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ABSTRACT

Many recent economic and inventory studies have included various types of fixed or lump-sum costs as important determinants of optimal behavior. In this paper, the classical inventory model is augmented to include fixed shortage costs. In general, the presence of fixed shortage costs can lead to complex optimal solutions. The purpose of this paper is to establish a set of sufficient conditions which guarantee the existence of an optimal ordering policy which is unique. The resulting optimal policy is described by a unique set of critical numbers which are bounded and decrease monotonically over the horizon for which the inventory system is to be operated.

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1. INTRODUCTION

A central problem in inventory theory has been the determination of the form of optimal policies for various forms of the cost functions. General surveys of such results may be found in Scarf [6] and Veinott [8]. In particular, cost functions involving fixed, or lump-sum, terms have received wide attention. Fixed costs enter the inventory problem principally with respect to ordering and to stockout or shortage. Recent papers by Mellon and Orr [5], Barro [1], and Brown and Lloyd [4] have employed this concept. Scarf [7] has shown under very general conditions the optimality of (s,S) policies when fixed ordering costs are included. Fixed ordering and shortage costs also represent a particular case of the models developed by Boylan [2], [3]. For a broad class of problems, it was shown that a multiple (s,S) policy would minimize costs over an n-period horizon. In the present paper, the classical inventory model with proportional ordering, holding, and shortage costs and backlogging of unsatisfied demand is augmented to include fixed shortage costs. In general, multiple solutions can occur in this case. The purpose of this paper is to establish sufficient conditions for the existence of an optimal ordering policy which is unique. Furthermore, the resulting optimal policy is described by a unique set of critical numbers which are shown to be bounded and to decrease monotonically over the horizon for which the inventory system is to be operated.

2. THE PROBLEM

The functional equation employed in the analysis is:

$$(1)\begin{cases} f_n(x) = \min\limits_{y \geq x}\Big[c(y-x) + \int_0^y h(y-\xi)\varphi(\xi)d\xi + \int_y^\infty p(\xi-y)\varphi(\xi)d\xi \\ \qquad + m\int_y^\infty \varphi(\xi)d\xi + \alpha\int_0^\infty f_{n-1}(y-\xi)\varphi(\xi)d\xi\Big] \text{ for } n = 1,2,\ldots,N. \\ f_0(x) \equiv 0 \end{cases}$$

where

x = beginning inventory level,

y = inventory level after ordering,

c = per unit ordering cost,

h = per unit holding charge per period,

p = per unit shortage charge per period,

m = fixed shortage charge.

We assume demands in every period to be independent and identically distributed random variables with probability density function $\varphi(\cdot)$. We make the further restriction that the density function of demand be pseudo-concave. That is, $\frac{d\varphi(\xi_1)}{d\xi}(\xi_2-\xi_1) \leq 0$ implies $\varphi(\xi_1) \geq \varphi(\xi_2)$. Unimodal density functions such as the normal, gamma, and exponential are among those that are pseudo-concave.

Define the functions

$$g(x,y) = c(y-x) + \int_0^y h(y-\xi)\varphi(\xi)d\xi + \int_y^\infty p(\xi-y)\varphi(\xi)d\xi + m\int_y^\infty \varphi(\xi)d\xi ,$$

$$G_n(x,y) = g(x,y) + \alpha \int_0^\infty f_{n-1}(y-\xi)\varphi(\xi)d\xi \quad \text{for } n = 1,2,\ldots,N .$$

For the last period in the planning horizon, $n = 1$, the functional equation is

$$f_1(x) = \min_{y\geq x} \left[G_1(x,y)\right]$$

$$= \min_{y\geq x} \left[g(x,y)\right]$$

where

(2) $$\frac{\partial g(x,y)}{\partial y} = c + h\,\Phi(y) - p\,[1-\Phi(y)] - m\,\varphi(y) .$$

We assume

(3) $$\left.\frac{\partial g(x,y)}{\partial y}\right|_{y=0} = c - p + [h+p]\,\Phi(0) - m\,\varphi(0) < 0$$

to eliminate a boundary solution. This restriction is weaker than the traditional assumption made for the classical model due to the inclusion of the negative term, $-m\varphi(0)$.

Setting the first derivative (2) equal to zero, we have

(4) $$\Phi(y) = \frac{p-c + m\varphi(y)}{h + p} ,$$

which can be solved for those values of y, y^*, which satisfy the first order minimization condition. The classical model assumes that $m = 0$, so that y^* is unique. The right and left hand sides of (4) are plotted in Figure 1 for a unimodal density function. For $m > 0$, there can be from one to at most three solutions to (4). Second order conditions depend in general upon $\frac{\partial\varphi(y)}{\partial y}$. We wish to determine sufficient conditions such that for each period a unique value of y minimizes the functional equation.

We first present two lemmas.

<u>Lemma 1</u>: Define $\bar{y}$ to be the mode of $\varphi(\cdot)$, a pseudo-concave density function. Then there exists at most one y, $y \geq \bar{y}$, such that equation (4) is satisfied.

<u>Proof</u>: Since $\Phi(y)$ is a non-decreasing function and $\frac{p-c+m\varphi(y)}{h+p}$ is a non-increasing function for $y \geq \bar{y}$, the lemma follows.

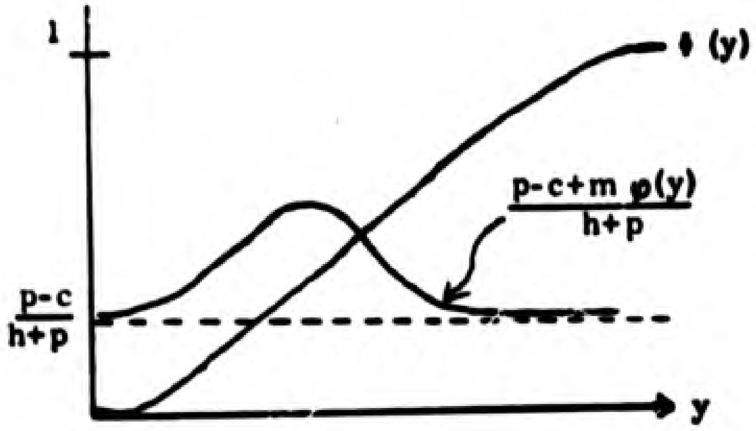
A sufficient condition for the existence of a unique y satisfying (4) is contained in the following lemma.

<u>Lemma 2</u>: A sufficient condition for (4) to have a unique solution is

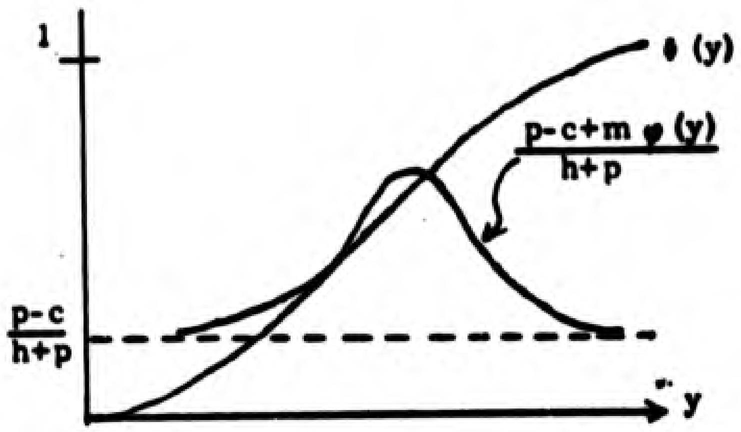
(5) $h\,\Phi(\bar{y}) - p\,[1-\Phi(\bar{y})] + c < 0$, or, equivalently,

(6) $\Phi(\bar{y}) < \frac{p-c}{h+p}$.

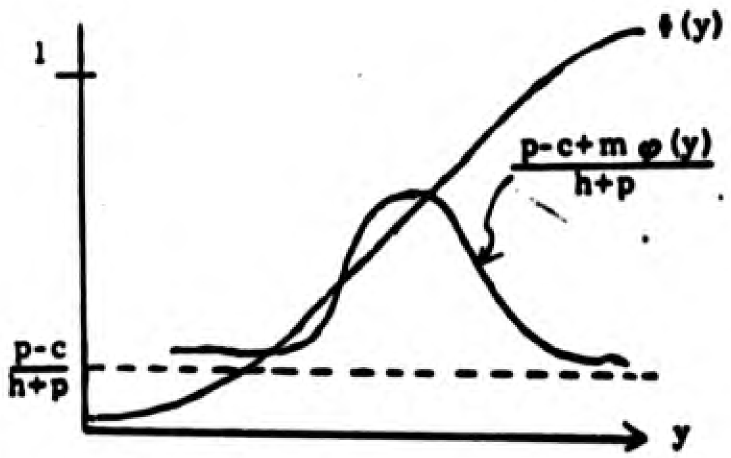
FIGURE I



CASE I: One Solution



CASE II: Two Solutions



CASE III: Three Solutions

Proof: If (5) holds for $\bar{y}$, then it holds for all $y \le \bar{y}$ and $\frac{\partial g(x,y)}{\partial y} < 0$ for all $y \le \bar{y}$. Then the first solution to (4) must occur at a value y^*, $y^* > \bar{y}$, which by Lemma 1 is unique. This is Case 1 of Figure 1.

We now prove the following theorem.

Theorem I: The form of the optimal policy for the inventory system specified by (1) is the same as for the classical model, as long as $\varphi(\cdot)$ is pseudo-concave and condition (5) holds. The optimal policy is characterized by a unique set of numbers y_n^*, such that

for $x_n < y_n^*$, order up to y_n^*

for $x_n \ge y_n^*$, do not order.

Furthermore,

$$0 < y^L \le y_n^* \le y_{n+1}^* < y^U < \infty \quad \text{for } n = 1,2,\ldots,N .$$

where the upper and lower bounds, y^U and y^L, respectively, are defined by

$$(7) \qquad (1-\alpha)c + h\,\Phi(y^U) - p\left[1-\Phi(y^U)\right] - m\,\varphi(y^U) = 0$$

$$(8) \qquad c + h\,\Phi(y^L) - p\left[1-\Phi(y^L)\right] - m\,\varphi(y^L) = 0 .$$

Finally, the derivative of the functional equation for $n = 1,2,\ldots,N$ is

$$\frac{\partial f_n(x)}{\partial x} = \begin{cases} -c \text{ for } x < y_n^* \\ \frac{\partial g(x,x)}{\partial x} + \alpha \int_0^\infty \frac{\partial f_{n-1}(x-\xi)}{\partial x} \varphi(\xi) d\xi \text{ for } x \geq y_n^* . \end{cases}$$

Proof: The proof is by the method of mathematical induction.

n = 1: The first order condition for a minimum in the last period is given by

$$(9) \qquad \frac{\partial g(x,y_1)}{\partial y_1} = c + h\,\Phi(y_1) - p[1-\Phi(y_1)] - m\varphi(y_1) = 0$$

and it has a unique solution y_1^* by pseudo-concavity, condition (5), and Lemma 2. Since (9) is identical to (8), $y_1^* = y^L$ and $y_1^* < y^U$. By assumption (3), $y^L > 0$. From equation (7), it follows that

$$\Phi(y^U) = \frac{p - (1-\alpha)c - m\varphi(y^U)}{h + p} .$$

Since

$$\Phi(y^U) < \frac{p - (1-\alpha)c}{h + p} < 1 ,$$

it follows directly that $y^U < \infty$. Since y_1^* is unique, the optimal policy is of the classical form as given in the theorem, so that the functional equation is

$$f_1(x) = \begin{cases} g(x,y_1^*) & \text{for } x < y_1^* \\ g(x,x) & \text{for } x \geq y_1^* . \end{cases}$$

Its derivative is

$$\frac{\partial f_1(x)}{\partial x} = \begin{cases} -c \text{ for } x < y_1^* \\ \frac{\partial g(x,x)}{\partial x} + \alpha \int_0^\infty \frac{\partial f_0(x-\xi)}{\partial x} \varphi(\xi)d\xi \text{ for } x \geq y_1^* . \end{cases}$$

Thus the theorem holds for $n = 1$.

__Arbitrary period__: Assume the theorem holds for periods 1,...,n-1. The functional equation for period n is

$$f_n(x) = \min_{y \geq x}\left[G_n(x,y)\right]$$

$$= \min_{y \geq x}\left[g(x,y) + \alpha \int_0^\infty f_{n-1}(y-\xi)\varphi(\xi)d\xi\right] .$$

The derivative of $G_n(x,y)$ with respect to y is

$$\frac{\partial G_n(x,y)}{\partial y} = \frac{\partial g(x,y)}{\partial y} + \alpha \int_0^\infty \frac{\partial}{\partial y}\left[f_{n-1}(y-\xi)\right]\varphi(\xi)d\xi ,$$

which upon substitution of the derivative of the functional equation for period n-1 is equivalent to

$$(10) \quad \frac{\partial G_n(x,y)}{\partial y} = \frac{\partial g(x,y)}{\partial y} + \alpha \int_0^{y-y^*_{n-1}} \frac{\partial f_{n-1}(y-\xi)}{\partial y} \varphi(\xi)d\xi - \alpha c\left[1-\Phi(y-y^*_{n-1})\right].$$

When $y < y^*_{n-1}$, (10) reduces to

$$(11) \qquad \frac{\partial G_n(x,y)}{\partial y} = \frac{\partial g(x,y)}{\partial y} - \alpha c ,$$

which is identical to equation (7). Equations (7) and (8) are graphed in Figure 2.

Since by hypothesis $y^*_{n-1} < y^U$, equation (11) must be negative for $y < y^*_{n-1}$. Therefore, y^*_n must be greater than or equal to y^*_{n-1}. If $y \geq y^*_{n-1}$, then the limits of integration in equation (10) imply that

$$y^*_{n-1} \leq y-\xi \leq y ,$$

so that from the definition of the derivative of $f_{n-1}(\cdot)$, equation (10) is

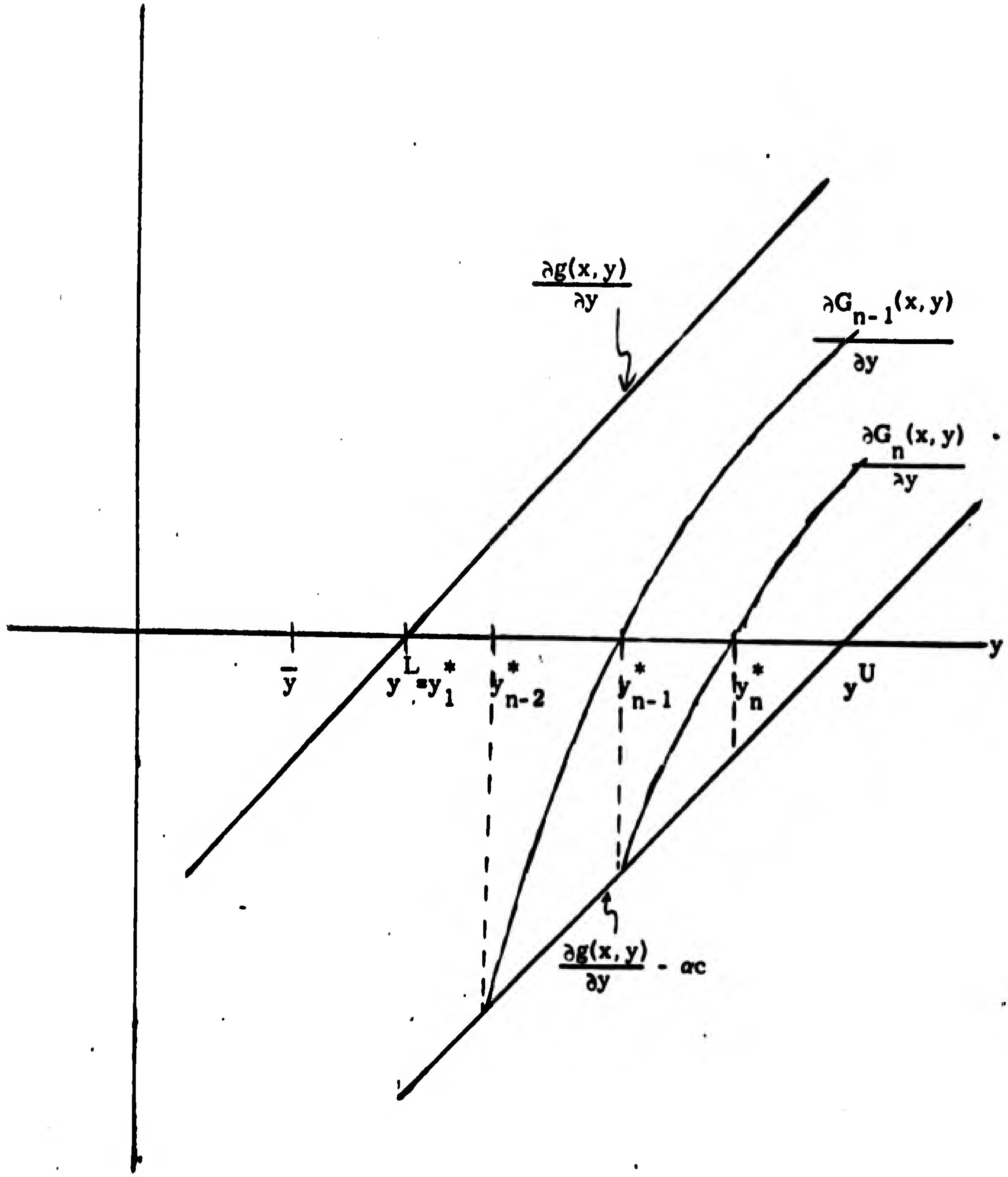
$$\frac{\partial G_n(x,y)}{\partial y} = \frac{\partial g(x,y)}{\partial y} + \alpha\int_0^{y-y^*_{n-1}}\left[\frac{\partial G_{n-1}(y-\xi,y-\xi)}{\partial y} - c\right]\varphi(\xi)d\xi - \alpha c\left[1-\Phi(y-y^*_{n-1})\right]$$

$$= \frac{\partial g(x,y)}{\partial y} - \alpha c + \alpha\int_0^{y-y^*_{n-1}} \frac{\partial G_{n-1}(y-\xi,y-\xi)}{\partial y}\varphi(\xi)d\xi .$$

The first two terms above are equation (11) in Figure 2. The third term above is always positive since y^*_{n-1} was assumed unique and

$$(12) \qquad \frac{\partial G_{n-1}(y-\xi,y-\xi)}{\partial y} > 0 \qquad \text{for } 0 \leq \xi \leq y-y^*_{n-1} .$$

FIGURE II



In the appendix, it is shown that $\frac{\partial G_n(x,y)}{\partial y}$ is monotonically increasing for $y \geq y^*_{n-1}$. Thus $y^*_n \geq y^*_{n-1}$ and is unique.

Finally, for $y = y^u$,

$$\frac{\partial G_n(x,y^u)}{\partial y} = \frac{\partial g(x,y^u)}{\partial y} - \alpha c + c\int_0^{y^u - y^*_{n-1}} \frac{\partial G_{n-1}(y^u-\xi, y^u-\xi)}{\partial y} \varphi(\xi)d\xi > 0$$

The first two terms above are zero by the definition of y^u from equation (7) and the third term is positive by (12) and the hypothesis $y^u > y^*_{n-1}$. Therefore, $y^*_n < y^u$.

The optimal policy is of the classical form as stated in the theorem, and is unique. By substitution of the form of the optimal policy into the functional equation, it can be easily shown that its derivative is of the form given in the theorem. Therefore, the theorem holds for any general period n .

APPENDIX

In this appendix, $\frac{\partial G_n(x,y)}{\partial y}$ is shown to be a monotonically increasing function for $y \geq y^*_{n-1}$. The proof is again by induction.

<u>n = 1</u>:

$$G_1(x,y) = g(x,y)$$

$$\frac{\partial G_1(x,y)}{\partial y} = \frac{\partial g(x,y)}{\partial y}$$

$$= c + h\Phi(y) - p[1-\Phi(y)] - m\varphi(y)$$

$$\frac{\partial^2 G_1(x,y)}{\partial y^2} = \frac{\partial^2 g(x,y)}{\partial y^2}$$

$$= h\varphi(y) + p\varphi(y) - m\frac{\partial\varphi(y)}{\partial y} .$$

Clearly $h\varphi(y) + p\varphi(y) > 0$ for all y. Furthermore, for $y \geq \bar{y}$, $\frac{\partial\varphi(y)}{\partial y} \leq 0$. Since $y^*_1 \geq \bar{y}$, we have $\frac{\partial^2 G_1(x,y)}{\partial y^2} > 0$ for $y \geq y^*_1$.

<u>Arbitrary period n</u>:

$$\frac{\partial G_n(x,y)}{\partial y} = \frac{\partial g(x,y)}{\partial y} - \alpha c + \int_0^{y-y^*_{n-1}} \frac{\partial G_{n-1}(y-\xi,\ y-\xi)}{\partial y}\ \varphi(\xi)d\xi$$

$$\frac{\partial^2 G_n(x,y)}{\partial y} = \frac{\partial^2 g(x,y)}{\partial y} + \alpha \int_0^{y-y^*_{n-1}} \frac{\partial^2 G_{n-1}(y-\xi,\ y-\xi)}{\partial y^2} \varphi(\xi) d\xi$$

$$+ \alpha(1) \frac{\partial G_{n-1}(y^*_{n-1}, y^*_{n-1})}{\partial y} \ .$$

For $y \geq y^*_{n-1}$, $\frac{\partial^2 g(x,y)}{\partial y^2} > 0$ as was seen from the first period equation. Since by hypothesis the theorem holds for

$n = 1,2,\ldots,n-1$, $\frac{\partial^2 G_{n-1}(y-\xi,y-\xi)}{\partial y} > 0$ for $y - \xi \geq y^*_{n-1}$. This restriction is met, since ξ is restricted to lie between zero and $y-y^*_{n-1}$. Finally, the third term is zero by the hypothesis that y^*_{n-1} is optimal for period $n-1$. Thus $\frac{\partial G_n(x,y)}{\partial y} > 0$ for $y \geq y^*_{n-1}$.

Q.E.D.

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