AD 720311

NRL Report 7213

Thermal Blooming of Laser Beams in Gases

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February 11, 1971

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37

CONTENTS

	Abstract	ii
	Problem Status	ii
	Authorization	ii
I.	STATEMENT AND FORMULATION OF THE PROBLEM	1
II.	SUMMARY AND DISCUSSION	4
	Background Survey	4
	Summary of Results	4
III.	DETERMINATION OF EQUATIONS FOR THE LIGHT RAYS AND THE INTENSITY DISTRIBUTION ALONG THE BEAM	5
	The Light Rays	5
	Intensity Profiles Downbeam	9
IV.	DETERMINATION OF DENSITY CHANGES	16
	The Linearized Hydrodynamic Equations and Initial Conditions	16
	Solution by Transforms	17
V.	TIME SCALES	21
	Convection	21
	Wind	22
VI.	TRACING AND LIGHT RAYS	23
	Qualitative Considerations	23
	Numerical Results for a Parabolic Power Distribution	25
	Very Smooth Monotonically Decreasing Power Distributions; the Gaussian Distribution	29
	Nonmonotonic Power Distributions	32
	A Variational Problem; Minimizing the Blooming	32
	ACKNOWLEDGMENTS	33
	REFERENCES	33

ABSTRACT

The trajectories of light rays and the intensity pattern of a laser beam as a function of time and of distance down the beam are determined. It is assumed in this calculation that (a) the medium is a homogeneous, isotropic, and initially quiescent gas, (b) convection, viscosity, and thermal conduction may be ignored at early times, (c) changes in total beam power as a function of distance downbeam may be ignored, (d) a specific model of energy deposition is valid, (e) the medium may be described in equations of hydrodynamics and thermodynamics and obeys the Lorentz-Lorenz Law, and (f) geometrical optics applies to the problem. It is shown that this model can be solved exactly; long- and short-time behavior of the solutions is discussed, and the times for the onset of convection are estimated.

The phenomenon of laser defocusing is shown to change rapidly with time; a definition of thermal blooming is given, and it is shown that the region of blooming moves up the beam toward the face of the laser. The intensity pattern at a fixed point in space is shown to change its profile, going over to a bright narrow annular ring whose radius increases with the passage of time.

Parameter combinations required for studies of various aspects of the blooming phenomenon are pointed out as the mathematical development progresses.

PROBLEM STATUS

This is a final report on one aspect of laser propogation studies; work on other aspects of the problem continues.

AUTHORIZATION

NRL Problem R05-31.303
Project ORD-0832-129/173-1/U1754 No. 2

Manuscript submitted October 12, 1970.

THERMAL BLOOMING OF LASER BEAMS IN GASES

I. INTRODUCTION; STATEMENT AND FORMULATION OF THE PROBLEM

The problem of the extent to which an intense laser beam passing through a gaseous medium is "defocused" by its heating effect upon the gas is discussed in this report, which covers only the classical aspects of this propagation problem.

As is well known, an initially parallel laser beam will propagate in vacuum and will retain its parallel character except for diffraction effects imposed upon it by the finite aperture of the source and the nonzero wavelength of the radiation. In a medium, however, such a beam is partially absorbed, causing that part of the medium through which it traverses to be heated; the index of refraction along the beam path is thereby reduced and the light rays of the beam are deflected into regions of higher index of refraction. The beam diameter thus increases as a function of distance from the source along the beam path. The beam is said to be "thermally defocused;" this phenomenon is also succinctly referred to as "thermal blooming."

We assume that the aperture from which the beam emanates is circular and that the power density (per unit area) is circularly symmetric. We let the axis of symmetry be the z axis. Then a quantitative measure of the thermal blooming of the beam will be the distance a light ray is from the z axis, at a distance z from the aperture, as compared to its initial distance r_0 from the axis. Furthermore, since the thermal blooming phenomenon is a dynamic one, we seek to calculate the trajectory of a given light ray as a function of time; hence our objective is the determination of the equations for all the light rays in the form $r = r(r_0; z, t)$ or the form $z = z(r_0; r, t)$ where r is the radial distance from the z axis and t is the time. Also, we seek to determine the power density distribution on a plane perpendicular to the axis of symmetry at any point down the beam as a function of time.

Because the problem is intrinsically very difficult, we are forced to introduce some simplifying assumptions, which are discussed below, together with our expectations of what the real situation would be if these restrictions were relaxed.

1. We assume that the gas is initially homogeneous, isotropic, and quiescent.

In real gases there are currents, density fluctuations, and temperature gradients, of course. In typical laboratory experiments these can be minimized by enclosing the beam path. Under ordinary conditions in the laboratory, however, these phenomena are quite pronounced. Their combined effect will be to lessen the blooming phenomenon. In the open atmosphere, winds and turbulence are uncontrollable phenomena which will indeed play an important role in determining whether or not blooming, among other things, occurs. Time scales for blooming will be determined under the idealized assumptions in this paper, and we shall estimate the times at which other gaseous effects become important.

2. We disregard convection, viscosity, and thermal conduction in the gas.

For gaseous media, neglecting viscosity for slow motions is not at all objectionable. Convection is quite another matter; by ignoring it we are in effect "turning off" the earth's gravitational field, which has important consequences. Therefore our solution of the problem here can only be approximate and must be limited to those times below which convection effects are very small. When convection does set in, a "steady state" for the "laser beam/medium" system is established. Our theory will not describe the system in the steady state but the transient effects only. We shall determine an estimate for the time required for convective effects to become pronounced.

Whether or not thermal conductivity may be ignored will depend upon the kind of gaseous medium used, the temperature gradients induced, and the time scales involved. The effect of a large thermal conductivity is to hasten the onset of a steady-state condition if it acts alone; in concert with convective effects, it could tend to prolong the time until the onset of the steady state. In any event, our expectation is that thermal conductivity in the gaseous case may be ignored for a wide variety of interesting physical systems.

3. We ignore the variation of the intensity of the initial laser beam down the beam axis due to absorption.

This assumption is equivalent to saying that, prior to any serious blooming, the beam is heating the gas uniformly in z. The purpose, and the effect, of this idealization becomes clear in the detailed calculation; it renders the problem tractable by causing the thermodynamic and hydrodynamic properties of the gas to be independent of z. The assumption is representative of the real situation provided one does not consider regions that are too far from the laser face; indeed this assumption sets the limit, in terms of the ***absorption length***, of applicability of the theory to the regions downbeam. In our calculations we shall ignore this limitation on distance, with the consequence that the heating effects at large distance, and subsequently the thermal blooming effect, will be overemphasized when compared to the real situation. This in no way invalidates the results for the regions where the approximation is good. (In applying the results of the theory to any given experimental situation, one must be aware, however, that this assumption is related to assumption 7 of geometrical optics, given below. We will devote a paragraph to this point at the end of this section.)

4. Model for the deposition of energy into the gas.

We replace the heating effect of the laser beam on the medium by a heat source which heats the gas locally in the same way the laser beam does at the time when the beam is turned on (t = 0). The effect of this assumption is to convert the problem to that of calculating the light ray trajectories in an atmosphere that is being heated up by an external fixed heat source. When the light rays begin to deviate ***significantly*** from their original (t = 0) trajectories, i.e., when blooming begins to set in seriously, it would seem as though our model of energy deposition is no longer realistic. However, we shall see later that this is not as serious an objection as might be thought, and our model will allow us to determine the times and locations where blooming becomes important to a good degree of accuracy.

5. Model for the medium.

We assume that the gas motions are described by the equations of hydrodynamics and thermodynamics, that conservation of energy holds, and that the index of refraction of air for the wavelengths at which the laser operates is related to the density by the Clausius-Mosotti (Lorentz-Lorenz) Law. Finally, the atmospheric motions will be studied for "short" times only, where "short" times will have to be determined by the characteristics of the medium and the rates of energy deposition.

6. Model for the distribution of energy in the initial laser beam.

We assume that the source of the laser radiation lies in the x-y plane and possesses rotational symmetry about the z axis. If r_0 be the distance of a point from the origin of coordinates, the intensity of the radiation is $I(r_0)$. It will be seen that no further specification of the power distribution in the source is necessary to solve the problem; in different words, the equations for the light rays and intensity profiles downbeam will be determined in terms of $I(r_0)$. For specific quantitative numerical results, we shall have to select a suitable form for the function $I(r_0)$, of course.

7. We work within the framework of geometrical optics.

By this assumption, we assume that any portion of the beam may be described locally by a light ray or set of light rays traversing the medium. Within this framework, we are thus ignoring the coherence aspect of the laser. If the fields become too strong (i.e., at too high a power density) nonlinear effects may become important and our analysis will be inappropriate. For low powers, this will not be a serious limitation. We also will lose fine interference details in the intensity profiles. For beams of moderate size this will not be serious either.

With the above assumptions, the problem of blooming will be seen to be capable of solution for a limited time span. We repeat that there are two considerations that limit the times for which the solutions we get are applicable—assumptions 4 and 5 must be met simultaneously. There are also two considerations that limit the distances and laser sizes—namely, assumptions 3 and 7 must both hold simultaneously. In those regions of space where diffraction effects due to finite aperture size and finite wavelengths manifest themselves, the assumptions may be inappropriate because the diffraction spreading may be comparable to or larger than the spreading of the beam due to blooming. The effect of assumption 3 is to limit the applicability of the theory to be developed to a downbeam distance $z_\alpha \leqslant \alpha^{-1}$, where α is the absorption coefficient of the medium. Ignoring thermal blooming, the downbeam distance to the point z_d where the diffraction effects have caused the beam radius to increase by, say, a factor of two, is of the order of $z_d = 2a^2/\lambda$, where a is the initial beam radius and λ is the wavelength. Therefore, our assumptions are applicable only to the cases for which $z < z_d$, or for which $\alpha > \lambda/2a^2$. Now in the laboratory almost any gas may be doped to increase its absorptivity α and render this inequality valid. Indeed this has been done to demonstrate the effect. Therefore there are experimental situations of interest in which all our assumptions are compatible and applicable.

Finally, we remark that any theoretical effort to quantitatively analyze the thermal blooming phenomenon cannot ignore the fact that much of the studies made thus far have been made with liquids which are both viscous and thermally conducting. Our assumptions have eliminated these important cases but we prefer to defer the treatment of liquids to a later paper. The physics essential to a quantitative study of the phenomenon is contained here, and the other factors, while important, add complications that do not serve to further illuminate the problem.

In Sect. II the results of the calculations will be summarized and discussed, together with numerical examples.

In Sect. III the basic equations for the light rays will be determined for any medium in which the above pertinent restrictions are imposed, i.e., any medium for which the density, velocity, pressure, and temperature are axially symmetric, independent of z, and for which the Lorentz-Lorenz law holds. The intensity of the defocused beam will also be determined. In Sec. IV, the response of the medium to the heating is determined. In Sec. V, a rough estimate is made of the times required for convection to set in and for wind effects to be severe. In Sec. VI, the results of Secs. III and IV are combined to determine the equations for the light rays, and numerical computations are performed for particular models of a laser.

II. SUMMARY AND DISCUSSION

Background Survey

Observations of the thermal defocusing phenomenon have occurred for several lasers and media combinations. Gordon et al.(1) studied the blooming using a helium-neon laser and several liquids. The phenomenon was put to use to measure the absorption coefficient of several materials by Leite and collaborators (2). Rieckhoff (3) has reported observations of blooming and studied its relation to beam power. Studies of the temperal evolution of the phenomenon in liquids have been published as well (4,5). Since these early observations the phenomenon has come under increasing scrutiny from both the experimental and theoretical aspects. Brueckner and Jorna (6,7) have studied the problem by assuming the beam to be coupled to the fluid by the mechanisms of electrostriction, the high-frequency Kerr effect, and thermal energy deposition. Their principal results are to show that the laser/medium system is unstable in the sense that the coupling mechanisms will tend to amplify small-scale inhomogeneities in the beam and in the density of the fluid. Akhmanov et al (8) presented both a theoretical and experimental study of thermal defocusing in solids, liquids, and gases. Their study included the effects of thermal conductivity, fluid flow, and the steady state at which the system arrives in the case of liquids when convection is present. Blooming in gases has been detected only more recently (9,10). Consequently, there is little, so far, in the literature with which our study here may be compared. From the discussion of Sect. I, it is clear that we are presuming at the outset that the power density is too weak for the coupling mechanisms of Brueckner and Jorna to play a significant role, and the effects of convection and fluid motion as discussed by Akhmanov have not yet appeared.

Summary of Results

Qualitatively, the principle results of the present calculation may be summarized as follows. Both thermal defocusing and thermal focusing may occur in a gaseous medium, depending upon the shape of the power distribution across the face of the laser. For a power distribution that is monotonically decreasing, such as a Gaussian or parabolic density, and for times large compared to the time required for sound to cross the laser diameter, the beam defocuses; the amount of defocusing at a given distance down the beam increases with time. If z_s is the distance from the laser down the beam to the point where the beam, by some measure, has doubled in size, then z_s depends upon time and becomes shorter as time increases. The blooming may be said to be traveling up the beam toward the laser; the velocity dz_s/dt is negative and increasing, i.e., although the point z_s is moving toward the laser, it is decelerating and will never, in principle, reach the laser. In practice, of course, convection will set in to vitiate our hypotheses and the deductions made therefrom, so our description fails to be accurate after a time t_{conv} that is characteristic of the system for convection. While the defocusing is occurring, the intensity distribution down the beam must clearly be changing with time. The results are illustrated in Figs. 4, 6, and 7 for a parabolic distribution; in Fig. 4, the light rays are plotted in terms of a reduced radial coordinate $x = r/a$ and a reduced coordinate ζ that measures the distance down the beam; ζ is related to the actual beam distance by the relation $\zeta = \sqrt{2(n_0 - 1)}\ (z/a)\ \sqrt{t/t_c}$ where t is time that has elapsed since the laser has been switched on, t_c is a time constant composed of constants of the medium and the laser, and n_0 is the ambient index of refraction. With these scaled or reduced coordinates, only one set of trajectories need be drawn; from these all trajectories at all times may be derived. Also, besides a scaling in time, there is complete scaling in beam sizes and lengths, and certain results hold for all lasers and media (provided our model remains valid). For example, we note from Fig. 4 that the size of the beam doubles at a value of $\zeta \approx 2.3$. Similarly, the reduced intensity $I(r)/(W/\pi a^2)$ may be expressed in terms of $x = r/a$, with ζ as a parameter. Figures 4, 5, and 6 show that as ζ grows (or, if z is held fixed, we may say as time increases) the beam spreads out, hollowing in the center, and a bright thin annulus of light develops. Also, if t is held fixed, then z increases as ζ increases, so we may simultaneously assert that at a given time the beam spreads out into an intense annulus with increasing distance downbeam.

This behavior of the blooming with z and t makes it clear that to observe the blooming phenomenon in gases one must, for a given value of z, wait for the passage of time until the beam diameter grows, or go down the beam a considerable distance to detect the beam spreading at very early times. The latter method will probably be more reliable than the former method because convection will set in after a time t_{conv} which will alter the beam pattern completely. It is shown in Sect. V that t_{conv} is completely independent of z (provided z is not too large) so that it is possible in practice to detect the blooming at a time $t < t_{conv}$ by going downbeam far enough, provided this distance does not put the observer in the diffraction-dominated zone. (As noted in Sect. I, this distance depends upon the absorption constant α, the wavelength λ, and aperture radius a; for most laboratory dimensions, therefore, one will not get into the diffraction-dominated zones.)

Equivalent plots of light rays and intensity for a Gaussian intensity profile are shown in Figs. 8 and 10. Clearly, the qualitative features are the same as those of the parabolic case. The result brings up the question of the effects of beam shaping on the thermal blooming. It may be shown that if the laser intensity profile is not monotonic, one may have both focusing and defocusing. It is also shown that the intensity profile that minimizes the blooming is a rectangular function; indeed for that case, there is no blooming in our model. There is no unique intensity profile that maximizes the blooming effect.

The response of the gas to the passage of the laser beam is simple to describe. Partial absorption of the beam causes the air to heat up; at early times, the ensuing air motion gives rise to pressure changes, velocity flows, and density changes. When time exceeds that required for sound to traverse the beam, the gas tends towards an isobaric change. The resultant density changes then becomes identical to those that one would obtain by simple calorimetric arguments applied locally to each point in the medium.

III. DETERMINATION OF EQUATIONS FOR THE LIGHT RAYS AND THE INTENSITY DISTRIBUTION ALONG THE BEAM

The Light Rays

Let **R** be the position vector from the origin of coordinates to a point P on a given light trajectory; the origin is placed at the center of the aperture of the laser beam and the z axis is perpendicular to that opening. The light path will be parametrized by the distance s along that path from the z = 0 plane. In general, $\mathbf{R} = \mathbf{R}(s, t)$, i.e., **R** will depend both upon s and upon the time. The beam will be assumed to be switched on at t = 0. Cylindrical coordinates are appropriate to the problem as it has been formulated (see Fig. 1). Hence

$$\mathbf{R}(s,t) = \mathbf{r}(s,t) + z(s,t)\hat{\mathbf{k}} . \tag{1}$$

The equation for the light ray (11) is

$$\frac{d}{ds}\left(n \frac{d\mathbf{R}}{ds}\right) = \nabla n \tag{2}$$

where $n = n(\mathbf{R},t)$ is the index of refraction. Because of the symmetry of the problem it is clear that if $\hat{\mathbf{r}}$ be the unit vector parallel to **r**, then

$$\left.\begin{aligned}\frac{d\hat{\mathbf{r}}(s)}{ds} &= 0 \\ \text{and} \qquad\qquad & \\ n(\mathbf{R},t) &= n(r,z,t)\end{aligned}\right\} \tag{3}$$

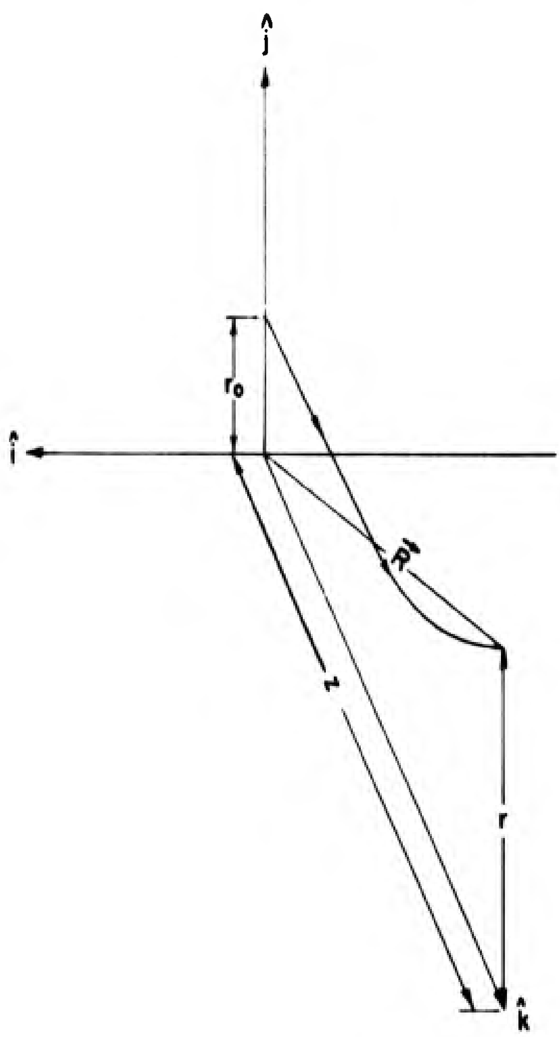

Fig. 1 - Cylindrical coordinate system centered at the aperture of the laser beam, with the z axis perpendicular to the aperture

Using Eqs. (1) and (3), the vector Eq. (2) may be rewritten as the pair of equations

$$\frac{d}{ds}\left(n\frac{dr}{ds}\right) = \frac{\partial n}{\partial r}, \tag{4a}$$

and

$$\frac{d}{ds}\left(n\frac{dz}{ds}\right) = \frac{\partial n}{\partial z}. \tag{4b}$$

These equations may be integrated once by multiplying both sides of Eq. (4a) by 2ndr/ds, and both sides of Eq. (4b) by 2ndz/ds, noting that the result may be expressed more succinctly as

$$\frac{d}{ds}\left(n\frac{dr}{ds}\right)^2 = \frac{\partial n^2}{\partial r}\frac{dr}{ds}, \tag{5a}$$

and

$$\frac{d}{ds}\left(n\frac{dz}{ds}\right)^2 = \frac{\partial n^2}{\partial z}\frac{dz}{ds}. \tag{5b}$$

Each of these equations possess an integral, namely,

$$\frac{dr}{ds} = \pm\frac{1}{n(r,z,t)}\sqrt{\int_0^s ds'\,\frac{\partial n^2}{\partial r}\frac{dr}{ds'}}, \tag{6a}$$

and

$$\frac{dz}{ds} = \sqrt{\frac{n^2(r_0,0,t)}{n^2(r,z,t)} + \frac{1}{n^2(r,z,t)}\int_0^s ds'\,\frac{\partial n^2}{\partial z}\frac{dz}{ds'}} \tag{6b}$$

where we have incorporated the initial conditions on the trajectories that

$$\left.\frac{dr}{ds}\right|_{s=0} = 0, \tag{7a}$$

and

$$\left.\frac{dz}{ds}\right|_{s=0} = 1. \tag{7b}$$

Geometrically, the conditions given by Eqs. (7a) and (7b) state that the beam is leaving the laser perpendicularly to the aperture.

The integrals in Eq. (6a) and (6b) cannot be evaluated unless we know the dependence of r and z on s, i.e., unless we know the trajectory. But this, of course, is what we are attempting here to determine. In Eq. (6), the light rays are parametrized by s; it will prove more convenient to eliminate the parameter s entirely. We assume the relations $z = z(r_0;s,t)$ and $r = r(r_0;s,t)$ to be solved for s as a function of z; then the light ray may be written as $r = r(r_0;z,t)$ (with only a slight abuse of the functional notation) with the boundary condition that $r(r_0;0,t) = r_0$. Then with ds (dz/ds) = dz, the two equations may be combined to give

$$\frac{dr}{dz} = \pm\left[\frac{\displaystyle\int_0^z dz'\,\frac{\partial n^2[r(z'),z',t]}{\partial r}\frac{dr(z')}{dz'}}{\displaystyle n^2(r_0,0,t) + \int_0^z dz'\,\frac{\partial n^2[r(z'),z',t]}{\partial z'}}\right]^{1/2} \tag{8}$$

Equation (8) is our version of Eq. (2) for the trajectories, where the cylindrical symmetry has been explicitly included. Clearly it is a complicated integro differential equation and is exact. When the assumptions of our model are included, Eq. (8) will simplify considerably.

The next problem is to relate the index of refraction to the density of the medium; the Lorentz-Lorenz Law does this:

$$\frac{n^2 - 1}{n^2 + 2} = \frac{A\rho}{3}$$

Where A is a constant for all practical purposes. Because $n^2 - 1 \sim 0$, $A\rho$ will be a small quantity and the Lorentz-Lorenz Law may be taken as

$$n^2 = 1 + A\rho . \tag{9}$$

Inserting Eq. (9) into Eq. (8), we get

$$\frac{dr}{dz} = \pm \left[\frac{A \int_0^z dz' \, \frac{\partial\rho[r(z'),z',t]}{\partial r} \frac{dr(z')}{dz'}}{1 + A\rho(r_0,0,t) + A \int_0^z dz' \, \frac{\partial\rho[r(z'),z',t]}{\partial z'}} \right]^{1/2} \tag{10}$$

In any experiment where the change in density is due to absorption of energy from the beam, it is more frequently the case that the beam is studied over those distances for which the beam intensity is diminished *by absorption* by only very small amounts. Hence the quantity $\partial\rho/\partial z$ will be small and, in the denominator, $A\rho + A\int_0^z dz'(\partial\rho/\partial z')$ will be small compared to unity and may therefore be neglected. Next we note that

$$\int_0^z dz' \left[\frac{\partial\rho(r,z',t)}{\partial r} \frac{dr}{dz'}\right] = \int_0^z dz' \left[\frac{d\rho(r,z',t)}{dz'} - \frac{\partial\rho(r,z',t)}{\partial z'}\right]$$

$$= \rho(r,z,t) - \rho(r_0,0,t) - \int_0^z dz' \, \frac{\partial\rho[r(z'),z',t]}{\partial z'}$$

Therefore, the differential equation for the light rays becomes

$$\frac{dr}{dz} = \pm \sqrt{A\left[\rho(r,z,t) - \rho(r_0,0,t) - \int_0^z dz' \, \frac{\partial\rho(r,z',t)}{\partial z'}\right]} \tag{11}$$

In the problem considered in this paper, we regard the heat source (the laser) as turned off prior to time $t = 0$. The medium is, by assumption 1, homogeneous; hence, we may write $\rho(r,z,t)$ as

$$\rho(r,z,t) = \rho_0 + \rho_1(r,z,t) , \tag{12}$$

where $\rho_1 = 0$ for $t \leq 0$. This equation is exact here and need not be regarded as one restricted to small deviations from the initial ambient density. Then

$$\frac{dr}{dz} = \pm A^{1/2} \sqrt{\rho_1(r,z,t) - \rho_1(r_0,0,t) - \int_0^z dz \frac{\partial \rho_1(r,z',t)}{\partial z'}} . \tag{13}$$

Equation (13) is the one to use when assumption 3 of Sect. I does not apply. When that assumption does apply, Eq. (13) simplifies to

$$\frac{dr}{dz} = \pm A^{1/2} \sqrt{\rho_1(r,t) - \rho_1(r_0,t)} , \tag{14}$$

whose solution is obtained by simple quadratures:

$$z = \pm \frac{1}{\sqrt{A}} \int_{r_0}^{r} \frac{dr'}{\sqrt{\rho_1(r',t) - \rho_1(r_0,t)}} . \tag{15}$$

Thus, in our model, if we know the density variation in both space and time, Eq. (15) will give us the trajectories. For most practical cases the solution will have to be given numerically, of course. We make one final observation before concluding this section. In later sections we shall restrict our considerations to values of ρ_1 very small compared to ρ_0; under these limitations the constant A can be related to the index of refraction very closely, through Eq. (9), by

$$A = \frac{2(n_0 - 1)}{\rho_0}$$

where n_0 is the index appropriate to the density ρ_0. Then,

$$z = \pm \sqrt{\frac{\rho_0}{2(n_0 - 1)}} \int_{r_0}^{r} \frac{dr'}{\sqrt{\rho_1(r',t) - \rho_1(r_0,t)}} . \tag{16}$$

The ambiguity in the signature of the right-hand side is removed by considering that z is always a positive number, and the integrand has meaning only for those values of r' such that $\rho(r',t) - \rho(r_0,t) > 0$. For a given value of r_0, if the r' values are greater than r_0, we choose the positive sign; if the values of r' turn out to be less than r_0, we choose the negative sign.

Intensity Profiles Downbeam

Assuming we know the density distribution ρ as a function of both r and t, we may use Eq. (16) to determine the light rays whose trajectories we write in one of two forms:

$$r = r(r_0;z,t), \text{ with } r(r_0;0,t) = r_0 , \tag{17a}$$

or

$$z = z(r_0;r,t), \text{ with } z(r_0;r_0,t) = 0 . \tag{17b}$$

The solutions may also be regarded, as they were initially in this section, as parameterized by the arc length s from the laser aperture:

$$r = r(r_0;s,t), \tag{18a}$$

and

$$z = z(r_0;s,t) . \tag{18b}$$

The next problem is to determine the intensity distribution across a plane surface perpendicular to the beam axis at any distance z down the beam (i.e., in the positive z direction).*

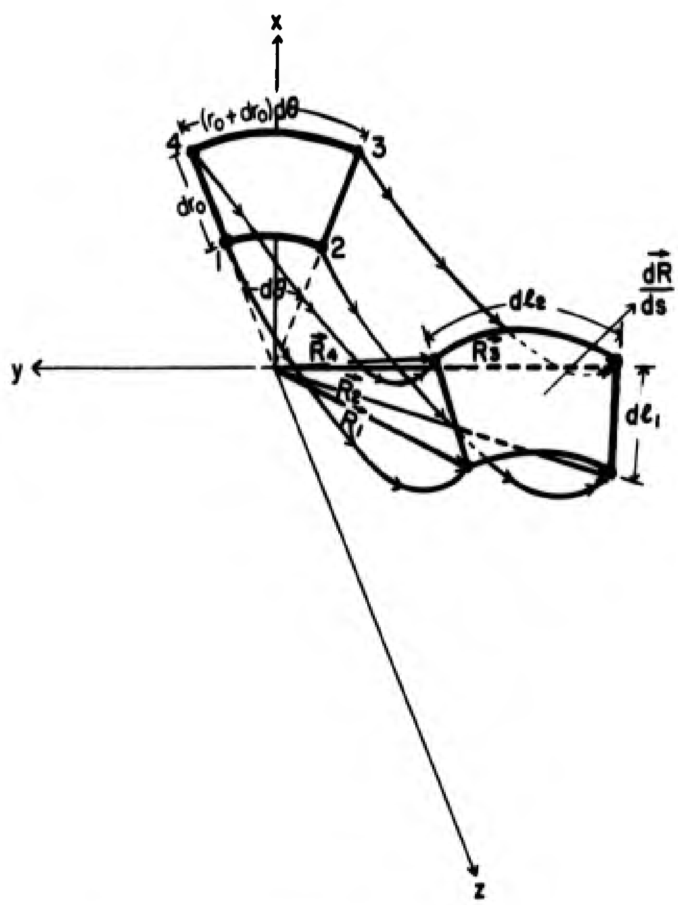

Fig. 2 - Sketch of some geometric quantities which enter into calculations of the total laser beam power

Referring to Fig. 2, we see that the total power which leaves the source located on the x-y plane and passes through the geometric figure with vertices labeled 1, 2, 3, and 4 is $I(r_0)r_0dr_0d\theta$. The light rays that leave each vertex of the plane geometric figure are illustrated. At some distance s_1 along the ray from vertex 1 there is a surface of constant phase which intercepts rays from vertices 2, 3, and 4. The radiation passes through this surface at right angles to it. Let **R** be the vector from the origin to the center of this surface. The power passing through this surface then is I(**R**)dA, where dA is the area of the surface element. Assuming no absorption by the medium (assumption 3 of Sect. I), then

*In this section we will suppress the time variable t to simplify our notation.

$$I(\mathbf{R}) = I(r_0)\,\frac{r_0 dr_0 d\theta}{dA} \tag{19}$$

If $\mathbf{R}_1$ is the vector from the origin to the point where the ray from vertex 1 intercepts the surface, $\mathbf{R}_2$ is the vector from the origin to the point where the ray from vertex 2 intercepts the surface, etc., then

$$dA = d\ell_1 \cdot d\ell_2$$

where

$$d\ell_1 \equiv |\mathbf{R}_4 - \mathbf{R}_1| \tag{20}$$

and

$$d\ell_2 \equiv |\mathbf{R}_2 - \mathbf{R}_1| \ .$$

Let s_4 be the path length from point 4, measured along the light ray from point 4, to the point whose position vector is $\mathbf{R}_4$, that is,

$$\mathbf{R}_4 = \mathbf{R}(r_0 + dr_0 ; s_4, \theta + 1/2 d\theta) \ .$$

Clearly, from the cylindrical symmetry of the problem, $s_3 = s_4$; thus

$$\mathbf{R}_3 = \mathbf{R}(r_0 + dr_0 ; s_4, \theta - 1/2 d\theta).$$

In a similar fashion

$$\mathbf{R}_1 = \mathbf{R}\left(r_0 ; s_1, \theta + \frac{1}{2} d\theta\right) ,$$

$$\mathbf{R}_2 = \mathbf{R}\left(r_0 ; s_1, \theta - \frac{1}{2} d\theta\right) .$$

To the first order in $d\theta$

$$\mathbf{R}_2 - \mathbf{R}_1 = \frac{\partial \mathbf{R}(r_0 ; s_1, \theta)}{\partial \theta}\, d\theta .$$

Now $\mathbf{R}(r_0 ; s) = r(r_0 ; s)\hat{r} + z(r_0 ; s)\hat{k}$, where $r(r_0 ; s)$ and $z(r_0 ; s)$ are independent of θ. This fact is of course the analytical formulation of the cylindrical symmetry requirement. Then

$$\frac{\partial \mathbf{R}}{\partial \theta} = r(r_0 ; s)\,\frac{\partial \hat{r}}{\partial \theta} \ .$$

The quantity $\partial\hat{r}/\partial\theta$ is a unit vector orthogonal to $\hat{r}$; hence we get the geometrically obvious result

$$d\ell_2 = |\mathbf{R}_2 - \mathbf{R}_1| = r(r_0 ; s) d\theta \ . \tag{21}$$

Next, if dr_0 is chosen small enough, then $s_4 - s_1 \equiv ds$ is a quantity of the same order as dr_0. To first order in these quantities,

$$\mathbf{R}_4 - \mathbf{R}_1 = \mathbf{R}(r_0+dr_0;s_1+ds) - \mathbf{R}(r_0;s_1) = \left.\frac{\partial \mathbf{R}}{\partial r_0}\right)_s dr_0 + \left.\frac{\partial \mathbf{R}}{\partial s}\right)_{r_0} ds \ .$$

Now $\mathbf{R}_4$ and $\mathbf{R}_1$ are two vectors which end on a surface of constant phase, that is to say, the optical length $\tau = \int n ds$ from vertex 1 to $\mathbf{R}_1$ is the same as the optical length from vertex 4 to $\mathbf{R}_4$. Hence the difference above may be written as

$$\mathbf{R}_4 - \mathbf{R}_1 = \left[\left.\frac{\partial \mathbf{R}}{\partial r_0}\right)_s + \left.\frac{\partial \mathbf{R}}{\partial s}\right)_{r_0} \left.\frac{ds}{dr_0}\right)_\tau\right] dr_0$$

where the notation $ds/dr_0)_\tau$ means the derivative is to be taken at constant phase. To make the meaning of this clearer, we note that Eqs. 18 may be inverted to read

$$s = s(r,z), \tag{22}$$

and

$$r_0 = r_0(r,z).$$

Thus given a point r,z we can compute the origin of the ray in the laser aperture from Eq. (22) and its path length as well. Now fix a value of r,z and choose another value at r+dr, z+dz such that this new point lies in the surface dA drawn in Fig. 2 (with the same values of θ of course). This leads to a new origin $r_0+\Delta r_0$ and a new length $s+\Delta s$; their ratio $\Delta s/\Delta r_0$, taken to the limit, is our quantity $\partial s/\partial r_0)_\tau$.

It is important to note that the inversion of Eq. (18) may not be unique, but that instead of leading to solutions such as Eq. (22), it may lead to sets of solutions

$$\left.\begin{aligned} s_i &= s_i(r,z) \\ r_{0i} &= r_{0i}(r,z) \end{aligned}\right\}, \ i = 1,2,\ldots,N \ . \tag{23}$$

In this event, several light rays (N of them) will be crossing through the point r,z,θ. When we are speaking of only one light ray, however, we shall ignore the subscript i.

From $\mathbf{R}_4 - \mathbf{R}_1$, and a little algebra, we get

$$d\ell_1 = dr_0 \left\{\left[\left.\frac{\partial r}{\partial r_0}\right)_s + \left.\frac{\partial r}{\partial s}\right)_{r_0} \left.\frac{\partial s}{\partial r_0}\right)_\tau\right]^2 + \left[\left.\frac{\partial z}{\partial r_0}\right)_s + \left.\frac{\partial z}{\partial s}\right)_{r_0} \left.\frac{\partial s}{\partial r_0}\right)_\tau\right]^2\right\}^{1/2} . \tag{24}$$

Therefore, the intensity in the direction of $d\mathbf{R}/ds$ at $\mathbf{R}$ is

$$I(\mathbf{R}) = \frac{I(r_0)r_0}{r} \left\{\left[\left.\frac{\partial r}{\partial r_0}\right)_s + \left.\frac{\partial r}{\partial s}\right)_{r_0} \left.\frac{\partial s}{\partial r_0}\right)_\tau\right]^2 + \left[\left.\frac{\partial z}{\partial r_0}\right)_s + \left.\frac{\partial z}{\partial s}\right)_{r_0} \left.\frac{\partial s}{\partial r_0}\right)_\tau\right]^2\right\}^{-1/2} . \tag{25}$$

Since we assume that the equations for the trajectory are known, the quantities $\partial r/\partial r_0)_s$, $\partial r/\partial s)_{r_0}$, $\partial z/\partial r_0)_s$, and $\partial z/\partial s)_{r_0}$ can be calculated. It remains to express $\partial s/\partial r_0)_\tau$ in terms of known quantities. Let δr and δz be increments in r and z due to changes in r_0 and s that take us from one point on a surface of constant phase to another point on the same surface. Then

$$\delta r = \left.\frac{\partial r}{\partial r_0}\right)_s dr_0 + \left.\frac{\partial r}{\partial s}\right)_{r_0} ds,$$

and

$$\delta z = \left.\frac{\partial z}{\partial r_0}\right)_s dr_0 + \left.\frac{\partial z}{\partial s}\right)_{r_0} ds.$$

The ratio of these last two equations gives

$$\left.\frac{\partial r}{\partial z}\right)_\tau = \frac{\left.\frac{\partial r}{\partial r_0}\right)_s + \left.\frac{\partial r}{\partial s}\right)_{r_0} \left.\frac{\partial s}{\partial r_0}\right)_\tau}{\left.\frac{\partial z}{\partial r_0}\right)_s + \left.\frac{\partial z}{\partial s}\right)_{r_0} \left.\frac{\partial s}{\partial r_0}\right)_\tau} .$$

Solving for the desired derivative,

$$\left.\frac{\partial s}{\partial r_0}\right)_\tau = - \frac{\left.\frac{\partial r}{\partial r_0}\right)_s - \left.\frac{\partial r}{\partial z}\right)_\tau \left.\frac{\partial z}{\partial r_0}\right)_s}{\left.\frac{\partial z}{\partial s}\right)_{r_0} - \left.\frac{\partial r}{\partial z}\right)_\tau \left.\frac{\partial z}{\partial s}\right)_{r_0}} . \qquad (26)$$

Now the differential vector $\delta\mathbf{R} = \delta r\hat{r} + \delta z\hat{k}$, where δr and δz are the increments above, lies on the surface of constant phase and is therefore orthogonal to the vector $d\mathbf{R}/ds$ where $\mathbf{R} = \mathbf{R}(r_0;s)$ is the vector from the origin to the trajectory. Thus,

$$\delta\mathbf{R} \cdot \frac{d\mathbf{R}}{ds} = 0 ,$$

or

$$\left.\frac{\partial r}{\partial s}\right)_{r_0} \delta r + \left.\frac{\partial z}{\partial s}\right)_{r_0} \delta z = 0 .$$

Therefore,

$$\left.\frac{\partial r}{\partial z}\right)_\tau = - \frac{\left.\frac{\partial z}{\partial s}\right)_{r_0}}{\left.\frac{\partial r}{\partial s}\right)_{r_0}} . \qquad (27)$$

Putting Eq. 27 into Eq. 26, we get

$$\left.\frac{\partial s}{\partial r_0}\right)_\tau = - \frac{\left.\frac{\partial r}{\partial r_0}\right)_s \left.\frac{\partial r}{\partial s}\right)_{r_0} + \left.\frac{\partial z}{\partial s}\right)_{r_0} \left.\frac{\partial z}{\partial r_0}\right)_s}{\left[\left.\frac{\partial r}{\partial s}\right)_{r_0}\right]^2 + \left[\left.\frac{\partial z}{\partial s}\right)_{r_0}\right]^2} \qquad (28)$$

Inserting Eq. 28 into 25, one obtains, after much algebra,

$$I(\mathbf{R}) = \frac{I(r_0)r_0}{r} \frac{1}{\left.\frac{\partial r}{\partial r_0}\right)_s \left.\frac{\partial z}{\partial s}\right)_{r_0} - \left.\frac{\partial r}{\partial s}\right)_{r_0} \left.\frac{\partial z}{\partial r_0}\right)_s} . \tag{29}$$

This is the intensity per unit area at $\mathbf{R}$ in the direction of $d\mathbf{R}/ds$ due to a ray which emanated from $\mathbf{R}(r_0) = r_0\hat{r}$. However, the above result is formulated in terms of the parameter s. As in the first part of this section, we prefer to eliminate s in favor of z. If the equation for the light ray is written as $r = r(r_0;z,t)$, we note that the quantity we have designated as dr/dz is in fact

$$\frac{dr}{dz} = \left.\frac{\partial r}{\partial z}\right)_{r_0} = \frac{\left.\frac{\partial r}{\partial s}\right)_{r_0}}{\left.\frac{\partial z}{\partial s}\right)_{r_0}} \tag{30}$$

and Eq. 29 immediately becomes

$$I(\mathbf{R}) = \frac{I(r_0)r_0}{r} \frac{\sqrt{1 + \left(\frac{dr}{dz}\right)^2}}{\left|\left.\frac{\partial r}{\partial r_0}\right)_s - \frac{dr}{dz}\left.\frac{\partial z}{\partial r_0}\right)_s\right|} .$$

Regarding r as a function of r_0 and, alternately, s or z, in the spirit of thermodynamics, we note that

$$\left.\frac{\partial r}{\partial r_0}\right)_s = \left.\frac{\partial r}{\partial r_0}\right)_z + \left.\frac{\partial r}{\partial z}\right)_{r_0} \left.\frac{\partial z}{\partial r_0}\right)_s ,$$

and therefore

$$I(\mathbf{R}) = \frac{I(r_0)r_0}{r} \frac{\sqrt{1 + \left(\frac{dr}{dz}\right)^2}}{\left|\left.\frac{\partial r}{\partial r_0}\right)_z\right|} . \tag{31}$$

As has been noted before, $I(\mathbf{R})$ given above represents the power crossing a unit area that is orthogonal to the light ray through the point whose position vector is $\mathbf{R}$. What can be observed experimentally is the power per unit area on a surface perpendicular to the beam axis (the z axis). Label this intensity by $\hat{I}(R)$ and note it is the above intensity multiplied by the cosine of the angle of incidence θ, which is related to the slope by

$$\cos\theta = \left[\sqrt{1 + \left(\frac{dr}{dz}\right)^2}\right]^{-1} .$$

Therefore

$$\hat{I}(r,z) = \frac{I(r_0)r_0}{r} \frac{1}{\left|\left.\frac{\partial r}{\partial r_0}\right)_z\right|} . \tag{32}$$

What remains to be determined is the quantity $\partial r/\partial r_0)_z$. In fact, Eq. (32) is a final answer, provided that $r = r(r_0,z)$ is in a known functional form. Our solution, Eq. (16), gives it in the opposite manner, namely $z = F(r_0,r)$. Under "normal" circumstances, this would cause no problem, for by the method of implicit functions we would simply write

$$\left.\frac{\partial F}{\partial r_0}\right)_z = \left.\frac{\partial F}{\partial r}\right)_{r_0} \left.\frac{\partial r}{\partial r_0}\right)_z + \left.\frac{\partial F}{\partial r_0}\right)_r = 0$$

which can be solved for the desired derivative $\partial r/\partial r_0)_z$. However a complication arises because of the form of the function $F(r,r_0)$; a formal application of the rules of calculus to obtain the partial derivative of F with respect to r_0 leads immediately to an infinity, so that $\partial r/\partial r_0)_z$ cannot be obtained in this way. However, on geometric grounds, it is easy to conclude that $\partial r/\partial r_0)_z$ must generally exist, for a slight change Δr in r_0 at the face of the laser must lead to a slight change Δr in the distance r of the light ray from the beam axis at a distance z down beam. Thus, $\lim \Delta r/\Delta r_0$ must exist generally. Therefore to circumvent the above difficulty, which stems from the fact that the integrand possesses a singularity (integrable of course) at the lower limit, we resort to a very natural artifice. We define

$$F_a(r_0,r) = \lambda \int_{r_0+a}^{r} \frac{dr'}{\sqrt{\rho_1(r') - \rho_1(r_0)}}, \text{ for } a > 0 .$$

Computing $\partial r/\partial r_0)_z$ with this function will not cause any problems with singularities. We will then take the limit as $a \to 0$. In this manner one gets, after an additional integration by parts, in the limit as a goes to zero,

$$\left.\frac{\partial r}{\partial r_0}\right)_z = + \frac{\rho_1'(r_0)}{\rho_1'(r)} + \rho_1'(r_0)\sqrt{\rho_1(r) - \rho_1(r_0)} \int_{r_0}^{r} dr' \frac{\rho_1''(r')}{[\rho_1'(r')]^2} \frac{1}{\sqrt{\rho_1(r') - \rho_1(r_0)}} . \tag{33}$$

Now we have already noted that for a given point r,z there may be several values of r_0 and s, i.e., several rays may cross through a given point r,z. Denoting the individual values of r_0 by r_{0i} $(i = 1,2,\ldots,N)$, the intensity at r,z will not be given by Eq. (32), but by

$$\hat{I}(r,z) = \sum_{i=1}^{N} \frac{I(r_{0i})r_{0i}}{r} \frac{1}{\left|\left.\frac{\partial r}{\partial r_{0i}}\right)_z\right|} \tag{34}$$

where

$$\left.\frac{\partial r}{\partial r_{0i}}\right)_z = + \frac{\rho_1'(r_{0i})}{\rho_1'(r)} + \rho_1'(r_{0i})\sqrt{\rho_1(r) - \rho_1(r_{0i})} \left[(\pm 1) \int_{r_{0i}}^{r} dr' \frac{\rho_1''(r')}{[\rho_1'(r')]^2} \frac{1}{\sqrt{\rho_1(r') - \rho_1(r_{0i})}}\right] \tag{35}$$

where we have reintroduced the signature ambiguity explicitly, the positive sign being taken for $r > r_0$, the negative sign for $r < r_0$.

In practice these equations will prove to be more difficult to use than their derivations indicate. Essentially the source of the difficulties arises from the fact that our trajectory [Eq. (16)] has the form $z = z(r_0, r; t)$ rather than $r = r(r_0, z; t)$.

IV. DETERMINATION OF DENSITY CHANGES

In this section we will study the impact of the heating effect on the medium due to the turning on of the laser. In the first subsection, we will write down the relevant hydrodynamic equations, their linearized form, and the initial conditions pertinent to our problem. In the second subsection the linearized equations will be solved exactly in terms of quadratures by using Laplace and Fourier transform method. The long- and short-time behavior of the solution for the density, pressure, and velocity are neatly delineated in this form of the solution.

The Linearized Hydrodynamic Equations and Initial Conditions

The hydrodynamic equations, together with the conservation of energy, are

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla p}{\rho} , \tag{36a}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0 , \tag{36b}$$

and

$$\rho \frac{d}{dt}\left(\frac{1}{2}v^2 + \frac{3}{2}\frac{p}{\rho}\right) + \nabla \cdot p\mathbf{v} = \rho \dot{Q} , \tag{36c}$$

where d/dt is the customary substantial derivative of hydrodynamics, and $3p/2\rho$ is the internal energy per gram of the gas, taken here to be an ideal gas. Here we have turned off the earth's gravitational field, i.e., we are ignoring convection, and we are also ignoring thermal conduction. For the small changes in temperature anticipated and for a gaseous medium, thermal conductivity will be small. Convection is another matter for the realistic problem. We shall have to estimate the times at which it becomes important to set limits on the times for which our solution here will be valid. This will be done in Sect. V. The quantity $\dot{Q}$ is the energy deposited per second *per gram* of medium by the laser. It will be related to the laser properties more explicitly later; for the present, we need not define it any further than to say that $\dot{Q}$ is a heat source that turns on at time $t = 0$.

We assume that the initial state (for time $t \leq 0$) is a steady state from which the desired solution deviates only slightly. Thus

$$\mathbf{v}(\mathbf{R},t) = \mathbf{v}_0(\mathbf{R},t) + \mathbf{v}_1(\mathbf{R},t) , \tag{37a}$$

$$\rho(\mathbf{R},t) = \rho_0(\mathbf{R},t) + \rho_1(\mathbf{R},t) , \tag{37b}$$

$$p(\mathbf{R},t) = p_0(\mathbf{R},t) + p_1(\mathbf{R},t) . \tag{37c}$$

The functions $\mathbf{v}_0, \rho_0$, and p_0 are thereby required to satisfy Eqs. (36) with $\dot{Q}$ set equal to zero. ρ_1 and p_1 are taken to be small quantities compared to ρ_0 and p_0. Furthermore, for the problem we are considering,

$v_0 = 0$, and ρ_0 and p_0 are constant in both space and time, while the quantities v_1, ρ_1, and p_1 of the first order vanish for times $t \leqslant 0$. Incorporating these conditions into the hydrodynamic equation and linearizing them in the usual way, we find that the first-order quantities must satisfy

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1 \ , \tag{38a}$$

$$\frac{\partial \rho_1}{\partial t} = -\rho_0 \nabla \cdot \mathbf{v}_1 \ , \tag{38b}$$

and

$$\frac{\partial p_1}{\partial t} - c^2 \frac{\partial \rho_1}{\partial t} = \frac{2}{3} \rho_0 \dot{Q} \ , \tag{38c}$$

where

$$c^2 \equiv \frac{5}{3} \frac{p_0}{\rho_0} \tag{39}$$

and c, here, is the velocity of sound in the gaseous medium.

Solution by Transforms

Let $\mathbf{v}_1(\mathbf{R};s)$, $\rho_1(\mathbf{R};s)$, $p_1(\mathbf{R};s)$ be the Laplace transforms of $\mathbf{v}_1(\mathbf{R},t)$, $\rho_1(\mathbf{R},t)$, and $p_1(\mathbf{R},t)$, respectively, with respect to the time variables. Let $\dot{Q}(\mathbf{R};s)$ be the Laplace transform of $\dot{Q}(\mathbf{R},t)$, again with respect to the time variable. Bearing in mind the initial conditions, the Laplace transforms of Eqs. (38) become

$$s\rho_0 \mathbf{v}_1(\mathbf{R};s) + \nabla p_1(\mathbf{R};s) = 0 \ , \tag{40a}$$

$$s\rho_1(\mathbf{R};s) + \rho_0 \nabla \cdot \mathbf{v}(\mathbf{R};s) = 0 \ , \tag{40b}$$

$$sp_1(\mathbf{R};s) - c^2 s\rho_1(\mathbf{R};s) = \frac{2}{3} \rho_0 \dot{Q}(\mathbf{R};s) \ . \tag{40c}$$

Let $\hat{\mathbf{v}}_1(\mathbf{k};s)$, $\hat{\rho}_1(\mathbf{k};s)$, $\hat{p}_1(\mathbf{k};s)$, $\dot{Q}(\mathbf{k};s)$ be the Fourier transforms with respect to the space variables of the quantities $\mathbf{v}_1(\mathbf{R};s)$, $\rho_1(\mathbf{R};s)$, $p_1(\mathbf{R};s)$, and $\dot{Q}(\mathbf{R};s)$, respectively. We choose the asymmetric form of the Fourier transform:

$$\rho_1(\mathbf{R};s) = \frac{1}{(2\pi)^3} \int d\mathbf{k}\, \hat{\rho}_1(\mathbf{k};s) e^{-i\mathbf{k}\cdot\mathbf{R}} \ , \tag{41}$$

similar equations holding for the other three variables. Inserting Eq. (41) into Eqs. (40), the Fourier transform quantities are required to satisfy

$$\rho_0 s\hat{\mathbf{v}}_1(\mathbf{k};s) - i\mathbf{k}\hat{p}_1(\mathbf{k};s) = 0 \ , \tag{42a}$$

$$s\hat{\rho}_1(\mathbf{k};s) - i\rho_0 \mathbf{k} \cdot \hat{\mathbf{v}}_1(\mathbf{k};s) = 0 \ , \tag{42b}$$

$$s[\hat{p}_1(\mathbf{k};s) - c^2 \hat{\rho}_1(\mathbf{k};s)] = \frac{2}{3} \rho_0 \hat{\dot{Q}}(\mathbf{k};s) \ . \tag{42c}$$

To proceed further, we need an expression for $\dot{Q}(\mathbf{R},t)$. Let $I(r_0)$ be the intensity distribution across the face of the laser. Let a be a characteristic length describing the distribution, and let W be the total power emitted by the laser. Then $I(r_0)$ may be expressed in terms of a dimensionless distribution function f by

$$I(r_0) = \frac{W}{\pi a^2} f\left(\frac{r_0}{a}\right) \tag{43}$$

where f is normalized so as to make

$$\int_0^\infty r_0 dr_0 \int_0^{2\pi} d\theta \; I(r_0) = W \; , \tag{44}$$

i.e.,

$$\int_0^\infty dx \; xf(x) = \frac{1}{2} \; . \tag{45}$$

Then in keeping with assumptions 3 and 4 of Sect. I,

$$\dot{Q}(\mathbf{R},t) = \frac{\alpha W}{\pi a^2 \rho_0} f\left(\frac{r}{a}\right) \Theta(t) \tag{46}$$

where $\Theta(t)$ is the Heaviside step function. Observe that the combination of constants $\alpha W \rho_0^{-1} a^{-2} c^{-2}$ has the dimension of reciprocal time; we thus define the first of a number of characteristic time parameters for this problem:

$$t_c \equiv \frac{3}{2} \frac{\pi \rho_0 a^2 c^2}{\alpha W} \; . \tag{47}$$

From these equations,

$$\dot{Q}(\mathbf{R};s) = \frac{\alpha W}{\rho_0 \pi a^2} f\left(\frac{r}{a}\right) \frac{1}{s} \; . \tag{48}$$

Solving Eqs. (42) for $\hat{\rho}_1(\mathbf{k};s)$ and using Eq. (48), one readily obtains, with $k = |\mathbf{k}|$,

$$\hat{\rho}_1(\mathbf{k};s) = -\frac{\rho_0}{t_c} \frac{k^2 c^2}{s^2(s^2 + k^2 c^2)} \hat{f}(\mathbf{k}) \tag{49}$$

where $\hat{f}(\mathbf{k})$ is the Fourier transform of $f(r/a)$. From the tables of Laplace transforms, Eq. (49) is easily inverted to give

$$\hat{\rho}_1(\mathbf{k};t) = \frac{-\rho_0}{t_c} \left(t - \frac{\sin kct}{kc}\right) \hat{f}(\mathbf{k}) \; . \tag{50}$$

From Eqs. (48) and (41), we get the density function

$$\rho_1(\mathbf{R},t) = -\rho_0 \left[\frac{t}{t_c} f\left(\frac{r}{a}\right) - \frac{1}{t_c} \frac{1}{(2\pi)^3} \int d\mathbf{k} \hat{f}(\mathbf{k}) \frac{\sin kct}{kc} e^{-i\mathbf{k}\cdot\mathbf{R}}\right] \; . \tag{51}$$

Because f(r/a) is known, the Fourier transform $\hat{f}(\mathbf{k})$ is known, and therefore the solution for the density has been reduced to quadratures. Solution (51) is exact for the linearized problem and the model we have proposed. The second term on the right-hand side of Eq. (51) includes the transient effects associated with the propagation of sound in the medium. We shall indicate here that it tends toward zero rapidly when t grows larger than a/c. For, if f(r/a) is a confined distribution in the plane, a two-dimensional wave packet if you like, then $\hat{f}(k)$ is likewise a two-dimensional wave packet in k-space with a typical width of, say, k_0; the wave packets satisfy the uncertainty principle, i.e., ka $\geqslant$ 1. In different words, $\hat{f}(k) \rightarrow 0$ rapidly for $k>a^{-1}$; thus for values of ct $>>$ a, sin kct is oscillating rapidly in the regions where $\hat{f}(\mathbf{k})$ differs significantly from zero. The Riemann-Lebesgue lemma tells us that this term goes to zero as t tends towards infinity; the above arguments show us that the long times involved here are those times for which t $>$ a/c. This result also justifies our statement above that this term is associated with the transient effects involving the propagation of density changes or sound. Thus the density in the long-time limit is

$$\rho_1(\mathbf{R},t) \approx -\rho_0 \frac{t}{t_c} f\left(\frac{r}{a}\right), \text{ for } t >> a/c . \tag{52}$$

However, the small terms that have been dropped are important from a theoretical point of view in that Eq. (52) alone will not solve the equation of continuity and the initial condition simultaneously, which is simply another way of saying that Eq. (52) is not an exact solution.

The short-time limit, under certain circumstances, can become important. Because of the "wave packet" characteristics of $\hat{f}(\mathbf{k})$, for t $<<$ a/c, an expansion of the sine function in the integrand of Eq. (51) will afford a good approximation to the value of the integrand, provided enough terms are taken. We take only the first two terms for purposes of illustration. *We get for the short-time approximation*

$$\rho_1(r,t) \approx +\rho_0 \frac{c^2 t^3}{6t_c} \nabla^2 f\left(\frac{r}{a}\right), \text{ for } t << a/c . \tag{53}$$

This exhibits a considerably different functional dependence upon t. If more accuracy is desired, higher powers of the expansion must be included. If t_c is so short that $\rho_1 \sim \rho_0$ for small times using Eq. (53), then the long-time regime is not reached before the whole linearization scheme becomes invalid.

Another form for ρ_1 may be obtained by using

$$\hat{f}(\mathbf{k}) = (2\pi)^2 \,\delta(z) \int_0^\infty dr' r' f\left(\frac{r'}{a}\right) J_0(kr') .$$

Putting this into Eq. (51), the density may be written as

$$\rho_1(\mathbf{R},t) = \frac{-\rho_0}{t_c} \left\{ t f\left(\frac{r}{a}\right) - \frac{1}{c} \int_0^\infty dr' r' f\left(\frac{r'}{a}\right) \int_0^\infty dk \; J_0(kr') \, J_0(kr) \, \sin kct \right\} . \tag{54}$$

The k integration can be done exactly, but the result is so complicated it will serve no purpose to write it down here. Suffice it to say that the k integration reduces the problem to a quadrature over f, which clearly exhibits the fact that the second term in Eq. (54) is associated with those facets of the changes in the medium that involve a finite time of propagation of the local effects. These integrals in general will be very difficult to perform even for reasonable f(r/a) and, unless one has a combination of laser and medium characteristics that will guarantee us that the long-time limit cannot be reached before the linearization approximation breaks down, there is no point in trying to evaluate the second term.

The pressure as a function of position and time becomes

$$p_1(\mathbf{R},t) = \frac{c\rho_0}{t_c} \int_0^\infty dr'r'f\left(\frac{r'}{a}\right) \int_0^\infty dk\ J_0(kr')\ J_0(kr)\ \sin kct\ . \tag{55}$$

Since we have already shown that this integral is tending toward zero when t begins to exceed a/c, we may state that *the expansion of the air becomes an isobaric process after times of the order of a/c.* For very short times, on the other hand,

$$p_1(k,t) \approx \frac{c^2\rho_0 t}{t_c} \int_0^\infty dr'r'f\left(\frac{r'}{a}\right) \int_0^\infty dkk\ J_0(kr')\ J_0(kr)$$

$$= \frac{c^2\rho_0 t}{t_c} \int_0^\infty dr'r'f\left(\frac{r'}{a}\right) \frac{\delta(r'-r)}{r'}\ ,$$

from which

$$p_1(\mathbf{R},t) = \frac{c^2\rho_0 t}{t_c} f\left(\frac{r}{a}\right)\ ,\ \text{for}\ t \ll a/c\ . \tag{56}$$

The velocity can be reduced to

$$v(\mathbf{R},t) = \frac{-c}{t_c}\hat{r} \int_0^\infty dr'r'f\left(\frac{r'}{a}\right) \int_0^\infty dk\ \frac{dJ_0(kr)\ J_0(kr')}{dr}\ \frac{\sin^2(kct/2)}{1/2\ kc}\ . \tag{57}$$

As $t \gg a/c$, this quantity does not vanish. For small times

$$v(\mathbf{R},t) \approx -\frac{c^2\hat{r}t^2}{t_c} \int_0^\infty dr'r'f\left(\frac{r'}{a}\right) \int_0^\infty dk\ k\ \frac{dJ_0(kr)}{dr}\ J_0(kr')\ .$$

But

$$\int_0^\infty dk\ k\ \frac{dJ_0(kr)}{dr}\ J_0(kr') = \frac{d}{dr}\int_0^\infty dkkJ_0(kr)\ J_0(kr') = \frac{d}{dr}\ \frac{\delta(r-r')}{r'}\ ,$$

so

$$v_1(\mathbf{R},t) \approx -\frac{c^2t^2}{at_c} f'\left(\frac{r}{a}\right)\hat{r},\ \text{for}\ t \ll a/c\ . \tag{58}$$

The results obtained above for ρ_1, p_1, v_1 for short times are also those obtained by a powers series expansion of these quantities in time.

The above results are consistent, in a rather negative fashion, with laboratory experience. For most laboratory experiments, laser beams are of the order of 1 mm in diameter, so that $a/c \sim 10^{-5}$ sec. With a total power of 1 watt and $\alpha \sim 10^{-6}$, then $t_c \sim 4 \times 10^3$ sec. Now assuming that $\rho_1/\rho_0 \approx 0.05$ is the maximum density change for which the linearized hydrodynamic equations are valid, the long-time limit is in the range of 10^{-5} sec $\geqslant t \geqslant 200$ sec. Since normal air circulation in the laboratory will certainly sweep the heated air from the path of the beam in times much shorter than 200 sec, no appreciable density changes occur due to heating under customary laboratory conditions, and hence no blooming will be seen. Thus, to see blooming and density changes, special precautions and special combinations of parameters must be taken, or high-power lasers must be used.

V. TIME SCALES

In the previous section we have solved the problem of an exact determination of the density of the gas as a function of time within the framework of our model. We have seen that two relevant time scales have emerged from the discussion, a time scale t_c characteristic of both the medium and the laser, and an essentially geometric time scale $t_g \equiv a/c$. Although we have not yet given a quantitative description of the blooming phenomenon, the theory of Sect. III shows the connection between the blooming phenomenon and the density changes. But the model on which these calculations are based is not wholly realistic because the effects of wind, convection, and random fluctuations are ignored, and the presence of these phenomena can affect the blooming quite extensively. Hence it is important to get an estimate of the times that are required for these manifestations to render our model inaccurate.

Convection

As the laser heats the gas, causing the gas to expand, these regions of lesser density will experience a bouyancy force that will cause them to rise. The effect of this is to bring into the laser beam unheated gas, which being more like the surroundings will cause the laser beam to bloom less. Clearly, after convection sets in, a steady state for the "beam/medium" system will have set in, with a small amount of thermal defocusing occurring in a noncylindrically symmetric fashion. Because the medium possesses inertia, a certain characteristic time, t_{conv}, will be required to pass before the effects of convection become dominant. Clearly, if the thermal blooming characteristic times t_c (or t_g) are much longer than t_{conv}, then convection will set in, requiring that special precautions be taken to be able to detect the blooming. If $t_c << t_{conv}$, on the other hand, the blooming will take place in a readily observable fashion. (Here we have assumed what will prove to be the usual case, i.e., that t_c, not t_g, governs the blooming.) Now we must estimate t_{conv}.

Consider the volume of air in the immediate vicinity of a point a distance r from the beam axis. Its density is $\rho_0 + \rho_1(r,t)$ where ρ_1, of course, is a negative quantity. If δV be the volume of this element of the medium, it experiences a buoyant force upward whose magnitude according to Archimedes' Principle, is given by $\delta F = -\rho_1 \delta V g$ where g is the acceleration due to gravity. Ignoring volume changes which are second-order effects, we calculate the motion of this element by Newton's second law. If s(t) be its displacement from its initial point at time t, then

$$s = \frac{f(r)g}{t_c} \frac{t^3}{6} .$$

We take the element initially at $r = 0$; for this position, the effect of convection is maximized. Then $f(0) \approx 1$. We shall define t_{conv} as that time by which the heated element will have moved upward by one-tenth the beam diameter, i.e., $s = a/5$. Then

$$t_{conv} = \left(\frac{3at_c}{5g}\right)^{1/3} = \left(\frac{9}{10}\ \frac{\pi a^3 c^2 \rho_0}{aWg}\right)^{1/2} . \qquad (59)$$

In a certain sense t_c as a characteristic time for the laser/medium combination is a misnomer, for the density change will be equal to ρ_0 at time $t = t_c$. Since this will certainly violate the linearization approximation, we should not compare t_{conv} with t_c, but rather with a suitable fraction thereof, say $\bar{t}_c = (1/20)\, t_c$; this means we are limiting density changes to no more than 5% of the ambient density. Then

$$\frac{t_{conv}}{\bar{t}_c} = \frac{20}{a}\left(\frac{4\,\alpha^2 W^2}{15\, g\pi^2 c^4 \rho_0{}^2}\right)^{1/3} . \qquad (60)$$

For a laser for which $\alpha \sim 10^{-6}$ cm, W = 1 watt, and a = 1 mm in air ($\rho_0 \sim 10^{-3}$), we obtain $t_{conv} \approx 10^{-2}\, \bar{t}_c$. Hence for an ordinary laser operating in the laboratory under ordinary conditions, convection sets in prior to any large density changes ($\delta\rho/\rho_0 \leqslant 10^{-3}$) so that the blooming will be difficult to see and a steady-state condition is rapidly established by the convecting air. Furthermore, the density changes are so slight before and after convection has set in, and usual laboratory distances are so small, that the beam passes through the air relatively undisturbed. Since $t_c \approx 4\times 10^3$ sec, then $t_{conv} \approx 10^{-2}$ $(5\times 10^{-2})\times 4\times 10^3 \approx 2$ sec. Air motions in the laboratory will probably mask any convective effects, as these time scales show.

The phrase "blooming will be difficult to see" in the above paragraph needs clarification. It will be seen in the following sections that even $\bar{t}_c$ turns out not to be too good a parameter to characterize the blooming phenomenon. The reason for this is simple–blooming is both a spatial and temporal phenomenon and no single time parameter can fully characterize it. Thus blooming, as we shall see, occurs at any time after the laser has been turned on. For very short times afterward, one has to go downbeam quite a distance before it becomes manifest, assuming it is not masked by diffraction. Under *usual* laboratory operations, this may not be feasible; special conditions such as long beam paths may have to be set up, or the gas may have to be doped to make it more absorptive to reduce these lengths. The latter case tends to accelerate the convective process, so one must therefore be prepared to observe the beam at correspondingly shorter intervals. It is with these precautions in mind that we have used the above-quoted phrase.

Wind

As with convection, mass motion of the air induced by any means will remove the heated air from the beam, causing the blooming phenomenon to be diminished. In the laboratory, ordinary air circulation can be a source of such motion, unless the beam is enclosed. For beams in the open atmosphere, winds and vertical convective currents associated with local meteorological conditions may sustain such mass motions. We term all these motions "wind." Let v_w be the local wind speed perpendicular to the beam axis. We define t_w as that time required for the wind to move a given element of air one-tenth the beam diameter (beam radius = a), a criterion we used for convective times. Then

$$t_w = \frac{a}{5v_w} . \qquad (61)$$

One effect of winds, apart from reducing the amount of blooming, is generally to cause a deflection of the beam into the wind. The amount of deflection will depend upon the wind speed; hence, fluctuations of wind speed along the beam axis will thus cause fluctuations in the amount of deflection, complicating a description of the beam. Hence the description of the blooming phenomenon given in this paper cannot possess validity for times significantly greater than t_w.

A second effect of wind, provided that conditions are right, is to cause turbulence, i.e., fluctuations in the parameters of the medium that are pertinent to the refraction of the beam. When turbulence occurs, it may profoundly affect the beam intensity profile, which in turn will alter the heating of the medium. When such conditions prevail independently of the time at which the laser is turned on, we cannot define any characteristic time before which the theory here can be valid. The density fluctuations are always present and occur at all points on the beam axis. The extent to which they affect the beam will depend upon the power absorbed, the size of the beam, the size of the density fluctuations, and their distribution along the beam axis, among other things. We do not here estimate any of these effects on the blooming of the beam. Therefore, the present theory only can be applied to beams in the open atmosphere when the atmosphere is quiet. But the theory can be used to place upper limits on the blooming phenomenon even when the atmosphere is not quiet.

VI. TRACING THE LIGHT RAYS

Qualitative Considerations

In Sect. III we derived the equation for the light rays, in terms of quadratures, as a function of density, using the approximations appropriate to our model for the system. Adjoining the linearization approximation, in Sect. IV we succeeded in obtaining the density changes in time and space in terms of the properties of the medium and the laser. Using the long-time form, which will describe most systems, the combined results may be written as

$$\zeta = \pm \int_{r_0/a}^{r/a} \frac{dx}{\sqrt{f\left(\frac{r_0}{a}\right) - f(x)}} \tag{62}$$

where

$$\zeta \equiv \sqrt{2(n_0 - 1)} \sqrt{\frac{t}{t_c}} \frac{z}{a} . \tag{63}$$

The virtue of putting the equations in this form lies in the fact that from a plot of ζ vs r/a and r_0/a one can read the coordinates of points of any light ray at *any* time from one fixed set of trajectories. Further, the ζ vs r/a plot will be valid for any laser of any size in any medium provided only that its intensity distribution be given by $f(r/a)$ and that considerations be restricted to the regions where diffraction effects are negligible. Hence a great deal of data can be compressed into one plot. (We are assuming, of course, that a numerical integration will be required here.)

A function $f(x)$ characteristic of many lasers is the Gaussian function $\exp(-x^2)$. However, before going to specific numerical results, a great deal may be said about the blooming process using only Eqs. (62) and (63) and the fact that the distribution is monotonically decreasing toward zero. First, the denominator will be positive only for $x > r_0/a$; hence we choose the positive sign and values of $r/a > r_0/a$. The slope of the curve given by Eq. (62) will always be positive (i.e., $d(r/a)/d\zeta > 0$); hence r will be an increasing function of ζ, as is to be expected on qualitative physical grounds. Further, for a ray which originates at a large distance r_0 from the z axis, $f(r_0/a)$ will be small; $f(x)$ will also be small, so that the integrand of necessity is very large for any given value of r. For the same r, but a smaller r_0, the integrand will not in general be so large. Thus, the rays closer to the beam axis diverge more than those further away, a result that has been obtained before. Thus, qualitatively, the ζ vs r/a plot will look like that of Fig. 3.

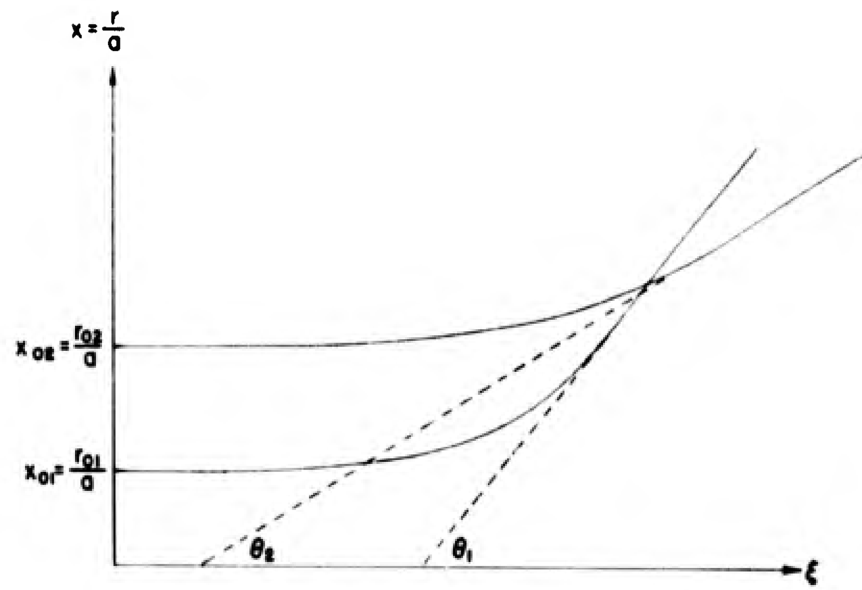

Fig. 3 - Typical light ray trajectories plotted in terms of the reduced coordinates ζ and x

Next, consider a light ray that emanates from a given point r_0/a and on the ζ vs r/a plot. There exists a value ζ_β of ζ where the ray will have deviated away from the ζ axis to a distance of, say, $\beta r_0/a$, where $\beta > 1$. This value ζ_β will correspond at a given time t to $z_\beta(t)$ where

$$z_\beta(t) = \frac{\zeta_\beta}{\sqrt{t}} \sqrt{\frac{a^2 t_c}{2(n_0 - 1)}} \, . \tag{64}$$

Equation (64) shows that the point z_β, at which the radial coordinate r of this light ray is βr_0, moves progressively closer toward the laser as time passes, and it moves with a velocity

$$\dot{z} = -\frac{1}{2} \frac{\zeta_\beta}{t^{3/2}} \sqrt{\frac{a^2 t_c}{2(n_0 - 1)}} = -\frac{z_\beta(t)}{2t} \, . \tag{65}$$

A second fact that can be discerned immediately is that the light ray ultimately becomes a straight line as $z \to \infty$, with a slope given by

$$\tan\theta = \left.\frac{dr}{dz}\right)_{z\to\infty} = \sqrt{2(n_0 - 1)}\, f\left(\frac{r_0}{a}\right) \sqrt{\frac{t}{t_c}} \, . \tag{66}$$

Typical angles are drawn in Fig. 3. Since the maximum value of f is unity, this slope has a maximum value, at a given time, of

$$\left(\left.\frac{dr}{dz}\right)_{z\to\infty}\right)_{max} = \sqrt{2(n_0 - 1)} \sqrt{\frac{t}{t_c}} \, . \tag{67}$$

Clearly, these slopes increase with time, as Eq. (66) indicates. The term "blooming" is apt, indeed.

Further, the inside rays "bloom" at earlier times and shorter distances than do the outer rays. This leads to a "hollowing out" of the beam. Thus, the intensity distribution of light on a surface perpendicular to the beam will begin to show a diminution of intensity in the center, and a peaking in an annular ring (convection being neglected still), and then a further and rapid diminution.

In the region where the blooming has become serious, that is, where the light rays are deviating seriously from their original path, and also further down the beam, the model of energy deposition that we have used breaks down. The light rays, being more dispersed, have ceased to heat these regions. But the blooming is due to the accumulation of small deflections upbeam where the model still holds good. Hence we may conclude that, to a first order, the breakdown of our model for energy deposition is not serious.

Excepting the last paragraph, our comments above have to be modified somewhat if f is not a monotonically decreasing function. Even if f is monotonically decreasing, if we look at the very-short-time behavior, there are differences in details because f gets replaced by $\nabla^2 f$, which is not monotonic any longer. However, the broad picture is still unchanged.

Next we look at $\hat{I}(r,z,t)$. Substituting Eq. (52) into Eq. (35) gives

$$\left.\frac{\partial r}{\partial r_{oi}}\right)_z = + \frac{f'\left(\frac{r_{oi}}{a}\right)}{f'\left(\frac{r}{a}\right)} + f'\left(\frac{r_{oi}}{a}\right)\sqrt{f\left(\frac{r_{oi}}{a}\right) - f\left(\frac{r}{a}\right)}$$

$$\int_{r_{oi}}^{r} dr' \frac{f''\left(\frac{r'}{a}\right)}{\left[f'\left(\frac{r'}{a}\right)\right]^2} \frac{1}{\sqrt{f\left(\frac{r_{oi}}{a}\right) - f\left(\frac{r'}{a}\right)}} \,. \tag{68}$$

This equation shows that the right-hand side of Eq. (34) may be computed once and for all without regard to time, just as the right-hand side of Eq. (62) may be computed. To translate these results into an intensity as a function of r and of t for a fixed value of z, however, becomes a rather more complicated problem. To see where the problem lies, we ask how, when Eqs. (68) and (34) are combined, does time enter the problem of coupling $\hat{I}$, for fixed z, for r and t? If we look at a fixed value of r, we then note that a light ray passing through r,z at time t will not pass through it at a later time, but instead a different light ray emanating from a different r_0 must be found. Thus, for r,z held fixed, r_{oi} will be functions of time. It is clear that the numerical problems grow as N, the number of rays crossing at r,z at time t, increases. For some choices of f(x), N may change with space and time, that is, in some regions of space only two rays will cross through each point for one time interval, and then later become three rays or more.

Numerical Results for a Parabolic Power Distribution

The most desirable choice for f(x) is the Gaussian function, but it does not yield results that are tractable analytically, and numerically the situation becomes complicated so quickly that the details quickly overshadow the general broad results. Thus, for didactic purposes, we select a simpler distribution for which many of the results may be obtained without extensive numerical computation. We take

$$f(x) = \begin{cases} 2(1 - x^2)\ , & x \leq 1 \\ 0\ , & x > 1 \end{cases} \tag{69}$$

where

$$\int_0^\infty dx \;\; xf(x) = \frac{1}{2} .$$

The relation between ζ and r is readily determined by simple integration to be

$$x = \begin{cases} x_0 \cosh(\sqrt{2}\,\zeta) \; , \; x \leq 1 & \text{(70a)} \\ 1 + \sqrt{1 - x_0^2}\left[\sqrt{2}\,\zeta - \ln\left(\frac{1 + \sqrt{1 - x_0^2}}{x_0}\right)\right], \; x > 1 & \text{(70b)} \end{cases}$$

where x and x_0 are the quantities r/a and r_0/a, respectively. These curves, for different choices of x_0, are plotted in Fig. 4. It can be seen that for $x \leq 1$ no light rays cross one another. This is clear from above, since Eq. (70a) shows that the curve that begins at αr_0 is always a multiple of that which begins at r_0, provided that $r < a$. For $x > 1$, only two light rays cross at a given ζ,x as can be seen by plotting x_0 versus x. This is done in Fig. 5. Figure 4 shows that the beam about doubles its size for those combinations of z,t such that $\sqrt{2}\,\zeta$ assumes the value 2.3. *We shall say that the blooming has become serious for these combinations of z and t.* That is, we regard this as a definition of the description "the blooming has become serious."

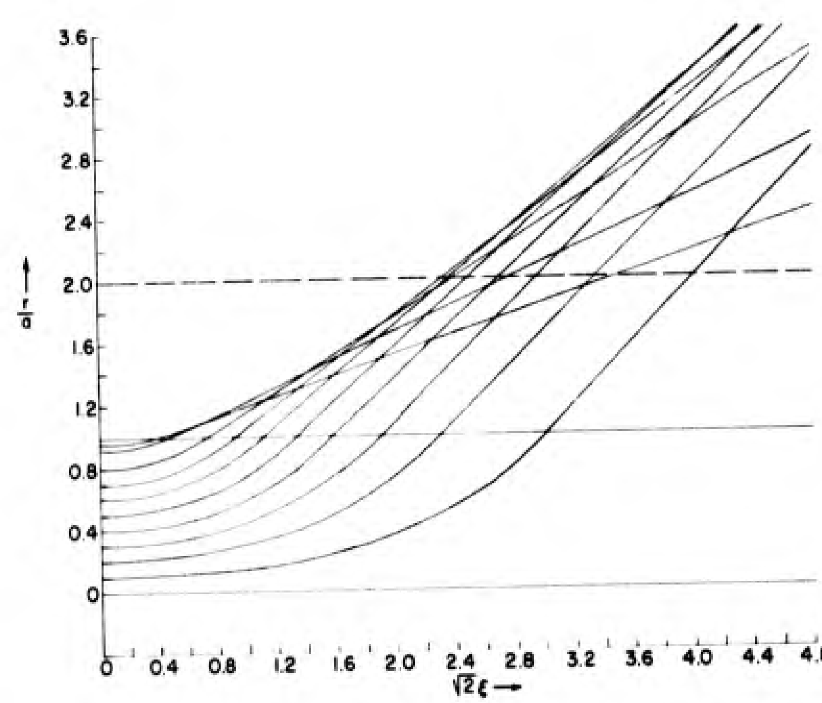

Fig. 4 - Light rays, in reduced coordinates, for the parabolic intensity distribution

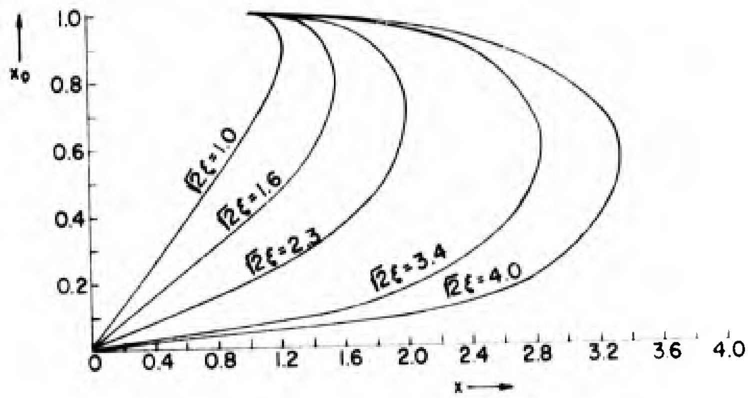

Fig. 5 - Plot of x_0 as a function of the reduced coordinates x and ζ

From Eqs. (70) we can readily compute the derivative, $\partial r/\partial r_0)_\zeta = \partial x/\partial x_0)_\zeta$ and thus obtain the intensity. One gets, after a bit of calculus,

$$\frac{\hat{I}(r,z,t)}{\frac{W}{\pi a^2}} = \begin{cases} \dfrac{2(\cosh^2\sqrt{2}\zeta - x^2)}{\cosh^4\sqrt{2}\zeta}, & r \leqslant a \\ \displaystyle\sum_{i=1}^{2} \frac{2(1-x_{0i}^2)^2 x_{0i}^2}{x|x_{0i}^2(1-x)+1-x_{0i}^2|}, & r > a \end{cases} \tag{71}$$

where, for a fixed value of ζ, that is, for a fixed distance z down the beam and at a given time t, x_{01} and x_{02} are the two values of x_0 which, when inserted into Eq. (70) give the same value for x.

From Eq. (71) we see that, at $t = 0, \zeta = 0$ and the intensity is $I(r,z,0) = W/\pi a^2\ f(r/a)$ for all values of z–i.e., the beam is unaffected at the instant the laser is turned on. Also, for $z = 0, \hat{I} = \hat{I}(r,0,t) = W/\pi a^2\ f(r/a)$, i.e., the power distribution across the face of the laser is unchanged in time. This must be so of course as it is essentially our boundary condition (see Sect. III.). The reduced intensity $\hat{I}/W/\pi a^2$ is plotted against r/a in Figs. 6 and 7 for various choices of the parameter ζ, i.e., for various choices of combinations of z t. If z is held fixed, then as ζ increases, the different curves of intensity distribution show the change of intensity on a fixed plane as time passes. The beam clearly is hollowing out into a ring of intense radiation whose diameter is growing with time as $t^{1/2}$.

The region of serious blooming occurs at those combinations of z,t such that $\sqrt{2}\,\zeta = 2.3$; i.e.,

$$z = \frac{2.3a}{2\sqrt{n_0-1}}\sqrt{\frac{t_c}{t}}\,.$$

Therefore

$$\frac{dz}{dt} = -\frac{2.3a}{4\sqrt{n_0-1}}\sqrt{\frac{t_c}{t}}\,\frac{1}{t} = -\frac{z}{2t}\,. \tag{72}$$

The speed with which the region of serious blooming approaches the laser is $-dz/dt$.

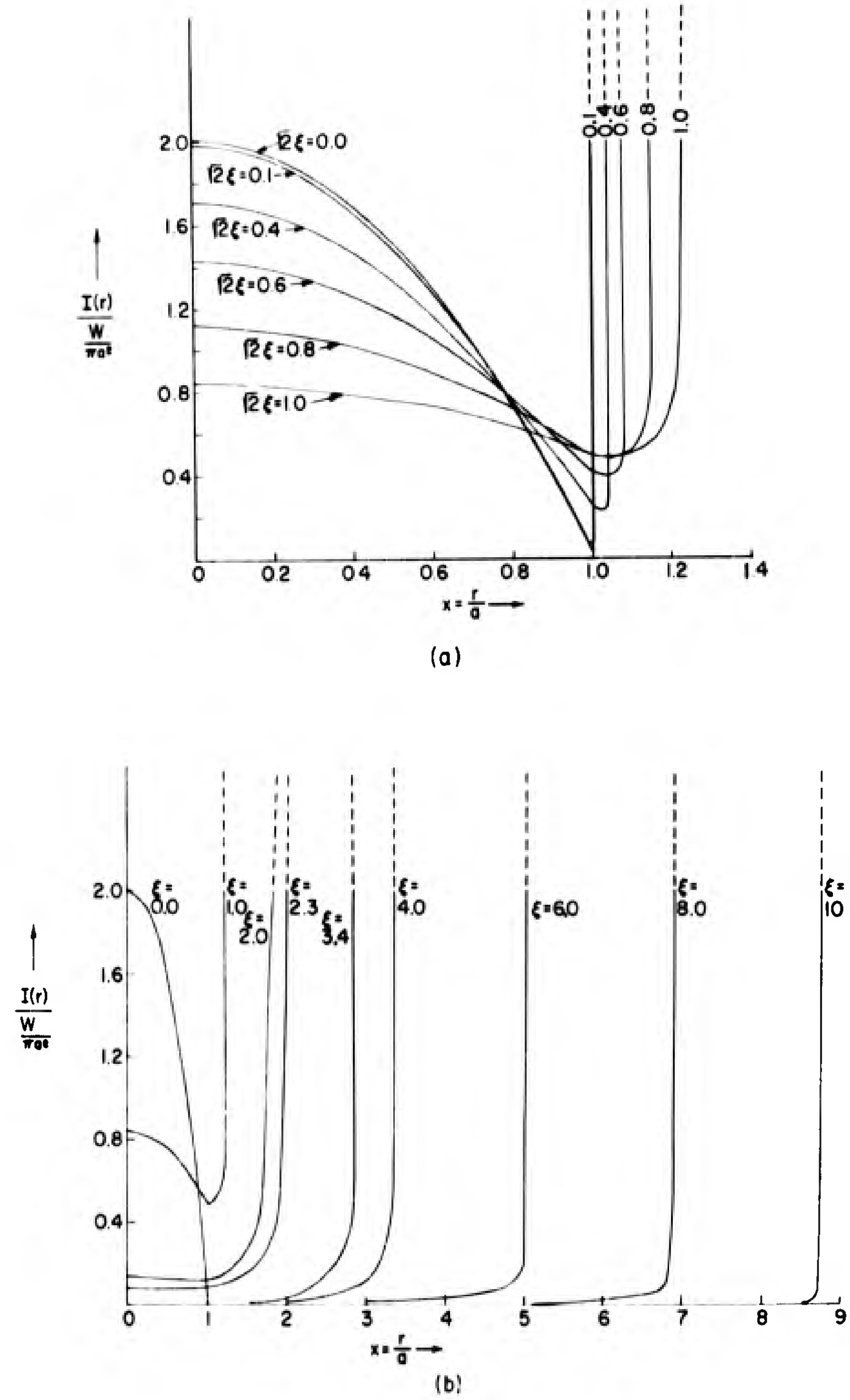

Fig. 6 - Reduced intensity as a function of reduced radial distance x for a parabolic intensity distribution. The parameter ξ has the range of values (a) $0.0 < \sqrt{2}\,\xi < 1.0$ and (b) $0.0 < \xi < 10$

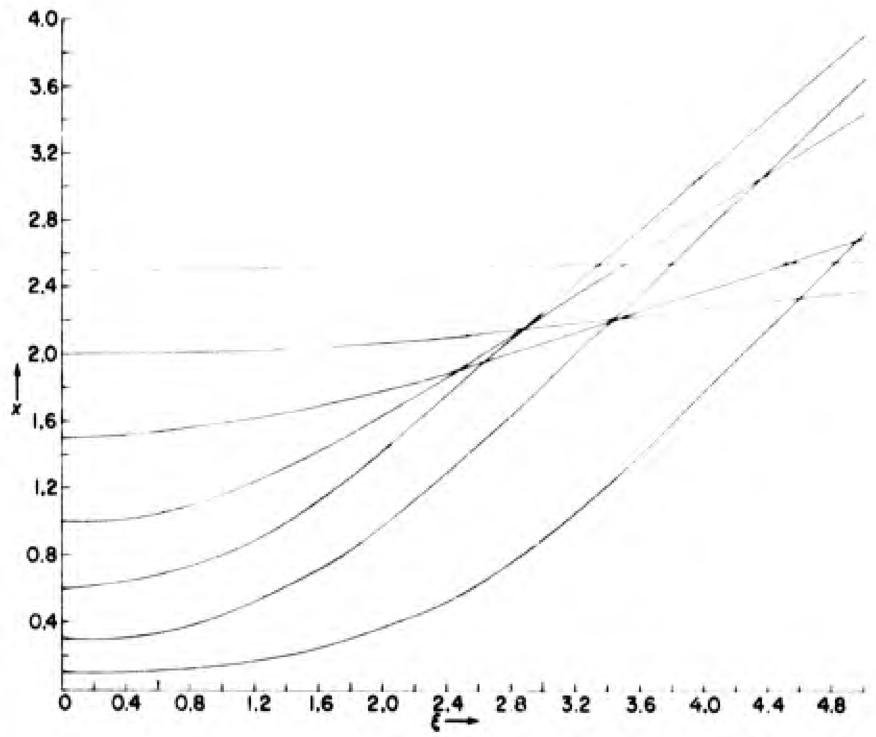

Fig. 7 - Light rays, in reduced coordinates, for the Guassian intensity distribution

Very Smooth Monotonically Decreasing Power Distributions; the Gaussian Distribution

The case of the parabolic distribution discussed above was singular in several senses of the word. First, the integrals could be calculated exactly so that the general development was not needed in full. Second, the first and second derivatives possess a discontinuity at $x = 1$, and indeed vanished for $x > 1$; hence the full apparatus of the general theory could not be applied.

In this section we treat a more general case, the case in which $f(x)$ is not only monotonically decreasing but possesses at least three continuous derivatives. A case in point is the Gaussian distribution which appears to be very realistic in many instances; numerical results will be given. Equations (62) and (68) thus apply, but since we have in mind the need for numerical computations, we cast these equations into a more readily computed form. The need for such a recasting is evident, since both Eqs. (62) and (68) contain integrals whose integrands possess singularities. Integrating by parts, Eqs. (62) and (68) become

$$\zeta = -2\,\frac{\sqrt{f(x_0) - f(x)}}{f'(x)} - 2\int_{x_0}^{x} dx'\,\frac{f''(x')\sqrt{f(x_0) - f(x')}}{[f'(x')]^2}\,, \qquad (73)$$

and

$$\left.\frac{\partial x}{\partial x_0}\right)_\zeta = +\frac{f'(x_0)}{f'(x)} - \frac{2f'(x_0)f''(x)}{[f'(x)]^3}\,[f(x_0) - f(x)]$$

$$+ 2\sqrt{f(x_0) - f(x)\, f'(x_0)} \int_{x_0}^{x} dx' \sqrt{f(x_0) - f(x)}\, \frac{f'''(x')f'(x') - 3[f''(x')]^2}{[f'(x')]^4}\,, \tag{74}$$

respectively, while the reduced intensity distribution becomes

$$\hat{I}(x,\zeta) = \sum_{x_0} \frac{f(x_0)x_0}{x\left|\left(\frac{\partial x}{\partial x_0}\right)_\zeta\right|} \tag{75}$$

where the sum is over all x_0 which yield the same ζ for the same value of x when inserted into Eq. (73).

For the Gaussian case,

$$f(x) = e^{-x^2} \tag{76}$$

The variable x is related to r by $x = r/a$, where a, here, is now a point at which the power decreases by a factor of e of its central value.

The trajectories and intensities for the Gaussian case were computed on a digital computer; several of the light rays are shown in Fig. 8. The expected phenomenon of blooming appears as in the parabolic case, but the ray crossing now can become more complicated.

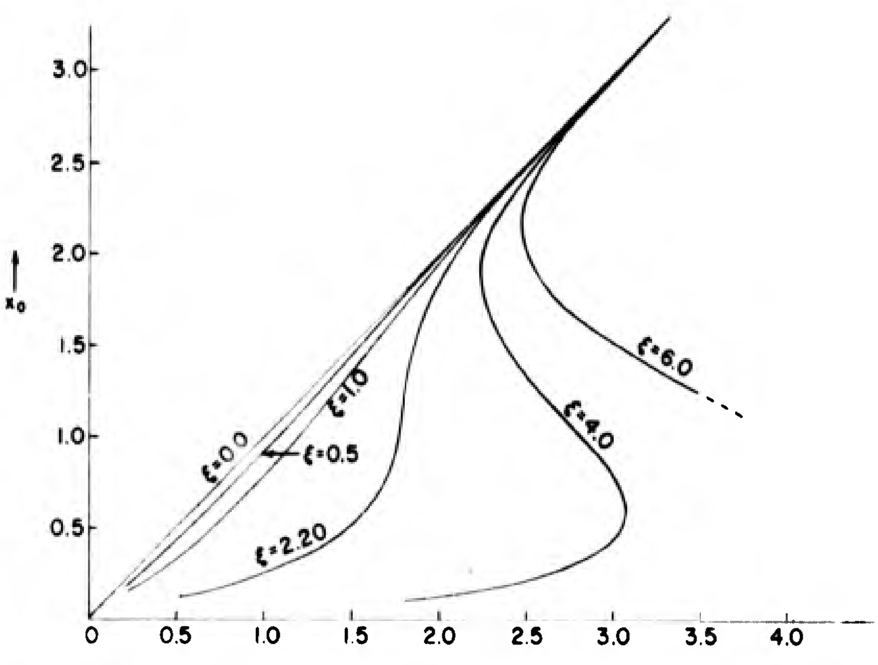

Fig. 8 - Plot of x_0 vs. x for the Gaussian intensity distribution with ζ as a parameter

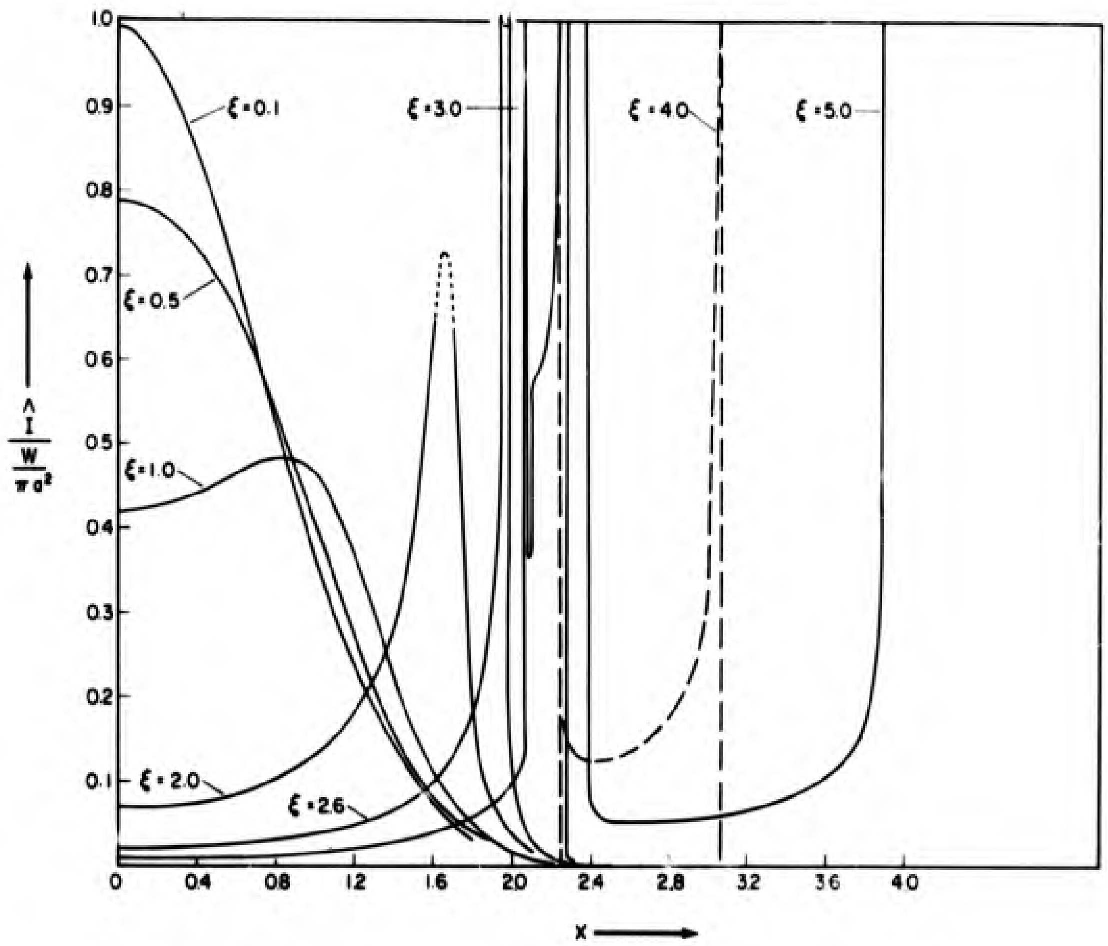

Fig. 9 - Reduced intensity as a function of reduced radial distance x for the Gaussian intensity distribution with ζ as a parameter

To discern how many light rays cross through a given point x,ζ (in reduced coordinates), refer to Fig. 9. The curves plotted there are the values of x that a given light ray starting at x_0 will assume when having gone downbeam a distance ζ. Thus, to find out how many light rays go through point x,ζ, we draw a straight line parallel to the ordinate axis going through the value x on the abscissa. The number of points of intercept equals the number of rays that go through x,ζ, and their ordinates x_0 will tell where, on the face of the laser, the rays originated. For values of ζ less than 3, approximately, only one ray goes through each point x,ζ. For values of ζ larger than 3 (we stress again that this number is only approximate since we had no analytical means of determining the exact number), there will be two points x_1,ζ and x_2,ζ through which two rays pass. For all points x,ζ such that $x_1 < x < x_2$, three rays cross at each point. The effect of these ray crossings on the intensity distribution curve is quite pronounced as an examination of Fig. 9 will show. The abscissa and ordinate of this graph are the same as those for the parabolic case graphed in Fig. 6. For $\zeta < 3.0$, we see that blooming proceeds in a rather smooth fashion. However, for $\zeta = 3.0$, 4.0, and 5.0 there is an abrupt infinite discontinuity at two points in each case, between which the intensity is quite high, and outside of the interval defined by these points the intensity is relatively quite low. The phenomenon becomes more pronounced as ζ grows, and the length between the points where the intensity becomes infinite increases as ζ increases. The infinite discontinuities owe their origin to the fact that two light rays from neighboring points on the laser face cross. In the full three dimensions, these rays form a caustic surface and the geometrical optics limit is no longer appropriate. Diffraction will play an important role here, causing the intensity to remain finite everywhere. Hence, the infinite peaks must be regarded as only indicative of a very high intensity.

The intensity profiles for the parabolic and Gaussian cases demonstrate quite dramatically how sensitive the details of the blooming phenomenon are to the original profile across the face of the laser. Since our model of energy deposition is only a first approximation to the correct expressions, our results here reflect this choice of model to some degree. In particular we expect the results to be applicable in those regions near the face of the laser or at early times. While it is true that the annular structure of the defocused beam has been seen experimentally, we cannot expect that our results will compare in all details with experimental results when the defocusing is severe. In contrast, we mention the case of the laser for which the intensity profile is rectangular; as will be shown in the section on "A Variational Problem" below, this laser beam will not bloom and, furthermore, our model of energy deposition is exact and the calculation is completely self-consistent.

Nonmonotonic Power Distributions

If $f(x)$ is not monotonic, the light rays evolve in a more complicated fashion. Easy qualitative descriptions of the light rays can be obtained by noting that Eq. (62) for the trajectories is identical in form to the energy integral for a particle in a one-dimensional conservative potential, with ζ being the analogue of time, $f(x_0)$ the analogue of the total energy, and $f(x)$ the potential energy. It can be shown qualitatively in this way that nonmonotonic distributions can lead to focusing, as well as defocusing.

A Variational Problem; Minimizing the Blooming

From the examples of the preceding paragraphs it becomes clear that the choice of the initial power distribution (or "beam shape" as it is sometimes called) will have rather profound effects on the spatio-temporal development of the laser beam in the medium. It becomes a reasonable question to ask if there exists any distribution f which will minimize the effects of blooming? (One could ask if there is an f which will maximize the effect of blooming. Since, in most uses of lasers, it is desirable to preserve the initial degree of collimation of the beam, this function f would be of interest for the purpose of avoiding this case.) Of course, we must find a mathematical characterization of the phrase "minimize the blooming effect." To this end, we assume we may use the long-time form for the light rays given by Eq. (62). The blooming will be minimized if, for a fixed deviation $x - x_0$ of a light ray from its initial distance from the beam axis, the "distance" ζ down the beam axis is maximized. Thus we want to choose an f such that for every x and x_0, ζ is maximized.

A solution to this problem is very simple: if $f(x) = \text{constant}$, ζ is always infinite, i.e. there is no blooming at all. Physically this is clear; if all the air is being heated by the same amount and the density changes are the same everywhere, light will propagate through without deflection. The trouble here is that such a beam possesses an infinite total energy and is therefore unacceptable as a physical situation. Another way of putting this is that Eq. (49) is violated. Thus, our problem is to vary f so as to maximize ζ and still have Eq. (49) satisfied. Again the solution is at hand:

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ \\ 0, & 1 < x < \infty \, . \end{cases} \tag{77}$$

It is clear that this choice of f causes no blooming at all and our model of energy deposition now becomes exact, although the whole problem becomes trivial after the fact. The interest in this result lies in its vivid illustration of the fact that a beam whose "shape" is as flat as possible will have the least amount of blooming. The blooming will be most pronounced for those rays in the outer parts of the beam. (We note that these comments are applicable only to the region where diffraction effects are small.)

The blooming effect is maximized if ζ takes on its smallest value for every pair x_0,x. Such a problem as this has no solution.

ACKNOWLEDGMENTS

The author would like to thank Dr. Alfred H. Aitken for several helpful and stimulating discussions, and Mr. Harrison Hancock for performing some very difficult numerical computations.

REFERENCES

1. Gordon, J.P., Leite, R.C.C., Moore, R.S., Porto, S.P.S., and Whinnery, J.R., J. Appl. Phys. 36:3 (1965)

2. Leite, R.C.C., Moore, R.S., Whinnery, J.R., Appl. Phys. Letters 5:141 (1964)

3. Rieckhoff, K.E., Appl. Phys. Letters 9:87 (1966)

4. Callen, W.R., Huth, B.G., Pantel, R.H., Appl. Phys. Letters 11:103 (1967)

5. Carman, R.L., and Kelley, P.L., Appl. Phys. Letters 12:241 (1968)

6. Brueckner, K.A., and Jorna, S., Phys. Rev. Letters 17:78 (1966)

7. Brueckner, K.A., and Jorna, S., Phys. Rev. 164:182 (1967)

8. Akhmanov, S.A., Krindach, D.P., Migulin, A.V., Sukhorukov, A.P., and Khokhlov, R.V., IEEE J. Quant. Elect., QE-4:568 (1968)

9. Gebhardt, F.G., and Smith, D.C., Appl. Phys. Letters 14:52 (1969)

10. McLean, E.A., and Sica, L., private communication (to be published)

11. Born, M., and Wolf, E., "Principles of Optics," 3rd ed., London:Pergamon, 1965

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Naval Research Laboratory Washington, D.C. 20390	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE
THERMAL BLOOMING OF LASER BEAMS IN GASES

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)
This is a final report on one aspect of laser propagation studies; work on other aspects continues.

5. AUTHOR(S) (First name, middle initial, last name)
J. N. Hayes

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
February 11, 1971	38	11

8a. CONTRACT OR GRANT NO NRL Problem R05-31.303 b. PROJECT NO ORD-0832-129/173-1/U1754 No. 2 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) NRL Report 7213 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT
Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Dept. of the Navy (Naval Ordnance Systems Command), Washington, D.C. 20360

13. ABSTRACT

The trajectories of light rays and the intensity pattern of a laser beam as a function of time and of distance down the beam are determined. It is assumed in this calculation that (a) the medium is a homogeneous, isotropic, and initially quiescent gas, (b) convection, viscosity, and thermal conduction may be ignored at early times, (c) changes in total beam power as a function of distance downbeam may be ignored, (d) a specific model of energy deposition is valid, (e) the medium may be described by equations of hydrodynamics and thermodynamics and obeys the Lorentz-Lorenz Law, and (f) geometrical optics applies to the problem. It is shown that this model can be solved exactly; long- and short-time behavior of the solutions is discussed, and the times for the onset of convection are estimated.

The phenomenon of laser defocusing is shown to change rapidly with time; a definition of thermal blooming is given, and it is shown that the region of blooming moves up the beam toward the face of the laser. The intensity pattern at a fixed point in space is shown to change its profile, going over to a bright narrow annular ring whose radius increases with the passage of time.

Parameter combinations required for studies of various aspects of the blooming phenomenon are pointed out as the mathematical development progresses.

Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Laser beams						
Propagation						
Light transmission						
Interaction with Matter						
Nonlinear optics						
Thermal blooming						

DD FORM 1 NOV 65 1473 (BACK)
(PAGE 2)

Security Classification