(i)}_j}{\partial p} + \sum_{n=0}^{3} z^{(i)}_{n,m} \frac{\partial \zeta^{(i)}_n}{\partial p} = 0,\\
&\sum_{j=1}^{14} y^{(i)}_{j,m} \frac{\partial \varphi^{(i)}_j}{\partial \lambda_p} + y^{(i)}_{16,m} \nu^{(i)}_p = 0, \quad 2y^{(i)}_{7,m} \nu^{(i)}_p - y^{(i)}_{16,m} \lambda^{(i)}_p = 0,\\
&\sum_{j=1}^{14} y^{(i)}_{j,m} \frac{\partial \varphi^{(i)}_j}{\partial \lambda_\Phi} + y^{(i)}_{17,m} f^{(i)}_\Phi = 0, \quad y^{(i)}_{8,m} f^{(i)}_\Phi = 0,\\
&z^{(i)}_{0,m} = \text{const}, \quad z^{(i)}_{1,m} = \text{const},\\
&z^{(i)}_{2,m} = \text{const}, \quad z_{3,m} = \text{const},\\
&\dot{z}^{(i)}_{4,m} = -\sum_{j=1}^{14} y^{(i)}_{j,m} \frac{\partial \varphi^{(i)}_j}{\partial b_{0l}} - \sum_{n=0}^{3} z^{(i)}_{n,m} \frac{\partial \zeta^{(i)}_n}{\partial b_{0l}},\\
&\dot{z}^{(i)}_{5,m} = -\sum_{j=1}^{14} y^{(i)}_{j,m} \frac{\partial \varphi^{(i)}_j}{\partial a_{0l}} - \sum_{n=0}^{3} z^{(i)}_{n,m} \frac{\partial \zeta^{(i)}_n}{\partial a_{0l}},\\
&\dot{z}^{(i)}_{6,m} = -\sum_{j=1}^{14} y^{(i)}_{j,m} \frac{\partial \varphi^{(i)}_j}{\partial J_{0l}} - \sum_{n=0}^{3} z^{(i)}_{n,m} \frac{\partial \zeta^{(i)}_n}{\partial J_{0l}},\\
&\dot{z}^{(i)}_{7,m} = -\sum_{j=1}^{14} y^{(i)}_{j,m} \frac{\partial \varphi^{(i)}_j}{\partial \mu_{k1l}} - \sum_{n=0}^{3} z^{(i)}_{n,m} \frac{\partial \zeta^{(i)}_n}{\partial \mu_{k1l}},
\end{aligned}
\right| \quad (2.\text{IV})
$$

$$\dot{z}^{(i)}_{8,m} = 0;$$

over $[t_k, t^{(1)}]$

$$
\begin{aligned}
&\dot{\tilde{y}}_{l.m} = -\sum_{j=1}^{13} \tilde{y}_{j.m} \frac{\partial \tilde{\varphi}_j}{\partial \tilde{x}_l} - \sum_{n=0}^{4} \tilde{z}_{n.m} \frac{\partial \tilde{\zeta}_n}{\partial \tilde{x}_l} \\
&(l = 1, \ldots, 5, 8, \ldots, 11, 16;\ m = 1, \ldots, 19; \\
&\tilde{x}_l = \{\tilde{V}, \tilde{\vartheta}, \tilde{H}, \tilde{L}, \tilde{\mu}, \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3, \tilde{\lambda}_4, \tilde{C}_0\}), \\
&\sum_{j=1}^{13} \tilde{y}_{j.m} \frac{\partial \tilde{\varphi}_j}{\partial \tilde{a}} + \sum_{n=0}^{4} \tilde{z}_{n.m} \frac{\partial \tilde{\zeta}_n}{\partial \tilde{a}} = 0, \\
&\sum_{j=1}^{13} \tilde{y}_{j.m} \frac{\partial \tilde{\varphi}_j}{\partial \tilde{\lambda}_s} + \tilde{y}_{14.m} \tilde{\nu}_s = 0, \quad 2\tilde{y}_{6.m} \tilde{\nu}_s - \tilde{y}_{14.m} \tilde{\lambda}_s = 0, \\
&\sum_{j=1}^{13} \tilde{y}_{j.m} \frac{\partial \tilde{\varphi}_j}{\partial \tilde{p}} + \sum_{n=0}^{4} \tilde{z}_{n.m} \frac{\partial \tilde{\zeta}_n}{\partial \tilde{p}} = 0, \\
&\sum_{j=1}^{13} \tilde{y}_{j.m} \frac{\partial \tilde{\varphi}_j}{\partial \tilde{\lambda}_p} + \tilde{y}_{15.m} \tilde{\nu}_p = 0, \quad 2\tilde{y}_{7.m} \tilde{\nu}_p - \tilde{y}_{15.m} \tilde{\lambda}_p = 0, \\
&\tilde{z}_{0.m} = \text{const}, \quad \tilde{z}_{1.m} = \text{const}, \\
&\tilde{z}_{2.m} = \text{const}, \quad \tilde{z}_{3.m} = \text{const}, \quad \tilde{z}_{4.m} = \text{const}, \\
&\dot{\tilde{z}}_{5.m} = -\sum_{j=1}^{13} \tilde{y}_{j.m} \frac{\partial \tilde{\varphi}_j}{\partial b_{01}} - \sum_{n=0}^{4} \tilde{z}_{n.m} \frac{\partial \tilde{\zeta}_n}{\partial b_{01}}, \\
&\dot{\tilde{z}}_{6.m} = -\sum_{j=1}^{13} \tilde{y}_{j.m} \frac{\partial \tilde{\varphi}_j}{\partial a_{01}} - \sum_{n=0}^{4} \tilde{z}_{n.m} \frac{\partial \tilde{\zeta}_n}{\partial a_{01}}, \\
&\dot{\tilde{z}}_{7.m} = -\sum_{j=1}^{13} \tilde{y}_{j.m} \frac{\partial \tilde{\varphi}_j}{\partial J_{01}} - \sum_{n=0}^{4} \tilde{z}_{n.m} \frac{\partial \tilde{\zeta}_n}{\partial J_{01}}, \\
&\dot{\tilde{z}}_{8.m} = -\sum_{j=1}^{13} \tilde{y}_{j.m} \frac{\partial \tilde{\varphi}_j}{\partial \mu_{\kappa 1}} - \sum_{n=0}^{4} \tilde{z}_{n.m} \frac{\partial \tilde{\zeta}_n}{\partial \mu_{\kappa 1}}, \\
&\dot{\tilde{z}}_{9.m} = -\sum_{j=1}^{13} \tilde{y}_{j.m} \frac{\partial \tilde{\varphi}_j}{\partial \mu_{\kappa}^{(\kappa)}} - \sum_{n=0}^{4} \tilde{z}_{n.m} \frac{\partial \tilde{\zeta}_n}{\partial \mu_{\kappa}^{(\kappa)}}, \\
&\dot{\tilde{z}}_{10.m} = 0.
\end{aligned}
\qquad (2.\tilde{IV})
$$

The initial conditions for the conjugate systems (2.IV) and ($2.\tilde{IV}$) are determined at moments $\psi_k^0 = 0$, $\tilde{\psi}_k^0 = 0$ and $\psi_1^{(1)} = 0$ and according to the appendix are equal to

where $\psi_k^0 = 0$ $(t = t_k)$

$$\left.\begin{aligned}
&y_{j,m}^{(\varkappa)} = \frac{\partial p_m}{\partial x_j} - \frac{\dot{p}_m}{\dot{\psi}_\varkappa^0}\frac{\partial \psi_\varkappa^0}{\partial x_j} \quad (m = 1, \ldots, 10,\ j = 1, \ldots, 4;\\
&\qquad\qquad x_1 = V, \ldots, x_4 = L),\\
&y_{8,m}^{(\varkappa)} = 0,\\
&y_{p,m}^{(\varkappa)} = \frac{\partial p_m}{\partial x_p} \quad (p = 9, \ldots, 12;\ x_9 = \lambda_1, \ldots, x_{12} = \lambda_4),\\
&y_{18,m}^{(\varkappa)} = \frac{\partial p_m}{\partial C_0^{(2)}},\\
&z_{0,m}^{(\varkappa)} = \frac{\partial p_m}{\partial \eta_{0\varkappa}^{(n)}},\quad z_{n,m}^{(\varkappa)} = 0 \quad (n = 1, 2),\\
&z_{3,m}^{(\varkappa)} = \frac{\partial p_m}{\partial \eta_{3\varkappa}},\\
&z_{4,m}^{(\varkappa)} = \frac{\partial p_m}{\partial \vartheta_{0\varkappa}},\quad z_{5,m}^{(\varkappa)} = 0,\quad z_{6,m}^{(\varkappa)} = 0,\\
&z_{7,m}^{(\varkappa)} = \frac{\partial p_m}{\partial \vartheta_{\varkappa 11}},\\
&z_{8,m}^{(\varkappa)} = \frac{\partial p_m}{\partial \alpha_{n.r}};
\end{aligned}\right\} \quad (2.3.11)$$

where $\tilde{\psi}_k^0 = 0$ $(t = \tilde{t}_k)$

$$\left.\begin{aligned}
&\tilde{y}_{j,\varkappa}^{(\varkappa)} = -\frac{\dot{p}_\varkappa}{\dot{\tilde{\psi}}_\varkappa^0}\frac{\partial \tilde{\psi}_\varkappa^0}{\partial \tilde{x}_j} \quad (k = 1, \ldots, 14)\\
&\qquad\qquad (j = 1, \ldots, 4;\ \tilde{x}_1 = \tilde{V}, \ldots, \tilde{x}_4 = \tilde{L}),\\
&\tilde{y}_{5,\varkappa}^{(\varkappa)} = 0,\\
&\tilde{y}_{p,\varkappa}^{(\varkappa)} = 0 \quad (p = 8, \ldots, 11, 18),\\
&\tilde{z}_{0,\varkappa}^{(\varkappa)} = \frac{\partial p_\varkappa}{\partial \tilde{\eta}_{0\varkappa}},\\
&\tilde{z}_{1,\varkappa}^{(\varkappa)} = \tilde{z}_{2,\varkappa}^{(\varkappa)} = 0,\\
&\tilde{z}_{3,\varkappa}^{(\varkappa)} = \frac{\partial p_\varkappa}{\partial \tilde{\eta}_{3\varkappa}},\quad \tilde{z}_{4,\varkappa}^{(\varkappa)} = \frac{\partial p_\varkappa}{\partial \tilde{\eta}_{4\varkappa}},
\end{aligned}\right\} \quad (2.3.12)$$

$$
\left.
\begin{aligned}
&z^{(\kappa)}_{l,k}=0 \quad (l=5,\ 6,\ 7),\\
&\tilde{z}^{(\kappa)}_{6,k}=\frac{\partial p_k}{\partial \mu_{\kappa 1}}, \quad \tilde{z}^{(\kappa)}_{8,k}=\frac{\partial p_k}{\partial \mu^{(\kappa)}_{\kappa}};\\
&\tilde{y}^{(\kappa)}_{j,n}=\frac{\partial \tilde{p}_n}{\partial \tilde{x}_j}-\frac{\dot{\tilde{p}}_n}{\dot{\tilde{\varphi}}^0_{\kappa}}\,\frac{\partial \tilde{\varphi}^0_{\kappa}}{\partial \tilde{x}_j}\ (n=1,\ldots,5;\ j=1,\ldots,4).\\
&\tilde{y}^{(\kappa)}_{5,n}=0,\\
&\tilde{y}^{(\kappa)}_{p,n}=\frac{\partial \tilde{p}_n}{\partial \tilde{x}_p}\ (p=8,\ldots,11;\ x_8=\tilde{\lambda}_1,\ldots,x_{11}=\tilde{\lambda}_4),\\
&\tilde{y}^{(\kappa)}_{16,n}=\frac{\partial \tilde{p}_n}{\partial \tilde{C}_0},\\
&\tilde{z}^{(\kappa)}_{l,n}=0 \quad (l=0,\ldots,4,\ 6,\ 7),\\
&\tilde{z}^{(\kappa)}_{5,n}=\frac{\partial \tilde{p}_n}{\partial b_{01}},\\
&\tilde{z}^{(\kappa)}_{6,n}=\frac{\partial \tilde{p}_n}{\partial \mu_{\kappa 1}},\\
&\tilde{z}^{(\kappa)}_{8,n}=\frac{\partial \tilde{p}_n}{\partial \mu^{(\kappa)}_{\kappa}}.
\end{aligned}
\right| \qquad (2.3.12)
$$

The branching conditions lead to certain peculiarities in the determination of the discontinuity conditions of the conjugate coefficients for conjugate system (2.IV) and (2.IV̌) at points $t^{(1)}$ and $\tilde{t}_k^{(1)}$. Therefore, let us analyze the conclusion of the discontinuity conditions of the conjugate coefficients in more detail. Due to the properties of the conjugate systems analyzed in the appendix we produce[1]

$$
\begin{aligned}
\Delta p_m = &-\left(\sum_{j=1}^{4} y_{j.m}\delta x_j^{(\kappa)}+\sum_{p=9}^{12} y_{p.m}\delta x_p^{(\kappa)}+y_{16.m}\delta C_0^{(2)}+z_{0.m}\delta\eta_{00}^{(\kappa)}+\right.\\
&\left.+z_{3.m}\delta\eta_{30}+\sum_{l=4}^{8} z_{l.m}\delta a_l^{(2)}\right)\Bigg|_{t^{(2)}}+\left[\sum_{j=1}^{4}\frac{\partial p_m}{\partial x_{j\kappa}^{(2)}}\delta x_{j\kappa}^{(2)}+\right.\\
&+\left(\frac{\partial p_m}{\partial \mu_{\kappa}^{(2)}}-\frac{\dot{p}_m|_{t^{(2)}}}{\dot{\varphi}_{\mu}^{(2)}}\right)\delta\mu_{\kappa}^{(2)}+\sum_{p=9}^{11}\frac{\partial p_m}{\partial x_{p\kappa}^{(2)}}\delta x_{p\kappa}^{(2)}+\sum_{n=1}^{2}\frac{\partial p_m}{\partial \eta_{n\kappa}^{(2)}}\delta\eta_{n\kappa}^{(2)}+
\end{aligned}
$$

(2.3.13)

---

[1] It should be considered here that integration is performed from t_k or $\tilde{t}_k$ to t_0.

$$
\begin{aligned}
&+\frac{\dot p_m|_{t_+^{(2)}}}{\dot\psi_\mu^{(2)}}\,\delta\mu_{\kappa 11}+\frac{\partial p_m}{\partial a_{02}}\,\delta a_{02}+\frac{\partial p_m}{\partial J_{02}}\,\delta J_{02}\Bigg]_{t_+^{(2)}}-\Bigg(\sum_{j=1}^{4} y_{j.m}\delta x_{j\kappa}^{(2)}+\\
&+y_{5.m}\delta\mu_\kappa^{(2)}+\sum_{p=9}^{12} y_{p.m}\delta x_{p\kappa}^{(2)}+y_{18.m}\delta C_0^{(2)}+\sum_{n=0}^{2} z_{n.m}\delta\eta_{n\kappa}^{(2)}+\\
&+\sum_{l=4}^{6} z_{l.m}\delta a_l^{(2)}+z_{8.m}\delta G_{n.r}\Bigg)\Bigg|_{t_+^{(2)}}+\Bigg(\sum_{j=1}^{4} y_{j.m}\delta x_{j0}^{(2)}+y_{5.m}\delta\mu_0^{(2)}+\\
&+\sum_{p=9}^{12} y_{p.m}\delta x_{p0}^{(2)}+y_{18.m}\delta C_0^{(2)}+\sum_{n=0}^{2} z_{n.m}\delta\eta_{n0}^{(2)}+\sum_{l=4}^{6} z_{l.m}\delta a_l^{(2)}+\\
&+z_{8.m}\delta G_{n.r}\Bigg)\Bigg|_{t_-^{(1)}}-\frac{\dot p_m|_{t_+^{(1)}}}{\dot\psi_\mu^{(1)}}\,\delta\mu_\kappa^{(1)}+\frac{\partial p_m}{\partial\eta_{0\kappa}^{(1)}}\,\delta\eta_{0\kappa}^{(1)}+\frac{\partial p_m}{\partial\eta_{1\kappa}^{(1)}}\,\delta\eta_{1\kappa}^{(1)}+\\
&+\frac{\partial p_m}{\partial\eta_{2\kappa}^{(1)}}\,\delta\eta_{2\kappa}^{(1)}+\sum_{l=4}^{6}\frac{\partial p_m}{\partial a_l^{(1)}}\,\delta a_l^{(1)}+\left(\frac{\partial p_m}{\partial\mu_{\kappa 1}}+\frac{\dot p_m|_{t_+^{(1)}}}{\dot\psi_\mu^{(1)}}\right)\delta\mu_{\kappa 1}- \qquad (2.3.13)\\
&-\Bigg(\sum_{j=1}^{4} y_{j.m}\delta x_{j\kappa}^{(1)}+y_{5.m}\delta\mu_\kappa^{(1)}+\sum_{p=9}^{12} y_{p.m}\delta x_{p\kappa}^{(1)}+y_{18.m}\delta C_0^{(1)}+\\
&+\sum_{n=0}^{2} z_{n.m}\delta\eta_{n\kappa}^{(1)}+\sum_{l=4}^{7} z_{l.m}\delta a_l^{(1)}+z_{8.m}\delta G_{p.l}\Bigg)\Bigg|_{t_+^{(1)}}+\\
&+\Bigg(\sum_{j=1}^{4}\tilde y_{j.m}\delta\tilde x_{j0}^{(\kappa)}+\sum_{p=8}^{11}\tilde y_{p.m}\delta\tilde x_{p0}^{(\kappa)}+\tilde y_{16.m}\delta\tilde C_0+\\
&+\tilde z_{0.m}\delta\tilde\eta_{00}^{(\kappa)}+\tilde z_{3.m}\delta\tilde\eta_{30}^{(\kappa)}+\tilde z_{4.m}\delta\tilde\eta_{40}^{(\kappa)}+\sum_{l=5}^{8}\tilde z_{l.m}\delta\tilde a_l^{(1)}+\\
&+\tilde z_{9.m}\delta\mu_\kappa^{(\kappa)}\Bigg)\Bigg|_{\tilde t_{\kappa-}^{(1)}}+\frac{\partial p_m}{\partial\tilde\eta_{1\kappa}}\,\delta\tilde\eta_{1\kappa}+\frac{\partial p_m}{\partial\tilde\eta_{2\kappa}}\,\delta\tilde\eta_{2\kappa}-\frac{\dot p_m|_{\tilde t_{\kappa+}^{(1)}}}{\dot{\tilde\psi}_\mu}\left(\delta\tilde\mu_\kappa^{(1)}-\right.\\
&\left.-\delta\mu_\kappa^{(\kappa)}\right)-\Bigg(\sum_{j=1}^{4}\tilde y_{j.m}\delta\tilde x_{j\kappa}^{(1)}+\tilde y_{5.m}\delta\tilde\mu_\kappa^{(1)}+\sum_{p=8}^{11}\tilde y_{p.m}\delta\tilde x_{p\kappa}^{(1)}+\tilde y_{16.m}\delta\tilde C_0+\\
&+\sum_{n=0}^{3}\tilde z_{n.m}\delta\tilde\eta_{n\kappa}+\sum_{l=5}^{8}\tilde z_{l.m}\delta\tilde a_l^{(1)}\Bigg)\Bigg|_{\tilde t_\kappa^{(1)}}+\Bigg(\sum_{j=1}^{4}\tilde y_{j.m}\delta\tilde x_{j0}+
\end{aligned}
$$

$$
+\tilde{y}_{9.m}\delta\tilde{\mu}_0+\sum_{p=5}^{11}\tilde{y}_{p.m}\delta\tilde{x}_{p0}+\tilde{y}_{10.m}\delta\tilde{C}_0+\sum_{n=0}^{3}\tilde{z}_{n.m}\delta\tilde{\eta}_{n0}+
$$

$$
+\sum_{l=5}^{8}\tilde{z}_{l.m}\delta\tilde{a}_l^{(1)}\Bigg)\Bigg|_{t_-^{(1)}}+\Bigg(\sum_{j=1}^{4}y_{j.m}^{(1)}\delta x_{j0}^{(1)}+y_{5.m}^{(1)}\delta\mu_0^{(1)}+\sum_{p=9}^{12}y_{p.m}^{(1)}\delta x_{p0}^{(1)}+
$$

$$
+y_{13.m}^{(1)}\delta C_0^{(1)}+\sum_{n=0}^{2}z_{n.m}\delta\eta_{n0}^{(1)}+\sum_{l=4}^{6}z_{l.m}\delta a_l^{(1)}\Bigg)\Bigg|_{t_0}, \qquad (2.3.13)
$$

where

$$
a_4^{(2)}=b_{02},\ \ldots,\ a_6^{(2)}=J_{02};
$$

$$
\tilde{a}_5^{(1)}=b_{01},\ \ldots,\ \tilde{a}_7^{(1)}=\mu_{\kappa 1};\quad a_4^{(1)}=b_{01},\ \ldots,\ a_6^{(1)}=J_{01}.
$$

$$
\mu_\kappa^{(2)}=\mu(t^{(2)}),\quad \mu_0^{(2)}=\mu(t_-^{(1)}),\quad \tilde{\mu}_\kappa^{(1)}=\tilde{\mu}(\tilde{t}_\kappa^{(1)}),\quad \tilde{\mu}_0^{(1)}=\mu(t_0).
$$

Furthermore, at points $\psi_\mu^{(i)} = 0$ and $\dot{\psi}_\mu = 0$, due to the continuity of the phase coordinates [see (2.1.7)] and the branching condition (2.2.19), the following relationships obtain:

where $\psi_\mu^{(2)} = 0$ $(t = t^{(2)})$

$$
\delta x_{j0}^{(\kappa)}=\delta x_{j\kappa}^{(2)}-\frac{\dot{x}_{j\kappa}^{(2)}-\dot{x}_{j0}^{(\kappa)}}{\dot{\psi}_\mu^{(2)}}\left(\delta\mu_\kappa^{(2)}-\delta\mu_{\kappa 11}\right)
$$

$$
(j=1,\ \ldots,\ 4,\ 9,\ \ldots,\ 12),
$$

$$
\delta\eta_{00}^{(\kappa)}=\delta\eta_{0\kappa}^{(2)}-\frac{\dot{\eta}_{0\kappa}^{(2)}-\dot{\eta}_{00}^{(\kappa)}}{\dot{\psi}_\mu^{(2)}}\left(\delta\mu_\kappa^{(2)}-\delta\mu_{\kappa 11}\right),
$$

$$
\delta\eta_{30}^{(\kappa)}=\frac{\dot{\eta}_{30}^{(\kappa)}}{\dot{\psi}_\mu^{(2)}}\left(\delta\mu_\kappa^{(2)}-\delta\mu_{\kappa 11}\right),
$$

$$
\delta C_0^{(2)}\Big|_{t_-^{(2)}}=\delta C_0^{(2)}\Big|_{t_+^{(2)}};
$$

where $\psi_\mu^{(1)} = 0$ $(t = t^{(1)})$

$$
\delta x_{j0}^{(2)}=\delta x_{j\kappa}^{(1)}-\frac{\dot{x}_{j\kappa}^{(1)}-\dot{x}_{j0}^{(2)}}{\dot{\psi}_\mu^{(1)}}\left(\delta\mu_\kappa^{(1)}-\delta\mu_{\kappa 1}\right)\qquad (j=1,\ \ldots,\ 4),
$$

$$
\delta\mu_0^{(2)}=\frac{\dot{\mu}_0^{(2)}}{\dot{\psi}_\mu^{(1)}}\left(\delta\mu_\kappa^{(1)}-\delta\mu_{\kappa 1}\right),
$$

$$\delta\eta_{n0}^{(2)} = \frac{\dot{\eta}_{n0}^{(2)}}{\dot{\psi}_{\mu}^{(1)}}\left(\delta\mu_{\kappa}^{(1)} - \delta\mu_{\kappa 1}\right) \qquad (n=0,\ 1,\ 2),$$

$$\delta\tilde{x}_{j0} = \delta x_{j\kappa}^{(1)} - \frac{\dot{x}_{j\kappa}^{(1)} - \dot{\tilde{x}}_{j0}}{\dot{\psi}_{\mu}^{(1)}}\left(\delta\mu_{\kappa}^{(1)} - \delta\mu_{\kappa 1}\right) \qquad (j=1,\ldots,4),$$

$$\delta\tilde{\mu}_0 = \frac{\dot{\tilde{\mu}}_0}{\dot{\psi}_{\mu}^{(1)}}\left(\delta\mu_{\kappa}^{(1)} - \delta\mu_{\kappa 1}\right) + \delta\mu_{\kappa 1},$$

$$\delta\tilde{\lambda}_{j0} = \delta\lambda_{j\kappa}^{(\kappa)} - \delta\lambda_{j0}^{(2)} - \frac{\dot{\lambda}_{j\kappa}^{(1)} - \dot{\lambda}_{j0}^{(2)} - \dot{\tilde{\lambda}}_{j0}}{\dot{\psi}_{\mu}^{(1)}}\left(\delta\mu_{\kappa}^{(1)} - \delta\mu_{\kappa 1}\right)$$

$$(j=1,\ldots,4),$$

$$\delta\tilde{C}_0 = \delta C_0^{(1)} - \delta C_0^{(2)},$$

$$\delta\tilde{\eta}_{n0} = \frac{\dot{\tilde{\eta}}_{n0}}{\dot{\psi}_{\mu}^{(1)}}\left(\delta\mu_{\kappa}^{(1)} - \delta\mu_{\kappa 1}\right) \qquad (n=0,\ldots,3);$$

<u>where $\tilde{\psi}_{\nu} = 0$ $(t = \tilde{t}_k^{(1)})$</u>

$$\delta\tilde{x}_{j0}^{(\kappa)} = \delta\tilde{x}_{j\kappa}^{(1)} - \frac{\dot{\tilde{x}}_{j\kappa}^{(1)} - \dot{\tilde{x}}_{j0}^{(\kappa)}}{\dot{\tilde{\psi}}_{\mu}}\left(\delta\tilde{\mu}_{\kappa}^{(1)} - \delta\mu_{\kappa}^{(\kappa)}\right) \qquad (j=1,\ldots,4),$$

$$\delta\tilde{\lambda}_{j0}^{(\kappa)} = \delta\tilde{\lambda}_{j\kappa}^{(1)} - \frac{\dot{\tilde{\lambda}}_{j\kappa}^{(1)} - \dot{\tilde{\lambda}}_{j0}^{(\kappa)}}{\dot{\tilde{\psi}}_{\mu}}\left(\delta\tilde{\mu}_{\kappa}^{(1)} - \delta\mu_{\kappa}^{(\kappa)}\right),$$

$$\delta\tilde{\eta}_{n0}^{(\kappa)} = \delta\eta_{n\kappa}^{(1)} - \frac{\dot{\tilde{\eta}}_{n\kappa}^{(1)} - \dot{\tilde{\eta}}_{n0}^{(\kappa)}}{\dot{\tilde{\psi}}_{\mu}}\left(\delta\tilde{\mu}_{\kappa}^{(1)} - \delta\mu_{\kappa}^{(\kappa)}\right) \qquad (n=0,\ 3),$$

$$\delta\tilde{\eta}_{40}^{(\kappa)} = \frac{\dot{\tilde{\eta}}_{40}^{(\kappa)}}{\dot{\tilde{\psi}}_{\mu}}\left(\delta\mu_{\kappa}^{(1)} - \delta\mu_{\kappa}^{(\kappa)}\right).$$

Furthermore, keeping in mind the weight relationships (2.1.2), we find

$$\delta\mu_{\kappa 1} = \alpha_b^{(1)}\delta b_{01} + \alpha_a^{(1)}\delta a_{01} + \alpha_J^{(1)}\delta J_{01} + \left(\alpha_{\mu}^{(1)} - 1\right)\delta\mu_{\kappa 1} +$$
$$+ \alpha_{\mu}^{(\kappa)}\delta\mu_{\kappa}^{(\kappa)} + \alpha_b^{(2)}\delta b_{02} + \alpha_a^{(2)}\delta a_{02} + \alpha_J^{(2)}\delta J_{02} + \alpha_{\mu}^{(2)}\delta\mu_{\kappa 11},$$

where

$$\alpha_{aj}^{(1)} = -\left[\frac{\frac{\partial\mu_{01}}{\partial G_0}}{\Delta_0}\left(\frac{\partial\xi_1^{(0)}}{\partial G_{011}}\,\frac{\partial\xi_1^{(1)}}{\partial a_j} - \frac{\partial\xi_1^{(0)}}{\partial a_j}\right) + \frac{\partial\mu_{01}}{\partial a_j}\right]$$

$$(a_j = \{b_{01},\ a_{01},\ J_{01},\ \mu_{\kappa 1},\ \mu_{\kappa}^{(\kappa)}\}),$$

$$\Delta_0 = \frac{\xi_1^{(0)}}{\partial G_0} - \frac{\partial\xi_1^{(1)}}{\partial G_0}\,\frac{\partial\xi_1^{(0)}}{\partial G_{011}},$$

$$u_{aj}^{(2)} = \frac{\partial\mu_{\kappa 1}/\partial G_0}{\Delta_0} \frac{\partial z_1^{(0)}}{\partial a_j}$$

$$(a_j = \{b_{02}, a_{02}, J_{02}, \mu_{\kappa 11}, G_{p.f}\}).$$

Using the relationships produced, we can convert (2.3.13) to

$$\begin{aligned}\Delta p_m = \Big[& \sum_{p=9}^{12} (y_{p.m}^{(2)} - \tilde{y}_{p-1.m})\delta x_{p0}^{(2)} + (y_{16.m}^{(2)} - \tilde{y}_{16.m})\delta C_0^{(2)} + \\ & + (z_{4.m}^{(2)} + \tilde{z}_{8.m}\alpha_b^{(2)})\delta b_{02} + (z_{5.m}^{(2)} + \tilde{z}_{8.m}\alpha_a^{(2)})\delta a_{02} + \\ & + (z_{6.m}^{(2)} + \tilde{z}_{8.m}\alpha_J^{(2)})\delta J_{02} + (z_{7.m0}^{(\kappa)} - \overline{y}_{5.m}^{(2)} + \tilde{z}_{8.m}\alpha_\mu^{(2)})\delta\mu_{\kappa 11}\Big]_{t_-^{(1)}} + \\ & + \Big[\sum_{p=10}^{12} y_{p.m}^{(1)}\delta x_{p0}^{(1)} + y_{13.m}^{(1)}\delta C_0^{(1)} + (z_{4.m}^{(1)} + \tilde{z}_{8.m}\alpha_b^{(1)})\delta b_{01} + \\ & + (z_{5.m}^{(1)} + \tilde{z}_{8.m}\alpha_a^{(1)})\delta a_{01} + (z_{6.m}^{(1)} + \tilde{z}_{8.m}\alpha_J^{(1)})\delta J_{01} + \\ & + [z_{7.m}^{(1)} + \tilde{z}_{8.m}(\alpha_\mu^{(1)} - 1)]\delta\mu_{\kappa 1} + (z_{8.m}^{(1)} + \tilde{z}_{8.m}\alpha_G)\delta G_{p.f} + \\ & + (\tilde{z}_{8.m} - \tilde{y}_{5.m}|_{t_-^{(1)}} + \tilde{z}_{8.m}\alpha_\mu^{(\kappa)})\delta\mu_\kappa^{(\kappa)}\Big]_{t_0} (m = 1, \dots, 10), \qquad (2.3.14)\end{aligned}$$

where

$$\overline{y}_{5.m}^{(2)} = \frac{1}{\dot{\psi}_\mu^{(2)}}\Big[\sum_{j=1}^{4} y_{j.m}^{(\kappa)}(\dot{x}_{j0}^{(\kappa)} - \dot{x}_{j\kappa}^{(2)}) + \sum_{p=9}^{12} y_{p.m}^{(\kappa)}(\dot{x}_{p0}^{(\kappa)} - \dot{x}_{p\kappa}^{(2)}) + \\ + z_{6.m}^{(\kappa)}(\dot{\eta}_{10.0}^{(\kappa)} - \dot{\eta}_{10\kappa}^{(2)}) + z_{4.m}^{(\kappa)}\dot{\eta}_{9.0}^{(\kappa)}\Big] - \frac{\dot{p}_m|_{t_+^{(2)}}}{\dot{\psi}_\mu^{(2)}}.$$

, we

In this connection, the discontinuity conditions of the conjugate coefficients of (2.IV) and (2.IV̌) at points $t^{(i)}$ can be represented as follows:

where $\psi_\mu^{(2)} = 0$ $(t = t^{(2)})$

$$\left.\begin{aligned} & y_{j.m}^{(2)} = y_{j.m}^{(\kappa)} + \frac{\partial p_m}{\partial x_{j\kappa}} \qquad (j = 1, \dots, 4, 9, \dots, 11; \\ & \qquad x_1 = V, \dots, x_4 = L, x_9 = \lambda_1, \dots, x_{11} = \lambda_3), \\ & y_{5.m}^{(2)} = \overline{y}_{5.m}^{(2)} + \frac{\partial p_m}{\partial \mu}\Big|_{t_+^{(2)}}, \\ & y_{12.m}^{(2)} = y_{12.m}^{(\kappa)}, \quad y_{15.m}^{(2)} = y_{15.m}^{(\kappa)}, \end{aligned}\right\} \qquad (2.3.15)$$

$$
\left.\begin{aligned}
&z_{0m}^{(2)} = z_{0,m}^{(\kappa)}, \quad z_{1,m}^{(2)} = \frac{\partial p_m}{\partial \eta_{1\kappa}^{(2)}}, \quad z_{2,m}^{(2)} = \frac{\partial p_m}{\partial \eta_{2\kappa}^{(2)}},\\
&z_{4,m}^{(2)} = z_{4,m}^{(\kappa)}, \quad z_{5,m}^{(2)} = \frac{\partial p_m}{\partial a_{2}}, \quad z_{6,m}^{(2)} = \frac{\partial p_m}{\partial J_{2}}, \quad z_{7,m}^{(2)} = 0,\\
&z_{8,m}^{(2)} = z_{8,m}^{(\kappa)};
\end{aligned}\right\} \qquad (2.3.15)
$$

where $\psi_{\nu}^{(1)} = 0 \ (t = t^{(1)})$

$$
\left.\begin{aligned}
&y_{j,m}^{(1)} = y_{j,m}^{(2)} + \tilde{y}_{j,m} \quad (j = 1, \ldots, 4),\\
&y_{p,m}^{(1)} = \tilde{y}_{p-1,m} \qquad (p = 9, \ldots, 12),\\
&y_{18,m}^{(1)} = \tilde{y}_{16,m},\\
&z_{n,m}^{(1)} = \frac{\partial p_m}{\partial \eta_{n\kappa}^{(1)}} \qquad (n = 0, 1, 2),\\
&z_{l,m}^{(1)} = \tilde{z}_{l+1,m} + \frac{\partial p_m}{\partial a_l^{(1)}} \ (l = 4, 5, 6;\ a_4^{(1)} = b_{01},\ a_5^{(1)} = a_{01},\\
&\qquad\qquad\qquad\qquad\qquad\qquad a_6^{(1)} = J_{01}),\\
&z_{8,m}^{(1)} = z_{8,m}^{(2)},\\
&y_{5,m}^{(1)} = \frac{1}{\dot{\mu}_{\kappa}^{(1)}} \left[\sum_{j=1}^{4} y_{j,m}^{(2)} (\dot{x}_{j0}^{(2)} - \dot{x}_{j\kappa}^{(1)}) + y_{5,m}^{(2)} \dot{\mu}_0^{(2)} + \right.\\
&\quad + \sum_{n=0}^{2} z_{n,m}^{(2)} \dot{\eta}_{0n}^{(2)} + \sum_{j=1}^{4} \tilde{y}_{j,m} (\dot{\tilde{x}}_j - \dot{x}_{j\kappa}^{(1)}) + \tilde{y}_{5,m} \dot{\tilde{\mu}}_0 +\\
&\left. + \sum_{p=9}^{12} \tilde{y}_{p-1,m} (\dot{x}_{p0}^{(2)} + \dot{\tilde{x}}_{p-1} - \dot{x}_{p\kappa}^{(1)}) + \sum_{n=0}^{3} \tilde{z}_{n,m} \dot{\tilde{\eta}}_{n0} \right] - \frac{\dot{p}_m \big|_{t_+^{(1)}}}{\dot{\mu}_{\kappa}^{(1)}}\\
&z_{7,m}^{(1)} = - y_{5,m}^{(1)} + \frac{\partial p_m}{\partial \mu_{\kappa 1}} \bigg|_{t_-^{(1)}};
\end{aligned}\right\} \qquad (2.3.16)
$$

where $\tilde{\psi}_{\nu} = 0 \ (t = t_{k}^{(1)})$

$$
\left.\begin{aligned}
&\tilde{y}_{j,m-} = \tilde{y}_{j,m-} \qquad (j = 1, \ldots, 4),\\
&\tilde{y}_{p,m-} = \tilde{y}_{p,m-} \qquad (p = 8, \ldots, 11, 16),\\
&\tilde{y}_{5,m-} = \frac{1}{\dot{\tilde{\mu}}} \left[\sum_{j=1}^{4} \tilde{y}_{j,m} (\dot{\tilde{x}}_{j0}^{(\kappa)} - \dot{\tilde{x}}_{j\kappa}^{(1)}) - \sum_{p=8}^{11} \tilde{y}_{p,m} (\dot{\tilde{x}}_{p0}^{(\kappa)} - \right.\\
&\quad - \dot{\tilde{x}}_{p\kappa}^{(1)}) + \tilde{z}_{0,m-} (\dot{\tilde{\eta}}_{00}^{(\kappa)} - \dot{\tilde{\eta}}_{0\kappa}^{(1)}) + \tilde{z}_{3m-} (\dot{\tilde{\eta}}_{30}^{(\kappa)} - \dot{\tilde{\eta}}_{3\kappa}^{(1)}) -
\end{aligned}\right| \qquad (2.3.17)
$$

$$+\tilde{z}_{4.m-}\dot{\tilde{\eta}}_{40}^{(\kappa)}\Big] + \frac{\dot{p}_m|_{t_{k-}^{(1)}}}{\dot{\tilde{\psi}}_\mu},$$

$$\tilde{z}_{l.m+} = \tilde{z}_{l.m-} \qquad (l = 0,\ 3,\ 5, \ldots,\ 8),$$

$$\tilde{z}_{1.m+} = \frac{\partial p_m}{\partial \tilde{\eta}_{1\kappa}}, \quad \tilde{z}_{2.m+} = \frac{\partial p_m}{\partial \tilde{\eta}_{2\kappa}}. \qquad (2.3.17)$$

In the same manner, we can produce the discontinuity conditions of the conjugate coefficients of system (2.IV) corresponding to the functionals p_n $(n = 1, \ldots, 5)$ in the form

where $\psi_1 = 0$ $(t = t_k^{(1)})$

$$\tilde{y}_{j.n+} = \tilde{y}_{j.n-} \qquad (j = 1, \ldots, 4),$$

$$\tilde{y}_{p.n+} = \tilde{y}_{p.n-} \qquad (p = 8, \ldots, 11,\ 16),$$

$$\tilde{y}_{5.n+} = \frac{1}{\dot{\tilde{\psi}}_\mu}\Big[\sum_{j=1}^{4}\tilde{y}_{j.n-}(\dot{\tilde{x}}_{j0}^{(\kappa)} - \dot{\tilde{x}}_{j\kappa}^{(1)}) + \sum_{p=8}^{11}\tilde{y}_{p.n-}(\dot{\tilde{x}}_{p0}^{(\kappa)} -$$

$$- \dot{\tilde{x}}_{p\kappa}^{(1)}) + \tilde{z}_{0.n-}(\dot{\tilde{\eta}}_{00}^{(\kappa)} - \dot{\tilde{\eta}}_{0\kappa}^{(1)}) + \tilde{z}_{3.n-}(\dot{\tilde{\eta}}_{30}^{(\kappa)} - \dot{\tilde{\eta}}_{3\kappa}^{(1)}) +$$

$$+ \tilde{z}_{4.n-}\dot{\tilde{\eta}}_{40}^{(\kappa)}\Big],$$

$$\tilde{z}_{m.l+} = \tilde{z}_{m.l-} \qquad (m = 0,\ 3,\ 5, \ldots,\ 8),$$

$$\tilde{z}_{1.l+} = 0, \qquad \tilde{z}_{2.l+} = 0; \qquad (2.3.18)$$

where $\psi_\nu^{(1)} = 0$ $(t = t^{(1)})$

$$y_{j.l}^{(1)} = \tilde{y}_{j.l} \qquad (j = 1, \ldots, 4),$$

$$y_{p.l}^{(1)} = \tilde{y}_{p-1.l} \qquad (p = 9, \ldots, 12),$$

$$y_{18.l}^{(1)} = \tilde{y}_{16.l},$$

$$z_{m.l}^{(1)} = \tilde{z}_{m+1.l} \qquad (m = 4,\ 5,\ 6),$$

$$z_{n.l}^{(1)} = 0 \qquad (n = 0,\ 1,\ 2),$$

$$z_{3.l}^{(1)} = 0,$$

$$y_{5.l}^{(1)} = \frac{1}{\dot{\tilde{\psi}}_\mu^{(1)}}\Big[\sum_{j=1}^{4}\tilde{y}_{j.l}(\dot{\tilde{x}}_{j0} - \dot{x}_{j\kappa}^{(1)}) + \sum_{p=9}^{12} y_{p-1.l}(\dot{\tilde{x}}_{p-1.0} + \qquad (2.3.19)$$

$$+\dot{x}_{p0}^{(2)}-\dot{x}_{p\kappa}^{(1)})+\tilde{y}_{5.l}\dot{\tilde{\mu}}_0+\sum_{n=0}^{3}\tilde{z}_{n.l}\dot{\tilde{\eta}}_{n0}\Bigg].$$

$$z_{7.l}^{(1)}=-y_{5.l}^{(1)}+\frac{\partial\tilde{p}_n}{\partial\mu_{\kappa 1}}. \qquad (2.3.19)$$

Then, keeping in mind (2.3.12), (2.3.18) and (2.3.19), we produce

$$\Delta\tilde{p}_n=\Bigg(-\sum_{p=9}^{12}\tilde{y}_{p-1.n}\delta x_{p0}^{(2)}-\tilde{y}_{16.n}\delta C_0^{(2)}+\tilde{z}_{8.n}a_b^{(2)}\delta b_{02}+\tilde{z}_{8.n}a_a^{(2)}\delta a_{02}+$$

$$+\tilde{z}_{8.n}a_J^{(2)}\delta J_{02}+\tilde{z}_{8.n}a_\mu^{(2)}\delta\mu_{\kappa 11}\Bigg)\Bigg|_{t_-^{(1)}}+\Bigg[\sum_{p=10}^{12}y_{p.n}^{(1)}\delta\dot{x}_{p0}^{(1)}+y_{18.n}^{(1)}\delta C_0^{(1)}+$$

$$+(z_{4.n}^{(1)}+\tilde{z}_{8.n}a_b^{(1)})\delta b_{01}+(z_{5.n}^{(1)}+\tilde{z}_{8.n}a_a^{(1)})\delta a_{01}+(z_{6.n}^{(1)}+\tilde{z}_{8.n}a_J^{(1)})\delta J_{01}+$$

$$+[z_{7.n}^{(1)}+\tilde{z}_{8.n}(a_\mu^{(1)}-1)]\delta\mu_{\kappa 1}+\tilde{z}_{8.n}^{(1)}a_G\delta G_{p.l}+$$

$$+(\tilde{z}_{9.n}-\tilde{y}_{5.n}|_{t_-^{(1)}}+\tilde{z}_{8.n}a_\mu^{(n)})\delta\mu_\kappa^{(n)}\Bigg]_{t_0}. \qquad (2.3.20)$$

Performing similar calculations, we can find at point $t^{(1)}$ the following initial conditions of the conjugate system (2.IV), corresponding to the functionals p_l (l = 11,..., 14):

$$y_{j.l}^{(1)}=\tilde{y}_{j.l}+\frac{\partial p_l}{\partial x_j}\ (j=1,\ldots,4;\ l=11,\ldots,14;$$

$$x_1=V_1,\ldots,x_4=L_1),$$

$$y_{p.l}^{(1)}=\tilde{y}_{p-1.l}+\frac{\partial p_l}{\partial x_p}\ (p=9,\ldots,12;\ x_9=\lambda_1^{(1)},\ldots,x_{12}=\lambda_4^{(1)}),$$

$$y_{5.l}^{(1)}=\frac{1}{\dot{\psi}_\mu^{(1)}}\Bigg[\sum_{j=1}^{4}\tilde{y}_{j.l}(\dot{\tilde{x}}_{j0}-\dot{x}_{j\kappa}^{(1)})+$$

$$+\sum_{p=9}^{12}\tilde{y}_{p-1.l}(\dot{\tilde{x}}_{p-1.0}+\dot{x}_{p.0}^{(2)}-\dot{x}_{p\kappa}^{(1)})+ \qquad (2.3.21)$$

$$+\tilde{y}_{6.l}\dot{\tilde{\mu}}_0+\sum_{m=0}^{3}\tilde{z}_{m.l}\dot{\tilde{\eta}}_{m0}\Bigg]+\frac{\partial p_l}{\partial\mu}-\frac{\dot{p}_l}{\dot{\psi}_\mu^{(1)}},$$

$$y_{18.l}^{(1)}=\frac{\partial p_l}{\partial C_0^{(1)}}+\tilde{y}_{16.l},\qquad z_{m.l}^{(1)}=\frac{\partial p_l}{\partial\eta_{m\kappa}^{(1)}}\ (m=0,1,2),$$

while the
ψ_ν = 0 are

$$
\begin{aligned}
&z_{3,l}^{(1)} = 0, \qquad z_{4,l}^{(1)} = \frac{\partial p_l}{\partial b_{(\cdot)}} + \tilde{z}_{5,l}, \\
&z_{5,l}^{(1)} = \frac{\partial p_l}{\partial a_{(\cdot)}} + \tilde{z}_{6,l}, \qquad z_{6,l}^{(1)} = \frac{\partial p_l}{\partial l_{(\cdot)}} + \tilde{z}_{7,l}, \\
&z_{7,l}^{(1)} = \frac{\partial p_l}{\partial \nu_{\kappa 1}} + \frac{\dot{p}_l}{\dot{\xi}_{\mu}^{(1)}} - \frac{1}{\dot{\xi}_{\mu}^{(1)}} \Bigg[\sum_{j=1}^{4} \tilde{y}_{j,l} (\dot{x}_{j0} - \dot{x}_{j\kappa}^{(1)}) + \\
&\quad + \sum_{p=9}^{12} \tilde{y}_{p-1,l} (\dot{x}_{(p-1)0} + \dot{x}_{p0}^{(2)} - \dot{x}_{p\kappa}^{(1)}) + \\
&\quad + \tilde{y}_{5,l} \dot{\mu}_0 + \sum_{m=0}^{3} \tilde{z}_{m,l} \dot{\eta}_{m0} \Bigg], \\
&z_{8,l}^{(1)} = \frac{\partial p_l}{\partial \alpha_{p,l}},
\end{aligned}
\tag{2.3.21}
$$

while the discontinuity conditions of the conjugate coefficients for $\psi_{\nu} = 0$ are produced in the form

$$
\begin{aligned}
&\tilde{y}_{j,l+} = \tilde{y}_{j,l-} \quad (j = 1, 2, 3, 4; \; l = 1, 2, 3), \\
&\tilde{y}_{j,4+} = \tilde{y}_{j,4-} + \frac{\partial p_4}{\partial \tilde{x}_j} \quad (\tilde{x}_1 = \tilde{V}, \ldots, \tilde{x}_4 = \tilde{L}), \\
&\tilde{y}_{p,l+} = \tilde{y}_{p,l-} \quad (p = 8, \ldots, 11), \\
&\tilde{y}_{p,4+} = \tilde{y}_{p,4-} + \frac{\partial p_4}{\partial \tilde{x}_p} \quad (\tilde{x}_8 = \tilde{i}_1, \ldots, \tilde{x}_{10} = \tilde{i}_3), \\
&\tilde{y}_{11,4+} = \tilde{y}_{11,4-}, \\
&\tilde{y}_{5,l+} = \frac{1}{\dot{\xi}_{\mu}} \Bigg[\sum_{j=1}^{4} \tilde{y}_{j,l-} (\dot{x}_{j0}^{(\kappa)} - \dot{x}_{j\kappa}^{(1)}) + \sum_{p=8}^{11} \tilde{y}_{p,l-} (\dot{x}_{p0}^{(\kappa)} - \\
&\quad - \dot{x}_{p\kappa}^{(1)}) + \tilde{z}_{0,l-} (\dot{\eta}_{00}^{(\kappa)} - \dot{\eta}_{0\kappa}^{(1)}) + \tilde{z}_{3,l-} (\dot{\eta}_{30}^{(\kappa)} - \dot{\eta}_{3\kappa}^{(1)}) + \\
&\quad + \tilde{z}_{4,l-} \dot{\eta}_{40}^{(\kappa)} \Bigg] - \frac{\dot{p}_l \tau_{\kappa+}^{(1)}}{\dot{\xi}_{\mu}}, \\
&\tilde{y}_{5,4} = \left(\frac{\partial p_4}{\partial \mu} - \frac{\dot{p}_4}{\dot{\xi}_{\mu}} \right) \Bigg|_{t_{\kappa-}^{(1)}} + \frac{1}{\dot{\xi}_{\mu}} \Bigg[\sum_{j=1}^{4} \tilde{y}_{j,4-} (\dot{x}_{j0}^{(\kappa)} - \dot{x}_{j\kappa}^{(1)}) + \\
&\quad + \sum_{p=8}^{11} \tilde{y}_{p,4-} (\dot{x}_{p0}^{(\kappa)} - \dot{x}_{p\kappa}^{(1)}) + \tilde{z}_{0,4-} (\dot{x}_{p0}^{(\kappa)} - \dot{x}_{p\kappa}^{(1)}) +
\end{aligned}
\tag{2.3.22}
$$

$$+\tilde{z}_{3.l-}(\dot{\tilde{\eta}}_{30}^{(\kappa)}-\dot{\eta}_{3\kappa}^{(1)})+\tilde{z}_{4.l-}\dot{\tilde{\eta}}_{40}^{(\kappa)}]-\frac{\dot{p}_l|_{\bar{t}_{\kappa+}^{(1)}}}{\dot{\tilde{\psi}}_\mu},$$

$$\left.\begin{aligned}&\tilde{z}_{0.l+}=\tilde{z}_{0.l-}\ (l=1,2,3,4),\ \tilde{z}_{1.l+}=\frac{\partial p_l}{\partial\tilde{\eta}_{1\kappa}},\quad \tilde{z}_{2.l+}=\frac{\partial p_l}{\partial\tilde{\eta}_{2\kappa}},\\&\tilde{z}_{m.l+}=z_{m.l-}\quad(m=3,5,\ldots,8).\end{aligned}\right|\qquad(2.3.22)$$

In connection with (2.3.21) and (2.3.22), we will have

$$\begin{aligned}\Delta p_l=&\Big(-\sum_{\rho=9}^{12}\tilde{y}_{\rho-1.l}\delta x_{\rho 0}^{(2)}-\tilde{y}_{16.l}\delta C_0^{(2)}+\tilde{z}_{8.l}a_b^{(2)}\delta b_{02}+\\&+\tilde{z}_{8.l}a_a^{(2)}\delta a_{02}+\tilde{z}_{8.l}a_J^{(2)}\delta J_{02}+\tilde{z}_{8.l}a_\mu^{(2)}\delta\mu_{\kappa 11}\Big)\Big|_{t_-^{(1)}}+\\&+\Big[\sum_{\rho=10}^{12}y_{\rho.l}^{(1)}\delta x_{\rho 0}^{(1)}+y_{18.l}^{(1)}\delta C_0^{(1)}+(z_{4.l}^{(1)}+\tilde{z}_{8.l}a_b^{(1)})\delta b_{01}+\\&+(z_{5.l}^{(1)}+\tilde{z}_{8.l}a_a^{(1)})\delta a_{01}+(\tilde{z}_{6.l}^{(1)}+z_{8.l}a_J^{(1)})\delta J_{01}+\\&+[z_{7.l}^{(1)}+\tilde{z}_{8.l}(a_\mu^{(1)}-1)]\delta\mu_{\kappa 1}+\tilde{z}_{8.l}^{(1)}a_G\delta G_{\rho.l}+\\&+(\tilde{z}_{9.l}-\tilde{y}_{5.l}|_{t_-^{(1)}}+\tilde{z}_{8.l}a_\mu^{(n)})\delta\mu_\kappa^{(n)}\Big]_{t_0}\quad(l=11,\ldots,14).\end{aligned}\qquad(2.3.23)$$

The discontinuity conditions of the conjugate coefficients at the points where the phase trajectory strikes the boundary of limitation of phase variables t_1^* and at the departure points t_1^{cx} will not be analyzed here, since they will differ depending on the form of limiting conditions (2.1.9). In principle, they are determined in the same manner as at points $t^{(i)}$ (the general expression for the discontinuity conditions is given in the appendix).

System (2.3.14), (2.3.20) and (2.3.23), consisting of twelve linear equations, allows us to determine the twelve unknowns $\delta\lambda_{20}^{(1)},\ldots,\delta C_0^{(2)}$ and δa_{0i}, $\delta\nu_{kJ}$, δJ_{0i}, δb_{0i}, $\delta\nu_k^{(H)}$, δG_{pl} and thereby to find the direction of swiftest descent for improvement of the results of the preceding iteration. The discontinuity conditions (2.3.15), (2.3.19) and (2.3.22) make it possible to perform integration of conjugate system (2.IV) and (2.IV) from points $\psi_k^0=0$, $\psi_\nu^{(i)}=0$ and ψ_ν^0, $\psi_k^0=0$ to t_0.

In order to estimate the deviation of I from its maximum value near the extreme, we should additionally integrate system (2.IV) from $\psi_k^0 = 0$ to t_0 considering the discontinuity conditions of the conjugate coefficients such as (2.3.15)-(2.3.17) with the following initial conditions: where $\psi_k^0 = 0$

$$y_{j,J}^{(\kappa)} = \frac{\partial I}{\partial x_{j\kappa}} - \frac{j}{\dot{\psi}_\kappa^0} \frac{\partial \psi_\kappa^0}{\partial x_{j\kappa}} \quad (j = 1, \ldots, 4; \quad x_{1\kappa} = V_\kappa, \ldots, x_{4\kappa} = L_\kappa),$$

$$y_{5,J}^{(\kappa)} = 0,$$

$$y_{p,J}^{(\kappa)} = 0 \quad (p = 9, \ldots, 12, 18), \quad z_{n,J}^{(\kappa)} = 0 \quad (n = 0, 1, 2, 3),$$

$$z_{4,J}^{(\kappa)} = \frac{\partial I}{\partial b_2}, \quad z_{5,J}^{(\kappa)} = z_{6,J}^{(\kappa)} = 0, \quad z_{7,J}^{(\kappa)} = \frac{\partial I}{\partial \mu_{\kappa 1}}, \quad z_{8,J}^{(\kappa)} = \frac{\partial I}{\partial G_{pJ}}.$$

We should keep in mind here that where $\tilde{\psi}_k^0 = 0$, all conjugate coefficients of system (2.IV̆) are equal to zero.

Therefore, using the discontinuity conditions of the conjugate coefficients of system (2.IV) in the form:

<u>where $\psi_1^{(2)} = 0 \ (t = t^{(2)})$</u>

$$y_{j,J}^{(2)} = y_{j,J}^{(\kappa)} \quad (j = 1, \ldots, 4, 9, \ldots, 12, 18),$$

$$z_{0,J}^{(2)} = z_{0,J}^{(\kappa)}, \quad z_{1,J}^{(2)} = 0, \quad z_{2,J}^{(2)} = 0, \quad z_{4,J}^{(2)} = z_{4,J}^{(\kappa)},$$

$$z_{5,J}^{(2)} = \frac{\partial I}{\partial a_{c2}}, \quad z_{6,J} = \frac{\partial I}{\partial I_{02}}, \quad z_{7,J}^{(2)} = 0, \quad z_{8,J}^{(2)} = z_{8,J}^{(\kappa)},$$

$$y_{5,J}^{(2)} = \frac{1}{\dot{\psi}_\mu^{(2)}} \left[\sum_{j=1}^{4} y_{j,J}^{(\kappa)} (\dot{x}_{j0}^{(\kappa)} - \dot{x}_{j\kappa}^{(2)}) + \sum_{p=9}^{12} y_{p,J}^{(\kappa)} (\dot{x}_{p0}^{(\kappa)} - \dot{x}_{p\kappa}^{(2)}) + \right.$$

$$\left. + z_{0,J}^{(\kappa)} (\dot{\eta}_{00}^{(\kappa)} - \dot{\eta}_{0\kappa}^{(2)}) + z_{4,J}^{(\kappa)} \dot{\eta}_{30}^{(\kappa)} \right];$$

<u>where $\psi_1^{(1)} = 0 \ (t = t^{(1)})$</u>

$$y_{j,J}^{(1)} = y_{j,J}^{(2)} \quad (j = 1, \ldots, 4), \quad y_{p,J}^{(1)} = 0 \quad (p = 9, \ldots, 12, 18),$$

$$z_{n,J}^{(1)} = 0 \quad (n = 0, 1, 2), \quad z_{4,J}^{(1)} = \frac{\partial I}{\partial b_{01}}, \quad z_{5,J}^{(1)} = \frac{\partial I}{\partial a_{c1}},$$

$$z_{6,J}^{(1)} = \frac{\partial I}{\partial I_{01}}, \quad z_{8,J}^{(1)} = z_{8,J}^{(2)}, \quad z_{7,J}^{(1)} = -y_{5,J}^{(1)} + \frac{\partial I}{\partial \mu_{\kappa 1}},$$

$$y_{5,J}^{(1)} = \frac{1}{\dot{\psi}_\mu^{(1)}} \left[\sum_{j=1}^{4} y_{j,J}^{(2)} (\dot{x}_{j0}^{(2)} - \dot{x}_{j\kappa}^{(1)}) + y_{5,J}^{(2)} \dot{\mu}_0^{(2)} + \sum_{n=0}^{2} z_{n,J}^{(2)} \dot{\eta}_{n0}^{(2)} \right].$$

we produce

$$dI = \left[\sum_{p=9}^{12} y^{(2)}_{p,J}\delta x^{(2)}_{p0} + y^{(2)}_{13,J}\delta C_0^{(2)} + z^{(2)}_{4,J}\delta b_{02} + z^{(2)}_{5,J}\delta a_{02} + z^{(2)}_{6,J}\delta J_{02} \right]_{t^{(1)}} +$$

$$+ \left(z^{(H)}_{1,J} - y^{(2)}_{5,J}\right)_{t^{(2)}} \delta\mu_{k11} + \left[\sum_{p=10}^{12} y^{(1)}_{p,J}\delta x^{(1)}_{p0} + y^{(1)}_{13,J}\delta C_0^{(1)} + z^{(1)}_{4,J}\delta b_{01} + \right.$$

$$\left. + z^{(1)}_{5,J}\delta a_{01} + z^{(1)}_{6,J}\delta J_{01} + \left(z^{(1)}_{7,J}\delta\mu_{k1} + z^{(1)}_{8,J}\delta G_{p1}\right) \right]_{t_0} + \frac{\partial I}{\partial \mu_k^{(H)}}\delta\mu_k^{(H)}.$$

This relationship after solution of systems (2.3.14), (2.3.20) and (2.3.23) for $\delta\lambda_{20}^{(1)}, \ldots, \delta C_0^{(2)}$ and δb_{0i}, δa_{0i}, δJ_{0i}, $\delta\mu_{kJ}$, δG_{p1}, $\delta\mu_k^{(H)}$, allows us to find the value of dI near the extreme.

The further course of the computational procedure can be determined from the flow chart (see Figure 1.22).

As an example, Appendix 1 to Chapter II analyzes the algorithm for optimization of the parameters and control of a hypothetical three stage missile with liquid fueled motor, capable of delivering a known payload into a fixed orbit, while Appendix 2 to Chapter II analyzes the algorithm for optimization of the parameters and control of a hypothetical two stage ballistic missile with a liquid fueled motor, capable of delivering a known payload over a fixed range considering limitations on the trajectory angle.

APPENDIX I TO CHAPTER II

Algorithm for Variational Method of Optimization of Parameters and Movement Modes of Three Stage Liquid Fueled Rocket Capable of Placing a Known Payload in a Fixed Orbit

Let us analyze the algorithm for the variational method for optimization of the main plan parameters a_{0i}, μ_{kJ} and G_{0J} ($i = 1, 2, 3$; $J = I, II, III$) and control functions p and α of a hypothetical three stage liquid fueled rocket, capable of placing a known payload G_{pl} into a fixed orbit (V_k, θ_k, H_k fixed) with minimum launch weight G_0.

The solution of the problem formulated can be performed on the basis of the investigations performed in Chapter II.

In the plan equations (2.1.2), we represent μ_{0I}, μ_{0II} and μ_{pl} as follows:

$$\left.\begin{aligned}
\mu_{0I} &= \frac{1}{1 - B^{(1)} + D^{(1)}}\left[\mu_{kI} - \left(B^{(1)} + a_1 C^{(1)} + \frac{A^{(1)}}{G_{0I}}\right)\right],\\
\mu_{0II} &= \frac{1}{1 - B^{(2)} + D^{(2)}}\left[\mu_{kII} - \left(B^{(2)} + a_2 C^{(2)} + \frac{A^{(2)}}{G_{0II}}\right)\right],\\
\mu_{pl} &= \frac{\left(\mu_{kI} - B^{(1)} - a_1 C^{(1)} - \frac{A^{(1)}}{G_{0I}}\right)\left(\mu_{kII} - B^{(2)} - a_2 C^{(2)} - \frac{A^{(2)}}{G_{0II}}\right)}{(1 - B^{(1)} + D^{(1)})(1 - B^{(2)} + D^{(2)})(1 - B^{(3)} + D^{(3)})} \times\\
&\quad \times \left(\mu_{kIII} - B^{(3)} - a_3 C^{(3)} - \frac{A^{(3)}}{G_{0III}}\right).
\end{aligned}\right\} \quad (1)$$

where $A^{(i)}$, $B^{(i)}$, $C^{(i)}$ and $D^{(i)}$ are constant statistical coefficients.

The initial conditions are assumed as follows:

$$t_0=0,\ \mu(t_0)=1,\ V(t_0)=0,\ \theta(t_0)=\frac{\pi}{2},\ H(t_0)=0. \tag{2}$$

Furthermore, we will consider that up to $t = t^{cx}$, the rocket flies vertically $\theta = \pi/2$ and its departure from vertical lift can be performed only at $t > t^{cx}$.

Thus, in sector $t_0 \leq t \leq t^{cx}$, there is a limitation on phase variable θ.

The final conditions are fixed in the form

$$V(t^{(2)})=V_k^{(2)},\ \theta(t^{(2)})=\theta_k^{(2)},\ H(t^{(2)})=H_k^{(2)}. \tag{3}$$

The problem is analyzed in consideration of the fact that during the flight, angle of attack α may change within fixed limits.

Then, the main system of equations (coupling equations and Euler-Lagrange equations), according to conditions (2.I) and (2.II) become

$$\left.\begin{aligned}
&V'=\varphi_1,\ \theta'=\varphi_2,\ H'=\varphi_3,\ \mu'=\varphi_4=-\frac{a_{0i}p_0}{P^n_{yd\,i}}\\
&\lambda_1'=\varphi_5=-\sum_{j=1}^{4}\frac{\partial\varphi_j}{\partial V}\lambda_j-\frac{\partial\varphi_2}{\partial V}\lambda_5,\\
&\lambda_2'=\varphi_6=-\sum_{j=1}^{4}\frac{\partial\varphi_j}{\partial\theta}\lambda_j-\frac{\partial\varphi_2}{\partial\theta}\lambda_5,\\
&\lambda_3'=\varphi_7=-\sum_{j=1}^{4}\frac{\partial\varphi_j}{\partial H}\lambda_j-\frac{\partial\varphi_2}{\partial H}\lambda_5,\\
&\eta_0'=\varphi_8=-\sum_{j=1}^{4}\frac{\partial\varphi_j}{\partial a_{1i}}\lambda_j-\frac{\partial\varphi_2}{\partial a_{1i}}\lambda_5,\\
&\eta_1'=\varphi_9=-\sum_{j=1}^{4}\frac{\partial\varphi_j}{\partial a_{0i}}\lambda_j-\frac{\partial\varphi_2}{\partial a_{0i}}\lambda_5,\\
&\lambda_4'=\frac{P^n_{yd\,i}}{a_{0i}}\sum_{j=1}^{4}\varphi_j\lambda_j,
\end{aligned}\right\} \tag{4}$$

where

1) In the case $t \leq t^{cx}$ and $\theta = \pi/2$

$$\lambda_5 = -\lambda_2, \quad \sin\alpha = 0, \quad p_0 = 1,$$

but if $t > t^{cx}$, we should go over to case 2.

2) In case $t > t^{cx}$, $\lambda_5 = 0$ and where $\alpha_{min}^{(i)} \leq \alpha \leq \alpha_{max}^{(i)}$

$$\sin\alpha = \sin\alpha_{opt} = -\frac{\lambda_2}{\sqrt{\lambda_2^2 + \left(\frac{\lambda_1 V}{1 + \frac{k_u}{a_{0i}}\gamma^{(i)}}\right)^2}},$$

while where $\alpha_{opt} > \alpha_{max}^{(i)}$ or $\alpha_{opt} < \alpha_{min}^{(i)}$

$$\sin\alpha = \sin\alpha_{max}^{(i)} \quad \text{or} \quad \sin\alpha = \sin\alpha_{min}^{(i)},$$

where

$$\alpha_{max}^{(i)} = \text{const} \quad \text{and} \quad \alpha_{min}^{(i)} = \text{const}.$$

Furthermore, if $H_p \leq 0$, then $p_0 = 1$, while otherwise $p_0 = 0$.

Integration of system (4) can be performed by one of the numerical methods if we fix, in addition to the values of the main plan parameters a_{0i}, l_{kJ} and G_{0J}, the values of the Lagrange coefficients λ_{10}, λ_{20} and λ_{30} at t_0.

These values should satisfy the final conditions (5) and the conditions of optimality of the main plan parameters. As stop functions, we will analyze:

where $t_0 < t \leq t^{(1)}$

$$\varphi_\mu^{(1)} = \mu - \mu_{kI} = 0,$$

where $t^{(1)} < t \leq t^{(2)}$

$$\varphi_\mu^{(2)} = \mu - \mu_{kII} = 0,$$

where $t^{(2)} < t \leq t^{(3)}$

$$\varphi_\kappa^0 = \mu - \mu_{\kappa III} = 0.$$

Using the homogeneity of system (4) relative to the Lagrange coefficient, we assume λ_{10} = -I and thereby reduce the solution of the multipoint boundary problem to determination of the main plan parameters a_{0i}, μ_{kJ} and the values of Lagrange coefficients λ_{20} and λ_{30} at point t_0, in which case the functionals:

at point $t^{(3)}$

$$p_1 = \left(\frac{V_3}{V_\kappa^{\prime\prime}} - 1\right)^2, \quad p_2 = \theta_3^2, \quad p_3 = \left(\frac{H_3}{H_\kappa} - 1\right)^2,$$

$$p_4 = \left(1 + \frac{A_\mu^{(3)}}{A_a^{(3)}}\right)^2,$$

$$p_5 = \left(1 + \frac{A_\mu^{(3)} m_a^{(1)}}{A_a^{(1)} m_\mu^{(3)}}\right)^2;$$

at point $t^{(2)}$

$$p_6 = \left(1 + \frac{A_\mu^{(2)}}{A_a^{(2)}}\right)^2,$$

$$p_7 = \left(1 + \frac{A_\mu^{(2)} m_a^{(1)}}{A_a^{(1)} m_\mu^{(2)}}\right)^2;$$

at point $t^{(1)}$

$$p_8 = \left(1 + \frac{A_\mu^{(1)}}{A_a^{(1)}}\right)^2$$

are equal to zero.

Here

$$m_a^{(1)} = \left(\mu_{011} + \frac{\partial\mu_{011}}{\partial G_{011}} G_{011}\right)\left(\frac{G_{0111}^2}{G_{n.r}} \frac{\partial\mu_{111}}{\partial G_{0111}} + 1\right) \frac{\partial\mu_{01}}{\partial a_{01}} \mu_{0111}.$$

$$m_{\mu}^{(2)}=\left(\mu_{0111}+G_{0111}\frac{\partial\mu_{0111}}{\partial G_{0111}}\right)\mu_{01}\frac{\partial\mu_{011}}{\partial\mu_{\kappa 11}},\quad m_{\mu}^{(3)}=\mu_{01}\mu_{011}\frac{\partial\mu_{0111}}{\partial\mu_{\kappa 111}},$$

$$A_{a}^{(1)}=\eta_{1\kappa}^{(1)}+G_{0}\frac{\partial\mu_{01}}{\partial a_{01}}\left[\eta_{0\kappa}^{(2)}+\left(\mu_{011}+G_{011}\frac{\partial\mu_{011}}{\partial G_{011}}\right)\eta_{0\kappa}^{(3)}\right],$$

$$A_{\mu}^{(1)}=-\left\{G_{0}\frac{\partial\mu_{01}}{\partial\mu_{\kappa 1}}\left[\eta_{0\kappa}^{(2)}+\left(\mu_{011}+G_{011}\frac{\partial\mu_{011}}{\partial G_{011}}\right)\eta_{0\kappa}^{(3)}\right]+\lambda_{4\kappa}^{(1)}\right\}\frac{\dfrac{\partial\mu_{01}}{\partial a_{01}}}{\dfrac{\partial\mu_{01}}{\partial\mu_{\kappa 1}}},$$

$$A_{a}^{(2)}=\left(\eta_{1\kappa}^{(2)}+G_{011}\frac{\partial\mu_{011}}{\partial a_{02}}\eta_{0\kappa}^{(3)}\right)\frac{\dfrac{\partial\mu_{011}}{\partial\mu_{\kappa 11}}}{\dfrac{\partial\mu_{011}}{\partial a_{02}}},$$

$$A_{\mu}^{(2)}=-G_{011}\frac{\partial\mu_{011}}{\partial\mu_{\kappa 11}}\eta_{0\kappa}^{(3)}-\lambda_{4\kappa}^{(2)},$$

$$A_{a}^{(3)}=\eta_{1\kappa}^{(3)}\frac{\dfrac{\partial\mu_{0111}}{\partial\mu_{\kappa 111}}}{\dfrac{\partial\mu_{0111}}{\partial a_{03}}},$$

$$A_{\mu}^{(3)}=-\lambda_{4\kappa}^{(3)}.$$

Functionals $p_4, \ldots, p_8$, which are the optimization conditions of the main plan parameters a_{0i} and μ_{kJ}, were produced from formulas (2.2.15).

The values of G_{0J} are determined from dependences (1) unambiguously by solving the plan equations. They are

$$G_{0111}=\frac{A^{(3)}+(1-B^{(3)}+D^{(3)})G_{p.d}}{\mu_{\kappa 111}-B^{(3)}-a_{03}C^{(3)}},$$

$$G_{011}=\frac{A^{(2)}+(1-B^{(2)}+D^{(2)})G_{0111}}{\mu_{\kappa 11}-B^{(2)}-a_{02}C^{(2)}},$$

$$G_{0}=\frac{A^{(1)}+G_{011}(1-B^{(1)}+D^{(1)})}{\mu_{\kappa 1}-B^{(1)}-a_{01}C^{(1)}}.$$

The conjugate equation system, making up the supplementary system of equations together with equations (4), has the following form in this problem:

$$\dot{y}_{m.l} = -\sum_{j=1}^{9} \frac{\partial \varphi_j}{\partial x_m} y_{j.l} - \frac{\partial \varphi_{10}}{\partial x_m} y_{12.l} - \frac{\partial \varphi_2}{\partial x_m} y_{13.l}{}^{1)} \qquad (5)$$

$$(x_m = V,\ \theta,\ H,\ \mu,\ \lambda_1,\ \lambda_2,\ \lambda_3,\ r_0^{(l)},\ \eta_1^{(l)},\ G_{0l},\ a_{0l};$$
$$m = j = 1, \ldots, 11;\ \ l = 1, \ldots, 8),$$

where:

<u>in case 2</u>

$$y_{13.l} = 0$$

and where $\alpha^{(i)}_{max} > \alpha > \alpha^{(i)}_{min}$

$$y_{12.l} = -\frac{1}{\frac{\partial \varphi_{10}}{\partial \alpha}} \sum_{j=1}^{9} \frac{\partial \varphi_j}{\partial \alpha} y_{j.l},$$

and where $\alpha = \alpha^{(i)}_{max}$ or $\alpha = \alpha^{(i)}_{min}$

$$y_{12.l} = 0;$$

<u>in case 1</u>

$$y_{12.l} = -\frac{1}{\frac{\partial \varphi_2}{\partial \alpha}} \left(\frac{\partial \varphi_2}{\partial V} y_{5.l} + \frac{\partial \varphi_2}{\partial \theta} y_{6.l} + \frac{\partial \varphi_2}{\partial H} y_{7.l} + \frac{\partial \varphi_2}{\partial G_{0l}} y_{8.l} + \frac{\partial \varphi_2}{\partial a_{0l}} y_{9.l} \right),$$

$$y_{13.l} = -\frac{1}{\frac{\partial \varphi_2}{\partial \alpha}} \left(\sum_{j=1}^{9} \frac{\partial \varphi_j}{\partial \alpha} y_{j.l} + \frac{\partial \varphi_{10}}{\partial \alpha} y_{12.l} \right).$$

The initial conditions of the conjugate system (point $t^{(3)}$) are determined according to the following relationships:

$$y_{1.1} = \frac{\partial p_1}{\partial V}, \qquad y_{4.l} = \frac{P^{n}_{sp}}{a_{03}} \frac{dp_l}{dt} \quad (l = 1,\ 2,\ 3),$$

---

[1] Here $\phi_{10} = (\lambda_5 + \lambda_2) \frac{\partial \phi_2}{\partial \alpha} + \lambda_1 \frac{\partial \phi_1}{\partial \alpha} = 0.$

$$y_{2.2}=\frac{\partial p_2}{\partial \theta}, \quad y_{3.3}=\frac{\partial p_3}{\partial H},$$

$$y_{m.l}=0 \; (m=1,\ 2,\ 3,\ 5,\dots,\ 11;\ l=1,\ 2,\ 3;\ m \neq l),$$

$$y_{m.4}=\frac{\partial p_4}{\partial x_m} \quad (x_1=V,\dots,\ x_3=H;\ x_5=\lambda_1,\dots;\ x_{11}=a_{03};\ m=1,\ 2,\ 3,\ 5,\dots,\ 11),$$

$$y_{m.5}=\frac{\partial p_5}{\partial x_m},$$

$$y_{4.4}=\frac{\partial p_4}{\partial \mu_{\kappa}^{(3)}}-\frac{P^{n}_{y_{p_3}}}{a_{03}}\frac{dp_4}{dt}, \quad y_{4.5}=\frac{\partial p_5}{\partial \mu_{\kappa}^{(3)}}+\frac{P^{n}_{y_{p_3}}}{a_{03}}\frac{dp_5}{dt},$$

$$y_{m.l}=0 \quad (l+6,\ 7,\ 8;\quad m=1,\dots,\ 7),$$

$$y_{8.6}=\frac{\partial p_6}{\partial \eta_0^{(3)}},\ y_{9.6}=0,\quad y_{10.6}=0,\quad y_{11.6}=0,$$

$$y_{8.7}=\frac{\partial p_7}{\partial \eta_0^{(3)}},\quad y_{9.7}=0,\ y_{10.7}=\frac{\partial p_7}{\partial G_{0111}},\ y_{11.7}=\frac{\partial p_7}{\partial a_{03}},$$

$$y_{8.8}=\frac{\partial p_8}{\partial \eta_0^{(3)}},\quad y_{9.8}=0,\ y_{10.8}=\frac{\partial p_8}{\partial G_{0111}},\ y_{11.8}=\frac{\partial p_8}{\partial a_{01}}.$$

At points t_2 and t_1, the discontinuity conditions should be fulfilled:

<u>point t_2</u>

$$y_{4.l}(t_+^{(2)})=-\frac{P^{n}_{y_{p_2}}}{a_{02}}\Big[-y_{4.l}(t_-^{(2)})\frac{a_{02}}{P^{n}_{y_{p_2}}}+y_{1.l}(t_-^{(2)})(\varphi_1(t_-^{(2)})-\varphi_1(t_+^{(2)}))+$$
$$+y_{2.l}(t_-^{(2)})(\varphi_2(t_-^{(2)})-\varphi_2(t_+^{(2)}))+y_{5.l}(t_-^{(2)})(\varphi_5(t_-^{(2)})-\varphi_5(t_+^{(2)}))+$$
$$+y_{7.l}(t_-^{(2)})(\varphi_7(t_-^{(2)})-\varphi_7(t_+^{(2)}))+y_{8.l}(t_-^{(2)})\varphi_8(t_-^{(2)})+y_{9.l}(t_-^{(2)})\varphi_9(t_-^{(2)})\Big]+$$
$$+\left(\frac{\partial p_l}{\partial \mu_{\kappa}^{(2)}}+\frac{P^{n}_{y_{p_2}}}{a_{02}}\frac{dp_l}{dt}\right)\Bigg|_{t_+^{(2)}}=\bar{y}_{4.l}(t_+^{(2)})+\frac{\partial p_l}{\partial \mu_{\kappa}^{(2)}}\ (l=1,\dots,\ 8),$$

$$y_{m.l}(t_+^{(2)})=y_{m.l}(t_-^{(2)}) \quad (m=1,\ 2,\ 3,\ 5,\ 6,\ 7;\quad l=1,\dots,\ 5),$$

$$y_{m.l}(t_+^{(2)})=0 \quad (m=8,\dots,\ 11;\ l=1,\dots,\ 4),$$

$$y_{8.5}(t^{(2)})=\frac{\partial p_5}{\partial \eta_0^{(2)}},\ y_{9.5}(t_-^{(2)})=0,$$

$$y_{10.5}(t_{+}^{(2)})=\frac{\partial p_5}{\partial G_{011}},\quad y_{11.5}(t_{+}^{(2)})=\frac{\partial p_5}{\partial a_{02}},$$

$$y_{m.6}(t_{+}^{(2)})=y_{m.6}(t_{-}^{(2)})+\frac{\partial p_6}{\partial x_m}\ (x_1=V,\ x_2=\theta,\ x_3=H,$$
$$x_5=\lambda_1,\ \ldots,\ x_{11}=a_{02}),$$

$$y_{m.7}(t_{+}^{(2)})=y_{m.7}(t_{-}^{(2)})+\frac{\partial p_7}{\partial x_m},$$

$$y_{m.8}(t_{+}^{(2)})=y_{m.8}(t_{-}^{(2)})\quad (m=1,\ \ldots,\ 7),$$

$$y_{8.8}(t_{+}^{(2)})=\frac{\partial p_8}{\partial \eta_0^{(2)}},\ y_{9.8}(t_{+}^{(2)})=0,$$

$$y_{10.8}(t_{+}^{(2)})=\frac{\partial p_8}{\partial G_{011}},\ y_{11.8}(t_{+}^{(2)})=\frac{\partial p_8}{\partial a_{02}};$$

point t_1

$$y_{m.l}(t_{+}^{(1)})=y_{m.l}(t_{-}^{(1)})\ (m=1,\ 2,\ 3,\ 5,\ 6,\ 7;\ l=1,\ \ldots,\ 7),$$

$$y_{4.l}(t_{+}^{(1)})=-\frac{P_{5\rho_1}^{n}}{a_{01}}\Big[-y_{4.l}(t_{-}^{(1)})\frac{a_{r2}}{P_{5\rho_0}^{n}}+y_{1.l}(t_{-}^{(1)})(\varphi_1(t_{-}^{(1)})-\varphi_1(t_{+}^{(1)}))+$$
$$+y_{2.l}(t_{-}^{(1)})(\varphi_2(t_{-}^{(1)})-\varphi_2(t_{+}^{(1)}))+y_{5.l}(t_{-}^{(1)})(\varphi_5(t_{-}^{(1)})-\varphi_5(t_{+}^{(1)}))+$$
$$+y_{7.l}(t_{-}^{(1)})(\varphi_7(t_{-}^{(1)})-\varphi_7(t_{+}^{(1)}))+y_{8.l}(t_{-}^{(1)})\varphi_8(t_{-}^{(1)})+$$
$$+y_{9.l}(t_{-}^{(1)})\varphi_9(t_{-}^{(1)})\Big]+\left(\frac{\partial F_l}{\partial \mu_k^{(1)}}+\frac{P_{5\rho_1}^{n}}{a_{01}}\frac{d p_l}{dt}\right)\Bigg|_{t_{+}^{(1)}}=\bar{y}_{4.l}(t_{+}^{(1)})+$$
$$+\frac{\partial p_l}{\partial \mu_k^{(1)}}\ (l=1,\ \ldots,\ 8),$$

$$y_{m.l}(t_{+}^{(1)})=0\quad (m=8,\ \ldots,\ 11;\ l=1,\ \ldots,\ 4),$$

$$y_{8.5}(t_{+}^{(1)})=0,\ y_{9.5}(t_{+}^{(1)})=\frac{\partial p_5}{\partial \eta_1^{(1)}},$$

$$y_{10.5}(t_{+}^{(1)})=\frac{\partial p_5}{\partial G_0},\ y_{11.5}(t_{+}^{(1)})=\frac{\partial p_5}{\partial a_{01}},$$

$$y_{m.6}(t_{+}^{(1)})=0\quad (m=8,\ \ldots,\ 11);$$

$$y_{8.7}(t_{+}^{(1)})=0,\ y_{9.7}(t_{+}^{(1)})=\frac{\partial p_7}{\partial \eta_1^{(1)}},$$

$$y_{10.7}(t_{+}^{(1)})=\frac{\partial p_7}{\partial G_0},\ y_{11.7}(t_{+}^{(1)})=\frac{\partial p_7}{\partial a_{01}},$$

$$y_{m.8}(t_+^{(1)}) = y_{m.8}(t_-^{(1)}) + \frac{\partial p_8}{\partial x_m} (x_1 = V,\ x_2 = \theta,\ x_3 = H,\ x_5 = \lambda_1,$$

$$x_6 = \lambda_2,\ x_7 = \lambda_3,\ x_9 = \eta_1^{(1)},\ x_{10} = G_0,\ x_{11} = a_{01}),$$

$$y_{8.8}(t_+^{(1)}) = 0.$$

Assuming $\Delta p_1 = -c_1 p_1$, we produce after integration of the supplementary equation system from $t^{(3)}$ to t_0 the following system of heterogeneous linear equations:

$$y_{6.l}(t_0)\delta\lambda_{20} + y_{7.l}(t_0)\delta\lambda_{30} + Y_{\mu l}^{(1)}\delta\mu_{\kappa I} + Y_{\mu l}^{(2)}\delta\mu_{\kappa II} + Y_{\mu l}^{(3)}\delta\mu_{\kappa III} +$$
$$+ Y_{al}^{(1)}\delta a_{01} + Y_{al}^{(2)}\delta a_{02} + Y_{al}^{(3)}\delta a_{03} = \Delta p_l \quad (l = 1, \ldots, 8), \tag{6}$$

where

$$Y_{\mu l}^{(1)} = \frac{\partial p_l}{\partial \mu_{\kappa I}} - \frac{G_0}{\Delta_1} \frac{\partial \mu_{0I}}{\partial \mu_{\kappa I}} y_{10.l}(t_0) -$$
$$- \bar{y}_{4.l}(t_+^{(1)}) \quad (l = 1, \ldots, 8),$$

$$Y_{\mu l}^{(2)} = \frac{\partial p_l}{\partial \mu_{\kappa II}} - G_{0II} \frac{\partial \mu_{0II}}{\partial \mu_{\kappa II}} \sum_{i=1}^{2} y_{10.l}(t_-^{(i-1)}) \omega_2^{(i)} - \bar{y}_{4.l}(t_+^{(2)}),$$

$$Y_{\mu l}^{(3)} = \frac{\partial p_l}{\partial \mu_{\kappa III}} - G_{0III} \frac{\partial \mu_{0III}}{\partial \mu_{\kappa III}} \sum_{i=1}^{3} y_{10.l}(t_-^{(i-1)}) \omega_3^{(i)} - \frac{P_{1\kappa_3}^{0}}{a_{03}} \dot{p}|_{t_0}$$

$$Y_{al}^{(1)} = y_{11.l}(t_0) - \frac{G_0}{\Delta_1} \frac{\partial \mu_{01}}{\partial a_{01}} y_{10.l}(t_0),$$

$$Y_{al}^{(j)} = y_{11.l}(t_-^{(j-1)}) - G_{0J} \frac{\partial \mu_{0J}}{\partial a_{0J}} \sum_{i=1}^{j} y_{10.l}(t_-^{(i-1)}) \omega_j^{(i)}$$
$$(J = j = 2,\ 3).$$

Here

$$\Delta_i = \mu_{0J} + G_{0J} \frac{\partial \mu_{0J}}{\partial a_{0J}} \quad (i = J = 1,\ 2,\ 3),$$

$$\omega_2^{(1)} = \frac{1}{\Delta_1 \Delta_2}, \quad \omega_2^{(2)} = \frac{1}{\Delta_2},$$

$$\omega_3^{(1)} = \frac{1}{\Delta_1 \Delta_2 \Delta_3}, \quad \omega_3^{(2)} = \frac{1}{\Delta_2 \Delta_3}, \quad \omega_3^{(3)} = \frac{1}{\Delta_3}.$$

The solution of system (6) after calculation of $Y^{(i)}_{\nu l}$ and $Y^{(i)}_{al}$ allows us to determine the values of $\delta\lambda_{20}$, $\delta\lambda_{30}$ and $\delta\alpha_{0i}$, $\delta\nu_{kJ}$.

After this, we must go over to the BOSH, the algorithm of which is presented in Chapter I, § 4, while the flow chart is shown on Figure 1.24.

This algorithm was run on the BESM-3M computer. The program allowed computation of the optimal phase trajectory, optimal controls p(t) and α(t) and optimal values of the main plan parameters of a three stage rocket, placing a fixed payload into a selected orbit with minimum launch weight G_0.

The calculations were performed for a homogeneous, plane-parallel and central gravitational field of the earth, in the last case with and without consideration of the atmosphere. The results of the calculations show that without optimizing the thrust as a function of control, the optimal phase trajectory and the optimal values of the main plan parameters of the rocket, determined with the homogeneous plane-parallel gravitational field of the earth without considering the atmosphere are similar to the optimal phase trajectory and the optimal values of the main plan parameters determined with a central field, with and particularly without considering the atmosphere.

It is important that the optimal phase trajectory and optimal values of the main plan parameters of a multistage rocket with a homogeneous plane-parallel terrestrial gravitational field, the calculation of which can be performed during the course of solution of the multipoint boundary problem, are a good zero approximation for the algorithm for optimization of control and the main plan parameters of a multistage rocket, placing a fixed payload into a selected orbit with minimum launch weight.

Figures 2.3 and 2.4 present nomograms for determination of the optimal values of the main plan parameters of a hypothetical three stage rocket and values of Lagrange coefficients λ_{20} and λ_{30} calculated in the case of a homogeneous plane-parallel terrestrial gravitational field without considering the atmosphere for an orbit with a height H_k = 350 km (or H_k = 200 km for a "flat" earth) and $\theta_k = 0$ as functions of the values of V_k.

Table 2.1 is presented in order to illustrate the course of the iterational process of this algorithm for optimization of the three stage rocket. It shows how p_Σ and p_Σ^{kin} changed from iteration to iteration; up to $p_\Sigma^{kin} = 0.02$, only the functionals related to the fixed boundary conditions $V_k = V_{kp}$, $\theta_k = 0$ and $H_k = 200$ km were considered. The initial solution (N = 0) for these calculations consisted of the optimal values of the main plan parameters of the three stage rocket, λ_{20} and λ_{30}, taken from the nomograms shown on Figures 2.3 and 2.4. The course of the iterational process shown in Table 2.1 is typical for the algorithm presented in Chapter I, § 4 and shows the fully satisfactory convergence of this process.

At the same time, it must be noted that as a result of the calculations we noted a difference in principle in the optimal operating modes of the liquid fueled power plant in the optimal phase trajectory in the case of a homogeneous plane-parallel and central gravitational fields. Analytic investigations demonstrate and calculations confirm that the optimal operating mode of the liquid fueled power plant with the actual choking characteristics used to place a fixed payload at a fixed point in the phase space (V_k, $\theta_k = 0$, H_k) in a homogeneous plane-parallel field with minimum launch weight is only the maximum thrust mode. This sort of unambiguous answer to the question of the optimal operating mode of a liquid fuel engine with realistic choking characteristic as a rocket moves through a central gravitational field cannot be analytically produced. In this case, the answer can be found only by calculation.

Table 2.1

Value of Summary Functional	Iteration Number					
	0	1	2	3	4	5
$\bar{p}_\Sigma^{kin}$	1	0,29	$1{,}9 \cdot 10^{-3}$	—	—	—
$\bar{p}_\Sigma$	1	1,23	0,82	0,69	0,62	0,27

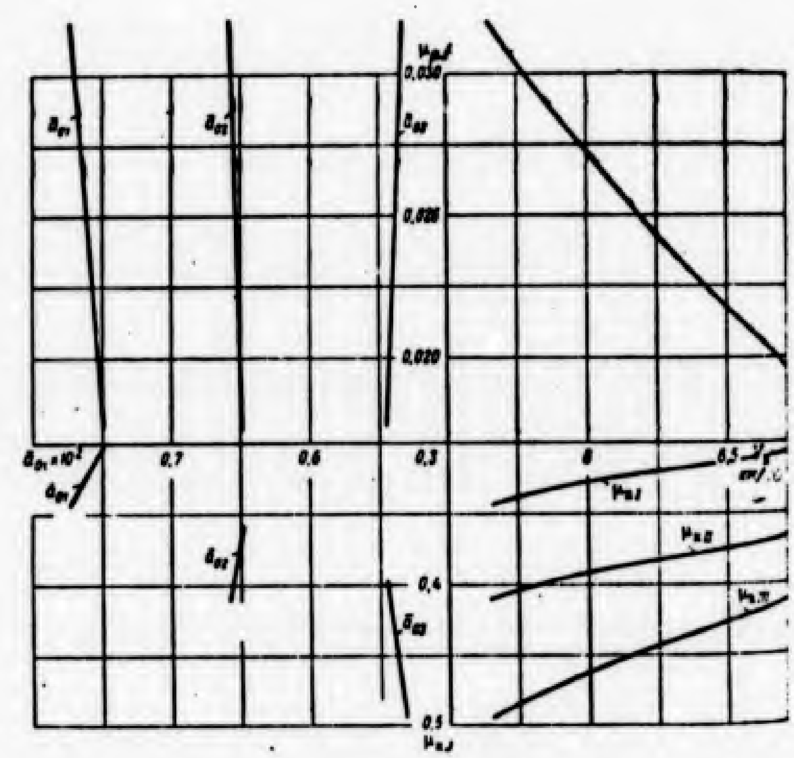

Figure 2.3. Nomogram for Determination of Optimal Values of Main Plan Parameters of Hypothetical Three Stage Rocket as Functions of Final Flight Velocity:

$\bar{a}_{0i} = a_{0i}/P^{-1}_{y1}$, $H_K = 350$ km, $\theta_K = 0$

It has been determined by calculations that with the optimal flight modes of a multistage rocket in a central gravitational field, an intermediate passive sector generally arises in the motion of the last stage, beginning after some acceleration of the last stage and ending with negative values of θ.

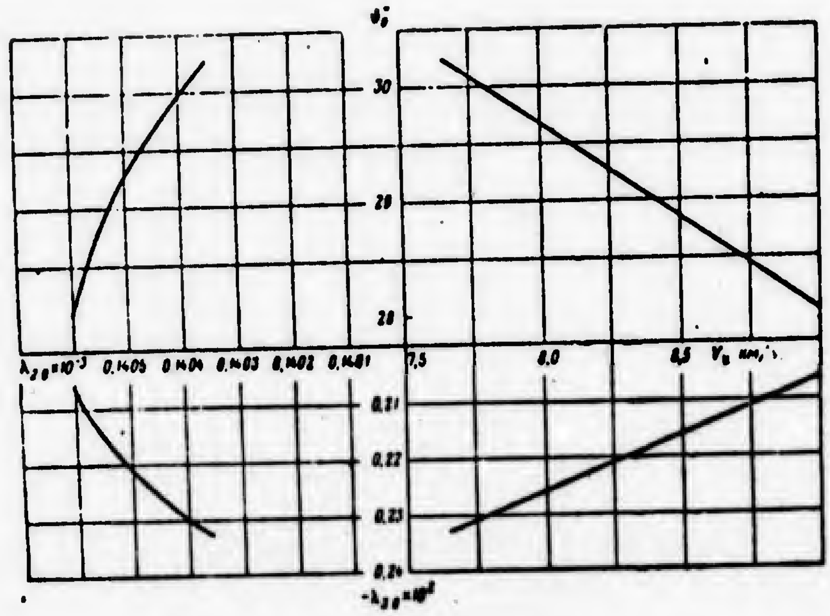

Figure 2.4. Nomogram for Determination of Initial Values of Lagrange Coefficients λ_{20} and λ_{30} as Functions of Final Flight Velocity:

$H_к = 260$ км, $\theta_к = 0$

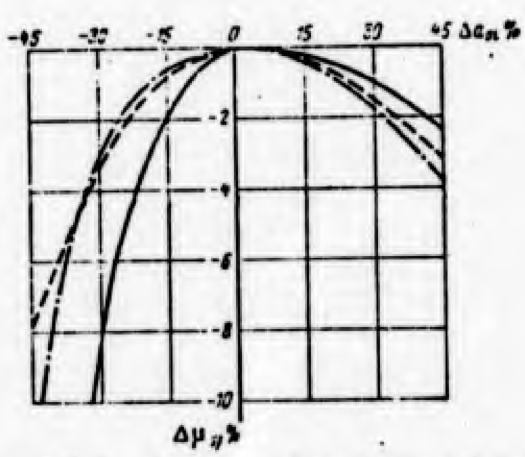

Figure 2.5. Deviation from Maximum Relative Payload as a Function of Deviation of Initial Thrust to Weight Ratio of ith Stage from Optimal Value:

This difference in the optimal operating modes of a liquid fueled power plant apparently results from differences in the nature of the central and the homogeneous plane-parallel gravitational fields.

Various deviations from the optimal values of thrust to weight ratio of the stages are possible during the process of planning an optimal multistage rocket. Furthermore, there is independent interest in estimating the steepness of the surface $G_0 = G_0\ (a_{0i},\ \nu_{kJ})$ near the point of the global minimum of launch weight $G_{0\ min}$ (or the point of the maximum relative payload $\mu_{pl\ max}$). Therefore, calculations were performed to determine the influence of deviations in thrust to weight ratio of the stages of a three stage rocket from the optimal values on the deviation in the relative payload from the maximum value. The calculations were performed so that as the thrust to weight ratio deviated from the optimal value for any stage, the fixed boundary conditions $V_k = V_{kp}$, θ_k, H_k were fulfilled in the class of optimal equations p(t) and α(t), satisfying all necessary conditions.

The results of these calculations are shown on Figure 2.5. We can see that with the selected $a_{to}^{(i)}$ and $\gamma_{pp}^{(i)}$, a deviation in thrust to weight ratio of any stage from the optimal value of $\pm 15\%$ or less causes a decrease in payload of less than 2%. Therefore, we can state that the surface $G_0 = G_0\ (a_{oi})$ does not slope sharply near the point of the global minimum $G_{0\ min}$. True, this surface is asymmetrical: as the thrust weight ratio increases over the optimal value, the relative payload decreases to a lesser extent than when the thrust weight ratio changes in the direction of lower a_{0i}.

Usually the specific gravity of the power plant of a stage $\gamma_{pp}^{(i)}$ and the coefficient of the fuel sector of the stage $a_{to}^{(i)}$ are used as the determining coefficients of a plan control. In the initial stage of planning of a multistage rocket, these coefficients are assumed constant and fixed due to statistical considerations. This fixation of the specific gravity of the power plant and fuel sector coefficient of the stage may result in deviations of the initial values of $\gamma_{pp}^{(i)}$ and $a_{to}^{(i)}$ from the actual values achieved as a result of planning. Therefore, it is important to estimate the influence of deviations of $\gamma_{pp}^{(i)}$ and $a_{to}^{(i)}$ from their initial values on themaximum value of relative payload and the optimal values of the main plan parameters. Calculations were performed so that as $\gamma_{pp}^{(i)}$ or $a_{to}^{(i)}$ deviated from the initial values, the conditions of optimization of the main plan parameters and the boundary conditions were once more fulfilled in the class of optimal controls $p(t)$ and $\alpha(t)$.

These calculation results, shown on Figures 2.6-2.11, show the possible values of deviations of maximum relative payload and optimal values of main plan parameters as $\gamma_{pp}^{(i)}$ and $\gamma_{to}^{(i)}$ vary from their initial values. The deviations of the maximum value of relative payload and optimal values of the main plan parameters do not exceed $\pm 2\%$ as $\gamma_{pp}^{(i)}$ or $a_{to}^{(i)}$ vary by $\pm 15\%$ or less.

In this connection, for the initial stage of planning of a multistage rocket we can consider the results of solution of the algorithm for optimization of the main plan parameters and control of the multistage rocket to place the required payload in the selected orbit with the minimum value of launch weight quite reliable with constant values of power plant specific gravity and stage fuel section coefficient.

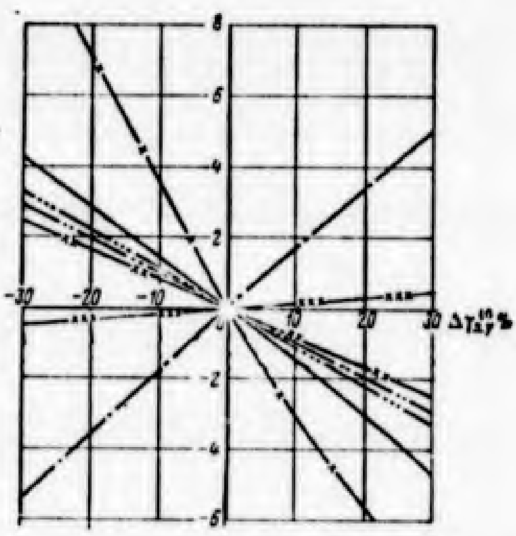

Figure 2.6. Change in Maximum Relative Payload and Optimal Values of Main Plan Parameters of Hypothetical Three Stage Rocket as Functions of Deviations in the Specific Gravity of the Power Plant of the First Stage from its Initial Value:

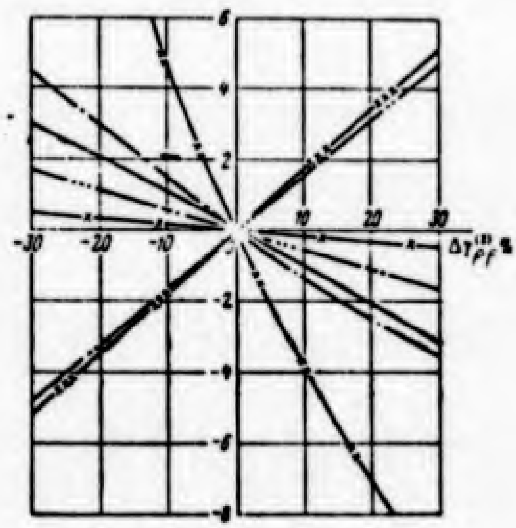

Figure 2.7. Change in Maximum Relative Payload and Optimal Values of Main Plan Parameters of Hypothetical Three Stage Rocket as a Function of Deviations of Specific Gravity of Power Plant of Second Stage from its Initial Value:

—— $\Delta p_{p.l}$ % —- × —— Δa_{01} %
—·— $\Delta p_{k\,I}$ % —×× —— Δa_{02} %
—··— $\Delta p_{k\,II}$ % —××× —— Δa_{03} %
—···— $\Delta p_{k\,III}$ %

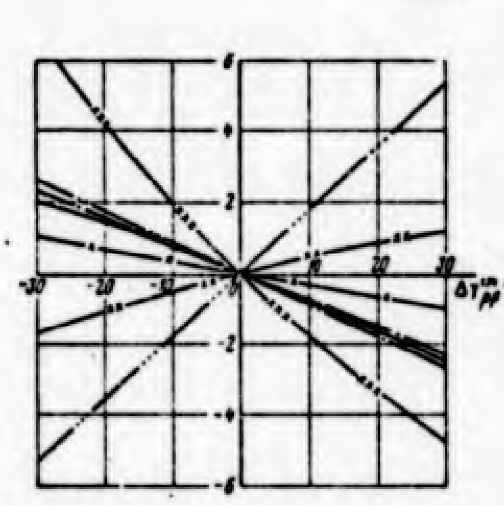

Figure 2.8. Change in Maximum Relative Payload and Optimal Values of Main Plan Parameters of Hypothetical Three Stage Rocket as a Function of Deviations of Specific Gravity of Power Plant of Third Stage from its Initial Value:

—— $\Delta p_{p.l}$ % —- × —— Δa_{01} %
—·— $\Delta p_{k\,I}$ % —×× —— Δa_{02} %
—··— $\Delta p_{k\,II}$ % —××× —— Δa_{03} %
—···— $\Delta p_{k\,III}$ %

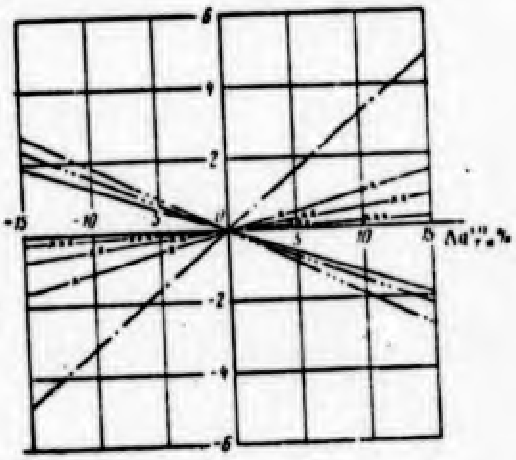

Figure 2.9. Change in Maximum Relative Payload and Optimal Values of Main Plan Parameters of Hypothetical Three Stage Rocket as a Function of Deviation of First Stage Fuel Sector Coefficient from Initial Value:

—— $\Delta\mu_{p.d}$ % · —·— × —·—· Δn_{01} %
—·— $\Delta\mu_{k\,I}$ % —— × × —— Δn_{02} %
—··— $\Delta\mu_{k\,II}$ % —·× × ×— Δn_{03} %
—···— $\Delta\mu_{k\,III}$ %

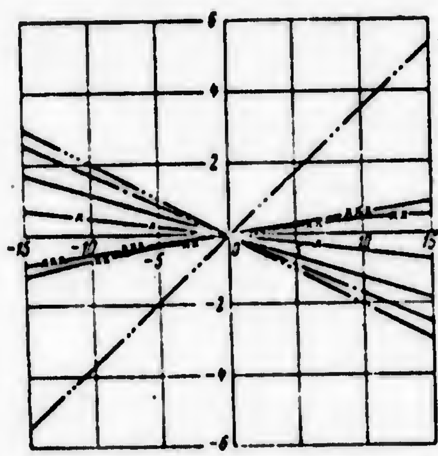

Figure 2.10. Change in Maximum Relative Payload and Optimal Values of Main Plan Parameters of Hypothetical Three Stage Rocket as a Function of Deviation of Second Stage Fuel Sector Coefficient from Initial Value:

—— $\Delta\mu_{p.d}$ % —— × —— Δn_{01} %
—·— $\Delta\mu_{k\,I}$ % —— × × —— Δn_{02} %
—··— $\Delta\mu_{k\,II}$ % —× × ×— Δn_{03} %
—···— $\Delta\mu_{k\,III}$ %

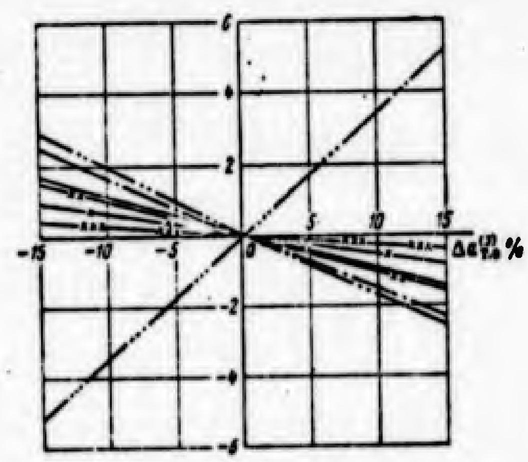

Figure 2.11. Change in Maximum Relative Payload and Optimal Values of Main Plan Parameters of Hypothetical Three Stage Rocket as a Function of Deviation of Third Stage Fuel Sector Coefficient from Initial Value:

—— $\Delta\mu_{p.d}$ % —— × —— Δn_{01} %
—·— $\Delta\mu_{k\,I}$ % —— × × —— Δn_{02} %
—··— $\Delta\mu_{k\,II}$ % —× × ×— Δn_{03} %
—···— $\Delta\mu_{k\,III}$ %

APPENDIX 2 TO CHAPTER II

Algorithm of Variational Method of Optimization of Parameters and Modes of Motion of Two Stage, Liquid Fuel Ballistic Missile Capable of Delivering Known Payload at Known Range Considering Limitations on Trajectory Angle

We will analyze the algorithm for the variational method of optimization of the main plan parameters a_{01}, a_{02}, μ_{kI}, μ_{kII}, $G_{0\,II}$ and control functions p and α of a hypothetical liquid fueled two stage ballistic missile, capable of delivering a known payload G_{pl} over fixed range L_k considering limitations on trajectory angle θ, with minimum launch weight $G_{0\,min}$.

In solving this problem, we will consider a number of assumptions usually made in planning of long range ballistic missiles.

After completion of the powered flight sector (point $t^{(2)}$), the missile (warhead) travels the greater portion of its path in practically airless space. Therefore, we need not consider the influence of the atmosphere over the passive flight sector to achieve accuracy sufficient for the initial stages of planning. This allows us to use known relationships between kinematic parameters at point $t^{(2)}$ in evaluating the passive portion of the flight [1]

$$\varphi_{12} = \chi_2 \,\mathrm{tg}^2 \beta/2 - 2\,\mathrm{tg}\,\theta_2\,\mathrm{tg}\,\beta/2 - \overline{H}_2 = 0, \qquad (1)$$

$$\varphi_{22} = L_k - (L_2 + R_3\beta) = 0, \qquad (2)$$

---

[1] See [2] and [27].

where

$$\gamma_2 = \frac{1}{n_2 \cos^2 \theta_2} - \overline{H}_2 - 2,$$

$$n_2 = \frac{V_2^2 (1 + \overline{H}_2)}{2 g_0 R_3},$$

$$\overline{H}_2 = \frac{H_2}{R_3},$$

thereby optimizing the phase trajectory in the sector $[t_0, t^{(2)}]$.

At point $t_0 = 0$, the following are usually fixed:

$$\mu(0) = 1,\ \theta_0 = \frac{\pi}{2},\ V_0 = 0,\ H_0 = 0,\ L_0 = 0. \qquad (3)$$

Thus, relationships (1)-(3) are the boundary conditions (2.1.5) in expanded form.

In connection with conditions (1), (2) and the assumption that the values of θ_2 are optimal, within its limits equation (2.2.27) takes on the following form:

$$C_0 dt^{(2)} + \lambda_{12} dV_2 + \lambda_{22} d\theta_2 + \lambda_{32} dH_2 + \lambda_4 dL_2 + e_{12} d\psi_{12} + e_{22} d\psi_{22} = 0.$$

From this we find

$$\lambda_{12} + e_{12} \frac{\partial \psi_{12}}{\partial V_2} = 0, \qquad \lambda_{22} + e_{12} \frac{\partial \psi_{12}}{\partial \theta_2} = 0,$$

$$\lambda_{32} + e_{12} \frac{\partial \psi_{12}}{\partial H_2} = 0, \quad \lambda_4 - e_{22} = 0,$$

$$e_{12} \frac{\partial \psi_{12}}{\partial \beta} - e_{22} R_3 = 0,\ C_0 = 0.$$

Since

$$\frac{\partial \psi_{12}}{\partial \beta} = \frac{\gamma_2 \operatorname{tg} \beta_2 - \operatorname{tg} \theta_2}{\cos^2 \beta_2},$$

then where $\theta_2 > 0$, since $R_2 > R_3$ [27], we always have

$$\frac{\partial \psi_{12}}{\partial \beta} \geq 0,$$

in which the equality is possible for ranges for which the trajectory contacts the surface of the earth. In the last case, we produce $\lambda_4 = 0$. In the following, we will analyze only ranges corresponding to the condition

$$\chi_2 \operatorname{tg} \beta/2 - \operatorname{tg} \theta_2 > 0.$$

Therefore we can write

$$\lambda_{12} = -\lambda_4 R_3 \frac{\partial \psi_{12}/\partial V_2}{\partial \psi_{12}/\partial \beta} = \lambda_4 R_3 \frac{\partial \beta}{\partial V_2},$$

$$\lambda_{22} = -\lambda_4 R_3 \frac{\partial \psi_{12}/\partial \theta_2}{\partial \psi_{12}/\partial \beta} = \lambda_4 R_3 \frac{\partial \beta}{\partial \theta_2},$$

$$\lambda_{32} = -\lambda_4 R_3 \frac{\partial \psi_{12}/\partial H_2}{\partial \psi_{12}/\partial \beta} = \lambda_4 R_3 \frac{\partial \beta}{\partial H_2}.$$

If at point $t^{(2)}$ the optimal stable control α is possible, we have

$$\operatorname{tg} \alpha_2 = \frac{\partial \psi_{12}/\partial \theta_2}{V_2 \, \partial \psi_{12}/\partial V_2}.$$

The denominator of this fraction is finite. Then equation $\alpha_2 = 0$ is possible only where

$$\frac{\partial \psi_{12}}{\partial \theta_2} = 0.$$

This equation corresponds to the passive sector of the trajectories of maximum range for the set of trajectories of identical velocity V_2 or trajectories of minimum velocity for the set of trajectories of constant range $L_{\pi ac}$ [2, 27].

Thus, the trajectories of maximum range for the set of trajectories of identical velocity or trajectories of minimum velocity for the set of trajectories of constant range known from the theory of Kepler motion can be related to the optimal trajectories in the passive sector only when the condition of transversality is fulfilled with $\lambda_{2k} = 0$ or with zero value of the optimal angle of attack at the end of the active flight sector.

After transformations, we produce

$$\lambda_{12}=\lambda_4\frac{2R_3\sin^2\beta/2}{n_2V_2\cos^2\theta_2(\chi_2\operatorname{tg}\beta/2-\operatorname{tg}\theta_2)},$$

$$\lambda_{22}=-\lambda_4\frac{R_3\sin\beta\left(\frac{1}{n_2}\operatorname{tg}\theta_2\operatorname{tg}\beta/2-1\right)}{(\chi_2\operatorname{tg}\beta/2-\operatorname{tg}\theta_2)\cos^2\theta_2},$$

$$\lambda_{32}=\frac{\lambda_4}{\chi_2\operatorname{tg}\beta/2-\operatorname{tg}\theta_2}\left[\frac{\sin^2\beta/2}{n_2(1+\bar{H}_2)\cos^2\theta_2}+1\right].$$

With the optimal stable control α (within the limits $-\pi/2 < \alpha < \pi/2$), according to Weierstrass condition (2.2.8), we have $\lambda_{12} < 0$. In this case we produce

$$\lambda_4 < 0.$$

Using the homogeneity of the Euler-Lagrange equations relative to the Lagrange coefficients, we assume $\lambda_4 = -1$. We then will have

$$\left.\begin{aligned}
\lambda_{12}&=-\frac{2R_3\sin^2\beta/2}{n_2V_2\cos^2\theta_2(\chi_2\operatorname{tg}\beta/2-\operatorname{tg}\theta_2)},\\
\lambda_{22}&=\frac{R_3\sin\beta\left(\frac{1}{n_2}\operatorname{tg}\beta/2\operatorname{tg}\theta_2-1\right)}{(\chi_2\operatorname{tg}\beta/2-\operatorname{tg}\theta_2)\cos^2\theta_2},\\
\lambda_{32}&=-\frac{1}{(\chi_2\operatorname{tg}\beta/2-\operatorname{tg}\theta_2)}\left[1+\frac{\sin^2\beta/2}{n_2\cos^2\theta_2(1+\bar{H}_2)}\right].
\end{aligned}\right\} \quad (4)$$

Due to the first integral of (2.2.10) and the condition of transversality $C_0 = 0$, we can write

$$H_p=\lambda_{12}\frac{g_0R_3^2}{(R_3+H_2)^2}\sin\theta_2+\lambda_{22}\frac{\cos\theta_2}{V_2(R_3+H_2)}\left(\frac{g_0R_3^2}{R_3+H_2}-V_2^2\right)-$$
$$-\lambda_{32}V_2\sin\theta_2+\frac{R_3V_2\cos\theta_2}{R_3+H_2}.$$

Substituting the values of λ_{12}, λ_{22} and λ_{32} in this equation according to (4), we produce

$$H_p = 0$$

Thus, at point t_2, the condition of optimal disconnection of the power plant occurs.

The main system of equations -- the coupling equations and Euler-Lagrange equations according to (2.I), (2.II) is as follows:

$$
\left.
\begin{aligned}
& V'=\varphi_1,\ \theta'=\varphi_2,\ H'=\varphi_3,\ L'=\varphi_4,\ \mu'=\varphi_5, \\
& \lambda_1'=\varphi_6=-\sum_{j=1}^{5}\frac{\partial\varphi_j}{\partial V}\lambda_j-\frac{\partial\varphi_2}{\partial V}\lambda_6, \\
& \lambda_2'=\varphi_7=-\sum_{j=1}^{5}\frac{\partial\varphi_j}{\partial\theta}\lambda_j-\frac{\partial\varphi_2}{\partial\theta}\lambda_6, \\
& \lambda_3'=\varphi_8=-\sum_{j=1}^{5}\frac{\partial\varphi_j}{\partial H}\lambda_j-\frac{\partial\varphi_2}{\partial H}\lambda_6, \\
& \eta_0'=\varphi_9=-\sum_{j=1}^{5}\frac{\partial\varphi_j}{\partial G_{0i}}\lambda_j-\frac{\partial\varphi_2}{\partial G_{0i}}\lambda_6, \\
& \eta_1'=\varphi_{10}=-\sum_{j=1}^{5}\frac{\partial\varphi_j}{\partial a_{0i}}\lambda_j-\frac{\partial\varphi_2}{\partial a_{0i}}\lambda_6,
\end{aligned}
\right| \qquad (5)
$$

where

$$\lambda_5=\frac{P^{n}_{y_i\, i}}{a_{0i}}\sum_{j=1}^{4}\varphi_j\lambda_j,\ \lambda_4=-1.$$

Here

1) where $\theta^{(i)}_{max}>\theta>\theta^{(i)}_{min}$

$$\lambda_6=0$$

while where $\alpha_{max}>\alpha_{opt}>\alpha_{min}$

$$\sin\alpha=\sin\alpha_{opt}=-\frac{\lambda_2}{\sqrt{\lambda_2^2+\left(\dfrac{\lambda_1 V}{1+\dfrac{b_{0i}}{a_{0i}}\,\gamma^{(i)}}\right)^2}},$$

while where $\alpha^{(1)}_{max} \leq \alpha_{opt}$ or $\alpha^{(1)}_{min} \geq \alpha_{opt}$

$$\sin\alpha = \sin\alpha^{(l)}_{max} \quad \text{or} \quad \sin\alpha = \sin\alpha^{(l)}_{min},$$

where we assume $\alpha^{(1)}_{max}$ = const and $\alpha^{(1)}_{min}$ = const;

Furthermore, if $H_p \leq 0$, then $p_0 = 1$, while otherwise $p_0 = 0$;

2) where $\theta = \theta^{(i)}_{max}$ or $\theta = \theta^{(i)}_{min}$

$$\varphi_{11} = (\lambda_6 + \lambda_2)\frac{\partial\varphi_2}{\partial\alpha} + \lambda_1 \frac{\partial\varphi_1}{\partial\alpha} = 0$$

or

$$\lambda_6 = -\left(\lambda_2 + \lambda_1 \frac{\partial\varphi_1/\partial\alpha}{\partial\varphi_2/\partial\alpha}\right),$$

$$\sin\alpha = \sin\alpha_{cb} = \frac{\left[\frac{g_0 R_3^2}{(R_3 + H)^2} - \frac{V^2}{R_3 + H}\right]\mu\cos\theta}{g_0(a_{0i}p + b_{0i}\bar{Y}^{(i)})},$$

where in the case $\theta = \theta^{(i)}_{min}$ and $\sin\alpha_{opt} > \sin\alpha_{cb}$ we should go over to condition 1, while in the case $\theta = \theta^{(i)}_{max}$ we should go over to condition 1 with $\sin\alpha_{opt} < \sin\alpha_{cb}$.

In order to perform integration of system (5) by one of the numerical methods, we must know, in addition to the values of the main plan parameters a_{0i}, ν_{kJ} and G_{0J}, the values at the initial point t_0 of the Lagrange coefficients λ_{10}, λ_{20} and λ_{30}, which would satisfy conditions (1), (4) (after exclusion of parameter β from them using (2)) and the optimality conditions of the main plan parameters, or the values of any three phase coordinates V_2, θ_2, H_2, L_2 at point $t^{(2)}$, which would satisfy the initial conditions (3) and the conditions of optimality of the main plan parameters.

It is formally insignificant at which point t_0 or t_2 the initial conditions are fixed for solution of the multipoint boundary problem produced. However, on the basis of the physical conditions of flight, the area of search for the unknown phase coordinates at point $t^{(2)}$, V_2, θ, H_2, L_2 can be more precisely determined than the area of search for the Lagrange coefficients λ_{10}, λ_{20}, λ_{30} at initial point t_0.

Therefore, the solution of the multipoint boundary problem is reduced to determination of the main plan parameters a_{0i}, μ_{kJ} and the phase coordinates at point $t^{(2)}$, θ_2, H_2 and L_2, for which the functionals

at point t_0

$$\left.\begin{aligned} &p_1=V_0^2,\ p_2=\left(\frac{\theta_0}{\theta_0^{(1)}}-1\right)^2,\ p_3=H_0^2,\\ &p_4=L_0^2,\ p_5=(\eta_{10}^{(1)})^2,\end{aligned}\right\} \tag{6}$$

at point $t^{(1)}$

$$p_6=(\eta_{10}^{(2)})^2,\ p_7=\left[\frac{\left(\lambda_{5x}^{(1)}+\frac{\partial\varphi_1^{(1)}}{\partial\lambda_{x1}}\eta_{0\,0}^{(2)}\right)\frac{m_p^{(2)}}{m_p^{(1)}}}{\lambda_{5x}^{(2)}}-1\right]^2 \tag{7}$$

are equal to zero.

The functionals p_5, p_6 and p_7 reflect the conditions of optimization of the main plan paramcters.

The values of G_0 and $G_{0\ II}$ will be determined from the plan equations.

The stop functions will be taken as

$$\varphi_1^0=\mu-1=0,\ \varphi_2^0=\mu-1=0. \tag{8}$$

The supplementary system of equations, in addition to equations (5), includes the conjugate system

$$y'_{m.l} = -\sum_{j=1}^{10} \frac{\partial \varphi_j}{\partial x_m} y_{j.l} - \frac{\partial \varphi_{11}}{\partial x_m} y_{13.l} - \frac{\partial \varphi_2}{\partial x_m} y_{14.l}$$

$$(x_m = V, \theta, H, L, \mu, \lambda_1, \lambda_2, \lambda_4, \eta_0, \eta_1, G_{0l}, a_{0l}; \quad m = j = 1, \ldots, 12; \; l = 1, \ldots, 7), \tag{9}$$

where:

<u>in case 1</u>

$$y_{14.l} = 0,$$

where $\alpha^{(1)}_{max} > \alpha > \alpha^{(i)}_{min}$

$$y_{13.l} = -\frac{1}{\frac{\partial \varphi_{11}}{\partial \alpha}} \sum_{j=1}^{10} \frac{\partial \varphi_j}{\partial \alpha} y_{j.l},$$

wjere $\alpha = \alpha^{(i)}_{max}$ or $\alpha = \alpha^{(i)}_{min}$

$$y_{13.l} = 0;$$

<u>in case 2</u>

$$y_{13.l} = -\frac{1}{\frac{\partial \varphi_2}{\partial \alpha}} \left(\frac{\partial \varphi_2}{\partial V} y_{6.l} + \frac{\partial \varphi_2}{\partial \theta} y_{7.l} + \frac{\partial \varphi_2}{\partial H} y_{8.l} + \frac{\partial \varphi_2}{\partial G_{0l}} y_{9.l} + \right.$$

$$\left. + \frac{\partial \varphi_2}{\partial a_{0l}} y_{10.l} \right), \quad y_{14.l} = -\frac{1}{\frac{\partial \varphi_2}{\partial \alpha}} \left(\sum_{j=1}^{10} \frac{\partial \varphi_j}{\partial \alpha} y_{j.l} + \frac{\partial \varphi_{11}}{\partial \alpha} y_{13.l} \right).$$

The initial conditions (point t_0) of the conjugate system are determined as follows:

$$
\left.\begin{aligned}
&y_{1.1}=\frac{\partial p_1}{\partial V},\; y_{2.2}=\frac{\partial p_2}{\partial \theta},\; y_{3.3}=\frac{\partial p_3}{\partial H},\\
&y_{4.4}=\frac{\partial p_4}{\partial L},\; y_{10.5}=\frac{\partial p_5}{\partial \eta^{(1)}},\\
&y_{5.l}=\frac{P^{*}_{sp_1}}{a_{01}}\,\frac{dp_l}{dt}\; (l=1,\dots,5),\\
&y_{j.l}=0 (j=1,\dots,4,6,\dots,12;\; l=1,\dots,5;\\
&j\neq l \text{ и, кроме } j=10,\; l=5),\\
&y_{5.6}=0,\\
&y_{j.l}=0 (j=1,\dots,12;\; l=6,7).
\end{aligned}\right\} \qquad (10)
$$

At point $t^{(1)}$, the discontinuity conditions must be fulfilled in the form

$$
\left.\begin{aligned}
&y_{j.l}(t^{(1)}_{-})=y_{j.l}(t^{(1)}_{+})\; (j=1,\dots,4,6,\dots,8;\; l=1,\dots,4),\\
&y_{j.5}(t^{(1)}_{+})=y_{j.5}(t^{(1)}_{-})+y_{10.5}(t^{(1)}_{-})\,\partial\eta^{(1)}_{1k}/\partial x_j\, (x_j=V,\dots,\\
&L,\lambda_1,\lambda_2,\lambda_3),\\
&y_{5.l}(t^{(1)}_{+})=-\frac{P_{sp_2}}{a_{02}}\left[y_{1.l}(t^{(1)}_{+})(\varphi_1(t^{(1)}_{-})-\varphi_1(t^{(1)}_{+}))+\right.\\
&+y_{2.l}(t^{(1)}_{+})(\varphi_2(t^{(1)}_{-})-\varphi_2(t^{(1)}_{+}))+y_{6.l}(t^{(1)}_{+})(\varphi_6(t^{(1)}_{-})-\\
&\left.-\varphi_6(t^{(1)}_{+}))+y_{8.l}(t^{(1)}_{+})(\varphi_8(t^{(1)}_{-})-\varphi_8(t^{(1)}_{+}))\right]\; (l=1,\dots,5),\\
&y_{j.l}(t^{(1)}_{+})=0\; (j=9,\dots,12;\; l=1,\dots,5)\\
&y_{5.6}(t^{(1)}_{+})=\frac{P^{*}_{sp2}}{a_{02}}\,\frac{dp_6}{dt},\quad y_{10.6}(t^{(1)}_{+})=\frac{\partial p_6}{\partial \eta^{(2)}_1},\\
&y_{j.6}(t^{(1)}_{+})=0\; (j=1,\dots,4,6,\dots,9,11,12),\\
&y_{1.7}(t^{(1)}_{+})=\frac{\partial p_7}{\partial V},\; y_{2.7}(t^{(1)}_{+})=\frac{\partial p_7}{\partial \theta},\\
&y_{3.7}(t^{(1)}_{+})=\frac{\partial p_7}{\partial H},\; y_{4.7}(t^{(1)}_{+})=0,\\
&y_{5.7}(t^{(1)}_{-})=\frac{P^{*}_{sp2}}{a_{02}}\,\frac{dp_7}{dt},\\
&y_{6.7}(t^{(1)}_{+})=\frac{\partial p_7}{\partial \lambda_1},\; y_{7.7}(t^{(1)}_{+})=\frac{\partial p_7}{\partial \lambda_2},\; y_{8.7}(t^{(1)}_{+})=\frac{\partial p_7}{\partial \lambda_3},\\
&y_{9.7}(t^{(1)}_{+})=\frac{\partial p_7}{\partial \eta_0},\; y_{10.7}(t^{(1)}_{+})=\frac{\partial p_7}{\partial \eta_1},\; y_{11.7}(t^{(1)}_{+})=\frac{\partial p_7}{\partial G_{011}},\\
&y_{12.7}(t^{(1)}_{+})=\frac{\partial p_7}{\partial a_{02}}.
\end{aligned}\right\} \qquad (11)
$$

After integration of the supplementary system, we can go over to solution of the linear system (2.3.14) and (2.3.23), which in this problem is

$$\Delta p_l = A_{1.l}\delta\theta_2 + A_{2.l}\delta H_2 + A_{3.l}\delta L_2 + A_{4.l}\delta\mu_{\kappa 11} + A_{5.l}\delta a_{02} + \\ + A_{6.l}\delta\mu_{\kappa 1} + A_{7.l}\delta a_{01} \quad (l = 1, \ldots, 7). \tag{12}$$

Here

$$A_{1.l} = y_{2.l}(t^{(2)}) + \sum_{j=1}^{3} \omega_{j.l} \frac{\partial \lambda_{j2}}{\partial \theta_2} - y_{V.l} \frac{\lambda_{22}}{\lambda_{12}},$$

$$A_{2.l} = y_{3.l}(t^{(2)}) + \sum_{j=1}^{3} \omega_{j.l} \frac{\partial \lambda_{j.2}}{\partial H_2} - y_{V.l} \frac{\lambda_{32}}{\lambda_{12}},$$

$$A_{3.l} = y_{4.l}(t^{(2)}) + y_{V.l} \frac{1}{\lambda_{12}},$$

$$y_{V.l} = y_{1.l}(t^{(2)}) + z_l \frac{\partial \lambda_{5\kappa}^{(2)}}{\partial V_2} + \sum_{j=1}^{3} \omega_{j.l} \frac{\partial \lambda_{j2}}{\partial V_2},$$

$$\omega_{1.l} = y_{6.l}(t^{(2)}) + z_l \frac{\partial \lambda_{5\kappa}^{(2)}}{\partial \lambda_{12}},$$

$$\omega_{2.l} = y_{7.l}(t^{(2)}) + z_l \frac{\partial \lambda_{5\kappa}^{(2)}}{\partial \lambda_{22}},$$

$$\omega_{3.l} = y_{8.l}(t^{(2)}),$$

$$z_l = y_{10.l}(t^{(2)}) \frac{\partial \mu_{011}}{\partial a_{02}} \frac{1}{\partial \mu_{011}/\partial \mu_{\kappa 11}},$$

$$A_{4.l} = y_{5.l}(t^{(2)}) + y_{10.l}(t^{(2)}) \frac{\partial \eta_{1\kappa}^{(2)}}{\partial \mu_{\kappa 11}} - \left(z_{G.l}^{(1)} + \mu_{011} y_{11.l}(t^{(2)})\right) \frac{G_0}{\mu_{011}} \frac{\partial \mu_{011}}{\partial \mu_{\kappa 11}},$$

$$A_{5.l} = y_{12.l}(t^{(2)}) + y_{10.l}(t^{(2)}) \frac{\partial \eta_{1\kappa}^{(2)}}{\partial a_{02}} - \left(z_{G.l}^{(1)} + \mu_{011} y_{11.l}(t^{(2)})\right) \frac{G_0}{\mu_{011}} \frac{\partial \mu_{011}}{\partial a_{02}},$$

$$A_{6.l} = y_{5.l}(t^{(1)}) + y_{10.l}(t^{(1)}) \frac{\partial \eta_{1\kappa}^{(1)}}{\partial \mu_{\kappa 1}} + y_{10.l}(t^{(2)}) \frac{\partial \eta_{1\kappa}^{(1)}}{\partial \mu_{\kappa 1}} + z_{G.l}^{(1)} \frac{G_0}{\mu_{01}} \frac{\partial \mu_{01}}{\partial \mu_{\kappa 1}},$$

$$A_{7.l} = y_{12.l}(t^{(2)}) + y_{10.l}(t^{(1)}) \frac{\partial \eta_{1\kappa}^{(1)}}{\partial a_{01}} + z_{G.l}^{(1)} \frac{G_0}{\mu_{01}} \frac{\partial \mu_{01}}{\partial a_{01}},$$

$$z_{G.l}^{(1)} = y_{11.l}(t^{(1)}) + y_{10.l}(t^{(1)}) \frac{\partial \eta_{1\kappa}^{(1)}}{\partial G_0}.$$

As a result of solution of the system of linear equations (12), we find $\delta\theta_2$, δH_2, δL_2, δa_{0i}, $\delta\mu_{kJ}$. Further calculations should be performed according to the BOSH algorithm (flow chart shown on Figure 1.24).

A program was composed for the BESM-3M computer for this algorithm for calculation of the optimal values of the main plan parameters and the optimal mode of motion of a two stage LRBM capable of delivering a known payload over a known range with minimum launch weight and limitations on trajectory angle.

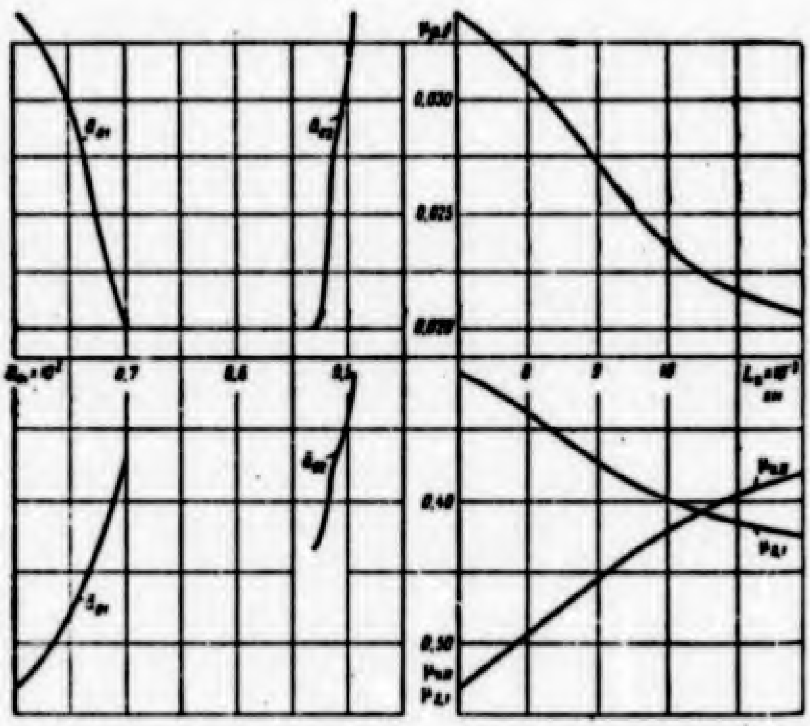

Figure 2.12. Nomogram for Determination of Optimal Values of Plan Parameters of Hypothetical Two Stage Ballistic Missile as a Function of Final Flight Range:

$$\bar{a}_{0I} = \frac{a_{0I}}{P^0_{y\,I}}, \quad \mu_{2,1} = \frac{P_{\kappa\,II}}{P_{\kappa\,I}}$$

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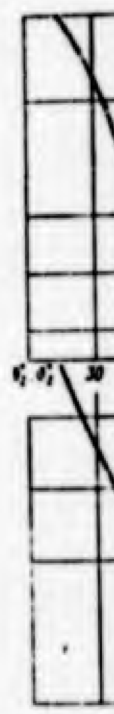

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Calculations performed using this program allowed us to determine the optimal phase trajectory of the active sector, optimal flight modes and optimal values of main plan parameters of the two stage LRBM.

Calculations were performed for a homogeneous plane-parallel and central terrestrial gravitational field over the active flight sector, in the latter case with and without consideration of the atmosphere. The final passive sector was always calculated considering the central gravitational field. Analysis of the calculation results produced show that the optimal values of the main plan parameters and the values of phase coordinates at the end of the powered flight sector V_2, θ_2, H_2 and L_2, determined with the homogeneous plane-parallel field over the powered flight sector without considering the atmosphere can be used as a good zero approximation.

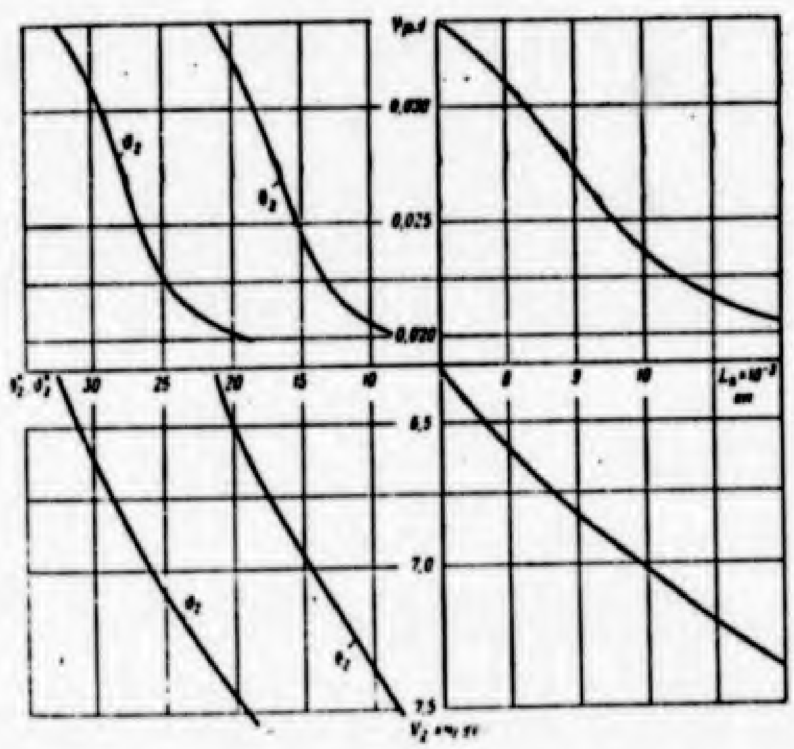

Figure 2.13. Nomogram for Determination of Optimal Values of Kinematic Parameters at the End of Powered Flight Sector V_2 and θ_2 of Hypothetical Two Stage Ballistic Missile as a Function of Final Flight Range

The optimal values of the main plan parameters of a two stage ballistic missile and values of phase coordinates V_2, θ_2, H_2 and L_2 at the end of the optimal powered flight sector, calculated for the case of a homogeneous plane-parallel gravitional field over the powered sector without considering the atmosphere are shown on Figures 2.12-2.14 as functions of the final flight range.

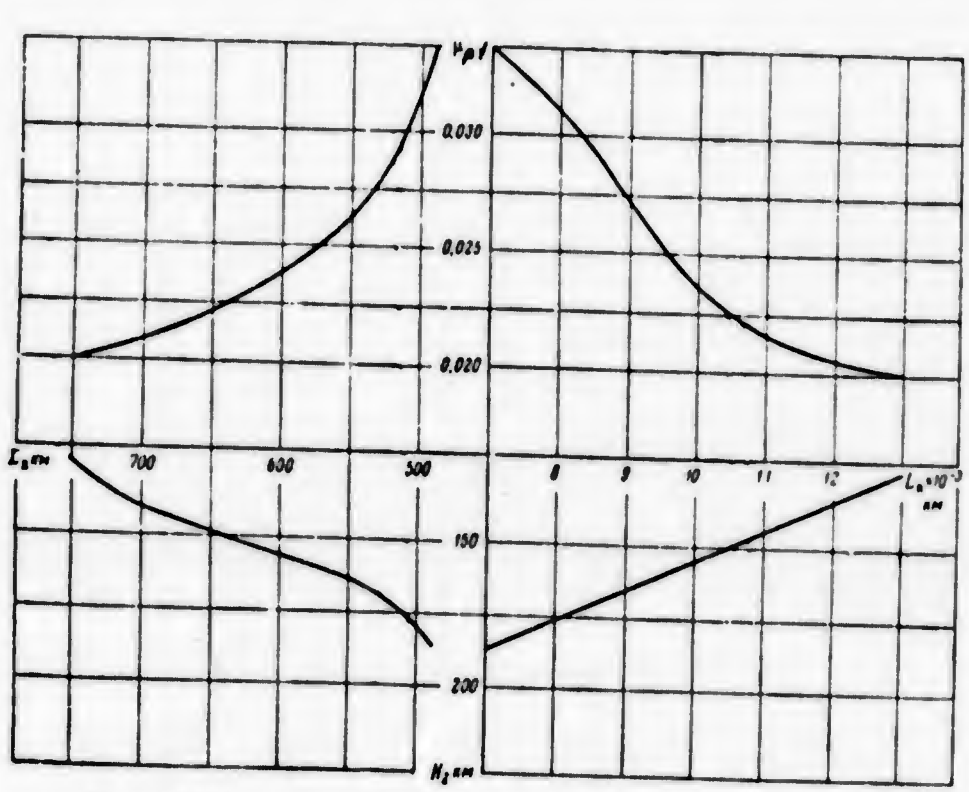

Figure 2.14. Nomogram for Determination of Optimal Values of Kinematic Parameters at End of Powered Flight Sector H_2 and L_2 for Hypothetical Two Stage Ballistic Missile as a Function of Final Flight Range.

Table 2.2

Value of Summary Functional	Iteration Number				
	0	1	2	3	4
kin P_Σ	$V_0 = 124.6$ m/sec $\vartheta_0 = \pi/2$ $H_0 = 45.5$ km $L_0 = [illegible]04$ km	$V_0 = 16.9$ m/sec $\vartheta_0 = \pi/2$ $H_0 = [illegible]4$ m $L_0 = 13$ m	$V_0 = 12.7$ m/sec $\vartheta_0 = \pi/2$ $H_0 = 123$ m $L_0 = [illegible]37$ m	$V_0 = 0.13$ m/sec $\vartheta_0 = \pi/2$ $H_0 = 51$ m $L_0 = 111$ m	$V_0 = 1.09$ m/sec $\vartheta_0 = \pi/2$ $H_0 = 0.88$ m $L_0 = 8.1$ m
$P_\Sigma^{rap} - P_1 - + P_2 + P_3$	0,83	1,14	0,35	0,47	[illegible]0,123

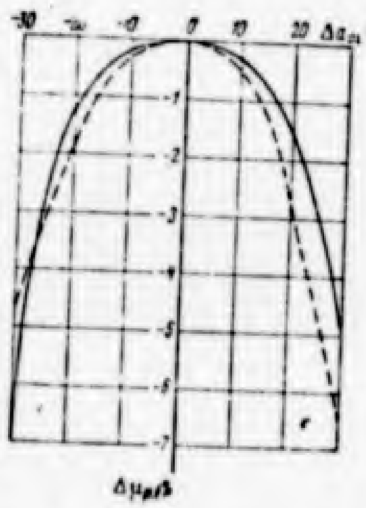

Figure 2.15. Deviation of Maximum Relative Payload as a Function of Deviation of Initial Thrust to Weight Ratio of ith Stage from Optimal Value

------- Δa_{01}

——— Δa_{02}

Table 2.2 can be used to determine the course of the iterational process of this algorithm for the multipoint boundary problem considering a central gravitational field, when the initial solution (N = 0) was the data taken from the nomograms shown on Figures 2.12-2.14 for a final range L_k = 10,000 km. The data of Table 2.2 confirm the conclusion made earlier in Appendix 1 to Chapter II that the iterational processes performed by the computer according to the algorithm for optimal planning of the flight vehicle and presented in detail in Chapter I, § 4, produce satisfactory convergence.

Calculations were also performed to determine the influence of deviations in thrust to weight ratio of the first and second stages of a rocket from the optimal values on the deviation of the relative payload from its maximum value. Calculations were performed so that as the thrust to weight ratio of any stage deviated from the optimal value, the fixed final flight range was always fulfilled in the class of optimal controls $p(t)$ and $\alpha(t)$.

The results of these calculations are shown on Figure 2.15.

CHAPTER III. VARIATIONAL METHOD OF OPTIMIZATION OF MODES OF MOTION AND MAIN PLAN PARAMETERS OF MULTISTAGE SPACECRAFT

§ 1. Statement of Variational Problem of Optimization of Modes of Motion and Principal Plan Parameters

In the initial stage of rough planning of a spacecraft, we must determine the main plan parameters and modes of motion allowing us to create a vehicle satisfying the requirements placed upon it in the optimal manner.

A spacecraft must perform the following maneuvers: separation from the planet, interplanetary flight, approach to the target planet, entry into orbit around the target planet or landing on it, adjustment of trajectories, etc. Each of these maneuvers has its own specific peculiarities.

The evaluation of these maneuvers, in addition to the other requirements, should always call for the fulfillment of the requirement of delivery of the fixed payload to the target planet or to some designated orbit. Therefore, these maneuvers have the task either of delivery of the required payload to the designated planet or to some orbit with the minimum launch weight of the spacecraft, or delivery of the maximum payload with a fixed launch weight. The importance of this problem is obvious.

In this chapter, we will state the problem of determination of the modes of motion and main plan parameters of a multistage spacecraft capable of delivering either a fixed useful mass M_{pl} with minimum launch mass $M_{0\,min}$ or the maximum useful mass $M_{pl\,max}$ with fixed launch mass to the target planet with intermediate orbiting around the earth (before travel to the target planet). These modes of motion and the main plan parameters will be referred to as the optimal modes of motion and the optimal main plan parameters of the multistage spacecraft.

As power plants for the stages of the spacecraft, we shall analyze single-mode power plants, i. e. power plants operating only at maximum thrust.

There are a number of factors speaking in favor of the use of single-mode power plants in solving the present problem. For example, in preceding chapters we demonstrated that if a power plant has a choking characteristic which is near linear with a certain drop in specific thrust as the fuel flow rate per second is decreased, only the maximum thrust mode of the power plant, in combination with inertial flight, will appear in the optimal modes of motion of the flight vehicle. Therefore, power plants with these characteristics will act as single-mode power plants in the optimal modes of motion.

In this connection, the primary single-mode power plants are liquid fuel rockets.

Theoretical and experimental investigations of electric rocket power plants have shown that these power plants are technically feasible and that their application in spacecraft is expedient. They have indicated the main principles of the design of electric rocket power plants, allowing comprehensive evaluation of spacecraft using electric rocket motors to be performed. By "electric rocket motors" we refer to three types of electric rocket power plants, currently in the development stage: electrostatic or ion motors, motors using an electric arc and magnetohydrodynamic motors, sometimes called "plasma" motors. The simpler organization of the stable operating process forces us at the present time to use primarily single-mode electric rocket motors (ERM). Therefore, we should analyze various types of ERM as possible single-mode power plants.

Thus, the investigation of the optimal modes of motion of a multistage spacecraft is performed on the assumption that the permissible operating modes of the power plants of the stages (liquid fuel motors or ERM) are the maximum thrust mode and $p = 0$. No limitations will be placed on the dynamic characteristics of the motors, and therefore the control function P (or $p = P/P_{max}$) will be analyzed in the class of piecewise continuous functions with finite number of first order discontinuities.

The spacecraft as an object of control can be characterized by the control functions α, β and γ (guiding cosines of reactive force), the values of which are not limited.

The time of the transient process in the case of maximum possible rate of change of thrust vector and consequently of α, β and γ is very brief in comparison with the time of the powered sector, allowing us to assume it equal to zero with great accuracy, thereby eliminating consideration of limitations based on inertia of the control system and to analyze control functions α, β and γ as piecewise continuous functions with a finite number of first order discontinuities.

Planning and theoretical studies of spacecraft with ERM have shown that the main plan parameters for a spacecraft stage with ERM are the specific power of the power plant N_{sp}, the thrust to weight ratio of the stage a_{0i}, the exhaust velocity of the products of the working medium W_{0i}^{e} or the specific thrust P_{spi}^{e}, mass M_{0J} and the relative final mass $\nu_{kJ} = M_{kJ}/M_{0J}$ of the stage. We should discuss specifically the selection of N_{sp}. Its "nature" is such that N_{sp} should be as great as possible. However, its magnitude depends on the level of development of space science at the given stage [41]. Therefore, in the following we will consider the value of N_{sp} fixed.

For a spacecraft stage with a liquid fuel rocket motor (LRM), the main plan parameters are the thrust to weight ratio a_{0i}, mass M_{0J} and ν_{kJ}.

Thus, the main plan parameters of a multistage spacecraft with single-mode power plants for each stage, including LRM and ERM, will be M_{0J} ($J = 1, \ldots, N$), M_{pl}, ν_{kJ}, a_{0i} ($i = 1, \ldots, n$), P_{maxi}, W_{0i}^{e} or P_{spi}^{e}.

Therefore, the plan equation and other weight relationships for the multistage spacecraft with single-mode power plants are

$$\left.\begin{array}{ll} \beta_1^{(0)} \equiv M_{p,l} - M_0\mu_{p,l} = 0, & (3.1.1) \\ \beta_1^{(J)} \equiv M_{0(J+1)} - M_{kJ}\mu_{0J} = 0 & \\ (J = l, \ldots, N;\ i = 1, \ldots, n). & (3.1.2) \end{array}\right\} \quad (3.1)$$

In the following for generality we will assume

$$\mu_{kJ} = f_{\kappa p}^{(i)}(a_{0i}, \nu_{kJ}, M_{0J}, W_{0i}^{e}),$$
$$\mu_{pJ} = f_{np}(a_{0i}, \nu_{kJ}, M_{0J}, W_{0i}^{e}).$$

According to the definition of thrust to weight ratio, we must also consider the coupling equation of plan parameters in the form

$$\xi_2^{(i)} = a_{0i} - \frac{P_{\max i}}{M_{0i}} = 0. \tag{3.1.3}$$

Since the range of change of the main plan parameters for the corresponding types of spacecraft and power plants fall within definite limits, the following conditions of limitation of the plan parameters obtain:

$$\left.\begin{aligned} & M_{0i\,\min} \leqslant M_{0i} \leqslant M_{0i\,\max}, \\ & a_{0i\,\min} \leqslant a_{0i} \leqslant a_{0i\,\max}, \\ & W^{e}_{0i\,\min} \leqslant W^{e}_{0i} \leqslant W^{e}_{0i\,\max}, \\ & (P_{\max i})_{\min} \leqslant P_{\max i} \leqslant (P_{\max i})_{\max}, \\ & M_{p.f\,\min} \leqslant M_{p.f} \leqslant M_{p.f\,\max}. \end{aligned}\right\} \tag{3.1.4}$$

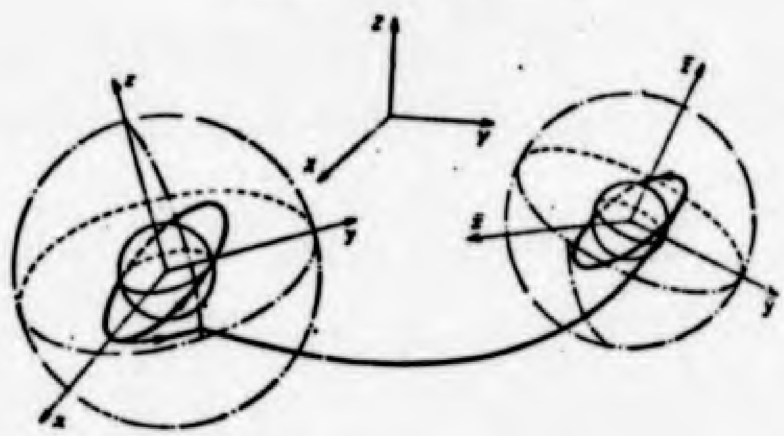

Figure 3.1. Diagram of Course of Inter-Planetary Flight

In studying the optimal modes of motion of a multistage spacecraft, we will consider that the launch is performed into a predetermined intermediate orbit, located at a distance from the earth such that it allows the aerodynamic drag of the atmosphere to be ignored (Figure 3.1).

Furthermore, in order to make the problem more general, we assume that the interplanetary flight will be completed with arrival of the spacecraft at the target planet (Venus, Mars, etc.). We will assume that arrival of the spacecraft at the target planet will involve either a probe flyby at a predetermined distance from this planet or entry into an orbit around the target planet, etc. It is assumed that these maneuvers are performed at a distance from the target planet sufficient that the drag of its atmosphere can be ignored.

Generally speaking, in calculating the trajectory of a spacecraft, it is necessary to consider the force of gravity of the Sun and all planetary systems. However, in the stage of preliminary planning of a spacecraft, involving preliminary selection of the main plan parameters and modes of motion, consideration of all forces of gravity leads to unnecessary complexity of the calculations. Therefore, the problem of determination of the course of the interplanetary flight of the spacecraft with optimal modes of motion from the intermediate orbit around the earth to the target planet will be divided into internal and external problems.

The internal problems are divided into two: the first problem is the problem of separation, involved with investigation of the motion of the spacecraft from the intermediate orbit to the sphere of influence of the Earth, while the second problem is the capture problem, involving investigation of the motion of the spacecraft from the sphere of influence of the target planet to the final maneuver. In solving the problem of separation, we use geocentric coordinates, while the solution of the capture problem is performed in planetocentric coordinates.

The external problem is related to the investigation of the motion of the spacecraft from the sphere of influence of the Earth to the sphere of influence of the target planet. In solving the external problem, we will use heliocentric coordinates.

In an analysis of the optimal flight modes of a spacecraft traveling to the moon, the over-all problem is divided into two problems. One problem is related to the investigation of the motion of the spacecraft from the intermediate orbit to the sphere of influence of the moon, the other -- from the sphere influence of the moon to the final maneuver, which could be performed using selenocentric coordinates.

However, the division of the problem of determining the course of an interplanetary flight into three problems (the external and two internal problems) does not result in separation of the problem of optimization of the main plan parameters and modes of motion into three variational problems. In this sense, in our investigation of the problem of optimization of the main plan parameters and modes of motion for interplanetary flights, the external and internal problems are analyzed jointly. Therefore, the result of the solution of the variational problem is the determination of a single optimal phase trajectory. This is one of the primary distinctions of this work from works published in the scientific and technical literature (see [12] and the bibliography which it contains).

Analysis of the external and internal problems is performed on the assumption that the only force of gravity acting on the spacecraft is the force of gravity of the central body Earth, Sun, etc.). The origin of the geocentric and planetocentric systems of coordinates is located at the center of the planet (Earth, target planet). The system of coordinate axes will be selected as required for the time, related to the fixed launch date.

Suppose the fixed intermediate orbit around the Earth, lying in a plane fixed in space, is elliptical. Then, assuming the geocentric system of coordinates to be rectangular and equatorial, we have [14]

$$\left.\begin{aligned}
&\psi_{01} \equiv v_{x0} - \frac{c}{p}\Big[\vartheta_1 e \sin(u_0 - \omega) + \\
&+ \frac{\partial\vartheta_1}{\partial u_0}(1 + e\cos(u_0 - \omega))\Big] = 0, \\
&\psi_{02} \equiv v_{y0} - \frac{c}{p}\Big[\vartheta_2 e \sin(u_0 - \omega) + \\
&+ \frac{\partial\vartheta_2}{\partial u_0}(1 + e\cos(u_0 - \omega))\Big] = 0, \\
&\psi_{03} \equiv v_{z0} - \frac{c}{p}\Big[\vartheta_3 e \sin(u_0 - \omega) + \\
&+ \frac{\partial\vartheta_3}{\partial u_0}(1 + e\cos(u_0 - \omega))\Big] = 0, \\
&\psi_{04} \equiv x_0 - r_0\vartheta_1 = 0,\ \psi_{05} \equiv y_0 - \\
&- r_0\vartheta_2 = 0,\ \psi_{06} \equiv z_0 - r_0\vartheta_3 = 0,
\end{aligned}\right\} \tag{3.1.5}$$

where

$$\vartheta_1 = \cos u_0 \cos\Omega - \sin u_0 \sin\Omega \cos i,$$
$$\vartheta_2 = \cos u_0 \sin\Omega + \sin u_0 \cos\Omega \cos i,$$
$$\vartheta_3 = \sin u_0 \sin i.$$

The final maneuver of the spacecraft near the target planet, let us suppose, will involve fulfillment of the following final conditions:

$$\psi_{\kappa s}(\bar{v}_{x\kappa}, \bar{v}_{y\kappa}, \bar{v}_{z\kappa}, \bar{x}_{\kappa}, \bar{y}_{\kappa}, \bar{z}_{\kappa}, t_{\kappa}) = 0 \quad (s = 1, \ldots, m \leqslant 7). \tag{3.1.6}$$

Note. If the task of the final maneuver of the spacecraft is entry into a fixed final orbit around the target planet, then, assuming the planetocentric system of coordinates to be rectangular and equatorial, the final orbit eliptical, we produce the final conditions (3.1.6) as

$$\left.\begin{aligned}
\psi_{\kappa 1} &\equiv \bar{v}_{x\kappa} - \frac{\bar{c}}{\bar{p}}\Big[\bar{\theta}_1 \bar{e}\sin(\bar{u}_\kappa - \bar{\omega}) + \\
&+ \frac{\partial\bar{\theta}_1}{\partial\bar{u}_\kappa}(1 + \bar{e}\cos(\bar{u}_\kappa - \bar{\omega}))\Big] = 0, \\
\psi_{\kappa 2} &\equiv \bar{v}_{y\kappa} - \frac{\bar{c}}{\bar{p}}\Big[\bar{\theta}_2 \bar{e}\sin(\bar{u}_\kappa - \bar{\omega}) + \\
&+ \frac{\partial\bar{\theta}_2}{\partial\bar{u}_\kappa}(1 + \bar{e}\cos(\bar{u}_\kappa - \bar{\omega}))\Big] = 0, \\
\psi_{\kappa 3} &\equiv \bar{v}_{z\kappa} - \frac{\bar{c}}{\bar{p}}\Big[\bar{\theta}_3 \bar{e}\sin(\bar{u}_\kappa - \bar{\omega}) + \\
&+ \frac{\partial\bar{\theta}_3}{\partial\bar{u}_\kappa}(1 + \bar{e}\cos(\bar{u}_\kappa - \bar{\omega}))\Big] = 0, \\
\psi_{\kappa 4} &\equiv \bar{x}_\kappa - \varrho_\kappa\bar{\theta}_1 = 0, \quad \psi_{\kappa 5} \equiv \bar{y}_\kappa - \varrho_\kappa\bar{\theta}_2 = 0, \\
\psi_{\kappa 6} &\equiv \bar{z}_\kappa - \varrho_\kappa\bar{\theta}_3 = 0.
\end{aligned}\right\} \tag{3.1.6N}$$

where

$$\begin{aligned}
\bar{\theta}_1 &= \cos\bar{u}_\kappa\cos\bar{\Omega} - \sin\bar{u}_\kappa\sin\bar{\Omega}\cos\bar{i}, \\
\bar{\theta}_2 &= \cos\bar{u}_\kappa\sin\bar{\Omega} + \sin\bar{u}_\kappa\cos\bar{\Omega}\cos\bar{i}, \\
\bar{\theta}_3 &= \sin\bar{u}_\kappa\sin\bar{i}.
\end{aligned}$$

Here the system of elements of the intermediate orbit is related to the plane of the equator of the Earth, while the system of elements of the final orbit around the target planet is related to the plane of the equator of that planet.

If we assume the geocentric and planetocentric systems of coordinates to be rectangular and ecliptical, then, relating the system of elements of the intermediate and final orbits to the ecliptic plane, we can retain the formal inscription of equations (3.1.5) and (3.1.6N).

Further, let us assume the heliocentric system of coordinates to be rectangular and ecliptical. In this case, at the moment when the spacecraft passes through the sphere of influence of the earth ($t = t_k^{\oplus}$), corresponding to the condition

$$\psi_{\Delta 1} \equiv \Delta_{\odot} - \sqrt{x_k^2 + y_k^2 + z_k^2} = 0, \tag{3.1.7}$$

the geocentric rectangular equatorial coordinates should be transformed to heliocentric rectangular ecliptical coordinates, using the following formulas for conversion of the phase coordinates:

$$\left.\begin{aligned}
&\Pi_1 \equiv X_0 - (x_k + X^{\oplus}(t_k^{\oplus})) = 0, \quad \Pi_2 \equiv Y_0 - \\
&- (y_k \cos \varepsilon + z_k \sin \varepsilon + Y^{\oplus}(t_k^{\oplus})) = 0, \\
&\Pi_3 \equiv Z_0 - (z_k \cos \varepsilon - y_k \sin \varepsilon) = 0, \\
&\Pi_4 \equiv V_{x0} - (v_{xk} + V_x^{\oplus}(t_k^{\oplus})) = 0, \\
&\Pi_5 \equiv V_{y0} - (v_{yk} \cos \varepsilon + v_{zk} \sin \varepsilon + V_y^{\oplus}(t_k^{\oplus})) = 0, \\
&\Pi_6 \equiv V_{z0} - (v_{zk} \cos \varepsilon - v_{yk} \sin \varepsilon) = 0.
\end{aligned}\right\} \tag{3.1.8}$$

At the moment of passage of the spacecraft through the sphere of influence of the target planet ($t = t_k^{\odot}$ or $t = t_0^{pl}$), corresponding to the condition

$$\psi_{\Delta 2} \equiv \Delta_{pl} - \sqrt{(X_k^{pl} - X_k)^2 + (Y_k^{pl} - Y_k)^2 + (Z_k^{pl} - Z_k)^2} = 0, \tag{3.1.9}$$

the heliocentric rectangular ecliptical coordinates should be converted to planetocentric rectangular equatorial coordinates, using the following conversion formulas:

$$
\left.
\begin{aligned}
&\bar{\Pi}_1 \equiv \bar{x}_0-(X_\kappa - X_\kappa^{pl})=0,\\
&\bar{\Pi}_2 \equiv \bar{y}_0-(Y_\kappa \cos\bar{\varepsilon} - Z_\kappa \sin\bar{\varepsilon} - Y_\kappa^{pl})=0,\\
&\bar{\Pi}_3 \equiv \bar{z}_0-(Z_\kappa \cos\bar{\varepsilon} + Y_\kappa \sin\bar{\varepsilon} - Z_\kappa^{pl})=0,\\
&\bar{\Pi}_4 \equiv \bar{v}_{x0}-V_{x\kappa} - V_{x\kappa}^{pl}=0,\\
&\bar{\Pi}_5 \equiv \bar{v}_{y0}-(V_{y\kappa} \cos\bar{\varepsilon} - V_{z\kappa} \sin\bar{\varepsilon} - V_{y\kappa}^{pl})=0,\\
&\bar{\Pi}_6 \equiv \bar{v}_{z0}-(V_{z\kappa} \cos\bar{\varepsilon} + V_{y\kappa} \sin\bar{\varepsilon} - V_{z\kappa}^{pl})=0.
\end{aligned}
\right\}
\qquad (3.1.10)
$$

Note. The dependence of the ecliptical coordinates and velocity components of the earth on $t_k^{\oplus}$ and the dependence of the ecliptical coordinates and velocity components of the planet on $t_k^{\odot}$ in the heliocentric system can be expressed as follows [38]:

$$
\left.
\begin{aligned}
&X_\kappa^{\oplus} = a_{\oplus}\left[P_x^{\oplus}(\cos E_\kappa^{\oplus} - e_{\oplus}) + \frac{\partial P_x^{\oplus}}{\partial \omega_{\oplus}} \cos\varphi_{\oplus} \sin E_\kappa^{\oplus}\right],\\
&Y_\kappa^{\oplus} = a_{\oplus}\left[P_y^{\oplus}(\cos E_\kappa^{\oplus} - e_{\oplus}) + \frac{\partial P_y^{\oplus}}{\partial \omega_{\oplus}} \cos\varphi_{\oplus} \sin E_\kappa^{\oplus}\right],\\
&V_{x\kappa}^{\oplus} = \frac{a_{\oplus} n_{\oplus}}{1-e_{\oplus}\cos E_\kappa^{\oplus}}\left[-P_x^{\oplus}\sin E_\kappa^{\oplus} + \frac{\partial P_x^{\oplus}}{\partial \omega_{\oplus}} \cos\varphi_{\oplus} \cos E_\kappa^{\oplus}\right],\\
&V_{y\kappa}^{\oplus} = \frac{a_{\oplus} n_{\oplus}}{1-e_{\oplus}\cos E_\kappa^{\oplus}}\left[-P_y^{\oplus}\sin E_\kappa^{\oplus} + \frac{\partial P_y^{\oplus}}{\partial \omega_{\oplus}} \cos\varphi_{\oplus} \cos E_\kappa^{\oplus}\right],\\
&X_\kappa^{pl} = a_{pl}\left[P_x^{pl}(\cos E_\kappa^{pl} - e_{pl}) + \frac{\partial P_x^{pl}}{\partial \omega_{pl}} \cos\varphi_{pl} \sin E_\kappa^{pl}\right],\\
&Y_\kappa^{pl} = a_{pl}\left[P_y^{pl}(\cos E_\kappa^{pl} - e_{pl}) + \frac{\partial P_y^{pl}}{\partial \omega_{pl}} \cos\varphi_{pl} \sin E_\kappa^{pl}\right],\\
&Z_\kappa^{pl} = a_{pl}\left[P_z^{pl}(\cos E_\kappa^{pl} - e_{pl}) + \frac{\partial P_z^{pl}}{\partial \omega_{pl}} \cos\varphi_{pl} \sin E_\kappa^{pl}\right],\\
&V_{x\kappa}^{pl} = \frac{a_{pl} n_{pl}}{1-e_{pl}\cos E_\kappa^{pl}}\left[-P_x^{pl}\sin E_\kappa^{pl} +\right.\\
&\qquad \left.+ \frac{\partial P_x^{pl}}{\partial \omega_{pl}} \cos\varphi_{pl} \cos E_\kappa^{pl}\right],
\end{aligned}
\right\}
\qquad (3.1.11)
$$

$$
\left.
\begin{aligned}
&V_{y\kappa}^{pl} = \frac{a_{pl} n_{pl}}{1-e_{pl}\cos E_\kappa^{pl}}\left[-P_y^{pl}\sin E_\kappa^{pl} + \frac{\partial P_y^{pl}}{\partial \omega_{pl}} \cos\varphi_{pl} \cos E_\kappa^{pl}\right],\\
&V_{z\kappa}^{pl} = \frac{a_{pl} n_{pl}}{1-e_{pl}\cos E_\kappa^{pl}}\left[-P_z^{pl}\sin E_\kappa^{pl} + \frac{\partial P_z^{pl}}{\partial \omega_{pl}} \cos\varphi_{pl} \cos E_\kappa^{pl}\right].
\end{aligned}
\right\}
$$

where

$$P_x = \cos\omega\sin\Omega - \sin\omega\sin\Omega\cos i,$$
$$P_y = \cos\omega\sin\Omega + \sin\omega\cos\Omega\cos i,$$
$$P_z = \sin\omega\sin i,$$
$$\cos\varphi = \sqrt{1-e^2}.$$

The differential equations of motion and kinematic couplings are expressed in the form

$$\left.\begin{array}{ll}
\varphi_1 \equiv V'_x - H_1 = 0, & (3.1.12)\\
\varphi_2 \equiv V'_y - H_2 = 0, & (3.1.13)\\
\varphi_3 \equiv V'_z - H_3 = 0, & (3.1.14)\\
\varphi_4 \equiv x' - H_4 = 0, & (3.1.15)\\
\varphi_5 \equiv y' - H_5 = 0, & (3.1.16)\\
\varphi_6 \equiv z' - H_6 = 0, & (3.1.17)\\
\varphi_8 \equiv \alpha^2 + \beta^2 + \gamma^2 - 1 = 0, & (3.1.18)
\end{array}\right\} \quad (3.I)$$

where in the geocentric system of coordinates

$$H_1^{\oplus} = \frac{a_{0}P}{\mu}\alpha - k_{\oplus}\frac{x}{r^3},\quad H_2^{\oplus} = \frac{a_{0}P}{\mu}\beta - k_{\oplus}\frac{y}{r^3},$$
$$H_3^{\oplus} = \frac{a_{0}P}{\mu}\gamma - k_{\oplus}\frac{z}{r^3},\quad H_4^{\oplus} = v_x,\quad H_5^{\oplus} = v_y,$$
$$H_6^{\oplus} = v_z;$$

in the heliocentric system of coordinates

$$H_1^{\odot} = \frac{a_{0}P}{\mu}\alpha - k_{\odot}\frac{X}{R^3},\quad H_2^{\odot} = \frac{a_{0}P}{\mu}\beta - k_{\odot}\frac{Y}{R^3},$$
$$H_3^{\odot} = \frac{a_{0}P}{\mu}\gamma - k_{\odot}\frac{z}{R^3},\quad H_4^{\odot} = V_x,\quad H_5^{\odot} = V_y,\quad H_6^{\odot} = V_z;$$

in the planetocentric system of coordinates

$$H_1^{pl} = \frac{a_{0i} p}{\mu} \alpha - k_{pl} \frac{\bar{x}}{\rho^3}, \quad H_2^{pl} = \frac{a_{0i} p}{\mu} \beta - k_{pl} \frac{\bar{y}}{\rho^3},$$

$$H_3^{pl} = \frac{a_{0i} p}{\mu} \gamma - k_{pl} \frac{\bar{z}}{\rho^3}, \quad H_4^{pl} = \bar{v}_x, \quad H_5^{pl} = \bar{v}_y, \quad H_6^{pl} = \bar{v}_z.$$

Equations (3.1.12)-(3.1.17) are written on the assumption that the Sun, Earth and target planet are material points or spheres with spherical distribution of density, attracting the spacecraft with a force which is inversely proportional to the square of the distan between the spacecraft and the corresponding body. The spacecraft is taken as a body of "zero" mass.

The right portions of equations (3.12)-(3.1.17) undergo first order discontinuities at the spheres of influence of the Earth and the planet due to the first order discontinuities of the phase coordinates resulting from the planned coordinate conversions. However, within the spheres of influence and within the heliocentric system of coordinates, the phase variables are continuous, while the right portions of equations (3.1.12)-(3.1.17) undergo first first order discontinuities only in connection with possible piecewise linear changes in the control functions and the main plan parameters.

The relationship between the flow rate of working medium per second, thrust and parameters is expressed by the flow characteristic

$$\varphi_7 \equiv \mu' - H_7 = 0, \qquad (3.1.19)\ (3.1)$$

where the piecewise continuous change in thrust is fixed by equation

$$\varphi_9 = p(1-p) = 0. \qquad (3.1.20)\ (3.1)$$

Here we assume $H_7 = a_{0i} p/W_{0i}$ for an LRM or $H_7 = -a_{0i} p/W$ for an ERM and it is arbitrarily assumed that where $p = 0$, $W = W_0$, i. e. the exhaust velocity of the products of the working medium where $p = 1$ is equal to the exhaust velocity of the products of the working medium where p = 0. This assumption assumes the possibility of immediate discontinuation of thrust (or for ERM of the power level) while retaining maximum exhaust velocity of the products of the working medium. Although actually, this type of regulation of the power plant is hardly possible, it does simplify the problem and introduces no errors since it is related to inertial flight.

At the beginning and end of operation of each stage, the following relationships obtain:

$$\left.\begin{aligned} &\mu(t_+^{(i)})=1,\\ &\psi_\mu^{(i)}=\mu(t_-^{(i)})-\mu_{kJ}=0. \end{aligned}\right\} \qquad (3.1.21)$$

The equations $\psi_\mu^{(i)} = 0$ reflect the limitation on the phase variable μ, which is

$$\mu^{(i)}(t)>\mu_{kJ}.$$

Therefore, when equality $\mu(t^{(n)}) = \mu_{kN}$ is reached, the maneuver of the spacecraft should be completed or must be completed with a passive sector.

In connection with the above, we can more precisely formulate the problem analyzed in this chapter. It consists of determining main plan parameters a_{0i}, μ_{kJ}, W_{0i}^e, M_{0J} and M_{pl} and controls $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ of the multistage spacecraft such that the solution of system (3.I) is performed while satisfying the boundary conditions (3.1.5), (3.1.6), (3.1.21) and therefore also the conditions for travel of the spacecraft to the sphere of influence of the Earth (3.1.7) and on to the sphere of influence of the target planet (3.1.9) such that with the fixed M_{pl}, we achieve M_0 = inf or with the fixed M_0, we achieve M_{pl} = sup.

This problem is a variational problem, the mathematical theory of which is presented in the appendix. Then, according to the theory, we go over from the closed to the open area of change of the main plan parameters by introducing the following relationships:

$$
\left.
\begin{aligned}
&\xi_{01}^{(i)} \equiv (M_{0j} - M_{0j\,\min})(M_{0j\,\max} - M_{0j}) - w_{1i}^2 = 0, \\
&\xi_{02}^{(i)} \equiv (a_{0i} - a_{0i\,\min})(a_{0i\,\max} - a_{0i}) - w_{2i}^2 = 0, \\
&\xi_{03}^{(i)} \equiv (W_{0i}^{\prime\,\circ} - W_{0i\,\min}^{\prime\,\circ})(W_{0i\,\max}^{\prime\,\circ} - W_{0i}^{\prime\,\circ}) - w_{3i}^2 = 0, \\
&\xi_{04}^{(i)} \equiv [P_{\max i} - (P_{\max i})_{\min}]\,[(P_{\max i})_{\max} - P_{\max i}] - w_{4i}^2 = 0, \\
&\xi_{05}^{(i)} \equiv (M_{p.l} - M_{p.l\,\min})(M_{p.l\,\max} - M_{p.l}) - w_5^2 = 0.
\end{aligned}
\right| \qquad (3.1.22)\ (3.1)
$$

The variational problem following from this problem can now be formulated as follows:

in the class of phase variables

$$
\left.
\begin{aligned}
&x(t),\ y(t),\ z(t),\ v_x(t),\ v_y(t),\ v_z(t), \\
&X(t),\ Y(t),\ Z(t),\ V_x(t),\ V_y(t),\ V_z(t), \\
&\bar{x}(t),\ \bar{y}(t),\ \bar{z}(t),\ \bar{v}_x(t),\ \bar{v}_y(t),\ \bar{v}_z(t), \rho,
\end{aligned}
\right\} \qquad (3.1.23)
$$

control functions

$$\alpha(t),\ \beta(t),\ \gamma(t)\ \text{and p}, \qquad (3.1.24)$$

main plan parameters

$$M_{0j},\ M_{p.l},\ a_{0i},\ \mu_{xj},\ W_{0i}^{\prime\,\circ},\ P_{\max}^{(i)} \qquad (3.1.25)$$

and arbitrary parameters

$$w_{1i},\ w_{2i},\ w_{3i},\ w_{4i},\ w_{5i}, \qquad (3.1.26)$$

permissible in the interval $t_0 \le t \le t_k$ and satisfying couplings (3.1), boundary conditions (3.1.5), (3.1.6), (3.1.21), conditions of exit from the sphere of influence (3.1.7), (3.1.9) and conditions of conversion of coordinates (3.1.8) and (3.1.10), find phase variables (3.1.23), control functions (3.1.24) and main plan parameters (3.1.25) such that the criterion of effectiveness

$$I(M_0,\ M_{n.r}) \qquad (3.1.27)$$

reaches its maximum value.

This variational problem has a number of peculiarities differentiating it from those analyzed in Chapters I and II. First of all, we are studying the motion of a vehicle in different phase spaces, as a result of which functions H_m ($m = 1, \ldots, 6$) have first order discontinuities due to the conversions of the phase coordinates. Therefore, the phase variables are analyzed in the class of piecewise continuous functions. Secondly, within interval (t_0, t_k), certain boundary conditions are fixed in the form of conditions of contact with the sphere of influence of the earth and the target planet and equations for conversion of the phase coordinates.

Due to these specifics, this variational problem differs from the other variational problems studied in the scientific and technical literature (see bibliography in [12].

Let us compose the expression

$$\Phi = I + e_1^{(0)}\varphi_1^{(0)} + \sum_{i=1}^{n} e_1^{(i+1)}\varphi_1^{(i+1)} + \sum_{i=1}^{n} e_2^{(i)}\varphi_2^{(i)} + \sum_{i=1}^{n}\sum_{j=1}^{4} e_{0j}^{(i)}\varphi_{0j}^{(i)} +$$

$$+ e_{05}\varphi_{05} + \sum_{Q=1}^{6} e_Q\psi_{0Q} + \sum_{\varkappa=1}^{m} e_{\kappa\varkappa}\psi_{\kappa\varkappa} + \sum_{i=1}^{n} e_{\mu}^{(i)}\psi_{\mu}^{(i)} + e_{\Delta 1}\psi_{\Delta 1} +$$

$$+ e_{\Delta 2}\psi_{\Delta 2} + \sum_{s=1}^{6} l_s\Pi_s + \sum_{s=1}^{6} \bar{l}_s\bar{\Pi}_s + \int_{t_0}^{t_k^{\oplus}} F_{\oplus}dt + \int_{t_0^{\odot}}^{t_k^{\odot}} F_{\odot}dt + \int_{t_0^{pl}}^{t_k^{pl}} F_{pl}dt, \quad (3.1.28)$$

where

$$F_{\oplus} = \sum_{m=1}^{9} \varphi_m^{\oplus}\lambda_m, \quad F_{\odot} = \sum_{m=1}^{9} \varphi_m^{\odot}\Lambda_m, \quad F_{pl} = \sum_{m=1}^{9} \varphi_m^{pl}\bar{\lambda}_m;$$

λ_m, Λ_m, $\bar{\lambda}_m$ are the variable Lagrange factors;

$e_1^{(0)}$, $e_1^{(i)}$, ..., $\bar{l}_s$ are the constant Lagrange factors.

Expression (3.1.28) defines $\Phi = I$. Therefore, according to the appendix we can state that the conditional extreme (maximum) of criterion of effectiveness I corresponds to the unconditional extreme (maximum) of functional Φ.

Let us now go over to determination of the necessary conditions of the maximum of functional Φ.

§ 2. Necessary Conditions for Optimization of Main Plan Parameters and Control (Condition of Stability, Weierstrass Condition and Maximum Principle Considering Control Condition)

In this section, we will study the necessary conditions for optimization of the main plan parameters and control, in which we include the condition of stability, Weierstrass condition and maximum principle, studied considering the control condition.

The condition of stability includes the Euler-Lagrange equation, discontinuity condition, condition of conversion of Lagrange coefficients at points of contact with the sphere of influence of the earth and the target planet and conditions of transversality. All of these conditions can be found from the equation

$$
\begin{aligned}
d\Phi = {} & dI + e_1^{(0)} d\beta_1^{(0)} + \sum_{i=1}^{n} e_1^{(i+1)} d\beta_1^{(i+1)} + \sum_{i=1}^{n} e_2^{(i)} d\beta_2^{(i)} + \\
& + \sum_{i=1}^{n} \sum_{j=1}^{4} e_{0j}^{(i)} d\beta_{0j}^{(i)} + e_{05} d\beta_{05} + \sum_{\varrho=1}^{6} e_\varrho d\psi_{0\varrho} + \sum_{\sigma=1}^{m} e_{\kappa\sigma} d\psi_{\kappa\sigma} + \sum_{i=1}^{n} e_\mu^{(i)} d\psi_\mu^{(i)} + \\
& + e_{\Delta 1} d\psi_{\Delta 1} + e_{\Delta 2} d\psi_{\Delta 2} + \sum_{s=1}^{6} l_s d\Pi_s + \sum_{s=1}^{6} \bar{l}_s d\bar{\Pi}_s - \Big(-H^{\oplus} dt + \\
& + \sum_{m=1}^{7} \lambda_m d\xi_m \Big)_{t_0} + \int_{t_0}^{t_\kappa^{\oplus}} \sum_{m=1}^{7} \left(\lambda_m' + \frac{\partial H^{\oplus}}{\partial \xi_m} \right) \Delta\xi_m dt + \int_{t_0}^{t_\kappa^{\oplus}} \sum_{p=1}^{4} \frac{\partial F_{\oplus}}{\partial u_p} \Delta u_p dt - \\
& - \int_{t_0}^{t_\kappa^{\oplus}} \sum_{i=1}^{n} \frac{\partial H^{\oplus}}{\partial M_{0i}} \Delta M_{0i} dt - \int_{t_0}^{t_\kappa^{\oplus}} \sum_{i=1}^{n} \frac{\partial H^{\oplus}}{\partial a_{0i}} \Delta a_{0i} dt - \int_{t_0}^{t_\kappa^{\oplus}} \sum_{i=1}^{n} \frac{\partial H^{\oplus}}{\partial W_{0i}^{\varrho}} \times \\
& \times \Delta W_{0i}^{\varrho} dt + \Big(-H^{\oplus} dt + \sum_{m=1}^{7} \lambda_m d\xi_m \Big)_{t_\kappa^{\oplus}} - \Big(-H^{\odot} dt + \\
& + \sum_{m=1}^{7} \Lambda_m d\Xi_m \Big)_{t_0^{\odot}} + \int_{t_0^{\odot}}^{t_\kappa^{\odot}} \sum_{m=1}^{7} \left(\Lambda_m' + \frac{\partial H^{\odot}}{\partial \Xi_m} \right) \Delta\Xi_m dt + \int_{t_0^{\odot}}^{t_\kappa^{\odot}} \sum_{p=1}^{4} \frac{\partial F_{\odot}}{\partial u_p} \Delta u_p \times
\end{aligned}
$$

$$\times dt - \int_{t_0^{\odot}}^{t_k^{\odot}} \sum_{i=1}^{n} \frac{\partial H^{\odot}}{\partial M_{0i}} \Delta M_{0i} dt - \int_{t_0^{\odot}}^{t_k^{\odot}} \sum_{i=1}^{n} \frac{\partial H^{\odot}}{\partial a_{0i}} \Delta a_{0i} dt - \int_{t_0^{\odot}}^{t_k^{\odot}} \sum_{i=1}^{n^{\circ}} \frac{\partial H^{\odot}}{\partial W_{0i}^{\circ}} \times$$

$$\times \Delta W_{0i}^{\circ} dt + \left(-H^{\odot} dt + \sum_{m=1}^{7} \Lambda_m d\Xi_m \right)_{t_k^{\odot}} - \left(-H^{pl} dt + \right.$$

$$\left. + \sum_{m=1}^{7} \bar{\lambda}_m d\bar{\xi}_m \right)_{t_0^{pl}} + \int_{t_0^{pl}}^{t_k} \sum_{m=1}^{7} \left(\bar{\lambda}'_m + \frac{\partial H^{pl}}{\partial \bar{\xi}_m} \right) \Delta \bar{\xi}_m dt + \int_{t_0^{pl}}^{t_k} \sum_{p=1}^{4} \frac{\partial F_{pl}}{\partial u_p} \times$$

$$\times \Delta u_p dt - \int_{t_0^{pl}}^{t_k} \sum_{i=1}^{n} \frac{\partial H^{pl}}{\partial M_{0i}} \Delta M_{0i} dt - \int_{t_0^{pl}}^{t_k} \sum_{i=1}^{n} \frac{\partial H^{pl}}{\partial a_{0i}} \Delta a_{0i} dt -$$

$$- \int_{t_0^{pl}}^{t_k} \sum_{i=1}^{n^{\xi}} \frac{\partial H^{pl}}{\partial W_{0i}^{\xi}} \Delta W_{0i}^{\xi} dt + \left(-H^{pl} dt + \sum_{m=1}^{7} \bar{\lambda}_m d\bar{\xi}_m \right)_{t_k} = 0, \qquad (3.2.1)$$

where

$$\xi_m = x,\ y,\ z,\ v_x,\ v_y,\ v_z,\ \mu;$$

$$\Xi = X,\ Y,\ Z,\ V_x,\ V_y,\ V_z,\ \mu;$$

$$\bar{\xi}_m = \bar{x},\ \bar{y},\ \bar{z},\ \bar{v}_x,\ \bar{v}_y,\ \bar{v}_z,\ \mu;$$

$$u_p = \alpha,\ \beta,\ \gamma,\ p;$$

$$H^{\odot} = \sum_{m=1}^{7} H_m^{\odot} \lambda_m, \quad H^{\odot} = \sum_{m=1}^{7} H_m^{\odot} \cdot \Lambda_m, \quad H^{pl} = \sum_{m=1}^{7} H_m^{pl} \bar{\lambda}_m.$$

Keeping in mind (3.1), the system of Euler-Lagrange equations according to condition (3.2.1) can be represented as follows:

$$\left.\begin{aligned} &\lambda'_1 = H_8 = -\lambda_4,\ \lambda'_2 = H_9 = -\lambda_5,\ \lambda'_3 = H_{10} = -\lambda_6, \\ &\lambda'_4 = H_{11} = -\frac{\partial H}{\partial x},\ \lambda'_5 = H_{12} = -\frac{\partial H}{\partial y},\ \lambda'_6 = H_{13} = -\frac{\partial H}{\partial z}, \end{aligned}\right\} \qquad (3.11)$$

$$\lambda_7' + \frac{a_N p}{\mu^2}(\lambda_1 \alpha + \lambda_2 \beta + \lambda_3 \gamma) = 0,$$

$$\lambda_1 \frac{a_N p}{\mu} - 2\alpha\lambda_8 = 0, \quad \lambda_2 \frac{a_N p}{\mu} - 2\beta\lambda_8 = 0, \quad \lambda_3 \frac{a_N p}{\mu} - 2\gamma\lambda_8 = 0,$$

$$\frac{a_N}{\mu}(\lambda_1 \alpha + \lambda_2 \beta + \lambda_3 \gamma) - \lambda_7 \frac{a_N}{W_a} - \lambda_9 (1 - 2p) = 0.$$

Here for brevity, we use λ_m, x, y and z to represent the Lagrange coefficients and phase variables in the various systems of coordinates.

The last five equations, after certain transformations, become

$$\lambda_7' - \lambda \frac{a_N p}{\mu^2} = 0, \tag{3.2.2}$$

$$\alpha = \frac{\lambda_1}{\lambda}, \quad \beta = \frac{\lambda_2}{\lambda}, \quad \gamma = \frac{\lambda_3}{\lambda}, \tag{3.2.3}$$

$$\lambda_p = \lambda \frac{1}{\mu} - \lambda_7 \frac{1}{W_a}, \tag{3.2.4}$$

$$\lambda = \pm\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}, \tag{3.2.5}$$

where when p = 1, $\lambda_p = -\frac{\lambda_9}{a_N}$: when p = 0, $\lambda_p = \frac{\lambda_9}{a_N}$. (3.2.6)

The Euler-Lagrange equations (3.2.3) are related to determination of the direction of the force of gravity, while (3.2.4) are related to its magnitude. However, keeping in mind the dependence of (3.2.5) and (3.2.6), these equations have no unambiguous solution which might be achieved by introducing additional necessary conditions.

Since H does not contain the functions H_m (m = 1,...6), which depend explicitly on time, system (3.II) has a first integral of the form

$$H^{(1)(i)} = -C_0^{(1)(i)}, \; H^{(2)(i)} = -C_0^{(2)(i)}, \; H^{(3)(i)} = -C^{(3)(i)}. \tag{3.2.7}$$

A spent stage of the spacecraft can be either switched off or also separated. At the moment of cutoff of a stage, the phase variables are continuous. In studying the conditions of stage separation, we will make a number of assumptions which will not reduce the accuracy of the solutions, determined by assumptions already made. As our main assumption, we will assume instantaneous separation of a spent stage, corresponding to the condition $t_-^{(i)} = t_+^{(i)}$, and will assume that separation of a stage causes no perturbation in motion of the spacecraft and does not change the orientation of the spacecraft. All of this allows us to consider the phase variables $x, \ldots, V_z$ continuous at the moment of stage separation.

Then, due to the independence of the variations of phase coordinates at the moment of separation (cutoff) of a stage, their coefficients should be equal to zero. Therefore we produce

$$[C_0^{(i)}]_- = [C_0^{(i)}]_+,$$
$$\lambda_{1-}^{(i)} = \lambda_{1+}^{(i)}, \ldots, \lambda_{6-}^{(i)} = \lambda_{6+}^{(i)}. \qquad (3.2.8)$$

These equations indicate that at points of separation (cutoff) of stages, continuity should be maintained for the Lagrange coefficients $\lambda_1, \ldots, \lambda_6$ and first integral C_0 should be maintained constant. Lagrange coefficient λ_7 may undergo first order discontinuities at points of staging, and its values at the right of the point of discontinuity can be determined from (3.2.7) according to (3.2.8). Conditions (3.2.8) in the following will be referred to as the discontinuity conditions of the Lagrange coefficients upon stage separation.

Furthermore, stability condition (3.2.1) indicates that first order discontinuities of the control functions do not disrupt continuity of the Lagrange coefficients $\lambda_1, \ldots, \lambda_7$ or constancy of first integral C_0.

In order to fulfill (3.2.1), we select Lagrange factors $e_{\Delta 1}$, $\bar{e}_\Delta$, l_s, $\bar{l}_s$ $(s = 1, \ldots, 6)$ such that the coefficients in the coupled variations of the phase coordinates at the spheres of influence of the earth and the target planet are equal to zero. The coefficients with free variations should also be equal to zero. Then, when the geocentric system of coordinates is converted to a heliocentric system of coordinates (at the sphere of influence of the earth) at point $t = t_k^+$, we produce

$$\lambda_{1\kappa}+l_4\frac{\partial\Pi_4}{\partial v_{x\kappa}}=0,\quad \lambda_{2\kappa}+l_5\frac{\partial\Pi_5}{\partial v_{y\kappa}}+l_6\frac{\partial\Pi_6}{\partial v_{y\kappa}}=0,$$

$$\lambda_{3\kappa}+l_5\frac{\partial\Pi_5}{\partial v_{z\kappa}}+l_6\frac{\partial\Pi_6}{\partial v_{z\kappa}}=0,$$

$$\lambda_{4\kappa}+e_{s1}\frac{\partial\vartheta_{\Delta 1}}{\partial x_\kappa}+l_1\frac{\partial\Pi_1}{\partial x_\kappa}=0,\quad \lambda_{5\kappa}+e_{s1}\frac{\partial\vartheta_{\Delta 1}}{\partial y_\kappa}+l_2\frac{\partial\Pi_2}{\partial y_\kappa}+l_3\frac{\partial\Pi_3}{\partial y_\kappa}=0,$$

$$\lambda_{6\kappa}+e_{s1}\frac{\partial\vartheta_{s1}}{\partial z_\kappa}+l_2\frac{\partial\Pi_2}{\partial z_\kappa}+l_3\frac{\partial\Pi_3}{\partial z_\kappa}=0,$$

$$\Lambda_{10}-l_4\frac{\partial\Pi_4}{\partial V_{x0}}=0,\quad \Lambda_{20}-l_5\frac{\partial\Pi_5}{\partial V_{y0}}=0,\quad \Lambda_{30}-l_6\frac{\partial\Pi_6}{\partial V_{z0}}=0,$$

$$\Lambda_{40}-l_1\frac{\partial\Pi_1}{\partial X_0}=0,\quad \Lambda_{50}-l_2\frac{\partial\Pi_2}{\partial Y_0}=0,\quad \Lambda_{60}-l_3\frac{\partial\Pi_3}{\partial Z_0}=0,$$

$$\lambda_{7\kappa}=\Lambda_{70},$$

$$C_0^{\oplus}-C_0^{\odot}+\sum_{s=1}^{6} l_s\frac{\partial\Pi_s}{\partial t_\kappa^{\oplus}}=0,$$

while when the heliocentric system of coordinates is converted to the planetocentric system of c-ordinates at point $t = t_\kappa^{\odot}$ we find

$$\Lambda_{1\kappa}+\bar{l}_4\frac{\partial\bar{\Pi}_4}{\partial V_{x\kappa}}=0,\quad \Lambda_{2\kappa}+\bar{l}_5\frac{\partial\bar{\Pi}_5}{\partial V_{y\kappa}}+\bar{l}_6\frac{\partial\bar{\Pi}_6}{\partial V_{y\kappa}}=0,$$

$$\Lambda_{3\kappa}+\bar{l}_5\frac{\partial\bar{\Pi}_5}{\partial V_{z\kappa}}+\bar{l}_6\frac{\partial\bar{\Pi}_6}{\partial V_{z\kappa}}=0,$$

$$\Lambda_{4\kappa}+e_{s2}\frac{\partial\vartheta_{s2}}{\partial X_\kappa}+\bar{l}_1\frac{\partial\bar{\Pi}_1}{\partial X_\kappa}=0,$$

$$\Lambda_{5\kappa}+e_{s2}\frac{\partial\vartheta_{s2}}{\partial Y_\kappa}+\bar{l}_2\frac{\partial\bar{\Pi}_2}{\partial Y_\kappa}+\bar{l}_3\frac{\partial\bar{\Pi}_3}{\partial Y_\kappa}=0,$$

$$\Lambda_{6\kappa}+e_{s2}\frac{\partial\vartheta_{s2}}{\partial Z_\kappa}+\bar{l}_2\frac{\partial\bar{\Pi}_2}{\partial Z_\kappa}+\bar{l}_3\frac{\partial\bar{\Pi}_3}{\partial Z_\kappa}=0,$$

$$\bar{\lambda}_{10}-\bar{l}_4\frac{\partial\bar{\Pi}_4}{\partial\bar{v}_{x0}}=0,\quad \bar{\lambda}_{20}-\bar{l}_5\frac{\partial\bar{\Pi}_5}{\partial\bar{v}_{y0}}=0,\quad \bar{\lambda}_{30}-\bar{l}_6\frac{\partial\bar{\Pi}_6}{\partial\bar{v}_{z0}}=0,$$

$$\bar{\lambda}_{40}-\bar{l}_1\frac{\partial\bar{\Pi}_1}{\partial\bar{x}_0}=0,\quad \bar{\lambda}_{50}-\bar{l}_2\frac{\partial\bar{\Pi}_2}{\partial\bar{y}_0}=0,\quad \bar{\lambda}_{60}-\bar{l}_3\frac{\partial\bar{\Pi}_3}{\partial\bar{z}_0}=0,$$

$$\Lambda_{7\kappa}=\bar{\lambda}_{70},$$

$$C_0^{\odot}-C_0^{\sigma l}+\sum_{s=1}^{6}\bar{l}_s\frac{\partial\bar{\Pi}_s}{\partial t_\kappa^{\odot}}+e_{s_1}\frac{\partial\vartheta_{\Delta_1}}{\partial t_\kappa^{\odot}}=0.$$

After excluding the constant Lagrange factors $c_{\Delta 1}$, $c_{\Delta 2}$, l_s and $\bar{l}_s$ ($s = 1, \ldots, 6$) from these relationships, we produce the following conditions for conversion of the Lagrange coefficients and constant C_0:

<u>where $t = t_k^{\oplus}$</u>

$$\left.\begin{aligned}
&\Lambda_{10} = \lambda_{1\kappa},\\
&\Lambda_{20} = \lambda_{2\kappa}\cos\varepsilon + \lambda_{3\kappa}\sin\varepsilon,\\
&\Lambda_{30} = -\lambda_{2\kappa}\sin\varepsilon + \lambda_{3\kappa}\cos\varepsilon,\\
&\Lambda_{50} = \lambda_{5\kappa}\cos\varepsilon + \lambda_{6\kappa}\sin\varepsilon + \frac{\Lambda_{40} - \lambda_{4\kappa}}{x_\kappa}(y_\kappa\cos\varepsilon + z_\kappa\sin\varepsilon),\\
&\Lambda_{60} = -\lambda_{5\kappa}\sin\varepsilon + \lambda_{6\kappa}\cos\varepsilon + \frac{\Lambda_{40} - \lambda_{4\kappa}}{x_\kappa}(-y_\kappa\sin\varepsilon + z_\kappa\cos\varepsilon),\\
&\Lambda_{70} = \lambda_{7\kappa},\\
&\psi_{11} = C_0^{\odot} - C_0^{\circ} - \Lambda_{10}A_x^{\circ} - \Lambda_{20}A_y^{\bullet} - \Lambda_{40}V_{x\kappa}^{\bullet} - \Lambda_{50}V_{y\kappa}^{\bullet} = 0;
\end{aligned}\right\} \qquad (3.2.9)$$

<u>where $t = t_k^{\odot}$</u>

$$\left.\begin{aligned}
&\bar{\lambda}_{10} = \Lambda_{1\kappa},\\
&\bar{\lambda}_{20} = \Lambda_{2\kappa}\cos\bar{\varepsilon} - \Lambda_{3\kappa}\sin\bar{\varepsilon},\\
&\bar{\lambda}_{30} = \Lambda_{2\kappa}\sin\bar{\varepsilon} + \Lambda_{3\kappa}\cos\bar{\varepsilon},\\
&\bar{\lambda}_{50} = \Lambda_{5\kappa}\cos\bar{\varepsilon} - \Lambda_{6\kappa}\sin\bar{\varepsilon} + \frac{\bar{\lambda}_{40} - \Lambda_{4\kappa}}{X_\kappa - X_\kappa^{pd}}\left[(Y_\kappa - Y_\kappa^{pd})\times\right.\\
&\left.\times\cos\bar{\varepsilon} - (Z_\kappa - Z_\kappa^{pd})\sin\bar{\varepsilon}\right],\\
&\bar{\lambda}_{60} = \Lambda_{5\kappa}\sin\bar{\varepsilon} + \Lambda_{6\kappa}\cos\bar{\varepsilon} + \frac{\bar{\lambda}_{40} - \Lambda_{4\kappa}}{X_\kappa - X_\kappa^{pd}}\left[(Y_\kappa - Y_\kappa^{pd})\sin\bar{\varepsilon} +\right.\\
&\left.+(Z_\kappa - Z_\kappa^{pd}\cos\bar{\varepsilon})\right], \quad \bar{\lambda}_{70} = \Lambda_{7\kappa},\\
&\psi_{12} = C_0^{pd} - C_0^{\odot} - \frac{\bar{\lambda}_{40} - \Lambda_{4\kappa}}{X_\kappa - X_\kappa^{pd}}\left[(X_\kappa - X_\kappa^{pd})V_{x\kappa}^{pd} +\right.\\
&\left.+(Y_\kappa - Y_\kappa^{pd})\times V_{y\kappa}^{pd} + (Z_\kappa - Z_\kappa^{pd})V_{z\kappa}^{pd}\right] + \bar{\lambda}_{10}A_x^{pd} +\\
&+\bar{\lambda}_{20}A_y^{pd} + \bar{\lambda}_{30}A_z^{pd} + \bar{\lambda}_{40}V_{x\kappa}^{pd} + \bar{\lambda}_{50}V_{y\kappa}^{pd} + \bar{\lambda}_{60}V_{z\kappa}^{pd},
\end{aligned}\right\} \qquad (3.2.10)$$

where

$$A_x^{\odot}=\frac{\partial V_x^{\odot}}{\partial t_k^{\odot}},\quad A_y^{\odot}=\frac{\partial V_y^{\odot}}{\partial t_k^{\odot}}.$$

$$A_x^{pl}=\frac{\partial V_x^{pl}}{\partial t_k^{\odot}},\quad A_y^{pl}=\frac{\partial V_y^{pl}}{\partial t_k^{\odot}},\quad A_z^{pl}=\frac{\partial V_z^{pl}}{\partial t_k^{\odot}}.$$

The conditions of conversion of the Lagrange coefficients and constant C_0 (3.2.9) and (3.2.10) allow us to determine the Lagrange coefficients at point $t_k^{\oplus}$ in the heliocentric ecliptical system of coordinates, and at point $t_k^{\odot}$ in the planetocentric equatorial system of coordinates. They show that when the system of coordinates is converted at the moment of passage through the sphere of influence of the earth and the sphere of influence of the planet, the Lagrange coefficients $\lambda_2, \ldots, \lambda_6$ and integration constant C_0 may undergo first order discontinuities, while the Lagrange coefficients λ_1 and λ_7 are continuous. Furthermore, we should note that at points $t_0^{\odot}$ and t_0^{pl}, the values of some one of the Lagrange coefficients Λ_{mo} and $\bar{\lambda}_{mo}$ ($m = 1, \ldots, 6$), for example Λ_{40} and $\bar{\lambda}_{40}$, is not determined and may take on an arbitrary value. This fact will help in the solution of the boundary problem.

Furthermore, for fulfillment of the stability condition (3.2.1) we select the factors $e_1^{(0)}$, $e_1^{(i+1)}$, etc. so that the coefficients with the variations of certain parameters are equal to zero, while with others they are equal to zero due to the independence of the variations of these parameters.

We will then have

$$\frac{\partial I}{\partial M_{n.d}}+e_1^{(0)}\frac{\partial \varphi_1^{(0)}}{\partial M_{n.d}}+e_{05}\frac{\partial \varphi_{05}}{\partial M_{n.d}}=0, \qquad (3.2.11)$$

$$\frac{\partial I}{\partial M_0}+e_1^{(0)}\frac{\partial \varphi_1^{(0)}}{\partial M_0}+e_1^{(i+1)}\frac{\partial \varphi_1^{(i+1)}}{\partial M_0}+e_2^{(1)}\frac{\partial \varphi_2^{(1)}}{\partial M_0}+e_{01}^{(1)}\frac{\partial \varphi_{01}^{(1)}}{\partial M_0}=0$$
$$(i=1,\ldots,n), \qquad (3.2.12)$$

$$
\left.
\begin{aligned}
&e_1^{(0)} \frac{\partial \varphi_1^{(0)}}{\partial a_{0i}} + e_1^{(i)} \frac{\partial \varphi_1^{(i)}}{\partial a_{0i}} + e_2^{(i)} \frac{\partial \varphi_2^{(i)}}{\partial a_{0i}} + e_{02}^{(i)} \frac{\partial \varphi_{02}^{(i)}}{\partial a_{0i}} - \int_{t_0}^{t_\kappa^{\oplus}} \frac{\partial H^{\oplus}}{\partial a_{0i}}\, dt - \\
&- \int_{t_0^{\odot}}^{t_\kappa^{\odot}} \frac{\partial H^{\odot}}{\partial a_{0i}}\, dt - \int_{t_0^{pl}}^{t_\kappa} \frac{\partial H^{pl}}{\partial a_{0i}}\, dt = 0, \\
&e_1^{(0)} \frac{\partial \varphi_1^{(0)}}{\partial \mu_{\kappa J}} + e_1^{(i)} \frac{\partial \varphi_1^{(i)}}{\partial \mu_{\kappa J}} + \lambda_{7\kappa}^{(i)} \frac{\partial \psi_{\mu}^{(i)}}{\partial \mu_{\kappa J}} = 0 \quad (J = 2, \ldots, N), \\
&e_1^{(0)} \frac{\partial \varphi_1^{(0)}}{\partial W_{0i}^{\mu}} + e_1^{(i)} \frac{\partial \varphi_1^{(i)}}{\partial W_{0i}^{\mu}} + e_{03}^{(i)} \frac{\partial \varphi_{03}^{(i)}}{\partial W_{0i}^{\kappa}} - \int_{t_0}^{t_\kappa^{\oplus}} \frac{\partial H^{\oplus}}{\partial W_{0i}^{\mu}}\, dt - \\
&- \int_{t_0^{\odot}}^{t_\kappa^{\odot}} \frac{\partial H^{\odot}}{\partial W_{0i}^{\mu}}\, dt - \int_{t_0^{pl}}^{t_\kappa} \frac{\partial H^{pl}}{\partial W_{0i}^{\mu}}\, dt = 0, \\
&\qquad e_2^{(i)} \frac{\partial \varphi_2^{(i)}}{\partial P_{\max i}} + e_{04}^{(i)} \frac{\partial \varphi_{04}^{(i)}}{\partial P_{\max i}} = 0, \\
&\qquad \lambda_{7\kappa}^{(i)} + e_{\mu}^{(i)} = 0,
\end{aligned}
\right\} \quad (3.2.13)
$$

$$
\left.
\begin{aligned}
&e_{01}^{(i)} w_{1i} = 0,\ e_{02}^{(i)} w_{2i} = 0,\ e_{03}^{(i)} w_{3i} = 0, \\
&e_{04}^{(i)} w_{4i} = 0,\ e_{05} w_5 = 0.
\end{aligned}
\right\} \quad (3.2.14)
$$

If the optimal value of the main plan parameter reaches the boundary, the corresponding $e_{0j}^{(i)} \neq 0$, which follows from (3.2.14). Otherwise, $e_{0j}^{(i)} = 0$. In this sense, equation (3.2.14) allows the limiting values of the main plan parameters to be separated from the optimal values located within the area of limitations.

Excluding the constant Lagrange factors $e_1^{(0)}$, $e_1^{(i+1)}$, $e_2^{(i)}$ from (3.2.11)-(3.2.13), we produce the conditions of optimality of the main plan parameters. We will not present the calculations here, since we have not defined concretely the number of stages and the type of power plant in each stage.

Due to the mutual independence of the phase variables of the intermediate orbit around the earth and the phase variables of the final conditions, from (3.2.1) we find

$$C_0^- dt_0 + \lambda_{10} dv_{x0} + \lambda_{20} dv_{y0} + \lambda_{30} dv_{z0} + \lambda_{40} dx_0 + \\ + \lambda_{50} dy_0 + \lambda_{60} dz_0 - \sum_{q=1}^{6} e_q d\xi_{0q} = 0, \tag{3.2.15}$$

$$C_0^{nl} dt_k + \bar{\lambda}_{1k} d\bar{v}_{xk} + \bar{\lambda}_{2k} d\bar{v}_{yk} + \bar{\lambda}_{3k} d\bar{v}_{zk} + \bar{\lambda}_{4k} d\bar{x}_k + \\ + \bar{\lambda}_{5k} d\bar{y}_k + \bar{\lambda}_{6k} d\bar{z}_k + \sum_{s=1}^{m} e_{ks} d\xi_{ks} = 0. \tag{3.2.16}$$

Considering the initial conditions (3.1.5) and the fixation of t_0, from (3.2.15) we produce

$$\lambda_{10} - e_1 = 0, \; \lambda_{20} - e_2 = 0, \; \lambda_{30} - e_3 = 0,$$
$$\lambda_{40} - e_4 = 0, \; \lambda_{50} - e_5 = \;, \; \lambda_{60} - e_6 = 0,$$
$$(e_4\vartheta_1 + e_5\vartheta_2 + e_6\vartheta_3) \frac{\partial r_0}{\partial u_0} + r_0 \left(e_4 \frac{\partial \theta_1}{\partial u_0} + e_2 \frac{\partial \theta_2}{\partial u_0} + e_3 \frac{\partial \vartheta_3}{\partial u_0} \right) + \\ + \frac{c}{p} \Big[(e_1\theta_1 + e_2\theta_2 + e_3\theta_3) e \cos(u_0 - \omega) + \\ + \left(e_1 \frac{\partial^2 \theta_1}{\partial u_0^2} + e_2 \frac{\partial^2 \theta_2}{\partial u_0^2} + e_3 \frac{\partial^2 \theta_3}{\partial u_0^2} \right) (1 + e\cos(u_0 - \omega) \Big].$$

From this, we will have the following condition of transversality t_0:

$$\varphi_{07} = \frac{c}{p} \Big\{ \lambda_{10} \Big[\theta_1 e \cos(u_0 - \omega) + \frac{\partial^2 \theta_1}{\partial u_0^2} (1 + e\cos(u_0 - \omega)) \Big] + \\ + \lambda_{20} \Big[\theta_2 e \cos(u_0 - \omega) + \frac{\partial^2 \theta_2}{\partial u_0^2} (1 + e\cos(u_0 - \omega)) \Big] + \\ + \lambda_{30} \Big[\vartheta_3 e \cos(u_0 - \omega) + \frac{\partial^2 \theta_3}{\partial u_0^2} (1 + e\cos(u_0 - \omega)) \Big] \Big\} + \lambda_{40} \left(\vartheta_1 \frac{\partial r_0}{\partial u_0} + \\ + r_0 \frac{\partial \theta_1}{\partial u_0} \right) + \lambda_{50} \left(\vartheta_2 \frac{\partial r_0}{\partial u_0} + r_0 \frac{\partial \theta_2}{\partial u_0} \right) + \lambda_{60} \left(\vartheta_3 \frac{\partial r_0}{\partial u_0} + r_0 \frac{\partial \theta_3}{\partial u_0} \right) = 0. \tag{3.2.17}$$

Considering (3.2.16) and the final conditions (3.1.6), we produce the following relationship at point t_k:

$$\left.\begin{aligned}
&C_6^{d}+\sum_{\sigma=1}^{m} e_{\kappa\sigma}\frac{\partial\psi_{\kappa\sigma}}{\partial M_\kappa}=0,\\
&\bar{\lambda}_{1\kappa}+\sum_{\sigma=1}^{m} e_{\kappa\sigma}\frac{\partial\psi_{\kappa\sigma}}{\partial\bar{v}_{r\kappa}}=0, \quad \bar{\lambda}_{2\kappa}+\sum_{\sigma=1}^{m} e_{\kappa\sigma}\frac{\partial\psi_{\kappa\sigma}}{\partial\bar{v}_{y\kappa}}=0,\\
&\bar{\lambda}_{3\kappa}+\sum_{\sigma=1}^{m} e_{\kappa\sigma}\frac{\partial\psi_{\kappa\sigma}}{\partial\bar{v}_{z\kappa}}=0, \quad \bar{\lambda}_{4\kappa}+\sum_{\sigma=1}^{m} e_{\kappa\sigma}\frac{\partial\psi_{\kappa\sigma}}{\partial\bar{x}_\kappa}=0,\\
&\bar{\lambda}_{5\kappa}+\sum_{\sigma=1}^{m} e_{\kappa\sigma}\frac{\partial\psi_{\kappa\sigma}}{\partial\bar{y}_\kappa}=0, \quad \bar{\lambda}_{6\kappa}+\sum_{\sigma=1}^{m} e_{\kappa\sigma}\frac{\partial\psi_{\kappa\sigma}}{\partial\bar{z}_\kappa}=0,
\end{aligned}\right\} \qquad (3.2.18)$$

excluding the constant Lagrange coefficients $e_{k\sigma}$ from this relationship, we produce the corresponding conditions of transversality at the end point.

Note. If an orbit around the target planet is the final condition, the conditions of transversality at point t_k are as follows:

$$\begin{aligned}
\psi_{\kappa 7} \equiv \frac{\bar{e}}{\bar{p}}\Big\{&\bar{\lambda}_{1\kappa}\Big[\bar{\theta}_1\bar{e}\cos(\bar{u}_\kappa-\bar{\omega})+\frac{\partial^2\bar{\theta}_1}{\partial\bar{u}_\kappa^2}(1+\bar{e}\cos(\bar{u}_\kappa-\bar{\omega}))\Big]+\\
&+\bar{\lambda}_{2\kappa}\Big[\bar{\theta}_2\bar{e}\cos(\bar{u}_\kappa-\bar{\omega})+\frac{\partial^2\bar{\theta}_2}{\partial\bar{u}_\kappa^2}(1+\bar{e}\cos(\bar{u}_\kappa-\bar{\omega}))\Big]+\\
&+\bar{\lambda}_{3\kappa}\Big[\bar{\theta}_3\bar{e}\cos(\bar{u}_\kappa-\bar{\omega})+\frac{\partial^2\bar{\theta}_3}{\partial\bar{u}_\kappa^2}(1+\bar{e}\cos(\bar{u}_\kappa-\bar{\omega}))\Big]\Big\}+\\
&+\bar{\lambda}_{4\kappa}\Big(\bar{\theta}_1\frac{\partial\varrho_\kappa}{\partial\bar{u}_\kappa}+\varrho_\kappa\frac{\partial\bar{\theta}_1}{\partial\bar{u}_\kappa}\Big)+\bar{\lambda}_{5\kappa}\Big(\bar{\theta}_2\frac{\partial\varrho_\kappa}{\partial\bar{u}_\kappa}+\varrho_\kappa\frac{\partial\bar{\theta}_2}{\partial\bar{u}_\kappa}\Big)+\\
&+\lambda_{6\kappa}\Big(\bar{\theta}_3\frac{\partial\varrho_\kappa}{\partial\bar{u}_\kappa}+\varrho_\kappa\frac{\partial\bar{\theta}_3}{\partial\bar{u}_\kappa}\Big)=0.
\end{aligned} \qquad (3.2.18N)$$

Conditions of transversality (3.2.17) and (3.2.18) are the final necessary conditions for the maximum of the criterion of effectiveness I, following from stability condition (3.2.1).

We noted above [see formulas (3.2.5) and (3.2.6)] that the condition of stability gives no unambiguous answer concerning the sign of the Lagrange coefficients λ and λ_p. This is caused by the fact that the conditions of stability give a broad class of extremes which are maximizing, minimizing or saddle curves. Therefore, we must have additional necessary conditions to be certain that functional I has reached its maximum value. The Weierstrass condition is such a condition, and it has the following form for this problem (see appendix):

$$H \leqslant \tilde{H}, \tag{3.2.19}$$

whre $\tilde{H}$ corresponds to permissible control.

Then, considering (3.2.3), we find the following for the control functions α(t), β(5) and γ(t):

$$\lambda[1-(\alpha\tilde{\alpha}+\beta\tilde{\beta}+\gamma\tilde{\gamma})] \leqslant 0,$$

which is possible only where

$$\lambda \leqslant 0.$$

Thus, in order to satisfy the Weierstrass condition, Lagrange coefficient λ must be nonpositive, so that equation (3.2.5) should be written in the form

$$\lambda = -\sqrt{\lambda_1^2+\lambda_2^2-\lambda_3^2} \tag{3.2.20}$$

Since there are only two permissible modes of operation of the power plant ($p = p_{max}$ and $p = 0$), in order to determine the optimal operating mode of the power plant we must turn to the control condition (see Chapter I, § 3) which in this case, following the Weierstrass condition (3.2.19), will correspond formally with the maximum (minimum) principle. With the maximum thrust mode, according to this condition, we should have

$$\frac{\lambda}{\mu} - \frac{\lambda_7}{W_{ef}} \leqslant 0 \quad \text{or} \quad \lambda_p \leqslant 0. \tag{3.2.21}$$

Otherwise, passive flight of the spacecraft should be realized.

Thus, in order to satisfy the minimum principle and the control condition where $\lambda_p < 0$, the optimal mode of operation of the power plant must be the maximum thrust mode, while where $\lambda_p > 0$, the optimal mode of motion is passive flight. Switching of the power plant either off or on ($p = p_{max}$) should be performed where $\lambda_p = 0$.

Fulfillment of the necessary conditions (3.2.20) and (3.2.21) allows the production of optimal control modes $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $p(t)$, corresponding to attainment of the maximum value of I.

Determination of the condition of stability [including the Euler-Lagrange equations (3.II) and the condition of optimization of the main plan parameters (3.2.11)-(3.2.13), condition of conversion of the Lagrange coefficients and quantity C_0 (3.2.9) and (3.2.10), conditions of transversality (3.2.17) and (3.2.18)], Weierstrass condition (3.2.20) and maximum principle (3.2.21) results in the production of all first order necessary conditions. In this sense, the variational problem stated in § 1 is solved.

§ 3. Computational Algorithm for Variational Method of Optimization of Main Plan Parameters and Modes of Motion

If the solution produced in § 2 for the variational problem formulated in § 1 is to be used to determine the concrete values of optimal main plan parameters and optimal modes of motion of the spacecraft, we must perform numerical integration of system (3.I)-(3.II), satisfying the conditions for arrival at the sphere of influence of the earth and target planet (3.1.7) and (3.1.9) at points $t_k^{\oplus}$ and $t_k^{\odot}$, satisfying conditions (3.1.21) at points $t^{(i)}$ and satisfying the final conditions (3.1.6), the conditions of optimization of the main plan parameters (3.2.11)-(3.2.13) and the conditions for transversality (3.2.18) at point t_k. However, the values of the main plan parameters M_{0I} or M_{pl}, for example M_{pl} and $M_{0II}, \ldots, M_{0N}$, a_{0i}, μ_{kJ} and W_{0i}^e are unknown, the argument of the lattitude of the launch point u_0 has not been determined, the values of the Lagrange coefficients $\lambda_{10}, \ldots, \lambda_{60}$, λ_{70} or $C_0^{\oplus}$ have not been defined at point t_0 and the values of the Lagrange coefficients Λ_{40} and $\overline{\lambda}_{40}$ have not been fixed at points $t_0^{\odot}$ and t_0^{pl}.

Therefore, concrete solution of the variational problem is reduced to solution of a multipoint boundary problem. Numerical integration of the variational problem is therefore solution of the computational problem of the multipoint boundary problem, formulated as follows.

We must determine values of the main plan parameters M_{0J} (without M_{0I}), M_{pl}, a_{0i}, μ_{kJ} and W_{0i}^e, the argument of the lattitude of the launch point u_0, Lagrange coefficients $\lambda_{10}, \ldots, \lambda_{60}, \Lambda_{40}, \bar{\lambda}_{40}$ and C_0^+ for which the numerical integration of the system

over $[t_0, t_k^+]$

$$\left.\begin{array}{l}
\dot{v}_x = H_1^{\oplus},\ \dot{v}_y = H_2^{\oplus},\ \dot{v}_z = H_3^{\oplus} \\
\dot{x} = H_4^{\oplus},\ \dot{y} = H_5^{\oplus},\ \dot{z} = H_6^{\oplus},\ \dot{\mu} = H_7^{\oplus}, \\
\dot{\lambda}_1 = H_8^{\oplus},\ \dot{\lambda}_2 = H_9^{\oplus},\ \dot{\lambda}_3 = H_{10}^{\oplus}, \\
\dot{\lambda}_4 = H_{11}^{\oplus},\ \dot{\lambda}_5 = H_{12}^{\oplus},\ \dot{\lambda}_6 = H_{13}^{\oplus}, \\
\dot{C}_0^{\oplus} = H_{14}^{\oplus} = 0,\ \dot{\eta}_1^{(i)} = H_{15}^{\oplus(i)},\ \dot{\eta}_2^{(i)} = H_{16}^{\oplus(i)};
\end{array}\right| \qquad (3.III)$$

over $[t_0^{\odot}, t_k^{\odot}]$

$$\left.\begin{array}{l}
\dot{V}_x = H_1^{\odot},\ \dot{V}_y = H_2^{\odot},\ \dot{V}_z = H_3^{\odot}, \\
\dot{X} = H_4^{\odot},\ \dot{Y} = H_5^{\odot},\ \dot{Z} = H_6^{\odot},\ \dot{\mu} = H_7^{\odot}, \\
\dot{\Lambda}_1 = H_8^{\odot},\ \dot{\Lambda}_2 = H_9^{\odot},\ \dot{\Lambda}_3 = H_{10}^{\odot}, \\
\dot{\Lambda}_4 = H_{11}^{\odot},\ \dot{\Lambda}_5 = H_{12}^{\odot},\ \dot{\Lambda}_6 = H_{13}^{\odot},\ \dot{C}^{\odot} = H_{14}^{\odot} = 0, \\
\dot{\eta}_1^{\odot(i)} = H_{15}^{\odot(i)},\ \dot{\eta}_2^{\odot(i)} = H_{16}^{\odot(i)};
\end{array}\right| \qquad (3.III)$$

over $[t_0^{pl}, t_k^{pl}]$

$$\left.\begin{array}{l}
\dot{\bar{v}}_x = H_1^{pl},\ \dot{\bar{v}}_y = H_2^{pl},\ \dot{\bar{v}}_z = H_3^{pl}, \\
\dot{\bar{x}} = H_4^{pl},\ \dot{\bar{y}} = H_5^{pl},\ \dot{\bar{z}} = H_6^{pl},\ \dot{\mu} = H_7^{pl}, \\
\dot{\bar{\lambda}}_1 = H_8^{pl},\ \dot{\bar{\lambda}}_2 = H_9^{pl},\ \dot{\bar{\lambda}}_3 = H_{10}^{pl}, \\
\dot{\bar{\lambda}}_4 = H_{11}^{pl},\ \dot{\bar{\lambda}}_5 = H_{12}^{pl},\ \dot{\bar{\lambda}}_6 = H_{13}^{pl}, \\
\dot{C}_0^{pl} = H_{14}^{pl} = 0,\ \dot{\eta}_1^{pl(i)} = H_{15}^{pl(i)},\ \dot{\eta}_2^{pl(i)} = H_{16}^{pl(i)}
\end{array}\right| \qquad (3.III)$$

leads to satisfaction of the following conditions:

$$\gamma_1^{(0)}=0,\ \beta_1^{(l)}=0,\ \psi_{0\varrho}=0\ (\varrho=1,\dots,7), \tag{3.3.1}$$

$$\psi_{\mu}^{(l)}=0\ (l=1,\dots,n-1), \tag{3.3.2}$$

$$\psi_{\Delta 1}=0,\ \psi_{\Delta 2}=0,\ \psi_{f1}=0,\ \psi_{f2}=0, \tag{3.3.3}$$

$$\psi_{\kappa\sigma}=0\ (\sigma=1,\dots,7), \tag{3.3.4}$$

$$R_{\varkappa}=0\ (\varkappa=1,\dots,2n+n^{e}), \tag{3.3.5}$$

where n is the number of stages

n^e is the number of stages with ERM,

$$\eta_{1\kappa}^{(l)}=-\int_{t_0}^{t_\kappa^{\bullet}}\frac{\partial H^{\bullet}}{\partial a_{0l}}\,dt-\int_{t_0^{\circledcirc}}^{t_\kappa^{\circledcirc}}\frac{\partial H^{\circledcirc}}{\partial a_{0l}}\,dt-\int_{t_0^{pd}}^{t_\kappa}\frac{\partial H^{pd}}{\partial a_{0l}}\,dt,$$

$$\eta_{2\kappa}^{(l)}=-\int_{t_0}^{t_\kappa^{\bullet}}\frac{\partial H^{\bullet}}{\partial W_{0l}^{e}}\,dt-\int_{t_0^{\circledcirc}}^{t_\kappa^{\circledcirc}}\frac{\partial H^{\circledcirc}}{\partial W_{0l}^{e}}\,dt-\int_{t_0^{pd}}^{t_\kappa}\frac{\partial H^{pd}}{\partial W_{0l}^{e}}\,dt.$$

Here

$$H_1^{\bullet}=\frac{a_{0l}p}{\mu}\,\frac{\lambda_1}{\lambda}-\frac{k_{\oplus}x}{r^3},\quad H_2^{\bullet}=\frac{a_{0l}p}{\mu}\,\frac{\lambda_2}{\lambda}-k_{\oplus}\,\frac{y}{r^3},$$

$$H_3^{\bullet}=\frac{a_{0l}p}{\mu}\,\frac{\lambda_3}{\lambda}-k_{\oplus}\,\frac{z}{r^3},$$

$$H_4^{\bullet}=v_x,\ H_5^{\bullet}=v_y,\ H_6^{\bullet}=v_z,\ H_7^{\bullet}=-\frac{a_{0l}p}{W_{0l}},$$

$$H_8^{\bullet}=-\lambda_4,\ H_9^{\bullet}=-\lambda_5,\ H_{10}^{\bullet}=-\lambda_6,$$

$$H_{11}^{\bullet}=\frac{k_{\oplus}}{r^3}\left[\lambda_1\left(1-\frac{3x^2}{r^2}\right)-\lambda_2\,\frac{xy}{r^2}-\lambda_3\,\frac{xz}{r^2}\right],$$

$$H_{12}^{\bullet}=\frac{k_{\oplus}}{r^3}\left[\lambda_2\left(1-\frac{3y^2}{r^2}\right)-\lambda_1\,\frac{xy}{r^2}-\lambda_3\,\frac{yz}{r^2}\right],$$

$$H_{13}^{\oplus}=\frac{k_{\oplus}}{r^3}\left[\lambda_3\left(1-\frac{3z^2}{r^2}\right)-\lambda_2\,\frac{yz}{r^2}-\lambda_1\,\frac{xy}{r^2}\right],$$

(3.3.1)

$$H_{14}^{\oplus(l)}=-p\left(\frac{\lambda}{\mu}-\frac{\lambda_7}{W_{0l}}\right),\quad H_{15}^{\oplus(l)}=-\frac{a_{0l}p}{(W_{0l}^{\bullet})^2}\,\lambda_7,$$

(3.3.2)

$$H_1^{\odot}=\frac{a_{0l}p}{\mu}\,\frac{\Lambda_1}{\Delta}-k_{\odot}\,\frac{X}{R^3},\quad H_2^{\odot}=\frac{a_{0l}p}{\mu}\,\frac{\Lambda_2}{\Delta}-k_{\odot}\,\frac{Y}{R^3},$$

(3.3.3)

(3.3.4)

(3.3.5)

$$H_3^{\odot}=\frac{a_{0l}p}{\mu}\,\frac{\Lambda_3}{\Delta}-k_{\odot}\,\frac{Z}{R^3},\quad H_4^{\odot}=V_x,\quad H_5^{\odot}=V_y,$$

$$H_6^{\odot}=V_z,\quad H_7^{\odot}=-\frac{a_{0l}p}{W_{0l}},\ H_8^{\odot}=-\Lambda_4,\ H_9^{\odot}=-\Lambda_5,\ H_{10}^{\odot}=-\Lambda_6,$$

$$H_{11}^{\odot}=\frac{k_{\odot}}{R^3}\left[\Lambda_1\left(1-\frac{3X^2}{R^2}\right)-\Lambda_2\frac{XY}{R^2}-\Lambda_3\frac{XZ}{R^2}\right],$$

$$H_{12}^{\odot}=\frac{k_{\odot}}{R^3}\left[\Lambda_2\left(1-\frac{3Y^2}{R^2}\right)-\Lambda_1\frac{XY}{R^2}-\Lambda_3\frac{YZ}{R^2}\right],$$

$$H_{13}^{\odot}=\frac{k_{\odot}}{R^3}\left[\Lambda_3\left(1-\frac{3Z^2}{R^2}\right)-\Lambda_2\frac{YZ}{R^2}-\Lambda_1\frac{XZ}{R^2}\right],$$

$$H_{14}^{\odot(l)}=-p\left(\frac{\Lambda}{\mu}-\frac{\Lambda_7}{W_{0l}}\right),\quad H_{15}^{\odot(l)}=-\frac{a_{0l}p}{(W_{0l})^2}\,\Lambda_7,$$

$$H_1^{pd}=\frac{a_{0l}p}{\mu}\,\frac{\bar\lambda_1}{\bar\lambda}-k_{pd}\frac{\bar x}{\varrho^3},\quad H_2^{pd}=\frac{a_{0l}p}{\mu}\,\frac{\bar\lambda_2}{\bar\lambda}-k_{pd}\,\frac{\bar y}{\varrho^3},$$

$$H_3^{pd}=\frac{a_{0l}p}{\mu}\,\frac{\bar\lambda_3}{\bar\lambda}-k_{pd}\,\frac{\bar z}{\varrho^3},\quad H_4^{pd}=\bar v_x,\ H_5^{pd}=\bar v_y,\ H_6^{pd}=\bar v_z,$$

$$H_7^{pd}=-\frac{a_{0l}p}{W_{0l}},\ H_8^{pd}=-\bar\lambda_4,\ H_9^{pd}=-\bar\lambda_5,\ H_{10}^{pd}=-\bar\lambda_6,$$

$$H_{11}^{pd}=\frac{k_{pd}}{\varrho^3}\left[\bar\lambda_1\left(1-\frac{3\bar x^2}{\varrho^2}\right)-\bar\lambda_2\,\frac{\bar x\bar y}{\varrho^2}-\bar\lambda_3\,\frac{\bar x\bar z}{\varrho^2}\right],$$

$$H_{12}^{pd}=\frac{k_{pd}}{\varrho^3}\left[\bar\lambda_2\left(1-\frac{3\bar y^2}{\varrho^2}\right)-\bar\lambda_1\,\frac{\bar x\bar y}{\varrho^2}-\bar\lambda_3\,\frac{\bar y\bar z}{\varrho^2}\right],$$

$$H_{13}^{pd}=\frac{k_{pd}}{\varrho^3}\left[\bar\lambda_3\left(1-\frac{3\bar z^2}{\varrho^2}\right)-\bar\lambda_2\,\frac{\bar y\bar z}{\varrho^2}-\bar\lambda_1\,\frac{\bar x\bar z}{\varrho^2}\right],$$

$$H_{14}^{pd(l)}=-\varrho\left(\frac{\bar\lambda}{\mu}-\frac{\bar\lambda_7}{W_{0l}}\right),\quad H_{15}^{pd(l)}=-\frac{a_{0l}p}{(W_{0l})^2}\,\bar\lambda_7,$$

where Lagrange coefficient λ_7 is determined from the expression of the first integral which replaces Euler-Lagrange equation (3.2.2) as follows:

$$\lambda_7 = \frac{W_{0i}}{a_{0i}}\left[\frac{a_{0i}p}{\mu}\lambda - \frac{k_{\oplus}}{r^3}(\lambda_1 x + \lambda_2 y + \lambda_3 z) + \lambda_4 v_x + \lambda_5 v_y + \lambda_6 v_z + C_0^{\oplus}\right],$$

$$\Lambda_7 = \frac{W_{0i}}{a_{0i}}\left[\frac{a_{0i}p}{\mu}\Lambda - \frac{k_{\odot}}{R^3}(\Lambda_1 X + \Lambda_2 Y + \Lambda_3 Z) + \Lambda_4 V_x + \Lambda_5 V_y + \Lambda_6 V_z + C_0^{\odot}\right],$$

$$\bar{\lambda}_7 = \frac{W_{0i}}{a_{0i}}\left[\frac{a_{0i}p}{\mu}\bar{\lambda} - \frac{k_{pl}}{\rho^3}(\bar{\lambda}_1 \bar{x} + \bar{\lambda}_2 \bar{y} + \bar{\lambda}_3 \bar{z}) + \bar{\lambda}_4 \bar{v}_x + \bar{\lambda}_5 \bar{v}_y + \bar{\lambda}_6 \bar{v}_z + C_0^{pl}\right].$$

Equations (3.3.5) are the conditions of optimization of the main plan parameters after exclusion of the constant Lagrange coefficients from (3.2.11)-(3.2.15).

In the following we will consider that equation (3.3.1) can always be satisfied, determining the corresponding values of M_{0J} when the values of M_{0I} or M_{pI} are fixed and expressing the seven unknown quantities using the remaining seven independent quantities, for example the Lagrange coefficients $\lambda_{10}, \ldots, \lambda_{60}$ and u_0. Equation (3.3.2) will be used as the stop functions, determining the moments of separation (cutoff) of the stages. Thus, it remains to satisfy only conditions (3.3.3)-(3.3.5). Furthermore, in order to decrease the number of conditions fulfilled, we will use homogeneity of system (3.III) relative to the Lagrange coefficients. In this connection, we divide system (3.III) and equations (3.3.5) by one of the Lagrange coefficients $\lambda_{10}, \ldots, \lambda_{60}$, for example by $|\lambda_{10}|$. Then, excluding Lagrange coefficient λ_{10} from (3.3.5), we produce $(2n + n^c - 1)$ conditions for optimization of the plan parameters and six independent unknowns at the initial point.

After the conversions performed, the computational problem is reduced to determination of main plan parameters a_{0i}, λ_{kJ} and W^e_{0i}, lattitude argument u_0 and Lagrange coefficients $\lambda_{30}, \ldots, \lambda_{60}$, Λ_{40}, $\bar{\lambda}_{40}$ and C^*_0 for which the numerical integration of system (3.III) leads to satisfaction of conditions (3.3.3)-(3.3.5), where $\gamma = 2n + n^e - 1$.

We represent

$$p_s = \psi^2_{k\sigma} \qquad (s = 1, \ldots, 7), \tag{3.3.6}$$

$$p_s = R^2_s \qquad (s = 7 + \chi = 7, \ldots, 2n + n^e + 6), \tag{3.3.7}$$

$$p_l = \psi^2_{\Delta 2} \qquad (l = 2n + n^e + 7),\ p_r = \psi^2_{t2}\ (r = l + 1). \tag{3.3.8}$$

$$p_m = \psi^2_{\Delta 1} \qquad (m = r + 1),\ p_k = \psi^2_{t1}\ (k = m + 1). \tag{3.3.9}$$

In this connection, constructing the algorithm for the multipoint boundary problem according to the appendix and Chapter I, § 4 and using it on a computer, we can calculate main plan parameters a_{0i}, λ_{kJ} and W^e_{0i}, lattitude argument u_0 and Lagrange coefficients $\lambda_{30}, \ldots, \lambda_{60}$, Λ_{40}, $\bar{\lambda}_{40}$ and C^*_0 for which the solution of system (3.III) leads to zero values of functionals (3.3.6)-(3.3.9).

In constructing the algorithm of the multipoint boundary problem which we have formulated, it is very important to select properly the stop functions for estimation of the functionals (3.3.6), (3.3.8) and (3.3.9).

Thus, for their estimation as stop functions it will be more correct (in the sense of generality of the solution and freedom from any possible confusion in the realization of the logic of calculation) to select relationships such as

$$\psi^0_1 = t^*_k - t = 0,$$

$$\psi^0_2 = t^{\odot}_k - t = 0,$$

$$\psi^0_3 = t^{pl}_k - t = 0,$$

where t^*_k, $t^{\cdot}_k$ are the defined time of arrival of the spacecraft at the sphere of influence of the earth and of the target planet; t^{pl}_k is the defined time of termination of the flight, where with fixed t_k, $t^{pl}_k = t_k$.

Of course, in this case conversion of the coordinates will occur not at the moment of arrival of the spacecraft at the sphere of influence of the Earth and at the sphere of influence of the target planet, but rather where $t = t_k^{\oplus}$ and $t = t_k^{♂}$ respectively. However, as a result of the solution the transition from one system of coordinates to the other will occur only at the sphere of influence.

Let us compose, following the mathematical theory of the algorithm, the conjugate system, the equations of which will be conjugate to the equations in variations (3.III) as well, in the form

$$
\left.
\begin{aligned}
&\dot{\vartheta}_{j\alpha} = -\sum_{j=1}^{14} \vartheta_{j\alpha} \frac{\partial H_j}{\partial \xi_j} - \sum_{i=1}^{n} \nu_{1\alpha}^{(i)} \frac{\partial H_{15}^{(i)}}{\partial \xi_j} - \sum_{i=1}^{n} \nu_{2\alpha}^{(i)} \frac{\partial H_{16}^{(i)}}{\partial \xi_j} \\
&(j=1, \ldots, 14;\ \alpha=1, \ldots, l+3), \\
&\dot{\nu}_{1\alpha}^{(i)}=0,\quad \dot{\nu}_{2\alpha}^{(i)}=0 \quad (i=1, \ldots, n), \\
&\dot{\nu}_{3\alpha}^{(i)} = -\sum_{j=1}^{14} \nu_{j\alpha} \frac{\partial H_j}{\partial a_{0i}} - \sum_{i=1}^{n} \nu_{1\alpha}^{(i)} \frac{\partial H_{15}^{(i)}}{\partial a_{0i}} - \sum_{i=1}^{n} \nu_{2\alpha}^{(i)} \frac{\partial H_{16}^{(i)}}{\partial a_{0i}}, \\
&\dot{\nu}_{4\alpha}^{(i)} = -\sum_{j=1}^{14} \nu_{j\alpha} \frac{\partial H_j}{\partial W_{0i}^{\varepsilon}} - \sum_{i=1}^{n} \nu_{1\alpha}^{(i)} \frac{\partial H_{15}^{(i)}}{\partial W_{0i}^{\varepsilon}} - \sum_{i=1}^{n} \nu_{2\alpha}^{(i)} \frac{\partial H_{16}^{(i)}}{\partial W_{0i}^{\varepsilon}}, \\
&\dot{\nu}_{5\alpha}^{(i)}=0,
\end{aligned}
\right\} \qquad (3.IV)
$$

where

over $[t_0, t_k^{\oplus}]$

$$\xi_j = v_x;\ v_y;\ v_z;\ x;\ y;\ z;\ \mu;\ \lambda_1;\ \lambda_2;\ \lambda_3;\ \lambda_4;\ \lambda_5;\ \lambda_6;\ C_0^{\oplus};$$

$$\vartheta_{j\alpha} = \vartheta_{1\alpha}^{\oplus}, \ldots, \vartheta_{14\alpha}^{\oplus};\quad \nu_{j\alpha}^{(i)} = \nu_{1\alpha}^{\oplus(i)}, \ldots, \nu_{5\alpha}^{\oplus(i)};$$

over $[t_0^{\odot}, t_k^{\odot}]$

$$\xi_j = V_x;\ V_y;\ V_z;\ X;\ Y;\ Z;\ \mu;\ \Lambda_1;\ \Lambda_2;\ \Lambda_3;\ \Lambda_4;\ \Lambda_5;\ \Lambda_6;\ C_0^{\odot};$$

$$\vartheta_{j\alpha} = \vartheta_{1\alpha}^{\odot}, \ldots, \vartheta_{14\alpha}^{\odot};\quad \nu_{j\alpha}^{(i)} = \nu_{1\alpha}^{\odot(i)}, \ldots, \nu_{5\alpha}^{\odot(i)};$$

over $[t_0^{pl}, t_k^{pl}]$

$$\xi_j = \bar{v}_x;\ \bar{v}_y;\ \bar{v}_z;\ \bar{x};\ \bar{y};\ \bar{z};\ \bar{\mu};\ \bar{\lambda}_1;\ \bar{\lambda}_2;\ \bar{\lambda}_3;\ \bar{\lambda}_4;\ \bar{\lambda}_5;\ \bar{\lambda}_6;\ C_0^{\alpha};$$
$$\vartheta_{j\varkappa} = \bar{\vartheta}_{1\varkappa}, \ldots, \bar{\vartheta}_{14\varkappa};\ \nu_{l\varkappa}^{(l)} = \bar{\nu}_{1\varkappa}^{(l)}, \ldots, \bar{\nu}_{5\varkappa}^{(l)}.$$

According to the appendix, the initial conditions for the conjugate system (3.IV) at moments $\psi_j^0 = 0$ ($j = 1,\ldots, 3$) and $\psi_\mu^{(i)} = 0$ ($i = 1,\ldots, n - 1$) can be represented as follows:

$$\left.\begin{array}{l}
\bar{\vartheta}_{j\varkappa o} = \left.\dfrac{\partial p_o}{\partial \xi_{j\varkappa}}\right|_{\xi_\varkappa^{pl}} \qquad (j=1, \ldots, 14;\ o=1, \ldots, 7), \\
\bar{\nu}_{1\varkappa o}^{(l)} = 0,\ \bar{\nu}_{2\varkappa o}^{(l)} = 0 \qquad (l=1, \ldots, n), \\
\bar{\nu}_{3\varkappa o}^{(l)} = \left.\dfrac{\partial p_o}{\partial a_{0l}}\right|_{\xi_\varkappa^{pl}},\ \bar{\nu}_{4\varkappa o}^{(l)} = \left.\dfrac{\partial p_o}{\partial W_{0l}^{\varepsilon}}\right|_{\xi_\varkappa^{pl}} \qquad (l=1, \ldots, n^*), \\
\bar{\nu}_{5\varkappa o}^{(l)} = \left.\dfrac{\partial p_o}{\partial \mu_{\varkappa l}}\right|_{\xi_\varkappa^{pl}};
\end{array}\right\} \qquad (3.3.10)$$

$$\left.\begin{array}{l}
\vartheta_{j\varkappa s}^{(l)} = \left.\dfrac{\partial p_s}{\partial \xi_{j\varkappa}}\right|_{\psi_\mu^{(l)}} \qquad (j=1, \ldots, 6, 8, \ldots, 14) \\
\qquad\qquad\qquad\qquad (s=8, \ldots, 2n+n^*+6) \\
\vartheta_{7\varkappa s} = \left.\dfrac{\partial p_s}{\partial \mu} - \dfrac{\dot{p}_s}{\dot{\psi}_\mu^{(l)}}\right|_{\psi_\mu^{(l)}}, \\
\nu_{1\varkappa s}^{(l)} = \left.\dfrac{\partial p_s}{\partial \eta_{1\varkappa}^{(l)}}\right|_{\psi_\mu^{(l)}},\ \nu_{2\varkappa s}^{(l)} = \left.\dfrac{\partial p_s}{\partial \eta_{2\varkappa}^{(l)}}\right|_{\psi_\mu^{(l)}}, \\
\nu_{3\varkappa s}^{(l)} = \left.\dfrac{\partial p_s}{\partial a_{0l}}\right|_{\psi_\mu^{(l)}},\ \nu_{4\varkappa s}^{(l)} = \left.\dfrac{\partial p_s}{\partial W_{0l}^{\varepsilon}}\right|_{\psi_\mu^{(l)}}, \\
\nu_{5\varkappa s}^{(l)} = \left.\dfrac{\partial p_s}{\partial \mu_{\varkappa l}} + \dfrac{\dot{p}_s}{\dot{\psi}_\mu^{(l)}}\right|_{\psi_\mu^{(l)}};
\end{array}\right\} \qquad (3.3.11)$$

$$\left.\begin{aligned}&\vartheta^{\odot}_{j\kappa l}=\left.\frac{\partial f_l}{\partial \xi^{\odot}_{j\kappa}}\right|_{t^{\odot}_{\kappa}} \quad (j=1,\ldots,14;\ l=\chi+8,\ \chi+9);\\ &\nu^{\odot(i)}_{1\kappa l}=\nu^{\odot(i)}_{2\kappa l}=\nu^{\odot(i)}_{5\kappa l}=0,\ \nu^{\odot(i)}_{3\kappa l}=\frac{\partial f_l}{\partial a_{0i}},\ \nu^{\odot(i)}_{4\kappa l}=\frac{\partial f_l}{\partial \Psi_{0i}};\end{aligned}\right\} \qquad (3.3.12)$$

$$\left.\begin{aligned}&\vartheta^{\oplus}_{j\kappa m}=\left.\frac{\partial p_m}{\partial \xi^{\oplus}_{j\kappa}}\right|_{t^{\oplus}_{\kappa}} \quad (j=1,\ldots,14;\ m=r+1,\ r+2);\\ &\nu^{\oplus(i)}_{1\kappa m}=\nu^{\oplus(i)}_{2\kappa m}=\nu^{\oplus(i)}_{5\kappa m}=0,\ \nu^{\oplus(i)}_{3\kappa m}=\frac{\partial p_m}{\partial a_{0i}},\ \nu^{\oplus(i)}_{4\kappa m}=\frac{\partial p_m}{\partial \Psi^{p}_{0i}}.\end{aligned}\right\} \qquad (3.3.13)$$

The properties of the conjugate equations allow us to compose the following system of linear equations:

$$\begin{aligned}\Delta p_\alpha = &-\left(\sum_{j=1}^{6}\vartheta^{\oplus}_{j0\alpha}\frac{\partial\psi_{0j}}{\partial u_0}+\vartheta^{\oplus}_{90\alpha}\frac{\partial\psi_{07}/\partial u_0}{\partial\psi_{07}/\partial\lambda_{20}}\right)\delta u_0+\\ &+\left(-\vartheta^{\oplus}_{90\alpha}\frac{\partial\psi_{07}/\partial\lambda_{30}}{\partial\psi_{07}/\partial\lambda_{20}}+\vartheta^{\oplus}_{100\alpha}\right)\delta\lambda_{30}+\left(-\vartheta^{\oplus}_{90\alpha}\frac{\partial\psi_{07}/\partial\lambda_{40}}{\partial\psi_{07}/\partial\lambda_{20}}+\vartheta^{\oplus}_{110\alpha}\right)\delta\lambda_{40}+\\ &+\left(-\vartheta^{\oplus}_{90\alpha}\frac{\partial\psi_{07}/\partial\lambda_{50}}{\partial\psi_{07}/\partial\lambda_{20}}+\vartheta^{\oplus}_{120\alpha}\right)\delta\lambda_{50}+\left(-\vartheta^{\oplus}_{90\alpha}\frac{\partial\psi_{07}/\partial\lambda_{60}}{\partial\psi_{07}/\partial\lambda_{20}}+\vartheta^{\oplus}_{130\alpha}\right)\delta\lambda_{60}+\\ &+\vartheta^{\oplus}_{140\alpha}\delta C^{\oplus}_{0}+\bar\vartheta_{\lambda_{4\alpha}}\delta\bar\lambda_{40}+\vartheta^{\odot}_{\lambda_{4\alpha}}\delta\Lambda_{40}+\dot p_\alpha\delta t^{pf}_{\kappa}+\dot p_l\delta t^{\odot}_{\kappa}+\dot p_m\delta t^{\odot}_{\kappa}+\\ &+\sum_{i=1}^{n}\nu^{(i)}_{30\alpha}\delta a_{0i}+\sum_{i=1}^{n^{*}}\nu^{(i)}_{40\alpha}\delta W^{p}_{0i}+\sum_{i=1}^{n}\nu^{(i)}_{\mu\alpha}\delta\mu_{\kappa i}+\\ &+\sum_{j=1}^{n}\frac{\partial p_\alpha}{\partial M_{0j}}\delta M_{0j}+\frac{\partial p_\alpha}{\partial M_{pf}}\delta M_{pf}\end{aligned} \qquad (3.3.14)$$

where $(\alpha=1,\ldots,2n+n^{*}+10)$,

$$\bar\vartheta_{\lambda_{4\alpha}}=\bar\vartheta_{120\alpha}\frac{\partial\bar\varphi_1}{\partial\bar\lambda_{40}}+\bar\vartheta_{130\alpha}\frac{\partial\bar\varphi_1}{\partial\bar\lambda_{40}}-\bar\vartheta_{140\alpha}\frac{\partial\bar\varphi_2}{\partial\bar\lambda_{40}},$$

$$\vartheta^{\odot}_{\lambda_{4\alpha}}=\vartheta^{\odot}_{140\alpha}\frac{\partial\bar\varphi_1}{\partial\Lambda_{40}}+\vartheta^{\odot}_{\cdot\alpha}\frac{\partial\varphi_{\cdot}}{\partial\Lambda_{40}}-\vartheta^{\odot}_{140\alpha}\frac{\partial\bar\varphi_2}{\partial\Lambda_{40}},$$

$$\begin{aligned}\nu^{(i)}_{\mu\alpha}=\nu^{(i)}_{50\alpha}-\big[&\nu^{(i+1)}_{10\alpha}\dot\mu^{(i+1)}_{0}+\vartheta^{(i+1)}_{10\alpha}\big(\dot V^{(i+1)}_{x0}-\dot V^{(i)}_{x\kappa}\big)+\vartheta^{(i+1)}_{20\alpha}\big(\dot V^{(i+1)}_{y0}-\\ &-\dot V^{(i)}_{y\kappa}\big)+\vartheta^{(i+1)}_{30\alpha}\big(\dot V^{(i+1)}_{z0}-\dot V^{(i)}_{z\kappa}\big)+\nu^{(i+1)}_{10\alpha}\dot\eta^{(i+1)}_{10}+\nu^{(i+1)}_{20\alpha}\dot\eta^{(i+1)}_{20}\big]\frac{1}{\dot\mu^{(i)}_{\kappa}},\end{aligned}$$

$$\bar\varphi_1=\frac{\Lambda_{40}-\lambda_{4\kappa}}{x_\kappa}(z_\kappa\cos\varepsilon-y_\kappa\sin\varepsilon),$$

$$\bar\varphi_2=\Lambda_{10}A^{c}_{x}+\Lambda_{20}A^{c}_{y}+\Lambda_{40}V^{2}_{x\kappa}+\Lambda_{50}V^{2}_{y\kappa},$$

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$$\bar{\varphi}_1 = \frac{\bar{\lambda}_{40} - \Lambda_{1\kappa}}{X_\kappa - X_\kappa^{pd}} \left[(Y_\kappa - Y_\kappa^{pd}) \sin \bar{\varepsilon} + (Z_\kappa - Z_\kappa^{pd}) \cos \bar{\varepsilon} \right],$$

$$\bar{\varphi}_2 = \frac{\bar{\lambda}_{40} - \Lambda_{1\kappa}}{X_\kappa - X_\kappa^{pd}} \left[(X_\kappa - X_\kappa^{pd}) V_{x\kappa}^{pd} + (Y_\kappa - Y_\kappa^{pd}) V_{y\kappa}^{pd} + \right.$$

$$\left. + (Z_\kappa - Z_\kappa^{pd}) V_{z\kappa}^{pd} \right] + \bar{\lambda}_{10} A_x^{pd} + \bar{\lambda}_{20} A_y^{pd} + \bar{\lambda}_{30} A_z^{pd} + \bar{\lambda}_{40} V_{x\kappa}^{pd} +$$

$$+ \bar{\lambda}_{50} V_{y\kappa}^{pd} + \bar{\lambda}_{60} V_{z\kappa}^{pd},$$

$$\Delta p_a = - c_a p_a,$$

c_a are arbitrary weight coefficients.

In system (3.3.14) we must exclude δM_{0I} or δM_{pI} and $\delta M_{0II}, \ldots, \delta M_{0N}$ using the equations $dR_I^{(0)} = 0$ and $dR_I^{(i)} = 0$. After this, we can solve system (3.3.14) for δu_0, $\delta t_k^{(+}$, $\delta t_k^{(-}$, $\delta \lambda_{30}, \ldots, \delta C_0^{(+)}$ and δa_{0i}, $\delta \lambda_{kJ}$, δW_{0i}^e.

Integration of conjugate system (IV) becomes possible due to fulfillment at points $t_k^{(-}$, $t_k^{(+}$ and $t^{(i)}$ of the conditions of conversion and discontinuity of the conjugate coefficients in the form:

where $t = t_k^{(\cdot)}$

$$\vartheta_{1\kappa 2}^{\odot} = \bar{\vartheta}_{10 2}, \; \vartheta_{2\kappa 2}^{\odot} = \bar{\vartheta}_{20 2} \cos \bar{\varepsilon} + \bar{\vartheta}_{30 2} \sin \bar{\varepsilon}, \; \vartheta_{3\kappa 2}^{\odot} = \bar{\vartheta}_{30 2} \cos \bar{\varepsilon} - \bar{\vartheta}_{40 2} \sin \bar{\varepsilon},$$

$$\vartheta_{4\kappa 2}^{\odot} = \bar{\vartheta}_{40 2}^{\odot} + \bar{\vartheta}_{120 2} \frac{\partial \bar{\varphi}_1}{\partial X_\kappa} + \bar{\vartheta}_{130 2} \frac{\partial \bar{\varphi}_1}{\partial X_\kappa} - \bar{\vartheta}_{140 2} \frac{\partial \bar{\varphi}_2}{\partial X_\kappa},$$

$$\vartheta_{5\kappa 2}^{\odot} = \bar{\vartheta}_{50 2} \cos \bar{\varepsilon} + \bar{\vartheta}_{60 2} \sin \bar{\varepsilon} + \bar{\vartheta}_{120 2} \frac{\partial \bar{\varphi}_1}{\partial Y_\kappa} + \bar{\vartheta}_{130 2} \frac{\partial \bar{\varphi}_1}{\partial Y_\kappa} - \bar{\vartheta}_{140 2} \frac{\partial \bar{\varphi}_2}{\partial Y_\kappa},$$

$$\vartheta_{6\kappa 2}^{\odot} = \bar{\vartheta}_{60 2} \cos \bar{\varepsilon} - \bar{\vartheta}_{50 2} \sin \bar{\varepsilon} + \bar{\vartheta}_{120 2} \frac{\partial \bar{\varphi}_1}{\partial Z_\kappa} + \bar{\vartheta}_{130 2} \frac{\partial \bar{\varphi}_1}{\partial Z_\kappa} - \bar{\vartheta}_{140 2} \frac{\partial \bar{\varphi}_2}{\partial Z_\kappa},$$

$$\vartheta_{7\kappa 2}^{\odot} = \bar{\vartheta}_{70 2}, \; \vartheta_{8\kappa 2}^{\odot} = \bar{\vartheta}_{80 2} - \bar{\vartheta}_{140 2} \frac{\partial \bar{\varphi}_2}{\partial \Lambda_{1\kappa}}, \; \vartheta_{9\kappa 2}^{\odot} = \bar{\vartheta}_{90 2} \cos \bar{\varepsilon} + \bar{\vartheta}_{100 2} \sin \bar{\varepsilon} -$$

$$- \bar{\vartheta}_{140 2} \frac{\partial \bar{\varphi}_2}{\partial \Lambda_{2\kappa}},$$

$$\vartheta^{\odot}_{10\kappa n} = \bar{\vartheta}_{100n}\cos\bar{\varepsilon} - \bar{\vartheta}_{90n}\sin\bar{\varepsilon} - \bar{\vartheta}_{140n}\frac{\partial\bar{\varphi}_2}{\partial\Lambda_{3\kappa}},$$

$$\vartheta^{\odot}_{11\kappa n} = \bar{\vartheta}_{120n}\frac{\partial\bar{\varphi}_1}{\partial\Lambda_{4\kappa}} + \bar{\vartheta}_{130n}\frac{\partial\bar{\varphi}_1}{\partial\Lambda_{6\kappa}} - \bar{\vartheta}_{140n}\frac{\partial\bar{\varphi}_2}{\partial\Lambda_{6\kappa}},$$

$$\bar{\vartheta}^{\odot}_{12\kappa n} = \bar{\vartheta}_{120n}\cos\bar{\varepsilon} + \bar{\vartheta}_{130n}\sin\bar{\varepsilon} - \bar{\vartheta}_{140n}\frac{\partial\bar{\varphi}_2}{\partial\Lambda_{5\kappa}},$$

$$\vartheta^{\odot}_{13\kappa n} = \bar{\vartheta}_{130n}\cos\bar{\varepsilon} - \bar{\vartheta}_{120n}\sin\bar{\varepsilon} - \bar{\vartheta}_{140n}\frac{\partial\bar{\varphi}_2}{\partial\Lambda_{6\kappa}}, \quad \vartheta^{\odot}_{14\kappa n} = \bar{\vartheta}_{140n},$$

$$\nu^{\odot(i)}_{1\kappa n} = \bar{\nu}^{(i)}_{10n}, \quad \nu^{\odot(i)}_{2\kappa n} = \bar{\nu}^{(i)}_{20n}, \quad \nu^{\odot(i)}_{3\kappa n} = \bar{\nu}^{(i)}_{30n},$$

$$\nu^{\odot(i)}_{4\kappa n} = \bar{\nu}^{(i)}_{40n}, \quad \nu^{\odot(i)}_{5\kappa n} = \bar{\nu}^{(i)}_{50n};$$

<u>where $t = t_k^+$</u>

$$\vartheta^{\oplus}_{1\kappa n} = \vartheta^{\odot}_{10n}, \quad \vartheta^{\oplus}_{2\kappa n} = \vartheta^{\odot}_{20n}\cos\varepsilon - \vartheta^{\odot}_{30n}\sin\varepsilon, \quad \vartheta^{\oplus}_{3\kappa n} = \vartheta^{\odot}_{30n}\cos\varepsilon + \vartheta^{\odot}_{20n}\sin\varepsilon,$$

$$\vartheta^{\oplus}_{4\kappa n} = \vartheta^{\odot}_{40n} + \vartheta^{\odot}_{120n}\frac{\partial\varphi_1}{\partial x_\kappa} + \vartheta^{\odot}_{120n}\frac{\partial\varphi_1}{\partial x_\kappa},$$

$$\vartheta^{\oplus}_{5\kappa n} = \vartheta^{\odot}_{50n}\cos\varepsilon - \vartheta^{\odot}_{60n}\sin\varepsilon + \vartheta^{\odot}_{120n}\frac{\partial\varphi_1}{\partial y_\kappa} + \vartheta^{\odot}_{130n}\frac{\partial\varphi_1}{\partial y_\kappa},$$

$$\vartheta^{\oplus}_{6\kappa n} = \vartheta^{\odot}_{60n}\cos\varepsilon + \vartheta^{\odot}_{50n}\sin\varepsilon + \vartheta^{\odot}_{120n}\frac{\partial\varphi_1}{\partial z_\kappa} + \vartheta^{\odot}_{130n}\frac{\partial\varphi_1}{\partial z_\kappa},$$

$$\vartheta^{\oplus}_{7\kappa n} = \vartheta^{\odot}_{70n}, \quad \vartheta^{\bullet}_{8\kappa n} = \vartheta^{\odot}_{80n} - \vartheta^{\odot}_{140n}\frac{\partial\varphi_2}{\partial\lambda_{1\kappa}},$$

$$\vartheta^{\oplus}_{9\kappa n} = \vartheta^{\odot}_{90n}\cos\varepsilon - \vartheta^{\odot}_{100n}\sin\varepsilon, \quad \vartheta^{\oplus}_{10\kappa n} = \vartheta^{\odot}_{100n}\cos\varepsilon + \vartheta^{\odot}_{90n}\sin\varepsilon,$$

$$\vartheta^{\oplus}_{11\kappa n} = \vartheta^{\odot}_{120n}\frac{\partial\varphi_1}{\partial\lambda_{4\kappa}} + \vartheta^{\odot}_{130n}\frac{\partial\varphi_1}{\partial\lambda_{6\kappa}} - \vartheta^{\odot}_{140n}\frac{\partial\varphi_2}{\partial\lambda_{6\kappa}},$$

$$\vartheta^{\oplus}_{12\kappa n} = \vartheta^{\odot}_{120n}\cos\varepsilon - \vartheta^{\odot}_{130n}\sin\varepsilon - \vartheta^{\odot}_{140n}\frac{\partial\varphi_2}{\partial\lambda_{5\kappa}},$$

$$\vartheta^{\bullet}_{13\kappa n} = \vartheta^{\odot}_{130n}\cos\varepsilon + \vartheta^{\odot}_{120n}\sin\varepsilon - \vartheta^{\odot}_{140n}\frac{\partial\varphi_2}{\partial\lambda_{6\kappa}}, \quad \vartheta^{\bullet}_{140n} = \vartheta^{\odot}_{14\kappa n},$$

$$\nu^{\bullet}_{\kappa n} = \nu^{\odot}_{10n}, \quad \nu^{\bullet}_{2\kappa n} = \nu^{\odot}_{20n}, \quad \nu^{\bullet}_{3\kappa n} = \nu^{\odot}_{30n},$$

$$\nu^{\bullet}_{4\kappa n} = \nu^{\odot}_{40n}, \quad \nu^{\bullet}_{5\kappa n} = \nu^{\odot}_{50n};$$

<u>where $t = t_k^{(i)}$</u>

$$\vartheta^{(i)}_{j\kappa n} = \vartheta^{(i+1)}_{j0n} \qquad (j = 1, \ldots, 6, 8, \ldots, 14),$$

$$\vartheta^{(i)}_{7\kappa n} = \Big[\vartheta^{(i+1)}_{70n}\dot{\mu}^{(i+1)}_0 + \vartheta^{(i-1)}_{10n}\big(\dot{V}^{(i+1)}_{x0} - \dot{V}^{(i)}_{x\kappa}\big) + \vartheta^{(i+1)}_{20n}\big(\dot{V}^{(i+1)}_{y0} -$$

$$- \dot{V}^{(i)}_{y\kappa}\big) + \vartheta^{(i+1)}_{30n}\big(\dot{V}^{(i+1)}_{z0} - \dot{V}^{(i)}_{z\kappa}\big) + \nu^{(i+1)}_{10n}\dot{\eta}^{(i+1)}_{10} +$$

$$+ \nu^{(i+1)}_{20n}\dot{\eta}^{(i-1)}_{20}\Big]\frac{1}{\dot{\mu}^{(i)}_\kappa} + \frac{\partial p_0}{\partial\mu^{(i)}_\kappa} - \dot{p}_1/\dot{\mu}^{(i)}_\kappa.$$

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It is now possible, using initial conditions (3.3.10)-(3.3.13) and the conditions of transformation and discontinuity of the conjugate coefficients, to perform integration of system (3.IV) together with (3.III) from t_k to t_0. In order to determine the coefficients in the system of linear equations (3.3.14), this must be performed $(2n + n^2 + 10)$ times.

Further construction of the computational procedure should be organized as in § 4 of Chapter I.

APPENDIX. MATHEMATICAL THEORY OF VARIATIONAL METHOD OF OPTIMAL PLANNING OF AN OBJECT

§ 1. Definitions and Statement of Problem of Optimal Planning of Object

The object of study is the process of selection of the controlled motion of an object and its structural state, represented as planning of the object. Furthermore, we formulate certain definitions and present a mathematical statement of the problem of planning of an object, leading to the variational problem of optimal planning of the object.

Suppose there are real vector spaces Y_n, X_m, Π_l^u and Π_r^a respectively with elements $y = (y_1, \ldots, y_n)$, $x = (x_1, \ldots, x_m)$, $u = (u_1, \ldots, u_l)$, $a = (a_1, \ldots, a_r)$. Let us take a sector on the time axis $[t_0, t_k]$ during which we will study the motion of the object as set T, where $t \in T$.

The components of the vectors y and x will be referred to as the phase coordinates of the object. Therefore, spaces Y_n and X_m are called phase spaces. The phase coordinates determine a phase point or phase state of the object and thereby fully characterize the position of the object in the phase space. The motion of the object is fixed in space $T \times Y_n$ and $T \times X_m$. It will be defined if for each $t \in T$ we define $y = y(t)$ in Y_n or $x = x(t)$ in X_m. The curve described by the phase point as it moves is referred to as the phase trajectory of the motion of the object being studied. In the following, the components of vector function $y(t)$ $y_1(t), \ldots, y_n(t)$ and vector function $x(t)$ $x_1(t), \ldots, x_m(t)$ will be called phase variables.

Vectors u and a are called the controlling action and the parameter. Using vector u we can represent the controlling action on the motion of the object. Using parameter a we can influence the structural state and nature of motion of the object.

In spaces Y_n, X_m, Π_l^u and Π_r^a we fix areas $\tilde{B}$, $\tilde{C}$, $\tilde{U}$ and $\tilde{A}$ respectively. Generally speaking, $\tilde{B}$, $\tilde{C}$, $\tilde{U}$ and $\tilde{A}$ are closed **areas**, since in applied problems the course of the phase trajectories in the corresponding phase spaces can be limited to a definite area and the values of control and parameters of the object have certain limits. However, considering the boundaries of closed area $\tilde{B}$ and $\tilde{C}$ in spaces Y_n and X_m smooth and piecewise smooth hypersurfaces, and areas $\tilde{U}$ and $\tilde{A}$ compact, without reducing generality of the problem, we can study the motion of an object in open areas B, C, U and A of spaces Y_n, X_m, Π_l^u and Π_r^a respectively. However, in this latter case additional couplings appear [24, 30, 33], limiting the motion of the object. Keeping this note in mind, we will analyze the motion and structural state of an object in open areas B, C, U and A of spaces Y_n, X_m, Π_l^u and Π_r^a respectively.

Function $u = u(t)$, defined in T and taking on its value in the area of control U will be referred to as the control or control function. Each control

$$u(t)=(u_1(t),\ldots, u_l(t))$$

is a vector function fixed in T and the values of which lie in the area of control U. Each control u(t) defined in T and taking on values in U will be called a permissible control if it relates to the class of piecewise continuous functions satisfying the following condition for definition at points of discontinuity

$$u(\tau)=u(\tau-0),$$

and is continuous at points on the sector $[t_0, t_k]$.

Function a(t), defined in set T and taking on values in A, takes on the same value a for all t from T, i. e. it is constant in T. Therefore, function a(t) is defined by the following functional relationship $a(t) = a$. This is the formal distinction of a parameter from a phase variable. Any parameter defined in T as a constant function and taking on values in A will be called a permissible parameter. A permissible parameter determines the structural state of the object.

If in sets $T \times B \times U$ and $T \times C \times U$ we define the vector functions y(t), u(t) and x(t), u(t) respectively, where $t \in T$ and $y \in B$, $x \in C$ and $u \in U$, each ordered pair (y(t), u(t)) and (x(t), u(t)) will be called a mode of motion of the object, meaning that a mode of motion of an object consists of control u(t) and two pieces of the phase trajectory y(t) and x(t). The set of pairs (y(t), u(t)) (x(t), u(t)) make up subset M which is the set of modes of motion.

Suppose there is a set $P = M \times A$ such that $p = (z(t), a) \in P$ and $z = (y(t), u(t)) \in M$ and $z = (x(t), u(t)) \in M$ and $a \in A$. The process of selection of an element $p = (z, a)$, $p \in P$ corresponding to the fixed requirements and limitations is called planning of the object. This is the mathematical definition of the essence of planning. In other words, planning of an object refers to the process of selection of its mode of motion and structural state, corresponding to fixed requirements and limitations.

Let us now explain the requirements and limitations on the mode of motion and structural state of the object which may be encountered in planning of an object.

The hypersurface (manifold) S_τ, fixed in $T \times B \times A$ by the equations

$$\phi_\tau(t_q, y_q(t_q^-), y_q(t_q^+), a) = 0 \quad (\phi_\tau = (\phi_\tau^1, \ldots, \phi_\tau^\eta), \quad \eta \leq n + r + 1), \tag{1.1}$$

is distinguished in advance and is smooth with respect to its elements t_q, y_q^-, y_q^+ and a. An object reaching this hyperplane in its movement can change its structural state and the number of control actions. The change in the properties of the object is planned in advance. Therefore, it is assumed that sets U and A have two subsets $U_1 \subset U$, $U_2 \subset U$ and $A_1 \subset A$, $A_2 \subset A$ with elements $u^{(1)} = (u_1, \ldots, u_{s'}, \ldots, u_s)$, $u^{(2)} = (u_{s'+1}, \ldots, u_l)$ and $a^{(1)} = (a_1, \ldots, a_{m'}, \ldots, a_m)$, $a^{(2)} = (a_{m'+1}, \ldots, a_r)$ respectively, where $U_1 \cap U_2$ and $A_1 \cap A_2$ cannot be calculated. Thus, hypersurface S_τ subdivides subset $P_y = T \times B \times U \times A \subset P$ into two subsets: $P_y^{(1)} = T \times B \times U_1 \times A_1 \subset P_y$ and $P_y^{(2)} = T \times B \times U_2 \times A_2 \subset P_y$.

Area P is divided into two areas $P_y = T \times B \times U \times A$ and $P_x = T \times C \times U \times A$ by the smooth hypersurface (manifold) S_π fixed in P by the equation

$$\phi_\pi(t_\nu, y(t_\nu)) = 0 \quad (\phi_\pi = (\phi_\pi^1, \ldots, \phi_\pi^\varepsilon); \ \varepsilon < n). \tag{1.2}$$

The phase trajectory of an object should intersect hypersurface S_π. After the object "passes through" hypersurface S_π it enters area P_x with its new system of phase coordinates in space X_m. Conversion of coordinates of phase space Y_n to coordinates of phase space X_m is performed according to the equations

$$Q_j(t_\tau, y_{i\tau}, x_{j\nu})=0 \quad (j=1, \ldots, m;\ i=1, \ldots, n). \tag{1.3}$$

Suppose set P is such that with all $t \in T$ we have:

<u>over $[t_0, t_q^-]$</u>

$$\dot{y}=\varphi(y, u^{(1)}, a^{(1)}), \tag{1.4}$$

$$\dot{a}^{(1)}=0, \tag{1.5}$$

$$\xi(y, u^{(1)}, a^{(1)})=0; \tag{1.6}$$

<u>over $[t_q^+, t_v^-]$</u>

$$\dot{y}=f(y, u^{(2)}, a^{(2)}), \tag{1.7}$$

$$\dot{a}=0, \tag{1.8}$$

$$\gamma(y, u^{(2)}, a^{(2)})=0; \tag{1.9}$$

<u>over $[t_v^+, t_k]$</u>

$$\dot{x}=g(x, u^{(2)}, a^{(2)}), \tag{1.10}$$

$$\dot{a}=0, \tag{1.11}$$

$$\zeta(x, u^{(2)}, a^{(2)})=0, \tag{1.12}$$

where

$$\varphi=(\varphi_1, \ldots, \varphi_n), \quad \xi=(\xi_1, \ldots, \xi_d) \quad d<s;$$
$$f=(f_1, \ldots, f_n), \quad \gamma=(\gamma_1, \ldots, \gamma_h) \quad h<l-s';$$
$$g=(g_1, \ldots, g_m), \quad \zeta=(\zeta_1, \ldots, \zeta_\kappa) \quad \kappa<l-s'.$$

The real scalar functions ϕ_i and f_i $(i = 1, \ldots, n)$, g_j $(j = 1, \ldots, m)$ are defined and continuous together with the partial derivatives with respect to their arguments to the order necessary for further contributions respectively in $P_y^{(1)}$, $P_y^{(2)}$ and P_x. The phase variables $y_i(t)$ in the intervals $[t_0, t_q^-]$, $[t_q^+, t_v^-]$ and $x_j(1)$ at $[t_v^+, t_k]$ are assumed continuous and piecewise differentiable respectively on the strength of (1.4), (1.7) and (1.10).

The motion of an object is described in the phase space Y_n by equations (1.4), (1.7), and in phase space X_m by equation (1.10).

Introduction of the dependences (1.6), (1.9) and (1.12) involves a transition from the closed area of permissible control to the open area of permissible control U and with certain additional conditions [28], of permissible values of phase variables of area P. The vector functions ξ and γ are defined and continuous together with their partial derivatives at $P_y^{(1)}$ and $P_y^{(2)}$ respectively, while vector function ζ is defined and continuous at P_x.

The introduction of the dependences

$$\left.\begin{array}{l} \beta_{1l}(a_1, \ldots, a_{m'}, \ldots, a_m)=0 \quad a_l \in A_1 \\ (l=1, \ldots, m; \; i=1, \ldots, e<m), \\ \beta_{2k}(a_{m'+1}, \ldots, a_r)=0 \quad a_\chi \in A_2 \\ (\chi=m'+1, \ldots, r; \; k=1, \ldots, f+e<r) \end{array}\right\} \tag{1.13}$$

allows us to look upon the permissible area of parameters A as open.

It is assumed that vector functions ξ and γ, ζ and β_1, β_2 satisfy all requirements of the theorem for existence of an implicit function.

Suppose S_0 and S_k are smooth manifolds fixed in spaces Y_n and X_m respectively by the equations

$$\psi_0(t_0, y_0, a^{(1)})=0 \quad (\psi_0=(\psi_{01}, \ldots, \psi_{0\delta}); \; \delta<n+r-e+1), \tag{1.14}$$

$$\psi_k(t_k, x_k)=0 \quad (\psi_k=(\psi_{k1}, \ldots, \psi_{k\vartheta}); \; \vartheta<m+1). \tag{1.15}$$

Acceptance of manifolds S_0, S_τ, S_V and S_k as smooth allows us to state that the ranks of the matrices

$$\left\| \frac{\partial\psi_\tau}{\partial t_q} \; \frac{\partial\psi_\tau}{\partial y_{iq}^-} \; \frac{\partial\psi_\tau}{\partial y_{iq}^+} \; \frac{\partial\psi_\tau}{\partial a} \right\|, \tag{1.16}$$

$$\left\| \frac{\partial \psi_v}{\partial t_v} \frac{\partial \psi_v}{\partial y_{iv}} \right\|, \quad \left\| \frac{\partial Q_j}{\partial x_{jv}} \frac{\partial Q_j}{\partial y_{iv}} \frac{\partial Q_j}{\partial t_v} \right\|, \tag{1.17}$$

$$\left\| \frac{\partial \psi_0}{\partial t_0} \frac{\partial \psi_0}{\partial y_{i0}} \frac{\partial \psi_0}{\partial a^{(1)}} \right\|, \quad \left\| \frac{\partial \psi_k}{\partial t_k} \frac{\partial \psi_k}{\partial x_{jk}} \right\| \tag{1.18}$$

are n, ε, m and θ, ϑ respectively.

Suppose further functional J is defined and continuous together with its partial derivatives over $P(t_k)$, such that

$$J = J(a, x_k, t_k). \tag{1.19}$$

Functional J must be analyzed as the criterion of planning perfection or the criterion of effectiveness of the object: the greater J, the better the planning of the object and the more fully the capabilities of the object are therefore used.

Thus, each element $p = (z(t), a)$ $p \in P$ should be such that its projections $y(t) \in B$, $x(t) \in C$, $u(t) \in U$ and $a \in A$ are interrelated by equations (1.4)-(1.12) and satisfy the relationships (1.1) and (1.2), (1.14) and (1.15), defining the manifolds S_q and S_v, S_0 and S_k, and conditions (1.3) and (1.13). Any such element p, $p \in P$ unambiguously defining the value of functional J will be referred to as permissible element p. We can now clarify the mathematical definition of planning of an object given earlier: planning of an object refers to the process of determination of the permissible element $p = (z(t), a)$, where $p \in P$ and $z(t) = (y(t), u(t)) \in M$ and $z(t) = (x(t), u(t)) \in M$ and $a \in A$, unambiguously defining the value of functional J.

In other words, each mode of motion and each structural state of the object must satisfy the following requirements.

The initial and intermediate phase and structural states of the object are determined by equations (1.14), (1.13) and (1.1) respectively, while the final phase state of the object is determined by equation (1.15); the motion of the object must be such that the object reaches hypersurface (1.2) and passes from phase space Y_n to the new phase space X_m, the conversion of coordinates occurring according to relationships (1.3); the phase trajectory of motion of the object y(t) and x(t), permissible control u(t) and permissible parameter a must be interrelated by the equations (1.4)-(1.12). Any mode of motion and any structural state of the object satisfying these conditions and unambiguously determining the criterion of effectiveness of the object J will be referred to as a permissible mode of motion and permissible structural state of the object. These notes and the mathematical definition of planning of an object allow us to state that planning of an object is the process of determination of the permissible mode of motion and permissible structural state of the object unambiguously defining the criterion of effectiveness of the object J.

There is a set of permissible elements $p = (z(t), a)$, where $p \in P$, $z(t) \in M$ and $a \in M$, each of which unambiguously defines functional J in $P(t_k)$.

The process of determination of the permissible element p, $p \in P$, for which functional J reaches its maximum value in $P(t_k)$ will be referred to as optimal planning of an object. Permissible element $p \in P$ for which functional J reaches its greatest possible value in $P(t_k)$ will be called the optimal element p^0, while its projections $(y^0(t), x^0(t)) \in B \times C$ will be called the optimal phase trajectory, $u^0(t) \in U$ the optimal control and $a^0 \in A$ the optimal parameter or optimal structural state of the object.

In other words, optimal planning means the process of determination of a permissible mode of motion and permissible structural state of the object for which the criterion of effectiveness of the object J reaches its greatest possible value.

These definitions reveal the essence of optimal planning of an object. Optimal planning is a process consisting of investigation of plans for the object and determination of the plan for which the criterion of plan improvement J reaches its greatest possible value.

These definitions of permissible element p, $p \in P$ and therefore of the permissible mode of motion and permissible structural state of the object, within the framework of the requirements and limitations formulated, are rather broad. Therefore, these definitions of planning and optimal planning of an object are rather broad. It is possible in principle, using some one of the known methods of construction of optimal processes, to construct a rather general mathematical theory of optimal planning on the basis of these definitions. In this work, we have not set ourselves the task of selecting an optimal method for construction of optimal processes; this is a significant independent problem requiring independent analysis. The mathematical theory of optimal planning of an object analyzed in this work is based on indirect methods of variational calculus, including methods of classical variational calculus and the new method of the mathematical theory of optimal processes -- the L. S. Pontryagin maximum principle. In this sense, the mathematical theory of optimal planning which we present is a mathematical theory of variational methods of optimal planning of an object. The development of the mathematical theory of variational methods of optimal planning of an object has required a certain modification of the theoretical basis of known indirect methods of variational calculus. Furthermore, the indirect methods of variational calculus generally lead to a multipoint boundary problem. Therefore, the mathematical theory of variational methods of optimal planning of an object can be considered complete only when it also includes theoretical development of a definite method of solution of the multipoint boundary problem. The mathematical theory of variational methods of optimal planning analyzed in this work combines the theory of indirect methods of variational calculus and the theory of a definite method of solution of the multipoint boundary problem and thereby attains logical completeness.

For convenience in our further presentation, we shall now present definitions of optimal planning of an object in terms convenient for use of the methods of variational calculus. It is stated that curve C in an n + m + r + 1-dimensional space lies in the area $R = T \times B \times C \times A$ and is permissible if phase trajectory $(y(t), x(t))$, the structural state of the object a and the corresponding control u(t) are permissible. We represent by D the class of permissible curves C. Then, the definition of optimal planning of an object of equivalent form is formulated as follows: optimal planning of an object means determination of curve C in class D, considering the numerical realization, on which J(C) attains its maximum possible value.

This variational problem of optimal planning of an object differs quite a bit from the variational problems of classical variational calculus such as the problem of Mayer and Boltz and from the known problems of the mathematical theory of optimal processes using the L. S. Pontryagin maximum principle. For example, the functional depends not only on the phase coordinates of the final point, but also on the parameters; the right portions of the coupling equations (1.4), (1.7) and (1.10), which undergo first order discontinuities, are functions not only of the phase variables, but also of control -- functions having no derivatives in the coupling equations, and of the parameters; the phase variables undergo first order discontinuities at manifolds S_τ and S_π, fixed by equations (1.1) and (1.2). The control functions which undergo a finite number of first order discontinuities over the interval $t_0 \leq t < t_k$ and the parameters are assumed fixed in closed areas, etc.

However, analysis of the mathematical theory of this variational problem in optimal planning of an object will be based on the theoretical principles of classical variational calculus and the L. S. Pontryagin maximum principle, most fully expressed in [7] and [28], although use will also be made of [30, 33, 34]. Considering these works, we provide a theoretical basis for the necessary conditions of optimal planning of an object, which are also sufficient in combination. Proof is further given for the method of solution of the multipoint boundary problem, one peculiarity of which is that solution of the multipoint boundary problem is combined with satisfaction of the second order necessary conditions.

Thus, the mathematical theory of variational methods of optimal planning of an object includes the theoretical principles for the necessary and sufficient conditions of the variational method and the method of solution of the multipoint boundary problem. The results of this theory make it possible to algorithmize the variational methods of optimal planning, allowing us to use digital computers for the solution of practical problems in optimal planning.

VARIATIONAL METHOD

§ 2. Variations and Equations in Variations

The corner points of curve C will refer to points of first order discontinuities of functions y(t), x(t), u(t) and the point of change of a.

Suppose the following "noncontact" conditions are correct on curve C[1]

$$\frac{\partial\psi_0}{\partial t_0}+\frac{\partial\psi_0}{\partial y_{i0}}\dot{y}_{i0}\neq 0,\quad \frac{\partial\psi_\varkappa}{\partial t_\varkappa}+\frac{\partial\psi_\varkappa}{\partial x_{j\varkappa}}\dot{x}_{j\varkappa}=0$$

$$(i=1,\ldots,n;\quad j=1,\ldots,m).$$

$$\frac{\partial\psi_\tau}{\partial t_q}+\frac{\partial\psi_\tau}{\partial y^-_{iq}}\dot{y}^-_{iq}+\frac{\partial\psi_\tau}{\partial y^+_{iq}}\dot{y}^+_{iq}\neq 0,$$

$$\frac{\partial\psi_\varkappa}{\partial t_0}+\frac{\partial\psi_\varkappa}{\partial y_{i\varkappa}}\dot{y}_{i\varkappa}\neq 0.$$

---

[1] It is stated that a curve of class D satisfies the noncontact condition if the hypersurface described in area R by its ends and the hypersurface defined in area R by the conditions for the ends have no common tangent directions.

Here and in the following, indices encountered twice will be added.

The set

$$
\begin{gathered}
y_i(t, b), x_j(t, b), \quad u_k(t, b), a_n(b) \; [t_0(b) < t \leqslant t_\kappa(b); \; |b| < \varepsilon] \\
(i = 1, \ldots, n; \; j = 1, \ldots, m, \; k = 1, \ldots, l; \; n = 1, \ldots, r)
\end{gathered} \tag{2.1}
$$

is such that $y_i(t, b)$ in $P_y(t)$ and $x_j(t, b)$ in $P_x(t)$ have the derivatives $\dot{y}_i(t, b)$, $\dot{x}_j(t, b)$ between the corner points, and these derivatives and functions $t_0(b)$, $t_q(b)$, $t_v(b)$, $t_k(b)$, $y_i(t, b)$, $x_j(t, b)$, $u_k(t, b)$, $a_n(b)$ have continuous derivatives between the corner points with respect to parameter b in the area of the set of points (t, b), defined by the inequalities in the brackets. This set consists of a finite number of successively adjacent elementary sets fixed between the corner points in the adjacent intervals t', t'', $t''t'''$, ..., $t^{(n-1)}t^{(n)}$, where

$$t' < t_0(b) < t'', \quad t^{(n-1)} < t_\kappa(b) < t^{(n)}.$$

Set (2.1), having these properties and consisting only of permissible curves, is called a permissible set.

The differentials of set (2.1) can be represented as

$$
\left.
\begin{gathered}
dt_0 = t_{0b}\,db, \quad dt_q = t_{qb}\,db, \quad dt_v = t_{vb}\,db, \quad dt_\kappa = t_{\kappa b}\,db, \\
dy_i = \dot{y}_i\,dt + \delta y_i\,db, \quad dx_j = \dot{x}_j\,dt + \delta x_j\,db, \\
da_n = \delta a_n\,db, \\
d^2 y_i = \dot{y}_i d^2 t + \ddot{y}_i dt^2 + 2\delta \dot{y}_i\,db\,dt + \delta^2 y_i\,db^2, \\
p^2 x_j = \dot{x}_j d^2 t + \ddot{x}_j dt^2 + 2\delta \dot{x}_j\,db\,dt + \delta^2 x_j\,db^2.
\end{gathered}
\right\} \tag{2.2}
$$

Any permissible set has differentials placed in tne first seven equations of (2.2). It may not have d^2y_i and d^2x_j, since the definition of this set does not include the requirement for existence of $\ddot{y}_i$ and $\ddot{x}_j$.

The variations of a set along curve E refer to the quantities

$$\left.\begin{aligned}\delta t_0=t_{0b}(0),\quad \delta t_q=t_{qb}(0),\quad \delta t_v=t_{vb}(0),\quad \delta t_\kappa=t_{\kappa b}(0),\\ \delta y_i=y_{ib}(t,0),\ \delta x_j=x_{jb}(t,0),\\ \delta u_k=u_{kb}(t,0),\ \delta a_n=a_{nb}(0).\end{aligned}\right\}\tag{2.3}$$

It follows from the properties of set (2.1) that functions $\delta y_i(t)$, $\delta x_j(t)$, $\delta u_i(t)$, δa_n have the same properties of continuity over the intervals

$$t_0(0)\leqslant t\leqslant t_q^-(0),\quad t_q^+(0)\leqslant t\leqslant t_v^-(0)\ \text{и}\ t_v^+(0)\leqslant t\leqslant t_\kappa(0),$$

corresponding to curve E as the functions $y_i(t)$, $x_j(t)$, $u_k(t)$ and a_n, defining permissible curves. The set of quantities δt_0, δt_q, δt_v, δt_k, δy_i, δx_j, δu_k, δa_n, having these properties, is referred to as the permissible set of variations along E.

Since all curves of the permissible set satisfy equations (1.4)-(1.12), variations δy_i, δu_k, δx_j, δa_n along curve E of the set corresponding to the value of the parameter $b = b_0$ satisfy the following linear system of equations:

over $[t_0, t_q^-]$

$$\delta\dot{y}_i=\frac{\partial\varphi_i}{\partial y_i}\delta y_i+\frac{\partial\varphi_i}{\partial u_q}\delta u_q+\frac{\partial\varphi_i}{\partial a_l}\delta a_l$$

$$(i=1,\ldots,n;\quad q=1,\ldots,s',\ldots,s;\quad l=1,\ldots,m',\ldots,m),\tag{2.4}$$

$$\delta\dot{a}_l=0,\tag{2.5}$$

$$\frac{\partial\xi_\alpha}{\partial y_i}\delta y_i+\frac{\partial\xi_\alpha}{\partial u_q}\delta u_q+\frac{\partial\xi_\alpha}{\partial a_l}\delta a_l=0\quad(\alpha=1,\ldots,d);\tag{2.6}$$

over $[t_q^+, t_v]$

$$\delta\dot{y}_i=\frac{\partial f_i}{\partial y_i}\delta y_i+\frac{\partial f_i}{\partial u_p}\delta u_p+\frac{\partial f_i}{\partial a_\gamma}\delta a_\gamma,\tag{2.7}$$

$$(p=s'+1,\ldots,l;\quad \gamma=m'+1,\ldots,r),$$

$$\delta\dot{a}_\pi = 0, \tag{2.8}$$

$$\frac{\partial\gamma_\beta}{\partial y_l}\delta y_l + \frac{\partial\gamma_\beta}{\partial u_p}\delta u_p + \frac{\partial\gamma_\beta}{\partial a_\chi}\delta a_\chi = 0, \quad (\beta = 1, \ldots, h) \tag{2.9}$$

over <u>$[t_v^+, t_k]$</u>

$$\delta\dot{x}_j = \frac{\partial g_j}{\partial x_j}\delta x_j + \frac{\partial g_j}{\partial u_p}\delta u_p + \frac{\partial g_j}{\partial a_\chi}\delta a_\chi \quad (j = 1, \ldots, m), \tag{2.10}$$

$$\delta\dot{a}_\pi = 0, \tag{2.11}$$

$$\frac{\partial\zeta_\nu}{\partial u_p}\delta u_p + \frac{\partial\zeta_\nu}{\partial x_j}\delta x_j + \frac{\partial\zeta_\nu}{\partial a_\chi}\delta a_\chi = 0 \quad (\nu = 1, \ldots, \kappa). \tag{2.12}$$

Here the arguments in the derivatives are functions $y_i(t, b_0)$, $u_q(t, b_0)$, $a_m(b_0)$, $u_p(t, b_0)$, $a_x(b_0)$, $x_j(t, b_0)$, defining curve E.

Linear equation system (2.4)-(2.12) is referred to as the system of equations in variations along E. The coefficients of these equations are fully defined by curve E, regardless of the set containing it.

Since the curves of the permissible set satisfy equations (1.1)-(1.3) and (1.13)-(1.15), the variations of the set along E satisfy the equations

$$\Psi_0 = \left(\frac{\partial\psi_0}{\partial t_0} + \dot{y}_{i0}\frac{\partial\psi_0}{\partial y_{i0}}\right)\delta t_0 + \frac{\partial\psi_0}{\partial y_{i0}}\delta y_{i0} + \frac{\partial\psi_0}{\partial a_s}\delta a_s = 0, \tag{2.13}$$

$$\Psi_\kappa = \left(\frac{\partial\psi_\kappa}{\partial t_\kappa} + \dot{x}_{j\kappa}\frac{\partial\psi_\kappa}{\partial x_{j\kappa}}\right)\delta t_\kappa + \frac{\partial\psi_\kappa}{\partial x_{j\kappa}}\delta x_{j\kappa} = 0, \tag{2.14}$$

$$\Psi_\tau = \left(\frac{\partial\psi_\tau}{\partial t_q} + \dot{y}_{iq}^-\frac{\partial\psi_\tau}{\partial y_{iq}^-} + \dot{y}_{iq}^+\frac{\partial\psi_\tau}{\partial y_{iq}^+}\right)\delta t_q + \frac{\partial\psi_\tau}{\partial y_{iq}^+}\delta y_{iq}^+ + \frac{\partial\psi_\tau}{\partial y_{iq}^-}\delta y_{iq}^- + \frac{\partial\psi_\tau}{\partial a_\chi}\delta a_\chi = 0, \tag{2.15}$$

$$\Psi_\pi = \left(\frac{\partial\psi_\pi}{\partial t_v} + \dot{y}_{iv}\frac{\partial\psi_\pi}{\partial y_{iv}}\right)\delta t_v + \frac{\partial\psi_\pi}{\partial y_{iv}}\delta y_{iv} = 0, \tag{2.16}$$

$$\bar{Q}_j = \left(\frac{\partial Q_j}{\partial t_v} + \dot{y}_{iv}\frac{\partial Q_j}{\partial y_{iv}} + \dot{x}_{jv}\frac{\partial Q_j}{\partial x_{jv}}\right)\delta t_v + \frac{\partial Q_j}{\partial y_{iv}}\delta y_{iv} + \frac{\partial Q_j}{\partial x_{jv}}\delta x_{jv} = 0, \tag{2.17}$$

$$B_{1i} = \frac{\partial \beta_{1i}}{\partial a_l} \delta a_l = 0, \quad B_{2\kappa} = \frac{\partial \beta_{2\kappa}}{\partial a_\chi} \delta a_\chi = 0. \tag{2.18}$$

Equations (2.13)-(2.18) are the equations in variations of the final and intermediate conditions.

It should be noted that the results produced are directly extended to the case of set (2.1), when $b = \{b_1, \ldots, b_p\}$.

§ 3. Inclusion Lemma

Suppose permissible curve E is fixed, satisfying equations (1.4)-(1.12) and conditions (1.1)-(1.2), and (1.13)-(1.15). If there are no other such curves, the problem as formulated is trivial. Therefore, we must use the following lemma.

Lemma 1. If permissible curve E satisfies equations (1.4)-(1.12) and δt_0, δt_q, δt_v, δt_k, δy_i, δx_j, δu_k and δa_n is the permissible set of variations satisfying the variational equations (2.4)-(2.12) on E, then there is a permissible single-parameter set (2.1) containing curve E with b = 0 consisting of curves satisfying equations (1.2)-(1.12), such that δt_0, δt_q, δt_v, δt_k, $\delta u_k(t)$, $\delta y_i(t)$, $\delta x_j(t)$, δa_n are variations of the set along E.

For proof, let us expand equation system (1.4)-(1.6), (1.7)-(1.9) and (1.10)-(1.12) respectively with equations such as

$$F_\gamma = v_\gamma, \quad a_l = a_l^0 \quad (\gamma = n + d + 1, \ldots, n + l), \tag{3.1}$$

$$G_\alpha = w_\alpha, \quad a_\chi = a_\chi^0 \quad (\alpha = n + h + 1, \ldots, n + l), \tag{3.2}$$

$$H_\beta = z_\beta, \quad a_\chi = a_\chi^0 \quad (\beta = m + k + 1, \ldots, n + l). \tag{3.3}$$

The functions $F_\gamma(y_i, u_q, a_l)$, $G_\alpha(y_l, u_p, a_x^0)$, $H_\beta(x_j, u_p, a_x^0)$ between the corner points have continuous partial derivatives in the vicinity of the set of elements (t, y_i, x_j, u_k, a_n), belonging to curve E. The functional determinant of the system of functions ξ_α, F_γ, γ_β, G_α, ζ_ν, H_β with respect to variables u_k does not vanish at E. Substituting the functions $y_i(t)$, $x_j(t)$, $u_k(t)$ and a_n^0, determining curve E, into equations (1.4)-(1.12) and (3.1)-(3.3), we can determine the functions v_γ, w_γ and z_β, corresponding to curve E. They will be continuous everywhere with the exception of the corner points of curve E. Furthermore, the systems of variational equations (2.4)-(2.6), (2.7)-(2.9) and (2.10)-(2.12) are supplemented by the following variational equations:

$$\frac{\partial F_\gamma}{\partial y_i}\delta y_i + \frac{\partial F_\gamma}{\partial u_q}\delta u_q + \frac{\partial F_\gamma}{\partial a_l}\delta a_l = \delta v_\gamma, \tag{3.4}$$

$$\frac{\partial G_\alpha}{\partial y_i}\delta y_i + \frac{\partial G_\alpha}{\partial u_p}\delta u_p + \frac{\partial G_\alpha}{\partial a_\chi}\delta a_\chi = \delta w_\alpha, \tag{3.5}$$

$$\frac{\partial H_\beta}{\partial x_j}\delta x_j + \frac{\partial H_\beta}{\partial u_p}\delta u_p + \frac{\partial H_\beta}{\partial a_\chi}\delta a_\chi = \delta z_\beta. \tag{3.6}$$

The system of variational equations (2.4)-(2.12) and (3.4)-(3.6) defines functions $\delta v_\gamma(t)$, $\delta w_\alpha(t)$ and $\delta z_\beta(t)$, corresponding to curve E and functions $\delta y_i(t)$, $\delta x_j(t)$, $\delta u_k(t)$ and δa_n. Functions $\delta v_\gamma(t)$, $\delta w_\alpha(t)$ and $\delta z_\beta(t)$ are continuous everywhere except for values of t corresponding to the corner points of curve E. According to the theorem of implicit control functions (1.6), (3.1) or (1.9), (3.2) or (1.12), (3.3) can be solved for u_k, and therefore we produce:

over $[t_0, t_q^-]$

$$\left.\begin{aligned} &\dot{y}_i = \tilde{f}_i(t, y_i, v_\gamma, a_l^0), \\ &\dot{a}_l = 0, \\ &u_q = \gamma_q(t, y_i, v_\gamma, a_l^0); \end{aligned}\right\} \tag{3.7}$$

over $[t_q^+, t_v^-]$

$$\left.\begin{aligned} &\dot{y}_i = \tilde{f}_i(t, y_i, w_\alpha, a_\chi^0), \\ &\dot{a}_n = 0, \\ &u_p = \theta_p(t, y_i, w_\alpha, a_\chi^0); \end{aligned}\right\} \tag{3.8}$$

over $[t_v^+, t_k]$

$$\left.\begin{aligned} &\dot{x}_j = \tilde{g}_j(t, x_j, z_\beta, a_\chi^0), \\ &\dot{a}_n = 0, \\ &u_p = \vartheta_p(t, x_j, z_\beta, a_\chi^0). \end{aligned}\right\} \tag{3.9}$$

where the functions $\tilde{\phi}_i$, χ_s, $\tilde{f}_i$, θ_p, $\tilde{g}_j$, ϑ_p have continuous partial derivatives between the corner points in the area of the set of values (t, y_i, x_j, v_γ, w_α, z_β, a_n^0) of curve E, since the functions ϕ_i, ξ_α, [illegible], G_α and g_j, ζ_ν and H_β have continuous partial derivativ[illegible] [illegible]r points.

Suppose $t^{(1)}$ is the first value of variable t following t_0, corresponding to a corner point of curve E. Supose further E_1 is another curve E corresponding to the interval (t_0, $t^{(1)}$). Functions v_γ and δv_γ, if analyzed only in the interval (t_0, $t^{(1)}$), can be arbitrarily continued so that they remain continuous over a slightly greater interval. Then the right portions in equations

$$\dot{y}_i = \tilde{\varphi}_i(t, y_i, v_\gamma(t) + b\delta v_\gamma(t), a_l^0), \tag{3.10}$$

$$u_q = \chi_q(t, y_i, v_\gamma(t) + b\delta v_\gamma(t), a_l^0) \tag{3.11}$$

are continuous with respect to t, y_i, a_1^0, b and have continuous partial derivatives with respect to variables y_i, a_1^0, b in the vicinity of the values (t, y_i, a_1^0, b = 0) corresponding to arc E_1. It follows from the theorem of existence for differential equations that equations (3.10) have the solution

$$y_i = Y_i(t, t_0, y_{i0}, a_l^0, b),$$

and therefore

$$u_q = \chi_q(t, t_0, u_{q0}, a_l^0, b).$$

Functions Y_i, $\dot{Y}_i$ are continuous and have continuous partial derivatives with respect to variables y_{i0}, a_1^0, b in the area of the set of points (t, t_0, y_{i0}, a_1^0, b), corresponding to arc E_1. The functions

$$\left.\begin{aligned} y_i &= Y_i(t, t_0, y_i(t_0) + b\delta y_i(t_0), a_l^0 + b\delta a_l, b) = y_i(t, b), \\ u_q &= \chi_q(t, t_0, u_q(t_0) + b\delta u_q(t_0), a_l^0 + b\delta a_l, b) = u_q(t, b), \\ a_l &= a_l^0 + b\delta a_l = a_l(b) \end{aligned}\right\} \tag{3.12}$$

then determine the elementary set, the curves of which satisfy equations (1.4)-(1.6) in the interval containing interval $(t_0, t^{(1)})$.

The functions $y_i(t, b)$, $u_q(t, b)$ and $a_l(b)$ from (3.12) where $t = t_0$ take on the following initial values:

$$y_i(t_0, b) = y_i(t_0) + b\delta y_i(t_0),$$
$$u_q(t, b) = u_q(t_0) + b\delta u_q(t_0),$$
$$a_l = a_l^0 + b\delta a_l,$$

and consequently their variations $y_{ib}(t, 0)$, $u_{qb}(t, 0)$, $a_{lb}(0)$ along E where $t = t_0$ have the initial values $\delta y_i(t_0)$, $\delta u_q(t_0)$, δa_l. Further, functions (3.12) satisfy equations (3.10), (3.11) and therefore also equations

$$\dot{y}_i = \varphi_i(y_i(t) + b\delta y_i(t),\ u_q(t) + b\delta u_q(t),\ a_l^0 + b\delta a_l),$$
$$\dot{a}_l = 0,$$
$$\Xi_s(y_i(t) + b\delta y_i(t),\ u_q(t) + b\delta u_q(t)) = 0,$$
$$F_\gamma = v_\gamma(t) + b\delta v_\gamma(t),\ a_l = a_l^0 + b\delta a_l.$$

Thus, derivatives $y_{ib}(t, 0)$, $u_{qb}(t, 0)$, $a_{lb}(0)$ satisfy equations (2.4)-(2.6), (3.4)-(3.6) on E_1 and should be respectively identical with variations $\delta y_i(t)$, $\delta u_q(t)$, δa_l, since these variations make up the unique solution of equations (2.4)-(2.6), (3.4)-(3.6) with the initial values of $\delta y_i(t_0)$, $\delta u_q(t_0)$, δa_l. Consequently, an elementary set is defined, fixed over interval $(t_0, t^{(1)})$, the curves of which satisfy equations (1.4)-(1.6), and the variations of this set along E_1 correspond to the fixed functions $\delta y_i(t)$, $\delta u_q(t)$, δa_l.

Continuing this process of construction of elementary sets, we can produce the single parameter set (2.1) of set F satisfying all requirements of the lemma. We must particularly mention the construction of the new elementary set in the intervals with origins t_q^+ and t_v^+.

According to the condition of the problem, the functional determinant

$$\left|\frac{\partial\psi_\tau}{\partial y_{iq}^{+}}\right| \neq 0.$$

Then we can represent in the area of solution (1.1)

$$y_i(t_q^{+}) = \tilde{\varphi}_\tau(y_i^{-}(t_q^{-}), a_l, t_q). \qquad (3.13)$$

and therefore along E we have

$$\delta y_i(t_q^{+}) = \left(-\dot{y}_{iq}^{+} + \frac{\partial\tilde{\varphi}_\tau}{\partial t_q} + \dot{y}_{iq}^{-}\right)\delta t_q + \\ + \frac{\partial\tilde{\varphi}_\tau}{\partial y_i(t_q^{-})}\delta y_i(t_q^{-}) + \frac{\partial\tilde{\varphi}_\tau}{\partial a_l}\delta a_l. \qquad (3.14)$$

By analogy with (3.12), we write the functions

$$y_i = Y_i(t, t^{(k)}, y_i(t^{(k)}) + b\delta y_i(t^{(k)}), a_l^0 + b\delta a_l, b),$$
$$u_q = \chi_q(t, t^{(k)}, u_q(t^{(k)}) + b\delta u_q(t^{(k)}); a_l^0 + b\delta a_l, b),$$
$$a_l = a_l^0 + b\delta a_l,$$

defining the elementary set, the curves of which satisfy equations (1.4)-(1.6) in the interval containing interval $(t^{(k)}, t_q^{-})$. They take on the following values where $t = t_q^{-}$:

$$\left.\begin{aligned} & y_i(t_q^{-}, b) = y_i(t_q^{-}) + b\delta y_i(t_q^{-}), \\ & u_q(t_q^{-}, b) = u_s(t_q^{-}) + b\delta u_q(t_q^{-}), \\ & a_l = a_l^0 + b\delta a_l. \end{aligned}\right\} \qquad (3.15)$$

Substituting the values of (3.15) in the right portion of (3.13), we have

$$y_{iq}(t_q^{-}, b) = \tilde{\varphi}_\tau(y_{iq}(t_q^{-}) + b\delta y_{iq}(t_q^{-}), a_l^0 + b\delta a_l, t_q, b).$$

From this, after differentiation of this equation, we find that $y_{ib}(t_q^-, 0)$, $a_{lb}(0)$ and $y_{ib}(t_q^+, 0)$ along E_k are identical to variations from (3.14), since there is a unique solution of equations (1.4)-(1.9).

According to a condition of the problem, functional determinant

$$\left|\frac{\partial Q_j}{\partial x_j}\right| \neq 0.$$

Therefore, we can write

$$x_{jv} = \tilde{Q}_j(y_{iv}, t_v). \qquad (3.16)$$

and along E we have

$$\delta x_{jv} = \left(-\dot{x}_{jv} + \dot{y}_{iv} + \frac{\partial \tilde{Q}_j}{\partial t_v}\right)\delta t_v + \frac{\partial \tilde{Q}_j}{\partial y_{iv}}\delta y_{iv}. \qquad (3.17)$$

By analogy with (3.12), we write the functions

$$y_i = Y_i(t, t^{(s)}, y_i(t^{(s)}) + b\delta y_i(t^{(s)}), a_l^0 + b\delta a_l, b),$$
$$u_p = \vartheta_p(t, t^{(s)}, u_p(t^{(s)}) + b\delta u_p(t^{(s)}), a_l^0 + b\delta a_l, b),$$
$$a_l = a_l^0 + b\delta a_l,$$

determining the elementary set, the curves of which satisfy equations (1.6)-(1.9) in the interval containing interval $(t^{(s)}, t_v^-)$. They take on the following values where $t = t_v^-$:

$$\left.\begin{aligned} y_i(t_v^-, b) &= y_i(t_v^-) + b\delta y_i(t_v^-), \\ u_p(t_v^-, b) &= u_p(t_v^-) + b\delta u_p(t_v^-), \\ a_l &= a_l^0 + b\delta a_l. \end{aligned}\right\} \qquad (3.18)$$

Substituting the values of (3.18) into the right portion of (3.16), we find

$$x_{jv}(t_v^-, b) = \tilde{Q}_j(y_i(t_v^-) + b\delta y_i(t_v^-), t_v^-, b).$$

Then after differentiation of this equation with respect to b, we find that $y_{ib}(t_v^-, 0)$, $a_{nb}(0)$ and $x_{jb}(t_v^+, 0)$ along E_s are identical to the corresponding variations of (3.17), since there is a unique solution to equations (1.4)-(1.12).

The functions $t_0(b)$, $t_q(b)$, $t_v(b)$ and $t_k(b)$ can be determined by the equations

$$t_0(b)=t_0+b\delta t_0,\quad t_q(b)=t_q+b\delta t_q,\quad t_v(b)=t_v+b\delta t_v,$$
$$t_k(b)=t_k+b\delta t_k,$$

where t_0, t_q, t_v and t_k correspond to the points of curve E.

Result 1. If permissible curve E satisfies equations (1.4)-(1.12) and if variations $\delta t_{0\alpha}$, $\delta t_{q\alpha}$, $\delta t_{v\alpha}$, $\delta t_{k\alpha}$, $\delta y_{i\alpha}(t)$, $\delta x_{j\alpha}(t)$, $\delta u_{k\alpha}(t)$, $\delta a_{n\alpha}$ represent the p permissible sets of variations satisfying variational equations (2.4)-(2.12) along E, there is a permissible p-parametric set (2.1), containing curve E with values of parameters $b_\alpha = 0$ ($\alpha = 1,\ldots, p$), consisting of curves satisfying equations (1.4)-(1.12) and such that for any $\alpha = 1,\ldots, p$, the values of $\delta t_{0\alpha}$, $\delta t_{q\alpha}$, $\delta t_{v\alpha}$, $\delta t_{k\alpha}(t)$, $\delta x_{j\alpha}(t)$, $\delta u_{k\alpha}(t)$, $\delta a_{n\alpha}$ are variations of this set along E with respect to parameter b_α.

The proof of this statement is quite similar to the proof of the preceding lemma. Therefore we will omit it.

§ 4. Stability Condition

Suppose $\theta + \vartheta + \eta + \varepsilon - e - f + 2$ is the parametric permissible set of curves

$$y_i(t, b_1, \ldots, b_p)\quad (i=1,\ldots,n) \tag{4.1}$$
$$(p=1,\ldots,\theta+\vartheta+\eta+\varepsilon-e-f+2),$$
$$x_j(t, b_1, \ldots, b_p)\ (j=1,\ldots,m), \tag{4.2}$$
$$u_k(t, b_1, \ldots, b_p)\ (k=1,\ldots,l), \tag{4.3}$$
$$a_n(b_1, \ldots, b_p)\quad (n=1,\ldots,r). \tag{4.4}$$

Then for this set of curves we produce

$$J(b)=J(b_1,\ldots,b_p) \tag{4.5}$$

Substituting the values of (4.1)-(4.4) into the left portions of equations (1.1), (1.2) and (1.14), (1.15) and considering (4.5), we find

$$
\left.\begin{array}{l}
J(b_1, \ldots, b_p) = J(0) + \eta, \\
\varphi_0(b_1, \ldots, b_p) = 0, \quad \varphi_\kappa(b_1, \ldots, b_p) = 0, \\
\varphi_\tau(b_1, \ldots, b_p) = 0, \quad \varphi_\pi(b_1, \ldots, b_p) = 0, \\
\beta_{1i}(b_1, \ldots, b_p) = 0, \quad \beta_{2k}(b_1, \ldots, b_p) = 0 \\
\qquad\qquad (i = 1, \ldots, e), \\
\qquad\qquad (k = 1, \ldots, f).
\end{array}\right\} \tag{4.6}
$$

These equations have the solution (b, η) = (0, 0), corresponding to the curve E which gives the functional its maximum. The functional determinant of the left portions of these equations with respect to parameters b_α

$$
\left|\begin{array}{c}
\partial J/\partial b_\alpha \\
\partial\varphi_0/\partial b_\alpha \\
\cdots \\
\partial\varphi_\pi/\partial b_\alpha \\
\cdots \\
\partial\beta_{2k}/\partial b_\alpha
\end{array}\right| = \left|\begin{array}{c}
\delta J(\delta t, \delta y_i, \delta x_j, \ldots, \delta a_n) \\
\Psi_0(\delta t, \delta y_i, \ldots, \delta a_l) \\
\cdots\cdots\cdots\cdots \\
\Psi_\pi(\delta t, \delta y_i, \ldots, \delta a_n) \\
\cdots\cdots\cdots\cdots \\
B_{2k}(\delta a_1, \ldots, \delta a_n)
\end{array}\right| \tag{4.7}
$$

for curve E is equal to zero at point (b, η) = (0, 0) with any selection of variations. Otherwise, according to the theorem of implicit functions, there is a unique system of functions $b_\alpha(\eta)$, continuous in the area η = 0, satisfying conditions $b_\alpha(0) = 0$ and converting equation (4.6) to an identity. Then curve E cannot give a maximum to J(0), since J(η) > J(0) where η > 0.

Thus, if the greatest possible rank of the matrix of the determinant (4.7) is q, then q < p. Therefore, the system of constants l_0, e_0, e_τ, e_π, e_k, e_{1i}, e_{2k} is found, not simultaneously equal to zero, satisfying the system of homogeneous linear equations

$$
l_0\delta J_\alpha + e_0\Psi_{0\alpha} + e_\tau\Psi_{\tau\alpha} + e_\pi\Psi_{\pi\alpha} + e_\kappa\Psi_{\kappa\alpha} + e_{1i}B_{1i\alpha} + e_{2k}B_{2k\alpha} = 0
$$
$$
(\alpha = 1, \ldots, p).
$$

the coefficients of which correspond to the columns of determinant (4.7).

For these constants, the equation

$$l_0 \delta J + e_0 \Psi_0 + e_\tau \Psi_\tau + e_\varepsilon \Psi_\varepsilon + e_\kappa \Psi_\kappa + e_{1l} B_{1l} + e_{2k} B_{2k} = 0 \tag{4.8}$$

should obtain with arbitrary permissible set of variations δt_0, δt_q, δt_v, δt_k, δy_i, δx_j, δu_k, δa_n, satisfying variational equations (2.4)-(2.12) along F.

If we multiply each of the variational equations (2.4)-(2.6) in turn by the Lagrange factors $\lambda_i^{(1)}(t)$, $\eta_e^{(1)}(t)$, $\mu_\alpha^{(1)}(t)$ and add the products produced, then after integration within limits from t_0 to t_q^- we will have

$$\int_{t_0}^{t_q^-} \left[\sum_{i=1}^{n} \left(\delta \dot{y}_i - \frac{\partial \varphi_i}{\partial y_i} \delta y_i - \frac{\partial \varphi_i}{\partial u_q} \delta u_q - \frac{\partial \varphi_i}{\partial a_l} \delta a_l \right) \lambda_i^{(1)}(t) + \right.$$

$$\left. + \sum_{m=1}^{l} \delta \dot{a}_l \eta_l^{(1)}(t) + \sum_{\alpha=1}^{s} \left(\frac{\partial \xi_\alpha}{\partial y_i} \delta y_i + \frac{\partial \xi_\alpha}{\partial u_q} du_q + \frac{\partial \xi_\alpha}{\partial a_l} \delta a_l \right) \mu_\alpha^{(1)}(t) \right] dt =$$

$$= \int_{t_0}^{t_q^-} \left[\sum_{i=1}^{n} \left(\frac{\partial F_1}{\partial y_i} \delta y_i + \frac{\partial F_1}{\partial \dot{y}_i} \delta \dot{y}_i \right) + \sum_{l=1}^{m} \left(\frac{\partial F_1}{\partial a_l} \delta a_l + \frac{\partial F_1}{\partial \dot{a}_l} \delta \dot{a}_l \right) + \right.$$

$$\left. + \sum_{q=1}^{s} \frac{\partial F_1}{\partial u_q} \delta u_q \right] dt = 0 \quad (l = 1, \ldots, m \leqslant r;\ q = 1, \ldots, s \quad l), \tag{4.9}$$

where

$$F_1 = \sum_{i=1}^{n} (\dot{y}_i - \varphi_i) \lambda_i^{(1)}(t) + \sum_{l=1}^{m} \dot{a}_l \eta_l^{(1)}(t) + \sum_{\alpha=1}^{s} \xi_\alpha \mu_\alpha^{(1)}(t).$$

In the same manner we produce

$$\int_{t_q^+}^{t_v} \left[\sum_{i=1}^{n} \left(\frac{\partial F_2}{\partial y_i} \delta y_i + \frac{\partial F_2}{\partial \dot{y}_i} \delta \dot{y}_i \right) + \right.$$

$$\left. + \sum_{n=1}^{r} \left(\frac{\partial F_2}{\partial a_n} \delta a_n + \frac{\partial F_2}{\partial \dot{a}_n} \delta \dot{a}_n \right) + \sum_{p=s'+1}^{l} \frac{\partial F_2}{\partial u_p} \delta u_p \right] dt = 0, \qquad (4.10)$$

$$\int_{t_v^+}^{t_k} \left[\sum_{j=1}^{m} \left(\frac{\partial F_3}{\partial x_j} \delta x_j + \frac{\partial F_3}{\partial \dot{x}_j} \delta \dot{x}_j \right) + \right.$$

$$\left. + \sum_{n=1}^{r} \left(\frac{\partial F_3}{\partial a_n} \delta a_n + \frac{\partial F_3}{\partial \dot{a}_n} \delta \dot{a}_n \right) + \sum_{p=s'+1}^{l} \frac{\partial F_3}{\partial u_p} \delta u_p \right] dt = 0, \qquad (4.11)$$

where

$$F_2 = \sum_{i=1}^{n} (\dot{y}_i - f_i) \lambda_i^{(2)}(t) + \sum_{n=1}^{r} \dot{a}_n \eta_n^{(2)}(t) + \sum_{\xi=1}^{h} \gamma_\xi \mu_\xi^{(2)}(t),$$

$$F_3 = \sum_{j=1}^{m} (\dot{x}_j - g_j) \lambda_j^{(3)}(t) + \sum_{n=1}^{r} \dot{a}_n \eta_n^{(3)}(t) + \sum_{\nu=1}^{k} \zeta_\nu \mu_\nu^{(3)}(t).$$

Since

$$J = \int_{t_0}^{t_k} \dot{J}\, dt,$$

we have the equation

$$\delta J = \int_{t_0}^{t_k} \delta \dot{J}\, dt.$$

If now after multiplication by l_0, we add the left portions of equations (4.9)-(4.11) and the left portions of expressions (2.17), multiplied by the constant Lagrange factor e_{qj} to the right portion of this equation, which does not change the value of the right portion, then, after substituting the value of $l_0 \delta J$ produced in (4.8), we will have

$$
\begin{aligned}
l_1\delta J + e_0\Psi_0 + e_1\Psi_1 + e_2\Psi_2 + e_3\Psi_3 + e_{1i}B_{1i} + \\
+ e_{2k}B_{2k} + e_{qj}\bar{Q}_j + \int\limits_{t_0}^{t_\Phi^-} \Bigg[\sum_{i=1}^{n} \left(\frac{\partial F_1}{\partial y_i} \delta y_i + \frac{\partial F_1}{\partial \dot{y}_i} \delta \dot{y}_i \right) + \\
+ \sum_{l=1}^{m} \left(\frac{\partial F_1}{\partial a_l} \delta a_l + \frac{\partial F_1}{\partial \dot{a}_l} \delta \dot{a}_l \right) + \sum_{q=1}^{s} \frac{\partial F_1}{\partial u_q} \delta u_q \Bigg] dt + \\
+ \int\limits_{t_\Phi^+}^{t_\Psi^-} \Bigg[\sum_{i=1}^{n} \left(\frac{\partial F_2}{\partial y_i} \delta y_i + \frac{\partial F_2}{\partial \dot{y}_i} \delta \dot{y}_i \right) + \\
+ \sum_{n=1}^{r} \left(\frac{\partial F_2}{\partial a_n} \delta a_n + \frac{\partial F_2}{\partial \dot{a}_n} \delta \dot{a}_n \right) \cdot \sum_{p=s'+1}^{l} \frac{\partial F_2}{\partial u_p} \delta u_p \Bigg] dt + \\
+ \int\limits_{t_\Psi^+}^{t_k} \Bigg[\sum_{j=1}^{m} \left(\frac{\partial F_3}{\partial x_j} \delta x_j + \frac{\partial F_3}{\partial \dot{x}_j} \delta \dot{x}_j \right) - \sum_{n=1}^{r} \left(\frac{\partial F_3}{\partial a_n} \delta a_n + \frac{\partial F_3}{\partial \dot{a}_n} \delta \dot{a}_n \right) + \\
+ \sum_{p=s'+1}^{l} \frac{\partial F_3}{\partial u_p} \delta u_p \Bigg] dt = 0.
\end{aligned}
$$

This equation is equivalent to condition

where

$$d\Phi=0, \tag{4.12}$$

$$\Phi=l_0J+e_0\psi_0+e_1\psi_1+e_s\psi_s+e_\kappa\psi_\kappa+e_{1i}\varphi_{1i}+e_{2\kappa}\varphi_{2\kappa}+e_{qj}Q_j+ \\ +\int_{t_0}^{t_q^-}F_1\,dt+\int_{t_q^+}^{t_\varphi^-}F_2\,dt+\int_{t_\varphi^+}^{t_\kappa}F_3\,dt. \tag{4.13}$$

For each permissible curve E giving functional J its maximum and satisfying equations (1.4)-(1.12) and conditions (1.1), (1.2), (1.13)-(1.15), (1.19), condition (4.12) should be fulfilled. It is referred to as the stability condition.

In expanded form, stability condition (4.12) can be written as follows:

$$d\Phi=l_0dJ+e_0d\psi_0+e_1d\psi_1+e_sd\psi_s+e_\kappa d\psi_\kappa+e_{1i}d\varphi_{1i}+e_{2\kappa}d\varphi_{2\kappa}+ \\ +e_{qj}dQ_j+\int_{t_0}^{t_q^-}\left[\left(\frac{\partial F_1}{\partial y_l}-\frac{d}{dt}\frac{\partial F_1}{\partial \dot{y}_l}\right)\Delta y_l\,dt+\frac{\partial F_1}{\partial u_q}\Delta u_q+\right. \\ \left.+\left(\frac{\partial F_1}{\partial a_l}-\frac{d}{dt}\frac{\partial F_1}{\partial \dot{a}_l}\right)\Delta a_l\right]dt+\left[\left(F_1-\dot{y}_l\frac{\partial F_1}{\partial \dot{y}_l}\right)dt+\frac{\partial F_1}{\partial \dot{y}_l}dy_l+\right. \\ \left.+\frac{\partial F_1}{\partial \dot{a}_l}da_l\right]_{t_0}^{t_q^-}+\int_{t_q^+}^{t_\varphi^-}\left[\left(\frac{\partial F_2}{\partial y_l}-\frac{d}{dt}\frac{\partial F_2}{\partial \dot{y}_l}\right)\Delta y_l+\right. \\ \left.+\frac{\partial F_2}{\partial u_p}\Delta u_p+\left(\frac{\partial F_2}{\partial a_n}-\frac{d}{dt}\frac{\partial F_2}{\partial \dot{a}_n}\right)\Delta a_n\right]dt+ \\ +\left[\left(F_2-\dot{y}_l\frac{\partial F_2}{\partial \dot{y}_l}\right)dt+\frac{\partial F_2}{\partial \dot{y}_l}dy_l+\frac{\partial F_2}{\partial \dot{a}_n}da_n\right]_{t_q^+}^{t_\varphi^-}+ \\ +\int_{t_\varphi^+}^{t_\kappa}\left[\left(\frac{\partial F_3}{\partial x_j}-\frac{d}{dt}\frac{\partial F_3}{\partial \dot{x}_j}\right)\Delta x_j+\left(\frac{\partial F_3}{\partial a_n}-\frac{d}{dt}\frac{\partial F_3}{\partial \dot{a}_n}\right)\Delta a_n+\right. \\ \left.+\frac{\partial F_3}{\partial u_p}\Delta u_p\right]dt+\left[\left(F_3-\dot{x}_j\frac{\partial F_3}{\partial \dot{x}_j}\right)dt+\frac{\partial F_3}{\partial \dot{x}_j}\Delta x_j+\right. \\ \left.+\frac{\partial F_3}{\partial \dot{a}_n}\Delta a_n\right]_{t_\varphi^+}^{t_\kappa}=0. \tag{4.14}$$

Here we have kept in mind that

$$\int_{t_0}^{t_q^-} \frac{\partial F_1}{\partial \dot{y}_i}\Delta \dot{y}_i\,dt = \frac{\partial F_1}{\partial \dot{y}_i}\Delta y_i \Bigg|_{t_0}^{t_q^-} - \int_{t_0}^{t_q^-} \frac{d}{dt}\frac{\partial F_1}{\partial \dot{y}_i}\Delta y_i\,dt,$$

$$dy_i = \Delta y_i + \dot{y}_i\,dt, \quad \Delta y_i = \delta y_i\,db.$$

$$\int_{t_0}^{t_q^-} \frac{\partial F_1}{\partial \dot{a}_l}\Delta \dot{a}_l\,dt = \frac{\partial F_1}{\partial \dot{a}_l}\Delta a_l \Bigg|_{t_0}^{t_q^-} - \int_{t_0}^{t_q^-} \frac{d}{dt}\frac{\partial F_1}{\partial \dot{a}_l}\Delta a_l\,dt,$$

$$da_l = \Delta a_l, \quad \Delta u_q = \delta u_q\,db.$$

For the other time intervals we can write expressions similar to those presented above.

In order for equation (4.12) to be true, a number of requirements must be fulfilled. First of all, the Lagrange coefficients $\lambda_i^{(1)}(t)$, $\eta_l^{(1)}(t)$, $\lambda_i^{(2)}(t)$, $\eta_\alpha^{(2)}(t)$, $\lambda_j^{(3)}(t)$, $\eta_\alpha^{(3)}(t)$ will be selected so that they satisfy the following differential equations:

$$\frac{d}{dt}\frac{\partial F_1}{\partial \dot{y}_i} - \frac{\partial F_1}{\partial y_i} = 0 \quad \text{or} \quad \dot{\lambda}_i^{(1)} = -\frac{\partial H_1}{\partial y_i}, \tag{4.15}$$

$$\frac{d}{dt}\frac{\partial F_1}{\partial \dot{a}_l} - \frac{\partial F_1}{\partial a_l} = 0 \quad \text{or} \quad \dot{\eta}_l^{(1)} = -\frac{\partial H_1}{\partial a_l}, \tag{4.16}$$

$$\frac{d}{dt}\frac{\partial F_2}{\partial \dot{y}_i} - \frac{\partial F_2}{\partial y_i} = 0 \quad \text{or} \quad \dot{\lambda}_i^{(2)} = -\frac{\partial H_2}{\partial y_i}, \tag{4.17}$$

$$\frac{d}{dt}\frac{\partial F_2}{\partial \dot{a}_\alpha} - \frac{\partial F_2}{\partial a_\alpha} = 0 \quad \text{or} \quad \dot{\eta}_\alpha^{(2)} = -\frac{\partial H_2}{\partial a_\alpha}, \tag{4.18}$$

$$\frac{d}{dt}\frac{\partial F_3}{\partial \dot{x}_j} - \frac{\partial F_3}{\partial x_j} = 0 \quad \text{or} \quad \dot{\lambda}_j^{(3)} = -\frac{\partial H_3}{\partial x_j}, \tag{4.19}$$

$$\frac{d}{dt}\frac{\partial F_3}{\partial \dot{a}_\alpha} - \frac{\partial F_3}{\partial a_\alpha} = 0 \quad \text{or} \quad \dot{\eta}_\alpha^{(3)} = -\frac{\partial H_3}{\partial a_\alpha}. \tag{4.20}$$

Here

$$\left.\begin{aligned} H_1 &= \varphi_i\lambda_i^{(1)} + \xi_s\mu_s^{(1)} + \dot{a}_l\eta_l^{(1)},\\ H_2 &= f_i\lambda_i^{(2)} + \gamma_\beta\mu_\beta^{(2)} + \dot{a}_\alpha\eta_\alpha^{(2)},\\ H_3 &= g_j\lambda_j^{(3)} + \zeta_\nu\mu_\nu^{(3)} + \dot{a}_\alpha\eta_\alpha^{(3)}.\end{aligned}\right| \tag{4.21}$$

Further, as a result of the independence $(l - d)$ of variations Δu_q, $(l - h)$ of variations $\Delta u_p^{(2)}$ and $(l - k)$ of variations $\Delta u_p^{(3)}$, the coefficients with them should be equal to zero according to the main lemma of variational calculus. The Lagrange factors $\mu_\alpha^{(1)}$, $\mu_\beta^{(2)}$ and $\mu_v^{(3)}$ will be selected so that the coefficients with dependent d of variations Δu_q, h of variations $\Delta u_p^{(2)}$ and k of variations $\Delta u_p^{(3)}$ are equal to zero. Therefore we produce

$$\frac{\partial F_1}{\partial u_q} = 0 \quad \text{or} \quad \frac{\partial H_1}{\partial u_q} = 0, \tag{4.22}$$

$$\frac{\partial F_2}{\partial u_p} = 0 \quad \text{or} \quad \frac{\partial H_2}{\partial u_p} = 0, \tag{4.23}$$

$$\frac{\partial F_3}{\partial u_p} = 0 \quad \text{or} \quad \frac{\partial H_3}{\partial u_p} = 0. \tag{4.24}$$

Then condition (4.14) becomes as follows:

$$\begin{aligned}
&l_0 dJ - \left(F_1 - \dot{y}_{i0} \frac{\partial F_1}{\partial \dot{y}_{i0}} - e_0 \frac{\partial \psi_0}{\partial t_0}\right) dt_0 - \left(\frac{\partial F_1}{\partial \dot{y}_{i0}} - e_0 \frac{\partial \psi_0}{\partial y_{i0}}\right) dy_{i0} + \\
&\quad + e_0 \frac{\partial \psi_0}{\partial a_l} + e_{1l} \frac{\partial \beta_{1l}}{\partial a_l} + \frac{\partial F_1}{\partial \dot{a}_l}\Big|_{t_q^-} - \frac{\partial F_1}{\partial \dot{a}_l}\Big|_{t_0} + \frac{\partial F_2}{\partial \dot{a}_l}\Big|_{t_v^-} - \\
&\quad - \frac{\partial F_2}{\partial \dot{a}_l}\Big|_{t_q^+} + \frac{\partial F_3}{\partial \dot{a}_l}\Big|_{t_n} - \frac{\partial F_3}{\partial \dot{a}_l}\Big|_{t_v^+}\Bigg) da_e + \\
&\quad + \left(F_1 - \dot{y}_{iq}^- \frac{\partial F_1}{\partial \dot{y}_{iq}^-} + e_\tau \frac{\partial \psi_\tau}{\partial t_q}\right) dt_q^- + \\
&+ \left(\frac{\partial F_1}{\partial \dot{y}_{iq}^-} + e_\tau \frac{\partial \psi_\tau}{\partial y_{iq}^-}\right) dy_{iq}^- - \left(F_1 - \dot{y}_{iq}^+ \frac{\partial F_1}{\partial \dot{y}_{iq}^+}\right) dt_q^+ - \\
&- \left(\frac{\partial F_2}{\partial \dot{y}_{iq}^+} - e_\tau \frac{\partial \psi_\tau}{\partial y_{iq}^+}\right) dy_{iq}^+ + \left(F_2 - \dot{y}_{iv} \frac{\partial F_2}{\partial \dot{y}_{iv}} + e_\kappa \frac{\partial \psi_\kappa}{\partial t_v} + \right. \\
&\left. + e_{qj} \frac{\partial Q_j}{\partial t_v}\right) dt_v^- + \left(\frac{\partial F_2}{\partial \dot{y}_{iv}} + e_\kappa \frac{\partial \psi_\kappa}{\partial y_{iv}} + e_{qj} \frac{\partial Q_j}{\partial y_{iv}}\right) dy_{iv} - \\
&- \left(F_3 - \dot{x}_{jv} \frac{\partial F_3}{\partial \dot{x}_{jv}}\right) dt_v^+ - \left(\frac{\partial F_3}{\partial \dot{x}_{jv}} - e_{qj} \frac{\partial Q_j}{\partial x_{jv}}\right) dx_{jv} +
\end{aligned}$$

$$+\left(e_\tau \frac{\partial \psi_\tau}{\partial a_2} + e_{2k} \frac{\partial \vartheta_{2k}}{\partial a_2} + \frac{\partial F_2}{\partial \dot{a}_2} + \frac{\partial F_2}{\partial \dot{a}_2}\bigg|_{t_v^-} - \frac{\partial F_2}{\partial \dot{a}_2}\bigg|_{t_q^+} + \right.$$

$$\left. + \frac{\partial F_2}{\partial \dot{a}_2}\bigg|_{t_\kappa} - \frac{\partial F_2}{\partial \dot{a}_2}\bigg|_{t_v^+}\right) da_2 + \left(F_3 - \dot{x}_{j\kappa} \frac{\partial F_3}{\partial \dot{x}_{j\kappa}} + e_\kappa \frac{\partial \psi_\kappa}{\partial t_\kappa}\right) dt_\kappa +$$

$$+\left(\frac{\partial F_3}{\partial \dot{x}_{j\kappa}} + e_\kappa \frac{\partial \psi_\kappa}{\partial x_{j\kappa}}\right) dx_{j\kappa} = 0.$$

(4.25)

Here for simplification we have assumed set $A_1 \cap A_2$ to be empty.

The coefficients with θ + ' + η + ε + e + f of the dependent sets dt_0, dt_k, $dy_i(t_0)$, $dy_i(t_q^-)$, $dy_i(t_q^+)$, $dy_i(t_v)$, $dx_j(t_v)$, etc. will be set equal to zero by the corresponding selection of Lagrange factors l_0, e_0, $e_\tau, \ldots, e_k$. The coefficients with the remaining independent variations must be equal to zero.

Thus, we produce

$$F_1 - \dot{y}_{i0} \frac{\partial F_1}{\partial \dot{y}_{i0}} - e_0 \frac{\partial \psi_0}{\partial t_0} = 0 \quad \text{or} \quad H_{10} + e_0 \frac{\partial \psi_0}{\partial t_0} = 0, \tag{4.26}$$

$$\frac{\partial F_1}{\partial \dot{y}_{i0}} - e_0 \frac{\partial \psi_0}{\partial y_{i0}} = 0 \quad \text{or} \quad \lambda_{i0}^{(1)} - e_0 \frac{\partial \psi_0}{\partial y_{i0}} = 0 \quad (i = 1, \ldots, n), \tag{4.27}$$

$$F_3 - \dot{x}_{j\kappa} \frac{\partial F_3}{\partial \dot{x}_{j\kappa}} + e_\kappa \frac{\partial \psi_\kappa}{\partial t_\kappa} + l_0 \frac{\partial J}{\partial t_\kappa} = 0$$

or

$$-H_{3\kappa} + e_\kappa \frac{\partial \psi_\kappa}{\partial t_\kappa} + l_0 \frac{\partial J}{\partial t_\kappa} = 0;$$

$$\frac{\partial F_3}{\partial x_{j\kappa}} + e_\kappa \frac{\partial \psi_\kappa}{\partial x_{j\kappa}} + l_0 \frac{\partial J}{\partial x_{j\kappa}} = 0 \quad \text{or} \quad \lambda_{j\kappa}^{(3)} + e_\kappa \frac{\partial \psi_\kappa}{\partial x_{j\kappa}} + l_0 \frac{\partial J}{\partial x_{j\kappa}} = 0, \tag{4.28}$$

(4.29)

$$l_0 \frac{\partial J}{\partial a_l} + e_0 \frac{\partial \psi_0}{\partial a_l} + e_\tau \frac{\partial \psi_\tau}{\partial a_l} + e_{1l} \frac{\partial \vartheta_{1l}}{\partial a_l} + \frac{\partial F_2}{\partial \dot{a}_l}\bigg|_{t_v^+}^{t_\kappa} + \frac{\partial F_1}{\partial \dot{a}_l}\bigg|_{t_q^+}^{t_v^-} +$$

$$+ \frac{\partial F_1}{\partial \dot{a}_l}\bigg|_{t_0}^{t_q^-} = 0 \quad \text{or} \quad l_0 \frac{\partial J}{\partial a_l} + e_0 \frac{\partial \psi_0}{\partial a_l} + e_{1l} \frac{\partial \vartheta_{1l}}{\partial a_l} +$$

$$+ \eta_{l\kappa}^{(3)} - \eta_{l0}^{(1)} = 0 \qquad (l = 1, \ldots, m \leqslant r), \tag{4.30}$$

$$l_0 \frac{\partial I}{\partial a_\chi} + e_\tau \frac{\partial \psi_\tau}{\partial a_\chi} + e_{2\varkappa} \frac{\partial \vartheta_{2\varkappa}}{\partial a_\chi} + \frac{\partial F_2}{\partial a_\chi}\bigg|_{t_v^-} - \frac{\partial F_2}{\partial a_\chi}\bigg|_{t_q^+} + \frac{\partial F_3}{\partial a_\chi}\bigg|_{t_\varkappa} -$$

$$- \frac{\partial F_3}{\partial a_\chi}\bigg|_{t_v^+} = 0 \text{ or } l_0 \frac{\partial I}{\partial a_\chi} + e_\tau \frac{\partial \psi_\tau}{\partial a_\chi} + e_{2\varkappa} \frac{\partial \vartheta_{2\varkappa}}{\partial a_\chi} + \eta_{\chi\varkappa}^{(3)} - \eta_{\chi q}^{(2)} = 0 \quad (4.31)$$

$$(\chi = m + 1, \ldots, r),$$

$$F_1 - \dot{y}_{iq}^- \frac{\partial F_1}{\partial \dot{y}_{iq}^-} + e_\tau \frac{\partial \psi_\tau}{\partial t_q} - F_2 + \dot{y}_{iq}^+ \frac{\partial F_2}{\partial \dot{y}_{iq}^+} = 0$$

or

$$H_{2q} - H_{1q} + e_\tau \frac{\partial \psi_\tau}{\partial t_q} = 0, \quad (4.32)$$

$$\frac{\partial F_1}{\partial \dot{y}_{iq}^-} + e_\tau \frac{\partial \psi_\tau}{\partial \dot{y}_{iq}^-} = 0 \text{ or } \lambda_{iq}^{(1)} + e_\tau \frac{\partial \psi_\tau}{\partial y_{iq}^-} = 0 \ (i = 1, \ldots, n), \quad (4.33)$$

$$- \frac{\partial F_2}{\partial \dot{y}_{iq}^+} + e_\tau \frac{\partial \psi_\tau}{\partial y_{iq}^+} = 0 \text{ or } \lambda_{iq}^{(2)} - e_\tau \frac{\partial \psi_\tau}{\partial y_{iq}^+} = 0, \quad (4.34)$$

$$F_2 - \dot{y}_{iv} \frac{\partial F_2}{\partial \dot{y}_{iv}} + e_\varkappa \frac{\partial \psi_\varkappa}{\partial t_v} + e_{qj} \frac{\partial Q_j}{\partial t_v} - F_3 + \dot{x}_{jv} \frac{\partial F_3}{\partial \dot{x}_{jv}} = 0$$

or

$$H_{3v} - H_{2v} + e_\varkappa \frac{\partial \psi_\varkappa}{\partial t_v} + e_{qj} \frac{\partial Q_j}{\partial t_v} = 0, \quad (4.35)$$

$$\frac{\partial F_2}{\partial \dot{y}_{iv}} + e_\varkappa \frac{\partial \psi_\varkappa}{\partial y_{iv}} + e_{qj} \frac{\partial Q_j}{\partial y_{iv}} = 0 \text{ or } \lambda_{iv}^{(2)} + e_\varkappa \frac{\partial \psi_\varkappa}{\partial y_{iv}} + e_{qj} \frac{\partial Q_j}{\partial y_{iv}} = 0, \quad (4.36)$$

$$\frac{\partial F_3}{\partial \dot{x}_{jv}} - e_{qj} \frac{\partial Q_j}{\partial x_{jv}} = 0 \text{ or } \lambda_{jv}^{(3)} - e_{qj} \frac{\partial Q_j}{\partial x_{jv}} = 0. \quad (4.37)$$

Here it is considered that $t_q^- = t_q^+$, $t_v^- = t_v^+$,

$$\eta_{lq}^{(1)} = \eta_l^{(2)} = \eta_l^{(3)}, \ \eta_{\chi v}^{(2)} = \eta_{\chi v}^{(3)}. \quad (4.38)$$

Furthermore, due to the possibility of free selection of Lagrange coefficients $\eta_{l_0}^{(1)}$, $\eta_{\chi 0}^{(2)}$ from (4.30) and (4.31), we have

$$\eta_{l_0}^{(1)} - e_0 \frac{\partial \psi_0}{\partial a_l} - e_{1l} \frac{\partial \vartheta_{1l}}{\partial a_l} = 0, \tag{4.39}$$

$$\eta_{l\theta}^{(2)} - e_\tau \frac{\partial \psi_\tau}{\partial a_\chi} = 0, \tag{4.40}$$

$$\eta_{l\kappa}^{(3)} + l_0 \frac{\partial J}{\partial a_l} = 0, \tag{4.41}$$

$$\eta_{\chi\kappa}^{(3)} + l_0 \frac{\partial J}{\partial a_\chi} + e_{2\kappa} \frac{\partial \vartheta_{2\kappa}}{\partial a_\chi} = 0. \tag{4.42}$$

Relationships (4.26)-(4.29) make up the transversality conditions, related to the phase coordinates at the ends. Equations (4.30) and (4.31) or (4.39)-(4.42) can be called the conditions of optimality of the structural states of the object or the conditions of optimality of a parameter. They allow us to determine the optimal parameters a_n. Equations (4.32)-(4.38) can be called the conditions for discontinuity and conversion of Lagrange coefficients. They make it possible to determine the Lagrange factors to the right (or the left) of the discontinuity point in t_q and t_v.

It is demonstrated in [24, 30] that at the first order discontinuity point of the control function, Lagrange factors λ ,n and the expression $F = \dot{y}_i \partial F/\partial \dot{y}_i$ should be equal left and right. This should always be kept in mind.

Equation system (4.15)-(4.20) and (4.22)-(4.24) will be referred to as the Euler-Lagrange equation system.

Equation system (4.15)-(4.20), (4.22)-(4.24) and system of relationships (4.26)-(4.37) express the stability condition of functional J. In solving the problem of optimization of the functional, the coupling equations (1.4)-(1.12) and dependences (1.1)-(1.3), (1.13)-(1.15) should be attached to it. Thus, in order to calculate the 2n + 2m + r constants appearing as a result of integration of the 2n + 2m + r first order equations (1.4), (1.7), (1.10) and (4.15)-(4.20), the r parameters a_n and $\theta + \vartheta + \eta + \varepsilon + e + f + m + 1$ Lagrange factors e_0, e_k,..., l_0 together with quantities t_0, t_q, t_v, t_k, we have $2n + 2m + 2r + \theta + \vartheta + \eta + \varepsilon + e + f + m + 5$ conditions (1.1)-(1.3), (1.13)-(1.15) and (4.26)-(4.37). Equations (4.26)-(4.37) are homogeneous relative to the Lagrange factors l_0, e_0,..., e_k, and therefore we can assume $l_0 = 1$. We then produce that the number of unknowns is equal to the number of conditions. Therefore, solution of the problem of the maximum of J is reduced to integration of equation system (1.4)-(1.12) and (4.15)-(4.20) considering conditions (4.22)-(4.24) and the solution of the multipoint boundary problem for satisfaction of conditions (1.1)-(1.3), (1.13)-(1.15) and (4.25)-(4.37).

Furthermore, we should note that due to the independence of F_1 in explicit form on t, we have

from which from $[t_0, t_q^-]$ $\frac{d}{dt}\left(F_1 - \dot{y}_i \frac{\partial F_1}{\partial \dot{y}_i}\right) = 0,$

$$F_1 - \dot{y}_i \frac{\partial F_1}{\partial \dot{y}_i} = c_1 \quad \text{or } H_1 = -c_1. \tag{4.43}$$

By analogy we produce

$$H_2 = -c_2 \quad \text{at} \quad [t_q^+, t_v^-]. \tag{4.44}$$

$$H_3 = -c_3 \quad \text{at} \quad [t_v^+, t_k]. \tag{4.45}$$

In conclusion, let us produce a definition of the extreme for the variational problem stated. Curve E, giving the maximum to functional J, consists of piecewise-smooth arcs E_{0q}, E_{qv} and E_{vk}. In connection with this, we designate by the extreme the set of permissible piecewise-smooth arcs E_{0q}, E_{qv} and E_{vk} of curve E and the corresponding Lagrange factors l_0, $\lambda_1^{(1)}$, $\eta_1^{(1)}$, $\lambda_1^{(2)}$, $\eta_n^{(2)}$, $\lambda_j^{(3)}$, $\eta_n^{(3)}$ and e_τ, e_q and e_π, satisfying equations (1.4)-(1.13) and (4.15)-(4.23) and the discontinuity conditions and transformation (1.1)-(1.3), (4.32)-(4.37). The extreme is called nonsingular if the determinant $\left|\frac{\partial^2 H_\gamma}{\partial u_i \partial u_k}\right|$ (γ = 1, 2, 3) does not vanish on it between corner points for values of u located within the closed area.

We formulate the following theorem.

Theorem 1. Inclusion Theorem. Any nonsingular extreme E with fixed factor l_0 is contained with values of

$$t_0 \leqslant t \leqslant t_k,\ \alpha_{10}, \ldots, \alpha_{p0};\ \beta_{10}, \ldots, \beta_{p0}$$

in a 2p-parametric set of extremes

$$y_i(t, \alpha, \beta),\ a_s(\alpha, \beta),\ \lambda_i^{(1)}(t, \alpha, \beta),\ \eta_s^{(1)}(t, \alpha, \beta)$$

$$\text{over} \quad t_0 \leqslant t \leqslant \bar{t}_q;$$

$$y_i(t, \alpha, \beta),\ a_s(\alpha, \beta),\ \lambda_i^{(2)}(t, \alpha, \beta),\ \eta_s^{(2)}(t, \alpha, \beta)$$

$$\text{over} \quad t_q^+ \leqslant t \leqslant \bar{t}_v;$$

$$x_j(t, \alpha, \beta),\ a_s(\alpha, \beta),\ \lambda_j^{(3)}(t, \alpha, \beta),\ \eta_s^{(3)}(t, \alpha, \beta)$$

$$\text{over} \quad t_v^+ \leqslant t \leqslant t_k,$$

where

$$\alpha = (\alpha_1, \ldots, \alpha_p),\ \beta = (\beta_1, \ldots, \beta_p).$$

The functions y_i, y_{it}, a_s, $\lambda_i^{(\gamma)}$, $\lambda_{it}^{(\gamma)}$, $\eta_s^{(\gamma)}$, $\eta_{st}^{(\gamma)}$ ($\gamma = 1,2$) and x_j, x_{jt}, $\lambda_j^{(3)}$, $\lambda_{jt}^{(3)}$, $\eta_s^{(3)}$, $\eta_{st}^{(3)}$ have continuous derivatives in the area of the set of values (t, α, β) corresponding to arcs E_{0q}, E_{qv} and E_{vk} of curve E. The determinants

$$\begin{vmatrix} y_{i\alpha_k} & y_{i\beta_k} & a_{s\alpha_k} & a_{s\beta_k} \\ \lambda_{i\alpha_k}^{(1)} & \lambda_{i\beta_k}^{(1)} & \eta_{s\alpha_k}^{(1)} & \eta_{s\beta_k}^{(1)} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} x_{j\alpha_k} & x_{j\beta_k} & a_{s\alpha_k} & a_{s\beta_k} \\ \lambda_{j\alpha_k}^{(3)} & \lambda_{j\beta_k}^{(3)} & \eta_{s\alpha_k}^{(3)} & \eta_{s\beta_k}^{(3)} \end{vmatrix} \tag{4.47}$$

do not vanish on arcs of E.

Since the extreme of E is nonsingular and due to the conditions of the problem related to vector functions ξ, γ and ζ, the determinants

$$\begin{vmatrix} H_{1\alpha_i\beta_k} & \xi_{\alpha_i} \\ \xi_{\alpha_k} & 0 \end{vmatrix}, \quad \begin{vmatrix} H_{2\alpha_i\beta_k} & \psi_{\alpha_i} \\ \psi_{\alpha_k} & 0 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} H_{3\alpha_i\beta_k} & \zeta_{\alpha_i} \\ \zeta_{\alpha_k} & 0 \end{vmatrix}$$

are not equal to zero on the arcs of extreme E. In this connection, the right portions of differential equations (4.15)-(4.20) can be represented as dependent only on the phase variables, the parameter and the Lagrange coefficients $\lambda^{(\gamma)}$ and $\eta^{(\gamma)}$ ($\gamma = 1, 2, 3$) respectively. Then, application of the theorems of existence for differential equations of systems (4.15) and (4.16), (4.17) and (4.18), (4.19) and (4.20) of the theorem with the exception of the affirmation concerning the determinants is proven. Transition from one arc of the extreme to another is achieved by satisfaction of equations (1.1)-(1.3) and the discontinuity equations (4.32)-(4.37).

Differentiating the identities

$$y_{i0} = y_i(t_0, \alpha, \beta), \ \lambda_{i0}^{(1)} = \lambda_i^{(1)}(t_0, \alpha, \beta),$$
$$a_s = a_s(\alpha, \beta) \text{ and } \eta_{i0}^{(1)} = \eta_i^{(1)}(t_0, \alpha, \beta)$$

with respect to variables $\alpha_i = y_{i0}$ ($i = 1, \ldots, n$), $\beta_i = \lambda_{i0}^{(1)}$, $\alpha_l = a_s$ ($s = 1, \ldots, r$; $l = n + s$) and $\beta_l = \eta_{s0}^{(1)}$ we find that the first determinant from (4.47) is equal to one where $t = t_0$.

It can be shown that this determinant is not equal to zero throughout $[t_0, t_q^-]$ on E_1 if we differentiate system (4.15), (4.16) with respect to α_k and use a known theorem from the theory of linear differential equations concerning the Vronskiy determinant in the partial interval. Similar proofs can be given for the other determinants (4.47). Thus, the theorem of inclusion is fully proven.

§ 5. The Necessary Weierstrass Condition and the Minimum Principle

Suppose the permissible curve E is a normal curve realizing the maximum of functional J. This curve satisfies the stability condition with the unique system of factors $l_0 = 1$, $\lambda_i^{(1)}(t)$, $\lambda_i^{(2)}(t)$, $\lambda_j^{(3)}(t)$, $\eta_1^{(1)}(t), \ldots, e_0, e_\tau, \ldots, e_{qj}$. Let us select an arbitrary point on curve E, t_3, lying in the interval $t_v^* \leq t_3 \leq t_k$, for example. Suppose ϑ_p is a series of number $\vartheta_p \in U$, such that element $(t_3, x_j(t_3), a_\gamma, \vartheta_p)$ is permissible and satisfies equations (1.10)-(1.12). Using § 3, we can affirm that the following set of permissible curves exists:

$$y_i(t, b),\ x_j(t, b),\ u_k(t, b),\ a_n(b)\ (t_0 - \delta < t < t_3;\ |b| < \varepsilon), \tag{5.1}$$

$$X_j(t, b),\ \vartheta_p(t),\ a_\gamma(b)\ (t_3 \leqslant t \leqslant t_3 + e;\ |b| < \varepsilon,\ |e| < \varepsilon), \tag{5.2}$$

$$x_j(t, b),\ u_p(t, b),\ a_\gamma(b)\ (t_3 + e \leqslant t \leqslant t_k + \delta,\ |b| < \varepsilon,\ |e| < \varepsilon), \tag{5.3}$$

satisfying differential equations of the form (3.7)-(3.9) in the first and third of these time intervals, and the following equations in the second interval:

$$\left.\begin{aligned} \dot{X}_j &= \tilde{g}_j(t, X_j, Z_\beta, a_\gamma), \\ \dot{a}_\gamma &= 0, \\ \vartheta_p &= \gamma_p(t, \dot{X}_j, Z_\beta, a_\gamma). \end{aligned}\right\} \tag{5.4}$$

where ϑ_p refers to any permissible control. Furthermore, sets (5.1)-(5.3) were constructed so that they must satisfy the following initial conditions:

$$\left.\begin{aligned} y_i(b, t_0) &= y_i(t_0) + \sum_{\sigma=1}^{p} b_\sigma \delta y_i(t_0), \\ a_n(b) &= a_n^0 + \sum_{\sigma=1}^{p} b_\sigma \delta a_n, \\ x_j(b, t_0) &= x_j(t_0) + \sum_{\sigma=1}^{p} b_\sigma \delta x_j(t_0), \\ X_j(b, t_3) &= x_j(b, t_3),\ x_j(b, e, t_3 + e) = X_j(b, t_3 + e). \end{aligned}\right\} \tag{5.5}$$

Where $e > 0$, sets (5.1)-(5.3) together make up one permissible ($p + 1$)-parameter set, consisting of a finite sequence of adjacent elementary sets. Based on the proof of lemma 1 (§3), we can similarly prove that the functions defining sets (5.1)-(5.3), as well as their derivatives with respect to t have continuous partial derivatives not only with respect to b, but also with respect to e between the corner points.

Where $b = e = 0$, sets (5.1) and (5.3) produce functions defining curve E. Actually, like functions $y_i(t)$, $u_k(t)$, $x_j(t)$, these functions satisfy equations (3.7)-(3.9) where $b = 0$, are continuous between corner points and on the basis of conditions (5.5), take on the corresponding initial values where $t = t_0$, while the solution of equations (3.7)-(3.9) satisfying the fixed initial conditions is defined, as we have shown, unambiguously. The variations of sets (5.1), (5.3) along E with respect to parameter b_α satisfy the variational equations (2.4)-(2.12) and (3.4)-(3.6). Differentiating relationships (5.5), we find that the variations produced make up the solution of variational equations (2.4)-(2.12) and (3.4)-(3.6).

Suppose

$$\frac{\partial y_i}{\partial b_\alpha}=\delta y_{i\alpha},\ \frac{\partial x_j}{\partial b_\alpha}=\delta x_{j\alpha},\ \frac{\partial u_k}{\partial b_\alpha}=\delta u_{k\alpha},\ \frac{\partial a_n}{\partial b_\alpha}=\delta a_{n\alpha}$$

represents the variations of sets (5.1) and (5.3) with respect to parameters b_α ($\alpha = 1,\ldots, p$), while

$$\frac{\partial y_i}{\partial e}=\delta y_i=0,\ \frac{\partial u_k}{\partial e}=\delta u_k=0,\ \frac{\partial a_n}{\partial e}=\delta a_n=0,\ \frac{\partial x_j}{\partial e}=\delta x_j$$

represents the analogous variations with respect to parameter E.

Based on (5.1), (5.3) and (5.5), we have

$$\delta x_j=0(t_0-\delta<t<t_3),\ \dot{x}_j(t_3)+\delta x_j(t_3)=\dot{X}_j(t_3). \tag{5.6}$$

Further, we introduce

$$\left.\begin{aligned} t_0(b)=t_0+\sum_{\alpha=1}^{p}b_\alpha\delta t_{0\alpha},\ t_q(b)=t_q+\sum_{\alpha=1}^{p}b_\alpha\delta t_{q\alpha},\\ t_v(b)=t_v+\sum_{\alpha=1}^{p}b_\alpha\delta t_{v\alpha},\ t_{\kappa b}=t_\kappa+\sum_{\alpha=1}^{p}b_\alpha\delta t_{\kappa\alpha},\end{aligned}\right\}. \tag{5.7}$$

where t_0, t_q, t_v and t_k correspond to curve E. If we substitute functions y_i, x_j, u_k, a_n and t_0, t_q, t_v, t_k according to (5.1), (5.3) and (5.7) into conditions (1.1), (1.2) and (1.13)-(1.15), they will appear as follows:

$$\varphi_0(b)=0,\ \varphi_1(b)=0,\ \varphi_2(b)=0,\ \varphi_{11}(b)=0,\ \varphi_{22}(b)=0, \tag{5.8}$$

$$\varphi_\alpha(b,\ e)=0. \tag{5.9}$$

Since curve E realizing the maximum satisfies conditions (1.15), equation (5.9) has partial solution (b, e) = (0, 0), for which the functional determinant

$$|\partial\varphi_\gamma/\partial b_\alpha| \tag{5.10}$$

is not equal to zero, since the curve is assumed normal. Therefore, equations (5.9) determine the functions

$$b_\alpha=B_\alpha(e), \tag{5.11}$$

which vanish where e = 0 and have continuous partial derivatives in the area of this value. Here for brevity, functions (5.8)-(5.9) are represented by the corresponding ψ_μ. Then where e = 0, the following relationships can be fulfilled:

$$\sum_{\alpha=1}^{p}\left(\frac{\partial\psi_\mu}{\partial b_\alpha}\right)_0 B'_\alpha(0)+\left(\frac{\partial\psi_\mu}{\partial e}\right)_0=0, \tag{5.12}$$

where the subscript 0 indicates calculation of derivatives where e = 0.

If the values of b_α according to (5.11) are substituted into functions (5.1)-(5.3), (5.7) and (5.8), we produce a single-parameter set of curves containing curve E where e = 0. With sufficiently small $e \geq 0$, all curves of this set are permissible curves, satisfying equations (1.4)-(1.12) and conditions (1.1), (1.2), (1.13)-(1.15). Thus, they all belong to class D. Curves (5.1)-(5.3) are not permissible with small, negative e. Actually, each value from the interval $t_3 + e < t < t_3$ corresponds in this case to three points on this curve. If functional J reaches its maximum on curve E, the value of J on the curves of this set cannot increase as e increases from zero. Therefore, the necessary condition for the maximum of functional J will be the inequality

$$\left(\frac{dI}{de}\right)_0 = \sum_{\alpha=1}^{p} \left(\frac{\partial I}{\partial b_\alpha}\right)_0 B'_\alpha(0) + \left(\frac{\partial I}{\partial e}\right)_0 < 0, \tag{5.13}$$

where

$$I(b, e) = J. \tag{5.14}$$

Adding expressions identical to zero to the left portion of (5.14), we produce

$$I = J + \int_{t_0(0)}^{t_q(0)} F_1 dt + \int_{t_q(0)}^{t_p(0)} F_2 dt + \int_{t_p(0)}^{t_1} F_3 dt + \int_{t_0}^{t_0+e} \bar{F}_3 dt + \int_{t_0+e}^{t_1} F_3 dt.$$

Differentiating this expression with respect to b_α, we find

$$\begin{aligned}
\frac{\partial I}{\partial b_\alpha} = \delta J_\alpha - \left(F_1 \delta t_\alpha + \frac{\partial F_1}{\partial \dot{y}_i} \delta y_{i\alpha}\right)_{t_0} + \left(F_1 \delta t_\alpha + \frac{\partial F_1}{\partial \dot{y}_i^-} \delta y_{i\alpha}^- + \frac{\partial F_1}{\partial \dot{a}_q} \delta a_{1\alpha} - \right. \\
\left. - F_2 \delta t_\alpha - \frac{\partial F_2}{\partial \dot{y}_i^+} \delta y_{i\alpha}^+ \right)_{t_q} + \left(F_2 \delta t_\alpha + \frac{\partial F_2}{\partial \dot{y}_i} \delta y_{i\alpha} + \frac{\partial F_2}{\partial \dot{a}_s} \delta a_{s\alpha} - F_3 \delta t_\alpha - \right. \\
\left. - \frac{\partial F_3}{\partial \dot{x}_j} \delta x_{j\alpha}\right)_{t_p} + \left(F_3 \delta t_\alpha + \frac{\partial F_3}{\partial \dot{x}_j} \delta x_{j\alpha} + \frac{\partial F_3}{\partial \dot{a}_s} \delta a_{s\alpha}\right)_{t_0} + \left(\frac{\partial F_3}{\partial \dot{x}_j} \delta x_{j\alpha} + \right. \\
\left. + \frac{\partial F_3}{\partial \dot{a}_s} \delta a_{s\alpha}\right)_{t_0} - \left(\frac{\partial F_3}{\partial \dot{x}_j} \delta \dot{x}_{j\alpha} + \frac{\partial F_3}{\partial \dot{a}_s} \delta a_{s\alpha}\right)_{t_0+e} + \\
+ \int_{t_0}^{t_q} \left[\left(\frac{\partial F_1}{\partial y_i} - \frac{d}{dt} \frac{\partial F_1}{\partial \dot{y}_i}\right) \delta y_{i\alpha} + \left(\frac{\partial F_1}{\partial a_q} - \frac{d}{dt} \frac{\partial F_1}{\partial \dot{a}_q}\right) \delta a_{1\alpha} + \frac{\partial F_1}{\partial u_q} \delta u_{q\alpha}\right] dt + \\
+ \int_{t_q}^{t_p} \left[\left(\frac{\partial F_2}{\partial y_i} - \frac{d}{dt} \frac{\partial F_2}{\partial \dot{y}_i}\right) \delta y_{i\alpha} + \left(\frac{\partial F_2}{\partial a_2} - \frac{d}{dt} \frac{\partial F_2}{\partial \dot{a}_2}\right) \delta a_{2\alpha} + \right.
\end{aligned}$$

$$+\frac{\partial F_2}{\partial u_p}\delta u_{p\mu}\Big]dt+\int\limits_{t_0}^{t_2}\Big[\Big(\frac{\partial F_3}{\partial x_j}-\frac{d}{dt}\frac{\partial F_3}{\partial \dot{x}_j}\Big)\delta x_{j\mu}+\Big(\frac{\partial F_3}{\partial a_z}-\frac{d}{dt}\frac{\partial F_3}{\partial \dot{a}_z}\Big)\delta a_{z\mu}+$$

$$+\frac{\partial F_3}{\partial u_p}\delta u_{p\mu}\Big]dt+\int\limits_{t_0}^{t_0+e}\Big[\Big(\frac{\partial F_3}{\partial X_j}\delta X_{j\mu}+\frac{\partial F_3}{\partial \dot{X}_j}\delta \dot{X}_{j\mu}\Big)+$$

$$+\Big(\frac{\partial F_3}{\partial a_z}\delta a_z+\frac{\partial F_3}{\partial \dot{a}_z}\delta a_z\Big)\Big]dt+\int\limits_{t_0+e}^{t_k}\Big[\Big(\frac{\partial F_3}{\partial x_j}-\frac{d}{dt}\frac{\partial F_3}{\partial \dot{x}_j}\Big)\delta x_{j\mu}+$$

$$+\Big(\frac{\partial F_3}{\partial a_z}-\frac{d}{dt}\frac{\partial F_3}{\partial \dot{a}_z}\Big)\delta a_{z\mu}+\frac{\partial F_3}{\partial u_p}\delta u_{p\mu}\Big]dt.$$

In the case of fulfillment of the stability condition and where $e = 0$, the equation produced can be represented as

$$\frac{\partial I}{\partial \vartheta_\kappa}+e_\mu\frac{\partial \psi_\mu}{\partial \vartheta_\kappa}=0, \tag{5.15}$$

where e_μ represents the Lagrange coefficients $e_0, \ldots, e_k$ corresponding to ψ_μ.

Now we can similarly determine the derivative

$$\frac{\partial I}{\partial e}=[F_3(\dot{X}_j, x_j, \vartheta_p, a_z, \lambda_j^{(3)}, \eta_z^{(3)}, \mu_\cdot^{(3)})\delta t-F_3(\dot{x}_j, x_j,$$

$$u_p, a_z, \lambda_j^{(3)}, \eta_z^{(3)}, \mu_\cdot^{(3)})\delta t]_{t_0+e}-\frac{\partial F_3}{\partial \dot{x}_j}\delta x_j\Big|_{t_0+e}+\Big(F_3\delta t+\frac{\partial F_3}{\partial \dot{x}_j}\delta x_j\Big)_{t_k}+$$

$$+\int\limits_{t_0}^{t_0+e}\Big(\frac{\partial F_3}{\partial X_j}\delta X_j+\frac{\partial F_3}{\partial \dot{X}_j}\delta \dot{X}_j\Big)dt+\int\limits_{t_0+e}^{t_k}\Big(\frac{\partial F_3}{\partial x_j}-\frac{d}{dt}\frac{\partial F_3}{\partial \dot{x}_j}\Big)\delta x_j dt.$$

Keeping in mind the stability condition and deperdences (5.6)-(5.7), where $e = 0$ we will have

$$\Big(\frac{\partial I}{\partial e}\Big)_0+e_\mu\Big(\frac{\partial \psi_\mu}{\partial e}\Big)_0=[F_3(\dot{X}_j, x_j, \vartheta_p, a_z, \lambda_j^{(3)}, \eta_z^{(3)}, \mu_\cdot^{(3)})-$$

$$-F_3(\dot{x}_j, x_j, u_p, a_z, \lambda_j^{(3)}, \eta_z^{(3)}, \mu_\cdot^{(3)})-(\dot{X}_j-\dot{x}_j)\frac{\partial F_3}{\partial \dot{x}_j}\Big]_{t_0}=E|_{t_0}; \tag{5.16}$$

where

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$$E = F_3(\dot{X}_j, x_j, \vartheta_p, a_s, \lambda_j^{(3)}, \eta_i^{(3)}, \mu_\nu^{(3)}) -$$
$$- F_3(\dot{x}_j, x_j, u_p, a_s, \lambda_j^{(3)}, \eta_i^{(3)}, \mu_\nu^{(3)}) - (\dot{X}_j - \dot{x}_j)\frac{\partial F_3}{\partial \dot{x}_j} \qquad (5.17)$$

is the Weierstrass condition.

Expression (5.16) considering (5.12) and (5.15) can be represented as

$$F|_{t_3} = \sum_{\nu=1}^{p} \left(\frac{\partial I}{\partial b_\nu}\right)_0 B_\nu^{\cdot}(0) + \left(\frac{\partial I}{\partial \varepsilon}\right)_0 .$$

Then according to (5.13) we produce

$$E|_{t_3} \leq 0. \qquad (5.18)$$

Thus, we have proven the necessity of fulfillment of condition (5.18) for the normal curve E which realizes the maximum at any point t_3 lying between the corner points of curve E. At points t_q and t_v, function E may undergo first order discontinuities as a result of first order discontinuities in the right portions of equations (1.4), (1.7), (1.10). Therefore, at these points the inequality should be tested twice for the left and right limits of function E.

Theorem 2. Necessary Weierstrass Condition. Permissible curve E, satisfying the stability condition with factors $l_0 = 1$, $\lambda(t)$, $\eta(t)$ and equations (1.1)-(1.13), satisfies the necessary Weierstrass condition if for any element (t, y, a, u) or (t, x, a, u) of curve E between the corner points, the inequality

$$E \leq 0$$

is fulfilled with all possible, permissible (t, y, a, ϑ) ≠ (t, y, a, u) or (t, x, a, ϑ) ≠ (t, x, a, u), satisfying equations (1.1)-(1.13). Any normal curve E realizing the maximum satisfies this condition.

Expanding the Weierstrass function E, let us represent it for the corresponding time interval as

$$E_1 = H_1(y_i, u_q, a_l, \lambda_i^{(1)}, \eta_i^{(1)}, \mu_\nu^{(1)}) - \tilde{H}_1(y_i, \vartheta_q, a_l, \lambda_i^{(1)}, \eta_i^{(1)}, \mu_\nu^{(1)})$$

or

$$E_1 = H_1^\lambda - \tilde{H}_1^\lambda \quad (t_0 < t < t_q^-). \tag{5.19}$$

or

$$E_2 = H_2(y_i, u_p, a_l, \lambda_i^{(2)}, \eta_l^{(2)}, \mu_\beta^{(2)}) - \tilde{H}_2(y_i, \vartheta_p, a_l, \lambda_i^{(2)}, \eta_l^{(2)}, \mu_\beta^{(2)})$$
$$E_2 = H_2^\lambda - \tilde{H}_2^\lambda, \ (t_q^+ < t < t_v^-). \tag{5.20}$$

or

$$E_3 = H_3(x_j, u_p, a_l, \lambda_j^{(3)}, \eta_l^{(3)}, \mu_\nu^{(3)}) - \tilde{H}_3(x_j, \vartheta_p, a_l, \lambda_j^{(3)}, \eta_l^{(3)}, \mu_\nu^{(3)}).$$
$$E_3 = H_3^\lambda - \tilde{H}_3^\lambda (t_v^+ < t < t_k). \tag{5.21}$$

where

$$H_1^\lambda = \lambda_i^{(1)} \varphi_i, \ H_2^\lambda = \lambda_i^{(2)} f_i, \ H_3^\lambda = \lambda_j^{(3)} g_j. \tag{5.22}$$

We have kept in mind the identity to zero of $\dot{a}_n = 0$ and functions ξ_α, γ_β and ζ_ν.

Then inequality (5.18) can be represented as

$$H_s^\lambda \leq \tilde{H}_s^\lambda \quad (s = 1, 2, 3). \tag{5.23}$$

Fixing the phase coordinates y_i or x_j, parameter a_n and Lagrange coefficients $\lambda_i^{(s)}$ in (5.23) and placing arbitrary permissible controls ϑ_k in the right portion of the inequality ($\vartheta_k \in U$), we can interpret the Weierstrass condition as the L. S. Pontryagin maximum principle (in this case a minimum principle), described in considerable detail in [28].

Suppose $u_s(t)$ (where $t_0 \leq t \leq t_q^-$, $s = 1,\ldots, l = d$; where $t_q^+ \leq t \leq t_v^-$, $s = 1,\ldots, l - h$; where $t_v^+ \leq t \leq t_k$, $s = 1,\ldots, l - k$) is the permissible control of the system, where $u_s \in U$. The area of control is an arbitrary compact set of all points $u_s(t)$, $t_0 \leq t \leq t_k$. With fixed values of y_i or x_j, a_n and $\lambda_i^{(s)}$, function H_s^{λ} (s = 1, 2, 3) becomes a function of parameter $u \in U$; the precise lower bound of the values of this function will be represented by

$$m_s(y,\ x,\ a,\ \lambda^{(s)}) = \inf_{u \in U} H_s^{\lambda}(y,\ x,\ a,\ \lambda^{(s)},\ u)$$

We represent by the symbols I and I′ the full stability condition and the stability condition without equations (4.22)-(4.24). We can then formulate the following result.

Result 2. Minimum Principle. The permissible curve E satisfying arbitrarily I′ with sets $l_0 = 1$, $\lambda_i^{(s)}(t)$ and equations (1.1)-(1.13), satisfying the necessary Weierstrass condition everywhere if for any element (t, x, y, a, u) of curve E between corner points, the following condition is fulfilled:

$$H_s^{\lambda}(y,\ x,\ a,\ \lambda^{(s)},\ u) = m\,(y,\ x,\ a,\ \lambda^{(s)}). \quad u \in U \tag{5.24}$$

The control functions of the system are autonomous. Therefore, condition (5.24) must be tested for each control function $u \in U_1$ individually, thus producing from (5.23) as many relationships as there are control functions of the system.

Assuming

$$\vartheta(t) = u\,(t) + \Delta u(t) \quad (\vartheta,\ u \in U) \tag{5.25}$$

and considering Δu infinitely small, on the basis of the Tailor formula we can write relationships (5.19)-(5.21) with an accuracy to infinitely small higher order terms as

$$e_1 = -\frac{\partial^2 H_1}{\partial u_s \partial u_k}\Delta u_s \Delta u_k;\quad e_2 = -\frac{\partial^2 H_2}{\partial u_p \partial u_m}\Delta u_p \Delta u_m;$$

$$e_3 = -\frac{\partial^2 H_3}{\partial u_p \partial u_m}\Delta u_p \Delta u_m$$

with any $(\Delta u_1, \ldots, \Delta u_l) \neq (0, \ldots, 0)$, satisfying the following equations according to (5.25) and (1.6), (1.9), (1.12):

$$\frac{\partial \xi_s}{\partial u_s} \Delta u_s = 0, \quad \frac{\partial \gamma_\beta}{\partial u_p} \Delta u_p = 0, \quad \frac{\partial \zeta_\nu}{\partial u_p} \Delta u_p = 0. \tag{5.26}$$

Thus, the following theorem occurs.

Theorem 3. Necessary Clebsch condition. Permissible curve E, satisfying the stability condition with factors $l_0 = 1$, $\lambda^{(s)}(t)$ and equations (1.1)-(1.13), satisfies the necessary Clebsch condition with these factors if for any element (t, y, a, u) or (t, x, a, u) of curve E, the following inequality is fulfilled between corner points:

$$e \geqslant 0 \tag{5.27}$$

with any $(\Delta u_1, \ldots, \Delta u_l) \neq (0, \ldots, 0)$, satisfying equations (5.26).

Any normal curve realizing the maximum should satisfy this condition, tested at points of discontinuity twice for the left and right limits of function e.

§ 6. Sufficient Condition

Suppose there is a set of extremes fixed by functions such as

$$\left.\begin{array}{c} y_i(t, \alpha, \beta),\ \lambda_i^{(1)}(t, \alpha, \beta);\ \lambda_i^{(2)}(t, \alpha, \beta), \\ \alpha = \{\alpha_1, \ldots, \alpha_{p'}\},\ \beta = \{\beta_{p'+1}, \ldots, \beta_p\}; \\ x_j(t, \alpha, \beta),\ \lambda_j^{(3)}(t, \alpha, \beta); \\ \eta_m^{(1)}(t, \alpha, \beta),\ \eta_n^{(2)}(t, \alpha, \beta),\ \eta_n^{(3)}(t, \alpha, \beta),\ a_n(\alpha, \beta), \end{array}\right\} \tag{6.1}$$

having between corner points, together with their derivatives y_t, x_t, $\lambda_t^{(1)}$, $\lambda_t^{(2)}$, ..., $\eta_t^{(3)}$, continuous partial derivatives at least to the second order inclusively at all points (t, α, β) for which (α, β) lies within G and $t_0(\alpha, \beta) \leq t \leq t_k(\alpha, \beta)$, while the determinants

$$\begin{vmatrix} y_{i\alpha} & y_{i\beta} \\ a_{n\alpha} & a_{n\beta} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} x_{j\alpha} & x_{j\beta} \\ a_{n\alpha} & a_{n\beta} \end{vmatrix} \tag{6.2}$$

are not equal to zero.

Here G is an area on hyperplane $\alpha\beta$, while concerning functions $t_0(\alpha, \beta)$, $t_q(\alpha, \beta)$, $t_v(\alpha, \beta)$ and $t_k(\alpha, \beta)$, it is assumed that they are unambiguous, continuous and different in this area.

Set (6.1) corresponds to the set of controls

$$u_k(t, \alpha, \beta) \quad (k = 1, \ldots, l). \tag{6.3}$$

Suppose set (6.1) covers the singly coupled area Π in space $T \times B \times A$ and $T \times C \times A$ one time. This corresponds to the condition that for values of (t, α, β) satisfying the inequalities

$$t_0 - \varepsilon \leqslant t \leqslant t_\kappa + \varepsilon, \quad |\alpha - \alpha_0| \leqslant \varepsilon, \quad |\beta - \beta_0| \leqslant \varepsilon,$$

one and only one extreme of the set passes through each point (t, x, a) or (t, y, a) from Π.

Let us now go over to proof of the supplementary lemmas which must be affirmed for proof of the sufficient condition. If $z(\alpha, \beta)$ is an unambiguous, continuous function, having continuous first order partial derivatives in area G, the functions

$$\left.\begin{array}{c} z(\alpha, \beta),\ y_i[z(\alpha, \beta), \alpha, \beta],\ \lambda_i^{(1)}[z(\alpha, \beta), \alpha, \beta], \\ \lambda_i^{(2)}[z(\alpha, \beta), \alpha, \beta], \\ x_j[z(\alpha, \beta), \alpha, \beta],\ \lambda_j^{(3)}[z(\alpha, \beta), \alpha, \beta];\ \eta_i^{(1)}[z(\alpha, \beta), \alpha, \beta], \\ \eta_n^{(2)}[z(\alpha, \beta), \alpha, \beta],\ \eta_n^{(3)}[z(\alpha, \beta), \alpha, \beta],\ a_n(\alpha, \beta) \end{array}\right\} \tag{6.4}$$

define hypersurface S. This hypersurface intersects each extreme of (6.1) at a point defined by the condition t = z. Let us compose a curved integral of the form

$$I^* = \int_{C_1}\left[\left(F_1 - \dot{y}_l \frac{\partial F_1}{\partial \dot{y}_l}\right)dt + \frac{\partial F_1}{\partial \dot{y}_l}dy_l - \frac{\partial F_1}{\partial \dot{a}_m}da_m + dJ\right] +$$
$$+\int_{C_2}\left[\left(F_2 - \dot{y}_l \frac{\partial F_2}{\partial \dot{y}_l}\right)dt + \frac{\partial F_2}{\partial \dot{y}_l}dy_l + \frac{\partial F_2}{\partial \dot{a}_n}da_n + dJ\right] +$$
$$+\int_{C_3}\left[\left(F_3 - \dot{x}_j \frac{\partial F_3}{\partial \dot{x}_j}\right)dt + \frac{\partial F_3}{\partial \dot{x}_j}dx_j + \frac{\partial F_3}{\partial \dot{a}_n}da_n + dJ\right]. \qquad (6.5)$$

where C_1, C_2, C_3 are arcs of permissible curve C in $[t_0, t_q^-]$, $[t_q^+, t_v^-]$ and $[t_v^+, t_k]$ respectively. On the hyperplane $t = t^*$ $(t_v < t^* < t_k)$, considering (6.4), interval (6.5) can be brought to the form

$$I^* = \int_{C_3}\left[\left(\frac{\partial F_3}{\partial \dot{x}_j}\frac{\partial x_j}{\partial \alpha} + \frac{\partial F_3}{\partial \dot{a}_n}\frac{\partial a_n}{\partial \alpha} + \frac{\partial J}{\partial \alpha}\right)d\alpha + \left(\frac{\partial F_3}{\partial \dot{x}_j}\frac{\partial x_j}{\partial \beta} + \frac{\partial F_3}{\partial \dot{a}_n}\frac{\partial a_n}{\partial \beta} + \frac{\partial J}{\partial \beta}\right)d\beta\right].$$

In order for this integral in area Π to be independent of the path, it is necessary and sufficient that the following condition be fulfilled:

$$\frac{\partial x_j}{\partial \alpha}\frac{\partial}{\partial \beta}\frac{\partial F_3}{\partial \dot{x}_j} + \frac{\partial a_n}{\partial \alpha}\frac{\partial}{\partial \beta}\frac{\partial F_3}{\partial \dot{a}_n} - \frac{\partial x_j}{\partial \beta}\frac{\partial}{\partial \alpha}\frac{\partial F_3}{\partial \dot{x}_j} - \frac{\partial a_n}{\partial \beta}\frac{\partial}{\partial \alpha}\frac{\partial F_3}{\partial \dot{a}_n} = 0.$$

Generalizing this result, we find that the necessary and sufficient condition for independence of integral (6.5) on the path on hyperplane $t = t^*$ $(t_0 \le t^* \le t_k)$ consists of the inequalities

$$\frac{\partial y_l}{\partial \alpha}\frac{\partial \lambda_l^{(1)}}{\partial \beta} + \frac{\partial a_l}{\partial \alpha}\frac{\partial \eta_l^{(1)}}{\partial \beta} - \frac{\partial y_l}{\partial \beta}\frac{\partial \lambda_l^{(1)}}{\partial \alpha} - \frac{\partial a_l}{\partial \beta}\frac{\partial \eta_l^{(1)}}{\partial \alpha} = 0 \quad (t_0 \leqslant t^* \leqslant t_q^-). \qquad (6.6)$$

$$\frac{\partial y_l}{\partial \alpha}\frac{\partial \lambda_l^{(2)}}{\partial \beta} + \frac{\partial a_n}{\partial \alpha}\frac{\partial \eta_n^{(2)}}{\partial \beta} - \frac{\partial y_l}{\partial \beta}\frac{\partial \lambda_l^{(2)}}{\partial \alpha} - \frac{\partial a_n}{\partial \beta}\frac{\partial \eta_n^{(2)}}{\partial \alpha} = 0 \quad (t_q^+ \leqslant t \leqslant t_v^-). \qquad (6.7)$$

$$\frac{\partial x_j}{\partial \alpha}\frac{\partial \lambda_j^{(3)}}{\partial \beta} + \frac{\partial a_n}{\partial \alpha}\frac{\partial \eta_n^{(3)}}{\partial \beta} - \frac{\partial x_j}{\partial \beta}\frac{\partial \lambda_j^{(3)}}{\partial \alpha} - \frac{\partial a_n}{\partial \beta}\frac{\partial \eta_n^{(3)}}{\partial \alpha} = 0 \quad (t_v^+ \leqslant t \leqslant t_k). \qquad (6.8)$$

It is not difficult to see that conditions (6.6)-(6.8) are where t = t* an analog of the conditions

$$\frac{\partial B_l^{(1)}}{\partial y_k}=\frac{\partial B_k^{(1)}}{\partial y_l},\quad \frac{\partial C_l^{(1)}}{\partial a_m}=\frac{\partial C_m^{(1)}}{\partial a_l},\quad \frac{\partial B_l^{(1)}}{\partial a_m}=\frac{\partial C_m^{(1)}}{\partial y_l}\ (t_0\leqslant t\leqslant t_q^-), \tag{6.9}$$

$$\frac{\partial B_l^{(2)}}{\partial y_k}=\frac{\partial B_k^{(2)}}{\partial y_l},\quad \frac{\partial C_l^{(2)}}{\partial a_\chi}=\frac{\partial C_\chi^{(2)}}{\partial a_l},\quad \frac{\partial B_l^{(2)}}{\partial a_\chi}=\frac{\partial C_\chi^{(2)}}{\partial y_l}\ (t_q^+\leqslant t\leqslant t_v^-), \tag{6.10}$$

$$\frac{\partial B_j^{(3)}}{\partial x_k}=\frac{\partial B_k^{(3)}}{\partial x_j},\quad \frac{\partial C_l^{(3)}}{\partial a_\chi}=\frac{\partial C_\chi^{(3)}}{\partial a_l},\quad \frac{\partial B_j^{(3)}}{\partial a_\chi}=\frac{\partial C_\chi^{(3)}}{\partial x_j}\ (t_v^-\leqslant t\leqslant t_k), \tag{6.11}$$

where

$$B_l^{(1)}=\frac{\partial F_1}{\partial \dot y_l},\quad C_m^{(1)}=\frac{\partial F_1}{\partial \dot a_m},\quad B_l^{(2)}=\frac{\partial F_2}{\partial \dot y_l},\quad C_\chi^{(2)}=\frac{\partial F_2}{\partial \dot a_\chi},$$

$$B_l^{(3)}=\frac{\partial F_3}{\partial \dot x_j},\quad C_\chi^{(3)}=\frac{\partial F_3}{\partial \dot a_\chi}.$$

The necessary and sufficient condition for independence of I* on the path of integration in the entire singly coupled area G consists of (6.9)-(6.11) plus the equations

$$\left.\begin{aligned}
&\frac{\partial A^{(1)}}{\partial y_l}=\frac{\partial B_l^{(1)}}{\partial t},\quad \frac{\partial A^{(1)}}{\partial a_l}=\frac{\partial C_l^{(1)}}{\partial t}\ (t_0<t<t_q^-),\\
&\frac{\partial A^{(2)}}{\partial y_l}=\frac{\partial B_l^{(2)}}{\partial t},\quad \frac{\partial A^{(2)}}{\partial a_\chi}=\frac{\partial C_\chi^{(2)}}{\partial t}\ (t_q^+<t<t_v^-),\\
&\frac{\partial A^{(3)}}{\partial x_j}=\frac{\partial B_j^{(3)}}{\partial t},\quad \frac{\partial A^{(3)}}{\partial a_\chi}=\frac{\partial C_\chi^{(3)}}{\partial t}\ (t_v^-<t<t_k),
\end{aligned}\right\} \tag{6.12}$$

where

$$A^{(1)}=F_1-\dot y_l\frac{\partial F_1}{\partial \dot y_l},\quad A^{(2)}=F_2-\dot y_l\frac{\partial F_2}{\partial \dot y_l},\quad A^{(3)}=F_3-\dot x_j\frac{\partial F_3}{\partial \dot x_j}. \tag{6.13}$$

Expressions $\partial A^{(1)}/\partial y_l-\dfrac{\partial B_l^{(1)}}{\partial t}$ and $\dfrac{\partial A^{(1)}}{\partial a_l}-\dfrac{\partial C_l^{(1)}}{\partial t}$ will be written as follows:

$$\frac{\partial A^{(1)}}{\partial y_l}-\frac{\partial B_l^{(1)}}{\partial t}=\frac{\partial F_1}{\partial y_l}-\frac{\partial}{dt}\frac{\partial F_1}{\partial \dot{y}_l}-\dot{y}_k\frac{\partial}{\partial y_k}\frac{\partial F_1}{\partial y_k}+$$
$$+\dot{y}_k\left(\frac{\partial B_l^{(1)}}{\partial y_k}-\frac{\partial B_k^{(1)}}{\partial y_l}\right)=\frac{\partial F_1}{\partial y_l}-\frac{d}{dt}\frac{\partial F_1}{\partial \dot{y}_l}+\dot{y}_k\left(\frac{\partial B_l^{(1)}}{\partial y_k}-\frac{\partial B_k^{(1)}}{\partial y_l}\right),$$
$$\frac{\partial A^{(1)}}{\partial a_l}-\frac{\partial C_l^{(1)}}{dt}=\frac{\partial F_1}{\partial a_l}-\frac{\partial}{\partial t}\frac{\partial F_1}{\partial \dot{a}_l}-\dot{y}_i\frac{\partial}{\partial y_l}\frac{\partial F_1}{\partial \dot{a}_l}+$$
$$+\dot{y}_l\left(\frac{\partial C_l^{(1)}}{\partial y_l}-\frac{\partial B_l^{(1)}}{\partial a_l}\right)=\frac{\partial F_1}{\partial a_l}-\frac{d}{dt}\frac{\partial F_1}{\partial \dot{a}_l}+\dot{y}_i\left(\frac{\partial C_l^{(1)}}{\partial y_l}-\frac{\partial B_l^{(1)}}{\partial a_l}\right).$$

Further by analogy we produce

$$\frac{\partial A^{(2)}}{\partial y_i}-\frac{\partial B_i^{(2)}}{\partial t}=\frac{\partial F_2}{\partial y_l}-\frac{d}{dt}\frac{\partial F_1}{\partial \dot{y}_l}+\dot{y}_k\left(\frac{\partial B_i^{(2)}}{\partial y_k}-\frac{\partial B_k^{(2)}}{\partial y_l}\right),$$
$$\frac{\partial A^{(2)}}{\partial a_\chi}-\frac{\partial C_\chi^{(2)}}{dt}=\frac{\partial F_2}{\partial a_\chi}-\frac{d}{dt}\frac{\partial F_2}{\partial \dot{a}_\chi}+\dot{y}_i\left(\frac{\partial C_\chi^{(2)}}{\partial y_l}-\frac{\partial B_i^{(2)}}{\partial a_\chi}\right),$$
$$\frac{\partial A^{(3)}}{\partial x_j}-\frac{\partial B_j^{(3)}}{\partial t}=\frac{\partial F_3}{\partial x_j}-\frac{d}{dt}\frac{\partial F_3}{\partial \dot{x}_j}+\dot{x}_k\left(\frac{\partial B_j^{(2)}}{\partial x_k}-\frac{\partial B_k^{(2)}}{\partial x_j}\right),$$
$$\frac{\partial A^{(3)}}{\partial a_\chi}-\frac{\partial C_\chi^{(3)}}{\partial t}=\frac{\partial F_3}{\partial a_\chi}-\frac{d}{dt}\frac{\partial F_3}{\partial \dot{a}_\chi}+\dot{x}_j\left(\frac{\partial C_\chi^{(3)}}{\partial x_j}-\frac{\partial B_j^{(3)}}{\partial a_\chi}\right).$$

From this at the extremes where (6.9)-(6.11) or (6.6)-(6.8) are fulfilled, conditions (6.12) obtain.

Thus, the following lemma is correct.

Lemma 2. Suppose a p-parameter set of extremes of the form (6.1) is fixed, which singly covers the area Π of space T × B × A and T × C × A. If in area Π, equations (6.6)-(6.8) or (6.9)-(6.11) are fulfilled and area Π is singly coupled, integral I* is independent of the path in Π.

What we stated earlier confirms the correctness of this statement.

Lemma 3. At extreme E of the p-parameter set of extremes of area Π defined in lemma 2, I* (E) = Φ(E).

Actually, the integrands in I* at the extremes in area Π correspond to F_1, F_2 and F_3 respectively, so that the following equation obtains:

$$I^*(E)=\Phi(E). \tag{6.14}$$

Suppose there are p sets of variations:

over $[t_0, t_q^-]$

$$\delta t_{0\alpha},\ \delta y_\alpha(t),\ \delta u_\alpha(t),\ \delta\lambda_\alpha^{(1)}(t),\ \delta\eta_\alpha^{(1)}(t),\ \delta\mu_\alpha^{(1)}(t),\ \delta a_\alpha;$$

over $[t_q^+, t_v^-]$

$$\delta t_{q\alpha},\ \delta y_\alpha(t),\ \delta u_\alpha(t),\ \delta\lambda_\alpha^{(2)}(t),\ \delta\eta_\alpha^{(2)}(t),\ \delta\mu_\alpha^{(2)}(t),\ \delta a_\alpha; \qquad (6.15)$$

over $[t_v^+, t_k]$

$$\delta t_{v\alpha},\ \delta t_{k\alpha},\ \delta x_\alpha(t),\ \delta u_\alpha(t),\ \delta\lambda_\alpha^{(3)}(t),\ \delta\eta_\alpha^{(3)}(t),\ \delta\mu_\alpha^{(3)}(t),\ \delta a_\alpha,$$

which are related along nonsingular extreme E by the following system of equations in variations:

over $[t_0, t_q^-]$

$$\begin{aligned}
&\delta\dot{y}_\alpha = A_1\delta y_\alpha + B_1\delta a_\alpha + C_1\delta u_\alpha \quad (\alpha = 1, \ldots, p),\\
&\delta\dot{a}_\alpha = 0,\\
&D_1\delta y_\alpha + E_1\delta u_\alpha + P_1\delta a_\alpha = 0,\\
&\delta\dot{\lambda}_\alpha^{(1)} = -A_1'\delta\lambda_\alpha^{(1)} - F_1\delta a_\alpha - G_1\delta u_\alpha - K_1\delta y_\alpha - D_1'\delta\mu_\alpha^{(1)},\\
&\delta\dot{\eta}_\alpha^{(1)} = -F_1'\delta y_\alpha - B_1'\delta\lambda_\alpha^{(1)} - L_1\delta u_\alpha - M_1\delta a_\alpha - P_1'\delta\mu_\alpha^{(1)},\\
&G_1'\delta y_\alpha + C_1'\delta\lambda_\alpha^{(1)} + L_1'\delta a_\alpha + R_1\delta u_\alpha + E_1'\delta\mu_\alpha^{(1)} = 0;
\end{aligned} \qquad (6.16)$$

over $[t_q^+, t_v^-]$

$$\begin{aligned}
&\delta\dot{y}_\alpha = A_2\delta y_\alpha + B_2\delta a_\alpha + C_2\delta u_\alpha,\\
&\delta\dot{a}_\alpha = 0,\\
&D_2\delta y_\alpha + E_2\delta u_\alpha + P_2\delta a_\alpha = 0,\\
&\delta\dot{\lambda}_\alpha^{(2)} = -A_2'\delta\lambda_\alpha^{(2)} - F_2\delta a_\alpha - G_2\delta u_\alpha - k_2\delta y_\alpha - D_2'\delta\mu_\alpha^{(2)},\\
&\delta\dot{\eta}_\alpha^{(2)} = -F_2'\delta y_\alpha - B_2'\delta\lambda_\alpha^{(2)} - L_2\delta u_\alpha - M_2\delta a_\alpha - P_2'\delta\mu_\alpha^{(2)},\\
&G_2'\delta y_\alpha + C_2'\delta\lambda_\alpha^{(2)} + L_2'\delta a_\alpha + R_2\delta u_\alpha + E_2'\delta\mu_\alpha^{(2)} = 0;
\end{aligned} \qquad (6.17)$$

over $[t_v^+, t_k]$

$$
\left.\begin{aligned}
&\delta\dot{x}_\alpha = A_3\delta x_\alpha + B_3\delta a_\alpha + C_3\delta u_\alpha,\\
&\delta\dot{a}_\alpha = 0,\\
&D_3\delta y_\alpha + E_3\delta u_\alpha + P_3\delta a_\alpha = 0,\\
&\delta\dot{\lambda}_\alpha^{(3)} = -A_3'\delta\lambda_\alpha^{(3)} - F_3\delta a_\alpha - G_3\delta u_\alpha - K_3\delta x_\alpha - D_3'\delta\mu_\alpha^{(3)},\\
&\delta\dot{\eta}_\alpha^{(3)} = -F_3'\delta y_\alpha - B_3'\delta\lambda_\alpha^{(3)} - L_3\delta u_\alpha - M_3\delta a_\alpha - P_3'\delta\mu_\alpha^{(3)},\\
&G_3'\delta x_\alpha + C_3'\delta\lambda_\alpha^{(3)} + L_3'\delta a_\alpha + R_3\delta u_\alpha + E_3'\delta\mu_\alpha^{(3)} = 0,
\end{aligned}\right\} \qquad (6.18)
$$

where the index $'$ represents transposition of the matrix; δy_α, δa_α, δu_α, δx_α, $\delta\lambda_\alpha$, $\delta\eta_\alpha$, $\delta\mu_\alpha$ are matrix columns.

In equations (6.16)-(6.18), the matrices

$$
\begin{aligned}
&A_\gamma = \left\|\frac{\partial^2 H_\gamma}{\partial\lambda_j^{(1)}\partial y_l}\right\|, \quad B_\gamma = \left\|\frac{\partial^2 H_\gamma}{\partial\lambda_j^{(1)}\partial a_l}\right\|, \quad C_\gamma = \left\|\frac{\partial^2 H_\gamma}{\partial\lambda_j^{(1)}\partial u_q}\right\|,\\
&D_\gamma = \left\|\frac{\partial^2 H_\gamma}{\partial\mu_j^{(1)}\partial y_l}\right\|, \quad E_\gamma = \left\|\frac{\partial^2 H_\gamma}{\partial\mu_j^{(1)}\partial u_q}\right\|, \quad P_\gamma = \left\|\frac{\partial^2 H_\gamma}{\partial\mu_j^{(1)}\partial a_l}\right\|,\\
&F_\gamma = \left\|\frac{\partial^2 H_\gamma}{\partial y_i\partial a_l}\right\|, \quad G_\gamma = \left\|\frac{\partial^2 H_\gamma}{\partial y_i\partial u_q}\right\|, \quad K_\gamma = \left\|\frac{\partial^2 H_\gamma}{\partial y_i\partial y_k}\right\|,\\
&L_\gamma = \left\|\frac{\partial^2 H_\gamma}{\partial a_l\partial u_q}\right\|, \quad M_\gamma = \left\|\frac{\partial^2 H_\gamma}{\partial a_l\partial a_n}\right\|, \quad R_\gamma = \left\|\frac{\partial^2 H_\gamma}{\partial u_q\partial u_m}\right\|\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\gamma = 1,\ 2)
\end{aligned}
$$

and the similarly written matrices $A_3, \ldots, R_3$ are calculated on E. It should be kept in mind that the matrices E_γ and R_γ ($\gamma = 1, 2, 3$) are nonsingular, the former according to condition of the problem, the latter due to the assumption of existence of nonsingular extreme E.

Two arbitrary independent solutions of the p sets of variations (6.15) along E for the system of equations (6.16)-(6.18) will be represented where $\alpha = i$ and $\alpha = j$ ($i \neq j$) as and

$$
\left.\begin{aligned}
&\delta y(t),\ \delta\lambda^{(\gamma)}(t),\ \delta\eta^{(\gamma)}(t),\ \delta\mu^{(\gamma)}(t),\ \delta a,\ \delta x(t)\\
&\delta\tilde{y}(t),\ \delta\tilde{\lambda}^{(\gamma)}(t),\ \delta\tilde{\eta}^{(\gamma)}(t),\ \delta\tilde{\mu}^{(\gamma)}(t),\ \delta\tilde{a},\ \delta\tilde{x}(t),\\
&\qquad\qquad\qquad\qquad (\gamma=1,\ 2,\ 3)
\end{aligned}\right\} \tag{6.19}
$$

Since

$$
\frac{d}{dt}\left(\delta y'\delta\tilde{\lambda}^{(1)}+\delta a'\delta\tilde{\eta}^{(1)}-\delta\tilde{y}'\delta\lambda^{(1)}-\delta\tilde{a}'\delta\eta^{(1)}\right)=\delta\dot{y}'\delta\tilde{\lambda}^{(1)}+
$$

$$
+\delta y'\delta\dot{\tilde{\lambda}}^{(1)}+\delta a'\delta\dot{\tilde{\eta}}'-\delta\dot{\tilde{y}}'\delta\lambda^{(1)}-\delta\tilde{y}'\delta\dot{\lambda}^{(1)}-\delta\tilde{a}'\delta\dot{\eta}^{(1)},
$$

by substituting the values of $\delta\dot{y}$, $\delta\dot{\lambda}^{(1)}$ and $\delta\dot{\eta}^{(1)}$ into the right portion of the equation from the variational equations (6.16), we find

$$
\begin{aligned}
&\frac{d}{dt}\left(\delta y'\delta\tilde{\lambda}^{(1)}+\delta a'\tilde{\eta}^{(1)}-\delta\tilde{y}'\delta\lambda^{(1)}-\delta\tilde{a}'\delta\eta^{(1)}\right)=(A_1\delta y+B_1\delta a+\\
&+C_1\delta u)'\delta\tilde{\lambda}^{(1)}-\delta y'\,(A_1'\delta\tilde{\lambda}^{(1)}+F_1\delta\tilde{a}+G_1\delta\tilde{u}+K_1\delta\tilde{y}+D_1'\delta\tilde{\mu}^{(1)})-\\
&-\delta a'\,(F_1\delta\tilde{y}+B_1'\delta\tilde{\lambda}^{(1)}+L_1\delta\tilde{u}+M_1\delta\tilde{a}+P_1'\delta\tilde{\mu}^{(1)})-(A_1\delta\tilde{y}+\\
&+B_1\delta\tilde{a}+C_1\delta u)'\delta\lambda^{(1)}+\delta\tilde{y}'\,(A_1'\delta\lambda^{(1)}+F_1\delta a+G_1\delta u+K_1\delta y+\\
&+D_1'\delta\mu^{(1)})+\delta\tilde{a}'\,(F_1'\delta y+B_1'\delta\lambda^{(1)}+L_1\delta u+M_1\delta a+P_1'\delta\mu^{(1)})=\\
&=-\delta y'F_1\delta\tilde{a}+\delta\tilde{a}'F_1'\delta y-\delta y'K_1\delta\tilde{y}+\delta\tilde{y}'K_1\delta y-\delta a'F_1'\delta\tilde{y}+\\
&+\delta\tilde{y}'F_1\delta a-\delta a'M_1\delta\tilde{a}+\delta\tilde{a}'M_1\delta a+\delta u'C_1'\delta\tilde{\lambda}^{(1)}-\delta\tilde{u}C_1'\delta\lambda^{(1)}-\\
&-\delta y'G_1\delta\tilde{u}+\delta\tilde{y}'G_1\delta u-\delta a'L_1\delta\tilde{u}+\delta\tilde{a}'L_1\delta u-\delta y'D_1'\delta\tilde{\mu}^{(1)}+\\
&+\delta\tilde{y}'D_1\delta\mu^{(1)}-\delta a'P_1'\delta\tilde{\mu}^{(1)}+\delta\tilde{a}'P_1'\delta\mu^{(1)}=\delta u'C_1'\delta\tilde{\lambda}^{(1)}-\\
&-\delta\tilde{u}C_1'\delta\lambda^{(1)}-\delta y'G_1\delta\tilde{u}+\delta\tilde{y}'G_1\delta u-\delta a'L_1\delta\tilde{u}+\delta\tilde{a}L_1\delta u-\\
&\qquad-\delta y'D_1'\delta\tilde{\mu}^{(1)}+\delta\tilde{y}'D_1'\delta\mu^{(1)}.
\end{aligned}
$$

We have kept in mind here the equality of the rows

$$
(\delta y'F_1\delta\tilde{a})'=\delta y'F_1\delta\tilde{a},\quad (\delta\tilde{y}'F_1\delta a)'=\delta\tilde{y}'F_1\delta a,
$$

$$
(\delta y'K_1\delta\tilde{y})'=\delta y'K_1\delta\tilde{y},\quad (\delta a'M_1\delta\tilde{a})'=\delta a'M_1\delta\tilde{a}.
$$

Further, substituting the values of $C_1'\delta\lambda^{(1)}$ and $D_1\delta y$ in the right portion of the equation, according to the third and last equation of system (6.16) we will have

$$\frac{d}{dt}\left(\delta y'\delta\tilde{\lambda}^{(1)}+\delta a'\delta\tilde{\eta}^{(1)}-\delta\tilde{y}'\delta\lambda^{(1)}-\delta\tilde{a}'\delta\eta^{(1)}\right)=\delta\tilde{y}'G_1\delta u-\delta y'G_1\delta\tilde{u}+$$

$$\begin{aligned}&+\delta\tilde{a}'L_1\delta u-\delta a'L_1\delta\tilde{u}-\delta u'G_1'\delta\tilde{y}-\delta u'L_1'\delta\tilde{a}-\delta uR_1\delta\tilde{u}-\\&-\delta u'E_1'\delta\tilde{\mu}^{(1)}-\delta a'P_1'\delta\tilde{\mu}^{(1)}+\delta\tilde{a}'P_1'\delta\mu^{(1)}+\delta\tilde{u}'G_1'\delta y+\delta\tilde{u}'L_1'\delta a+\\&+\delta\tilde{u}'R_1\delta u+\delta\tilde{u}'E_1'\delta\mu^{(1)}+\delta u'E_1'\delta\tilde{\mu}^{(1)}-\delta\tilde{u}'E_1'\delta\mu^{(1)}+\\&\qquad+\delta a'P_1'\delta\tilde{\mu}^{(1)}-\delta\tilde{a}'P_1'\delta\mu^{(1)}=0,\end{aligned}\tag{6.20}$$

since the rows of the matrix are equal

$$\begin{aligned}&(\delta y'G_1\delta u)'=\delta\tilde{y}'G_1\delta u,\quad(\delta u'G_1\delta\tilde{u})'=\delta y'G_1\delta\tilde{u},\\&(\delta\tilde{a}'L_1\delta u)'=\delta\tilde{a}'L_1\delta u,\quad(\delta a'L_1\delta\tilde{u})'=\delta a'L_1\delta\tilde{u},\\&(\delta\tilde{u}'R_1\delta u)'=\delta\tilde{u}'R_1\delta a.\end{aligned}$$

Thus we have

$$\frac{d}{dt}\left(\delta y'\delta\tilde{\lambda}^{(1)}+\delta a'\delta\tilde{\eta}^{(1)}-\delta\tilde{y}'\delta\lambda^{(1)}-\delta\tilde{a}'\delta\eta^{(1)}\right)=0$$

or

$$\delta y'\delta\tilde{\lambda}^{(1)}+\delta a'\delta\tilde{\eta}^{(1)}-\delta\tilde{y}'\delta\lambda^{(1)}-\delta\tilde{a}'\delta\eta^{(1)}=d_1,\tag{6.21}$$

where d_1 is a constant quantity.

Similarly according to (6.17) and (6.18) we find

$$\delta y'\delta\tilde{\lambda}^{(2)}+\delta a'\delta\tilde{\eta}^{(2)}-\delta\tilde{y}'\delta\lambda^{(2)}-\delta\tilde{a}'\delta\eta^{(2)}=d_2,\tag{6.22}$$

$$\delta x'\delta\tilde{\lambda}^{(3)}+\delta a'\delta\tilde{\eta}^{(3)}-\delta\tilde{x}'\delta\lambda^{(3)}-\delta\tilde{a}'\delta\eta^{(3)}=d_3.\tag{6.23}$$

At the extremes at the discontinuity points defined by conditions (1.1)-(1.3), discontinuity conditions (4.32)-(4.38), (4.40) should be satisfied. Since conditions (1.1)-(1.3) and (4.32)-(4.37) are fulfilled identically, we can produce the following conditions for disruption of variations

$$d\psi_\tau=0, \tag{6.24}$$

$$\left(\frac{\partial F_1}{\partial y_{iq}^-}dy_{iq}^- - \dot y_{iq}^- d\frac{\partial F_1}{\partial \dot y_{iq}^-} + \frac{\partial F_1}{\partial a_1}da_1\right)-\left(\frac{\partial F_2}{\partial y_{iq}^+}dy_{iq}^+ - \dot y_{iq}^+ d\frac{\partial F_2}{\partial \dot y_{iq}^+} + \frac{\partial F_2}{\partial a_\chi}da_\chi\right)+$$

$$+e_\tau\frac{\partial^2\psi_\tau}{\partial t_q^2}dt_q+e_\tau\frac{\partial^2\psi_\tau}{\partial t_q\partial y_{iq}^-}dy_{iq}^-+e_\tau\frac{\partial^2\psi_\tau}{\partial t_q\partial y_{iq}^+}dy_{iq}^++$$

$$+e_\tau\frac{\partial^2\psi_\tau}{\partial t_q\partial a_\chi}da_\chi+\frac{\partial\psi_\tau}{\partial t_q}de_\tau=0, \tag{6.25}$$

$$d\lambda_{iq}^{(1)}+e_\tau\frac{\partial^2\psi_\tau}{\partial y_{iq}^-\partial y_{jq}^-}dy_{jq}^-+e_\tau\frac{\partial^2\psi_\tau}{\partial y_{jq}^-\partial y_{jq}^+}dy_{jq}^++e_\tau\frac{\partial^2\psi_\tau}{\partial y_{iq}^-\partial a_\chi}da_\chi+$$

$$+e_\tau\frac{\partial^2\psi_\tau}{\partial y_{iq}^-\partial t_q}dt_q+\frac{\partial\psi_\tau}{\partial y_{iq}^-}de_\tau=0\quad (i,\ j=1,\ldots,n), \tag{6.26}$$

$$-d\lambda_{iq}^{(2)}+e_\tau\frac{\partial^2\psi_\tau}{\partial y_{iq}^+\partial y_{jq}^-}dy_{jq}^-+e_\tau\frac{\partial^2\psi_\tau}{\partial y_{iq}^+\partial y_{jq}^+}dy_{jq}^++e_\tau\frac{\partial^2\psi_\tau}{\partial y_{iq}^+\partial a_\chi}da_\chi+$$

$$+e_\tau\frac{\partial^2\psi_\tau}{\partial y_{iq}^+\partial t_q}dt_q+\frac{\partial\psi_\tau}{\partial y_{iq}^+}de_\tau=0\quad (i,\ j=1,\ldots,n), \tag{6.27}$$

$$d\psi_\varkappa=0,\quad dQ_j=0, \tag{6.28}$$

$$\left(\frac{\partial F_2}{\partial y_{i\nu}}dy_{i\nu}-\dot y_{i\nu}d\frac{\partial F_2}{\partial \dot y_{i\nu}}+\frac{\partial F_2}{\partial a_\chi}da_\chi\right)-\left(\frac{\partial F_3}{\partial x_{j\nu}}dx_{j\nu}-\dot x_{j\nu}d\frac{\partial F_3}{\partial \dot x_{j\nu}}+\right.$$

$$\left.+\frac{\partial F_3}{\partial a_\chi}da_\chi\right)+e_\varkappa\frac{\partial^2\psi_\varkappa}{\partial t_\nu^2}dt_\nu+e_\varkappa\frac{\partial^2\psi_\varkappa}{\partial t_\nu\partial y_{i\nu}}dy_{i\nu}+\frac{\partial\psi_\varkappa}{\partial t_\nu}de_\varkappa+e_{qj}\frac{\partial^2 Q_j}{\partial t_\nu^2}dt_\nu+$$

$$+e_{qj}\frac{\partial^2 Q_j}{\partial t_\nu\partial y_{i\nu}}dy_{i\nu}+e_{qj}\frac{\partial^2 Q_j}{\partial t_\nu\partial x_{j\nu}}dx_{j\nu}+\frac{\partial Q_j}{\partial t_\nu}de_{qj}=0, \tag{6.29}$$

$$d\lambda_{i\nu}^{(2)}+e_\varkappa\frac{\partial^2\psi_\varkappa}{\partial y_{i\nu}\partial y_{k\nu}}dy_{k\nu}+e_\varkappa\frac{\partial^2\psi_\varkappa}{\partial y_{i\nu}\partial t_\nu}dt_\nu+\frac{\partial\psi_\varkappa}{\partial y_{i\nu}}de_\varkappa+$$

$$+e_{qj}\frac{\partial^2 Q_j}{\partial y_{i\nu}\partial y_{k\nu}}dy_{k\nu}+e_{qj}\frac{\partial^2 Q_j}{\partial y_{i\nu}\partial x_{j\nu}}dx_{j\nu}+e_{qj}\frac{\partial^2 Q_j}{\partial y_{i\nu}\partial t_\nu}dt_\nu+$$

$$+\frac{\partial Q_j}{\partial y_{i\nu}}de_{qj}=0\quad (i=k=1,\ldots,n), \tag{6.30}$$

$$-d\lambda_{j\nu}^{(3)}+e_{qj}\frac{\partial^2 Q_j}{\partial x_{j\nu}\partial y_{i\nu}}dy_{i\nu}+e_{qj}\frac{\partial^2 Q_j}{\partial x_{j\nu}\partial x_{s\nu}}dx_{s\nu}+$$

$$+e_{qj}\frac{\partial^2 Q_j}{\partial x_{j\nu}\partial t_\nu}dt_\nu+\frac{\partial Q_j}{\partial x_{j\nu}}de_{qj}=0. \tag{6.31}$$

Furthermore, at the discontinuity point defined by conditions (1.1), according to (4.38) we have

$$\eta_{lq}^{(1)} = \eta_{lq}^{(2)} \quad (\text{но } \dot{\eta}_{lq}^{(1)} \neq 0,\ \dot{\eta}_{lq}^{(2)} = 0) \quad (l = 1, \ldots, m'),$$

therefore

$$d\eta_{lq}^{(1)} - d\eta_{lq}^{(2)} = 0 \tag{6.32}$$

and according to (4.40)

$$-d\eta_{\chi q}^{(2)} + e_\gamma \frac{\partial^2 \psi_\gamma}{\partial a_\chi \partial y_{iq}^-} dy_{iq}^- + e_\gamma \frac{\partial^2 \psi_\gamma}{\partial a_\chi \partial y_{iq}^+} dy_{iq}^+ + e_\gamma \frac{\partial^2 \psi_\gamma}{\partial a_\chi \partial a_l} da_l + $$
$$+ e_\gamma \frac{\partial^2 \psi_\gamma}{\partial a_\chi \partial t_q} dt_q + \frac{\partial \psi_\gamma}{\partial a_\chi} de_\gamma = 0 \tag{6.33}$$
$$(\chi = l = m' + 1, \ldots, r).$$

In equations (6.24)-(6.33) we accept:

$$\left.\begin{array}{l}
\underline{\text{where } t = t_q} \\
dy_{qs}^- = \dot{y}_q^- b_s \delta t_{qs} + b_s \delta y_{qs}^-,\ da_s = b_s \delta a_s, \\
dy_{qs}^+ = \dot{y}_q^+ b_s \delta t_{qs} + b_s \delta y_{qs}^+,\ d\lambda_{qs}^{(1)} = \dot{\lambda}_q b_s \delta t_{qs} + b_s \delta \lambda_{qs}^{(1)}, \\
d\eta_{qs}^{(1)} = \dot{\eta}_q^{(1)} b_s \delta t_{qs} + b_s \delta \eta_{qs}^{(1)},\ d\lambda_{qs}^{(2)} = \dot{\lambda}_q b_s \delta t_{qs} + b_s \delta \lambda_{qs}^{(2)}, \\
d\eta_{lqs}^{(2)} = b_s \delta \eta_{lqs}^{(2)},\ d\eta_{\chi qs}^{(2)} = \dot{\eta}_{\chi qs}^{(2)} b_s \delta t_{qs} + b_s \delta \eta_{\chi qs}^{(2)}; \\
\underline{\text{where } t = t_v} \\
dy_{vs} = \dot{y}_v b_s \delta t_{vs} + b_s \delta y_{vs},\ da_s = b_s \delta a_s, \\
d\lambda_{vs}^{(2)} = \dot{\lambda}_v^{(2)} b_s \delta t_{vs} + b_s \delta \lambda_{vs}^{(2)}, \\
d\eta_{\chi vs}^{(2)} = \dot{\eta}_{\chi v}^{(2)} b_s \delta t_{vs} + b_s \delta \eta_{\chi vs}^{(2)},\ d\eta_{lvs}^{(2)} = b_s \delta \eta_{lvs}^{(2)}, \\
dx_{vs} = \dot{x}_v b_s \delta t_{vs} + b_s \delta x_{vs}, \\
d\lambda_{vs}^{(3)} = \dot{\lambda}_v^{(3)} b_s \delta t_{vs} + b_s \delta \lambda_{vs}^{(3)}, \\
d\eta_{\chi vs}^{(3)} = \dot{\eta}_{\chi vs}^{(3)} b_s \delta t_{vs} + b_s \delta \eta_{\chi vs}^{(3)}, \\
d\eta_{lvs}^{(3)} = b_s \delta \eta_{lvs}^{(3)}.
\end{array}\right\} \tag{6.34}$$

The set of variations (6.15) should be such that both independent solutions (6.19) along E for variational equations (6.16)-(6.18) satisfy the discontinuity conditions of the variations (6.24)-(6.33)

Let us multiply the left portions of equations (6.25)-(6.27) and (6.32), (6.33) which satisfy one solution (with the tilde) by dt_q, dy^-_{iq}, dy^+_{iq}, da_1 and da_χ respectively, then add the expressions produced, then multiply the left portions of the equations satisfying the other solution (without the tilde) by $d\tilde{t}_q$, $d\tilde{y}^-_{iq}$, $d\tilde{y}^+_{iq}$, $d\tilde{a}_1$, $d\tilde{a}_\chi$ and also add the expressions produced. Then, subtracting the second sum from the first sum, we produce

$$\begin{aligned}
&(dy^-_{iq}d\tilde{\lambda}^{(1)}_{iq} + da_1 d\tilde{\eta}^{(1)}_{1q} - d\tilde{y}^-_{iq}d\lambda^{(1)}_{iq} - d\tilde{a}_1 d\eta^{(1)}_{1q}) - \\
&-(dy^+_{iq}d\tilde{\lambda}^{(2)}_{iq} + da_1 d\tilde{\eta}^{(2)}_{1q} + da_\chi d\tilde{\eta}^{(2)}_{\chi q} - d\tilde{y}^+_{iq}d\lambda^{(2)}_{iq} - \\
&- d\tilde{a}_1 d\eta^{(2)}_{1q} - d\tilde{a}_\chi d\eta^{(2)}_{\chi q}) + (d\tilde{y}^-_{iq}dt_q - dy^-_{iq}d\tilde{t}_q)\frac{\partial F_1}{\partial y^-_{iq}} - \\
&- \dot{y}^-_{iq}(d\tilde{\lambda}^{(1)}_{iq}dt_q - d\lambda^{(1)}_{iq}d\tilde{t}_q) + (d\tilde{a}_1 dt_q - da_1 d\tilde{t}_q)\frac{\partial F_1}{\partial a_1} - \\
&-(d\tilde{y}^+_{iq}dt_q - dy^+_{iq}d\tilde{t}_q)\frac{\partial F_2}{\partial y^+_{iq}} + \dot{y}^+_{iq}(d\tilde{\lambda}^{(2)}_{iq}dt_q - d\lambda^{(2)}_{iq}\tilde{t}_q) - \\
&-(d\tilde{a}_\chi dt_q - da_\chi d\tilde{t}_q)\frac{\partial F_2}{\partial a_\chi} = 0.
\end{aligned}$$

Keeping expressions (6.34) in mind, the equation produced after simple calculations can be reduced to

$$\begin{aligned}
&\delta y^-_{iq}\delta\tilde{\lambda}^{(1)}_{iq} + \delta a_1\delta\tilde{\eta}^{(1)}_{1q} - \delta\tilde{y}^-_{iq}\delta\lambda^{(1)}_{iq} - \delta\tilde{a}_1\delta\eta^{(1)}_{1q} = \\
&= \delta y^+_{iq}\delta\tilde{\lambda}^{(2)}_{iq} + \delta a_\chi\delta\tilde{\eta}^{(2)}_{\chi q} - \delta\tilde{y}^+_{iq}\delta\lambda^{(2)}_{iq} - \delta\tilde{a}_\chi\delta\eta^{(2)}_{\chi q}.
\end{aligned} \qquad (6.35)$$

Thus, according to (6.21) and (6.22) we produce

$$d_1 = d_2. \qquad (6.36)$$

Conditions (6.28)-(6.31) can be supplemented according to (4.38) with the equations

$$d\eta^{(2)}_{1\nu} - d\eta^{(3)}_{1\nu} = 0,$$

$$d\eta^{(2)}_{\chi\nu} - d\eta^{(3)}_{\chi\nu} = 0,$$

where

$$\dot{\eta}_{1\nu} = 0, \quad \dot{\eta}_{\chi\nu} \neq 0.$$

Then, multiplying the left portions of these equations and equations (6.29)-(6.31) satisfying one of the solutions (with the tilde) by dt_v, dy_{iv}, dx_{jv}, da_1 and da_χ respectively and adding the expressions produced, then multiplying the left portions of the equations satisfying the other solution (without the tilde) by $d\tilde{t}_v$, $d\tilde{y}_{iv}$, $d\tilde{x}_{jv}$, $d\tilde{a}_1$, $d\tilde{a}_\chi$ respectively and also adding the expressions produced, after subtracting the second sum from the first sum and performing calculations similar to those preceding, we find

$$\delta y_{i v} \delta \tilde{\lambda}_{i v}^{(2)} + \delta a_n \delta \eta_{n v}^{(2)} - \delta \tilde{y}_{i v} \delta \lambda_{i v}^{(2)} - \delta \tilde{a}_n \delta \eta_{n v}^{(2)} = \delta x_{j v} \delta \tilde{\lambda}_{j v}^{(3)} + \\ + \delta a_n \delta \tilde{\eta}_{n v}^{(3)} - \delta \tilde{x}_{j v} \delta \lambda_{j v}^{(3)} - \delta \tilde{a}_n \delta \eta_{n v}^{(3)}.$$

Thus, according to (6.21)-(6.23) and condition (6.35), we produce

$$d_1 = d_2 = d_3 = d, \tag{6.37}$$

i.e. the right portions of equations (6.21)-(6.23) along E are equal to constant number d.

Conditions (6.35) and (6.37) are called the attached discontinuity conditions.

We can thus affirm the following.

Lemma 4. For any nonsingular extreme E, there are p independent sets of variations (6.15) coupled along E by variational equations (6.16)-(6.18) and satisfying the attached discontinuity conditions, while expressions $(\delta y' \delta \tilde{\lambda} + \tilde{\delta} a' \tilde{\delta} \eta - \delta \tilde{y}' \delta \lambda - \delta \tilde{a}' \delta \eta)$ and $(\delta x' \delta \tilde{\lambda} + \delta a' \tilde{\delta} \eta - \delta \tilde{x}' \delta \lambda - \delta \tilde{a}' \delta \eta)$ retain a constant value equal to d over the sector $t_0 \le t \le t_k$.

According to the inclusion theorem (theorem 1) along nonsingular extreme E there is a p-parameter set of variations satisfying the discontinuity variation conditions and coupled variational equations (6.16)-(6.18). Then, it follows from the preceding that each pair of p-parameter sets of variations has an attached discontinuity condition, while expressions (6.37) retain a constant value equal to d over $t_0 \le t \le t_k$. The lemma is proven.

Let us introduce the concept of attached solutions and attached systems of solutions of variational equation system (6.16)-(6.18). Two arbitrary independent solutions of the variational equation systems (6.16)-(6.18) will be called an attached solution if constant d is equal to zero. This system of linearly independent paired attached solutions will be referred to as the attached system of solutions of the system of variational equations.

Let us formulate the following lemma.

Lemma 5. Suppose E is a nonsingular extreme for which there is an attached system of solutions Z_{ik}, $\Lambda_{ik}^{(1)}$, $\Lambda_{ik}^{(2)}$, V_{jk}, $\Lambda_{jk}^{(3)}$, $H_{sk}^{(1)}$, $H_{sk}^{(2)}$, $H_{sk}^{(3)}$, A_{sk} $(i=1,\ldots,n;\ j=1,\ldots,m;\ s=1,\ldots,r;\ k=1,\ldots,n+r)$ of equations (6.16)-(6.18) along E, with determinants $\det_1 = |Z_{jk}A_{sk}|$ and $|V_{jk}A_{sk}| = \det_2$, not vanishing over E. Then E is an extremally singly coupled area Γ, singly covered with the (n + r) parameter set of extremes:

$$
\begin{aligned}
& y_i = y_i(t, \alpha_l) \quad (i=1,\ldots,n;\ l=1,\ldots,n+r). \\
& x_j = x_j(t, \alpha_l) \quad (j=1,\ldots,m), \\
& a_s = a_s(\alpha_l), \\
& \lambda_i^{(p)} = \lambda_i^{(p)}(t, \alpha_l) \quad (p=1, 2), \\
& \lambda_j^{(3)} = \lambda_j^{(3)}(t, \alpha_l), \\
& \eta_s^{(v)} = \eta_s^{(v)}(t, \alpha_l) \quad (v=1, 2, 3),
\end{aligned}
$$

containing E with the following values (t, α):

$$t_0 \leqslant t \leqslant t_k,\ \alpha_l = 0 \ (l=1,\ldots,n+r).$$

Functions y_i, y_{it}, x_j, x_{jt}, a_s, $\lambda_i^{(p)}$, $\lambda_{it}^{(p)}$, $\lambda_j^{(3)}$, λ_{jt}, $\eta_s^{(v)}$, $\eta_{st}^{(v)}$ have continuous partial derivatives between the corner points in the area of the set of values (t, α) corresponding to curve E, and the variations of the set along E are

$$
\left.
\begin{aligned}
& y_{i\alpha_k}(t, 0) = Z_{ik}(t), \quad x_{j\alpha_k}(t, 0) = V_{jk}(t), \\
& \lambda_{i\alpha_k}^{(p)}(t, 0) = \Lambda_{ik}^{(p)}(t), \ \lambda_{j\alpha_k}^{(3)}(t, 0) = \Lambda_{jk}^{(3)}(t), \\
& \eta_{s\alpha_k}^{(v)}(t, 0) = H_{sk}^{(v)}(t), \ a_{s\alpha_k} = A_{sk}.
\end{aligned}
\right\} \qquad (6.38)
$$

For proof, let us analyze the 2(n + r)-parameter set of extremes

$$y_i = y_i(t, \gamma, \beta),\ a_s(\gamma, \beta),\ \lambda_i^{(p)} = \lambda_i^{(p)}(t, \gamma, \beta),\ \eta_s^{(p)} = \eta_s^{(p)}(t, \gamma, \beta), \qquad (6.39a)$$

$$x_j = x_j(t, \gamma, \beta),\ \lambda_j^{(3)} = \lambda_j^{(3)}(t, \gamma, \beta),\ \eta_s^{(3)} = \eta_s^{(3)}(t, \gamma, \beta), \qquad (6.39b)$$

containing E with

$$t_{00} < t < t_{\kappa 0},\ \gamma_i = \gamma_{i0},\ \beta_i = \beta_{i0}\ (i = 1, \ldots, n+r)$$

and having the properties discussed in the inclusion theorem (theorem 1).

Without limiting generality, we can consider that the following values have been taken as the constants γ_1 and β_1:

$$\left.\begin{array}{ll} \gamma_l = y_l(t_0, \gamma, \beta) & (l = 1, \ldots, n), \\ \gamma_p = a_s(\gamma, \beta) & (s = 1, \ldots, r;\ p = n+s), \\ \beta_l = \lambda_l^{(1)}(t_0, \gamma, \beta) & (l = 1, \ldots, n); \\ \beta_p = \eta_s^{(1)}(t_0, \gamma, \beta) & (s = 1, \ldots, r;\ p = n+s), \end{array}\right\}. \qquad (6.40)$$

where t_0 is generally a certain number of the t-interval corresponding to E.

From equations (6.40) we find

$$\left.\begin{array}{ll} \delta_{lk} = y_{l\gamma_k}(t_0, \gamma, \beta), & 0 = \lambda_{l\gamma_k}^{(1)}(t_0, \gamma, \beta) \\ & (l = 1, \ldots, n;\ k = 1, \ldots, n+r), \\ \delta_{pk} = a_{s\gamma_k}(\gamma, \beta), & 0 = \eta_{s\gamma_k}^{(1)}(t_0, \gamma, \beta) \\ & (s = 1, \ldots, r;\ p = n+s), \\ 0 = y_{l\beta_k}(t_0, \gamma, \beta), & \delta_{lk} = \lambda_{l\beta_k}(t_0, \gamma, \beta), \\ 0 = a_{s\beta_k}(\gamma, \beta), & \delta_{pk} = \eta_{s\beta_k}^{(1)}(t_0, \gamma, \beta), \end{array}\right\} \qquad (6.41)$$

where $\delta_{ll} = 1,\ \delta_{pp} = 1\ \&\ \delta_{lk} = 0,\ \delta_{pk} = 0$ if $l \neq k\ \&\ p \neq k$.

Let us define the two functions C(α) and B(α) as follows:

$$\left.\begin{array}{l} 2C(\alpha) = 2\gamma_{l_0}\alpha_l + Z_{lk}(t_0)\alpha_l\alpha_k + A_{sk}\alpha_p\alpha_s \\ \qquad (l = 1, \ldots, n+r), \\ 2B(\alpha) = 2\beta_{l_0}\alpha_l + \Lambda_{lk}^{(1)}(t_0)\alpha_l\alpha_k + H_{sk}^{(1)}\alpha_p\alpha_s. \end{array}\right\} \qquad (6.42)$$

Then set (6.39a) where $\gamma_l = C_{\alpha_l}$, $\beta_l = B_{\beta_l}$ $(l = 1, \ldots, n + r)$ will be an $(n + r)$-parameter set of extremes

$$\left.\begin{aligned} & y_i = y_i(t, C_\alpha, B_s) = y_i(t, \alpha_l) \quad (l = 1, \ldots, n + r), \\ & a_s = a_s(C_\alpha, B_s) = a_s(\alpha_l), \\ & \lambda_i^{(1)} = \lambda_i^{(1)}(t, C_\alpha, B_s) = \lambda_i^{(1)}(t, \alpha_l), \\ & \eta_s^{(1)} = \eta_s^{(1)}(t, C_\alpha, B_s) = \eta_s^{(1)}(t, \alpha_l), \end{aligned}\right\} \tag{6.43}$$

containing E where $(\alpha_1, \ldots, \alpha_{n+r}) = (0, \ldots, 0)$, the variations of which with respect to parameter α_k are

$$\begin{aligned} y_{i\alpha_k} &= y_{i\gamma_j} Z_{jk}(t_0) + y_{i\gamma_p} A_{sk} + y_{i\beta_j} \Lambda_{jk}^{(1)}(t_0) + y_{i\beta_p} H_{sk}^{(1)}(t_0), \\ a_{s\alpha_k} &= a_{s\gamma_j} Z_{jk}(t_0) + a_{s\gamma_p} A_{sk} + a_{s\beta_j} \Lambda_{jk}^{(1)}(t_0) + a_{s\beta_p} H_{sk}^{(1)}(t_0), \\ \lambda_{i\alpha_k}^{(1)} &= \lambda_{i\gamma_j}^{(1)} Z_{jk}(t_0) + \lambda_{i\gamma_p}^{(1)} A_{sk} + \lambda_{i\gamma_p}^{(1)} \Lambda_{jk}^{(1)}(t_0) + \lambda_{i\beta_p}^{(1)} H_{sk}^{(1)}(t_0), \\ \eta_{s\alpha_k}^{(1)} &= \eta_{s\gamma_j}^{(1)} Z_{jk}(t_0) + \eta_{s\gamma_p}^{(1)} A_{sk} + \eta_{s\beta_j}^{(1)} \Lambda_{jk}^{(1)}(t_0) + \eta_{s\beta_p}^{(1)} H_{sk}^{(1)}(t_0). \end{aligned}$$

For curve E, these variations where $t = t_0$ have the values

$$Z_{ik}(t_0),\ \Lambda_{ik}^{(1)}(t_0)\,(i = 1, \ldots, n;\ k = 1, \ldots, n + r),\ A_{sk}\ \&\ H_{sk}^{(1)}(t_0)\,(s = 1, \ldots, r),$$

which follows from relationships (6.41), and they satisfy variational equation system (6.15). The determinant

$$\left\| \begin{matrix} Z_{ik}(t_0) A_{sk} \\ \Lambda_{ik}^{(1)}(t_0) H_{sk}^{(1)}(t_0) \end{matrix} \right\| = \left\| \begin{matrix} y_{i\alpha_k} a_{s\alpha_k} \\ \lambda_{i\alpha_k}^{(1)} \eta_{s\alpha_k} \end{matrix} \right\|$$

according to (6.41) is equal to one.

Thus, along E they are identical to $Z_{ik}(t)$, $\Lambda_{ik}^{(1)}(t)$, $H_{sk}^{(1)}(t)$ and A_{sk}, as was stated in the lemma.

Since the determinant $|y_{i\alpha_k} a_{s\alpha_k}| = |Z_{ik}(t_0)A_{sk}|$ is equal to one and the determinant $|Z_{ik}(t)A_{sk}|$ does not vanish anywhere in the interval $t_0 \leq t \leq t_k$ on curve E, according to the theorem on implicit functions in space $T \times B \times A$, there is an area Γ of curve E in which equation (6.43) has a unique solution $\alpha_k(t, y, a)$, vanishing on E and having continuous first order partial derivatives in area Γ. Then area Γ is singly coupled and singly covered by the set of extremes (6.43).

It remains to be shown that in sufficiently small area Γ integral I* formed by functions (6.43) is independent of the path of integration. On hyperplane $t = t_0$

$$I^* = \int (F_{y_i'} dy_i + F_{a_s'} da_s) = \int \{\lambda_i^{(1)}[t_0, \alpha(t_0, y, a)]\, dy_i [t_0, \alpha(t_0, y, a)] + $$
$$+ \eta_s^{(1)}[t_0, \alpha(t_0, y, a)]\, da_s [t_0, \alpha(t_0, y, a)]\}.$$

It also follows from equations (6.40), (6.41) and (6.43) that on this hyperplane

$$I^* = \int \{[\gamma_{l0} + \Lambda_{lj}(t_0)\alpha_j] Z_{lk}(t_0) + [\beta_{p0} + H_{sj}(t_0)\alpha_j] A_{sk}\}\, d\alpha_k =$$
$$= \int d\left[\gamma_{l0} Z_{lk}(t_0)\alpha_k + \frac{1}{2} \Lambda_{lj}(t_0) Z_{jk}(t_0)\alpha_j\alpha_k + \beta_{p0} A_{sk} + \right.$$
$$\left. + \frac{1}{2} H_{sj}(t_0) A_{sk}\alpha_j\alpha_k\right].$$

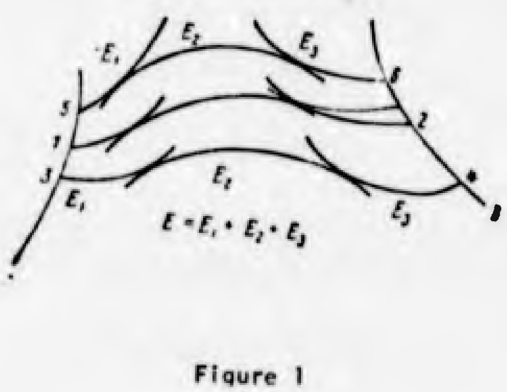

Figure 1

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In converting the integrand, it was considered that for the attached system Z_{ik}, Λ_{ik} and A_{sk}, H_{sk}, the equation

$$Z_{ij}\Lambda_{ik}+A_{sj}H_{sk}-\Lambda_{ij}Z_{ik}-H_{sj}A_{sk}=0$$

becomes an identity.

Thus, I^* is independent of the path of integration on hyperplane $t = t_0$. Similar discussions can be performed for any value of $t \geq t_q^+$ from the t-interval corresponding to curve E.

The arbitrary curve B_{46}, lying in the sufficiently small area Γ of curve E, defines the single-parameter set of extremes intersecting hyperplane $t = t_0$ at the points of curve C_{35}, forming a configuration as shown on Figure 1. We then have

$$\int_B \left[\left(F_3-\dot{x}_j\frac{\partial F_3}{\partial \dot{x}_j}\right)dt+\frac{\partial F_3}{\partial \dot{x}_j}dx_j+\frac{\partial F_3}{\partial \dot{a}_s}da_s\right]=$$

$$=I^*(B_{46})=I^*(C_{35})+I(E_{56})-I(E_{34}).$$

In the right portion of this solution, each term is fully defined if we fix points 4 and 6, since I^* is independent of the path of integration on the hyperplane $t = t_0$. Therefore, the integral I^* is independent of the path of integration on the entire area Γ. The lemma is proven.

If we use the conditions and results of lemma 5, no necessary conditions other than those analyzed are required for formulation of the sufficient condition and its proof.

Before stating and proving the sufficient condition, let us formulate the necessary conditions -- I, II_N, II_N' and II_π, III. The symbols I, II' and II_π, III represent the following conditions: the stability condition, Weierstrass condition (5.23) for the open area of permissible controls and the maximum principle of L. Pontryagin (in this case the minimum principle) (5.24) for the closed area of permissible controls, plus the necessary Clebsch condition (5.27).

It is said that function E satisfies the Weierstrass condition II_N if for each element (t, y, x, a, u, $\lambda^{(s)}$) (s = 1, 2, 3), satisfying equations (1.1)-(1.13) and lying in the area N of the set of elements corresponding to curve E, the inequality

$$E(t, y, x, \lambda^{(\alpha)}, u, \vartheta) \geqslant 0$$

is fulfiled between corner points with arbitrary permissible (t, y, x, $\dot{\vartheta}$) ≠ (t, y, x, u), satisfying equations (1.1)-(1.13). If the equality sign is excluded, the preceding condition is called the reinforced Weierstrass condition II'_N.

III′ represents the reinforced Clebsch condition, produced from III by exclusion of the equality sign.

The theory of the main sufficient condition for the strong relative maximum can now be formulated as follows.

Theorem 4. Sufficient condition for strong relative maximum. Suppose E is a normal curve of class D. If curve E satisfies the conditions I, II′ and III′ and if there is no attached system of solutions satisfying the conditions of lemma 5 for it, then E is a nonsingular extreme and in space T × B × A and T × C × A there is an area Γ of curve E such that for any curve C of class D lying in Γ and not corresponding with E, the ends of which lie sufficiently close to the ends of E, the inequality J(E) > J(C) is fulfilled.

Since according to a condition of the theorem curve E is a normal class D curve, satisfying conditions I and III′, by definition (see § 4) it is a nonsingular extreme. The normal nonsingular extreme E can be included in the 2p-parameter set of extremes, in which case the determinants (4.47) on arcs of E are not equal to zero. This follows from the inclusion theorem (theorem 1).

According to lemma 4 for the nonsingular extreme E, there are p independent sets of variations (6.15) coupled along E by equations (6.16)-(6.18) and satisfying the attached discontinuity conditions, where the expressions (6.21)-(6.23) are equal to the constant value of d. Due to the existence, according to the condition of the problem, of an attached system of solutions, this constant d is equal to zero. Then, keeping in mind the conditions of the theorem and lemma 5, we can affirm the existence of a singly coupled area Γ, covered by a p-parameter set of extremes, containing the given nonsingular extreme E. Therefore, there is an integral I* (E), independent of the path of integration over the entire area Γ. Since according to lemma 3, I*(E)= Φ(E), for any curve C having the properties stated in the theorem we have

$$I(C)-I(E)=[\Phi(C_1)+\Phi(C_2)+\Phi(C_3)]-[\Phi(E_1)+$$
$$+\Phi(E_2)+\Phi(E_3)]=[\Phi(C_1)+\Phi(C_2)+\Phi(C_3)]-[I^*(E_1)+I^*(E_2)+$$
$$+I^*(E_3)]=\int_{t_1}^{t_\theta} E_1(t,y,a,\lambda^{(1)},u,\vartheta)\,dt+\int_{t_\theta}^{t_\varphi} E_2(t,y,a,\lambda^{(2)},u,\vartheta_\sigma)\,dt+$$
$$+\int_{t_\varphi}^{t_\pi} E_3(t,x,a,\lambda^{(3)},u,\vartheta)\,dt.$$

Each integral is negative and vanishes only when C is an extreme in area Γ. Therefore the difference I(C)-(I(E) is negative, except for the case of coincidence of curve C with curve E, since only one curve can pass through the point corresponding to the ends of curve E. The theorem is thus proven.

It is shown in [7] (see lemma 88.1) that conditions II_N' and III′ are equivalent to condition II_N and the condition that curve E is nonsingular. For any necessary extreme, according to result 2, condition II_N' is equivalent to condition II_π. Therefore, the following result obtains.

Result 3. Suppose E is a normal curve of class D. If nonsingular curve E satisfies conditions I and II_π and there exists an attached system of solutions for it, satisfying the conditions of lemma 5, then E is a nonsingular extreme and in space T × B × A and T × C × A there is an area Γ of curve E such that for any curve of class D lying in Γ and not corresponding with E, the ends of which are sufficiently close to the ends of E, the inequality J(E) > J(C) is fulfilled.

Suppose on the normal curve E, which is an extreme and therefore satisfies the discontinuity conditions (1.1)-(1.3) and (4.32)-(4.37), the condition at the end (1.15) and transversality conditions (4.28), (4.29) and (4.41), (4.42) are fulfilled. Let us assume further that along E there exist p-permissible independent solutions of the equations (6.16)-(6.18), satisfying the variational discontinuity conditions and thereby the attached discontinuity conditions, and also satisfying the following final variational conditions

$$d\psi_\pi=0, \tag{6.44}$$

$$\frac{\partial F_3}{\partial x_{jk}} dx_{jk} - \dot{x}_{jk} d\frac{\partial F_3}{\partial \dot{x}_{jk}} + \frac{\partial F_3}{\partial a_\chi} da_\chi + e_k \frac{\partial^2 \psi_k}{\partial t_k \partial x_{jk}} dx_{jk} +$$

$$+ e_k \frac{\partial^2 \psi_k}{\partial t_k^2} dt_k + \frac{\partial \psi_k}{\partial t_k} de_k + \frac{\partial^2 J}{\partial t_k \partial x_{jk}} dx_{jk} + \frac{\partial^2 J}{\partial t_k \partial a_l} da_l +$$

$$+ \frac{\partial^2 J}{\partial t_k \partial a_\chi} da_\chi + \frac{\partial^2 J}{\partial t_k^2} dt_k = 0, \tag{6.45}$$

$$d\lambda_{jk}^{(3)} + e_k \frac{\partial^2 \psi_k}{\partial x_{jk} \partial x_{sk}} dx_{sk} + e_k \frac{\partial^2 \psi_k}{\partial x_{jk} \partial t_k} dt_k + \frac{\partial \psi_k}{\partial x_{jk}} de_k +$$

$$+ \frac{\partial^2 J}{\partial x_{jk} \partial x_{sk}} dx_{sk} + \frac{\partial^2 J}{\partial x_{jk} \partial a_l} da_l + \frac{\partial^2 J}{\partial x_{jk} \partial a_\chi} da_\chi + \frac{\partial^2 J}{\partial x_{jk} \partial t_k} dt_k = 0 \tag{6.46}$$

$$(j, s = 1, \ldots, m),$$

$$d\eta_{lk}^{(3)} + \frac{\partial^2 J}{\partial a_l \partial x_{jk}} dx_{jk} + \frac{\partial^2 J}{\partial a_l \partial a_p} da_p + \frac{\partial^2 J}{\partial a_l \partial a_\chi} da_\chi + \frac{\partial^2 J}{\partial a_l \partial t_k} dt_k = 0 \tag{6.47}$$

$$(l, p = 1, \ldots, m < r),$$

$$d\eta_{\chi k}^{(3)} + \frac{\partial^2 J}{\partial a_\chi \partial x_{jk}} dx_{jk} + \frac{\partial^2 J}{\partial a_\chi \partial a_s} da_s + \frac{\partial^2 J}{\partial a_\chi \partial a_l} da_l + \frac{\partial^2 J}{\partial a_\chi \partial t_k} dt_k +$$

$$+ e_{2k} \frac{\partial^2 \varphi_{2k}}{\partial a_\chi \partial a_s} da_s + \frac{\partial \varphi_{2k}}{\partial a_\chi} de_{2k} = 0 \tag{6.48}$$

$$(\chi, s = m+1, \ldots r),$$

produced for the conditions from the ends (1.15) and the transversality conditions at the end point (4.28), (4.29) and (4.41), (4.42) due to their identical fulfillment. Furthermore, it should be kept in mind that according to (1.13), $d\beta_{2k} = 0$.

Let us multiply the left portions of equations (6.45)-(6.48), satisfying one of the solutions (with the tilde) by dt_k, d_{xj_k}, d_{a_l}, d_{a_χ}, $d\lambda_{jk}^{(3)}$, $d\eta_{nk}^{(3)}$ respectively and add the expressions produced, multiplying the parts of the equations satisfying the other solution (without the tilde) by $d\tilde{t}_k$, $d\tilde{x}_{jk}$, $d\tilde{a}_l$, $d\tilde{a}_\chi$, $d\tilde{\lambda}_{jk}^{(3)}$, $d\tilde{\eta}_{nk}^{(3)}$ respectively and also adding the expressions produced. Then, subtracting the second sum from the first sum, we find

$$dx_{jk} d\tilde{\lambda}_{jk}^{(3)} + da_n d\tilde{\eta}_{nk}^{(3)} - d\tilde{x}_{jk} d\lambda_{jk} - d\tilde{a}_n d\eta_{nk}^{(3)} + \frac{\partial F_3}{\partial x_{jk}} (d\tilde{x}_{jk} dt_k -$$

$$- dx_{jk} d\tilde{t}_k) - \dot{x}_{jk} (d\tilde{\lambda}_{jk}^{(3)} dt_k - d\lambda_{jk}^{(3)} d\tilde{t}_k) + (d\tilde{a}_\chi dt_k - da_\chi d\tilde{t}_k) \frac{\partial F_3}{\partial a_\chi} = 0.$$

If we now go over to variations, we produce

$$\delta x_{jk} \delta\tilde{\lambda}_{jk}^{(3)} + \delta a_n \delta\tilde{\eta}_{nk}^{(3)} - \delta\tilde{x}_{jk} \delta\lambda_{jk}^{(3)} - \delta\tilde{a}_n \delta\eta_{nk}^{(3)} = 0.$$

From this, keeping in mind (6.23) and (6.37), we have

$$d = 0 \qquad (6.49)$$

i. e. along the given curve E the right portions of equations (6.21)-(6.23) are equal to zero. Thus, if on the normal curve E, a nonsingular extreme and therefore satisfying the discontinuity conditions (1.1)-(1.3) and (4.32)-(4.37) by definition, the conditions for the ends and the transversality conditions (4.28), (4.29) and (4.41), (4.42) are fulfilled at the end point, then in the case of existence of p-linearly independent solutions of the conditions for the ends and transversality conditions in variations (6.45)-(6.48) at the end point, the variational equations (6.16)-(6.18) analyzed as initial conditions, there is an attached system of solutions of variational equations (6.16)-(6.18) along E, satisfying the attached discontinuity conditions. Similarly, it can be shown that if on normal curve E, a nonsingular extreme and therefore satisfying discontinuity conditions (1.1)-(1.3) and (4.32)-(4.37) by definition, the conditions for the ends (1.15) and the transversality conditions (4.26), (4.27) and (4.39) are fulfilled at the initial point, then in the case of existence of p-linearly independent solutions of the conditions for the ends and transversality conditions, there exists along E an attached system of solutions of variational equations (6.16)-(6.18), satisfying the attached discontinuity conditions.

Generally speaking, we cannot find a solution to variational equations (6.16)-(6.18) along E such that it simultaneously satisfies the conditions for the ends and the variational transversality conditions at the initial and final points.

We have thus proven the following lemma.

Lemma 6. Suppose E is a normal, nonsingular extreme, in class D, the ends of which fulfill the transversality conditions (4.26)-(4.29) and (4.39), (4.41) and (4.42). If there exist p-linearly independent solutions of the conditions for the ends and variational transversality conditions of the initial and final points respectively, then, taking them as the initial conditions, we produce along E two attached systems of solutions of the variational equations (6.16)-(6. 18), satisfying the attached discontinuity conditions.

Based on this lemma and on the theorem for the main sufficient condition for a strong relative maximum, we can formulate the following theorem for the sufficient condition for the strong relative maximum.

Theorem 5. Suppose E is a normal curve of class D. If curve E satisfies conditions I, II_N' and III′ and there exist p-linearly independent solutions of the conditions for the ends and the variational transversality conditions at the initial or final points respectively with $\det_1 \neq 0$ and $\det_2 \neq 0$, then E is a nonsingular extreme and in space $T \times B \times A$ and $T \times C \times A$ there exists an area Γ of curve E such that for any curve C of class D lying in Γ not corresponding with E, the ends of which lie sufficiently close to the ends of E, the inequality $J(E) \geq J(C)$ is fulfilled.

Lemma 6, theorem 5 and result 3 confirm the correctness of the following.

Result 4. Suppose E is a normal curve of class D. If nonsingular curve E satisfies conditions I′ and II_π and there exist p-linearly independent solutions of the conditions for the ends and variational transversality conditions at the initial or end points respectively, with $\det_1 \neq 0$ and $\det_2 \neq 0$, E is a nonsingular extreme and in space $T \times B \times A$ and $R \times C \times A$ there exists an area Γ of curve E such that for any curve of class C lying in Γ and not corresponding with E and the ends of which lie sufficiently close to the ends of E, the inequality $J(E) \geq J(C)$ is fulfilled.

MULTIPOINT BOUNDARY PROBLEM

§ 7. Mathematical Theory of Algorithm of Multipoint Boundary Problem

In order to determine the extreme and optimal controls providing the maximum value for functional J, we must solve system (1.4)-(1.12) and (4.15)-(4.20) considering (4.22)-(4.24), satisfying the conditions at the initial point (1.14), (4.26), (4.27) and (4.39), the conditions at intermediate points (1.1)-(1.3) and (4.32)-(4.37), (4.40), the conditions at the final point (1.15) and (4.28), (4.29) and (4.41), (4.42) and conditions (1.13). It was shown earlier (§ 4) that the number of conditions at the ends and in the intermediate points corresponds fully to that necessary for integration of system (1.4)-(1.12) and (4.15)-(4.20) considering (4.22)-(4.24). Therefore, determination of the extreme and optimal control giving functional J its maximum value with the fixed boundary and intermediate conditions is reduced to solution of a multipoint boundary problem.

For completeness of the solution of the variational problem which we have stated, it is desirable to find an algorithm for solution of the boundary problem such that, while providing rapid convergence of iterations, it allows fulfillment of the conditions of theorem 5 or result 4. Then the solution of the multipoint boundary problem considering fulfillment of the necessary Weierstrass condition allows us to affirm that the variational problem has been solved and that the necessary and sufficient conditions are fulfilled.

Suppose the solution

$$y(t)=y^0(t)+b\delta y(t)+0(b,t) \quad (|b|<\varepsilon, \lim \frac{0(b,t)}{b}=0$$

where $t\in[t_0,t_k]$),

$$x(t)=x^0(t)+b\delta x(t)+0(b,t),$$

$$u(t)=u^0(t)+b\delta u(t)+0(b,t),$$

$$a=a^0+b\delta a+0(b)$$

and the corresponding

$$\lambda(t)=\lambda^0(t)+b\delta\lambda(t)+0(b,t),$$

$$\eta(t)=\eta^0(t)+b\delta\eta(t)+0(b,t),$$

$$\mu(t)=\mu^0(t)+b\delta\mu(t)+0(b,t)$$

satisfy equations (1.1)-(1.12) and the stability conditions. Here the vector functions $y^0(t)$, $x^0(t)$, $u^0(t)$, a^0, $\lambda^0(t)$, $\mu^0(t)$ are a possible initial or "zero" solution of equation system (1.4)-(1.12) and (4.15)-(4.20) considering (4.22)-(4.24).

The intial solution satisfies the equations

$$\psi_1^0=0,\ \psi_2^0=0,\ \psi_3^0=0, \tag{7.1}$$

which are equations of the system of equations (1.1)-(1.2) and (1.15) respectively, where the function ψ_1^0 is selected from the components of vector functions ψ_τ which depend only on t_q, y_q^- and a_1, where $a_1 \in A_1$. We will call functions (7.1) the stop functions. Generally speaking, the left portion of the remaining equations except for (7.1) of system (1.1), (1.2), (1.15) and equations (4.28), (4.29) and (4.32)-(4.37), (4.40)-(4.42) may not be equal to zero.

Vector function $\delta y(t)$, $\delta x(t)$, $\delta u(t)$, δa, $\delta\lambda(t)$, $\delta\eta(t)$, $\delta\mu(t)$, independent of b, will be determined by the system of variational equations (6.16)-(6.18) along the curve of the "zero" solution C^0. Let us write the system of equations conjugate with the system of variational equations (6.16)-(6.18) as follows:

at $[t_0, t_q^-]$

$$\left.\begin{aligned}
&\dot z^{(1)} = -A_1' z^{(1)} - D_1'\chi^{(1)} + K_1'\nu^{(1)} + F_1\alpha^{(1)} - G_1\beta^{(1)},\\
&\dot l^{(1)} = -B_1' z^{(1)} + F_1'\nu^{(1)} + M_1'\alpha^{(1)} - L_1\beta^{(1)} - P_1'\chi^{(1)},\\
&\dot\nu^{(1)} = A_1\nu^{(1)} + B_1\alpha^{(1)} - C_1\beta^{(1)},\\
&\dot\alpha_{(1)} = 0,\\
&-C_1' z^{(1)} - E_1\chi^{(1)} + G_1'\nu^{(1)} + L_1'\alpha^{(1)} - R_1'\beta^{(1)} = 0,\\
&D_1\nu^{(1)} - E_1\beta^{(1)} + P_1\alpha^{(1)} = 0;
\end{aligned}\right\} \quad (7.2)$$

at $[t_q^+, t_v^-]$

$$\left.\begin{aligned}
&\dot z^{(2)} = -A_2' z^{(2)} - D_2'\chi^{(2)} + K_2'\nu^{(2)} + F_2\alpha^{(2)} - G_2\beta^{(2)},\\
&\dot l^{(2)} = -B_2' z^{(2)} + F_2'\nu^{(2)} + M_2'\alpha^{(2)} - L_2\beta^{(2)} - P_2'\chi^{(2)},\\
&\dot\nu^{(2)} = A_2\nu^{(2)} + B_2\alpha^{(2)} - C_2\beta^{(2)}\\
&\dot\alpha^{(2)} = 0,\\
&-C_2' z^{(2)} - E_2'\chi^{(2)} + G_2'\nu^{(2)} + L_2'\alpha^{(2)} - R_2'\beta^{(2)} = 0,\\
&D_2\nu^{(2)} - E_2\beta^{(2)} + P_2\alpha^{(2)} = 0;
\end{aligned}\right\} \quad (7.3)$$

at $[t_v^+, t_k]$

$$\left.\begin{aligned}
&\dot z^{(3)} = -A_3' Z^{(3)} - D_3'\chi^{(3)} + K_3'\nu^{(3)} + F_3\alpha^{(3)} - G_3\beta^{(3)},\\
&\dot l^{(3)} = -B_3' z^{(3)} + F_3\nu^{(3)} + M_3'\alpha^{(3)} - L_3\beta^{(3)} - P_3'\chi^{(3)},\\
&\dot\nu^{(3)} = A_3\nu^{(3)} + B_3\alpha^{(3)} - C_3\beta^{(3)},\\
&\dot\alpha^{(3)} = 0,\\
&-C_3' z^{(3)} - E_3'\chi^{(3)} + G_3'\nu^{(3)} + L_3'\alpha^{(3)} - R_3'\beta^{(3)} = 0,\\
&D_3\nu^{(3)} - E_3\beta^{(3)} + P_3\alpha^{(3)} = 0.
\end{aligned}\right\} \quad (7.4)$$

Systems (6.16)-(6.18) and (7.2)-(7.4) are conjugate, since the matrices with coefficients δy and z or δx and z, δa and l, δu and χ, $\delta\lambda$ and ν, $\delta\eta$ and α, $\delta\mu$ and β are produced from each other by transportation and sign changing.

Linear equation system (7.2)-(7.4) is called the conjugate system.

Let us now show that if $\delta y(t)$, $\delta x(t)$, δa, $\delta\lambda(t)$, $\delta\eta(t)$ is an arbitrary solution of system (6.16)-(6.18) along the curve of zero solution C^0, while $z(t)$, $l(t)$, $\nu(t)$ and α is an arbitrary solution of conjugate system (7.2)-(7.4), the scalar derivative

$$\left(\begin{Vmatrix} z(t) \\ l(t) \\ \nu(t) \\ \alpha \end{Vmatrix}, \begin{Vmatrix} \delta y(t) \\ \delta a \\ \delta\lambda(t) \\ \delta\eta(t) \end{Vmatrix}\right) = z'\delta y + l'\delta a + \nu'\delta\lambda + \alpha'\delta\eta$$

constantly correspondingly between t_0 and the hypersurface

$$\varphi_1^0=0,\ \psi_1^0=0 \quad \text{and} \quad \varphi_2^0=0,\ \psi_2^0=0 \quad \text{and} \quad \psi_3^0=0.$$

Actually, between t_0 and hypersurface $\psi_1^0 = 0$ we have

$$\frac{d}{dt}(z'\delta y + l'\delta a + \nu'\delta\lambda + \alpha'\delta\eta) = \dot z'\delta y + z'\delta\dot y + \dot l'\delta a + l'\delta\dot a +$$
$$+ \dot\nu'\delta\lambda + \nu'\delta\dot\lambda + \dot\alpha'\delta\eta + \alpha'\delta\dot\eta.$$

Substituting values of $\dot z$, $\delta\dot y$, $\dot l$, $\delta\dot a$, $\dot\nu$, $\delta\dot\lambda$, $\dot\alpha$, $\delta\dot\eta$ in the right portion of this equation according to variational equation system (6.16)-(6.18) and conjugate system (7 2)-(7.4), we produce

$$\frac{d}{dt}(z^{(1)'}\delta y + l^{(1)'}\delta a + \nu^{(1)'}\delta\lambda^{(1)} + \alpha^{(1)'}\delta\eta^{(1)}) = (-A_1'z^{(1)} - D_1'\chi^{(1)} +$$
$$+ K_1'\nu^{(1)} - F_1\alpha^{(1)} - G_1\beta^{(1)})'\delta y + z^{(1)'}(A_1\delta y + B_1\delta a + C_1\delta u) +$$
$$+ (-B_1'z^{(1)} - P_1'\chi^{(1)} + F_1\nu^{(1)} + M_1'\alpha^{(1)} - L_1\beta^{(1)})'\delta a + (A_1\nu^{(1)} +$$
$$+ B_1\alpha^{(1)} - C_1\beta^{(1)})'\delta\lambda^{(1)} + \nu^{(1)'}(-A_1\delta\lambda^{(1)} - F_1\delta a - G_1\delta u - K_1\delta y -$$
$$- D_1'\delta\mu^{(1)}) + \alpha^{(1)'}(-F_1'\delta y - B_1'\delta\lambda^{(1)} - L_1\delta u - M_1\delta a - P_1'\delta\mu^{(1)}) =$$
$$= -\chi^{(1)'}D_1'\delta y - \beta^{(1)'}G_1'\delta y + z^{(1)'}C_1\delta u - \beta^{(1)'}C_1'\delta\lambda^{(1)} + \nu^{(1)'}G_1\delta u -$$
$$- \nu^{(1)'}D_1\delta\mu^{(1)} - \alpha^{(1)'}L_1\delta u - \beta^{(1)'}L_1'\delta a.$$

Keeping in mind the third equation from (6.16) and the third equation from (7.2), we have

$$-\chi^{(1)'}D_1\delta y=\chi^{(1)'}(E_1\delta u+P_1\delta a)=(E_1'\chi^{(1)})'\delta u+\chi^{(1)'}P_1\delta a=$$
$$=(-C_1'z^{(1)}+G_1'\nu^{(1)}+L_1'\alpha^{(1)}-R_1'\xi^{(1)})'\delta u+\chi^{(1)'}P_1\delta a.$$

Substituting the value of $\chi^{(1)'}D_1\delta y$ in the right portion of the preceding equation, we find

$$\frac{d}{dt}\left(z^{(1)'}\delta y+l^{(1)'}\delta a+\nu^{(1)'}\delta\lambda+\alpha^{(1)'}\delta\eta\right)=-\xi^{(1)'}R_1\delta u-\xi^{(1)'}G_1'\delta y-$$
$$-\xi^{(1)'}L_1'\delta a-\xi^{(1)'}C_1'\delta\lambda^{(1)}-\nu^{(1)'}D_1'\delta\mu^{(1)}-\alpha^{(1)'}P_1'\delta\mu^{(1)}.$$

Further, substituting the value of $R_1\delta u$ here, according to the last equation of (6.16) we produce

$$\frac{d}{dt}\left(z^{(1)'}\delta y+l^{(1)'}\delta a+\nu^{(1)'}\delta\lambda+\alpha^{(1)'}\delta\eta\right)=\xi^{(1)'}E_1'\delta\mu^{(1)}-\alpha^{(1)'}P_1'\delta\mu^{(1)}-$$
$$-\nu^{(1)'}D_1'\delta\mu^{(1)}=-(D_1\nu^{(1)}+P_1\alpha^{(1)}-E_1\xi^{(1)})'\delta\mu^{(1)}=0,$$

since the coefficient with matrix column $\delta\mu^{(1)}$ is zero according to the last equation of (7.2). Thus, between t_0 and hypersurface $\psi_1^0 = 0$ we have

$$z^{(1)'}\delta y+l^{(1)'}\delta a+\nu^{(1)'}\delta\lambda+\alpha^{(1)'}\delta\eta\,\Big|_{t_0}^{\psi_1^0}=0,$$

or the scalar product

$$\left(\begin{Vmatrix} z^{(1)}(t)\\ l^{(1)}(t)\\ \nu^{(1)}(t)\\ \alpha^{(1)} \end{Vmatrix},\ \begin{Vmatrix} \delta y(t)\\ \delta a\\ \delta\lambda^{(1)}(t)\\ \delta\eta^{(1)}(t) \end{Vmatrix}\right)\Bigg|_{t_0}^{\psi_1^0}=0. \qquad (7.5)$$

Similarly we produce

$$\left(\left\|\begin{matrix} z^{(2)}(t)\\ l^{(2)}(t)\\ \nu^{(2)}(t)\\ a^{(2)}(t)\end{matrix}\right\|, \left\|\begin{matrix}\delta y(t)\\ \delta a\\ \delta\lambda^{(2)}(t)\\ \delta\eta^{(2)}(t)\end{matrix}\right\|\right)\Bigg|_{\tau_1^0}^{\tau_2^0} = 0, \qquad (7.6)$$

$$\left(\left\|\begin{matrix} z^{(3)}(t)\\ l^{(3)}(t)\\ \nu^{(3)}(t)\\ a^{(3)}(t)\end{matrix}\right\|, \left\|\begin{matrix}\delta x(t)\\ \delta a\\ \delta\lambda^{(3)}(t)\\ \delta\eta^{(3)}(t)\end{matrix}\right\|\right)\Bigg|_{\tau_2^0}^{\tau_3^0} = 0. \qquad (7.7)$$

Therefore the following lemma is correct.

Lemma 7. Suppose $\delta y(t)$, $\delta a^{(1)}$, $\delta\lambda^{(1)}(t)$, $\delta\eta^{(1)}(t)$ and $\delta y(t)$, $\delta a^{(2)}$, $\delta\lambda^{(2)}(t)$, $\delta\eta^{2}(t)$ and $\delta x(t)$, $\delta a^{(2)}$, $\delta\lambda^{(3)}(t)$, $\delta\eta^{(3)}(t)$ is an arbitrary solution of variational equation system (6.6)-(6.18), while $z^{(i)}$ is an arbitrary solution of conjugate system (7.2)-(7.4) along the curve of the "zero" solution C^0, which satisfies equation system (1.4)-(1.12) and (4.15)-(4.20) considering equations (4.22)-(4.24) and the stop functions $\psi_s^0 = 0$ ($s = 1, 2, 3$), and the end of which satisfies the initial conditions (1.14) and (4.26), (4.27), (4.39),
0 and conditions (4.13). Then the scalar products

$$\left(\left\|\begin{matrix} z^{(1)}(t)\\ l^{(1)}(t)\\ \nu^{(1)}(t)\\ a^{(1)}\end{matrix}\right\|, \left\|\begin{matrix}\delta y(t)\\ \delta a^{(1)}\\ \delta\lambda^{(1)}(t)\\ \delta\eta^{(1)}(t)\end{matrix}\right\|\right), \quad \left(\left\|\begin{matrix} z^{(2)}(t)\\ l^{(2)}(t)\\ \nu^{(2)}(t)\\ a^{(2)}\end{matrix}\right\|, \left\|\begin{matrix}\delta y(t)\\ \delta a^{(2)}\\ \delta\lambda^{(2)}(t)\\ \delta\eta^{(2)}(t)\end{matrix}\right\|\right);$$

$$\left(\left\|\begin{matrix} z^{(3)}(t)\\ l^{3}(t)\\ \nu^{(3)}(t)\\ a^{(3)}\end{matrix}\right\|; \left\|\begin{matrix}\delta x(t)\\ \delta a^{(2)}\\ \delta\lambda^{(3)}(t)\\ \delta\eta^{(3)}(t)\end{matrix}\right\|\right)$$

are equal to the corresponding constant quantities in the corresponding sectors of the time number axis between the stop functions where $t \in T$.

Conditions (4.29), (4.32-(4.38), (4.40)-(4.42) after exclusion of the Lagrange constant factors e_0, e_τ, e_π, ..., e_k together with equations (1.1)-(1.3) and (1.15) without conditions (7.1) can be divided into two groups.

The first group includes the expressions

$$\left.\begin{aligned}&R^q_\zeta(t_q, y_i(t_q^-), a_l, \lambda_i^{(1)}(t_q^-), \eta_i^{(1)}(t_q^-)) = 0 \ (\zeta = 1,\dots,s; l = 1,\dots,p),\\ &R^v_\zeta(t_v, y_i(t_v^-), a_n, \lambda_i^{(2)}(t_v^-), \eta_n^{(2)}(t_v^-)) = 0 \ (\zeta = s+1,\dots,k),\\ &R^\kappa_m(t_\kappa, x_j(t_\kappa), a_n, \lambda_i^{(3)}(t_\kappa), \eta_n^{(3)}(t_\kappa)) = 0 \ (m = k+1,\dots,n+r),\end{aligned}\right\}, \qquad (7.8)$$

which should be fulfilled on the hypersurfaces fixed by equations (7.1) respectively.

The second group should include the equations

$$\left.\begin{aligned}&\Pi^q_k(t_q, y_i(t_q^-), y_i(t_q), a_\chi, \lambda_i^{(1)}(t_q), \lambda_j^{(2)}(t_q), \eta_i^{(2)}(t_q)) = 0\\ &\quad (k = 1,\dots,2n+r-p;\ i = j = 1,\dots,n;\\ &\quad \chi = p+1,\dots,r),\\ &\Pi^v_l(t_v, y_i(t_v), x_j(t_v), a_\chi, \lambda_i^{(2)}(t_v), \lambda_j^{(3)}(t_v)) = 0\\ &\quad (l = 1,\dots,2m;\ j = 1,\dots,m),\end{aligned}\right\} \qquad (7.9)$$

expressing the discontinuity conditions and transformation of the phase coordinates y_i, x_j and Lagrange coefficients λ_i respectively on surfaces $\psi_1^0 = 0$ and $\psi_2^0 = 0$.

On the curve of the "zero" solution C_0^0, conditions (7.8) are not fulfilled, although on hypersurfaces $\psi_1^0 = 0$ and $\psi_2^0 = 0$, discontinuity and transformation of the phase coordinates and Lagrange coefficients λ_i, η_n always occurs according to the corresponding condition (7.9). In connection with this, the left portion of equations (7.8) will be represented as discrepancies written as follows:

$$(R^q_\zeta)^2 = p_\zeta^{(1)},\ (R^v_\zeta)^2 = p_\zeta^{(2)},\ (R^\kappa_m)^2 = p_m^{(3)}. \qquad (7.10)$$

Functions $p_\zeta^{(1)}$, $p_\zeta^{(2)}$ and $p_m^{(3)}$ will be called the boundary problem functionals in the following.

Furthermore, we will assume that conditions (7.9) with known phase coordinates and Lagrange coefficients λ_i to the left of the discontinuity and with fixation of a_χ, t_q or t_v unambiguously determine the phase coordinates and Lagrange coefficients λ_i, η_n to the right of the discontinuity. Then according to the theorem of existence of implicit functions, the ranks of the matrices

$$A_q=\left\| \frac{\partial \Pi_k^q}{\partial y_{iq}^+} \; \frac{\partial \Pi_k^q}{\partial \lambda_{iq}^{(2)}} \; \frac{\partial \Pi_k^q}{\partial \eta_{\chi q}^{(2)}} \right\|, \quad A_v=\left\| \frac{\partial \Pi_l^v}{\partial r_{jv}} \; \frac{\partial \Pi_l^v}{\partial \lambda_{jv}^{(2)}} \right\|$$

should be 2n + (r - p) and 2 m respectively.

Let us now write the first equation of system (7.9) in variations

$$A_q \begin{Vmatrix} \delta y_q^+ \\ \delta\lambda_q^{(2)} \\ \delta\eta_q^{(2)} \end{Vmatrix} = -B_q \begin{Vmatrix} \delta y_q^- \\ \delta\lambda_q^{(1)} \end{Vmatrix} - C_q \delta a - \dot{\Pi}^q \delta t_q,$$

where
$$B_q=\left\| \frac{\partial \Pi_\kappa^q}{\partial y_{iq}^-} \; \frac{\partial \Pi_\kappa^q}{\partial \lambda_{iq}^{(1)}} \right\|; \quad C_q=\left\| \frac{\partial \Pi_\kappa^q}{\partial a_\chi} \right\|;$$

$\begin{Vmatrix} \delta y_q \\ \delta\lambda_q \end{Vmatrix}$; $\begin{Vmatrix} \delta a \\ \delta\eta_q \end{Vmatrix}$ are the matrix columns of variations of δy_{iq}, $\delta\lambda_{iq}$, δa_χ and $\delta\eta_{\chi q}$ respectively;

$\dot{\Pi}^q$ is the matrix column of the full derivatives of functions Π_k^q with respect to time.

From this, keeping in mind that matrix A_q is nonsingular and that

$$\delta t_q = -\frac{1}{\dot{\psi}_1^0}\left(F_q^{(1)}\delta y_q^- + F_q^{(2)}\delta a\right),$$

occurs, we produce

$$\begin{Vmatrix} \delta y_q^+ \\ \delta\lambda_q^{(2)} \\ \delta\eta_q^{(2)} \end{Vmatrix} = D_q \begin{Vmatrix} \delta y^- \\ \delta\lambda_q^{(1)} \end{Vmatrix} + E_q \delta a, \qquad (7.11)$$

where

$$D_q = A_q^{-1}\left(-B_q + \frac{1}{\dot{\psi}_1^0}\dot{\Pi}^q F_q^{(1)}\right) E_q = A_q^{-1}\left(-C_q + \frac{1}{\dot{\psi}_1^0}\dot{\Pi}^q F_q^{(2)}\right),$$

$$F_q^1 = \left\| \frac{\partial \psi_1^0}{\partial y_{iq}^-} \right\|, \quad F_q^{(2)} = \left\| \frac{\partial \psi_1^0}{\partial a_2} \right\|.$$

Similarly we find

$$\left\| \begin{matrix} \delta x_v \\ \delta \lambda_v^{(3)} \end{matrix} \right\| = D_v \left\| \begin{matrix} \delta y_v \\ \delta \lambda_v^{(2)} \end{matrix} \right\| + F_v \delta a, \tag{7.12}$$

where

$$D_v = A_v^{-1}\left(-B_v + \frac{1}{\dot{\psi}_2^0}\dot{\Pi}^v F_v\right), \quad E_v = -A_v^{-1} C_v,$$

$$B_v = \left\| \frac{\partial \Pi_i^v}{\partial y_{iv}} \; \frac{\partial \Pi_i^v}{\partial \lambda_{rv}^{(2)}} \right\|, \quad C_v = \left\| \frac{\partial \Pi_i^v}{\partial a_2} \right\|, \quad F_v = \left\| \frac{\partial \psi_2^0}{\partial y_{iv}} \right\|.$$

Thus, equations (7.11) and (7.12) allow us to determine the variations of the phase coordinates δy or δx and the variations of Lagrange factors $\delta\lambda$, $\delta\eta$ to the right of discontinuities arising at the moment of satisfaction of the stop functions $\psi_1^0 = 0$ and $\psi_2^0 = 0$, according to the values of variations of phase coordinates and Lagrange coefficients $\delta\lambda$ to the left of the discontinuity and δa.

Equations (7.11) are the altered conditions of discontinuity of variations, while equations (7.12) are the conditions of transformation of the variations.

If we assume that there are (n + r) independent sets δx, δa, $\delta\lambda^{(3)}$, $\delta\eta^{(3)} \mid_{\psi_3}^{0}$ and δy, $\delta\lambda^{(2)}$, $\delta\eta^{(2)} \mid_{\psi_2}^{0}$, then, integrating variational equation system (6.16)-(6.18) conditions (7.11) and (7.12) along C^0 (n + r) times, we can produce (n + r) independent relationships δy_0, δa, $\delta\lambda_0^{(1)}$ and $\delta\eta_0^{(1)}$, $\delta\eta_q^{(2)}$. However, for determination of the (n + r) independent sets δy_0, δa, $\delta\lambda_0^{(1)}$ and $\delta\eta_0^{(1)}$, $\delta\eta_q^{(2)}$, we can, using lemma 7, approach by a different path, integrating the system of conjugate equations (7.2)-(7.4) along C^0 (n + r) times considering relationships (7.5)-(7.7) and (7.11), (7.12).

We can now more precisely formulate the purpose of this problem. It consists in the determination of the deficient (n + r) values of the vector functions

$$\delta y(t_0),\ \delta\lambda(t_0),\ \delta a,\ \delta\eta^{(1)}(t_0),\ \delta\eta^{(2)}(t_\varphi),$$

for which the newly produced solutions

$$y(t)=y^0(t)+b\delta y(t) \text{ и } x(t)=x^0(t)+b\delta x(t),$$
$$a=a^0+b\delta a,\ \eta(t)=\eta^0(t)+b\delta\eta(t),\ \lambda(t)=\lambda^0(t)+b\delta\lambda(t)$$

give the (n + r) functionals $p_\xi^{(1)}$, $p_\zeta^{(2)}$, $p_m^{(3)}$ the minimum value.

To do this, considering variational equations (7.10), which have the form

$$dp_\xi^{(1)}\Big|_{\psi_1^0}=\left(\operatorname{grad} p_\xi^{(1)}\Big|_{\psi_1^0},\ \left\|\begin{matrix}\delta y\\ \delta a\\ \delta\lambda^{(1)}\\ \delta\eta^{(1)}\end{matrix}\right\|\right),$$

$$dp_\zeta^{(2)}\Big|_{\psi_2^0}=\left(\operatorname{grad} p_\zeta^{(2)}\Big|_{\psi_2^0},\ \left\|\begin{matrix}\delta y\\ \delta a\\ \delta\lambda^{(2)}\\ \delta\eta^{(2)}\end{matrix}\right\|\right),$$

$$dp_m^{(3)}\Big|_{\psi_3^0}=\left(\operatorname{grad} p_m^{(3)}\Big|_{\psi_3^0},\ \left\|\begin{matrix}\delta x\\ \delta a\\ \delta\lambda^{(3)}\\ \delta\eta^{(3)}\end{matrix}\right\|\right),$$

and the relationships

$$\operatorname{grad} p_\xi^{(1)}\Big|_{\psi_1^0}=\operatorname{grad} p_\xi^{(1)}\Big|_{t_\varphi^-}-\frac{\dot p_\xi^{(1)}}{\dot\psi_1^0}\operatorname{grad}\psi_1^0\Big|_{t_\varphi^-},$$

$$\operatorname{grad} p_\zeta^{(2)}\Big|_{\psi_2^0}=\operatorname{grad} p_\zeta^{(2)}\Big|_{t_\varphi^-}-\frac{\dot p_\zeta^{(2)}}{\dot\psi_2^0}\operatorname{grad}\psi_2^0\Big|_{t_\varphi^-},$$

$$\operatorname{grad} p_m^{(3)}\Big|_{\psi_3^0}=\operatorname{grad} p_m^{(3)}\Big|_{t_\kappa}-\frac{\dot p_m^{(3)}}{\dot\psi_3^0}\operatorname{grad}\psi_3^0\Big|_{t_\kappa}$$

which convert them to

$$
\left.\begin{aligned}
dp_{\xi}^{(1)}\Big|_{\psi_1^0} &= \left(\operatorname{grad} p_{\xi}^{(1)}\Big|_{t_q^-} - \frac{\dot{p}_{\xi}^{(1)}}{\dot{\psi}_1^0} \operatorname{grad} \psi_1^0\Big|_{t_q^-}, \left\| \begin{matrix} \delta y \\ \delta a \\ \delta \lambda^{(1)} \\ \delta \eta_i^{(1)} \end{matrix} \right\| \right), \\
dp_{\zeta}^{(2)}\Big|_{\psi_2^0} &= \left(\operatorname{grad} p_{\zeta}^{(2)}\Big|_{t_v^-} - \frac{\dot{p}_{\zeta}}{\dot{\psi}_2^0} \operatorname{grad} \psi_2^0\Big|_{t_v^-}, \left\| \begin{matrix} \delta y \\ \delta a \\ \delta \lambda^{(2)} \\ \delta \eta_i^{(2)} \end{matrix} \right\| \right), \\
dp_m^{(3)}\Big|_{\psi_3^0=0} &= \left(\operatorname{grad} p_m^{(3)}\Big|_{t_\kappa} - \frac{\dot{p}_m}{\dot{\psi}_3^0} \operatorname{grad} \psi_3^0\Big|_{t_\kappa}, \left\| \begin{matrix} \delta x \\ \delta a \\ \delta \lambda^{(3)} \\ \delta \eta_i^{(3)} \end{matrix} \right\| \right),
\end{aligned}\right\} \qquad (7.14)
$$

we should take

$$
\left.\begin{aligned}
z_{l\xi}^{(1)}(t_q) &= \left(\frac{\partial p_{\xi}^{(1)}}{\partial y_l} - \frac{\dot{p}_{\xi}^{(1)}}{\dot{\psi}_1^0} \frac{\partial \psi_1^0}{\partial y_l} \right)_{t_q^-}, \\
l_{l\xi}^{(1)}(t_q) &= \left(\frac{\partial p_{\xi}^{(1)}}{\partial a_l} - \frac{\dot{p}_{\xi}}{\dot{\psi}_1^0} \frac{\partial \psi_1^0}{\partial a_l} \right)_{t_q^-}, \\
\nu_{i\xi}^{(1)}(t_q) &= \frac{\partial p_{\xi}}{\partial \lambda_i^{(1)}}\Bigg|_{t_q^-}, \quad \alpha_{l\xi}^{(1)} = \frac{\partial p_{\xi}^{(1)}}{\partial \eta_l^{(1)}}\Bigg|_{t_q^-};
\end{aligned}\right\} \qquad (7.15)
$$

$$
\left.\begin{aligned}
z_{l\zeta}^{(2)}(t_v) &= \left(\frac{\partial p_{\zeta}^{(2)}}{\partial y_l} - \frac{\dot{p}_{\zeta}^{(2)}}{\dot{\psi}_2^0} \frac{\partial \psi_2^0}{\partial y_l} \right)_{t_v^-}, \\
l_{n\zeta}^{(2)}(t_v) &= \left(\frac{\partial p_{\zeta}^{(2)}}{\partial a_n} - \frac{\dot{p}_{\zeta}^{(2)}}{\dot{\psi}_2^0} \frac{\partial \psi_2^0}{\partial a_n} \right)_{t_v^-}, \\
\nu_{i\zeta}^{(2)}(t_v) &= \frac{\partial p_{\zeta}^{(2)}}{\partial \lambda_i^{(2)}}\Bigg|_{t_v^-}, \quad \alpha_{n\zeta}^{(2)} = \frac{\partial p_{\zeta}^{(2)}}{\partial \eta_n^{(2)}}\Bigg|_{t_v^-};
\end{aligned}\right\} \qquad (7.16)
$$

$$
\left.\begin{aligned}
z_{jm}^{(3)}(t_\kappa) &= \left(\frac{\partial p_m^{(3)}}{\partial x_j} - \frac{\dot{p}_m}{\dot{\psi}_3^0} \frac{\partial \psi_3^0}{\partial x_j} \right)_{t_\kappa}, \\
l_{nm}^{(3)}(t_\kappa) &= \left(\frac{\partial p_m^{(3)}}{\partial a_n} - \frac{\dot{p}_m^{(3)}}{\dot{\psi}_3^0} \frac{\partial \psi_3^0}{\partial a_n} \right)_{t_\kappa}, \\
\nu_{jm}^{(3)}(t_\kappa) &= \frac{\partial p_m^{(3)}}{\partial \lambda_j^{(3)}}\Bigg|_{t_\kappa}, \quad \alpha_{nm}^{(3)} = \frac{\partial p_m^{(3)}}{\partial \eta_n^{(3)}}.
\end{aligned}\right\} \qquad (7.17)
$$

Relationships (7.15)-(7.17) are the initial conditions for the system of conjugate equations.

We thus produce

$$
\left.\begin{aligned}
dp_1^{(1)} &= \left(\left\| \begin{matrix} z^{(1)}(t) \\ l^{(1)}(t) \\ \nu^{(1)}(t) \\ \alpha^{(1)} \end{matrix} \right\|_1, \left\| \begin{matrix} \delta y(t) \\ \delta a \\ \delta\lambda^{(1)}(t) \\ \delta\eta^{(1)}(t) \end{matrix} \right\| \right) \Bigg|_{\vartheta_1^0}, \\
dp_2^{(2)} &= \left(\left\| \begin{matrix} z^{(2)}(t) \\ l^{(2)}(t) \\ \nu^{(2)}(t) \\ \alpha^{(2)} \end{matrix} \right\|_2, \left\| \begin{matrix} \delta y(t) \\ \delta a \\ \delta\lambda^{(2)}(t) \\ \delta\eta^{(2)}(t) \end{matrix} \right\| \right) \Bigg|_{\vartheta_2^0},
\end{aligned}\right| \qquad (7.18)
$$

$$
dp_m^{(3)} = \left(\left\| \begin{matrix} z^{(3)}(t) \\ l^{(3)}(t) \\ \nu^{(3)}(t) \\ \alpha^{(3)} \end{matrix} \right\|_m, \left\| \begin{matrix} \delta x(t) \\ \delta a \\ \delta\lambda^{(3)}(t) \\ \delta\eta^{(3)}(t) \end{matrix} \right\| \right) \Bigg|_{\vartheta_3^0}. \qquad (7.19)
$$

Further, adding the left and right portions of equations (7.5)-(7.7) and keeping (7.19) in mind, we find

$$
\begin{aligned}
dp_m^{(3)} = {} & \left(\left\| \begin{matrix} z^{(3)}(t) \\ l^{(3)}(t) \\ \nu^{(3)}(t) \\ \alpha^{(3)} \end{matrix} \right\|_m, \left\| \begin{matrix} \delta x(t) \\ \delta a \\ \delta\lambda^{(3)}(t) \\ \delta\eta^{(3)}(t) \end{matrix} \right\| \right) \Bigg|_{\vartheta_{2+}^0} - \\
& - \left(\left\| \begin{matrix} z^{(2)}(t) \\ l^{(2)}(t) \\ \nu^{(2)}(t) \\ \alpha^{(2)} \end{matrix} \right\|_m, \left\| \begin{matrix} \delta y(t) \\ \delta a \\ \delta\lambda^{(2)}(t) \\ \delta\eta^{(2)}(t) \end{matrix} \right\| \right) \Bigg|_{\vartheta_{2-}^0} + \left(\left\| \begin{matrix} z^{(2)}(t) \\ l^{(2)}(t) \\ \nu^{(2)}(t) \\ \alpha^{(2)} \end{matrix} \right\|_m, \left\| \begin{matrix} \delta y(t) \\ \delta a \\ \delta\lambda^{(2)}(t) \\ \delta\eta^{(2)}(t) \end{matrix} \right\| \right) \Bigg|_{\vartheta_{1+}^0} - \\
& - \left(\left\| \begin{matrix} z^{(1)}(t) \\ l^{(1)}(t) \\ \nu^{(1)}(t) \\ \alpha^{(1)} \end{matrix} \right\|_m, \left\| \begin{matrix} \delta y(t) \\ \delta a \\ \delta\lambda^{(1)}(t) \\ \delta\eta^{(1)}(t) \end{matrix} \right\| \right) \Bigg|_{\vartheta_{1-}^0} + \left(\left\| \begin{matrix} z^{(1)}(t) \\ l^{(1)}(t) \\ \nu^{(1)}(t) \\ \alpha^{(1)} \end{matrix} \right\|_m, \left\| \begin{matrix} \delta y(t) \\ \delta a \\ \delta\lambda^{(1)}(t) \\ \delta\eta^{(1)}(t) \end{matrix} \right\| \right) \Bigg|_{t_0}.
\end{aligned}
$$

From this, considering relationships (7.11) and (7.12) and the equation

$$
\left(\begin{Vmatrix} l^{(3)}(t) \\ \alpha^{(3)} \end{Vmatrix}_m, \begin{Vmatrix} \delta a \\ \delta\eta^{(3)}(t) \end{Vmatrix}\right)\Big|_{\vartheta_{2+}}^{0} - \left(\begin{Vmatrix} l^{(2)}(t) \\ \alpha^{(2)} \end{Vmatrix}_m, \begin{Vmatrix} \delta a \\ \delta\eta^{(2)}(t) \end{Vmatrix}\right)\Big|_{\vartheta_{2-}}^{0} = 0,
$$

$$
\left(\begin{Vmatrix} l^{(2)}(t) \\ \alpha^{(2)} \end{Vmatrix}_m, \begin{Vmatrix} \delta a \\ \delta\eta^{(2)}(t) \end{Vmatrix}\right)\Big|_{\vartheta_{1+}}^{0} - \left(\begin{Vmatrix} l^{(1)}(t) \\ \alpha^{(1)} \end{Vmatrix}_m, \begin{Vmatrix} \delta a \\ \delta\eta^{(1)}(t) \end{Vmatrix}\right)\Big|_{\vartheta_{1-}}^{0} =
$$

$$
= \left(\begin{Vmatrix} l^{(2)}(t) \\ \alpha^{(2)} \end{Vmatrix}_m, \begin{Vmatrix} \delta \tilde{a} \\ \delta\tilde{\eta}^{(2)}(t) \end{Vmatrix}\right)\Big|_{\vartheta_{1+}}^{0},
$$

where

$$
\delta\tilde{a} = \{\delta a_{p+1}, \ldots, \delta a_r\}, \quad \delta\tilde{\eta}^{(2)}(t_\vartheta) = \{\delta\eta_{p+1}^{(2)}, \ldots, \delta\eta_r^{(2)}\},
$$

we produce

$$
d\eta_m^{(3)} = \left(\begin{Vmatrix} z^{(3)}(t) \\ v^{(3)}(t) \end{Vmatrix}_m^{\vartheta_{2+}^{0}} \left\| D_\vartheta \begin{Vmatrix} \delta y \\ \delta\lambda^{(2)} \end{Vmatrix} + E_\vartheta \delta a \right\|^{\vartheta_{2-}^{0}}\right) - \left(\begin{Vmatrix} z^{(2)}(t) \\ v^{(2)}(t) \end{Vmatrix}_m, \right.
$$

$$
\left.\begin{Vmatrix} \delta y(t) \\ \delta\lambda^{(2)}(t) \end{Vmatrix}\right)\Big|^{\vartheta_{2-}^{0}} + \left(\begin{Vmatrix} z^{(2)}(t) \\ v^{(2)}(t) \\ \alpha^{(2)} \end{Vmatrix}_m^{\vartheta_{1-}^{0}} \left\| D_\vartheta \begin{Vmatrix} \delta y \\ \delta\lambda^{(1)} \end{Vmatrix} + E_\vartheta \delta a \right\|^{\vartheta_{1-}^{0}}\right) -
$$

$$
- \left(\begin{Vmatrix} z^{(1)}(t) \\ v^{(1)}(t) \end{Vmatrix}_m, \begin{Vmatrix} \delta y(t) \\ \delta\lambda^{(1)}(t) \end{Vmatrix}\right)\Big|^{\vartheta_{1-}^{0}} + \left(\begin{Vmatrix} l^{(2)}(t) \\ \alpha^{2}(t) \end{Vmatrix}_m, \begin{Vmatrix} \delta\tilde{a} \\ \delta\tilde{\eta}^{(2)}(t) \end{Vmatrix}\right)\Big|^{\vartheta_{1+}^{0}} +
$$

$$
+ \left(\begin{Vmatrix} z^{(1)}(t_0) \\ v^{(1)}(t) \end{Vmatrix}_m, \begin{Vmatrix} \delta y_0(t_0) \\ \delta\lambda_0^{(1)}(t_0) \end{Vmatrix}\right) + \left(\begin{Vmatrix} l^{(1)}(t_0) \\ \alpha^{(1)} \end{Vmatrix}_m, \begin{Vmatrix} \delta a_0 \\ \delta\eta_0^{(1)}(t_0) \end{Vmatrix}\right).
$$

Here $\delta y_0(t_0)$, $\delta\lambda_0^{(1)}(t_0)$, $\delta a_0 = \{\delta a_1, \ldots, \delta a_p\}$, $\delta\eta_0^{(1)}(t_0)$ are the variations of the vector functions deficient at t_0.

Assuming

$$
\left.\begin{aligned}
D_\vartheta' \begin{Vmatrix} z^{(3)} \\ v^{(3)} \end{Vmatrix}_m \Big|^{\vartheta_{2+}^{0}} &= \begin{Vmatrix} z^{(2)} \\ v^{(2)} \end{Vmatrix}_m \Big|^{\vartheta_{2-}^{0}}, \\
D_\vartheta' \begin{Vmatrix} z^{(2)} \\ v^{(2)} \\ \alpha^{(2)} \end{Vmatrix}_m \Big|^{\vartheta_{1+}^{0}} &= \begin{Vmatrix} z^{(1)} \\ v^{(1)} \end{Vmatrix}_m \Big|^{\vartheta_{1-}^{0}},
\end{aligned}\right\} \qquad (7.20)
$$

we produce

$$
dp_m^{(3)} = \left(\begin{Vmatrix} z^{(2)}(t) \\ v^{(3)}(t) \end{Vmatrix}_m^{\vartheta_{2+}^{0}}, E_\vartheta \delta a \Big|^{\vartheta_{2-}^{0}}\right) + \left(\begin{Vmatrix} z^{(2)}(t) \\ v^{(2)}(t) \\ \alpha^{(2)} \end{Vmatrix}_m^{\vartheta_{1+}^{0}}, E_\vartheta \delta a \Big|^{\vartheta_{1-}^{0}}\right) +
$$

$$
+ \left(\begin{Vmatrix} l^{(2)}(t) \\ \alpha^{(2)}(t) \end{Vmatrix}_m, \begin{Vmatrix} \delta a \\ \delta\eta^{(2)}(t) \end{Vmatrix}\right)\Big|^{\vartheta_{1+}^{0}} + \left(\begin{Vmatrix} z^{(1)}(t_0) \\ v^{(1)}(t_0) \end{Vmatrix}_m, \begin{Vmatrix} \delta y_0(t_0) \\ \delta\lambda_0^{(1)}(t_0) \end{Vmatrix}\right) +
$$

$$
+ \left(\begin{Vmatrix} l^{(1)}(t_0) \\ \alpha^{(1)} \end{Vmatrix}_m, \begin{Vmatrix} \delta a_0 \\ \delta\eta_0^{(1)}(t_0) \end{Vmatrix}\right) \qquad (7.21)
$$

$$
(m = k+1, \ldots, n+r).
$$

Similarly in place of (7.18) we will have

$$dp_{\zeta}^{(2)} = \left(\left\| \begin{matrix} z^{(2)}(t) \\ \nu^{(2)}(t) \\ \alpha^{(2)} \end{matrix} \right\|_{\zeta}, E_q \delta a \Big|_{t_1-}^{t_1+} \right) + \left(\left\| \begin{matrix} l^{(2)}(t) \\ \alpha^{(2)}(t) \end{matrix} \right\|_{\zeta}, \left\| \begin{matrix} \delta a \\ \delta \eta^{(2)}(t) \end{matrix} \right\| \right) \Big|^{t_1+} +$$

$$+ \left(\left\| \begin{matrix} z^{(1)}(t_0) \\ \nu^{(1)}(t_0) \end{matrix} \right\|_{\zeta}, \left\| \begin{matrix} \delta y_0(t_0) \\ \delta \lambda_0^{(1)}(t_0) \end{matrix} \right\| \right) + \left(\left\| \begin{matrix} l^{(1)}(t_0) \\ \alpha^{(1)} \end{matrix} \right\|_{\zeta}, \left\| \begin{matrix} \delta a_0 \\ \delta \eta_0(t_0) \end{matrix} \right\| \right) \quad (7.22)$$

$$(\zeta = s+1, \ldots, k),$$

$$dp_{\xi}^{(1)} = \left(\left\| \begin{matrix} z^{(1)}(t_0) \\ \nu^{(1)}(t_0) \end{matrix} \right\|_{\xi}, \left\| \begin{matrix} \delta y_0(t_0) \\ \delta \lambda_0^{(1)}(t_0) \end{matrix} \right\| \right) + \left(\left\| \begin{matrix} l^{(1)}(t_0) \\ \alpha^{(1)} \end{matrix} \right\|_{\xi}, \left\| \begin{matrix} \delta a_0 \\ \delta \eta_0(t_0) \end{matrix} \right\| \right)$$

$$(\xi = 1, \ldots, s). \quad (7.23)$$

Generally speaking, system (7.21)-(7.23), consisting of (n + r) linear equations, has 2(n + r) unknowns $\delta y(t_0)$, $\delta\lambda^{(1)}(t_0)$, δa, $\delta\eta^{(1)}(t_0)$ and $\delta\eta^{(2)}(t_0)$. Using the initial conditions in variations produced from initial conditions (1.14), from the transversality conditions at the initial point (4.26) and (4.27) and from conditios (1.13), (4.39) and (4.40), we can exclude (n + r) independent variables from system (7.21)-(7.23). Therefore in the following we will consider that system (7.21)-(7.23) has (n + r) independent unknowns $\delta y(t_0)$, $\delta\lambda^{(1)}(t_0)$, δa, $\delta\eta^{(1)}(t_0)$ and $\delta\eta^{(2)}(t_q)$. The relationships produced (7.15)-(7.17), (7.20)-(7.23) allow us, with the initial or "zero" solution available, with which the sequence of iterations begins and which satisfies all limitations, to complete the solution of the problem.

Actually, based on the initial solution, we can fine (n + r) values of $dp_{\xi}^{(1)}$, $dp_{\zeta}^{(2)}$ and $dp_m^{(3)}$, setting them equal to

$$dp_{\xi}^{(1)} = -C_{\xi}^{(q)} p_{\xi}^{(1)}, \; dp_{\zeta}^{(2)} = -C_{\zeta}^{v} p_{\zeta}^{(2)}, \; dp_m^{(3)} = -C_m^k p_m^{(3)}, \quad (7.24)$$

where C_{ξ}^{q}, C_{ζ}^{v}, C_m^{k} are arbitrary weight coefficients.

We should then go over to integration along C^0 of the conjugate system of matrix equations corresponding to conjugate system (7.2)-(7.4):

$$
\left.\begin{aligned}
&\dot{Z}_i = -Z_i A_i' - X_i D_i' + N_i K_i' + A_i F_i - B_i G_i \\
&\qquad\qquad (i=1,2,3), \\
&\dot{H}_i = -Z_i B_i' - X_i P_i' + N_i F_i' + A_i M_i' - B_i L_i, \\
&\dot{N}_i = N_i A_i + A_i B_i - B_i C_i, \\
&\quad \dot{A}_i = 0, \\
&-Z_i G_i' - X_i E_i' + N_i G_i' + A_i L_i' - B_i R_i' = 0, \\
&N_i D_i + A_i P_i - B_i E_i = 0,
\end{aligned}\right\} \qquad (7.25)
$$

preliminarily having determined the initial values of the matrices at points $\psi_3^{0'} = 0$, $\psi_2^0 = 0$ and $\psi_1^0 = 0$ according to the following relationships:

$$
Z_i = \begin{Vmatrix} z_{jm}^{(i)} \\ z_{j\zeta}^{(i)} \\ z_{j\xi}^{(i)} \end{Vmatrix}, \quad H_i = \begin{Vmatrix} l_{nm}^{(i)} \\ l_{n\zeta}^{(i)} \\ l_{n\xi}^{(i)} \end{Vmatrix}, \quad N_i = \begin{Vmatrix} \nu_{jm}^{(i)} \\ \nu_{j\zeta}^{(i)} \\ \nu_{j\xi}^{(i)} \end{Vmatrix} \quad (i=1,2,3),
$$

where $z_{jm}^{(i)}$, $l_{nm}^{(i)}$, $\nu_{jm}^{(i)}, \ldots, \nu_{j\xi}^{(i)}$ are the conjugate coefficients calculated according to the relationships for initial conditions (7.15)-(7.17) and for the transformation and discontinuity conditions (7.20) of the conjugate system.

Thus, we produce system (7.21)-(7.23) of (n + r) linear equations with (n + r) independent unknowns

$$
\delta y(t_0),\ \delta\lambda^{(1)}(t_0),\ \delta a_0,\ \delta\eta_0^{(1)},\ (t_0),\ \delta\bar{\eta}^{(2)}(t_\varphi),\ \delta\bar{a},
$$

which allow us to determine the deficient vector functions for the new solution, leading to a lower value of functionals (7.10). This process continues as long as further decreases in the value of the functionals are possible. If the solution of the problem exists, the sequence of values of functionals (7.10) converges to the absolute minimal of the functionals, equal to zero, and thereby to satisfaction of conditions (7.8) and (7.9) corresponding to conditions (1.1)-(1.3), (1.15) and (4.29), (4.32)-(4.37), (4.40)-(4.42).

If determination by a sequence of iterations of the absolute minimal of functionals (7.10) is accompanied by checking of conditions II'_N or II_π and III', the result of the solution of the multipoint boundary problem will be determination of the normal nonsingular extreme E, which satisfies boundary conditions (1.14) and (1.15) and conditions I, II'_N or I', II_π and III'.

Thus, the following theorem obtains, in which the suggested algorithm for solution of the multipoint boundary problem is represented by A1.

Theorem 6. Suppose A1 leads to a solution in which functionals (7.1) become equal to zero. Then algorithm A1 determines normal curve E which is the nonsingular extreme which satisfies the boundary conditions (1.14) and (1.15) and conditions I and II'_N or I' and II_π.

Actually, if the iterational process leads to sequences of values of functionals (7.10) which converge on zero, it allows us to determine the normal curve E which satisfies condition I and equations (1.1)-(1.15). Therefore, by definition E is the extreme which satisfies boundary conditions (1.14) and (1.15), the transversality conditions and the optimality conditions for the parameters. Furthermore, if one normal curve E conditions II'_N or II_π and III' are fulfilled between the corner points, at the corner points to the left and right, by definition (see § 4) extreme E is nonsingular. The theorem is proven.

The solution of the conjugate system of matrix equations (7.25) corresponding to conjugate system (7.2)-(7.4) along any curve is equivalent to solution of the system of matrix equations in variations corresponding to the system of variational equations (6.16)-(6.18). Let us compose the matrix

$$Y_i = \left\| \begin{matrix} Z_i \\ H_i \end{matrix} \right\| (i = 1,2,3).$$

Matrix Y_i (i = 1, 2, 3) produced as a result of solution of the conjugate system of matrix equations (7.25) will be called the Y matrix of conjugate system (7.25) in the interval (ψ_3^0, t_0), if its determinant det Y ≠ 0 for all t ∈ T between the corner points. Any

Y matrix of the conjugate system of matrix equations (7.25) is determined as a result of determination of a certain fundamental system of solutions of the conjugate system (7.2)-(7.4). This leads to

Theorem 7. If the algorithm A1 leading to the solution is such that along the normal nonsingular extreme E we determine the existence of a Y matrix of the conjugate system of matrix equations (7.25), then J(E) reaches its maximum in the sense of theorem 5.

Algorithm A1 leading to the solution gives functional (7.10) its zero value. Then according to theorem 6 we can assume determination of normal curve E which satisfies conditions I, II_N' or I′, II_π and III′. The existence of the Y matrix, the conjugate system of matrix equations (7.25) indicates that the sign of $\det_1$ and $\det_2$ of the system of matrix equations in variations is unchanged and therefore that $\det_1 \neq 0$ and $\det_2 \neq 0$ for a certain fundamental system of solutions of the system of variational equations (6.16)-(6.18). Thus, the conditions of theorem 5 are observed. The theorem is proven.

When algorithm A1 is realized on a high-speed computer, a number of its specific features must be considered. They are related to the selection of the value of the optimal direction of increments

$$\delta y_h(t_0),\ \delta\lambda_h^{(1)}(t_0),\ \delta\eta_h^{(1)}(t_0),\ \delta a_h,\ \delta\tilde{a}_h,\ \delta\tilde{\eta}_h^{(2)}(t_q),$$

represented in the following by vector function s_h, and to the selection of the length of the increment step $\Delta y_h(t_0)$, $\Delta\lambda_h^{(1)}(t_0)$, $\Delta\eta_h^{(1)}(t_0)$, Δa_h, $\Delta\tilde{a}_h$, $\Delta\tilde{\eta}_h^{(2)}(t_q)$, represented in the following by vector function Δ_h, the vector functions deficient at t_0 and t_q for performance of the subsequent integration (h + 1), where h = 0, 1, 2,... etc. is the ordinal number of the iteration. The value of the direction of increment is related to the length of the step of an increment as follows:

$$\Delta_h = k_h s_h,$$

where k_h is an arbitrary weight coefficient for the selection of the length of an increment step or the weight coefficient for the increment step.

By analogy with k_h, the weight coefficients $C_h = \{C^q_{\theta h}, C^v_{\zeta h}, C^k_{mh}\}$, regulating the direction of the increment, will be called the weight coefficients of the direction of the increment.

Suppose

$$p_{\Sigma} = \sum_{1}^{s} p_i^{(1)} + \sum_{s}^{k} p_i^{(2)} + \sum_{k}^{n+r-2} p_m^{(3)}.$$

The iterational process will converge if with the corresponding weight coefficient of the increment step the final direction of the increment s_h is always such that

$$p_{\Sigma(h+1)} < p_{\Sigma h}, \tag{7.26}$$

since the vector functions deficient at t_0 and t_q, giving p_{h+1} its minimum value, have been found.

The computer program for A1 will lead to the minimum expenditure of machine time on the iterational process if in the program for each iteration selection of k_h and C_h are made to produce

$$p_{\Sigma(h+1)} = \inf_{\substack{k_h \in K \\ C_h \in C}} p_{\Sigma(h+1)}, \tag{7.27}$$

where K is the compact set of permissible k_h,
C is the compact set of permissible C_h.

Therefore, in composing a program for A1, a definite algorithm should be drawn up for the selection of k_h and C_h, allowing results close to condition (7.27) to be produced at each iteration.

§ 8. Estimation of Deviation of Functional from Maximum Value Near Extreme

Generally speaking, it is hardly possible to realize solution of a multipoint boundary problem on a computer, achieving $p = (p^{(1)}, p^{(2)}, p^{(3)}) = 0$. On the other hand, the question of whether it is expedient to achieve this precise solution by computer is justified. Therefore, the computational procedure for the multipoint boundary problem according to A1 will be more definite and the criterion for its termination will have a definite sense if we can estimate the deviation of functional J from the maximum value, when the solution is near the extreme, which sufficiently satisfies the boundary conditions. The solution of this problem can be produced on the basis of the formalism of AI already presented.

We have

$$dJ=\left(\operatorname{grad}J\Big|_{t_k}-\frac{\dot{J}}{\dot{\psi}_{3}^{0}}\operatorname{grad}\psi_{3}^{0}\Big|_{t_k},\ \left\|\begin{matrix}\delta x\\ \delta a\end{matrix}\right\|\right).$$

Then, assuming

$$\left.\begin{aligned}
&z_{jJ}^{(3)}(t_k)=\left(\frac{\partial J}{\partial x_j}-\frac{\dot{J}}{\dot{\psi}_{\tau 3}^{0}}\,\frac{\partial\psi_{\tau 3}^{0}}{\partial x_j}\right)_{t_k},\\
&l_{nJ}^{(3)}(t_k)=\left(\frac{\partial J}{\partial a_n}-\frac{\dot{J}}{\dot{\psi}_{\tau 3}^{0}}\,\frac{\partial\psi_{\tau 3}^{0}}{\partial a_n}\right)_{t_k},\\
&v_{jJ}^{(3)}(t_k)=0,\ \alpha_{nJ}^{(3)}=0,
\end{aligned}\right\}\qquad(8.1)$$

we find

$$dJ=\left(\left\|\begin{matrix}z^{(3)}(t)\\ v^{(3)}(t)\end{matrix}\right\|_{J}^{\tau_2+0},\ \left\|E_p\delta a\right\|^{\tau_2-0}\right)+\left(\left\|\begin{matrix}z^{(2)}(t)\\ v^{(2)}(t)\\ \alpha^{(2)}(t)\end{matrix}\right\|_{J}^{\tau_1+0},\ \left\|E_q\delta a\right\|^{\tau_1-0}\right)+$$
$$+\left(\left\|\begin{matrix}l^{(2)}(t)\\ \alpha^{(2)}(t)\end{matrix}\right\|_{J},\ \left\|\begin{matrix}\delta\tilde{a}\\ \delta\tilde{r}_1^{(2)}(t)\end{matrix}\right\|\right)^{\tau_1+0}+\left(\left\|\begin{matrix}z^{(1)}(t_0)\\ v^{(1)}(t_0)\end{matrix}\right\|_{J},\ \left\|\begin{matrix}\delta y_0(t_0)\\ \delta\lambda_0^{(1)}(t_0)\end{matrix}\right\|\right)+$$
$$+\left(\left\|\begin{matrix}l^{(1)}(t_0)\\ \alpha^{(1)}(t_0)\end{matrix}\right\|_{J},\ \left\|\begin{matrix}\delta a\\ \delta r_0^{(1)}(t_0)\end{matrix}\right\|\right).\qquad(8.2)$$

Here the conjugate coefficients $z_J^{(3)}(t)$, $v_J^{(3)}(t),\ldots,$ $z_J^{(1)}(t_0)$, $v_J^{(1)}(t_0)$, $l_J^{(1)}(t_0)$, $a_J^{(1)}$ are determined as a result of solution of the system of conjugate equations (7.2)-(7.4) with the initial conditions (8.1) along curve CJ near extreme E.

Relationships (8.2) allows us to estimate the deviation of functional J from the maximum value with slight deviation $y(t_0)$, $\lambda^{(1)}(t_0)$, $r^{(2)}(t_q)$, $r^{(1)}(t_0)$, a from their values corresponding to the extreme passing in the neighborhood of the fixed boundary conditions. It should be kept in mind here that if integration of system (6.46)-(6.48) is performed along C^J, then dJ > 0, if along E, then dJ > 0.

Equation (8.2) and system (7.21)-(7.23) can be represented as

$$dJ = \Omega_J \left\| \begin{matrix} \delta a \\ \delta y_0(t_0) \\ \delta\lambda_0^{(1)}(t_0) \\ \delta\eta \end{matrix} \right\|,$$

$$dp = \Omega_p \left\| \begin{matrix} \delta a \\ \delta y_0(t_0) \\ \delta\lambda_0^{(1)}(t_0) \\ \delta\eta \end{matrix} \right\|,$$

where Ω_J is a matrix row of 1 × (n + 4) order, equal to

$$\Omega_J = \left\| \left\| \begin{matrix} z^{(3)} \\ v^{(3)} \end{matrix} \right\|_J' E_v \Big|_{\tau_2+}^{0} + \left\| \begin{matrix} z^{(2)} \\ v^{(2)} \\ \alpha^{(2)} \end{matrix} \right\|_J' E_q \Big|_{\tau_1+}^{0} + \left\| l^{(2)} \right\|_J' + \right.$$
$$\left. + \left\| l_0^{(1)} \right\|_J' \;\vdots\; z_{0J}^{(1)} \;\vdots\; v_{0J}^{(1)} \;\vdots\; \alpha_J \right\|;$$

Ω_p is a matrix of (n + r) × (n + r) order, equal to

$$\Omega_p = \left\| \begin{matrix} \left\| \begin{matrix} z^{(3)} \\ v^{(3)} \end{matrix} \right\|_m' E_v \Big|_{\tau_2+}^{0} + \left\| \begin{matrix} z^{(2)} \\ v^{(2)} \\ \alpha^{(2)} \end{matrix} \right\|_m' E_q \Big|_{\tau_1-}^{0} + \left\| l^{(2)} \right\|_m' + \\ + \left\| l_0^{(1)} \right\|_m' \;\vdots\; \left\| z_0^{(1)} \right\|_m' \;\vdots\; \left\| v_0^{(1)} \right\|_m' \;\vdots\; \left\| \alpha^{(1)} \right\|_m' \\ \left\| \begin{matrix} z^{(2)} \\ v^{(2)} \\ \alpha^{(2)} \end{matrix} \right\|_\xi' E_q \Big|_{\tau_1+}^{0} + \left\| l^{(2)} \right\|_\xi' + \left\| l_0^{(1)} \right\|_\xi' \;\vdots\; \left\| z_0^{(1)} \right\|_\xi' \;\vdots\; \left\| v_0^{(1)} \right\|_\xi' \;\vdots\; \left\| \alpha^{(1)} \right\|_\xi' \\ \left\| l_0^{(1)} \right\|_\xi' \qquad \left\| z_0^{(1)} \right\|_\xi' \;\; \left\| v_0^{(1)} \right\|_\xi' \;\; \left\| \alpha \right\|_\xi' \end{matrix} \right\|;$$

dp is a matrix column of (n + r) × 1 order, equal to

$$dp = \left\| \begin{matrix} dp^{(3)} \\ dp^{(2)} \\ dp^{(1)} \end{matrix} \right\|.$$

From this we find

$$dJ = \Omega_J \Omega_p^{-1} dp = \Omega_J \Omega_p^{-1} \begin{Vmatrix} dp^{(3)} \\ dp^{(2)} \\ dp^{(1)} \end{Vmatrix}. \tag{8.3}$$

This relationship makes it possible to determine the deviation of a functional of J from the maximum value with slight deviations from the final conditions, from the transversality conditions and from the conditions of optimization of the parameters produced in any, jth iteration.

Thus, after any jth iteration and supplementary solution of system (7.2)-(7.4) along C^j with initial conditions (8.1), it is possible to use equation (8.2) or (8.3) to estimate the deviation of functional J from the maximum value either with slight changes in $y(t_0)$, $\lambda^{(1)}(t_0)$, $\eta^{(1)}(t_0)$, $\eta^{(2)}(t_q)$ and a on their values at the extreme or with any changes in final conditions, transversality conditions and optimization conditions of the parameters along C^j respectively.

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SYMBOLS USED

a_0 - thrust vector, component elements of which are thrust to weight ratio of stages;

b_{0i} - vector of specific mid-ship section, component elements of which are specific mid-ship sections of load-bearing elements of stages;

D^p - design parameter vector;

D - vector function of actual parameter;

d - vector of choking parameters of flow rate of working medium per second;

$e_0, e_k, e_q, e_\mu, e_1^0, \ldots, e_{10}$ - vectors of constant Lagrange factors;

f_{sf} - safety factor;

g_0 - gravitational acceleration at the surface of the earth;

G_0 - launch (initial) weight of flight vehicle;

G_k - vector of "dry" weight of booster or vector of dry weight. component elements of which are "dry" weights of stages of flight vehicle;

G_0 - vector of initial weight of booster or vector of initial weight, component elements of which are initial weights of stages of flight vehicle;

G_{0J} - vector of payload of booster or payload vector, component elements of which are payloads of stages of flight vehicle;

G_{wh} - vector of design working medium reserve, component elements of which are design working medium reserves of flight vehicle stages;

G_{wh}^e - vector of operational working medium reserve, component elements of which are operatioal reserves of working medium for stages;

G_{pl} - payload of flight vehicle;

H - flight altitude;

I - criterion of effectiveness;

J_0 - specific thrust vector corresponding to P_{max}, component elements of which are specific thrusts of stage power plants corrected to fixed conditions;

k - imperical coefficients vector;

L - flight range;

N^p - design load vector, component elements of which are design modes of load-bearing elements of flight vehicle;

N^e - operational load vector, component elements of which are operational loads of load-bearing elements of flight vehicle;

N - actual load vector, component elements of which are actual loads on load-bearing elements of flight vehicle;

p - relative thrust;

P_Σ - summary functional;

P_j - jth functional;

P_{max} - maximum thrust vector, component elements of which are maximum thrusts of power plants of flight vehicle stages, corrected to corresponding conditions;

Q - aerodynamic drag related to G_0;

Q_τ - vector required specific heat flux;

Q_τ^p - vector of specific heat flux, component elements of which are specific heat fluxes to characteristic points on flight vehicle surface;

$Q_{\tau k}^{(i)}$ - vector of actual specific heat fluxes to structural elements of stages;

r - vector of parameters of direct regulation of motor thrust;

R - structural element vector;

R_3 - radius of the earth;

s - length of flight trajectory;

T^p - vector of design characteristics of thermodynamic power plant parameters, component elements of which are thermodynamic parameters influencing weight and dimensions of fuel tanks, turbine-pump unit and other power plant units;

t - time of motion;

T_w - vector of actual temperature of structure elements;

V - flight vehicle velocity;

y(t), z(t) - vector function of conjugate factors;

Y - aerodynamic lift related to G_0;

w_π, w_α, w_δ, w_τ - vectors of arbitrary parameters;

a - angle of attack;

a_τ - heat transfer coefficient;

Γ - "geometric vector," component elements of which are parameters characterizing dimensions and form of flight vehicle, power

plant and

δ - strength v
characteri
elements;

$\bar{\delta}$ - relative s

θ - trajectory

λ(t), η(t) - vect

μ^* - relative w

μ_{rs} - relative f

μ_k - vector of
weight vec
weights of

μ_{wh} - vector of
elements o
flight veh

μ_0 - vector of
relative in

$\mu_k^{(i)}$ - vector of
weight vect

$\mu_{pl} = \frac{G_{pl}}{G_0}$ - relat

ν_u, ν - vector fun

Π - plan paramet

σ^p - vector of de
design stres

w - angle betwee

The letters

To save space, all

The quantities in t
difference between
the vector multipli
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Partial deri

The scalar o
are represented by
$f_1(x, a)$.

plant and individual units, their relative dimensions;

δ - strength vector, component elements of which are parameters characterizing strength properties of load-bearing structural elements;

$\bar{\delta}$ - relative strength vector;

θ - trajectory angle;

$\lambda(t)$, $\eta(t)$ - vector function of variable Lagrange factors;

μ^* - relative weight of flight vehicle;

μ_{rs} - relative fuel consumption per second;

μ_k - vector of relative "dry" weight of booster or relative dry weight vector, component elements of which are relative "dry" weights of stages of flight vehicle;

μ_{wh} - vector of relative design working fluid reserve, component elements of which are relative reserves of working fluid of flight vehicle stages;

μ_0 - vector of relative initial weight of booster for vector of relative initial weight;

$\mu_k^{(i)}$ - vector of relative final weight of booster or relative final weight vector;

$\mu_{pl} = \frac{G_{pl}}{G_0}$ - relative payload of flight vehicle;

ν_u, ν - vector functions of conditional controls;

Π - plan parameter vector;

σ^p - vector of design stresses, component elements of which are design stresses of load-bearing elements of flight vehicle;

w - angle between velocity vector and vector of force of gravity.

The letters without the * represent vectors: Π, a_0, x, u, etc. To save space, all vectors are vector-rows: $x = (x_1^*, x_2^*, \ldots, x_n^*)$. The quantities in the parentheses are the components of vector x. The difference between the row and column form is not considered. Usually the vector multiplied by the matrix to the left is a vector row, while a vector multiplied by the matrix to the right is a vector column.

Partial derivatives marked with the * are matrices:

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial u}.$$

The scalar or vector functions of scalar or vector variables are represented by placing the argument in parentheses: x(t), f(x, u), $f_1(x, a)$.

The arguments t_- and t_+ represent the left and right limiting values of the function of t: $x(t_-) = \lim\limits_{\substack{\varepsilon\to 0\\ \varepsilon<0}} x(t+\varepsilon)$ and $x(t_+) = \lim\limits_{\substack{\varepsilon\to 0\\ \varepsilon>0}} x(t+\varepsilon)$.

The symbol $\sup\limits_{u}$ is the supermum (least upperbound) of all u.

The symbol $\inf\limits_{u}$ is the infimum (greatest lowerbound) of all u.

The symbols sign is the sign operator: $\text{sign } a = \begin{cases} 1 & \text{where } a > 0 \\ 1 & \text{where } a < 0 \\ 0 & \text{where } a = 0 \end{cases}$

Appendix to Chapter 1

a - fraction of weight of structural elements (see indices) of $G^p_{\tau\Sigma}$;
a_0 - thrust to weight ratio;
b_0 - specific mid-ship section;
C_{kp} - critical cavitation coefficient (speed cavitation coefficient);
D - mid-ship section diameter; diameter of other elements (see indices);
E_t - modulus of elasticity at fixed temperature;
F - area (see indices);
f_a - relative area of output cross section of nozzle;
f_{sf} - safety factor;
$G_{\tau\Sigma}$ - total fuel reserve;
$G^p_{\tau\Sigma}$ - design summary fuel reserve;
G - weight of structural elements, portion of fuel (see indices);
$G_{\tau s}$ - fuel flow per second;
G'_{wh} - weight of tanks of main components and related design elements;
h_{bt} - relative height of tank bottom;
h^T - distance from liquid surface to design cross section;
h_x - distance from liquid surface to turbine pump;
k_m - weight ratio of main components in engine combustion chamber;
k_{gg} - weight ratio of main components in gas generator combustion chamber;
k - adiabatic indicator;
k^{ind} - statistical coefficients;

k_c - coefficient considering difference in midship section area from summary area of output cross section of nozzle;

l - length (see appendices);

l_b - relative tank length;

M_a - bending moment in design cross section from normal aerodynamic force component acting on lateral surface of rocket between section 0-0 and design section;

M_G - bending moment in design section from force of weight of portion of rocket between 0-0 section and design section;

M_{bnd} - actual bending moment;

N - actual axial force;

n_{x1} - axial load (coupled coordinate system);

n_{y1} - transverse load (coupled coordinate system);

n - rotating speed of turbine-pump unit, rpm;

P^n_{max} - maximum thrust in a vaccuum;

P^h - relative maximum thrust;

P^{π}_{sp} - specific thrust in a vaccuum;

p_k - pressure in combustion chamber;

p_a - pressure at nozzle output cross section;

p - pressure (see indices);

p_H - pressure at output of turbine-pump units;

p_s - saturated vapor pressure;

p_H - atmospheric pressure at altitude H;

p^p - design value of summary gauge pressure in design cross section of tank;

q - relative flow strength of combustion chamber;

$$\bar{Q} = \frac{\rho V^2}{2} C_x;$$

$$\bar{Y} = \frac{\rho V^2}{2} C_y;$$

R_H, r_H - external and internal radii of blow tank;

R - gas constant;

R_0 - axial component of aerodynamic forces acting on lateral surface of rocket between section 0-0 and design section (see indices);

S - midship section;

T_{wr}, T_{wo} - temperature of fuel, oxidizer tank shells at design cross section;

T - temperature (see indices);
V_6 - tank volume;
β_{th} - force of chamber pressure;
γ_{pp}, γ_{THA}, γ_{cc}, γ_c - specific weights of motor unit, turbine-pump unit, combustion chambers, nozzle;
γ_M - specific gravity of material of structural elements (see indices);
γ - specific gravity (see indices);
δ_g, δ_0 - thickness of fuel, oxidizer tank shells;
δ - thickness of structural elements (see indices);
Δp - hydraulic losses (see indices);
n_2 - stability reserve for tank;
n_1 - strength reserve;
μ_k - relative final weight;
μ - relative weight (see indices);
π_{ej} - degree of increase of pressure in ejector;
σ_{bt} - temporary drag at fixed temperature;
σ_{kp} - critical stress;
σ_m - meridianal stress;
σ_τ - tangential stress;
τ - time spent by fuel in chamber;
ϕ_p - coefficient of completeness of pressure in combustion chamber of engine not considering cooling;
ϕ_c - nozzle coefficient;
ϕ_{11} - loss coefficient to cooling;
χ_a - loss coefficient to dissipation of velocity at output of nozzle;

Indices

a - nozzle output cross section;
ap - equipment;
pp - control apparatus;
b - tank;
bt - blow tank;
BC - section at input pump;
g - fuel;
gg - gas generator;
np - nose po
pp - motor;
kp - critica
cc - combust
H - blow;
uc - unconsi
0 - oxidize
pl - payload
p - design;
c - nozzle;
Γ - fuel;
fg - gas gen
 - fuel se
 - tail se
b - beams;
ej - ejector

Chapter II

a_{0i} - thrust
b_{0i} - specific

$e_1^{(0)}$, $e_1^{(i)}$, $e_3^{(i)}$, e_2

$\tilde{e}_\mu^{(H)}$, $\tilde{e}_\mu$, e_v, ...
G_0 - launch (
G_{pl} - payload
H - flight a
$\tilde{H}$ - flight a
I - criterio
J_{0i} - specific
L - flight r
$\tilde{L}$ - flight r
$P_{max\ i}$ - maximu
p - relative
p_j - jth func
p_Σ - summary
$S^{(i)}$ - area of
t - flight t
V - flight v

np - nose portion;
pp - motor;
kp - critical cross section of nozzle;
cc - combustion chamber;
H - blow;
ue - unconsidered elements;
0 - oxidizer
pl - payload;
p - design;
c - nozzle;
Γ - fuel;
fg - gas generator fuel;
 - fuel section;
 - tail section;
b - beams;
ej - ejector;

Chapter II

a_{0i} - thrust to weight ratio of ith stage;

b_{0i} - specific midsection of ith stage;

$e_1^{(0)}, e_1^{(i)}, e_3^{(i)}, e_2, e_4, e_{0j}^{(i)}, e_{v}, e_0, e_{k0}, e_{\mu}^{(i)}, \tilde{e}_{k0},$

$\tilde{e}_{\mu}^{(H)}, \tilde{e}_{\nu}, e_V, \ldots, \tilde{e}_t$ - constant Lagrange factors;

G_0 - launch (initial) weight of flight vehicle;

G_{pl} - payload (load) on flight vehicle;

H - flight altitud

$\tilde{H}$ - flight altitude of booster with independent maneuver;

I - criterion of effectiveness;

J_{0i} - specific thrust of power plant at ith stage at $P_{max\ i}$;

L - flight range;

$\tilde{L}$ - flight range of booster with independent maneuver;

$P_{max\ i}$ - maximum thrust of power plant of ith stage;

p - relative thrust;

P_j - jth functional;

P_{Σ} - summary functional;

$S^{(i)}$ - area of characteristic surface of ith stage;

t - flight time;

V - flight velocity;

$\tilde{V}$ - flight velocity of booster with independent maneuver;

w_{1i}, w_{2i}, $\tilde{w}_i$ - arbitrary parameters;

$y_i(t)$, $z_1(t)$, $\tilde{y}_i(t)$, $\tilde{z}_1(t)$ - conjugate coefficients;

α - angle of attack;

μ - instantaneous relative weight of flight vehicle;

θ - trajectory angle;

$\tilde{\theta}$ - trajectory angle with independent maneuver of booster;

$\lambda_j(t)$, $\bar{\lambda}_j(t)$, $\eta(t)$, $\bar{\eta}(t)$ - temporary Lagrange factors;

$\mu_{pl} = \frac{G_{pl}}{G_0}$ - relative payload of flight vehicle;

G_{0II} - total weight of second stage;

$\mu_{0I} = \frac{G_{0II}}{G_0}$ - relative payload of booster;

μ_{kJ} - relative final weight of Jth stage;

μ_{kI} - relative weight of booster after separation of second stage;

$\mu_k^{(m)}$ - relative weight of booster at end of independent maneuver;

$u_\alpha^{(I)}$, $v_p^{(I)}$, $\tilde{v}_\alpha^{(I)}$, $\tilde{v}_p^{(I)}$ - arbitrary control functions.

Chapter III

a_0i - thrust to weight ratio of ith stage;

$e_1^{(0)}$, $e_1^{(I)}$, $e_2^{(I)}$, $e_{0J}^{(I)}$, e_{M}, e_v, $e_{\kappa 0}$, e_p, $e_{\Delta 1}$,

$e_{\Delta 2}$, e_s, $e_{\bar{s}}$ - constant Lagrange factors;

M_{pl} - mass of payload of spacecraft;

MO - initial mass of spacecraft;

M_{0i} - initial mass of ith stage;

N_{sp} - specific power of ERM;

P - thrust of power plant;

p - relative thrust of power plant;

p_Σ - summary functional;

p_j - jth functional;

v_x, v_y, v_z - component like velocities of spacecraft on x, y, z axis of geocentric equatorial coordinate system;

V_x, V_y, V_z - component velocities of spacecraft on X, Y, Z axes of heliocentric ecliptic coordinate system;

$\bar{v}_x, \bar{v}_y, \bar{v}_z$ - component velocities of spacecraft on $\bar{x}, \bar{y}, \bar{z}$ axes of planetocentric equatorial system;

$V_x^{\oplus}, V_y^{\oplus}, V_z^{\oplus}, X^{\oplus}, Y^{\oplus}, Z^{\oplus}$ - component velocities and coordinates of earth in heliocentric ecliptic system;

$V_x^{pl}, V_y^{pl}, V_z^{pl}, X^{pl}, Y^{pl}, Z^{pl}$ - component velocities and coordinates of target planet in heliocentric ecliptic system;

W_{0i}^{e} - exhaust velocity of working medium products of ERM, ith stage;

x, y, z - coordinates of spacecraft in geocentric equatorial system;

$\underline{X}, \underline{Y}, \underline{Z}$ - coordinates of spacecraft in heliocentric ecliptic system;

$\bar{x}, \bar{y}, \bar{z}$ - coordinates of spacecraft in planetocentric equatorial system;

α, β, γ - directing cosines of gravity vector;

a, p, e, ω, Ω, i - parameters of planet orbit (index ⊕ relates to earth, index pl relates to target planet);

ε $\bar{\varepsilon}$ - inclination of planes of equators of earth and target planet to ecliptic;

$$c^2 k_{\oplus} p, \quad \bar{c}^2 = k_{pl} p;$$

$\mu_m, \Lambda_m, \bar{\lambda}_m, \eta_j, \eta_{\bar{j}}$ - variable Lagrange factors;

μ - relative mass of spacecraft;

μ_{kJ} - relative final mass of ith stage;

v_m, v_m - conjugate coefficients;

Appendix

$\{a_i\}$ - set with generating element a_i;

= - equal;

≠ - not equal;

≡ - identical to;

≤ (≥) - less than (greater than) or equal to;

< (>) - less than (greater than);

∪ - union

∩ - intersection;

⊂ - is a subset of

∈ - is an element of

× - product

$[t', t'']$ - closed interval $t' \leq t \leq t''$;

(t', t'') - open interval $t' < t < t''$;

$(t', t'']$ - semiopen interval $t' < t \leq t''$.

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13. ABSTRACT
This book discusses algorithmic methods of planning of flight vehicles using electronic digital computers. One such method of optimal planning is developed and investigated, including a presentation of the variational problem (strict mathematical definition of the required conditions of optimization) considering analysis of the flight vehicle as a structure which receives various loads in flight, and as an object of control, plus an algorithm for its solution with a mathematical foundation for the algorithm itself. A multistage flight vehicle is used as an example to show the peculiarities of this algorithm when independent maneuver of the vehicle is possible after separation of initial stages or when the external and internal problems are analyzed together during interplanetary flights. In the algorithm, the criterion for improvement of the vehicle depends on the flight time, kinematic parameters of the vehicle at the end of the flight, its launch weight and the payload which it transports. Combining the theory of the variational problem and of the computer algorithm for its solution provides us with a logically complete algorithm for optimal planning of a vehicle. This book is designed for scientific workers and engineers in the rocket and aviation industries. Orig. art. has: 66 figures, 4 tables. [AM0046187]

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14.
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Hard
Wear
Seal
Ther
Dril
Flui
High

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14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Fluorinated Organic Compound						
Hardness						
Wear Resistant Material						
Seal						
Thermal Conductivity						
Drilling Machine						
Fluid Pump						
High Pressure Pump						

END

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