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THESIS

A REVIEW OF LITERATURE ON THE THEORY OF HIT AND KILL PROBABILITIES

by

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of Hit and Kill Probabilities

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ABSTRACT

Literature concerning the theory of hit and kill probabilities is readily available but is widely scattered. The paper presents a consolidation and a categorization of current literature and provides a general discussion of representative models in the field of hit and kill probabilities. A literature research matrix is presented to aid the researcher in locating existing models which meet his requirements.

TABLE OF CONTENTS

I. INTRODUCTION

An important problem faced by the weapons system analyist is that of determining the probability that a weapons system will effect a target hit or kill. The hit or kill probabilities are investigated initially by using a single shot or multiple shot hit or kill probability models which are further extended to include models which consider the entire engagement. Although two factors influence the kill probability function - the probability that the weapons system can deliver its ordnance to the target and the vunerability of the target - only the probability that a given system can deliver its ordnance to the target will be discussed in detail in this paper.

The author's purpose in presenting a survey of the literature reflecting hit and kill probability models is to provide the system analyist with a convenient reference for a detailed study using models which are pertinent to the weapons system being investigated, and where possible to eliminate redundant effort. The majority of the models discussed originated in technical memorandum and reports from various government agencies or activities and received only limited distribution. It is hoped that the consolidation and categorization of the models in this paper will enable an analyist to more easily locate literature that is relevant to his particular needs. Secondly, the paper will serve to familiarize the student or researcher with the terminology and assumptions related to hit and kill probability models.

Solution procedures for the models will not be discussed in detail as the interested reader can find the solution techniques by reading the original document. The paper focuses upon the logical categorization of the models as to the way in which the models are formulated and the presentation of relevant models within the literature. A convenient categorization of the hit and kill probability models is illustrated in figure 1. The literature research matrix found in Section III is indicative of the wide range and diversification of content of current literature. The terminology and symbology varys widely within the literature. To simplify the presentation, all models presented in Section IV have been standardized so as to conform to the definitions found in Section II.

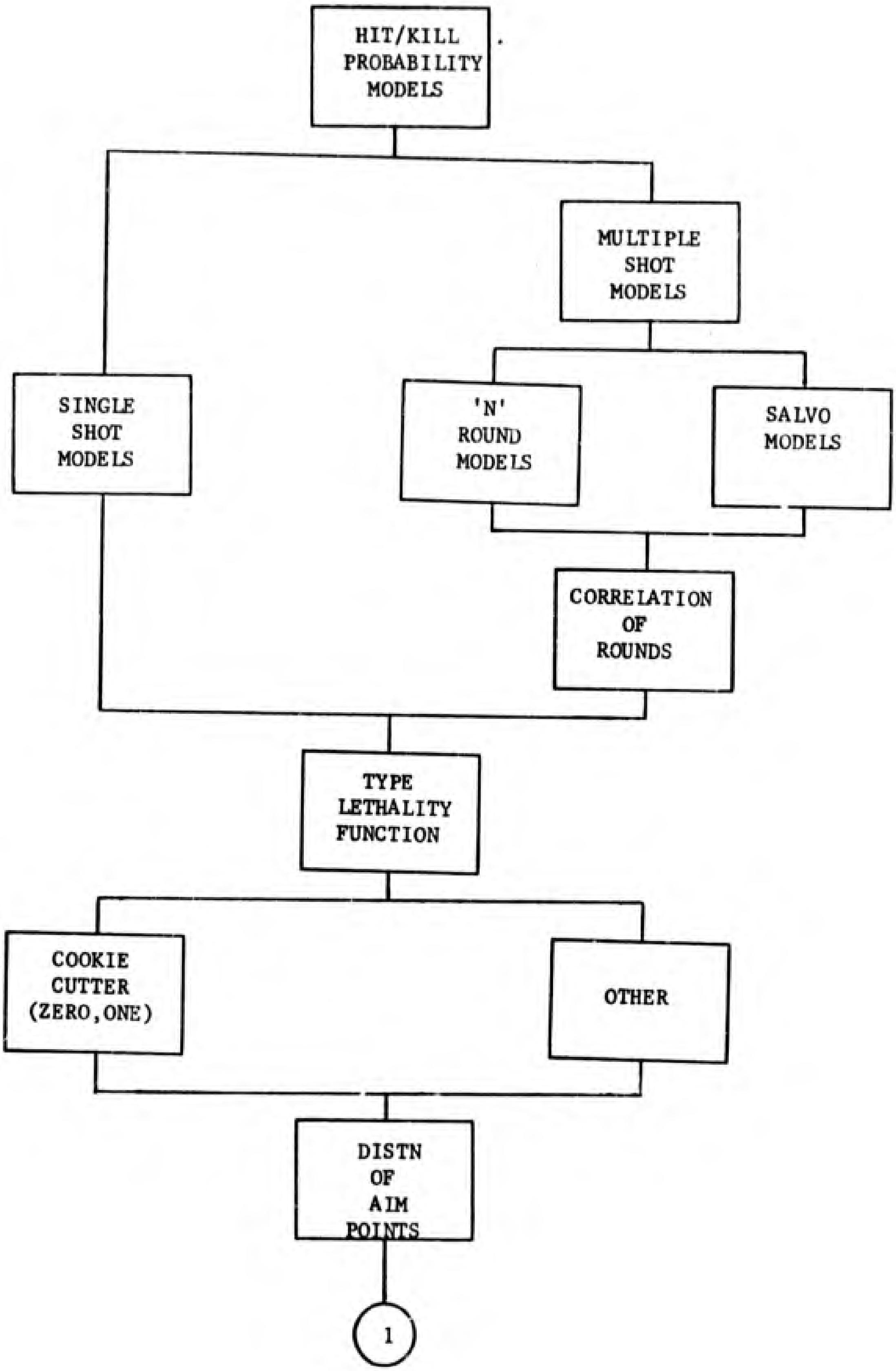

HIT/KILL PROBABILITY MODELS
MULTIPLE SHOT MODELS
SINGLE SHOT MODELS
'N' ROUND MODELS
SALVO MODELS
CORRELATION OF ROUNDS
TYPE LETHALITY FUNCTION
COOKIE CUTTER (ZERO,ONE)
OTHER
DISTN OF AIM POINTS
1

Figure 1

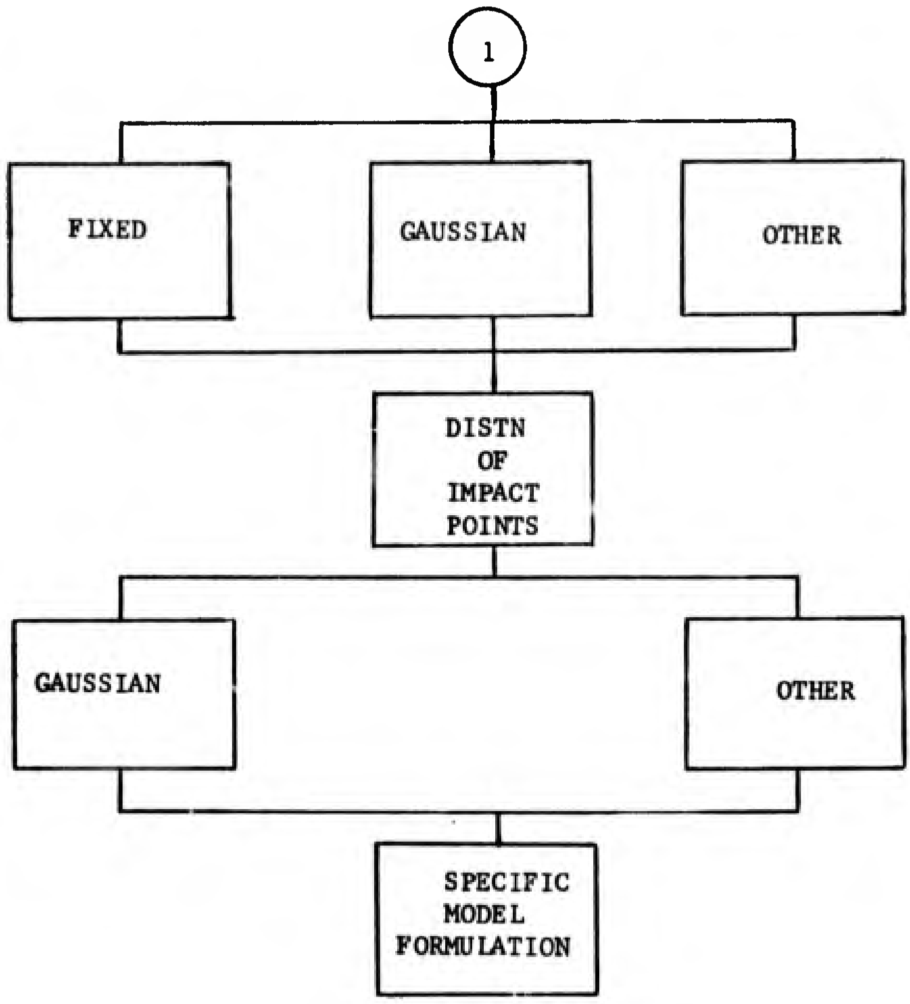


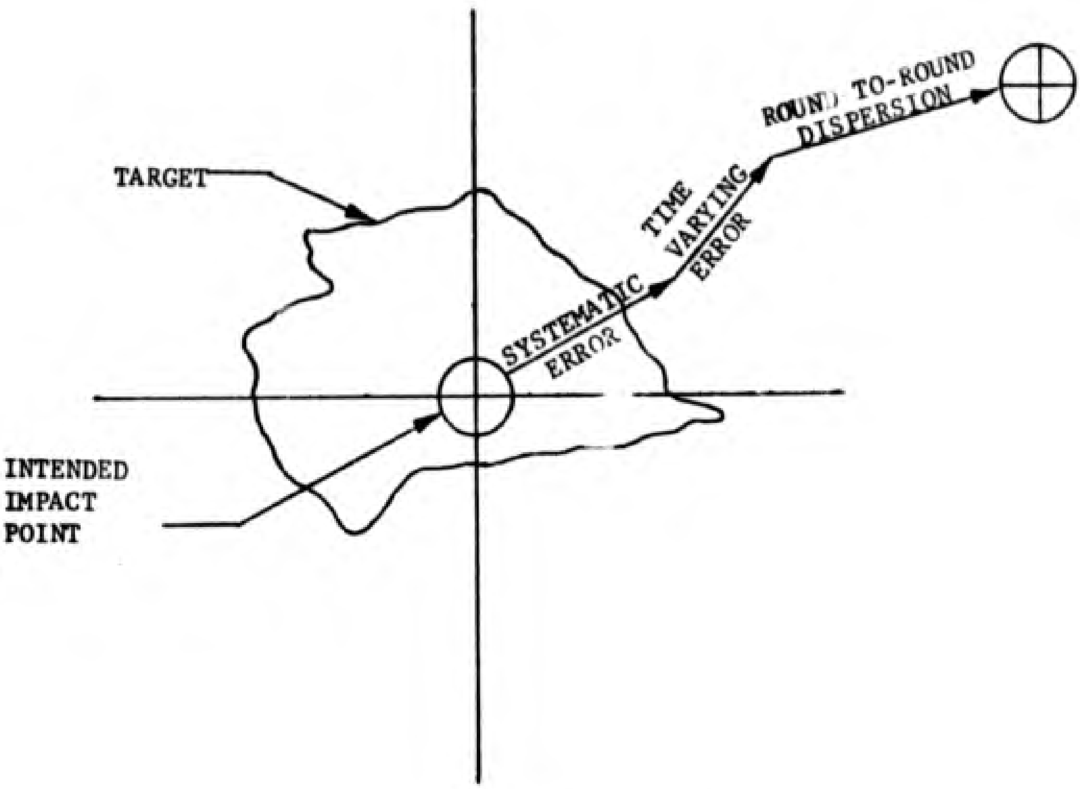
Figure 1 (continued)

II. DEFINITIONS AND TERMINOLOGY

This section of the paper will present the basic definitions and terminology which are used throughout the presentation. Unless otherwise specified, the definitions and notations in this section will apply to all models and subject material within this paper.

Hit probability is defined as the probability of a hit or hits being made on a target out of a given number of projectiles directed at the target. A hit is a blow or impact on a target by a bullet, bomb, or other projectile. It should be noted that a projectile could technically effect a hit if the force of its explosion created a blow or impact to the target. Single shot hit probability is the probability that a single projectile fired against a target will hit that target under a given set of conditions. Kill probability is defined as the conditional probability that a projectile will kill a target against which it is fired given that the projectile hits the target. As a noun, a kill is that level of damage which destroys a target. An aircraft would be considered a kill if the damage was such that the aircraft fell out of control immediately after sustaining the damage. As a verb, kill means to destroy the target to a specified level of damage. Military authorities have designated certain levels of damage required for a specific level of kill [Ref. 1].

Because of errors inherent in a weapons system the impact point for any projectile will in general differ from the intended aim point by an amount which is random from projectile to projectile. These differences between the intended and actual impact points are the result of three basic types of errors. Figure 2 illustrates these



Impact Point Error Geometry [Ref. 10]

Figure 2

errors which may be classified into systematic errors, time varying errors, and round-to-round dispersion errors. Systematic errors are those errors which may be considered constant for the duration of an engagement, but can vary from engagement to engagement. Time varying errors are errors which vary significantly during the engagement, but whose rate of variation is slow relative to the firing rate [Ref. 10]. Most models combine systematic and time varying errors into the general category of aiming errors. Round-to-round errors are often referred to in the literature as ballistic errors which are attributed to the ballistics of the projectile and which vary in an uncorrelated manner. Bias is a term used when describing errors inherent to a system and can result from any of the three error sources. Because bias has an explicit meaning within the framework of an individual model, the term will be individually defined whenever it is used.

The distribution of aim points may result from one or a combination of factors. Typical factors affecting the distribution of aim points are: (a) fire control errors; (b) an unknown target location; (c) uncontrolled wander due to the instability of a moving weapons carrier; (d) and flight roughness in the case of an aircraft target. The aim point may be considered as being fixed or as is the usual case, it may be considered as being normally distributed about the target center. For the two dimensional case, the aim points are most often assumed to have a bivariate normal distribution about the center of the target.

The distribution of rounds about the aim point (ballistic dispersion) is random and can be attributed to round imperfections or to meteorological conditions. For example, the interaction of temperature on propellents and air density causes uneven force vectors to operate on the projectile and thereby causes dispersion of the rounds about the aim point. The distribution most often used in describing the round-to-round dispersion is the normal distribution. For a large number of systems the normal distribution assumption has been verified by empirical data.

Many projectiles inflict damage to a target without having to impact the target directly. The blast or explosion effect of a projectile or schrapnel from the projectile may cause damage to the target if the target is within a lethal distance from the burst point of the projectile. A bullet, however, must impact the target to cause any degree of damage to the target. Lethality is the term used to describe the degree of damage a projectile inflicts on a given target. The lethality function is a function which relates projectile-target pairs in terms of the distance from the impact point to the projectile. The lethality function is the conditional probability of a target kill given a specific target is located at point P and the projectile impacts at point P-1. The "cookie-cutter" or "zero-one" lethality function is used to describe the situation where if the round impacts the target or is within a specified distance from the target at impact a kill is obtained with probability one; otherwise, the target does not sustain any damage. The exponential function is often used to model the lethality function of schrapnel producing projectiles.

The correlation coefficient is a number between plus and minus one that indicates the effect of the impact of projectile 'i' on the impact of projectile 'j', where 'i' and 'j' indicate individual projectiles. For independent rounds the correlation coefficient is zero which implies that projectile 'i' has no effect upon the impact of projectile 'j'. Similarly, perfect correlation of plus one means that round 'j' will have the same impact point as round 'i'. The introduction of the correlation coefficient into a mathematical model of hit probability is difficult because of the complexity of the resulting function. For certain specific models, it can be shown that if an 'n' round burst is fired, an increase in the correlation coefficient reduces the probability of a hit [Ref. 25]. This can be interpreted to mean that if the 'n' round burst has a fixed aiming error, an increasing number of rounds would miss the target as the correlation coefficient was increased from zero to one.

Models are developed which describe single shot and multiple shot hit and kill probabilities. The meaning of a single shot event is explicit in itself. A multiple shot event can be described in two ways. First, such an event is characterized by 'n' rounds being fired with one or more aim points. The rounds may or may not be independently aimed; however, the rounds must follow one another in sequence ($i = 1,2,\ldots,n$) over a period of time Δt. A salvo is the second type of multiple round event that requires 'n' rounds to be fired simultaneously at the same target. A salvo of 'n' projectiles may come from one weapon or may consist of one projectile fired from each of 'n' weapons. For either event the target must be the same for every round of the salvo. The salvo assumption of a single weapon firing 'n' rounds requires an extremely high rate of fire so that the rounds may be assumed to be fired simultaneously. A field artillery battery of six guns firing simultaneously at the same target exemplifies the salvo fire principle.

Problems in the development of models relating hit or kill probabilities have two basic conceptual approaches - the coverage problem and the vunerable area problem. The coverage problem is applicable when the exact location of the target is unknown; but it is known that the target is within a specified area. The weapons system directed against the target has a lethality function such that if the target is within a distance R from the impact point, the target will be damaged [Ref. 15]. This method has been used extensively in analyzing the effectiveness of a nuclear warhead against a target. The vunerable area problem describes the probability of kill in terms of damage to one or more vunerable components. The target is reduced to one or more vunerable areas (volumes for 3-dimensional models) which if hit, will cause a target kill. The vunerable area is always less than the presented target area. If the target has several vunerable areas, it is often possible to sum over these areas to get an average vunerable area. A typical example of such a target is an aircraft which is composed of several vunerable components, one of them being the pilot. The aircraft is modeled as a vunerable area that is much smaller than the presented area of the aircraft. If masking of one area by another occurs, then the problem of target representation becomes more difficult. For the two dimensional case, an average vunerable area of the target is projected onto a plane perpendicular to the trajectory of the oncoming projectile.

The description of the target in terms of vunerable components while being an important element of many models requires a detailed analysis of numerous target types with respect to specific types of fragments and blast effects. Ballistic Research Laboratories has conducted extensive testing to produce empirical data which forms the basis for vunerability input data [Ref. 7]. While the importance of such work is not meant to be deemphasized in this paper, the data is more relevant to the design of the warhead of a projectile than to the weapons system as a whole. The system analyist most often utilizes the vunerability data after it has been reduced to a vunerable area.

III. SUMMARY OF RESEARCH EFFORTS

To provide a reference guide to the numerous hit and kill probability models which exist within the current literature, the literature search matrix, figure 3, has been designed. While the author does not claim the references listed are a complete listing of all available material on the subject, those references listed will reduce the set of models the researcher needs to initially investigate when specific model characteristics are known. Many of the references listed include extensive bibliographies which will amplify the set of source material available.

The consolidation and categorization of documented material concerning subject matter has not been accomplished, or if accomplished, is unknown to the author. It was felt that a consolidation of the widely scattered material must provide a categorization of the models if it were to be of any value. The literature search matrix is the vehicle by which the categorization is accomplished. The matrix lists six major categories or descriptors of hit and kill probability models. By listing the descriptors, a set of applicable models can quickly be located by the researcher. The six categories are: (a) dimensionality of the model; (b) the number of rounds considered; (c) correlation of rounds; (d) lethality function; (e) distribution of aim points; (f) and the distribution of the rounds about the aim point. The references listed are those found in the bibliography of this paper. To use the matrix it is only necessary to locate the characteristic or set of characteristics which best describe the problem being investigated. The applicable reference or set of references are then easily identified.

REFERENCES	DIMENSION			NUMBER OF ROUNDS			ROUNDS CORRELATED		LETHALITY FUNCTION		DISTRIBUTION AIM POINT	
	1	2	3	single shot	n round	salvo	yes	no	zero - one	other	fixed	other
1		X	X	X	X			X	X			X
2		X				X		X		X	X	
3		X				X		X		X	X	
6	X				X		X		X		X	
7		X	X	X	X	X		X	X	X	X	X
8		X				X	X					X
9					X			X		X		X
10		X		X	X		X					X
11		X					X	X	X			
12		X				X		X	X			X
13		X	X	X				X	X		X	X
14		X				X		X		X		X
15		X	X	X				X	X	X	X	X
16	X	X	X	X	X	X	X	X	X	X	X	X
17		X	X	X				X	X		X	X
18		X	X	X				X	X	X		X
19		X		X	X			X	X	X		X
20		X		X				X	X	X	X	X
21		X		X	X	X			X	X		X
22		X	X			X		X	X			X
23		X				X	X	X		X		X
26		X			X			X	X	X		X
27		X			X			X	X		X	
28		X				X		X		X		X
29	X					X		X		X		X

The distribution or rounds about the aim point is assumed normal in all models except those in Ref. 7.

Figure 3

Throughout the literature there are several basic documents which are used continually as source documents for various papers. Reference 16, COLLECTION OF ARTICLES ON THE THEORY OF FIRING I, appears to be the foundation of many of the dispersion optimizing models. The document is a translation of a Russian work edited by Kolmogorov in 1948. The concepts presented in his document have been studied extensively in this country and have been applied when modeling non-nuclear projectiles. Two basic documents relating to the coverage problem are Refs. 15 and 24 which are current through 1963. Reference 15 is an extremely good review of the coverage problem as the authors present specific examples of coverage problems and the models used in analyzing the problems; whereas, Ref. 24 is a bibliography of coverage problems. Grubbs [Ref. 13] provides basic documentation for offset kill probabilities and discusses related works in his paper which was published in 1962. These documents are recommended for the individual who is doing initial work in the modeling of hit and kill probabilities and is desirous of investigating the basic theory for such models.

IV. RELATED MODELS

A. GENERAL

In this section of the paper models are presented which describe the probability of a target hit or kill assuming that a successful launch of the projectile has been accomplished. The models are categorized by the number of rounds represented within the model - single, multiple 'n' round, and multiple round salvo fire. Such a categorization minimizes the overlap between models and provides the researcher with a ready reference for a specific model. If a model can readily be extended to another category, it will be so indicated.

The most general model for hit probability is one which is equivalent to having no system bias or to having a fixed aim point. Let $F(\bar{r})$ be the probability density function describing the distribution of impact points of the projectiles, where $\bar{r}$ is a symbol for the coordinate system representing the location of the impact points. If 'A' denotes the area of the target, then the single shot hit probability P_h can be written

$$P_h = \int_A f(\bar{r})\, d\bar{r}. \tag{4-1}$$

To define the probability of a target kill, a function $p(\bar{r})$ is introduced. The function $p(\bar{r})$ is the probability that a target is killed given the projectile impacts or bursts at $\bar{r}$. The probability that a projectile kills the target is given by

$$P_k = \int p(\bar{r})\, f(\bar{r})\, d\bar{r}, \tag{4-2}$$

where the integration is taken over the entire burst region [Ref. 7].

The distribution of impact points is most often assumed to be normal or multivariate normal. For the individual desiring a concise review of the bivariate normal distribution, it is recommended that Ref. 4 be studied. The mathematical development of the bivariate normal distribution was given with special attention being focused upon the circular normal distribution.

B. SINGLE SHOT MODELS

McNolty has written three papers describing single shot kill probability models in which the functions of the weapon lethality, weapon bias, and target location are investigated. The lethality function of the projectile was modeled in Ref. 18. Expressions for the single shot kill probability are derived when the concept of a lethal circle of radius R (kill or no kill) cannot adequately represent the system being modeled. He defined four functions for the weapon lethality (conditional kill probability, $P_c(r)$) whose curves are monotonically decreasing from 1 to 0, depending upon the distance 'r' of the burst from the target. The specific lethality functions investigated were

$$P_c(r) = \begin{cases} 1 & 0 \leq r \leq R, \\ 0 & r > R, \quad \text{the zero-one function,} \end{cases} \tag{4-3}$$

$$P_c(r) = \exp(-r^2/2\,b^2), \quad (0 \leq r \leq \infty) \tag{4-4}$$

$$P_c(r) = \exp(-b/r), \quad (0 \leq r \leq \infty,\ b > 0) \tag{4-5}$$

$$P_c(r) = \begin{cases} b^2 - r^2/b, & (0 \leq r \leq b,\ b > 0) \\ 0, & (r > b) \end{cases} \tag{4-6}$$

$$P_c(r) = \begin{cases} 1 - r/b, & (0 \leq r \leq b,\ b > 0) \\ 0. & (r > b) \end{cases} \tag{4-7}$$

McNolty presented derivations of formulas for the probability of killing a point target and for the expected coverage of a population of targets when the bias of the weapon system was randomly distributed in accordance with a prescribed density function [Ref. 17]. He defined weapon bias as the offset distance from the aim point of the center of gravity of the weapon impact points. McNolty considered six cases in two and three dimensions in which the bias distributions were gamma, beta, Maxwell-Boltzmann, and Raleigh. Such models are useful if a sampling of the weapons system indicates the bias is randomly distributed.

The density of the target was the third basic variable studied by McNolty [Ref. 20]. He derived expressions for the probability of killing a randomly located point target and for the expected coverage of an area target of variable density. In this paper McNolty combined his two previous works concerning randomly distributed bias and lethal effects with the concept of a non-uniform target density. Two specific examples were provided which give insight into the use of his models.

Grubbs provided a useful analytic procedure for computing the circular and noncircular offset probabilities of hitting [Ref. 13]. He approximated the probabilities of hitting by the use of a central chi-square distribution with a fractional number of degrees of freedom or a transformation to approximate normality. The paper is significant in the field of offset probabilities of hitting in that he discussed related works and also provided six example problems using the techniques he presented. His work can be extended to kill probability models by incorporating lethality or vunerability data.

A review of the literature concerning a class of coverage problems was prepared by Guenther and Terrangrio in 1963 [Ref. 15]. The problem of killing a point target by a weapons system with a designated killing radius was discussed in detail. The paper was divided into three sections. The first section was devoted to probability content problems in which a spherical region C and a point, $B = (b_1, b_2, \ldots b_n)$, would be captured by a sphere of radius R whenever 'X' fell within or on

$$\Sigma_{i=1}^{i=n} (x_i - b_i)^2 = R^2. \qquad (4\text{-}8)$$

Models using the zero one damage function where the target was not a point target were discussed in section two of the paper. Section three was an extension of sections 1 and 2 where damage functions other than the zero one function were considered. This paper not only provided modeling techniques, but it also had an extensive bibliography which would be useful to the researcher.

C. MULTIPLE N ROUND MODELS

When the weapon system analyist must investigate the hit or kill probability associated with multiple rounds, mathematical simplicity is not usually ameanible to the problem. Except for very specific values for the model variables, the mathematical expressions for the probabilities often require digital treatment. McNolty derived integral expressions for the probabilities for multiple rounds when the target was randomly located according to an offset circular normal distribution and remained in its unknown positions throughout 'n' independent tosses (shots) of a lethal circle. In his paper, McNolty presented integral expressions for the probability of: (1) killing the target at least once in 'n' tosses (shots) of the lethal circle; (2) killing the target exactly 'n' times in 'N' tosses; (3) requiring less than or equal to 'm' shots to kill the target exactly once; (4) killing the target at least once in 'n' tosses when the bias (offset distance) was randomly distributed. It is interesting to note that McNolty derived an integral expression for killing the target at least once in 'n' tosses (shots) P_k^1, which was not equal to

$$\left[1 - (1 - P_k)^n\right] \tag{4-9}$$

where P_k was the single shot kill probability. His formulation was always

$$P_k^1 \leq 1 - (1 - P_k)^n, \tag{4-10}$$

with equality holding only when the target remained in its unknown (fixed) location throughout the 'n' shots of the lethal circle. This required integration over the entire X,Y plane to account for all possible target locations. McNolty also presented the distribution for the offset distance 'r' (system bias)

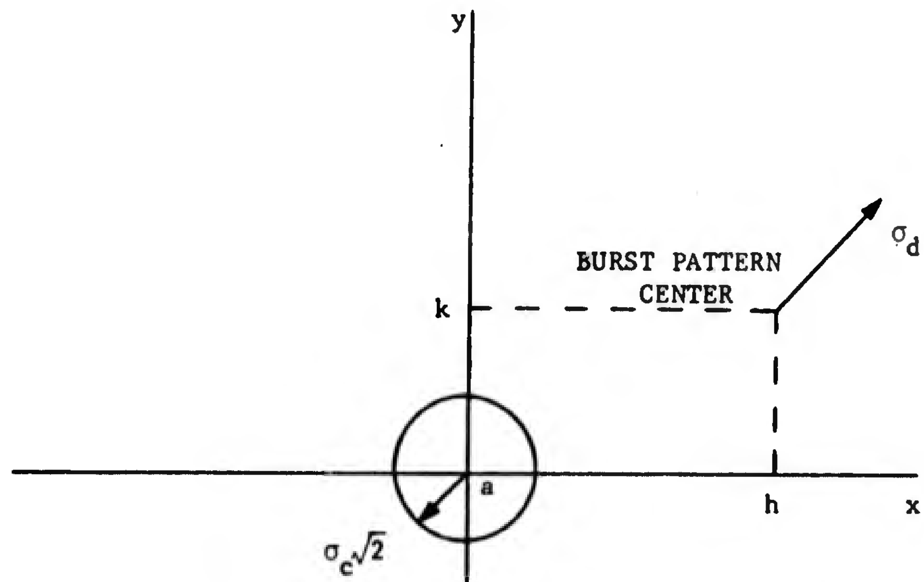
$$\begin{aligned} g(r)dr = {} & [2/\Gamma(\lambda)](\lambda/\alpha)^{\lambda}\ r^{2\lambda-1} \\ & \cdot \exp\ (-\lambda/\alpha r^2)\ dr, \end{aligned} \tag{4-11}$$

where $E(r) = \sqrt{\alpha/\lambda}\cdot[\Gamma(\lambda+1/2)/\Gamma(\lambda)]$, and $E(r^2) = \alpha$, which included the Raleigh, Maxwell-Boltzmann, and one sided Gaussian distributions as special cases. (In most cases for random bias application λ falls in the range $1/2 \leq \lambda \leq 3$ [Ref. 19].)

The diffused target concept was presented by Thompson for multiple shot kill probabilities [Ref. 27]. The technique is applicable as an event becomes increasingly likely with decreasing miss distance. The basic assumption for the concept was that the radial normal distribution was valid for the miss distance and the weapon bias, if it existed, was fixed. A more complete development of the concept was found in Ref. 1, where the problem was graphically presented and the detailed mathematical model was developed Figure 4 shows the most general two dimensional model for the diffused target concept. It can be shown that for systems having a fixed bias and a fixed target location, the single shot kill probability can be increased by increasing the bias up to a specific value. After the bias exceeds the lethal radius of the target, however, the kill probability drops off rapidly [Refs 1, 25].

Breaux investigated the diffused target concept and showed that the series solution in one case reduced to the incomplete beta function and for a second problem, he derived a new series solution [Ref. 9]. Breaux included in his presentation a discussion of computational considerations as the number of shots increased. It was found that the incomplete beta function could be used if the number of rounds fired was less than 50; whereas, the series Breaux developed was valid for up to 1,000 rounds.

The extension of a multiple shot model into an engagement kill model was done by Scheu in a paper investigating aircraft attrition by gun systems [Ref. 26]. In his development, he assumed that the single shot kill probability was a function of slant range, and then he averaged the single shot kill probability over the aircraft pass through the killing zone of the weapon. The engagement kill probability was developed to include the effect of terrain masking. The mathematical models were validated with empirical data. The validation of the model was not surprising since the empirical data was initially used in the assignment of specific values to constant factors which were part of the mathematical model. The target was represented by an average vunerable area and the engagement kill probability was



$$(Pr)_{ssk} = \frac{\sigma_c^2}{\sigma_c^2 + \sigma_d^2} \exp\left[-\frac{h^2+k^2}{2(\sigma_c^2 + \sigma_d^2)}\right]$$

$(Pr)_{ssk}$ is single-shot kill probability

h, k are bias components

σ_c is vunerable radius

σ_d is standard deviation of burst pattern dispersion

a is vunerable area $= 2\pi\, \sigma_c^2$

Two Dimensional Fixed Bias Burst Pattern

Figure 4

$$EXP = 1 - \exp(-\bar{Mp}), \tag{4-12}$$

where $\bar{Mp}$ was a function of the number of rounds fired, terrain, aircraft velocity, single shot kill probability, and the slant range between the gun and the aircraft.

The models presented thus far have all assumed that the rounds were independent. Peterson developed a method for finding the conditional hit and miss probabilities on a rectangular target given a hit or a miss on the previous shot when part of the random error was common to both shots but varied from engagement to engagement. The correlation coefficient depended upon the relative size of the two elements of the total dispersion. He provided two methods of achieving computational as well as tabular results [Ref. 6].

The effect of correlation between rounds was modeled by Helgert where consideration was also made for time varying errors [Ref. 10]. Although a detailed development of the model was not presented, the model was used in a detailed example with empirical data. Graphs and tables of cumulative hit probabilities were provided which showed the effect of a change in the error statistics for a given number of rounds fired (10 to 200). The target was assumed to be nearly circular or square, no aim point adjustment during the engagement, and equal error statistics in both coordinates. The error statistics considered were - target area, number of rounds fired, standard deviation of round-to-round dispersion errors, standard deviation of systematic errors, standard deviation of time varying errors, and the "average correlation between rounds".

D. MULTIPLE ROUND SALVO MODELS

Weapons systems which deliver a salvo of rounds to a target require a unique solution process to solve the kill probability problem. The most general assumption regarding the salvo model is to assume that the salvo aim point is distributed around the target according to a bivariate normal distribution with the individual rounds being subject to the bivariate normal distribution about the center of impact. Using these two basic assumptions, Rice and Bottero developed detailed kill probability formulae which can be conveniently used for hand computation [Ref. 22]. The weapon lethality was specified through the use of a near miss zone surrounding the two and three dimensional target and a conditional kill probability. The most general model in the work included the target orientation as a variable. Because of the clarity of the material and the logical presentation of the formula derivation, the model will be discussed in detail as an excellent example of the approach to salvo kill probability problems. The general derivation was:

a. Develop an expression for the probability that a single round of the salvo will impact within the target [p(x,y)].

b. Using $P_{hk} \cdot p(x,y)$ as the probability that the round will kill the target where P_{hk} is the conditional probability of a kill given a target hit, the probability that the target will survive the round is $1 - P_{hk} \cdot p(x,y)$. The probability that the target survives all 'n' rounds of the salvo is then $[1 - P_{hk} \cdot p(x,y)]^n$.

c. The probability of success A(x,y) can then be written as $1 - [1 - P_{hk} \cdot p(x,y)]^n$ which can be interpreted as the probability that at least one round of the salvo with aim point (x,y) will produce the desired level of damage.

d. The probability $P_1(n)$ that at least one damaging hit will be made by a salvo of 'n' rounds with intended aim point (x',y') is derived by averaging A(x,y) over all possible x and y.

e. The probability $P_q(n)$ of at least q damaging hits in a salvo of 'n' rounds can also be developed.

A computer program has been written for three dimensional targets which had the orientation angle assumed to be zero. The integrals were evaluated by using a Hasting approximation formula and Wedde's rule.

A comparative study of three mathematical models used in estimating salvo hit probabilities was conducted by Kline and Fergerson [Ref. 23]. The first model assumed that the landing position of the individual rounds of the salvo were independently and identically distributed. The second model assumed that the salvo of weapons was dispersed in a well defined rectangular array. The third model investigated assumed that the weapons which formed the salvo were randomly distributed within a randomly distributed circular region where the center of the circle was determined by a random selection from a circular normal distribution. The significant difference between the three models was the assumed statistical dependence between the landing positions of the various weapons of the salvo. The landing position of each weapon was assumed to be a circular normal distribution function in each of the three models. Strict statistical dependence between weapons was exemplified by model two with strict independence evident in model one. Model three represented a mixture of the two extremes. The paper graphically depicted the fact that the hit probabilities determined were directly affected by the dependence criteria (model) selected.

Banish developed a model for salvo kill probability using matrix algebra [Ref. 8]. His model took into consideration the time variational error within the system. His development assumed a normal distribution of the aim point and a normal distribution of the rounds about the aim position with a correlation between rounds being present. Banish's study was unique in that he utilized matrix algebra in the development of his model.

An expansion of the work done by Kolmogorov [Ref. 16] was accomplished by Walsh in his study of salvo kill probability models [Refs. 28, 29]. Walsh developed a general expression of the salvo kill probabilities which could be approximated using an explicit function of four parameters. The four parameters were the number of rounds 'n', the average target vunerability times the target size '$\bar{V}T$', the salvo aiming error dispersion 'σ_A', and a function of the round dispersion 'σ_R' and two dispersion type terms which depended on the target size, shape, and vunerability 'σ_{TX}^2, σ_{TY}^2'. The probability that exactly 'm' rounds of the salvo resulted in target kills could therefore be expressed as

$$P(m) = \Sigma_{i=0}^{i=n-m} \frac{[(-1)^i(\bar{V}T)^{m+1}n!]/[i!m!(n-m-i)!(2\pi)^{m+i}]}{[(m+i)\sigma_A^2 + \sqrt{(\sigma_R^2+\sigma_{TX}^2)(\sigma_R^2+\sigma_{TY}^2)}][(\sigma_R^2+\sigma_{TX}^2)(\sigma_R^2+\sigma_{TY}^2)]^{1/2(m+i-1)}} \quad (4\text{-}13)$$

The terms σ_{TX}^2 and σ_{TY}^2 were dependent upon the target size, shape, and vunerability such that

$$\sigma_{TX}^2 = 1/\bar{V}T \iint_T u^2 V(u,v)du\, dv \quad \text{and,} \quad (4\text{-}14)$$

$$\sigma^2_{TY} = 1/\bar{V}T \iint_T v^2 V(u,v)du\, dv, \qquad (4\text{-}15)$$

where V(u,v) was the conditional probability of a target kill given the round hits the target at (u,v) and $\bar{V}$ was the average target vunerability [Ref. 28]. Walsh also presented a procedure for evaluating the maximum salvo kill probability obtainable and for determining the optimum round hit-location-probability distribution that yields that maximum value. The method of approach to the problem of maximizing the salvo kill probability required the application of the calculus of variations. The determination of the approximately optimum form of the probability density function necessitated an iterative 'cut-and-try' method. The maximum value of the salvo kill probability $\bar{P}_k$ was

$$\bar{P}_k = \int \cdots \int \left\{ 1 - [1-\bar{p}(x)]^n \right\} A(x)dx_1 \cdots dx_d, \qquad (4\text{-}16)$$

where A(x) was the density function for the probability distribution of the aiming error x and $\bar{p}(x)$ was the optimal form of the conditional probability that a round with expected hit location (x) had enough individual effect to kill the target. The method developed for evaluating $\bar{P}_k$ and $\bar{p}(x)$, $(n > 1)$, was to first determine the value of C from

$$\int \cdots \int \left\{ 1 - [C/A(x)]^{1/(n-1)} \right\} dx_1 \cdots dx_d = \bar{V}T \qquad (4\text{-}17)$$

where $\bar{V}$ was the average target vunerability and T was the size of the target (area for d = 2, volume for d = 3). The integration was over values of x such that $A(x) > C$. The maximum salvo kill probability was expressed as

$$\bar{P}_k = 1 - C \left\{ [\text{size of x-region for which } A(x) \geq C] - \bar{V}T \right\} - \int \cdots \int_{A(x) < C} A(x) dx_1 \cdots dx_d, \quad (4\text{-}18)$$

$$\bar{p}(x) = \begin{cases} 1 - [C/A(x)]^{1/(n-1)} & x \text{ such that } A(x) \geq C \\ 0 & \text{otherwise.} \end{cases} \quad (4\text{-}19)$$

Walsh presented a numerical example to illustrate the 'cut-and-try' method for determining the optimum form for the probability density function of the round hit locations [Ref. 29].

A computational procedure for estimating the salvo hit probabilities for offset circular targets was developed by Groves and Smith in 1956 [Ref. 12]. The approach taken by the authors was to express the distribution of the impact (burst point) of the missiles in terms of the distance from the center of the target rather than from the center of impact of the salvo when the center of impact of the missiles was offset from the center of the target. Sample problems were illustrative of the proposed methodology and graphs were utilized to aid in the solution procedure.

Grubbs expanded his earlier work on circular and noncircular offset probabilities to include the development of an expression for the expected fraction of damage to a target [Ref. 14]. In general the average fraction of damage over a circular target was

$$\bar{f}(n) = \iint_{u^2 + v^2 \leq R_T} h(u,v)\, J(u,v)\, du\, dv, \quad (4\text{-}20)$$

where h(u,v) was the distribution of the target elements over a circular target of radius R_T whose center was at the origin and

J(u,v) was the probability that at least one of the 'n' rounds of the salvo would damage a target element at (u,v). The final result after several approximations was

$$\bar{f}(n) \cong \left[2\sqrt{(\sigma_{kx}^2+\sigma_x^2)(\sigma_{ky}^2+\sigma_y^2)}/R_T^2\right] \Sigma_{i=1}^{i=n} \binom{n}{i} (-1)^{i+1} \cdot 1/i$$
$$\cdot \left\{\sigma_{kx}^2\sigma_{ky}^2/\sqrt{(\sigma_{kx}^2+\sigma_x^2)(\sigma_{ky}^2+\sigma_y^2)}\right\}^i$$
$$\cdot \left\{1-\exp\left[-i\ R_T^2/2\ (\sigma_{kx}^2+\sigma_x^2+i\sigma_u^2)\right]\right\}^{1/2} \qquad (4\text{-}21)$$
$$\cdot \left\{1-\exp\left[-i\ R_T^2/2\ (\sigma_{ky}^2+\sigma_y^2+i\sigma_v^2)\right]\right\}^{1/2}$$

where σ_{kx}^2 and σ_{ky}^2 were the standard deviations of the noncircular damage function. The above formula was a general formula for the noncircular case and uniform target density. A computational procedure for the binomial coefficients using Jacobi Polynomials was developed by Breaux and Mohler which simplified the evaluation for the approximation [Ref. 3].

The development of an approximate formula for salvo kill probabilities by Walsh [Ref. 28] can be contrasted to the expected fractional damage approximation by Grubbs [Ref. 14]. Walsh used a general lethality function and the concept of target variances combined with the idea that the round to round variability strongly dominated the target variances. Circular normal delivery errors were also a basic assumption. Grubbs, however, used a lethality function that was treated as an elliptical normal fall-off type pattern. Noncircular delivery errors could easily be included in the theory developed by Grubbs.

For the analyist not having access to a high speed computer, Groves developed a method for the hand computation of the expected fractional kill 'F' for an elliptical target by a number of rounds 'n' all of which have the same lethal area 'A' and are all aimed at the same point $(\bar{x},\bar{y})$ relative to the target center [Ref. 5]. By letting a and b represent the semi-axis of the target with σ_x and σ_y being the delivery error standard deviations, the expected fractional kill could be denoted

$$F(a,b,x,y,\sigma_x,\sigma_y,A,n) = \frac{A}{2\pi ab}\sum_{i=1}^{i=n}\binom{n}{i}\left(\frac{-A}{2\pi\alpha\beta}\right)^{i-1}\left(\frac{1}{i}\right)\cdot P_h\left(\sqrt{i},\frac{\alpha}{a},\frac{\beta}{b},\frac{\sqrt{i}\,\bar{x}}{a},\frac{\sqrt{i}\,\bar{y}}{b}\right),$$

$$\text{where } \alpha = \sqrt{\frac{A}{2\pi}+\sigma_x^2},\ \beta = \sqrt{\frac{A}{2\pi}+\sigma_y^2};\ \text{and} \qquad (4\text{-}22)$$

P_h is a function that gives the probability of hitting a circular target of radius $\sqrt{i}$ centered at the origin of a coordinate system given a bivariate normal distribution of hits with mean $\frac{\bar{x}\sqrt{i}}{a}$ and standard deviation $\frac{\alpha}{a}$ in one direction; and mean $\frac{\bar{y}\sqrt{i}}{b}$ and standard deviation $\frac{\beta}{b}$ in the other direction. The values for $P_h\left(\sqrt{i},\frac{\alpha}{a},\frac{\beta}{b},\frac{\bar{x}\sqrt{i}}{a},\frac{\bar{y}\sqrt{i}}{b}\right)$ must be determined using approximation formulae or tabular values. Groves referenced both formulae and tables which may be used by the analyist and included tabular values for P_h for the special case when $\bar{x} = \bar{y} = 0$.

Because of the difficulty involved in using the series developed by Groves for the expected fractional kill of a circular target when the number of rounds 'n' is large, Breaux proposed an alternate series

$$\bar{f}(n) = C\sum_{i=1}^{i=n}\left[(1-xe^{-\lambda})^i - (1-x)^i\right]/i. \qquad (4\text{-}23)$$

The terms of the series decrease monotonically with the summation index i and were therefore readily summable [Ref. 2].

The models presented in this section are representative of the approaches which have been taken in hit and kill probability models. The models represent coverage problems, integral approximations, diffused target concepts, and techniques of numerical approximations. The relevancy of a specific model will be dependent upon the characteristics of the weapon system being analyzed. It was therefore felt that the detailed examination of each model was unnecessary and that the generalized presentation would give an insight into the major techniques and assumptions associated with hit and kill probability models.

V. AREAS FOR FUTURE RESEARCH

Areas for future research in hit and kill probability modeling are open both in the theoretical development of the models and in the application of existing models for the evaluation of specific weapons systems. Specific recommendations for future study include:

a. The optimization of the distribution of rounds about the aim point for salvo fire. Such a study should include the introduction of artificial dispersion.

b. The effect of time varying errors upon the kill probability by considering the relative movement between the target and the projectile.

c. The effect of round-to-round correlation upon the hit or kill probability for multiple round models.

d. The dynamic modeling of hit and kill probabilities when the probabilities are a function of time or range from the weapon to the target.

e. The development of a dynamic model that reflects the effect of varying the rate of fire of the weapons system upon hit or kill probabilities.

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13. ABSTRACT

Literature concerning the theory of hit and kill probabilities is readily available but is widely scattered. The paper presents a consolidation and a categorization of current literature and provides a general discussion of representative models in the field of hit and kill probabilities. A literature research matrix is presented to aid the researcher in locating existing models which meet his requirements.

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