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FINITE-DIMENSIONAL TERNARY ALGEBRAS

by

Howard Barry Ritea

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Mathematics at the University of California, Los Angeles

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TABLE OF CONTENTS

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ABSTRACT OF THE DISSERTATION

Finite-Dimensional Ternary Algebras

by

Howard Barry Ritea

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 1967

Professor Magnus R. Hestenes, Chairman

A ternary algebra furnishes a convenient structure within which rectangular matrices may be studied. If m and n represent positive integers, the system of all $m \times n$ matrices over the field of complex numbers is an example of a finite-dimensional ternary algebra. The present paper is devoted to a study of the algebraic structure of ternary algebras. In particular, it is shown that an arbitrary finite-dimensional ternary algebra over the complex numbers has a representation as a ternary algebra of rectangular matrices.

The aforementioned characterization is obtained by making use of an ideal theory in a ternary algebra. In addition to the concepts of a right and left ideal, a third type of subspace, called

a central ideal, is introduced and studied. An analysis is then made of minimal ideals. With the proper assumptions, a minimal ideal has a characterization as an inner-product space.

It is then shown that any finite-dimensional ternary algebra admits of a decomposition into a direct sum of minimal right ideals having certain orthogonality properties with one another. Moreover, a corresponding decomposition holds for minimal left ideals. Both direct sums then allow the algebra to be represented as a direct sum of central ideals, each having either dimension zero or dimension one.

From each one-dimensional central ideal in the above decomposition, a non-zero element is chosen and appropriately normalized. This provides a generalized orthonormal basis for the algebra. Every element then has a matrix representation with respect to this basis, and an additive and multiplicative isomorphism is established between the original ternary algebra and a ternary algebra of matrices.

The implications of the absence of the axiom called the positivity condition are then given for a finite-dimensional ternary algebra.

This is followed by a discussion and analysis of minimal central ideals in a finite-dimensional ternary algebra when the scalars are assumed to be real. The results obtained are used to find a characterization of such an algebra.

The paper concludes with applications of several of the results to ternary algebras of matrices and circulants. In particular, a characterization is found for an orthonormal basis of a matric algebra.

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1. INTRODUCTION

The concept of a ternary algebra was introduced and studied several years ago by M. R. Hestenes [1]. It was shown at the time that such an algebra provides a natural setting in which to study rectangular matrices. For positive integers m and n, the class of all $m \times n$ complex matrices is an example of a finite-dimensional ternary algebra.

In this paper we are concerned with a study of the algebraic structure of ternary algebras. Among other things, it is shown that a finite-dimensional complex ternary algebra $\mathfrak{A}$ can be characterized as a ternary algebra of rectangular matrices.

The concept of an ideal is of central importance to the theory. This theory does not require finite-dimensionality. Three types of ideals are introduced and studied in a ternary algebra, the first two being right and left ideals. The ternary composition leads to the definition of a third type of subspace, called a central ideal.

Following a discussion of some basic properties of right, left and central ideals, attention is concentrated on minimal ideals. It is shown that under mild restrictions, which are always satisfied in the finite-dimensional case, a minimal ideal is an inner-product space. An arbitrary finite-dimensional ternary algebra is then decomposed into a direct sum of minimal right ideals, and then into a direct sum of minimal left ideals. The decomposition is effected in such a way that the minimal right ideals possess certain orthogonality properties with one another, with a similar statement holding for the decomposition into minimal left ideals. Such decompositions are shown to be unique to within an isomorphism. Both direct sums are then combined to give a decomposition of the ternary algebra into central ideals, possessing the orthogonality properties of both the minimal right ideals and the minimal left ideals. Moreover, each of these central ideals has either dimension zero or dimension one. A non-zero element is then chosen from each of the one-dimensional central ideals, and after suitable normalization, a generalized orthonormal basis is obtained. Each element in the algebra then has a matrix representation relative to this basis, and the algebra is then shown to be isomorphic to a ternary algebra of matrices. The results are analogous to those of the Wedderburn theory for binary algebras.

A discussion of the axiom known as the positivity condition is given for a ternary algebra with special emphasis on the finite-dimensional case.

Attention is then turned to the theory of a real finite-dimensional ternary algebra, and a representation is achieved using the properties of minimal central ideals in such a system. Here it is shown that a minimal central ideal is of dimension one, two or four.

Finally, a study is made of matric algebras and circulants and a characterization is obtained for an orthonormal basis of a matric algebra.

The paper is divided into fourteen sections. The first two provide an introduction and a summary of several of the concepts discussed in [1]. In Section 3 the notion of an ideal in a ternary algebra is introduced, and the following three sections contain various characterizations of minimal ideals. In particular, it is shown that a minimal ideal is an inner-product space. Several decompositions of a finite-dimensional ternary algebra into direct sums of minimal ideals are given in Section 7, and uniqueness proofs appear in Section 8. These decompositions allow the construction of a generalized orthonormal basis in Section 9. In the following section, such a basis is used to obtain a representation of a finite-dimensional ternary algebra as an algebra of matrices. The positivity condition is discussed in Section 11. A characterization of a real finite-dimensional ternary algebra is given in the succeeding section. The last two sections contain applications to matrices and circulants.

2. FUNDAMENTAL CONCEPTS

Several definitions and concepts which will be of immediate use are given here. Others will be introduced as they are needed. Two types of ternary algebras will now be defined.

A generalized ternary algebra over the set $\mathbb{C}$ of all complex numbers is a linear space $\mathfrak{A}$ over $\mathbb{C}$ such that to any three elements A,B,C of $\mathfrak{A}$ there corresponds a unique element AB^*C in $\mathfrak{A}$, subject to the following four conditions:

1. $AA^*A = 0$ if and only if $A = 0$.
2. If $A,B,C,D,E \in \mathfrak{A}$, then
$$(AB^*C)D^*E = A(DC^*B)^*E = AB^*(CD^*E) .$$
3. If $\lambda \in \mathbb{C}$, then
$$(\lambda A)B^*C = \lambda(AB^*C) = AB^*(\lambda C) .$$
4. For any elements A,B,C,D in $\mathfrak{A}$ we have
$$(A + B)C^*D = AC^*D + BC^*D ,$$
$$DC^*(A + B) = DC^*A + DC^*B .$$

Condition (2) implies that we may use the symbol AB^*CD^*E to denote the element

$$AB^*CD^*E = (AB^*C)D^*E = AB^*(CD^*E) .$$

This generalized ternary algebra is the algebra considered by Hestenes in [1]. For our purposes, however, a more restrictive type of algebra will be considered, known as a ternary algebra.

By a ternary algebra will be meant a generalized ternary algebra subject to the following additional conditions:

5. If A is a non-zero element of $\mathcal{A}$ such that $AA^*A = \lambda A$ for some complex number λ, then $\lambda > 0$.

6. $A(B + C)^*D = AB^*D + AC^*D$ for all A,B,C,D in $\mathcal{A}$.

7. If $A,B,C \in \mathcal{A}$ and λ is a complex number, then

$$A(\lambda B)^*C = \bar{\lambda}(AB^*C) ,$$

where $\bar{\lambda}$ denotes the complex conjugate of λ.

In the case when all the scalars λ are real, $\mathcal{A}$ will be called a real ternary algebra or a real generalized ternary algebra if conditions (5),(6) and (7) are not fulfilled.

Suppose that $\mathcal{A}$ is a real or complex ternary algebra. A linear subspace $\mathcal{B}$ of $\mathcal{A}$ will be called a ternary subalgebra of $\mathcal{A}$ if whenever $A,B,C \in \mathcal{B}$, we have that $AB^*C \in \mathcal{B}$ also. Such a subspace $\mathcal{B}$ will occasionally be referred to as a subalgebra of $\mathcal{A}$.

Requirement (5) is called the positivity condition in $\mathcal{A}$. A discussion of this condition will be given in Section 11. It should be noted that (5) implies (1).

Axioms (6) and (7) are referred to as *-additivity and *-homogeneity respectively. Taken together they comprise the condition known as *-linearity.

The symbol B^* by itself has not been defined. Its main use is in applications to matrices and linear transformations. More precisely, let $\mathcal{A}$ be the class of all $m \times n$ matrices A with complex elements. Denote by A^* the conjugate transpose of A. $\mathcal{A}$ is then a linear space over the field of complex numbers and is closed under the triple product AB^*C. It is then easy to verify that $\mathcal{G}$ is a ternary algebra.

Before continuing, it should be stated that proofs of all results stated or alluded to in the remainder of this section are valid in both ternary algebras and generalized ternary algebras.

Two other basic concepts will be reviewed at this time, since they will be used repeatedly throughout.

The first is that of <u>orthogonality</u>. Let $\mathcal{G}$ be a ternary algebra and let A and B be in $\mathcal{G}$. B is said to be <u>left orthogonal</u> to A if $AA^*B = 0$ and to be <u>right orthogonal</u> to A if $AB^*B = 0$. B is called <u>orthogonal</u> to A if $AA^*B = BA^*A = 0$.

The following lemma will be of importance. A simple proof can be found in [1].

LEMMA 2.1. *The following statements are equivalent:*

(1) $AB^*C = 0$ *for all* C *in* $\mathcal{G}$;

(2) $AB^*B = 0$;

(3) $BA^*C = 0$ *for all* C *in* $\mathcal{G}$;

(4) $BA^*A = 0$.

Similarly, the following statements are equivalent:

(5) $CB^*A = 0$ *for all* C *in* $\mathcal{G}$;

(6) $BB^*A = 0$;

(7) $CA^*B = 0$ *for all* C *in* $\mathcal{G}$;

(8) $AA^*B = 0$.

The second concept is that of the <u>*-reciprocal</u>. Let A be an element of a ternary algebra $\mathcal{G}$. If there exists an element A'

in $\mathfrak{A}$ satisfying

$$A = A'A^*A = AA'^*A = AA^*A' ,$$

$$A' = AA'^*A' = A'A^*A' = A'A'^*A ,$$

then A' is called the *-reciprocal of A. For matrices A' is the conjugate transpose of the pseudo-inverse of A [3],[4],[5].

It can be shown [1] that if one of the two sets of relations

$$\text{(a)} \quad A = A'A^*A \quad , \quad A' = A'A'^*A$$

$$\text{(b)} \quad A = AA^*A' \quad , \quad A' = AA'^*A'$$

holds, they both hold, and A' is the *-reciprocal of A. If the *-reciprocal exists, it is unique.

Condition (1) in the definition of a generalized ternary algebra plays an important role in regard to the existence of the *-reciprocal in the finite-dimensional case.

THEOREM 2.1. _Let $\mathfrak{A}$ be a finite-dimensional linear space of elements satisfying conditions (2),(3) and (4) in the above definition of a generalized ternary algebra. Then a necessary and sufficient condition for the existence of the *-reciprocal of any element of $\mathfrak{A}$ is that condition (1) holds._

A proof of the sufficiency can be found in [1].

For the necessity, suppose A' exists for every A. If $A = 0$, condition (4) implies that $AA^*A = 0$, and if $AA^*A = 0$, then again using (4) we obtain $A'A'^*(AA^*A) = 0$. But

$$A'A'^*(AA^*A) = (A'A'^*A)A^*A = A'A^*A = A ,$$

thus proving the theorem.

Many other results on the *-reciprocal can be found in [1].

From among these results the following lemma will suffice for our later applications. For a proof, the reader is again referred to [1].

LEMMA 2.2. *Let* A *be an element possessing a *-reciprocal* A'. *The relations*

$$A'A*C = AA'*C \quad , \quad CA*A' = CA'*A$$

hold for every element C *in* $\mathfrak{a}$. *An element* B *is orthogonal to* A *if and only if the relations*

$$A'A*B = 0 \quad , \quad BA*A' = 0$$

hold. The element B *is left orthogonal to* A *if and only if* $A'A*B = 0$ *and is right orthogonal to* A *if and only if* $BA*A' = 0$.

3. IDEALS IN TERNARY ALGEBRAS

Right, left and central ideals will now be introduced and studied. The various concepts of orthogonality as given in Section 2 will lead to the construction of an orthogonal complement for each type of ideal.

In the theorems below the assumption of the existence of the *-reciprocal is superfluous in the finite-dimensional case.

A <u>right ideal</u> $\mathfrak{R}$ is a linear subspace of $\mathfrak{A}$ such that if $R \in \mathfrak{R}$, then $RR^*A \in \mathfrak{R}$ for every A in $\mathfrak{A}$. In other words, $\mathfrak{R}$ is closed relative to "right multiplication" by elements of $\mathfrak{A}$. By a <u>left ideal</u> $\mathfrak{L}$ will be meant a linear subspace of $\mathfrak{A}$ such that if $L \in \mathfrak{L}$, we have that $AL^*L \in \mathfrak{L}$ for every $A \in \mathfrak{A}$.

The formation of ternary products allows a third type of ideal to be defined: a <u>central ideal</u> $\mathfrak{C}$ is a ternary subalgebra of $\mathfrak{A}$ satisfying the condition that $CC^*AC^*C \in \mathfrak{C}$ for every element C in $\mathfrak{C}$ and A in $\mathfrak{A}$.

LEMMA 3.1. <u>Let</u> $\mathfrak{B}$ <u>be either a right, left, or central ideal in</u> $\mathfrak{A}$. <u>If</u> B <u>is an element of</u> $\mathfrak{B}$ <u>for which</u> B' <u>exists, then</u> $B' \in \mathfrak{B}$.

First suppose $\mathfrak{B}$ is a right ideal in $\mathfrak{A}$. Then $B' = BB^*(B'B'^*B')$, which is an element of $\mathfrak{B}$.

If $\mathfrak{B}$ is a left ideal, then $B' = (B'B'^*B')B^*B$, an element of $\mathfrak{B}$.

Finally, if $\mathfrak{B}$ is a central ideal, then $B' = BB^*(B'B'^*B'B'^*B')B^*B$, an element of $\mathfrak{B}$.

The following theorems give equivalent defining conditions for right, left and central ideals.

THEOREM 3.1. Let $\mathfrak{R}, \mathfrak{L}$ and $\mathfrak{C}$ be linear subspaces of $\mathfrak{A}$. Assume the existence of the *-reciprocal for each element of $\mathfrak{R}, \mathfrak{L}$ and $\mathfrak{C}$. Then:

(i) A necessary and sufficient condition that $\mathfrak{R}$ be a right ideal is that $R'R^*A$ be in $\mathfrak{R}$ for all elements $R \in \mathfrak{R}$ and $A \in \mathfrak{A}$.

(ii) $\mathfrak{L}$ is a left ideal if and only if $AL^*L' \in \mathfrak{L}$ for all elements $L \in \mathfrak{L}$ and $A \in \mathfrak{A}$.

(iii) $\mathfrak{C}$ is a central ideal if and only if $C'C^*AC^*C' \in \mathfrak{C}$ for all elements $C \in \mathfrak{C}$ and $A \in \mathfrak{A}$ and $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$.

To prove (i), suppose $\mathfrak{R}$ is a right ideal and that $R \in \mathfrak{R}$ and $A \in \mathfrak{A}$. By Lemma 2.2 and the defining equations for the *-reciprocal of R, we have

$$R'R^*A = RR'^*A = R(R'R'^*R)^*A = RR^*(R'R'^*A) ,$$

which is an element of $\mathfrak{R}$, since $R'R'^*A$ is in $\mathfrak{A}$.

Conversely, if $R \in \mathfrak{R}$ and $A \in \mathfrak{A}$, then since $R = R'R^*R$,

$$RR^*A = R'R^*(RR^*A)$$

is an element of $\mathfrak{R}$.

The proofs of (ii) and (iii) are similar.

THEOREM 3.2. *Let* $\mathcal{R}, \mathcal{L}$ *and* $\mathcal{C}$ *be linear subspaces of* $\mathcal{A}$ *with the added assumption that the *-reciprocal of each element of* $\mathcal{R}, \mathcal{L}$ *and* $\mathcal{C}$ *exists. Then:*

(i) *A necessary and sufficient condition that* $\mathcal{R}$ *be a right ideal is that* $RA^*B \in \mathcal{R}$ *for all* $R \in \mathcal{R}$ *and* $A, B \in \mathcal{A}$.

(ii) $\mathcal{L}$ *is a left ideal in* $\mathcal{A}$ *if and only if* $AB^*L \in \mathcal{L}$ *for all* $L \in \mathcal{L}$ *and* $A, B \in \mathcal{A}$.

(iii) $\mathcal{C}$ *is a central ideal in* $\mathcal{A}$ *if and only if* $CA^*C \in \mathcal{C}$ *for all* $C \in \mathcal{C}$ *and* $A \in \mathcal{A}$ *and* $\mathcal{C}$ *is a subalgebra of* $\mathcal{A}$.

For the proof of (i), assume $\mathcal{R}$ is a right ideal and let $R \in \mathcal{R}$ and $A, B \in \mathcal{A}$. Then since $R = RR^*R'$, we have

$$RA^*B = RR^*(R'A^*B) ,$$

which is an element of $\mathcal{R}$.

Conversely, if $RA^*B \in \mathcal{R}$ for all $R \in \mathcal{R}$, and $A, B \in \mathcal{A}$, then clearly $RR^*A \in \mathcal{R}$ for all $R \in \mathcal{R}$ and $A \in \mathcal{A}$.

The proof of (ii) is similar.

For statement (iii), suppose $\mathcal{C}$ is a central ideal and let $C \in \mathcal{C}$ and $A \in \mathcal{A}$. Then

$$CA^*C = CC^*(C'A^*C')C^*C$$

is in $\mathcal{C}$.

For the sufficiency, note that $CC^*AC^*C = C(CA^*C)^*C$ is an element of $\mathcal{C}$.

The above theorem shows that right and left ideals in $\mathcal{A}$

are subalgebras of $\mathfrak{G}$.

Certain subspaces of $\mathfrak{G}$ can now be constructed with regard to right, left and central ideals. They will be useful for various decomposition theorems to appear later. The first type to be discussed is that formed from the intersection of a collection of ideals. The result is stated in the form of a lemma whose use will be delayed until a further section.

LEMMA 3.2. The intersection of a finite number of right ideals is a right ideal. Similarly, the intersection of a finite number of left ideals is a left ideal. Finally, the intersection of a finite number of central ideals is a central ideal.

The proof is an easy verification and will be omitted.

Now let $\mathfrak{R}$ be a subclass of $\mathfrak{G}$. By the left orthogonal complement $\mathfrak{R}^{\lrcorner}$ of $\mathfrak{R}$ will be meant the set of all elements A in $\mathfrak{G}$ such that $RR^*A = 0$ for all $R \in \mathfrak{R}$. In other words, any element of $\mathfrak{R}^{\lrcorner}$ must be left orthogonal to every element in $\mathfrak{R}$.

THEOREM 3.3. $\mathfrak{R}^{\lrcorner}$ is a right ideal in $\mathfrak{G}$ for any right ideal $\mathfrak{R}$.

If A and B are in $\mathfrak{R}^{\lrcorner}$ and if $R \in \mathfrak{R}$, then

$$RR^*(A + B) = RR^*A + RR^*B = 0 + 0 = 0 .$$

Also, if $A \in \mathfrak{R}^{\lrcorner}$, $R \in \mathfrak{R}$ and λ is a complex number, then

$$RR^*(\lambda A) = \lambda(RR^*A) = 0 .$$

Hence, $\mathfrak{R}^{\lrcorner}$ is a linear subspace of $\mathfrak{G}$. Finally, let $A \in \mathfrak{R}^{\lrcorner}$ and $B \in \mathfrak{G}$. Then if R is any element in $\mathfrak{R}$, we have

$$RR^*(AB^*B) = (RR^*A)B^*B = 0 ,$$

so that $AB^*B \in \mathfrak{R}^\perp$. Thus $\mathfrak{R}^\perp$ is a right ideal in $\mathfrak{G}$.

In a corresponding fashion the *right orthogonal complement* of a subclass $\mathfrak{L}$ is defined to be the set $\mathfrak{L}^\perp$ of all elements A in $\mathfrak{G}$ satisfying $AL^*L = 0$ for all $L \in \mathfrak{L}$.

THEOREM 3.4. *If $\mathfrak{L}$ is a left ideal in $\mathfrak{G}$, then $\mathfrak{L}^\perp$ is also.*

The proof is entirely analogous to that of Theorem 3.3 and will be omitted.

For the sake of completeness we define the *orthogonal complement* of a subclass $\mathfrak{C}$ to be the set $\mathfrak{C}^\perp$ of all elements in $\mathfrak{G}$ which are orthogonal to every element in $\mathfrak{C}$.

THEOREM 3.5. *For any central ideal $\mathfrak{C}$ in $\mathfrak{G}$, $\mathfrak{C}^\perp$ is a central ideal also.*

Following a line of reasoning similar to that in the proof of Theorem 3.3, it is easily verified that $\mathfrak{C}^\perp$ is a ternary subalgebra of $\mathfrak{G}$. Suppose $B \in \mathfrak{C}^\perp$ and that $A \in \mathfrak{G}$. If C is any element of $\mathfrak{C}$, then using the orthogonality of B and C, we get

$$(BA^*B)C^*C = BA^*(BC^*C) = 0$$

and

$$CC^*(BA^*B) = (CC^*B)A^*B = 0,$$

so that $BA^*B \in \mathfrak{C}^\perp$. Hence, by Theorem 3.2 (iii), $\mathfrak{C}^\perp$ is a central ideal in $\mathfrak{G}$.

THEOREM 3.6. *Let $\mathfrak{B}$ and $\mathfrak{C}$ be subclasses of $\mathfrak{G}$ such* *that* $\mathfrak{B} \subseteq \mathfrak{C}$. *Then* $\mathfrak{C}^{\lrcorner} \subseteq \mathfrak{B}^{\lrcorner}$, $\mathfrak{C}^{\llcorner} \subseteq \mathfrak{B}^{\llcorner}$ *and* $\mathfrak{C}^{\perp} \subseteq \mathfrak{B}^{\perp}$.

To prove that $\mathfrak{C}^{\lrcorner} \subseteq \mathfrak{B}^{\lrcorner}$, we merely note that since $\mathfrak{B} \subseteq \mathfrak{C}$, any element which is left orthogonal to every element in $\mathfrak{C}$ is automatically left orthogonal to every element in $\mathfrak{B}$. Corresponding remarks prove the remaining statements.

It should be noted that the right orthogonal complement of a right ideal $\mathfrak{R}$, in general, is *not* a right ideal in $\mathfrak{A}$. An obvious corresponding statement is true of left ideals as well.

The section will conclude with a discussion of subideals. Let $\mathfrak{R}$ be a right ideal in $\mathfrak{A}$. A subspace $\mathfrak{S}$ of $\mathfrak{R}$ will be called a *subideal* of $\mathfrak{R}$ if $\mathfrak{S}$ is a right ideal *with respect to* $\mathfrak{R}$, i.e., if $S \in \mathfrak{S}$ and $R \in \mathfrak{R}$, then SS^*R is in $\mathfrak{S}$ also. Similarly, a subspace $\mathfrak{M}$ of a left ideal $\mathfrak{L}$ is said to be a subideal of $\mathfrak{L}$ if $\mathfrak{M}$ is a left ideal with respect to $\mathfrak{L}$. An analogous statement defines a subideal of a central ideal.

THEOREM 3.7. *Let* $\mathfrak{S}, \mathfrak{M}$ *and* $\mathfrak{D}$ *be subideals of the right ideal* $\mathfrak{R}$, *left ideal* $\mathfrak{L}$, *and central ideal* $\mathfrak{C}$ *respectively. Then if the *-reciprocal of each element of* $\mathfrak{S}, \mathfrak{M}$ *and* $\mathfrak{D}$ *exists,* $\mathfrak{S}$ *is a right ideal in* $\mathfrak{A}$, $\mathfrak{M}$ *is a left ideal in* $\mathfrak{A}$ *and* $\mathfrak{D}$ *is a central ideal in* $\mathfrak{A}$.

Let $S \in \mathfrak{S}$ and $A \in \mathfrak{A}$. It will be shown that $SS^*A \in \mathfrak{S}$. Since $\mathfrak{S} \subseteq \mathfrak{R}$, $S \in \mathfrak{R}$ and because $\mathfrak{R}$ is a right ideal in $\mathfrak{A}$, $SS^*A \in \mathfrak{R}$ also. Hence

$$SS^*A = (S'S^*S)S^*A = S'S^*(SS^*A) ,$$

an element of $\mathfrak{S}$ since $\mathfrak{S}$ is a right ideal in $\mathfrak{R}$.

A similar argument proves the result for the subideals $\mathfrak{M}$ and $\mathfrak{D}$.

4. MINIMAL IDEALS

In this section various properties and characterizations of an important class of right and left ideals will be given. It will be assumed that $\mathfrak{G}$ is a ternary algebra, not necessarily finite-dimensional.

A non-zero right ideal $\mathfrak{R}$ in $\mathfrak{G}$ is called minimal if $\mathfrak{R}$ contains no right ideals other than itself and the zero-ideal, i.e., the ideal whose only element is 0. Similarly, by a minimal left ideal will be meant a non-zero left ideal which contains no left ideals but itself and the zero-ideal. Finally, an obvious statement defines a minimal central ideal.

Clearly, any one-dimensional ideal is minimal. For another example, we return briefly to the case in which $\mathfrak{G}$ is the ternary algebra of all $m \times n$ matrices over the complex numbers. Suppose $\mathfrak{R}$ is a non-zero right ideal in $\mathfrak{G}$ such that any element R of $\mathfrak{R}$ has the form $R = uv^*$ where u is a fixed m-dimensional column vector and v is an n-dimensional column vector dependent upon R. Using results of this section it will be possible to show that $\mathfrak{R}$ is a minimal right ideal. This example will be further pursued later.

Hereafter the existence of the $*$-reciprocal will be assumed gratis. It is important to recall, however, that the existence is ensured in the case when $\mathfrak{G}$ is finite-dimensional.

The first theorem characterizes minimal right ideals as those right ideals which contain no non-zero left orthogonal elements.

THEOREM 4.1. Let $\mathfrak{R}$ be a right ideal in $\mathfrak{A}$. Then $\mathfrak{R}$ is minimal if and only if for any elements R, S in $\mathfrak{R}$, the relation $RR^*S = 0$ holds only in the event either $R = 0$ or $S = 0$.

Suppose that $\mathfrak{R}$ is a minimal right ideal and that $RR^*S = 0$ for R, S in $\mathfrak{R}$ and $R \neq 0$. Denote by $\mathfrak{J}$ the set of all elements T in $\mathfrak{R}$ such that $RR^*T = 0$. $\mathfrak{J}$ is non-empty, since $S \in \mathfrak{J}$. Also, it is easy to see that $\mathfrak{J}$ is a linear subspace of $\mathfrak{A}$. To show that $\mathfrak{J}$ is a right ideal in $\mathfrak{A}$, let $T \in \mathfrak{J}$ and $A \in \mathfrak{A}$. Then $RR^*(TT^*A) = (RR^*T)T^*A = 0$, so that $TT^*A \in \mathfrak{J}$. Thus, since $\mathfrak{J} \subseteq \mathfrak{R}$ and $\mathfrak{R}$ is minimal, either $\mathfrak{J} = \{0\}$, the zero-ideal, or $\mathfrak{J} = \mathfrak{R}$. But $\mathfrak{J} \neq \mathfrak{R}$ since $R \in \mathfrak{R}$ and $R \neq 0$ implies that $RR^*R \neq 0$. Hence, $\mathfrak{J} = \{0\}$, and since $S \in \mathfrak{J}$, $S = 0$.

Conversely, suppose $\mathfrak{R}$ contains no pair of non-zero left orthogonal elements. Let $\mathfrak{S}$ be a right ideal in $\mathfrak{A}$ such that $\mathfrak{S} \subseteq \mathfrak{R}$ and $\mathfrak{S} \neq \{0\}$. Choose $S \in \mathfrak{S}$, $S \neq 0$. If R is any element in $\mathfrak{R}$ and $T = S'S^*R - R$, then

$$SS^*T = SS^*(S'S^*R - R) = (SS^*S')S^*R - SS^*R = 0 \quad .$$

Since S and T are both in $\mathfrak{R}$ and $S \neq 0$, we conclude that $T = 0$. in other words, $R = S'S^*R$, which is in $\mathfrak{S}$, since $\mathfrak{S}$ is a right ideal. Hence, $\mathfrak{R} \subseteq \mathfrak{S}$ so that $\mathfrak{S} = \mathfrak{R}$ and $\mathfrak{R}$ is minimal.

The corresponding result for left ideals is as follows:

THEOREM 4.2. Let $\mathfrak{L}$ be a left ideal in $\mathfrak{a}$. Then $\mathfrak{L}$ is minimal if and only if $\mathfrak{L}$ contains no pair of non-zero right orthogonal elements.

The proof is the dual of that of Theorem 4.1.

THEOREM 4.3. Let $\mathfrak{R}$ and $\mathfrak{L}$ be a non-zero right ideal and a non-zero left ideal in $\mathfrak{a}$ respectively. Then the following two conditions are equivalent:

(i) $\mathfrak{R}$ contains no non-zero left orthogonal elements.

(ii) $R'R^*S = S$ for all $R,S \in \mathfrak{R}$ with $R \neq 0$.

Similarly, the following two conditions are equivalent:

(iii) $\mathfrak{L}$ contains no non-zero right orthogonal elements.

(iv) $LM^*M' = L$ for all $L,M \in \mathfrak{L}$ with $M \neq 0$.

We prove only the equivalence of (i) and (ii). Assume (i) and let $R,S \in \mathfrak{R}$, $R \neq 0$. Let $D = R'R^*S - S$. Then $D \in \mathfrak{R}$ and $RR^*D = 0$. Since $R \neq 0$, $D = 0$, that is, $R'R^*S = S$.

Now suppose (ii) holds and that $RR^*S = 0$ for $R,S \in \mathfrak{R}$ and $R \neq 0$. By Lemma 2.2, $R'R^*S = 0$ also. Hence,

$$S = R'R^*S = 0 .$$

The following theorem is an immediate consequence of the preceding results.

THEOREM 4.4. Suppose $\mathfrak{R}$ and $\mathfrak{L}$ are non-zero right and left ideals in $\mathfrak{a}$ respectively. Then the following statements are valid:

(i) $\mathfrak{R}$ <u>is minimal if and only if</u> $R'R^*S = S$ <u>for all</u> $R,S \in \mathfrak{R}$ <u>with</u> $R \neq 0$.

(ii) $\mathfrak{L}$ <u>is minimal if and only if</u> $LM^*M' = L$ <u>for all</u> $L,M \in \mathfrak{L}$ <u>with</u> $M \neq 0$.

Again let $\mathfrak{R}$ and $\mathfrak{L}$ be right and left ideals in $\mathfrak{A}$ respectively. For any $A \in \mathfrak{R}$, $A \neq 0$, denote by $\mathfrak{R}_A$ the set of all elements of the form $A'A^*B$ for all $B \in \mathfrak{A}$. Similarly, for any $A \in \mathfrak{L}$, $A \neq 0$, $\mathfrak{L}_A$ will represent all elements of the form BA^*A' for $B \in \mathfrak{A}$. With these conventions in mind we have the following

COROLLARY.

(i) $\mathfrak{R}_A$ <u>is a right ideal in</u> $\mathfrak{A}$ <u>and</u> $\mathfrak{R}$ <u>is minimal if and only if</u> $\mathfrak{R}_A = \mathfrak{R}$ <u>for all</u> $A \in \mathfrak{R}$ <u>with</u> $A \neq 0$.

(ii) $\mathfrak{L}_A$ <u>is a left ideal in</u> $\mathfrak{A}$ <u>and</u> $\mathfrak{L}$ <u>is minimal if and only if</u> $\mathfrak{L}_A = \mathfrak{L}$ <u>for all</u> $A \in \mathfrak{L}$ <u>with</u> $A \neq 0$.

Only the proof of (i) will be given. To show that $\mathfrak{R}_A$ is a right ideal, let $R,S \in \mathfrak{R}_A$ and let λ and μ be any complex numbers. Then there exist elements B,C in $\mathfrak{A}$ such that $R = A'A^*B$ and $S = A'A^*C$. Therefore, $\lambda R + \mu S = A'A^*D$ where $D = \lambda B + \mu C$, and so $\mathfrak{R}_A$ is a linear subspace. Also, if $R = A'A^*B \in \mathfrak{R}_A$ and $C,D \in \mathfrak{A}$, then

$$RC^*D = A'A^*(BC^*D) \in \mathfrak{R}_A$$

and $\mathfrak{R}_A$ is a right ideal in $\mathfrak{A}$.

Now suppose $\mathfrak{R}$ is minimal. For any $A \in \mathfrak{R}$, all elements of the form $A'A^*B$, where $B \in \mathfrak{A}$, are contained in $\mathfrak{R}$ since $\mathfrak{R}$ is a right ideal in $\mathfrak{A}$. In other words, $\mathfrak{R}_A \subseteq \mathfrak{R}$ for <u>any</u> $A \in \mathfrak{R}$. By the minimality of $\mathfrak{R}$, $\mathfrak{R}_A = \mathfrak{R}$ in the case when $A \in \mathfrak{R}$ and $A \neq 0$.

Conversely, assume $\mathfrak{R}_A = \mathfrak{R}$ for all $A \in \mathfrak{R}$, $A \neq 0$. Choose R, S in $\mathfrak{R}$ with $R \neq 0$. In particular, $\mathfrak{R}_R = \mathfrak{R}$ so that there exists $A \in \mathfrak{A}$ such that $S = R'R^*A$. Hence,

$$R'R^*S = R'R^*(R'R^*A) = R'R^*A = S$$

and by the theorem, $\mathfrak{R}$ is minimal.

We remark at this point that the notions of linear independence and linear dependence of a finite set of elements have the usual meanings in $\mathfrak{A}$ when considered as a linear space.

THEOREM 4.5. *If $\mathfrak{R}$ is a linear subspace of $\mathfrak{A}$ such that for each triple of elements A, B, C in $\mathfrak{R}$ for which $C \neq 0$ we have that $AB^*C = \lambda C$ for some complex number λ, then λ depends only upon A and B.*

Suppose $AB^*C = \lambda C$ and $AB^*D = \mu D$ for elements A, B, C, D where $C \neq 0$ and $D \neq 0$ and complex numbers λ and μ. We show that $\lambda = \mu$. There exists a number ν such that

$$AB^*(C + D) = \nu(C + D) .$$

But

$$AB^*C + AB^*D = \lambda C + \mu D .$$

Hence

$$(\lambda - \nu)C + (\mu - \nu)D = 0 .$$

If C and D are linearly independent, then $\lambda - \nu = \mu - \nu = 0$, so that $\lambda = \mu$. If C and D are linearly dependent, say $D = \alpha C$, then

$$AB^*D = \alpha AB^*C = \alpha\lambda C = \mu D = \alpha\mu C .$$

Since $C \neq 0$ and $\alpha \neq 0$, $\lambda = \mu$.

COROLLARY. *Let* $\mathfrak{R}$ *be a linear subspace of* $\mathfrak{A}$ *satisfying the assumptions of the theorem. For any elements* A, B *and* C *in* $\mathfrak{R}$, *let* λ *be the complex number such that* $AB^*C = \lambda C$. *Then* $BA^*C = \bar{\lambda}C$.

First note that if $AB^*C = 0$, then by the theorem $AB^*B = 0$ also. Lemma 2.1 then implies that $BA^*C = 0$ for all C in $\mathfrak{R}$. We thus assume that $AB^*C \neq 0$.

Let μ be the complex number such that $BA^*C = \mu C$. It will be shown that $\mu = \bar{\lambda}$. The theorem implies that $AB^*(BA^*C) = \lambda BA^*C = \lambda\mu C$, and also that $AB^*B = \lambda B$. Hence

$$AB^*BA^*C = A(AB^*B)^*C = A(\lambda B)^*C = \bar{\lambda}AB^*C = \bar{\lambda}\lambda C .$$

Therefore, $\lambda\mu C = \lambda\bar{\lambda}C$ and since $C \neq 0$, $\lambda\mu = \lambda\bar{\lambda}$. Since it must also be true that $\lambda \neq 0$, we have that $\mu = \bar{\lambda}$.

COROLLARY. *Suppose* $\mathfrak{R}$ *is a right ideal in* $\mathfrak{A}$. *If* $RS^*T = \lambda T$ *for all* $R, S, T \in \mathfrak{R}$ *for which* $T \neq 0$ *and some complex number* λ, *then* $\mathfrak{R}$ *is a minimal right ideal.*

Suppose $R \in \mathfrak{R}$, $R \neq 0$ and let $S \in \mathfrak{R}$. If $S = 0$, then $R'R^*S = 0 = S$. Hence, suppose $S \neq 0$. By assumption there exists a complex number λ such that $R'R^*S = \lambda S$. By the theorem, λ does not depend upon S, so that we also have $R'R^*R = \lambda R$.

Since $R'R^*R = R$, $\lambda = 1$. Hence, $R'R^*S = S$ and by Theorem 4.4 $\mathfrak{R}$ is a minimal right ideal.

The corresponding results for left ideals are as follows:

THEOREM 4.6. *If $\mathfrak{L}$ is a linear subspace of $\mathfrak{a}$ such that for each triple of elements A,B,C in $\mathfrak{L}$ for which $A \neq 0$ we have that $AB^*C = \lambda A$ for some complex number λ, then λ depends only upon B and C.*

COROLLARY. *Let $\mathfrak{L}$ be a linear subspace of $\mathfrak{a}$ satisfying the assumptions of the theorem. For any elements A,B and C in $\mathfrak{L}$, let λ be the complex number such that $AB^*C = \lambda A$. Then $AC^*B = \bar{\lambda}A$.*

COROLLARY. *Suppose $\mathfrak{L}$ is a left ideal in $\mathfrak{a}$. If $LM^*N = \lambda L$ for all $L,M,N \in \mathfrak{L}$ for which $L \neq 0$ and some complex number λ, then $\mathfrak{L}$ is a minimal left ideal.*

As particular instances of the last results we see that any right ideal $\mathfrak{R}$ or left ideal $\mathfrak{L}$ which is an inner-product space is minimal. If $A,B,C \in \mathfrak{R}$ the inner product as contained in the term AB^*C is taken between A and B. For $\mathfrak{L}$, B and C comprise the inner product. As an example, consider the case in which $\mathfrak{R}$ is the right ideal of matrices of rank one as described in the opening of this section. If $R = uv^*$, $S = uw^*$ and $T = ux^*$ are elements of $\mathfrak{R}$, then

$$RS^*T = uv^*wu^*ux^* = (v^*w)(u^*u)ux^* = \lambda T ,$$

where $\lambda = (v^*w)(u^*u)$. By the second corollary to Theorem 4.5,

$\mathfrak{R}$ is minimal.

In the next section it will be shown that the converse is also true, namely, that a minimal right or left ideal with a suitable restriction can be construed as an inner-product space.

LEMMA 4.1. *Let* $\mathfrak{R}$ *and* $\mathfrak{L}$ *be a right and a left ideal in* $\mathfrak{A}$ *respectively.*

(i) *If* $R \in \mathfrak{R}$ *and* $R = S + T$ *where* S *and* T *are right orthogonal, then* $S \in \mathfrak{R}$ *and* $T \in \mathfrak{R}$.

(ii) *If* $L \in \mathfrak{L}$ *and* $L = M + N$ *where* M *and* N *are left orthogonal, then* $M \in \mathfrak{L}$ *and* $N \in \mathfrak{L}$.

For a slight change of pace we prove (ii), the result for left ideals. Since $L = M + N$,

$$M'M^*L = M'M^*M + M'M^*N = M + M'M^*N$$

and

$$N'N^*L = N'N^*M + N'N^*N = N'N^*M + N \; .$$

Using the fact that M and N are left orthogonal, Lemma 2.2 implies that $M'M^*N = N'N^*M = 0$. Hence, $M = M'M^*L$ and $N = N'N^*L$, both of which are in $\mathfrak{L}$ since $\mathfrak{L}$ is a left ideal in $\mathfrak{A}$.

In order to continue, we introduce the notion of the *degree* of an element. Suppose A is in $\mathfrak{A}$ and let $A^{(1)} = A$, $A^{(3)} = AA^*A$ and in general $A^{(2k+1)} = AA^*A^{(2k-1)}$ for $k = 1,2,3,\ldots$. If there exists an integer n for which $A^{(1)},A^{(3)},\ldots,A^{(2n+1)}$ are linearly dependent, then A is said to be of *finite degree*. The smallest integer for which such a relation holds is called the <u>degree</u> of A. For an element A of degree one there exists a scalar α such that $AA^*A = \alpha A$. The positivity condition insures that $\alpha > 0$ if $A \neq 0$. If $\mathfrak{A}$ is finite-dimensional then every element is of finite degree. It should be noted, however, that the converse is <u>not</u> in general true. Consider, for example, the ternary algebra of infinite-dimensional matrices $A = (\alpha_{ij})$ with complex elements $\alpha_{ij}(i,j = 1,2,3,\ldots)$ of which at most a finite number are non-zero. Here, the $*$-operation is taken as conjugate transpose. Each element has finite degree, although $\mathfrak{A}$ is not finite-dimensional.

LEMMA 4.2. <u>Every non-zero element of finite degree in a minimal right ideal or minimal left ideal is of degree one.</u>

To fix the ideas, suppose $\mathfrak{R}$ is a minimal right ideal and that R is a non-zero element of finite degree n in $\mathfrak{R}$. Suppose that $n > 1$. In [1] it is proved (Theorem 12.4, page 175) that R may be written in the form $R = S + T$ where S and T are orthogonal and each is of degree at most $n - 1$. If either $S = 0$ or $T = 0$, then R is of degree at most $n - 1$, a contradiction. Hence, $S \neq 0$ and $T \neq 0$. Since S and T are right orthogonal, Lemma 4.1 (i) implies $S \in \mathfrak{R}$ and $T \in \mathfrak{R}$. Also, S and T are non-zero left orthogonal elements of $\mathfrak{R}$, contradicting Theorem 4.1. Hence, $n \leq 1$, and since $R \neq 0$ we have that $n = 1$ and the lemma is proved.

5. MINIMAL IDEALS AS INNER-PRODUCT SPACES

It will be shown in this section that any minimal right or left ideal in which each element is of finite degree may be considered an inner-product space. This, however, will be preceded by a result of a more general nature, namely, that a *-linear generalized ternary algebra $\mathfrak{a}$ in which each element is of degree one is an inner-product space. The result for minimal ideals will follow directly from this fact. We first prove a sequence of lemmas. To avoid repetition, it will be assumed in Lemmas 5.1 through 5.5 below that $\mathfrak{a}$ is a *-linear generalized ternary algebra in which each element is of degree one.

LEMMA 5.1. *If A and B are any two elements of $\mathfrak{a}$ which are orthogonal, then either $A = 0$ or $B = 0$.*

Suppose $A \neq 0$ and $B \neq 0$. Then using the orthogonality of A and B, it is easy to show that they are linearly independent. Because A and B are both of degree one, there exist real numbers α and β such that $AA^*A = \alpha A$ and $BB^*B = \beta B$. Since A and B are both non-zero, we have that $\alpha \neq 0$ and $\beta \neq 0$. Let γ be any non-zero real number such that $\gamma^2\beta \neq \alpha$. For this number γ there exists a scalar λ such that

$$(A + \gamma B)(A + \gamma B)^*(A + \gamma B) = \lambda(A + \gamma B) .$$

Using the orthogonality of A and B, this last equation becomes

$$AA^*A + \gamma^3 BB^*B = \lambda(A + \gamma B)$$

or,

$$\alpha A + \gamma^3 \beta B = \lambda A + \gamma\lambda B \ .$$

Hence,

$$(\lambda - \alpha)A + \gamma(\lambda - \gamma^2\beta)B = 0 \ .$$

Since A and B are linearly independent and $\gamma \neq 0$, we get that

$$\lambda = \alpha = \gamma^2\beta \ ,$$

a contradiction. Therefore, either $A = 0$ or $B = 0$.

LEMMA 5.2. <u>Let</u> A <u>and</u> B <u>be elements of</u> $\mathfrak{A}$ <u>for which</u> $A \neq 0$. <u>Then either</u> $A'A^*B = B$ <u>or</u> $BA^*A' = B$.

In [1] it is shown (Theorem 7.2, page 154) that B may be represented in the form

$$B = C + D + E + F \ ,$$

where

$$C = A'A^*BA^*A' ,$$

$$D = B + A'A^*BA^*A' - A'A^*B - BA^*A' \ ,$$

$$E = BA^*A' - A'A^*BA^*A' \ ,$$

$$F = A'A^*B - A'A^*BA^*A' \ ,$$

and D is orthogonal to A. Since $A \neq 0$, Lemma 5.1 implies that $D = 0$. Also, it is easy to check that the elements E and F are orthogonal. Hence, by Lemma 5.1, either $E = 0$ or $F = 0$. If $E = 0$, then $B = C + F = A'A^*B$ and if $F = 0$, then $B = C + E = BA^*A'$.

LEMMA 5.3. <u>If</u> A <u>and</u> B <u>are two elements of</u> $\mathfrak{a}$ <u>satisfying</u> $A'A^*B = B = BA^*A'$, <u>then there exists a complex number</u> μ <u>such that</u> $B = \mu A$.

If $A = 0$, then $B = A'A^*B = 0$, so that μ is arbitrary. If $B = 0$, then again $\mu = 0$. Thus, we may assume that $A \neq 0$ and $B \neq 0$. For some real numbers α and β we therefore have that $AA^*A = \alpha A$ and $BB^*B = \beta B$. Let $R = A + \lambda B$ and $S = A - \lambda B$ where λ is an arbitrary non-zero complex number. Since R and S are both of degree one, $RR^*R = \rho R$ and $SS^*S = \sigma S$ for some real numbers ρ and σ. From the equations $A'A^*B = B = BA^*A'$ and the fact that $A = \alpha A'$, we get that $AA^*B = \alpha B = BA^*A$. Also,

$$\rho R = RR^*R = \alpha A + 2\alpha\lambda B + \bar{\lambda}AB^*A + \lambda^2 BA^*B$$
$$+ \lambda\bar{\lambda}(AB^*B + BB^*A) + \lambda\bar{\lambda}\lambda\beta B$$

and

$$\sigma S = SS^*S = \alpha A - 2\alpha\lambda B - \bar{\lambda}AB^*A + \lambda^2 BA^*B$$
$$+\lambda\bar{\lambda}(AB^*B + BB^*A) - \lambda\bar{\lambda}\lambda\beta B .$$

Hence,

$$\tfrac{1}{2}(\rho R - \sigma S) = (2\alpha + \lambda\bar{\lambda}\beta)\lambda B + \bar{\lambda}AB^*A ,$$

so that $AB^*A = \gamma A + \delta B$, where

$$\gamma = \frac{\rho - \sigma}{2\bar{\lambda}} \quad \text{and} \quad \delta = \left(\frac{\rho + \sigma}{2} - 2\alpha - \beta\lambda\bar{\lambda} \right)\frac{\lambda}{\bar{\lambda}} .$$

Suppose that A and B are linearly independent. Then for any λ, we have that $R \neq 0$ and $S \neq 0$ so that $\rho \neq 0$ and $\sigma \neq 0$. Also, γ and δ are unique. Since we may choose $\lambda = 1$, γ and δ are real. Thus, $\gamma = (\rho - \sigma)/2\bar{\lambda}$ is real for all λ and hence $\gamma = 0$.

Now

$$\alpha^2 B = AA^*BA^*A = A(AB^*A)^*A$$
$$= A(\delta B)^*A = \delta AB^*A = \delta^2 B ,$$

so that $\delta^2 = \alpha^2 \neq 0$. Also

$$\alpha BA^*B = (AA^*B)A^*B = A(AB^*A)^*B = \delta AB^*B$$

and

$$\alpha BA^*B = BA^*(BA^*A) = B(AB^*A)^*A = \delta BB^*A .$$

Since $\delta \neq 0$ we have that $AB^*B = BB^*A$. By Lemma 5.2, either $B'B^*A = A$ or $AB^*B' = A$. The fact that $B = \beta B'$ implies that either $BB^*A = \beta A$ or $AB^*B = \beta A$. Hence,

$$AB^*B = BB^*A = \beta A .$$

In addition,

$$\tfrac{1}{2}(\rho R + \sigma S) = \alpha A + \lambda^2 BA^*B + \lambda\bar{\lambda}(AB^*B + BB^*A)$$
$$= (\alpha + 2\lambda\bar{\lambda}\beta)A + \lambda^2 BA^*B .$$

Therefore,

$$BA^*B = \left(\frac{\rho + \sigma}{2} - \alpha - 2\lambda\bar{\lambda}\beta\right)\frac{1}{\lambda^2} A + \left(\frac{\rho - \sigma}{2\lambda}\right) B$$

$$= \left(\rho - \alpha - 2\lambda\bar{\lambda}\beta\right)\frac{1}{\lambda^2} A ,$$

since $\gamma = 0$ implies that $\rho = \sigma$. But also

$$BA^*B = \frac{1}{\alpha}\,\delta AB^*B = \frac{\beta}{\alpha}\,\delta A\ ,$$

so that

$$(\rho - \alpha - 2\lambda\bar{\lambda}\beta)\,\frac{1}{\lambda^2} = \frac{\beta}{\alpha}\,\delta = \frac{\beta}{\alpha}(\rho - 2\alpha - \beta\lambda\bar{\lambda})\,\frac{\lambda}{\bar{\lambda}}\ ;$$

or,

$$\rho = 2\alpha + \beta\lambda\bar{\lambda} + \delta\,\frac{\bar{\lambda}}{\lambda} = \alpha + 2\beta\lambda\bar{\lambda} + \delta\,\frac{\beta}{\alpha}\,\lambda^2\ .$$

We therefore have that

$$\alpha - \beta\lambda\bar{\lambda} + \delta\,\frac{\bar{\lambda}}{\lambda} - \delta\,\frac{\beta}{\alpha}\,\lambda^2 = 0$$

for all $\lambda \neq 0$. Letting $\lambda = 1 + i$ yields

$$(\alpha - 2\beta) - i\,\frac{\delta}{\alpha}(\alpha + 2\beta) = 0\ ,$$

so that

$$\alpha - 2\beta = 0 \quad \text{and} \quad \alpha + 2\beta = 0$$

and therefore $\alpha = \beta = 0$, a contradiction.

Hence, A and B are dependent, so that there exist scalars η and ζ, not both zero, such that $\eta A + \zeta B = 0$. If $\eta = 0$, then $\zeta \neq 0$ and $B = 0$ and if $\zeta = 0$, then $\eta \neq 0$ and $A = 0$. Thus, either case leads to a contradiction, so that η and ζ are both non-zero. Therefore $B = \mu A$, where $\mu = -\,\eta\zeta^{-1}$.

LEMMA 5.4. <u>If</u> A <u>is a non-zero element of</u> $\mathfrak{G}$, <u>then either</u> $A'A^*B = B$ <u>for all</u> $B \in \mathfrak{G}$, <u>or</u> $BA^*A' = B$ <u>for all</u> $B \in \mathfrak{G}$.

Let B be an arbitrary element of $\mathfrak{G}$. Then by Lemma 5.2, either $A'A^*B = B$ or $BA^*A' = B$. Suppose $A'A^*B = B$. We may then write $B = C + D$, where $C = A'A^*BA^*A'$ and $D = A'A^*B - A'A^*BA^*A'$. Hence, $A'A^*C = C$ and $CA^*A' = C$, so that by Lemma 5.3 there exists a scalar μ such that $C = \mu A$. Note also that $DA^*A' = 0$. Therefore, $B = \mu A + D$, where $DA^*A' = 0$. In a similar way it can be shown that if $BA^*A' = B$, then $B = \mu A + D$, where $A'A^*D = 0$ and $DA^*A' = D$.

If $\mathfrak{G}$ is not one-dimensional, there exists a non-zero element B in $\mathfrak{G}$ such that A and B are linearly independent and either $A'A^*B = B$ or $BA^*A' = B$. Let us assume that $A'A^*B = B$. Then as shown above, $B = \mu A + D$, where $DA^*A' = 0$. Note also that $D \neq 0$ and that $A'A^*D = D$. Now suppose that E is any other element of $\mathfrak{G}$ such that $EA^*A' = E$ and $A'A^*E = 0$. Then

$$EE^*D = EE^*A'A^*D = E(AA'^*E)^*D = E(A'A^*E)^*D = 0$$

and

$$ED^*D = EA^*A'D^*D = E(DA'^*A)^*D = E(DA^*A')^*D = 0 ,$$

so that E and D are orthogonal. Since $D \neq 0$, Lemma 5.1 implies that $E = 0$. Thus, every element C of $\mathfrak{G}$ is of the form $C = \lambda A + F$, where λ is a scalar, $FA^*A' = 0$ and $A'A^*F = F$. Therefore, $A'A^*C = C$ for every element C in $\mathfrak{G}$. If we had assumed instead that $BA^*A' = B$, then a similar argument would show that $CA^*A' = C$ for every C in $\mathfrak{G}$.

LEMMA 5.5. <u>Either</u> $C'C^*D = D$ <u>for every pair of elements</u> C <u>and</u> D <u>in</u> $\mathfrak{G}$ <u>for which</u> $C \neq 0$, <u>or</u> $DC^*C' = D$ <u>for each pair</u> C,D <u>in</u> $\mathfrak{G}$ <u>with</u> $C \neq 0$.

Let A be a non-zero element of $\mathfrak{G}$. By Lemma 5.4 we have that either $A'A^*B = B$ for all $B \in \mathfrak{G}$ or $BA^*A' = B$ for all $B \in \mathfrak{G}$. Let us assume the first possibility. Suppose C is an arbitrary non-zero element of $\mathfrak{G}$. For the above element A, Lemma 5.2 implies that either $C'C^*A = A$ or $AC^*C' = A$. If $AC^*C' = A$, let $E = C'C^*A = C'C^*AC^*C'$. Then $C'C^*E = E = EC^*C'$. By Lemma 5.3 there exists a complex number μ such that $E = \mu C = C'C^*A$. If $\mu = 0$, then $C'C^*A = 0$ and Lemmas 2.1 and 2.2 combine to give that $A'A^*C = 0$. But by assumption, $A'A^*C = C$, so that $C = 0$, a contradiction. Hence, $\mu \neq 0$ and thus $C = \mu^{-1}C'C^*A$. Therefore, $CA^*A' = \mu^{-1}C'C^*AA^*A' = \mu^{-1}C'C^*A = C$. Lemma 5.3 again implies the existence of a complex number λ, necessarily non-zero, such that $C = \lambda A$. It then follows easily that $C'C^*A = A$. Hence, for the above element A we have that $C'C^*A = A$ for all $C \in \mathfrak{G}$ for which $C \neq 0$.

To complete the proof, let C and D be arbitrary elements of $\mathfrak{G}$ such that $C \neq 0$. Then

$$\begin{aligned} C'C^*D &= A'A^*C'C^*D = A'(CC'^*A)^*D \\ &= A'(C'C^*A)^*D = A'A^*D = D\,. \end{aligned}$$

If the initial assumption had been that $BA^*A' = B$, then a similar line of reasoning would have shown that $DC^*C' = D$ for every pair C,D in $\mathfrak{A}$ for which $C \neq 0$.

We are now able to prove the main result.

THEOREM 5.1. _Let_ $\mathfrak{A}$ _be a *-linear generalized ternary algebra in which each element is of degree one. Then one of the following two alternatives is true:_

(i) _If_ A,B _and_ C _are any three elements of_ $\mathfrak{A}$ _for which_ $C \neq 0$, _then there exists a unique complex number_ λ, _independent of_ C, _such that_

$$AB^*C = \lambda C \quad \text{and} \quad BA^*C = \bar{\lambda} C.$$

(ii) _For any triple of elements_ A,B,C _in_ $\mathfrak{A}$ _for which_ $A \neq 0$, _there exists a unique complex number_ λ, _independent of_ A, _such that_

$$AB^*C = \lambda A \quad \text{and} \quad AC^*B = \bar{\lambda} A .$$

By Lemma 5.5 we have that either $D'D^*E = E$ for each pair of elements D,E in $\mathfrak{A}$ for which $D \neq 0$ or $ED^*D' = E$ for any two elements D,E in $\mathfrak{A}$ with $D \neq 0$. We shall suppose that the first alternative holds, and show that this assumption leads to conclusion (i) of the present theorem. A proof analogous to the one below may be used to show that the second alternative implies conclusion (ii).

Let A,B,C by any triple of elements in $\mathfrak{A}$ for which $C \neq 0$. If either $A = 0$ or $B = 0$, λ will be set equal to zero.

Thus, assume $A \neq 0$ and $B \neq 0$.

If A and B are linearly dependent, there exists a complex number μ such that $A = \mu B$. Since B is a non-zero element of degree one, there exists a non-zero real number ν such that $BB^*B = \nu B$. It is easy to check that $B' = \frac{1}{\nu} B$ and $A' = \frac{1}{\overline{\mu}\nu} B$

Thus $A = \mu B = (\mu\nu)B'$ and $B = (\overline{\mu}\nu)A'$. Hence,

$$AB^*C = (\mu\nu)B'B^*C = (\mu\nu)C$$

and

$$BA^*C = (\overline{\mu}\nu)A'A^*C = (\overline{\mu}\nu)C ,$$

so that in this case $\lambda = \mu\nu$.

Now assume that A and B are linearly independent. Since A, B, $A + B$ and $A + iB$ (where $i^2 = -1$) are all $\neq 0$, we have that

$$A'A^*C = C , \tag{5.1}$$

$$B'B^*C = C , \tag{5.2}$$

$$(A + B)'(A + B)^*C = C , \tag{5.3}$$

$$(A + iB)'(A + iB)^*C = C . \tag{5.4}$$

Since A, B, $A + B$ and $A + iB$ are all of degree one, there exist real numbers α,β,γ and δ such that

$$AA^*A = \alpha A ,$$

$$BB^*B = \beta B ,$$

$$(A + B)(A + B)^*(A + B) = \gamma(A + B)$$

and

$$(A + iB)(A + iB)^*(A + iB) = \delta(A + iB) .$$

From these last equations it is not too difficult to check that $A = \alpha A'$, $B = \beta B'$, $A + B = \gamma(A + B)'$ and $A + iB = \delta(A + iB)'$. Equations (5.1) and (5.2) thus become $AA^*C = \alpha C$ and $BB^*C = \beta C$ and from equations (5.3) and (5.4) we get that

$$\begin{aligned}\gamma C = (A + B)(A + B)^*C &= AA^*C + AB^*C + BA^*C + BB^*C \\ &= \alpha C + AB^*C + BA^*C + \beta C ,\end{aligned}$$

and

$$\begin{aligned}\delta C = (A + iB)(A + iB)^*C &= AA^*C - iAB^*C + iBA^*C + BB^*C \\ &= \alpha C - iAB^*C + iBA^*C + \beta C .\end{aligned}$$

These last two equations are now written in the form

$$AB^*C + BA^*C = (\gamma - \alpha - \beta)C ,$$

$$AB^*C - BA^*C = i(\delta - \alpha - \beta)C .$$

Solving for AB^*C and BA^*C yields $AB^*C = \lambda C$ and $BA^*C = \overline{\lambda}C$, where

$$\lambda = \tfrac{1}{2}[(\gamma - \alpha - \beta) + i(\delta - \alpha - \beta)] .$$

To show that λ is unique, suppose that $AB^*C = \lambda C$ and $AB^*C = \mu C$. Since $C \neq 0$, $\lambda = \mu$. The fact that λ is independent of C follows from Theorem 4.5. This concludes the proof.

It should be noted that the above result need not be true in the case when $\mathfrak{a}$ is a <u>real</u> *-linear generalized ternary algebra in which each element is of degree one. Consider, for example, the collection $\mathbb{C}$ of all complex numbers regarded as a linear space over the field of real numbers. If B is a complex number, let $B^* = \overline{B}$, the complex conjugate of B. $\mathfrak{C}$ is then a real ternary algebra and each element of $\mathfrak{C}$ is of degree one. However, if A, B, C is any triple of elements in $\mathfrak{C}$; the product AB^*C is, in general, neither a real multiple of A nor a real multiple of C.

As a simple consequence of the last theorem we have the following

COROLLARY. Let $\mathfrak{R}$ be a minimal right ideal in a ternary algebra $\mathfrak{A}$ such that each element of $\mathfrak{R}$ is of finite degree. If R, S and T are any three elements of $\mathfrak{R}$, then there exists a unique complex number λ, independent of T, such that

$$RS^*T = \lambda T \quad \text{and} \quad SR^*T = \overline{\lambda} T .$$

For the proof, note that Lemma 4.2 ensures that each element of $\mathfrak{R}$ is of degree one. Also, by Theorem 4.4 we have that $R'R^*S = S$ for each pair R, S in $\mathfrak{R}$ for which $R \neq 0$. As was seen in the proof of Theorem 5.1, this property yields the desired result.

COROLLARY. Let $\mathfrak{L}$ be a minimal left ideal in a ternary algebra $\mathfrak{A}$ such that each element of $\mathfrak{L}$ is of finite degree. If L, M, N are any three elements of $\mathfrak{L}$, then there exists a unique complex number λ, independent of L, such that

$$LM^*N = \lambda L \quad \text{and} \quad LN^*M = \overline{\lambda} L .$$

The proof in this case depends upon the fact that each element of $\mathfrak{L}$ is of degree one and that $LM^*M' = L$ for each pair of elements L, M of $\mathfrak{L}$ for which $M \neq 0$.

Now let $\mathfrak{R}$ be a minimal right ideal and let R, S and T be any three elements of $\mathfrak{R}$. By the first corollary to Theorem 5.1 there exists a unique complex number λ such that $RS^*T = \lambda T$ and $SR^*T = \bar{\lambda}T$. We shall write $\lambda = \langle R,S \rangle$, so that $RS^*T = \langle R,S\rangle T$. Since λ is independent of T, we could equally well have defined $\langle R,S\rangle$ to be that number λ such that $RS^*S = \lambda S$. The same notation will be used in the case of a minimal left ideal $\mathfrak{L}$. If $L,M,N \in \mathfrak{L}$, for example, then $LM^*N = \lambda L$ and $LN^*M = \bar{\lambda}L$, so that here $\lambda = \langle M,N\rangle$. It is of interest to note that a minimal right ideal behaves like an inner-product space of row vectors and a minimal left ideal as an inner-product space of column vectors. These ideas will now be made precise with the next two theorems.

THEOREM 5.2. <u>Let</u> $\mathfrak{R}$ <u>be a minimal right ideal in which each element is of finite degree. Then</u> $\mathfrak{R}$ <u>is an inner-product space</u>.

Suppose that R and S are elements of $\mathfrak{R}$ and that $S \neq 0$. Then

$$\langle R,S\rangle S = RS^*S \quad \text{and} \quad SR^*S = \overline{\langle R,S\rangle} S .$$

But $SR^*S = \langle S,R\rangle S$ also. Hence, $\langle S,R\rangle = \overline{\langle R,S\rangle}$.

If $R_1, R_2, S \in \mathfrak{R}$ and $S \neq 0$, then

$$\langle \alpha_1 R_1 + \alpha_2 R_2, S\rangle S = (\alpha_1 R_1 + \alpha_2 R_2)S^*S$$

$$= \alpha_1 R_1 S^*S + \alpha_2 R_2 S^*S = \alpha_1 \langle R_1,S\rangle S + \alpha_2 \langle R_2,S\rangle S ,$$

so that

$$\langle \alpha_1 R_1 + \alpha_2 R_2, S\rangle = \alpha_1 \langle R_1, S\rangle + \alpha_2 \langle R_2, S\rangle \ .$$

Finally, using the positivity condition in $\mathfrak{R}$ it is easy to see that $\langle R,R\rangle \geq 0$ and $\langle R,R\rangle = 0$ if and only if $R = 0$.

THEOREM 5.3. Let $\mathfrak{L}$ be a minimal left ideal in which each element is of finite degree. Then $\mathfrak{L}$ is an inner-product space.

The above notation may also be used in the case when $\mathfrak{A}$ is a ternary algebra in which each element has degree one. For example, if A,B and C are in $\mathfrak{A}$ and $AB^*C = \lambda C$ and $BA^*C = \bar{\lambda} C$, we set $\langle A,B\rangle = \lambda$. With this convention $\mathfrak{A}$ is an inner-product space.

6. MINIMAL CENTRAL IDEALS

The results of the previous section will now be used to complete the study of minimal ideals and special reference will be made to minimal central ideals. It still will not be necessary to assume that the ternary algebra $\mathfrak{A}$ is finite-dimensional. We shall suppose only that every element of an ideal possesses a $*$-reciprocal.

THEOREM 6.1. *If* $\mathfrak{R}$ *and* $\mathfrak{L}$ *are right and left ideals in* $\mathfrak{A}$ *respectively, then* $\mathfrak{R} \cap \mathfrak{L}$ *is a central ideal.*

Let $\mathfrak{C} = \mathfrak{R} \cap \mathfrak{L}$. $\mathfrak{C}$ is clearly a linear subspace of $\mathfrak{A}$. Choose $C \in \mathfrak{C}$ and $A \in \mathfrak{A}$. Since $C \in \mathfrak{R}$, the element $CC^*AC^*C = CC^*(AC^*C)$ is in $\mathfrak{R}$, and because $C \in \mathfrak{L}$ the same element $CC^*AC^*C = (CC^*A)C^*C$ must belong to $\mathfrak{L}$. Hence,

$$CC^*AC^*C \in \mathfrak{R} \cap \mathfrak{L} = \mathfrak{C}$$

so that $\mathfrak{C}$ is a central ideal.

THEOREM 6.2. *Let* $\mathfrak{R}$ *be a minimal right ideal and let* $\mathfrak{L}$ *be a minimal left ideal in* $\mathfrak{A}$. *Then if each element in both* $\mathfrak{R}$ *and* $\mathfrak{L}$ *is of finite degree,* $\mathfrak{R} \cap \mathfrak{L}$ *is either the zero-ideal or is one-dimensional.*

Suppose $\mathfrak{R} \cap \mathfrak{L} \neq (0)$ and choose $A \in \mathfrak{R} \cap \mathfrak{L}$, $A \neq 0$. Let $B \in \mathfrak{R} \cap \mathfrak{L}$. Then since $\mathfrak{R}$ is a minimal right ideal, Theorem 4.4 (i) implies that $A'A^*B = B$. By Lemma 2.2, $A'A^*B = AA'^*B$. Finally, since $\mathfrak{L}$ is a minimal left ideal and $A, A', B \in \mathfrak{L}$, the second corollary to Theorem 5.1 implies that there exists a complex number λ such that $B = AA'^*B = \lambda A$, where $\lambda = \langle A', B\rangle$. Hence, $\mathfrak{R} \cap \mathfrak{L}$ is of dimension one.

As a result of the last two theorems, we have the following

COROLLARY. *Let* $\mathfrak{R}$ *and* $\mathfrak{L}$ *be minimal right and left ideals in* $\mathfrak{C}$ *respectively. Then if each element in both* $\mathfrak{R}$ *and* $\mathfrak{L}$ *is of finite degree,* $\mathfrak{R} \cap \mathfrak{L}$ *is a minimal central ideal.*

The next theorem will not be of use until later, but it will be appropriate to discuss it now.

THEOREM 6.3. *Let* $\mathfrak{R}$ *and* $\mathfrak{S}$ *be minimal right ideals and let* $\mathfrak{L}$ *and* $\mathfrak{M}$ *be minimal left ideals in* $\mathfrak{C}$. *Suppose* $A \in \mathfrak{R} \cap \mathfrak{L}$, $B \in \mathfrak{S} \cap \mathfrak{L}$ *and* $C \in \mathfrak{S} \cap \mathfrak{M}$. *Then:*

(i) $AB^*C \in \mathfrak{R} \cap \mathfrak{M}$.

(ii) *If* $AB^*C = 0$, *then either* $A = 0$ *or* $B = 0$ *or* $C = 0$.

To prove (i), we have that $A \in \mathfrak{R}$ and $\mathfrak{R}$ is a right ideal, so that $AB^*C \in \mathfrak{R}$ also. Since $C \in \mathfrak{M}$ and $\mathfrak{M}$ is a left ideal, $AB^*C \in \mathfrak{M}$. Thus, $AB^*C \in \mathfrak{R} \cap \mathfrak{M}$.

For the proof of (ii), suppose $AB^*C = 0$ and that $A \neq 0$ and $B \neq 0$. Then since A and B are elements of the minimal left ideal $\mathfrak{L}$, Theorem 4.4 (ii) implies that $BA^*A' = B$. Similarly, using the fact that B and C are in the minimal right ideal $\mathfrak{S}$, $B'B^*C = C$. Hence,

$$0 = B'A'^*(AB^*C) = B'(BA^*A')^*C = B'B^*C = C .$$

We now turn our attention to the study of minimal central ideals.

THEOREM 6.4. <u>If $\mathcal{C}$ is a minimal central ideal, then $\mathcal{C}$ contains no non-zero right orthogonal or left orthogonal elements.</u>

The theorem will be proved in the case of left orthogonal elements. Thus, suppose C and D are elements of $\mathcal{C}$ such that $CC^*D = 0$ and that $C \neq 0$. Let $\mathcal{E}$ denote the set of all elements E in $\mathcal{C}$ such that $CC^*E = 0$. $\mathcal{E}$ is a non-empty linear subspace of $\mathcal{G}$. Also, if $E \in \mathcal{E}$ and $A \in \mathcal{G}$, then

$$CC^*(EA^*E) = (CC^*E)A^*E = 0 ,$$

so that $EA^*E \in \mathcal{E}$. Hence, $\mathcal{E}$ is a central ideal in $\mathcal{G}$ and $\mathcal{E} \subseteq \mathcal{C}$. By the minimality of $\mathcal{C}$, either $\mathcal{E} = \{0\}$ or $\mathcal{E} = \mathcal{C}$. But since $C \in \mathcal{C}$ and $C \notin \mathcal{E}$ we have that $\mathcal{E} = \{0\}$ and thus $D = 0$. The proof for right orthogonality is similar.

An element B in $\mathcal{G}$ is called <u>centrally orthogonal</u> to another element A in $\mathcal{G}$ if $AB^*A = 0$. This relation is not symmetric in A and B.

THEOREM 6.5. <u>A central ideal $\mathcal{C}$ is minimal if $\mathcal{C}$ contains no non-zero centrally orthogonal elements.</u>

Suppose $\mathcal{D}$ is a central ideal in $\mathcal{G}$ such that $\mathcal{D} \subseteq \mathcal{C}$. If $\mathcal{D} \neq \{0\}$, choose $D \in \mathcal{D}$, $D \neq 0$. Let $C \in \mathcal{C}$. Then the element $C_0 = D'D^*CD^*D' - C$ is centrally orthogonal to D, i.e., $DC_0{}^*D = 0$. Since $D \neq 0$ and $\mathcal{C}$ contains no non-zero centrally orthogonal elements, we must have that $C_0 = 0$. Hence,

$C = D'D^*CD^*D'$, an element of $\mathfrak{D}$ since $\mathfrak{D}$ is a central ideal. Thus, $\mathfrak{C} \subseteq \mathfrak{D}$ so that $\mathfrak{D} = \mathfrak{C}$ and $\mathfrak{C}$ is therefore minimal.

For any central ideal $\mathfrak{C}$ and any element C in $\mathfrak{C}$, $\mathfrak{R}_C$ will denote as before the set of all elements of the form $C'C^*A$ where A varies over $\mathfrak{A}$. By $\mathfrak{L}_C$ will be meant all elements of the form AC^*C' for A ranging over $\mathfrak{A}$. As was shown previously, $\mathfrak{R}_C$ and $\mathfrak{L}_C$ are right and left ideals in $\mathfrak{A}$ respectively.

The following theorem gives a characterization of minimal central ideals akin to those obtained for right and left ideals.

THEOREM 6.6. *A necessary and sufficient condition that a non-zero central ideal* $\mathfrak{C}$ *be minimal is that* $\mathfrak{C} = \mathfrak{R}_C \cap \mathfrak{L}_C$ *for every non-zero element* C *in* $\mathfrak{C}$.

Suppose that $\mathfrak{C}$ is minimal and that $C \in \mathfrak{C}$, $C \neq 0$. If $D \in \mathfrak{R}_C \cap \mathfrak{L}_C$, then there exist elements A and B in $\mathfrak{A}$ such that $D = C'C^*A = BC^*C'$. This implies that $C'C^*D = D$ and $DC^*C' = D$. Hence, $D = C'C^*DC^*C'$, which is in $\mathfrak{C}$, so that $\mathfrak{R}_C \cap \mathfrak{L}_C \subseteq \mathfrak{C}$. By Theorem 6.1, $\mathfrak{R}_C \cap \mathfrak{L}_C$ is a central ideal in $\mathfrak{A}$. Since $\mathfrak{C}$ is minimal, either $\mathfrak{R}_C \cap \mathfrak{L}_C = \{0\}$ or $\mathfrak{R}_C \cap \mathfrak{L}_C = \mathfrak{C}$. But since $C = C'C^*C = CC^*C'$, we have that $C \in \mathfrak{R}_C \cap \mathfrak{L}_C$ and $C \neq 0$. Hence, $\mathfrak{C} = \mathfrak{R}_C \cap \mathfrak{L}_C$.

For the converse, assume that $\mathfrak{C} = \mathfrak{R}_C \cap \mathfrak{L}_C$ for every non-zero element C in $\mathfrak{C}$. It will be shown that $\mathfrak{C}$ contains no non-zero centrally orthogonal elements. Suppose that D and E are in $\mathfrak{C}$ and that $DE^*D = 0$. If $D \neq 0$, then $\mathfrak{C} = \mathfrak{R}_D \cap \mathfrak{L}_D$. Thus, there exist A and B in $\mathcal{G}$ such that $E = D'D^*A = BD^*D'$. As a result, $D'D^*E = E = ED^*D'$, so that

$$E = D'D^*ED^*D' = D'(DE^*D)^*D' = 0 .$$

By Theorem 6.5, then, $\mathcal{C}$ is minimal.

COROLLARY. <u>A central ideal</u> $\mathcal{C}$ <u>is minimal if and only if</u>

$$C'C^*D = D = DC^*C'$$

<u>for all elements</u> C <u>and</u> D <u>in</u> $\mathcal{C}$ <u>with</u> $C \neq 0$.

If $\mathcal{C}$ is minimal, then $\mathcal{C} = \mathcal{R}_C \cap \mathcal{L}_C$ for all non-zero elements C in $\mathcal{C}$. Hence, if $D \in \mathcal{C}$, there exist A and B in $\mathcal{G}$ such that $D = C'C^*A = BC^*C'$. Therefore,

$$C'C^*D = D = DC^*C' .$$

To prove the converse, suppose that $C'C^*D = D = DC^*C'$ for all C,D in $\mathcal{C}$, $C \neq 0$. If $CD^*C = 0$ for C,D in $\mathcal{C}$ and $C \neq 0$, then

$$D = C'C^*DC^*C' = C'(CD^*C)^*C' = 0 .$$

By Theorem 6.5, $\mathcal{C}$ is minimal.

LEMMA 6.1. <u>Let</u> $\mathcal{C}$ <u>be a central ideal and let</u> C <u>be an element of</u> $\mathcal{C}$ <u>such that</u> $C = D + E$ <u>for some</u> D <u>and</u> E <u>in</u> $\mathcal{G}$. <u>Then if</u> D <u>is orthogonal to</u> E, <u>both</u> D <u>and</u> E <u>are in</u> $\mathcal{C}$.

In [1] it is shown that the orthogonality of D and E implies that $C' = D' + E'$. Using the left and right orthogonality of D and E along with Lemmas 2.1 and 2.2, we get that

$$\begin{aligned} D'E^*D &= E'D^*D = E'E^*D \\ &= DD^*E' = DE^*D' = DE^*E' = 0 . \end{aligned}$$

Thus,

$$C'C^*D = D'D^*D + D'E^*D + E'D^*D + E'E^*D = D'D^*D = D ,$$

$$DC^*C' = DD^*D' + DD^*E' + DE^*D' + DE^*E' = DD^*D' = D .$$

Therefore, $D = C'C^*DC^*C'$ belongs to $\mathbf{C}$. By symmetry, $E \in \mathbf{C}$ also.

LEMMA 6.2. <u>Every non-zero element of finite degree in a minimal central ideal</u> $\mathbf{C}$ <u>is of degree one</u>.

Let C be a non-zero element of finite degree n in $\mathbf{C}$. Suppose that $n > 1$. As in the proof of Lemma 4.2, write $C = D + E$ where D and E are orthogonal and each is of degree at most $n - 1$. If either $D = 0$ or $E = 0$, then C is of degree at most $n - 1$, a contradiction. Hence, $D \neq 0$ and $E \neq 0$. Since D and E are orthogonal, Lemma 6.1 implies that both D and E belong to $\mathbf{C}$. Also, D and E are non-zero right orthogonal elements of $\mathbf{C}$, contradicting Theorem 6.4. Hence $n \leq 1$, and since $C \neq 0$, we have that $n = 1$, completing the proof.

We are now able to completely characterize any minimal central ideal in which each element is of finite degree.

THEOREM 6.7. <u>If</u> $\mathbf{C}$ <u>is a central ideal of dimension one, then</u> $\mathbf{C}$ <u>is minimal. Conversely, if</u> $\mathbf{C}$ <u>is a minimal central ideal in which each element is of finite degree, then</u> $\mathbf{C}$ <u>has dimension one</u>.

The first statement is obvious. To prove the second, we have by the corollary to Theorem 6.6 that $C'C^*D = D = DC^*C'$ for all C,D in $\mathbf{C}$ for which $C \neq 0$. Also, Lemma 6.2 implies that all elements of $\mathbf{C}$ are of degree one. Thus, Theorem 5.1 implies that for any three elements C,D and E in $\mathbf{C}$ there exists a complex number λ such that $CD^*E = \lambda C$. Hence, if C is any non-zero element of $\mathbf{C}$ and $D \in \mathbf{C}$ is arbitrary, then

$$D = C'C^*D = CC'^*D = \lambda C$$

for some complex number λ. The theorem is thus proved.

7. DECOMPOSITION THEOREMS

It will now be shown that a finite-dimensional ternary algebra admits a decomposition into left orthogonal minimal right ideals and into right orthogonal minimal left ideals. Together, these decompositions will allow the algebra to be represented as a direct sum of minimal central ideals possessing certain orthogonality properties.

The first result does not require finite-dimensionality. The symbol $\oplus$ will denote a direct sum.

LEMMA 7.1. *Let $\mathfrak{A}$ be a ternary algebra and suppose $\mathfrak{R}$ is a minimal right ideal and that $\mathfrak{L}$ is a minimal left ideal in $\mathfrak{A}$ such that the *-reciprocal of each element of $\mathfrak{R}$ and $\mathfrak{L}$ exists. Then $\mathfrak{A}$ has the decompositions $\mathfrak{A} = \mathfrak{R} \oplus \mathfrak{R}^{\perp}$ and $\mathfrak{A} = \mathfrak{L} \oplus \mathfrak{L}^{L}$.*

To prove the first decomposition, let A be an arbitrary element of $\mathfrak{A}$. Then if R is any non-zero element of $\mathfrak{R}$,

$$A = R'R^*A + (A - R'R^*A) .$$

Since $\mathfrak{R}$ is a right ideal in $\mathfrak{A}$, $R'R^*A \in \mathfrak{R}$. Let $S = A - R'R^*A$. It will be shown that $S \in \mathfrak{R}^{\perp}$. For this purpose, let T be any element of $\mathfrak{R}$. Since $R \in \mathfrak{R}$ and $R \neq 0$, Theorem 4.4 implies that $R'R^*T = T$. Hence,

$$\begin{aligned} TT^*S &= TT^*A - TT^*R'R^*A = TT^*A - T(RR'^*T)^*A \\ &= TT^*A - T(R'R^*T)^*A = TT^*A - TT^*A = 0 , \end{aligned}$$

where we have used Lemma 2.2. Thus, $S \in \mathfrak{R}^{\perp}$, so that $\mathfrak{R}$ and $\mathfrak{R}^{\perp}$ together generate $\mathfrak{A}$. Finally, if $D \in \mathfrak{R} \cap \mathfrak{R}^{\perp}$, then since $D \in \mathfrak{R}^{\perp}$, D is left orthogonal to every element of $\mathfrak{R}$. But since $D \in \mathfrak{R}$, D is left orthogonal to itself and is thus zero. Hence, $\mathfrak{A} = \mathfrak{R} \oplus \mathfrak{R}^{\perp}$.

An analogous argument proves the result for $\mathfrak{L}$.

THEOREM 7.1. <u>Let $\mathfrak{A}$ be a finite-dimensional ternary algebra. Then there exist mutually left orthogonal minimal right ideals $\mathfrak{R}_1,\ldots,\mathfrak{R}_m$ and mutually right orthogonal minimal left ideals $\mathfrak{L}_1,\ldots,\mathfrak{L}_n$ such that</u>

$$\mathfrak{A} = \mathfrak{R}_1 \oplus \cdots \oplus \mathfrak{R}_m = \mathfrak{L}_1 \oplus \cdots \oplus \mathfrak{L}_n .$$

For the proof, we note that $\mathfrak{A}$ is itself a right ideal. Either $\mathfrak{A}$ is minimal, in which case we are through, or $\mathfrak{A}$ contains a non-zero right ideal $\mathfrak{S}_1$. Either $\mathfrak{S}_1$ is minimal or $\mathfrak{S}_1$ contains a non-zero right ideal $\mathfrak{S}_2$. Continuing in this way, we are led to a descending chain of right ideals: $\mathfrak{A} \supseteq \mathfrak{S}_1 \supseteq \mathfrak{S}_2 \supseteq \cdots$, where the process must terminate after a finite number of steps, since $\mathfrak{A}$ is finite-dimensional. We thus arrive at a non-zero right ideal $\mathfrak{S}_k$ which contains no right ideals other than itself and the zero-ideal. Hence, $\mathfrak{S}_k$ is minimal. Let $\mathfrak{R}_1 = \mathfrak{S}_k$. By Lemma 7.1, $\mathfrak{A} = \mathfrak{R}_1 \oplus \mathfrak{R}_1^{\perp}$. Theorem 3.3 implies that $\mathfrak{R}_1^{\perp}$ is a right ideal in $\mathfrak{A}$. Either $\mathfrak{R}_1^{\perp}$ is minimal, in which case the argument is completed, or by a line of reasoning similar to that above we get that $\mathfrak{R}_1^{\perp}$ contains a minimal right ideal $\mathfrak{R}_2$. By Theorem 3.7, $\mathfrak{R}_2$ is a right ideal in $\mathfrak{A}$. It will now be shown that

$$\mathfrak{R}_1^{\perp} = \mathfrak{R}_2 \oplus (\mathfrak{R}_1^{\perp} \cap \mathfrak{R}_2^{\perp}) \ .$$

Let A be an arbitrary element of $\mathfrak{R}_1^{\perp}$. Then if R is any non-zero element of $\mathfrak{R}_2$, we have that

$$A = R'R^*A + (A - R'R^*A) \ .$$

Let $B = R'R^*A$ and $C = A - R'R^*A$. Clearly, $B \in \mathfrak{R}_2$. To show that $C \in \mathfrak{R}_1^{\perp} \cap \mathfrak{R}_2^{\perp}$, let $S \in \mathfrak{R}_1$ and $T \in \mathfrak{R}_2$. Since $A \in \mathfrak{R}_1^{\perp}$, we have that $SS^*A = 0$, and because $R \in \mathfrak{R}_2 \subseteq \mathfrak{R}_1^{\perp}$ and $S \in \mathfrak{R}_1$, $SS^*R = 0$. Hence,

$$SS^*C = SS^*A - SS^*R'R^*A = - SS^*RR'^*A = 0 \ ,$$

so that $C \in \mathfrak{R}_1^{\perp}$. Also, since $R,T \in \mathfrak{R}_2$ and $R \neq 0$, $R'R^*T = T$. Therefore,

$$\begin{aligned} TT^*C &= TT^*A - TT^*R'R^*A \\ &= TT^*A - T(R'R^*T)^*A = 0 \end{aligned}$$

and so $C \in \mathfrak{R}_2^{\perp}$. Finally, $\mathfrak{R}_1^{\perp} \cap \mathfrak{R}_2^{\perp} \subseteq \mathfrak{R}_2^{\perp}$ implies that

$$\mathfrak{R}_2 \cap (\mathfrak{R}_1^{\perp} \cap \mathfrak{R}_2^{\perp}) = \{0\}$$

so that

$$\mathfrak{R}_1^{\perp} = \mathfrak{R}_2 \oplus (\mathfrak{R}_1^{\perp} \cap \mathfrak{R}_2^{\perp})$$

as asserted.

Thus,

$$\mathfrak{A} = \mathfrak{R}_1 \oplus \mathfrak{R}_1^{\perp} = \mathfrak{R}_1 \oplus \mathfrak{R}_2 \oplus (\mathfrak{R}_1^{\perp} \cap \mathfrak{R}_2^{\perp}) \ .$$

Since $\mathfrak{R}_1^{\lrcorner} \cap \mathfrak{R}_2^{\lrcorner}$ is a right ideal, either it is minimal or the process continues. In the former case we have that $\mathfrak{R}_2 \subseteq \mathfrak{R}_1^{\lrcorner}$ so that $\mathfrak{R}_2$ is left orthogonal to $\mathfrak{R}_1$ and $\mathfrak{R}_1^{\lrcorner} \cap \mathfrak{R}_2^{\lrcorner}$ is contained in both $\mathfrak{R}_1^{\lrcorner}$ and $\mathfrak{R}_2^{\lrcorner}$ and is thus left orthogonal to both $\mathfrak{R}_1$ and $\mathfrak{R}_2$. In the latter case we arrive at a minimal left ideal $\mathfrak{R}_3$ and have that

$$\mathfrak{A} = \mathfrak{R}_1 \oplus \mathfrak{R}_2 \oplus \mathfrak{R}_3 \oplus (\mathfrak{R}_1^{\lrcorner} \cap \mathfrak{R}_2^{\lrcorner} \cap \mathfrak{R}_3^{\lrcorner}) .$$

Since $\mathfrak{A}$ is finite-dimensional, the procedure must terminate in a finite number of steps. Hence, there exist mutually left orthogonal minimal right ideals $\mathfrak{R}_1,\ldots,\mathfrak{R}_m$ such that

$$\mathfrak{A} = \mathfrak{R}_1 \oplus \cdots \oplus \mathfrak{R}_m .$$

The decomposition into left ideals is proved similarly.

THEOREM 7.2. <u>Let</u> $\mathfrak{A}$ <u>be a finite-dimensional ternary algebra</u>.

(i) <u>If</u> $\mathfrak{R}$ <u>is a right ideal in</u> $\mathfrak{A}$, <u>then</u> $\mathfrak{A} = \mathfrak{R} \oplus \mathfrak{R}^{\lrcorner}$.

(ii) <u>If</u> $\mathfrak{L}$ <u>is a left ideal in</u> $\mathfrak{A}$, <u>then</u> $\mathfrak{A} = \mathfrak{L} \oplus \mathfrak{L}^{\llcorner}$.

The proof of (i) will be given. If $\mathfrak{R}$ is minimal, then Lemma 7.1 implies that $\mathfrak{A} = \mathfrak{R} \oplus \mathfrak{R}^{\lrcorner}$ and we are through. Otherwise, by a process like that used in the proof of Theorem 7.1, $\mathfrak{R}$ contains a minimal right ideal $\mathfrak{R}_1$. Theorem 3.7 then implies that $\mathfrak{R}_1$ is a right ideal in $\mathfrak{A}$. Hence, $\mathfrak{A} = \mathfrak{R}_1 \oplus \mathfrak{R}_1^{\lrcorner}$. It will now be shown that $\mathfrak{R} = \mathfrak{R}_1 \oplus (\mathfrak{R}_1^{\lrcorner} \cap \mathfrak{R})$. Suppose R is an arbitrary element of $\mathfrak{R}$. Then $R = S + T$, where $S \in \mathfrak{R}_1$ and $T \in \mathfrak{R}_1^{\lrcorner}$. Since

$\mathfrak{R}_1 \subseteq \mathfrak{R}$ we have that $S \in \mathfrak{R}$ and $T = R - S \in \mathfrak{R}$. Hence, $T \in \mathfrak{R}_1^{\lrcorner} \cap \mathfrak{R}$. Also, it is clear that $\mathfrak{R}_1 \cap (\mathfrak{R}_1^{\lrcorner} \cap \mathfrak{R}) = \{0\}$. The assertion is thus proved.

Now note that if $\mathfrak{R}_1^{\lrcorner} \cap \mathfrak{R} = \{0\}$, then $\mathfrak{R} = \mathfrak{R}_1$, a minimal right ideal. This contradiction implies that $\mathfrak{R}_1^{\lrcorner} \cap \mathfrak{R} \neq \{0\}$, and $\mathfrak{R}_1^{\lrcorner} \cap \mathfrak{R}$ contains a minimal right ideal $\mathfrak{R}_2$. The possibility that $\mathfrak{R}_1^{\lrcorner} \cap \mathfrak{R} = \mathfrak{R}_2$ is not excluded. As was shown in the proof of Theorem 7.1, we have that

$$\mathfrak{A} = \mathfrak{R}_1 \oplus \mathfrak{R}_2 \oplus (\mathfrak{R}_1^{\lrcorner} \cap \mathfrak{R}_2^{\lrcorner}) .$$

Now let $R \in \mathfrak{R}$. Then $R = R_1 + R_2 + S$, where $R_1 \in \mathfrak{R}_1$, $R_2 \in \mathfrak{R}_2$ and $S \in \mathfrak{R}_1^{\lrcorner} \cap \mathfrak{R}_2^{\lrcorner}$. Since $\mathfrak{R}_1 \subseteq \mathfrak{R}$ and $\mathfrak{R}_2 \subseteq \mathfrak{R}$, $S = R - R_1 - R_2$ is an element of $\mathfrak{R}$ also. Therefore,

$$\mathfrak{R} = \mathfrak{R}_1 \oplus \mathfrak{R}_2 \oplus (\mathfrak{R} \cap \mathfrak{R}_1^{\lrcorner} \cap \mathfrak{R}_2^{\lrcorner}) .$$

If $\mathfrak{R} \cap \mathfrak{R}_1^{\lrcorner} \cap \mathfrak{R}_2^{\lrcorner} = \{0\}$, then $\mathfrak{R} = \mathfrak{R}_1 \oplus \mathfrak{R}_2$ and $\mathfrak{A} = \mathfrak{R} \oplus (\mathfrak{R}_1^{\lrcorner} \cap \mathfrak{R}_2^{\lrcorner})$. To complete this case, it will be shown that $\mathfrak{R}_1^{\lrcorner} \cap \mathfrak{R}_2^{\lrcorner} = \mathfrak{R}^{\lrcorner}$. For this purpose, let $S \in \mathfrak{R}_1^{\lrcorner} \cap \mathfrak{R}_2^{\lrcorner}$. Then if R is any element of $\mathfrak{R}$, we may write $R = R_1 + R_2$, where $R_1 \in \mathfrak{R}_1$ and $R_2 \in \mathfrak{R}_2$. Hence,

$$SS^*R = SS^*R_1 + SS^*R_2 = 0 + 0 = 0 ,$$

so that $S \in \mathfrak{R}^{\lrcorner}$. To prove the reverse inclusion, note that since $\mathfrak{R}_1 \subseteq \mathfrak{R}$ and $\mathfrak{R}_2 \subseteq \mathfrak{R}$, Theorem 3.6 implies that $\mathfrak{R}^{\lrcorner} \subseteq \mathfrak{R}_1^{\lrcorner}$ and $\mathfrak{R}^{\lrcorner} \subseteq \mathfrak{R}_2^{\lrcorner}$. Thus, $\mathfrak{R}^{\lrcorner} \subseteq \mathfrak{R}_1^{\lrcorner} \cap \mathfrak{R}_2^{\lrcorner}$ and the equality is established. Therefore, if $\mathfrak{R} \cap \mathfrak{R}_1^{\lrcorner} \cap \mathfrak{R}_2^{\lrcorner} = \{0\}$, then $\mathfrak{A} = \mathfrak{R} \oplus \mathfrak{R}^{\lrcorner}$.

If $\mathfrak{R} \cap \mathfrak{R}_1^{\lrcorner} \cap \mathfrak{R}_2^{\lrcorner} \neq \{0\}$, then $\mathfrak{R} \cap \mathfrak{R}_1^{\lrcorner} \cap \mathfrak{R}_2^{\lrcorner}$ contains a minimal right ideal $\mathfrak{R}_3$ and the above procedure is repeated until we arrive at a minimal right ideal $\mathfrak{R}_p$ such that

$$\mathfrak{R} \cap \mathfrak{R}_1^{\lrcorner} \cap \cdots \cap \mathfrak{R}_p^{\lrcorner} = \{0\} ,$$

$\mathfrak{R} = \mathfrak{R}_1 \oplus \cdots \oplus \mathfrak{R}_p$ and

$$\mathfrak{A} = \mathfrak{R} \oplus (\mathfrak{R}_1^{\lrcorner} \cap \cdots \cap \mathfrak{R}_p^{\lrcorner}).$$

As above, it is easy to show that $\mathfrak{R}_1^{\lrcorner} \cap \cdots \cap \mathfrak{R}_p^{\lrcorner} = \mathfrak{R}^{\lrcorner}$ and therefore that $\mathfrak{A} = \mathfrak{R} \oplus \mathfrak{R}^{\lrcorner}$. This completes the proof.

In the remainder of this section it will be assumed that $\mathfrak{A}$ is a finite-dimensional ternary algebra having decompositions $\mathfrak{A} = \mathfrak{R}_1 \oplus \cdots \oplus \mathfrak{R}_m$ and $\mathfrak{A} = \mathfrak{L}_1 \oplus \cdots \oplus \mathfrak{L}_n$, where $\mathfrak{R}_1,\ldots,\mathfrak{R}_m$ are mutually left orthogonal minimal right ideals and $\mathfrak{L}_1,\ldots,\mathfrak{L}_n$ are mutually right orthogonal minimal left ideals.

LEMMA 7.2. <u>Let</u> A <u>be an arbitrary element of</u> $\mathfrak{A}$.

(i) <u>If</u> R_i <u>is an arbitrary non-zero element of</u> $\mathfrak{R}_i$ <u>for</u> $i = 1,\ldots,m$, <u>then</u>

$$A = R_1'R_1^*A + \cdots + R_m'R_m^*A .$$

(ii) <u>If</u> L_j <u>is an arbitrary non-zero element of</u> $\mathfrak{L}_j$ <u>for</u> $j = 1,\ldots,n$, <u>then</u>

$$A = AL_1^*L_1' + \cdots + AL_n^*L_n'$$

Since $\mathfrak{A} = \mathfrak{R}_1 \oplus \cdots \oplus \mathfrak{R}_m$, we may write $A = A_1 + \cdots + A_m$, where $A_i \in \mathfrak{R}_i$ $(i = 1,\ldots,m)$. Therefore, if $1 \leq i \leq m$,

$R_i'R_i^*A = R_i'R_i^*A_i$, since R_i is left orthogonal to A_j if $i \neq j$. But since R_i and A_i are both in the minimal right ideal $\mathfrak{R}_i$ and $R_i \neq 0$, Theorem 4.4 implies that $R_i'R_i^*A_i = A_i$. Hence,

$$A = R_1'R_1^*A + \cdots + R_m'R_m^*A$$

as asserted.

For the proof of (ii), the decomposition $\mathfrak{A} = \mathfrak{L}_1 \oplus \cdots \oplus \mathfrak{L}_n$ is employed in an analogous manner.

For each $i = 1,\ldots,m$, the element $R_i'R_i^*A$ is called the <u>left projection</u> of A by R_i. It is a projection of A into the right ideal $\mathfrak{R}_i$. Similarly, each of the elements $AL_j^*L_j'$ for $j = 1,\ldots,n$ is called the <u>right projection</u> of A by L_j.

LEMMA 7.3. <u>Let</u> A <u>be an element of</u> $\mathfrak{A}$. <u>If</u> R_i <u>is an arbitrary non-zero element of</u> $\mathfrak{R}_i$ <u>for</u> $i = 1,\ldots,m$ <u>and if</u> L_j <u>is any non-zero element of</u> $\mathfrak{L}_j$ <u>for</u> $j = 1,\ldots,n$, <u>then</u> A <u>has the decomposition</u>

$$A = \sum_{i=1}^{m} \sum_{j=1}^{n} R_i'R_i^*AL_j^*L_j' .$$

By Lemma 7.2, $A = \Sigma_{i=1}^m R_i'R_i^*A$ and $A = \Sigma_{j=1}^n AL_j^*L_j'$. Hence,

$$A = \sum_{i=1}^{m} R_i' R_i^* A = \sum_{i=1}^{m} R_i' R_i^* \left(\sum_{j=1}^{n} A L_j^* L_j' \right)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} R_i' R_i^* A L_j^* L_j' \ ,$$

as was to be shown.

If $\mathfrak{B}_1, \ldots, \mathfrak{B}_k$ are subspaces of $\mathfrak{A}$, the notation $\overset{k}{\underset{i=1}{\oplus}} \mathfrak{B}_i$ will denote the direct sum of $\mathfrak{B}_1, \ldots, \mathfrak{B}_k$.

THEOREM 7.3. $\mathfrak{A}$ <u>has the decomposition</u>

$$\mathfrak{A} = \overset{m}{\underset{i=1}{\oplus}} \ \overset{n}{\underset{j=1}{\oplus}} (\mathfrak{R}_i \cap \mathfrak{L}_j) \ .$$

<u>The subspaces</u> $\mathfrak{R}_i \cap \mathfrak{L}_j$ <u>are central ideals which are either the zero-ideal or are one-dimensional</u>.

If A is any element of $\mathfrak{A}$, we have by Lemma 7.3 that

$$A = \sum_{i=1}^{m} \sum_{j=1}^{n} R_i' R_i^* A L_j^* L_j' \ ,$$

where R_i is an arbitrary non-zero element of $\mathfrak{R}_i$ $(i = 1, \ldots, m)$ and L_j is any non-zero element of $\mathfrak{L}_j$ $(j = 1, \ldots, n)$. Since the element $R_i' R_i^* A L_j^* L_j'$ belongs to both $\mathfrak{R}_i$ and $\mathfrak{L}_j$, we see that the subspaces $\mathfrak{R}_i \cap \mathfrak{L}_j$ generate $\mathfrak{A}$. Also, using the left orthogonality of $\mathfrak{R}_1, \ldots, \mathfrak{R}_m$ and the right orthogonality of $\mathfrak{L}_1, \ldots, \mathfrak{L}_n$ it is seen that if $C_{ij} \in \mathcal{R}_i \cap \mathcal{L}_j$ for $i = 1,\ldots,m$ and $j = 1,\ldots,n$ and $\Sigma_{i=1}^m \Sigma_{j=1}^n C_{ij} = 0$, then

$$0 = C'_{k\ell}C^*_{k\ell}\left(\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij}\right)C^*_{k\ell}C'_{k\ell}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} C'_{k\ell}C^*_{k\ell}C_{ij}C^*_{k\ell}C'_{k\ell}$$

$$= \sum_{j=1}^{n} C'_{k\ell}C^*_{k\ell}C_{kj}C^*_{k\ell}C'_{k\ell}$$

$$= C'_{k\ell}C^*_{k\ell}C_{k\ell}C^*_{k\ell}C'_{k\ell} = C_{k\ell} ,$$

for any $k = 1,\ldots,m$ and $\ell = 1,\ldots,n$. Hence, the subspaces $\mathcal{R}_i \cap \mathcal{L}_j$ are independent.

This proves the first statement. The second assertion is a direct result of Theorems 6.1 and 6.2.

The section will be concluded with a study of the relationships between the subspaces $\mathcal{R}_i \cap \mathcal{L}_j$.

THEOREM 7.4. <u>For each</u> $\mathcal{R}_i$ <u>there exists an</u> $\mathcal{L}_j$ <u>such that</u> $\mathcal{R}_i \cap \mathcal{L}_j \neq \{0\}$. <u>Also, for each</u> $\mathcal{L}_j$ <u>there exists an</u> $\mathcal{R}_i$ <u>such that</u> $\mathcal{R}_i \cap \mathcal{L}_j \neq \{0\}$.

The result will be proved by showing that for any fixed integer i,

$$\mathcal{R}_i = (\mathcal{R}_i \cap \mathcal{L}_1) \oplus \cdots \oplus (\mathcal{R}_i \cap \mathcal{L}_n) .$$

Let $R \in \mathcal{R}_i$. Then since

$$\mathcal{G} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$$

and $R \in \mathcal{G}$, there exist elements $L_j \in \mathcal{L}_j$ $(j = 1,\ldots,n)$ such that $R = L_1 + \cdots + L_n$. Therefore,

$$R = R'R^*R = R'R^*L_1 + \cdots + R'R^*L_n .$$

Now note that $R'R^*L_j \in \mathcal{R}_i \cap \mathcal{L}_j$ for $j = 1,\ldots,n$ so that the subspaces $\mathcal{R}_i \cap \mathcal{L}_1,\ldots,\mathcal{R}_i \cap \mathcal{L}_n$ generate $\mathcal{R}_i$. Also, these same spaces are independent, since they are mutually right orthogonal. This proves the assertion.

Hence, if $\mathcal{R}_i \cap \mathcal{L}_j = \{0\}$ for $j = 1,\ldots,n$, then $\mathcal{R}_i = \{0\}$, a contradiction.

The second statement of the theorem is proved by using a similar argument to show that

$$\mathcal{L}_j = (\mathcal{R}_1 \cap \mathcal{L}_j) \oplus \cdots \oplus (\mathcal{R}_m \cap \mathcal{L}_j) .$$

The minimal central ideals

$$C_{ij} = \mathcal{R}_i \cap \mathcal{L}_j \qquad (i = 1,\ldots,m \, ; \, j = 1,\ldots,n)$$

into which $\mathcal{G}$ has been decomposed may be considered in the form of an array as follows:

$$\begin{array}{cccc} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ C_{m1} & C_{m2} & \cdots & C_{mn} \end{array}$$

As noted in Theorem 7.3, each central ideal C_{ij} is either the zero-ideal or else is one-dimensional. Also, Theorem 7.4 guarantees that each row and each column in the above array contains at least one non-zero subspace C_{ij}. It is of interest to study the arrangement of the zero and non-zero subspaces.

LEMMA 7.4. Suppose that $\mathfrak{R}$ and $\mathfrak{S}$ are minimal right ideals and that $\mathfrak{L}$ and $\mathfrak{M}$ are minimal left ideals in $\mathfrak{A}$. Let $\mathcal{C} = \mathfrak{R} \cap \mathfrak{L}$, $\mathfrak{D} = \mathfrak{R} \cap \mathfrak{M}$, $\mathcal{E} = \mathfrak{S} \cap \mathfrak{L}$ and $\mathfrak{F} = \mathfrak{S} \cap \mathfrak{M}$. If any two of $\mathcal{C}, \mathfrak{D}, \mathcal{E}, \mathfrak{F}$ are non-zero subspaces, then the remaining two subspaces are either both non-zero or are both zero.

All of the possibilities that can occur are treated similarly. As illustrations, only two such possibilities will be considered.

For example, let us assume that $\mathcal{C} \neq \{0\}$ and $\mathcal{E} \neq \{0\}$. If $\mathfrak{F} \neq \{0\}$ also, choose $C \in \mathcal{C}$, $E \in \mathcal{E}$ and $F \in \mathfrak{F}$ such that C, E and F are all non-zero. Theorem 6.3 then implies that $CE^*F \in \mathfrak{D}$ and that $CE^*F \neq 0$.

As a second case, suppose that $\mathcal{C} \neq \{0\}$ and $\mathcal{E} \neq \{0\}$ but that $\mathfrak{F} = \{0\}$. Let D be an arbitrary element of $\mathfrak{D}$. Then if C is any non-zero element selected from the subspace $\mathcal{C}$, and if E

is a non-zero element of $\mathcal{E}$ we have by Theorem 6.3 that $EC^*D \in \mathcal{S} \cap \mathfrak{M} = \mathfrak{Z}$. Hence, $EC^*D = 0$ and since $E \neq 0$ and $C \neq 0$, the same theorem implies that $D = 0$.

Similar arguments employing the same theorem prove all remaining cases.

Consider again the decomposition

$$\mathcal{G} = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} C_{ij}$$

where $C_{ij} = \mathcal{R}_i \cap \mathcal{L}_j$ and recall that for any integer j there exists an integer i such that $C_{ij} \neq \{0\}$.

THEOREM 7.5. <u>Let</u> i <u>be an integer such that</u> $C_{i1} \neq \{0\}$. <u>Suppose that</u> $C_{ij} \neq \{0\}$ $(j = 1,\ldots,j_1)$, $C_{ij} = \{0\}$ $(j = j_1+1,\ldots,j_2)$, $C_{ij} \neq \{0\}$ $(j = j_2+1,\ldots,j_3)$, <u>etc. Then:</u>

(i) <u>If</u> k <u>is any other integer such that</u> $C_{k1} \neq \{0\}$, <u>we have that</u> $C_{kj} \neq \{0\}$ $(j = 1,\ldots,j_1)$, $C_{kj} = \{0\}$ $(j = j_1+1,\ldots,j_2)$, $C_{kj} \neq \{0\}$ $(j = j_2+1,\ldots,j_3)$, <u>etc</u>.

(ii) <u>If</u> h <u>is an integer such that</u> $C_{h1} = \{0\}$, <u>we have that</u> $C_{hj} = \{0\}$ $(j = 1,\ldots,j_1)$, $C_{hj} = \{0\}$ $(j = j_2+1,\ldots,j_3)$, <u>etc</u>.

To prove (i), let $1 \leq j \leq j_1$. Then since $C_{ij} \neq \{0\}$, $C_{i1} \neq \{0\}$ and $C_{k1} \neq \{0\}$, Lemma 7.4 implies that $C_{kj} \neq \{0\}$ also. If $j_1 + 1 \leq j \leq j_2$, then $C_{ij} = \{0\}$, $C_{i1} \neq \{0\}$ and $C_{k1} \neq \{0\}$. Hence, by Lemma 7.4 we have that $C_{kj} = \{0\}$. All remaining cases are treated similarly.

For the proof of (ii), let $1 \leq j \leq j_1$. Then $C_{ij} \neq \{0\}$, $C_{i1} \neq \{0\}$ and $C_{h1} = \{0\}$. Therefore, $C_{hj} = \{0\}$ by Lemma 7.4. The remaining cases are proved in the same way.

The last result makes clear the arrangement of the zero and non-zero subspaces C_{ij}. For suppose that row i in the array (C_{kj}) has the form

$$C_{i1} \cdots C_{ij_1} \{0\} \cdots \{0\}\, C_{i,j_2+1} \cdots C_{ij_3}\ \{0\} \cdots .$$

Then every other row is either of the form

$$C_{k1} \cdots C_{kj_1} \{0\} \cdots \{0\}\, C_{k,j_2+1} \cdots C_{kj_3} \{0\} \cdots$$

or

$$\{0\} \cdots \{0\}\, C_{k,j_1+1} \cdots C_{kj_2} \{0\} \cdots \{0\}\, C_{k,j_3+1} \cdots .$$

The right ideals $\mathcal{R}_1, \ldots, \mathcal{R}_m$ and the left ideals $\mathcal{L}_1, \ldots, \mathcal{L}_n$ can be rearranged so that the organization of the subspaces C_{kj} is of the form

$$\begin{array}{cccc} \mathcal{B}_1 & 0 & 0 \cdots & 0 \\ 0 & \mathcal{B}_2 & 0 \cdots & 0 \\ \cdot & \cdot & \cdot \cdots & \cdot \\ 0 & 0 & 0 \ldots & \mathcal{B}_p \end{array}$$

where each $\mathcal{B}_i$ is a block of non-zero subspaces (C_{kj}) and each 0 represents a rectangular matrix of zero-subspaces. This arrangement will be referred to as a <u>canonical decomposition</u> of $\mathfrak{C}$.

Either all of the subspaces C_{ij} are non-zero or else $\mathfrak{G}$ has such a decomposition.

The decompositions of this section may be used to obtain an alternate proof of the fact that any non-zero minimal central ideal in $\mathfrak{G}$ has dimension one.

THEOREM 7.6. Let $\mathfrak{C}$ be a non-zero minimal central ideal in $\mathfrak{G}$. Then there exist a minimal right ideal $\mathfrak{R}$ and a minimal left ideal $\mathfrak{L}$ in $\mathfrak{G}$ such that

$$\mathfrak{C} = \mathfrak{R} \cap \mathfrak{L} .$$

By Theorem 6.6, $\mathfrak{C}$ has the representation

$$\mathfrak{C} = \mathfrak{R}_C \cap \mathfrak{L}_C ,$$

for every non-zero element C in $\mathfrak{C}$, where $\mathfrak{R}_C$ and $\mathfrak{L}_C$ are the right and left ideals as defined in Section 6. If $\mathfrak{R}_C$ is not minimal, then Theorem 7.1 implies the existence of mutually left orthogonal minimal right ideals $\mathfrak{R}_1,\ldots,\mathfrak{R}_p$ in $\mathfrak{R}_C$ such that

$$\mathfrak{R}_C = \mathfrak{R}_1 \oplus \cdots \oplus \mathfrak{R}_p .$$

By Theorem 3.7, each of $\mathfrak{R}_1,\ldots,\mathfrak{R}_p$ is a right ideal in $\mathfrak{G}$. Now let D be any element of $\mathfrak{C}$. Then $D \in \mathfrak{R}_C$ and there exist elements $R_i \in \mathfrak{R}_i$ $(i = 1,\ldots,p)$ such that

$$\begin{aligned} D &= R_1 + \cdots + R_p \\ &= DC^*C' \\ &= R_1C^*C' + \cdots + R_pC^*C' , \end{aligned}$$

where we have used the corollary to Theorem 6.6. Since $R_iC^*C' \in \mathcal{R}_i \cap \mathcal{L}_C$ for $i = 1,\ldots,p$, the subspaces $\mathcal{R}_1 \cap \mathcal{L}_C,\ldots,\mathcal{R}_p \cap \mathcal{L}_C$ generate C. Because $\mathcal{R}_1 \cap \mathcal{L}_C,\ldots,\mathcal{R}_p \cap \mathcal{L}_C$ are left orthogonal,

$$C = (\mathcal{R}_1 \cap \mathcal{L}_C) \oplus \cdots \oplus (\mathcal{R}_p \cap \mathcal{L}_C) .$$

There exists at least one integer k such that $\mathcal{R}_k \cap \mathcal{L}_C \neq \{0\}$. $\mathcal{R}_k \cap \mathcal{L}_C$ is a central ideal properly contained in $\mathcal{R}_C \cap \mathcal{L}_C = C$, a contradiction. Therefore, $\mathcal{R}_C$ is minimal. A similar argument proves that $\mathcal{L}_C$ is minimal. If we now fix the non-zero element C and let $\mathcal{R} = \mathcal{R}_C$ and $\mathcal{L} = \mathcal{L}_C$, the result follows.

Theorem 6.7 is now a direct corollary of this last result and Theorem 6.2.

8. UNIQUENESS OF THE DECOMPOSITIONS

Let $\mathfrak{A}$ be a ternary algebra, not necessarily finite-dimensional. $\mathfrak{A}$ will be called simple if it is not possible to decompose $\mathfrak{A}$ in the form $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{C}$, where $\mathfrak{B}$ and $\mathfrak{C}$ are non-zero orthogonal ternary subalgebras of $\mathfrak{A}$.

As an example of a ternary algebra which is not simple, suppose that $\mathfrak{A}$ is finite-dimensional and has the decomposition

$$\mathfrak{A} = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} \mathfrak{C}_{ij}$$

as in the previous section. If at least one of the central ideals $\mathfrak{C}_{ij}$ is the zero-subspace, then as was shown at the end of Section 7, $\mathfrak{A}'$ has the canonical decomposition

$$\begin{matrix} \mathfrak{B}_1 & 0 & 0 \cdots 0 \\ 0 & \mathfrak{B}_2 & 0 \cdots 0 \\ \cdot & \cdot & \cdot \cdots \cdot \\ 0 & 0 & 0 \cdots \mathfrak{B}_p \end{matrix} .$$

It is clear that $\mathfrak{B}_1,\ldots,\mathfrak{B}_p$ are mutually orthogonal subalgebras and that $\mathfrak{A} = \mathfrak{B}_1 \oplus \cdots \oplus \mathfrak{B}_p$.

By a two-sided ideal in $\mathfrak{A}$ will be meant a linear subspace $\mathfrak{J}$ of $\mathfrak{A}$ such that $\mathfrak{J}$ is both a left and a right ideal in $\mathfrak{A}$.

LEMMA 8.1. If $\mathfrak{B}_1,\ldots,\mathfrak{B}_p$ are mutually orthogonal ternary subalgebras of $\mathfrak{A}$ such that $\mathfrak{A} = \mathfrak{B}_1 \oplus \cdots \oplus \mathfrak{B}_p$, then $\mathfrak{B}_1,\ldots,\mathfrak{B}_p$

<u>are all two sided ideals in</u> $\mathfrak{A}$.

Let i be a fixed integer between 1 and p inclusive, and choose $B_i \in \mathfrak{B}_i$. If A is an arbitrary element of $\mathfrak{A}$, then there exist elements $A_k \in \mathfrak{B}_k$ $(k = 1,\ldots,p)$ such that

$$A = A_1 + \cdots + A_p \ .$$

Using the orthogonality of $\mathfrak{B}_1,\ldots,\mathfrak{B}_p$ we get that $B_i B_i^* A = B_i B_i^* A_i$ and $AB_i^* B_i = A_i B_i^* B_i$. Since $\mathfrak{B}_i$ is a subalgebra of $\mathfrak{A}$, both $B_i B_i^* A_i$ and $A_i B_i^* B_i$ are elements of $\mathfrak{B}_i$. This completes the proof.

THEOREM 8.1. <u>If</u> $\mathfrak{A}$ <u>is a non-simple finite-dimensional ternary algebra, then there exist mutually orthogonal simple subalgebras</u> $\mathfrak{B}_1,\ldots,\mathfrak{B}_p$ <u>of</u> $\mathfrak{A}$ <u>such that</u> $\mathfrak{A} = \mathfrak{B}_1 \oplus \cdots \oplus \mathfrak{B}_p$. <u>Furthermore, this decomposition of</u> $\mathfrak{A}$ <u>is unique to within a re-ordering of</u> $\mathfrak{B}_1,\ldots,\mathfrak{B}_p$.

Since $\mathfrak{A}$ is non-simple, there exist orthogonal subalgebras $\mathfrak{B}$ and $\mathfrak{C}$ of $\mathfrak{A}$ such that $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{C}$. Either $\mathfrak{B}$ is simple or else $\mathfrak{B}$ contains orthogonal subalgebras $\mathfrak{D}$ and $\mathfrak{E}$ such that $\mathfrak{B} = \mathfrak{D} \oplus \mathfrak{E}$. This procedure can only be repeated a finite number of times, since $\mathfrak{A}$ is finite-dimensional. We therefore arrive finally at a decomposition

$$\mathfrak{A} = \mathfrak{B}_1 \oplus \cdots \oplus \mathfrak{B}_p$$

of $\mathfrak{A}$ into mutually orthogonal simple subalgebras.

To prove the uniqueness, suppose that $C_1,\ldots,C_q$ are also mutually orthogonal simple subalgebras of $\mathfrak{C}$ such that $\mathfrak{C} = \mathfrak{C}_1 \oplus \cdots \oplus \mathfrak{C}_q$. By Lemma 8.1 we have that $\mathfrak{B}_1, \ldots, \mathfrak{B}_p$ and $\mathfrak{C}_1, \ldots, \mathfrak{C}_q$ are two-sided ideals in $\mathfrak{C}$.

Let B_1 be an arbitrary element of $\mathfrak{B}_1$. Then $B_1 = C_1 + \cdots + C_q$, where $C_i \in \mathfrak{C}_i$ $(i = 1, \ldots, q)$. But also

$$B_1 = B_1^! B_1^* B_1 = B_1^! B_1^* C_1 + \cdots + B_1^! B_1^* C_q ,$$

where $B_1^! B_1^* C_j \in \mathfrak{B}_1 \cap \mathfrak{C}_j$ for $j = 1, \ldots, q$. Thus, the subalgebras $\mathfrak{B}_1 \cap \mathfrak{C}_1, \ldots, \mathfrak{B}_1 \cap \mathfrak{C}_q$ generate $\mathfrak{B}_1$. Also, since $\mathfrak{B}_1 \cap \mathfrak{C}_j \subseteq \mathfrak{C}_j$ for $j = 1, \ldots, q$ and $\mathfrak{C}_1, \ldots, \mathfrak{C}_q$ are mutually orthogonal, it is clear that $\mathfrak{B}_1 \cap \mathfrak{C}_1, \ldots, \mathfrak{B}_1 \cap \mathfrak{C}_q$ are orthogonal. Hence,

$$\mathfrak{B}_1 = (\mathfrak{B}_1 \cap \mathfrak{C}_1) \oplus \cdots \oplus (\mathfrak{B}_1 \cap \mathfrak{C}_q) ,$$

a decomposition of $\mathfrak{B}_1$ into mutually orthogonal subalgebras. Since $\mathfrak{B}_1$ is simple, all but one of $\mathfrak{B}_1 \cap \mathfrak{C}_1, \ldots, \mathfrak{B}_1 \cap \mathfrak{C}_q$ must be the zero-subspace. Therefore, there exists an integer i_1 such that $\mathfrak{B}_1 = \mathfrak{B}_1 \cap \mathfrak{C}_{i_1}$.

By an argument similar to that above, it can be shown that

$$\mathfrak{C}_{i_1} = (\mathfrak{C}_{i_1} \cap \mathfrak{B}_1) \oplus \cdots \oplus (\mathfrak{C}_{i_1} \cap \mathfrak{B}_p) ,$$

and because $\mathfrak{C}_{i_1}$ is simple and $\mathfrak{C}_{i_1} \cap \mathfrak{B}_1 = \mathfrak{B}_1$, a non-zero subalgebra, we have that $\mathfrak{C}_{i_1} = \mathfrak{C}_{i_1} \cap \mathfrak{B}_1 = \mathfrak{B}_1$. Repeating the argument for $\mathfrak{B}_2, \ldots, \mathfrak{B}_p$, we get that there exist integers $i_2, \ldots, i_p$ such that

$$\mathfrak{B}_2 = \mathfrak{C}_{i_2}, \ldots, \mathfrak{B}_p = \mathfrak{C}_{i_p} .$$

Also, since $\mathfrak{B}_i$ is orthogonal to $\mathfrak{B}_j$ if $i \neq j$, it is seen that the integers $i_1,\ldots,i_p$ are all distinct. By symmetry, there exist distinct integers $j_1,\ldots,j_q$ such that

$$C_1 = \mathfrak{B}_{j_1},\ldots,C_q = \mathfrak{B}_{j_q} .$$

Hence, there is a one-to-one correspondence between $\mathfrak{B}_1,\ldots,\mathfrak{B}_p$ and $C_1,\ldots,C_q$ so that $p = q$ and $C_1,\ldots,C_q$ is merely a rearrangement of $\mathfrak{B}_1,\ldots,\mathfrak{B}_p$.

It will be assumed throughout the remainder of this section that $\mathfrak{A}$ is a finite-dimensional ternary algebra. If $\mathfrak{B}$ is a subspace, then $\dim(\mathfrak{B})$ will denote the dimension of $\mathfrak{B}$.

LEMMA 8.2.

(i) Let $\mathfrak{R}$ and $\mathfrak{S}$ be minimal right ideals in $\mathfrak{A}$. If $\mathfrak{R}$ and $\mathfrak{S}$ are not left orthogonal, then $\mathfrak{R}$ and $\mathfrak{S}$ have the same dimension.

(ii) Let $\mathfrak{L}$ and $\mathfrak{M}$ be minimal left ideals in $\mathfrak{A}$. If $\mathfrak{L}$ is not right orthogonal to $\mathfrak{M}$, then $\dim(\mathfrak{L}) = \dim(\mathfrak{M})$.

Choose non-zero elements $R \in \mathfrak{R}$ and $S \in \mathfrak{S}$ such that $RR^*S \neq 0$. If T and U are non-zero elements of $\mathfrak{R}$ and $\mathfrak{S}$ respectively, such that $TT^*U = 0$, then

$$0 = T'T'^*(TT^*U)U'^*U' = T'T^*U' .$$

Hence,

$$0 = RR^*(T'T^*U')U^*S = R(T'T^*R)^*(U'U^*S) = RR^*S ,$$

since $T, R \in \mathfrak{R}$ and $T \neq 0$ and $U, S \in \mathfrak{S}$ and $U \neq 0$. Our supposition that $TT^*U = 0$ thus leads to a contradiction, so that $TT^*U \neq 0$ for all non-zero elements $T \in \mathfrak{R}$ and $U \in \mathfrak{S}$.

Since $\mathfrak{S}$ is an inner-product space, $\mathfrak{S}$ has a basis $S_1, \ldots, S_n$ such that $S_i S_j^* S_j = \delta_{ij} S_j$, where δ_{ij} is the Kronecker delta. If R is any non-zero element of $\mathfrak{R}$, then $RR^*S_1, \ldots, RR^*S_n$ are all non-zero and belong to $\mathfrak{R}$. They are also linearly independent, for if $\Sigma_{i=1}^n \lambda_i RR^*S_i = 0$, then

$$0 = \left(\sum_{i=1}^{n} \lambda_i RR^*S_i\right) S_j^*S_j = \sum_{i=1}^{n} \lambda_i RR^*\left(S_i S_j^* S_j\right)$$

$$= \sum_{i=1}^{n} \lambda_i \delta_{ij} RR^*S_j = \lambda_j RR^*S_j$$

for $j = 1, \ldots, n$. Hence, $\lambda_j = 0$. Therefore $\dim(\mathfrak{R}) \geq \dim(\mathfrak{S})$. By symmetry, $\dim(\mathfrak{S}) \geq \dim(\mathfrak{R})$, and part (i) is proved.

The proof of (ii) is based on the fact that $LM^*M \neq 0$ for any non-zero elements $L \in \mathfrak{L}$ and $M \in \mathfrak{M}$.

LEMMA 8.3.

(i) <u>If</u> $\mathfrak{R}$ <u>and</u> $\mathfrak{S}$ <u>are minimal right ideals in</u> $\mathfrak{a}$, <u>then if</u> $\mathfrak{R}$ <u>and</u> $\mathfrak{S}$ <u>are not orthogonal, they have the same dimension</u>.

(ii) <u>If</u> $\mathfrak{L}$ <u>and</u> $\mathfrak{M}$ <u>are minimal left ideals in</u> $\mathfrak{a}$, <u>then if</u>

$\mathfrak{L}$ <u>is not orthogonal to</u> $\mathfrak{M}$, <u>then</u> $\dim(\mathfrak{L}) = \dim(\mathfrak{M})$.

We may assume that $\mathfrak{R}$ is left orthogonal to $\mathfrak{S}$, since otherwise Lemma 8.2 implies that $\mathfrak{R}$ and $\mathfrak{S}$ have the same dimension.

Let $\mathfrak{C} = \mathfrak{L}_1 \oplus \cdots \oplus \mathfrak{L}_n$ be a decomposition of $\mathfrak{C}$ into mutually right orthogonal minimal left ideals. Using the same argument as in the proof of Theorem 7.4 it can be shown that

$$\mathfrak{R} = (\mathfrak{R} \cap \mathfrak{L}_1) \oplus \cdots \oplus (\mathfrak{R} \cap \mathfrak{L}_n)$$

and that

$$\mathfrak{S} = (\mathfrak{S} \cap \mathfrak{L}_1) \oplus \cdots \oplus (\mathfrak{S} \cap \mathfrak{L}_n) .$$

Since $\mathfrak{R} \neq \{0\}$, there exists an integer k such that $\mathfrak{R} \cap \mathfrak{L}_k \neq \{0\}$.

Suppose that $\mathfrak{S} \cap \mathfrak{L}_k = \{0\}$. If j is an integer such that $\mathfrak{R} \cap \mathfrak{L}_j \neq \{0\}$, we have by Lemma 7.4 that $\mathfrak{S} \cap \mathfrak{L}_j = \{0\}$. Hence, if we denote by I the index set for which $\mathfrak{R} \cap \mathfrak{L}_j \neq \{0\}$, then $\mathfrak{S} \cap \mathfrak{L}_j = \{0\}$ for $j \in I$. It will now be shown that $\mathfrak{R}$ and $\mathfrak{S}$ are orthogonal. Let $R \in \mathfrak{R}$ and $S \in \mathfrak{S}$. Then $R = \Sigma_{i\in I} R_i$ and $S = \Sigma_{i\in I'} S_i$, where $R_i \in \mathfrak{R} \cap \mathfrak{L}_i$, $S_i \in \mathfrak{S} \cap \mathfrak{L}_i$ and I' denotes the complement of I. Then

$$RS^*S = \sum_{i\in I} \sum_{j\in I'} R_i S_j^* S = 0 ,$$

since R_i and S_j belong to different right orthogonal left ideals. Also, $RR^*S = 0$ since $\mathcal{R}$ and $\mathcal{S}$ are left orthogonal. Because R and S were arbitrary, $\mathcal{R}$ is orthogonal to $\mathcal{S}$. This contradiction implies that $S \cap \mathcal{L}_k \neq \{0\}$. Lemma 7.4 then implies that if $1 \leq j \leq n$, then $\mathcal{R} \cap \mathcal{L}_j$ and $\mathcal{S} \cap \mathcal{L}_j$ are either both non-zero or are both zero-subspaces. Thus, if I is again the index set for which $\mathcal{R} \cap \mathcal{L}_j \neq \{0\}$, then $\mathcal{S} \cap \mathcal{L}_j \neq \{0\}$ for $j \in I$ also. In addition, if $j \in I$ then both $\mathcal{R} \cap \mathcal{L}_j$ and $\mathcal{S} \cap \mathcal{L}_j$ have dimension one. $\mathcal{R}$ and $\mathcal{S}$ therefore have the same dimension.

The proof of (ii) is accomplished by using the decomposition

$$\mathcal{A} = \mathcal{R}_1 \oplus \cdots \oplus \mathcal{R}_m$$

of $\mathcal{A}$ into mutually left orthogonal minimal right ideals.

THEOREM 8.2. Let $\mathcal{A}$ be simple. Then:

(i) If $\mathcal{A} = \mathcal{R}_1 \oplus \cdots \oplus \mathcal{R}_m$ is any decomposition of $\mathcal{A}$ into mutually left orthogonal minimal right ideals, then $\mathcal{R}_1, \ldots, \mathcal{R}_m$ all have the same dimension.

(ii) If $\mathcal{A} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$ is a decomposition of $\mathcal{A}$ into mutually right orthogonal minimal left ideals, then $\mathcal{L}_1, \ldots, \mathcal{L}_n$ all have the same dimension.

Let $\mathcal{A} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$ be a decomposition of $\mathcal{A}$ into mutually right orthogonal minimal left ideals. As in Theorem 7.4 we have that

$$\mathcal{R}_i = (\mathcal{R}_i \cap \mathcal{L}_1) \oplus \cdots \oplus (\mathcal{R}_i \cap \mathcal{L}_n)$$

for $i = 1,\ldots,m$. Since $\mathfrak{A}$ is simple, each of $\mathfrak{R}_i \cap \mathfrak{L}_1,\ldots,\mathfrak{R}_i \cap \mathfrak{L}_n$ is non-zero and is therefore of dimension one. Hence, $\dim(\mathfrak{R}_i) = n$ for $i = 1,\ldots,m$.

The proof of (ii) will be omitted.

COROLLARY. Suppose that $\mathfrak{A}$ is simple. Then:

(i) Any two minimal right ideals in $\mathfrak{A}$ have the same dimension.

(ii) Any two minimal left ideals in $\mathfrak{A}$ have the same dimension.

Let $\mathfrak{R}$ and $\mathfrak{S}$ be two minimal right ideals in $\mathfrak{A}$, and let $\mathfrak{A} = \mathfrak{R}_1 \oplus \cdots \oplus \mathfrak{R}_m$ be a decomposition of $\mathfrak{A}$ into mutually left orthogonal minimal right ideals. If $\mathfrak{R}$ were left orthogonal to each of $\mathfrak{R}_1,\ldots,\mathfrak{R}_m$, then $\mathfrak{R}$ would be left orthogonal to $\mathfrak{R}_1 \oplus \cdots \oplus \mathfrak{R}_m = \mathfrak{A}$. In particular, then, $\mathfrak{R}$ would be left orthogonal to itself, a contradiction. Thus, there exists a right ideal $\mathfrak{R}_i$ such that $\mathfrak{R}$ is not left orthogonal to $\mathfrak{R}_i$. By Lemma 8.1, $\dim(\mathfrak{R}) = \dim(\mathfrak{R}_i)$. The same argument may also be used to show that there exists a right ideal $\mathfrak{R}_j$ such that $\dim(\mathfrak{S}) = \dim(\mathfrak{R}_j)$. By the theorem, $\mathfrak{R}_i$ and $\mathfrak{R}_j$ have the same dimension, so that $\dim(\mathfrak{R}) = \dim(\mathfrak{S})$.

The proof of (ii) is of course a result of statement (ii) of the theorem.

We are now ready to prove the first uniqueness theorem.

THEOREM 8.3. Let $\mathcal{A} = \mathcal{R}_1 \oplus \cdots \oplus \mathcal{R}_m$ and $\mathcal{A} = \mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_p$ be any two decompositions of $\mathcal{A}$ into mutually left orthogonal minimal right ideals. Then $m = p$ and for each $\mathcal{R}_i$ there exists an $\mathcal{S}_j$ such that $\dim(\mathcal{R}_i) = \dim(\mathcal{S}_j)$. Also, for each $\mathcal{S}_i$ there exists an $\mathcal{R}_j$ such that $\dim(\mathcal{S}_i) = \dim(\mathcal{R}_j)$.

If $\mathcal{A}$ is simple, then the corollary to Theorem 8.2 implies that $\mathcal{R}_1, \ldots, \mathcal{R}_m$, $\mathcal{S}_1, \ldots, \mathcal{S}_p$ all have the same dimension. Let n denote this common dimension. Then $mn = pn$, so that $m = p$ and the first case is proved.

Now suppose that $\mathcal{A}$ is not simple. Then by Theorem 8.1, there exists a unique collection of mutually orthogonal simple subalgebras $\mathcal{B}_1, \ldots, \mathcal{B}_k$ such that

$$\mathcal{A} = \mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_k .$$

Let i denote a fixed integer between 1 and m and let A be a non-zero element of $\mathcal{R}_i$. Then

$$A = B_1 + \cdots + B_k$$

where $B_j \in \mathcal{B}_j$. Since $\mathcal{B}_j$ is right orthogonal to $\mathcal{B}_\ell$ if $j \neq \ell$, repeated use of Lemma 4.1 shows that $B_1, \ldots, B_k$ are all elements of $\mathcal{R}_i$. Also, B_j is left orthogonal to B_ℓ if $j \neq \ell$. Therefore, Theorem 4.1 implies that at most one of $B_1, \ldots, B_k$ is non-zero. If B_j denotes this non-zero element, then $A \in \mathcal{B}_j$. We now assert that all elements of $\mathcal{R}_i$ are contained in $\mathcal{B}_j$. For, if C is any other non-zero element of $\mathcal{R}_i$ such that $C \in \mathcal{B}_\ell$, where $\ell \neq j$, then A and C are two non-zero left orthogonal elements of $\mathfrak{R}_i$, a contradiction to Theorem 4.1. Hence, $\mathfrak{R}_i \subseteq \mathfrak{B}_j$. Each minimal right ideal $\mathfrak{R}_1,\ldots,\mathfrak{R}_m$ is therefore contained in one and only one of $\mathfrak{B}_1,\ldots,\mathfrak{B}_k$.

Note also that each subalgebra $\mathfrak{B}_j$ contains at least one right ideal $\mathfrak{R}_\ell$. For if this were not the case, each $\mathfrak{R}_\ell$ would be contained in a subalgebra other than $\mathfrak{B}_j$. Since $\mathfrak{B}_j$ is orthogonal to $\mathfrak{B}_\ell$ if $j \neq \ell$, $\mathfrak{B}_j$ would be orthogonal to each of $\mathfrak{R}_1,\ldots,\mathfrak{R}_m$ and therefore to $\mathfrak{R}_1 \oplus \cdots \oplus \mathfrak{R}_m = \mathfrak{C}$, a contradiction.

Suppose that $\mathfrak{R}_1,\ldots,\mathfrak{R}_{i_1}$ are contained in $\mathfrak{B}_1$, that $\mathfrak{R}_{i_1+1},\ldots,\mathfrak{R}_{i_2}$ are contained in $\mathfrak{B}_2$, etc. Let $B \in \mathfrak{B}_1$. Then since $\mathfrak{C} = \mathfrak{R}_1 \oplus \cdots \oplus \mathfrak{R}_m$ we have that $B = R_1 + \cdots + R_m$, where $B_j \in \mathfrak{R}_j$. Rearranging $R_1,\ldots,R_m$ yields

$$B = (R_1 + \cdots + R_{i_1}) + (R_{i_1+1} + \cdots + R_{i_2}) + \cdots .$$

Therefore,

$$B - (R_1 + \cdots + R_{i_1}) = (R_{i_1+1} + \cdots + R_{i_2}) + \cdots .$$

The left-hand side of this last equation is contained in $\mathfrak{B}_1$ and the right-hand side is an element of $\mathfrak{B}_2 \oplus \cdots \oplus \mathfrak{B}_k$. Since $\mathfrak{B}_1$ is orthogonal to each of $\mathfrak{B}_2,\ldots,\mathfrak{B}_k$, it is also true that $\mathfrak{B}_1$ is orthogonal to $\mathfrak{B}_2 \oplus \cdots \oplus \mathfrak{B}_k$. Therefore, $B - (R_1 + \cdots + R_{i_1})$ is orthogonal to itself and is thus zero. In other words, $B = R_1 + \cdots + R_{i_1}$, so that $\mathfrak{R}_1,\ldots,\mathfrak{R}_{i_1}$ generate $\mathfrak{B}_1$. Also, since $\mathfrak{R}_1,\ldots,\mathfrak{R}_{i_1}$ are left orthogonal, they are independent subspaces, so that $\mathfrak{B}_1 = \mathfrak{R}_1 \oplus \cdots \oplus \mathfrak{R}_{i_1}$.

The same type of argument proves that

$$\mathfrak{B}_1 = \mathfrak{S}_1 \oplus \cdots \oplus \mathfrak{S}_{j_1}$$

for some integer j_1. Since $\mathfrak{B}_1$ is simple, the first case is applicable and we get that $i_1 = j_1$ and that $\mathfrak{R}_1, \ldots, \mathfrak{R}_{i_1}$, $\mathfrak{S}_1, \ldots, \mathfrak{S}_{j_1}$ all have the same dimension.

Similar remarks apply to each of $\mathfrak{B}_2, \ldots, \mathfrak{B}_k$, thus completing the proof.

As expected, a corresponding result is valid for the decomposition of $\mathfrak{A}$ into minimal left ideals.

THEOREM 8.4. Let $\mathfrak{A} = \mathfrak{L}_1 \oplus \cdots \oplus \mathfrak{L}_n$ and $\mathfrak{A} = \mathfrak{M}_1 \oplus \cdots \oplus \mathfrak{M}_q$ be any two decompositions of $\mathfrak{A}$ into mutually right orthogonal minimal left ideals. Then $n = q$ and for each $\mathfrak{L}_i$ there exists an $\mathfrak{M}_j$ such that $\dim(\mathfrak{L}_i) = \dim(\mathfrak{M}_j)$ and for each $\mathfrak{M}_i$ there exists an $\mathfrak{L}_j$ such that $\dim(\mathfrak{M}_i) = \dim(\mathfrak{L}_j)$.

The proof is similar to that of Theorem 8.3 and will be omitted.

9. ORTHONORMAL BASES

The concept of orthonormality will now be generalized with regard to a finite-dimensional ternary algebra. It will be seen that every such algebra $\mathfrak{A}$ has an orthonormal basis. Certain expansions of elements of the algebra will be given relative to such a basis.

We shall assume throughout this section that $\mathfrak{A}$ is a finite-dimensional ternary algebra having the decompositions

$$\mathfrak{A} = \mathfrak{R}_1 \oplus \cdots \oplus \mathfrak{R}_m$$

and

$$\mathfrak{A} = \mathfrak{L}_1 \oplus \cdots \oplus \mathfrak{L}_n$$

into mutually left orthogonal minimal right ideals and mutually right orthogonal minimal left ideals respectively. Let

$$\mathfrak{A} = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} C_{ij}$$

be the corresponding decomposition of $\mathfrak{A}$ into central ideals $C_{ij} = \mathfrak{R}_i \cap \mathfrak{L}_j$ as in Theorem 7.3.

As before, the notation δ_{ij} will denote the Kronecker delta.

LEMMA 9.1. <u>If</u> $\mathfrak{A}$ <u>is simple, then</u> $\mathfrak{A}$ <u>has a basis consisting of non-zero elements</u> $c_{ij} \in C_{ij}$ <u>and satisfying</u>:

(i) $c_{ij}c^*_{ij}c_{i\ell} = c_{i\ell}$

(ii) $C_{g\ell}C^*_{i\ell}C_{i\ell} = C_{g\ell}$

(iii) $C_{gh}C^*_{ij}C_{k\ell} = \delta_{hj}\delta_{ik}\lambda C_{g\ell}$, where λ is a non-zero complex number dependent upon C_{gj}, C_{ij} and $C_{i\ell}$.

Since $\mathfrak{C}$ is simple, $\mathfrak{C}_{ij} \neq \{0\}$ for $i = 1,\ldots,m$ and $j = 1,\ldots,n$. From each central ideal $\mathfrak{C}_{ij}$ choose a non-zero element B_{ij}. B_{ij} is of degree one, since it belongs to the minimal right ideal $\mathfrak{R}_i$. Thus, there exists a positive scalar β such that

$$B_{ij}B^*_{ij}B_{ij} = \beta\, B_{ij} .$$

Let $C_{ij} = (1/\sqrt{\beta})\,B_{ij}$. Then

$$C_{ij}C^*_{ij}C_{ij} = C_{ij},$$

so that $C'_{ij} = C_{ij}$.

To prove (i), note that since C_{ij} and $C_{i\ell}$ are elements of the minimal right ideal $\mathfrak{R}_i$ and $C_{ij} \neq 0$, we have that

$$C'_{ij}C^*_{ij}C_{i\ell} = C_{i\ell} .$$

But since $C'_{ij} = C_{ij}$, this is the same as

$$C_{ij}C^*_{ij}C_{i\ell} = C_{i\ell} .$$

The equation

$$C_{g\ell}C^*_{i\ell}C_{i\ell} = C_{g\ell}$$

is a result of the fact that $C_{g\ell}$ and $C_{i\ell}$ both belong to the minimal left ideal $\mathfrak{L}_\ell$.

Since $C_{gh} = \mathcal{R}_g \cap \mathcal{L}_h \subseteq \mathcal{L}_h$ and $C_{ij} = \mathcal{R}_i \cap \mathcal{L}_j \subseteq \mathcal{L}_j$, we have that C_{gh} is right orthogonal to C_{ij} if $h \neq j$. Also, because $C_{ij} \subseteq \mathcal{R}_i$ and $C_{k\ell} \subseteq \mathcal{R}_k$, it is seen that C_{ij} is left orthogonal to $C_{k\ell}$ if $i \neq k$. Hence,

$$C_{gh}C^*_{ij}C_{k\ell} = 0$$

if either $h \neq j$ or $i \neq k$.

To complete the proof, let $h = j$ and $i = k$. We must consider the element $A = C_{gj}C^*_{ij}C_{i\ell}$. If $A = 0$, Theorem 6.3 implies that at least one of $C_{gj}, C_{ij}, C_{i\ell}$ is 0, a contradiction. The same theorem gives us that

$$A \in \mathcal{R}_g \cap \mathcal{L}_\ell = C_{g\ell} .$$

Since $C_{g\ell}$' has dimension one and the element $C_{g\ell}$ is a basis for $C_{g\ell}$, there exists a non-zero scalar λ such that

$$A = \lambda C_{g\ell} .$$

λ is necessarily dependent upon C_{gj}, C_{ij} and $C_{i\ell}$.

Let $\{C_{ij}\}$ be a basis of $\mathfrak{C}$ as in the preceding lemma. Then

$$C_{gh}C^*_{ij}C_{k\ell} = \delta_{hj}\delta_{ik}\,\lambda\, C_{g\ell} ,$$

where λ depends upon C_{gj}, C_{ij} and $C_{i\ell}$. We shall emphasize this dependency by writing $\lambda^{j\ell}_{gi}$, so that now

$$C_{gh}C^*_{ij}C_{k\ell} = \delta_{hj}\delta_{ik}\,\lambda^{j\ell}_{gi}\, C_{g\ell} .$$

The following results provide information about the scalars $\lambda^{j\ell}_{gi}$.

LEMMA 9.2. <u>If</u> $\mathfrak{C}$ <u>is simple and has a basis</u> $\{C_{ij}\}$ <u>satisfying conditions</u> (i),(ii) <u>and</u> (iii) <u>of Lemma</u> 9.1, <u>then</u> $|\lambda^{j\ell}_{gi}| = 1$. <u>Also,</u> $\lambda^{j\ell}_{gi} = 1$ <u>if either</u> $g = i$ <u>or</u> $j = \ell$.

As in the proof of Lemma 9.1, let $A = C_{gj}C^*_{ij}C_{i\ell}$. Then

$$\begin{aligned} AA^*A &= C_{gj}C^*_{ij}C_{i\ell}C^*_{i\ell}C_{ij}C^*_{gj}C_{gj}C^*_{ij}C_{i\ell} \\ &= C_{gj}(C_{i\ell}C^*_{i\ell}C_{ij})^*C_{ij}(C_{ij}C^*_{gj}C_{gj})^*C_{i\ell} \\ &= C_{gj}C^*_{ij}C_{ij}C^*_{ij}C_{i\ell} \\ &= C_{gj}C^*_{ij}C_{i\ell} \\ &= A\,, \end{aligned}$$

where conclusions (i) and (ii) of Lemma 9.1 have been used. It is also true that $A = \lambda^{j\ell}_{gi}\, C_{g\ell}$. Therefore,

$$\begin{aligned} A = AA^*A &= \lambda^{j\ell}_{gi}\,\bar{\lambda}^{j\ell}_{gi}\,\lambda^{j\ell}_{gi}\; C_{g\ell}C^*_{g\ell}C_{g\ell} \\ &= \lambda^{j\ell}_{gi}\,\bar{\lambda}^{j\ell}_{gi}\,\lambda^{j\ell}_{gi}\; C_{g\ell} = |\lambda^{j\ell}_{gi}|^2\, A\,, \end{aligned}$$

so that $|\lambda^{j\ell}_{gi}| = 1$.

Finally,

$$\lambda^{j\ell}_{ii}\, C_{i\ell} = C_{ij}C^*_{ij}C_{i\ell} = C_{i\ell}$$

and

$$\lambda_{gi}^{\ell\ell} C_{g\ell} = C_{g\ell}C^*_{i\ell}C_{i\ell} = C_{g\ell} .$$

Hence, $\lambda_{ii}^{j\ell} = 1$ and $\lambda_{gi}^{\ell\ell} = 1$.

LEMMA 9.3. _If_ G _is simple and has a basis_ $\{C_{ij}\}$ _satisfying conditions_ (i),(ii) _and_ (iii) _of Lemma_ 9.1, _then_

$$\lambda_{gi}^{j\ell} = \lambda_{gk}^{jh}\,\bar{\lambda}_{ik}^{jh}\,\lambda_{ik}^{\ell h}\,\bar{\lambda}_{gk}^{\ell h} ,$$

where k _and_ h _are arbitrary integers between_ 1 _and_ m _and_ 1 _and_ n _respectively_.

Again let $A = C_{gj}C^*_{ij}C_{i\ell}$. Then

$$\begin{aligned} A\,C^*_{k\ell}C_{kh} &= C_{gj}C^*_{ij}C_{i\ell}C^*_{k\ell}C_{kh} \\ &= \lambda_{ik}^{\ell h}\,C_{gj}C^*_{ij}C_{ih} \\ &= \lambda_{ik}^{\ell h}\,\lambda_{gi}^{jh}\,C_{gh} . \end{aligned}$$

But also $A = \lambda_{gi}^{j\ell}\,C_{g\ell}$. Therefore,

$$\begin{aligned} A\,C^*_{k\ell}C_{kh} &= \lambda_{gi}^{j\ell}\,C_{g\ell}C^*_{k\ell}C_{kh} \\ &= \lambda_{gi}^{j\ell}\,\lambda_{gk}^{\ell h}\,C_{gh} . \end{aligned}$$

Thus,

$$\lambda_{gi}^{j\ell}\,\lambda_{gk}^{\ell h} = \lambda_{ik}^{\ell h}\,\lambda_{gi}^{jh} ,$$

and since $|\lambda^{\ell h}_{gk}| = 1$, we have that

$$\lambda^{j\ell}_{gi} = \bar{\lambda}^{\ell h}_{gk}\,\lambda^{\ell h}_{ik}\,\lambda^{jh}_{gi}\,.$$

Let $\ell = j$. Then by Lemma 9.2,

$$1 = \bar{\lambda}^{jh}_{gk}\,\lambda^{jh}_{ik}\,\lambda^{jh}_{gi}\,,$$

so that

$$\lambda^{jh}_{gi} = \lambda^{jh}_{gk}\,\bar{\lambda}^{jh}_{ik}\,.$$

Hence,

$$\begin{aligned}\lambda^{j\ell}_{gi} &= \bar{\lambda}^{\ell h}_{gk}\,\lambda^{\ell h}_{ik}\,\lambda^{jh}_{gk}\,\bar{\lambda}^{jh}_{ik}\\ &= \lambda^{jh}_{gk}\,\bar{\lambda}^{jh}_{ik}\,\lambda^{\ell h}_{ik}\,\bar{\lambda}^{\ell h}_{gk}\,,\end{aligned}$$

as was to be proved.

In the factorization of $\lambda^{j\ell}_{gi}$ in the preceding lemma, let $h = k = 1$. Then

$$\lambda^{j\ell}_{gi} = \lambda^{j1}_{g1}\,\bar{\lambda}^{j1}_{i1}\,\lambda^{\ell 1}_{i1}\,\bar{\lambda}^{\ell 1}_{g1}\,.$$

Recall that

$$C_{gj}C^{*}_{ij}C_{i\ell} = \lambda^{j\ell}_{gi}\,C_{g\ell}\,.$$

Hence,

$$C_{gj}C^{*}_{ij}C_{i\ell} = \lambda^{j1}_{g1}\,\bar{\lambda}^{j1}_{i1}\,\lambda^{\ell 1}_{i1}\,\bar{\lambda}^{\ell 1}_{g1}\,C_{g\ell}\,,$$

or,

$$(\bar{\lambda}^{j1}_{g1}\, c_{gj})(\bar{\lambda}^{j1}_{i1}\, c_{ij})*(\bar{\lambda}^{\ell 1}_{i1}\, c_{i\ell}) = \bar{\lambda}^{\ell 1}_{g1}\, c_{g\ell}\ .$$

Letting $E_{ij} = \bar{\lambda}^{j1}_{i1}\, C_{ij}$ therefore yields

$$E_{gj}E^*_{ij}E_{i\ell} = E_{g\ell}\ .$$

Since each E_{ij} is a multiple of C_{ij}, the orthogonality relations which hold for the elements C_{ij} hold also for the elements E_{ij}. In other words,

$$E_{gh}E^*_{ij}E_{k\ell} = \delta_{hj}\delta_{ik}E_{g\ell}\ . \tag{9.1}$$

A basis $\{E_{ij}\}$ satisfying this last condition will be called an <u>orthonormal basis</u> of $\mathfrak{A}$. We have now proved

THEOREM 9.1. <u>Every simple finite-dimensional ternary algebra possesses an orthonormal basis</u>.

If $\mathfrak{A}$ is not simple, then Theorem 8.1 implies that we may write

$$\mathfrak{A} = \mathfrak{B}_1 \oplus \cdots \oplus \mathfrak{B}_p\ ,$$

where $\mathfrak{B}_1,\ldots,\mathfrak{B}_p$ are mutually orthogonal simple subalgebras of $\mathfrak{A}$. Each $\mathfrak{B}_k$ therefore has an orthonormal basis $\{E_{ij}\}$. The totality of all of these orthonormal bases forms such a basis for $\mathfrak{A}$. This proves

THEOREM 9.2. <u>Every finite-dimensional ternary algebra has an orthonormal basis</u>.

It is important to note that the definition of orthonormality as given here specializes to the usual notion of orthonormality of row or column vectors. For example, suppose that the elements E_{ij} are a set of n-dimensional column vectors e_i. The second subscript j is immaterial. Then if $*$ denotes conjugate transpose, condition (9.1) becomes

$$e_g e_i^* e_k = \delta_{hj}\delta_{ik}e_g = \delta_{ik}e_g ,$$

or,

$$e_i^* e_k = \delta_{ik} .$$

If each E_{ij} is an m-dimensional row vector f_j, then equation (9.1) yields

$$f_h f_j^* = \delta_{hj} .$$

The expansions which hold for orthonormal vectors are also valid for an orthonormal basis of a ternary algebra.

THEOREM 9.3. Let $\mathfrak{a}$ be a finite-dimensional ternary algebra and let $\{E_{ij}\}$ be an orthonormal basis of $\mathfrak{a}$. Then if A is any element of $\mathfrak{a}$, the following expansions hold true:

$$A = \sum_{g=1}^{m} E_{gh}E_{gh}^* A \qquad (h = 1,\ldots,n)$$

and

$$A = \sum_{\ell=1}^{n} A\, E_{k\ell}^* E_{k\ell} \qquad (k = 1,\ldots,m) .$$

Since $\{E_{ij}\}$ is a basis of $\mathcal{G}$, there exist complex numbers α_{ij} $(i = 1,\ldots,m\ ;\ j = 1,\ldots,n)$ such that

$$A = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij}E_{ij} .$$

Therefore,

$$E_{gh}E^*_{gh}\ A = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij}E_{gh}E^*_{gh}E_{ij}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij}\delta_{hh}\delta_{gi}E_{gj}$$

$$= \sum_{j=1}^{n} \alpha_{gj}E_{gj} ,$$

and so

$$\sum_{g=1}^{m} E_{gh}E^*_{gh}A = \sum_{g=1}^{m} \sum_{j=1}^{n} \alpha_{gj}E_{gj} = A .$$

For the second expansion, we have that

$$A\, E^*_{k\ell}E_{k\ell} = \sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_{ij}E_{ij}E^*_{k\ell}E_{k\ell}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_{ij}\delta_{j\ell}\delta_{kk}E_{i\ell}$$

$$= \sum_{i=1}^{m} \alpha_{i\ell}E_{i\ell} \;.$$

Hence,

$$\sum_{\ell=1}^{n} A\, E^*_{k\ell}E_{k\ell} = \sum_{i=1}^{m}\sum_{\ell=1}^{n} \alpha_{i\ell}E_{i\ell} = A \;.$$

To make clear the analogy between the immediately preceding expansions and those for a vector space, let $\mathcal{V}$ be an m-dimensional space of column vectors and let $e_1,\ldots,e_m$ be an orthonormal basis for $\mathcal{V}$. Then if a is any element of $\mathcal{V}$,

$$a = \sum_{g=1}^{m} (e^*_g\, a)\, e_g$$

$$= \sum_{g=1}^{m} e_g e^*_g\, a \;.$$

The correspondence between this last equation and the first expansion in Theorem 9.3 is obvious.

10. REPRESENTATIONS

The orthonormal bases as constructed in the previous section will now be used to show that every finite-dimensional ternary algebra may be characterized as a ternary algebra of matrices. This result is analogous to that of Wedderburn [6] in the binary case. The results of Section 8 will then permit a proof of the uniqueness of this representation.

THEOREM 10.1. _Let_ $\mathfrak{a}$ _be a simple finite-dimensional ternary algebra. Then there exist integers_ m _and_ n _such that_ $\mathfrak{a}$ _is isomorphic to the ternary algebra of all_ $m \times n$ _matrices over the field of complex numbers_.

Since $\mathfrak{a}$ is simple, there exist integers m and n such that $\mathfrak{a}$ has an orthonormal basis $\{E_{ij}\}$, where $i = 1,\ldots,m$ and $j = 1,\ldots,n$ and each $E_{ij} \neq 0$. Hence, if A is any element of $\mathfrak{a}$, then for some complex numbers α_{ij} it is true that

$$A = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij} E_{ij} .$$

(α_{ij}) is thus an $m \times n$ matrix associated with A. Clearly, this correspondence preserves sums and scalar multiples. It is also product-preserving. For, suppose that B and C are any other elements of $\mathfrak{a}$ with associated matrices (β_{ij}) and (γ_{ij}) respectively. Then

$$AB^*C = \sum_{g,i,k=1}^{m} \sum_{h,j,\ell=1}^{n} \alpha_{gh}\bar{\beta}_{ij}\gamma_{k\ell}E_{gh}E^*_{ij}E_{k\ell}$$

$$= \sum_{g,i,k=1}^{m} \sum_{h,j,\ell=1}^{n} \alpha_{gh}\bar{\beta}_{ij}\gamma_{k\ell}\delta_{hj}\delta_{ik}E_{g\ell}$$

$$= \sum_{g,k=1}^{m} \sum_{j,\ell=1}^{n} \alpha_{gj}\bar{\beta}_{kj}\gamma_{k\ell}E_{g\ell}$$

$$= \sum_{g=1}^{m} \sum_{\ell=1}^{n} \left(\sum_{k=1}^{m} \sum_{j=1}^{n} \alpha_{gj}\bar{\beta}_{kj}\gamma_{k\ell} \right) E_{g\ell} .$$

Therefore, the matrix associated with $P = AB^*C$ is

$$(\pi_{g\ell}) = \left(\sum_{k=1}^{m} \sum_{j=1}^{n} \alpha_{gj}\bar{\beta}_{kj}\gamma_{k\ell} \right) ,$$

the usual matrix product

$$(\alpha_{gh})(\beta_{ij})^*(\gamma_{k\ell}) .$$

If $\mathfrak{a}$ is not simple, then $\mathfrak{a}$ is the direct sum of mutually orthogonal simple subalgebras $\mathfrak{B}_1,\ldots,\mathfrak{B}_p$. For each $\mathfrak{B}_i$ there exist integers m_i and n_i such that $\mathfrak{B}_i$ is isomorphic to the ternary algebra of all $m_i \times n_i$ matrices over the complex numbers. We may of course assume that $\mathfrak{a}$ has the canonical decomposition

$$\begin{bmatrix} \mathfrak{B}_1 & 0 & 0 & \cdots & 0 \\ 0 & \mathfrak{B}_2 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & \mathfrak{B}_p \end{bmatrix}$$

as discussed at the end of Section 7. Therefore, with each element A of $\mathfrak{a}$ there is associated a matrix of the form

$$\begin{bmatrix} B_1 & 0 & 0 & \cdots & 0 \\ 0 & B_2 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & B_p \end{bmatrix} \tag{10.1}$$

where B_i is an $m_i \times n_i$ matrix of complex numbers and the "0" in the (i,j)-position is an $m_i \times n_j$ array of zeroes. Let $m = m_1 + \cdots + m_p$ and $n = n_1 + \cdots + n_p$. The above argument proves

THEOREM 10.2. *Let $\mathfrak{a}$ be a non-simple finite-dimensional ternary algebra. Then there exist integers m and n such that $\mathfrak{a}$ is isomorphic to a subalgebra of the ternary algebra of all $m \times n$ matrices over the complex numbers. Each matrix in this subalgebra has the form* (10.1).

It is possible to prove a converse to Theorem 10.1. This will be accomplished by finding an equivalent characterization of simple ternary algebras in the finite-dimensional case. We begin by proving the following result, which is a generalization of a well-known theorem on binary algebras.

THEOREM 10.3. _For any positive integers_ m _and_ n, _let_ $\mathfrak{M}_{mn}$ _denote the ternary algebra of all_ $m \times n$ _matrices over the field of complex numbers. Then_ $\mathfrak{M}_{mn}$ _contains no two-sided ideals other than itself and the zero-ideal._

The matrices E_{ij} $(i = 1,\ldots,m\ ;\ j = 1,\ldots,n)$ with a 1 in the (i,j)-position and zeroes elsewhere form a basis for $\mathfrak{M}_{mn}$. In fact, it is easy to verify that the elements E_{ij} are an orthonormal basis for $\mathfrak{M}_{mn}$.

Suppose $\mathfrak{J}$ is a non-zero two-sided ideal in $\mathfrak{M}_{mn}$. Let T be a non-zero element of $\mathfrak{J}$. Then there exist scalars τ_{gh} such that

$$T = \sum_{g=1}^{m} \sum_{h=1}^{n} \tau_{gh} E_{gh} \, .$$

Since $T \neq 0$, there exists a non-zero coefficient τ_{ij}. Hence, since $\mathfrak{J}$ is two-sided, it will contain

$$\tau_{ij}^{-1}E_{kj}E_{ij}^{*}T\,E_{ij}^{*}E_{i\ell} = \tau_{ij}^{-1}\sum_{g=1}^{m}\sum_{h=1}^{n}\tau_{gh}E_{kj}E_{ij}^{*}E_{gh}E_{ij}^{*}E_{i\ell}$$

$$= \tau_{ij}^{-1}\sum_{g=1}^{m}\sum_{h=1}^{n}\tau_{gh}\delta_{jj}\delta_{ig}E_{kh}E_{ij}^{*}E_{i\ell}$$

$$= \tau_{ij}^{-1}\sum_{h=1}^{n}\tau_{ih}E_{kh}E_{ij}^{*}E_{i\ell}$$

$$= \tau_{ij}^{-1}\sum_{h=1}^{n}\tau_{ih}\delta_{hj}\delta_{ii}E_{k\ell}$$

$$= \tau_{ij}^{-1}\tau_{ij}E_{k\ell}$$

$$= E_{k\ell}$$

for any k and ℓ. Therefore $\mathfrak{J} = \mathfrak{M}_{mn}$.

THEOREM 10.4. Let $\mathfrak{A}$ be a finite-dimensional ternary algebra. Then $\mathfrak{A}$ is simple if and only if $\mathfrak{A}$ contains no two-sided ideals other than itself and the zero-ideal.

First suppose that $\mathfrak{A}$ is simple. Then Theorem 10.1 implies that there exist integers m and n such that $\mathfrak{A}$ is isomorphic to the ternary algebra $\mathfrak{M}_{mn}$ of all $m \times n$ matrices over the complex numbers. By Theorem 10.3, $\mathfrak{M}_{mn}$ contains no two-sided ideals other than itself and the zero-ideal. The isomorphism then establishes that the same is true of $\mathfrak{A}$.

Now assume that $\mathfrak{A}$ is not simple and let $\mathfrak{B}$ and $\mathfrak{C}$ be non-zero orthogonal subalgebras of $\mathfrak{A}$ such that

$$\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{C} .$$

Clearly, $\mathfrak{B}$ and $\mathfrak{C}$ are properly contained in $\mathfrak{A}$. Also, by Lemma 8.1 we have that $\mathfrak{B}$ and $\mathfrak{C}$ are two-sided ideals in $\mathfrak{A}$. This completes the proof.

The following corollary is the desired converse to Theorem 10.1.

COROLLARY. <u>For any positive integers</u> m <u>and</u> n, <u>the ternary algebra</u> $\mathfrak{M}_{mn}$ <u>of all</u> $m \times n$ <u>matrices over the field of complex numbers is simple</u>.

COROLLARY. <u>Let</u> $\mathfrak{A}$ <u>be a finite-dimensional ternary algebra having the decomposition</u>

$$\mathfrak{A} = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} C_{ij}$$

<u>into central ideals</u> C_{ij} <u>as in Theorem</u> 7.3. <u>Then</u> $\mathfrak{A}$ <u>is simple if and only if</u> $C_{ij} \neq \{0\}$ <u>for</u> $i = 1,\ldots,m$ <u>and</u> $j = 1,\ldots,n$.

If there exist integers k and ℓ such that $C_{k\ell} = \{0\}$, then $\mathfrak{A}$ has the canonical decomposition into orthogonal subalgebras as discussed at the end of Section 7. Therefore, $\mathfrak{A}$ is non-simple.

Suppose now that $C_{ij} \neq \{0\}$ for $i = 1,\ldots,m$ and $j = 1,\ldots,n$. Then $\mathfrak{A}$ is isomorphic to the ternary algebra $\mathfrak{M}_{mn}$.

By the first corollary, $\mathfrak{M}_{mn}$ is simple, and so $\mathfrak{A}$ is simple also.

We shall now show that the aforementioned representations are unique.

LEMMA 10.1. *Let $\mathfrak{A}$ be a simple finite-dimensional ternary algebra and suppose that E_{ij} $(i = 1,\ldots,m\ ;\ j = 1,\ldots,n)$ and $F_{k\ell}$ $(k = 1,\ldots,p\ ;\ \ell = 1,\ldots,q)$ are two orthonormal bases of $\mathfrak{A}$. Then $m = p$ and $n = q$.*

For any integer i between 1 and m, denote by $\mathfrak{R}_i$ the set of all elements $R \in \mathfrak{A}$ which can be written in the form

$$R = \sum_{j=1}^{n} \rho_j E_{ij} \ .$$

$\mathfrak{R}_i$ is clearly a linear subspace of $\mathfrak{A}$. Also, each $\mathfrak{R}_i$ is a minimal right ideal in $\mathfrak{A}$. For, suppose that $R \in \mathfrak{R}_i$ and that A is an arbitrary element of $\mathfrak{A}$. Then there exist scalars ρ_j $(j = 1,\ldots,n)$ and $\alpha_{k\ell}$ $(k = 1,\ldots,m\ ;\ \ell = 1,\ldots,n)$ such that

$$R = \sum_{j=1}^{n} \rho_j E_{ij}$$

and

$$A = \sum_{k=1}^{m} \sum_{\ell=1}^{n} \alpha_{k\ell} E_{k\ell} \ .$$

Therefore,

$$RR^*A = \sum_{h=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{m} \sum_{\ell=1}^{n} \rho_h \bar{\rho}_j \alpha_{k\ell} E_{ih} E^*_{ij} E_{k\ell}$$

$$= \sum_{h=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{m} \sum_{\ell=1}^{n} \rho_h \bar{\rho}_j \alpha_{k\ell} \delta_{hj} \delta_{ik} E_{i\ell}$$

$$= \sum_{j=1}^{n} \sum_{\ell=1}^{n} \rho_j \bar{\rho}_j \alpha_{i\ell} E_{i\ell}$$

$$= \left(\sum_{j=1}^{n} |\rho_j|^2 \right) \sum_{\ell=1}^{n} \alpha_{i\ell} E_{i\ell} ,$$

so that $RR^*A \in \mathfrak{R}_i$.

To show that $\mathfrak{R}_i$ is minimal, suppose that R and S are two elements of $\mathfrak{R}_i$ such that $RR^*S = 0$ and that $R \neq 0$. There exist complex numbers ρ_j and σ_j $(j = 1,\ldots,n)$ such that

$$R = \sum_{j=1}^{n} \rho_j E_{ij}$$

and

$$S = \sum_{j=1}^{n} \sigma_j E_{ij} .$$

Hence,

$$0 = RR^*S$$

$$= \sum_{h,j,\ell=1}^{n} \rho_h \bar{\rho}_j \sigma_\ell E_{ih} E^*_{ij} E_{i\ell}$$

$$= \sum_{h,j,\ell=1}^{n} \rho_h \bar{\rho}_j \sigma_\ell \delta_{hj} \delta_{ii} E_{i\ell}$$

$$= \sum_{j,\ell=1}^{n} \rho_j \bar{\rho}_j \sigma_\ell E_{i\ell}$$

$$= \left(\sum_{j=1}^{n} |\rho_j|^2 \right) \sum_{\ell=1}^{n} \sigma_\ell E_{i\ell}$$

$$= \left(\sum_{j=1}^{n} |\rho_j|^2 \right) S .$$

Since $R \neq 0$, we have that $\Sigma_{j=1}^{n} |\rho_j|^2 \neq 0$, and therefore $S = 0$. By theorem 4.1, then, $\mathfrak{R}_i$ is minimal.

It is easy to verify that $\mathfrak{R}_i$ is left orthogonal to $\mathfrak{R}_j$ if $i \neq j$ and that

$$\mathfrak{A} = \mathfrak{R}_1 \oplus \cdots \oplus \mathfrak{R}_m .$$

For $i = 1,\ldots,p$, let $\mathcal{S}_i$ represent the totality of elements $S \in \mathfrak{G}$ which are of the form

$$S = \sum_{j=1}^{q} \sigma_j F_{ij} .$$

Then as above we can show that $\mathcal{S}_1,\ldots,\mathcal{S}_p$ are left orthogonal minimal right ideals in $\mathfrak{G}$ and that

$$\mathfrak{G} = \mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_p .$$

Since $\mathfrak{G}$ is simple, Theorem 8.3 implies that $m = p$.

To prove that $n = q$, we have only to let $\mathcal{L}_j$ denote all elements of the form

$$L = \sum_{i=1}^{m} \lambda_i E_{ij}$$

and designate by $\mathfrak{M}_k$ all elements of the form

$$M = \sum_{i=1}^{p} \mu_i F_{ik}$$

for $j = 1,\ldots,n$ and $k = 1,\ldots,q$. Then it is easy to show that $\mathcal{L}_1,\ldots,\mathcal{L}_n$ are mutually right orthogonal minimal left ideals in $\mathfrak{G}$ and that the same is true of $\mathfrak{M}_1,\ldots,\mathfrak{M}_q$. Also,

$$\mathfrak{G} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$$

and

$$\mathfrak{A} = \mathfrak{M}_1 \oplus \cdots \oplus \mathfrak{M}_q .$$

Therefore, by Theorem 8.4 we have that $n = q$.

THEOREM 10.5. *Let* $\mathfrak{A}$ *be a simple finite-dimensional ternary algebra. If* A *is any element of* $\mathfrak{A}$, *let* (α_{ij}) *be the matrix representation of* A *relative to some orthonormal basis* E_{ij} $(i = 1,\ldots,m ; j = 1,\ldots,n)$. *Then if* $(\tilde{\alpha}_{k\ell})$ *is the matrix representation of* A *relative to another orthonormal basis* $\{F_{k\ell}\}$, *then* (α_{ij}) *and* $(\tilde{\alpha}_{k\ell})$ *are of the same size*.

The proof is an immediate consequence of Lemma 10.1.

In the case when $\mathfrak{A}$ is not simple, $\mathfrak{A}$ has a unique decomposition into mutually orthogonal simple subalgebras:

$$\mathfrak{A} = \mathfrak{B}_1 \oplus \cdots \oplus \mathfrak{B}_p .$$

The matrix representation of any element of $\mathfrak{A}$ has the form (10.1). Theorem 10.5 then ensures that each submatrix B_i has the same size if any other orthonormal basis is used to obtain this representation. This proves

THEOREM 10.6. *Let* $\mathfrak{A}$ *be a non-simple finite-dimensional ternary algebra. If* A *is any element of* $\mathfrak{A}$, *let* (α_{ij}) *be the matrix associated with* A. *Assume further that* (α_{ij}) *has the canonical form* (10.1). *Then the size of each submatrix* B_i *is invariant for any orthonormal basis used to obtain the representation*.

11. THE POSITIVITY CONDITION

We shall now dispense with condition (5) in the definition of a ternary algebra and study the implications of the absence of this positivity condition within the framework of a finite-dimensional ternary algebra. However, condition (1) (that $AA^*A = 0$ if and only if $A = 0$) will be retained.

Throughout this section it will be assumed that $\mathfrak{a}$ is a *-linear finite-dimensional generalized ternary algebra.

If A is a non-zero element of $\mathfrak{a}$ of degree one, so that

$$AA^*A = \alpha A ,$$

then either $\alpha > 0$ or $\alpha < 0$.

An element P of $\mathfrak{a}$ such that $PP^*P = P$ is called a <u>positive unit</u>. An element N satisfying $NN^*N = - N$ is referred to as a <u>negative unit</u>. Note that $P' = P$ and $N' = -N$. A <u>unit</u> is defined to be an element R for which $R' = RR^*R$. It is shown in [1] that every unit may be represented as the sum

$$R = P + N$$

of a positive unit P and a negative unit N such that P and N are orthogonal.

LEMMA 11.1. *If P is a non-zero positive unit and N is a non-zero negative unit in $\mathfrak{a}$, then P and N are linearly independent.*

Suppose that there exist constants α and β, not both zero, such that

$$\alpha P + \beta N = 0 \ .$$

Clearly, α and β must both be non-zero. Thus, $P = \lambda N$, where $\lambda = -\alpha^{-1}\beta$. Hence

$$P = PP^*P = \lambda\bar{\lambda}\lambda NN^*N = -\lambda\bar{\lambda}\lambda N = -\lambda\bar{\lambda}P \ ,$$

and so $(1 + |\lambda|^2)P = 0$ which yields $P = 0$, a contradiction. Therefore, P and N are linearly independent.

THEOREM 11.1. *Let* $\mathfrak{B}$ *be either a minimal right ideal or a minimal left ideal in* $\mathfrak{A}$. *Then one and only one of the following two alternatives is true*:

(i) *For every non-zero element* B *in* $\mathfrak{B}$ *there exists a positive number* β *such that* $BB^*B = \beta B$.

(ii) *If* B *is any non-zero element of* $\mathfrak{B}$, *then* $BB^*B = \beta B$ *for some* $\beta < 0$.

To fix the ideas, let us assume that $\mathfrak{B}$ is a minimal right ideal in $\mathfrak{A}$. Each element in $\mathfrak{B}$ has degree one. Also, if A, B and C are any three elements of $\mathfrak{B}$ there exists a complex number λ such that

$$AB^*C = \lambda C \quad \text{and} \quad BA^*C = \bar{\lambda}C \ .$$

It must be stressed that this last fact was proved without making use of the positivity condition, which was only needed to show that $\mathfrak{B}$ is an inner-product space when this assumption is made.

Suppose that B is a non-zero element of $\mathfrak{B}$ such that $BB^*B = \beta B$ for some positive number β. Let C be some other element of $\mathfrak{B}$ such that $CC^*C = \gamma C$, where $\gamma \leq 0$. It will now be shown that the assumption $\gamma < 0$ leads to a contradiction.

Let $P = (1/\sqrt{\beta})B$ and $N = (1/\sqrt{-\gamma})C$. Then $PP^*P = P$ and $NN^*N = -N$. By Lemma 11.1, P and N are linearly independent, and so $\lambda P + N \neq 0$ for every scalar λ. Also, by the above remarks, there exists a complex number μ such that

$$PN^*A = \mu A \quad \text{and} \quad NP^*A = \bar{\mu}A$$

for every $A \in \mathfrak{B}$.

For any fixed real number λ, let $Q = \lambda P + N$. Then

$$\begin{aligned} 0 &\neq (\lambda P + N)(\lambda P + N)^*Q \\ &= \lambda^2 PP^*Q + \lambda PN^*Q + \lambda NP^*Q + NN^*Q \\ &= \lambda^2 Q + \lambda\mu Q + \lambda\bar{\mu}Q - Q \\ &= [\lambda^2 + 2(\mathrm{Re}\,\mu)\lambda - 1]Q \,. \end{aligned}$$

But the polynomial $\lambda^2 + 2(\mathrm{Re}\,\mu)\lambda - 1$ has the two real zeroes

$$(\mathrm{Re}\,\mu) \pm \sqrt{(\mathrm{Re}\,\mu)^2 + 1}\,,$$

thus yielding a contradiction. Therefore, $\gamma = 0$, and the theorem is proved.

In Theorems 11.2 through 11.7 below we shall assume that $\mathfrak{a}$ is simple and has the decompositions

$$\mathfrak{a} = \mathfrak{R}_1 \oplus \cdots \oplus \mathfrak{R}_m$$

and

$$\mathfrak{a} = \mathfrak{L}_1 \oplus \cdots \oplus \mathfrak{L}_n$$

into mutually left orthogonal minimal right ideals and mutually right orthogonal minimal left ideals respectively. These decompositions were obtained without making use of the positivity condition in $\mathfrak{a}$. A non-zero element A of degree one,

$$AA^*A = \alpha A ,$$

will be called _positive_ if $\alpha > 0$ and _negative_ if $\alpha < 0$.

THEOREM 11.2. _If there exists an_ $\mathfrak{R}_i$ _or an_ $\mathfrak{L}_j$ _containing a positive element_ A, _then every non-zero element in each of_ $\mathfrak{R}_1,\ldots,\mathfrak{R}_m$, $\mathfrak{L}_1,\ldots,\mathfrak{L}_n$ _is positive_.

Every element in each of $\mathfrak{R}_1,\ldots,\mathfrak{R}_m$, $\mathfrak{L}_1,\ldots,\mathfrak{L}_n$ is of degree one. Suppose that $\mathfrak{R}_i$ contains a positive element A. Then by Theorem 11.1, every non-zero element in $\mathfrak{R}_i$ is positive.

Since $\mathfrak{a}$ is simple we have that $\mathfrak{R}_i \cap \mathfrak{L}_j \neq \{0\}$ for $j = 1,\ldots,n$. For any j, choose a non-zero element $L_j \in \mathfrak{R}_i \cap \mathfrak{L}_j$. Since $L_j \in \mathfrak{R}_i$, L_j is positive. Because it is also true that $L_j \in \mathfrak{L}_j$, Theorem 11.1 implies that every non-zero element in $\mathfrak{L}_j$ is positive. Since j was arbitrary, every non-zero element in each of $\mathfrak{L}_1,\ldots,\mathfrak{L}_n$ is positive.

Now let j be fixed. Then $\mathfrak{R}_k \cap \mathfrak{L}_j \neq \{0\}$ for $k = 1,\ldots,m$. For any integer k, choose $R_k \in \mathfrak{R}_k \cap \mathfrak{L}_j$ such that

$R_k \neq 0$. Since $R_k \in \mathcal{L}_j$, R_k is positive, and because $R_k \in \mathcal{R}_k$, every non-zero element in $\mathcal{R}_k$ is positive. Since k was arbitrary, each non-zero element in each of $\mathcal{R}_1, \ldots, \mathcal{R}_m$ is positive.

A similar argument proves the theorem when it is assumed that $\mathcal{L}_j$ contains a positive element A.

THEOREM 11.3. _If there exists an_ $\mathcal{R}_i$ _or an_ $\mathcal{L}_j$ _containing a negative element_ A, _then every non-zero element in each of_ $\mathcal{R}_1, \ldots, \mathcal{R}_m$, $\mathcal{L}_1, \ldots, \mathcal{L}_n$ _is negative_.

The proof is like that of Theorem 11.2 and will be omitted.

THEOREM 11.4. _If there exists an_ $\mathcal{R}_i$ _or an_ $\mathcal{L}_j$ _containing a positive element, then every non-zero element of_ $\mathcal{G}$ _of degree one is positive_.

Suppose that A is a negative element of $\mathcal{G}$, and in fact, without loss of generality, that $AA^*A = -A$. Using the decomposition

$$\mathcal{G} = \mathcal{R}_1 \oplus \cdots \oplus \mathcal{R}_m ,$$

we may write

$$A = A_1 + \cdots + A_m ,$$

where $A_i \in \mathcal{R}_i$ and $A_iA_i^*A_j = 0$ for $i \neq j$.

Hence,

$$AA^*A = \sum_{i,j,k=1}^{m} A_i A_j^* A_k$$

$$= \sum_{i,j=1}^{m} A_i A_j^* A_j = - A = - \sum_{i=1}^{m} A_i .$$

Therefore, if $1 \leq k \leq m$,

$$\sum_{i,j=1}^{m} A_k A_k^* A_i A_j^* A_j = - \sum_{i=1}^{m} A_k A_k^* A_i ,$$

and using the left orthogonality of $A_1, \ldots, A_m$ this reduces to

$$\sum_{j=1}^{m} A_k A_k^* A_k A_j^* A_j = - A_k A_k^* A_k .$$

Since $A_k \in \mathcal{R}_k$ and $\mathcal{R}_k$ is a minimal right ideal, we have that for every non-zero element A_k there exists a non-zero scalar λ_k such that $A_k A_k^* A_k = \lambda_k A_k$. By Theorem 11.2, $\lambda_k > 0$. Thus,

$$\sum_{j=1}^{m} \lambda_k A_k A_j^* A_j = - \lambda_k A_k ,$$

and therefore

$$\sum_{j=1}^{m} A_k A_j^* A_j = - A_k .$$

Hence,

$$\sum_{j=1}^{m} A_k A_j^* A_j A_k^* A_k = - A_k A_k^* A_k = - \lambda_k A_k .$$

Also,

$$(A_k A_j^* A_j)(A_k A_j^* A_j)^* A_k = A_k A_j^* (A_j A_j^* A_j) A_k^* A_k$$

$$= \lambda_j A_k A_j^* A_j A_k^* A_k .$$

If $A_j \neq 0$, then $\lambda_j \neq 0$ and

$$A_k A_j^* A_j A_k^* A_k = \lambda_j^{-1} (A_k A_j^* A_j)(A_k A_j^* A_j)^* A_k .$$

The element $R_{kj} = A_k A_j^* A_j$ belongs to $\mathcal{R}_k$ and $R_{kj} = 0$ if and only if $A_k A_j^* A_j A_k^* A_k = 0$. If $R_{kj} \neq 0$ there exists a scalar $\mu_{kj} \neq 0$ such that

$$R_{kj} R_{kj}^* R_{kj} = \mu_{kj} R_{kj} .$$

By Theorem 11.2, $\mu_{kj} > 0$. Hence, $R_{kj} = \mu_{kj} R'_{kj}$ and

$$R_{kj} R_{kj}^* A_k = \mu_{kj} R'_{kj} R_{kj}^* A_k = \mu_{kj} A_k .$$

Therefore,

$$-\lambda_k A_k = \sum_{j=1}^{m} A_k A_j^* A_j A_k^* A_k$$

$$= \sum_{j=1}^{m} \lambda_j^{-1} R_{kj} R_{kj}^* A_k$$

$$= \left(\sum_{j=1}^{m} \lambda_j^{-1} \mu_{kj} \right) A_k$$

for $k = 1,\ldots,m$. If $A_k \neq 0$, then $-\lambda_k < 0$ and $\Sigma_{j=1}^{m} \lambda_j^{-1} \mu_{kj} > 0$, a contradiction. This completes the proof.

The following is the corresponding result for negative elements and is proved in an analogous manner:

THEOREM 11.5. <u>If there exists an</u> $\mathfrak{R}_i$ <u>or an</u> $\mathfrak{L}_j$ <u>containing a negative element, then every non-zero element of</u> $\mathfrak{A}$ <u>of degree one is negative</u>.

We are of course still retaining the assumption that $\mathfrak{A}$ is simple.

THEOREM 11.6. <u>If</u> $\mathfrak{A}$ <u>contains a positive element, then every other non-zero element of</u> $\mathfrak{A}$ <u>of degree one is positive</u>.

Denote by A this positive element. Using the decomposition

$$\mathfrak{A} = \mathfrak{R}_1 \oplus \cdots \oplus \mathfrak{R}_m$$

we may write

$$A = A_1 + \cdots + A_m$$

where $A_i \in \mathfrak{K}_i$ $(i = 1,\ldots,m)$. Also, for each A_i there exists a scalar λ_i such that $A_i A_i^* A_i = \lambda_i A_i$. Since $A \neq 0$, at least one of $A_1,\ldots,A_m$ must be non-zero. Hence, there exists some λ_j which is non-zero. If $\lambda_j < 0$, then A_j is negative and so Theorem 11.5 implies that A is a negative element, a contradiction. Thus, $\lambda_j > 0$, and by Theorem 11.4 we have the desired result.

THEOREM 11.7. <u>If</u> $\mathfrak{A}$ <u>contains a negative element, then every other non-zero element of</u> $\mathfrak{A}$ <u>of degree one is negative</u>.

The proof is the dual of that of Theorem 11.6.

The last two theorems imply that if $\mathfrak{A}$ is simple, then either every non-zero element of $\mathfrak{A}$ of degree one is positive or every such element is negative. Thus, if the positivity condition does not hold in $\mathfrak{A}$, it is possible to introduce a new rule of composition

$$AB^+C = - AB^*C \ ,$$

so that the positivity condition is now satisfied relative to the product AB^+C.

If $\mathfrak{A}$ is not simple, then there exist mutually orthogonal simple subalgebras $\mathfrak{B}_1,\ldots,\mathfrak{B}_p$ of $\mathfrak{A}$ such that

$$\mathfrak{A} = \mathfrak{B}_1 \oplus \cdots \oplus \mathfrak{B}_p \ .$$

For each β_i the positivity condition either holds for AB^*C or for the product AB^+C as discussed above.

12. REAL FINITE-DIMENSIONAL TERNARY ALGEBRAS

The present section will be devoted to a study of finite-dimensional ternary algebras in which the scalars are restricted to be real. Many of the results already proved for complex ternary algebras are also valid in the real case. In particular, if $\mathfrak{R}$ is a minimal right ideal in a real finite-dimensional ternary algebra $\mathfrak{A}$, then $A'A^*B = B$ for all elements A,B in $\mathfrak{R}$ for which $A \neq 0$. Similarly, if $\mathfrak{L}$ is a minimal left ideal in $\mathfrak{A}$, then $BA^*A' = B$ for each pair A,B in $\mathfrak{L}$ for which $A \neq 0$. If $\mathfrak{C}$ is a minimal central ideal in $\mathfrak{A}$, then $C'C^*DC^*C' = D$ for all elements C,D in $\mathfrak{C}$ for which $C \neq 0$. Every element in $\mathfrak{R}, \mathfrak{L}$ or $\mathfrak{C}$ has degree one. Also, the following decompositions of $\mathfrak{A}$ remain true:

$$\mathfrak{A} = \mathfrak{R}_1 \oplus \cdots \oplus \mathfrak{R}_m \tag{12.1}$$

and

$$\mathfrak{A} = \mathfrak{L}_1 \oplus \cdots \oplus \mathfrak{L}_n \tag{12.2}$$

where $\mathfrak{R}_1, \ldots, \mathfrak{R}_m$ are mutually left orthogonal minimal right ideals and $\mathfrak{L}_1, \ldots, \mathfrak{L}_n$ are mutually right orthogonal minimal left ideals. The resulting decomposition

$$\mathfrak{A} = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} \mathfrak{C}_{ij} \tag{12.3}$$

of $\mathfrak{A}$ into central ideals $\mathfrak{C}_{ij} = \mathfrak{R}_i \cap \mathfrak{L}_j$ will enable us to obtain a representation of $\mathfrak{a}$. As in the complex case, we have that $\mathfrak{a}$ is simple if and only if $C_{ij} \neq \{0\}$ for $i = 1,\ldots,m$ and $j = 1,\ldots,n$. However, it is not necessarily true in the real case that if $\mathfrak{a}$ is simple, then each C_{ij} has dimension one. Until further notice, it will be assumed in this section that $\mathfrak{a}$ is a simple, real finite-dimensional ternary algebra having the decompositions (12.1), (12.2) and (12.3).

LEMMA 12.1. *Suppose that* C *is a minimal central ideal in* $\mathfrak{a}$ *such that* $\dim(C) > 1$ *and that* A *and* B *are two linearly independent elements of* C *such that* $AA^*A = \alpha A$ *and* $BB^*B = \beta B$. *Then*:

(i) *There exists a real number* λ *such that*

$$AB^*A = \lambda A - \alpha B \quad \text{and} \quad BA^*B = \lambda B - \beta A .$$

(ii) $AB^*A = -\alpha B$ *if and only if* $BA^*B = -\beta A$.

Using the corollary to Theorem 6.6 and Lemma 6.2, we have that

$$AA^*C = \alpha A'A^*C = \alpha C ,$$

and

$$BB^*C = \beta B'B^*C = \beta C$$

for any element C in C. Moreover,

$$BA^*A = \alpha BA^*A' = \alpha B$$

and

$$AB^*B = \beta AB^*B' = \beta A\ .$$

Also, there exists a positive number σ such that

$$(A + B)(A + B)^*(A + B) = \sigma(A + B)\ .$$

Therefore,

$$(A + B)(A + B)^*C = \sigma(A + B)'(A + B)^*C = \sigma C$$

for any C in $\mathbf{C}$.

The above equations yield

$$\begin{aligned}\sigma C &= AA^*C + AB^*C + BA^*C + BB^*C\\ &= \alpha C + AB^*C + BA^*C + \beta C,\end{aligned}$$

and so

$$AB^*C + BA^*C = \lambda C\ ,$$

where $\lambda = \sigma - \alpha - \beta$. Letting $C = A$, we get that

$$AB^*A = \lambda A - \alpha B$$

and if $C = B$, then

$$BA^*B = \lambda B - \beta A\ .$$

To prove (ii), note that if $AB^*A = -\alpha B$, then $\lambda = 0$ and therefore $BA^*B = -\beta A$. The converse is similar.

LEMMA 12.2. If C [illegible]mal central ideal whose dimension is > 1, then there exist linearly independent elements U and V in C such that

$$UU^*U = U\ ,\ VV^*V = V\ ,$$

$$VU^*V = -U \quad \underline{\text{and}} \quad UV^*U = -V .$$

Let A and B be two linearly independent elements of C such that $AA^*A = \alpha A$ and $BB^*B = \beta B$. Then by Lemma 12.1 we have that for any real number μ,

$$\begin{aligned} A(B + \mu A)^*A &= AB^*A + \mu AA^*A \\ &= \lambda A - \alpha B + \mu\alpha A \\ &= (\lambda + 2\mu\alpha)A - \alpha(B + \mu A) . \end{aligned}$$

Hence, if $\mu = -\frac{1}{2}(\lambda/\alpha)$, then

$$A(B + \mu A)^*A = -\alpha(B + \mu A) .$$

It may be verified that for this μ,

$$(B + \mu A)(B + \mu A)^*(B + \mu A) = (\beta - \mu^2\alpha)(B + \mu A) ,$$

so that $\beta - \mu^2\alpha > 0$ and conclusion (ii) of Lemma 12.1 implies that

$$(B + \mu A)A^*(B + \mu A) = -(\beta - \mu^2\alpha)A .$$

Therefore, letting

$$U = \frac{1}{\sqrt{\beta - \mu^2\alpha}}(B + \mu A)$$

and $V = (1/\sqrt{\alpha})A$, we have the desired result.

THEOREM 12.1. <u>If</u> C <u>is a minimal central ideal whose dimension is</u> > 2, <u>then</u> $\dim(C) > 3$ <u>and there exist linearly</u>

<u>independent elements</u> U, I, J, K <u>in</u> C <u>such that if</u> A, B <u>and</u> C <u>are any three distinct elements from the set</u> $\{U, I, J, K\}$, <u>then</u>

$$AB^*A = -B$$

<u>and</u>

$$AB^*C = -BA^*C = -AC^*B = -CB^*A .$$

By Lemma 12.2, there exist linearly independent elements U and V in C such that $UU^*U = U$, $VV^*V = V$, $VU^*V = -U$ and $UV^*U = -V$. Let J be independent of U and V and assume without loss of generality that $JJ^*J = J$, $JU^*J = -U$ and $UJ^*U = -J$. By Lemma 12.1 there exists a real number λ such that for any real number μ,

$$\begin{aligned} J(V + \mu J)^*J &= JV^*J + \mu J \\ &= \lambda J - V + \mu J \\ &= (\lambda + 2\mu)J - (V + \mu J) . \end{aligned}$$

Thus, if we let $\mu = -(\lambda/2)$ and $W = V + \mu J$, then $JW^*J = -W$. Also,

$$\begin{aligned} UW^*U &= UV^*U + \mu UJ^*U \\ &= -V - \mu J \\ &= -W . \end{aligned}$$

A straightforward computation shows that $WW^*W = (1 - \mu^2)W$ for

$\mu = -(\lambda/2)$. Let $I = (1/\sqrt{1-\mu^2})W$. Then $II{*}I = I$, $UI{*}U = -I$ and by Lemma 12.1, $IU{*}I = -U$. In addition, $JI{*}J = -I$ and $IJ{*}I = -J$. These equations yield

$$\begin{aligned} IU{*}J &= -JI{*}JU{*}J = JI{*}U \\ &= -JI{*}IU{*}I = -JU{*}I \,. \end{aligned}$$

Let $K = IU{*}J$. It will now be shown that U, I, J and K are linearly independent. Suppose that

$$L = \alpha U + \beta I + \gamma J + \delta K = 0$$

and let

$$M = \alpha U - \beta I - \gamma J - \delta K.$$

Then

$$0 = LU{*}M = (\alpha^2 + \beta^2 + \gamma^2 + \delta^2)U \,,$$

so that $\alpha = \beta = \gamma = \delta = 0$.

The following equations may be easily verified.

$$KK{*}K = K \,,\ UK{*}U = -K \,,\ IK{*}I = -K \,,$$
$$KI{*}K = -I \,,\ JK{*}J = -K,\ KJ{*}K = -J \,.$$

Thus, if A and B are any two distinct elements from the set $\{U,I,J,K\}$, then $AB{*}A = -B$. Hence, if $A,B,C \in \{U,I,J,K\}$ and A,B,C are distinct, then

$$AB{*}C = AB{*}AA{*}C = -BA{*}C$$

and

$$AB*C = AC*CB*C = - AC*B .$$

These equations imply that

$$AB*C = - BA*C = BC*A = - CB*A .$$

This completes the proof.

THEOREM 12.2. <u>If</u> C <u>is any minimal central ideal in</u> G, <u>then the dimension of</u> C <u>is at most four</u>.

Suppose that C is a minimal central ideal in G such that $\dim(C) > 4$. Let U,I,J and $K = IU*J$ be linearly independent elements of C as described in Theorem 12.1. Assume that L is an element of C independent of U,I,J and K, and without loss of generality that $UL*U = - L$. By Lemma 12.1 there exist real numbers λ and μ such that

$$IL*I = \lambda I - L \quad \text{and} \quad JL*J = \mu J - L .$$

Therefore, if

$$M = L - \frac{\lambda}{2} I - \frac{\mu}{2} J ,$$

then

$$\begin{aligned} IM*I &= IL*I - \frac{\lambda}{2} II*I - \frac{\mu}{2} IJ*I \\ &= \lambda I - L - \frac{\lambda}{2} I + \frac{\mu}{2} J \\ &= - L + \frac{\lambda}{2} I + \frac{\mu}{2} J \\ &= - M \end{aligned}$$

and

$$\begin{aligned} JM*J &= JL*J - \frac{\lambda}{2} JI*J - \frac{\mu}{2} JJ*J \\ &= \mu J - L + \frac{\lambda}{2} I - \frac{\mu}{2} J \\ &= - L + \frac{\lambda}{2} I + \frac{\mu}{2} J \\ &= - M . \end{aligned}$$

Also,

$$\begin{aligned} UM*U &= UL*U - \frac{\lambda}{2} UI*U - \frac{\mu}{2} UJ*U \\ &= - L + \frac{\lambda}{2} I + \frac{\mu}{2} J \\ &= - M . \end{aligned}$$

Hence,

$$\begin{aligned} JM*I &= (JM*J)J*I = - MJ*I = - M(JI*I)*I \\ &= - MI*(IJ*I) = MI*J = II*MI*J \\ &= I(IM*I)*J = - IM*J . \end{aligned}$$

This last equation implies that

$$\begin{aligned} KM*K &= IU*JM*IU*J \\ &= - IU*IM*JU*J = - UM*U \\ &= M . \end{aligned}$$

But there exists a real number ν such that $KM*K = \nu K - M$. Thus,

$$\nu K - M = M ,$$

or,

$$\nu K - 2M = 0 \ ,$$

so that M and K are linearly dependent, and therefore L is not independent of I,J and K, a contradiction.

LEMMA 12.3. <u>Let $\mathfrak{R}$ and $\mathfrak{S}$ be minimal right ideals and let $\mathfrak{L}$ and $\mathfrak{M}$ be minimal left ideals in $\mathfrak{G}$. Let $\mathfrak{C} = \mathfrak{R} \cap \mathfrak{L}$, $\mathfrak{D} = \mathfrak{R} \cap \mathfrak{M}$, $\mathfrak{E} = \mathfrak{S} \cap \mathfrak{L}$ and $\mathfrak{F} = \mathfrak{S} \cap \mathfrak{M}$. If $\mathfrak{C},\mathfrak{D},\mathfrak{E}$ and $\mathfrak{F}$ are all non-zero subspaces, then $\mathfrak{C},\mathfrak{D},\mathfrak{E}$ and $\mathfrak{F}$ all have the same dimension.</u>

Note that $\mathfrak{C},\mathfrak{D},\mathfrak{E}$ and $\mathfrak{F}$ are all minimal central ideals in $\mathfrak{G}$. Suppose that $\dim(\mathfrak{C}) = p$ and let $C_1,\ldots,C_p$ be a basis for $\mathfrak{C}$. Choose non-zero elements $E \in \mathfrak{E}$ and $F \in \mathfrak{F}$. Then by Theorem 6.3 we have that $C_i E^* F \in \mathfrak{D}$ and that $C_i E^* F \neq 0$ for $i = 1,\ldots,p$. It will now be shown that $C_1 E^* F,\ldots,C_p E^* F$ are linearly independent. First of all, since E and F are non-zero elements of the minimal right ideal $\mathfrak{S}$, $F' F^* E = E$. Similarly, since C_i $(i = 1,\ldots,p)$ and E belong to $\mathfrak{L}$, $C_i E^* E' = C_i$ for $i = 1,\ldots,p$. Therefore, if

$$\sum_{i=1}^{p} \lambda_i C_i E^* F = 0 \ ,$$

then

$$0 = \sum_{i=1}^{p} \lambda_i C_i E^*FF'^*E' = \sum_{i=1}^{p} \lambda_i C_i (F'F^*E)^*E'$$

$$= \sum_{i=1}^{p} \lambda_i C_i E^*E' = \sum_{i=1}^{p} \lambda_i C_i ,$$

so that $\lambda_i = 0$ $(i = 1,\ldots,p)$. This implies that $\dim(\mathfrak{C}) \leq \dim(\mathfrak{D})$. The reverse inequality follows by symmetry, and so $\dim(\mathfrak{C}) = \dim(\mathfrak{D})$.

To prove that $\dim(\mathfrak{C}) = \dim(\mathfrak{E})$, choose non-zero elements $F \in \mathfrak{J}$ and $D \in \mathfrak{D}$. Then as above, FD^*C_i is a non-zero element of $\mathfrak{E}$ for $i = 1,\ldots,p$ and $FD^*C_1,\ldots,FD^*C_p$ are linearly independent. The remainder of the proof then follows immediately.

We emphasize that $\mathfrak{A}$ is still assumed to be simple.

THEOREM 12.3. <u>If</u> $\mathfrak{C}$ <u>and</u> $\mathfrak{J}$ <u>are minimal central ideals in</u> $\mathfrak{A}$, <u>then</u> $\mathfrak{C}$ <u>and</u> $\mathfrak{J}$ <u>have the same dimension</u>.

By Theorem 7.6, there exist minimal right ideals $\mathfrak{R}$ and $\mathfrak{S}$ and minimal left ideals $\mathfrak{L}$ and $\mathfrak{M}$ in $\mathfrak{A}$ such that

$$\mathfrak{C} = \mathfrak{R} \cap \mathfrak{L} \quad \text{and} \quad \mathfrak{J} = \mathfrak{S} \cap \mathfrak{M} .$$

Three cases will be considered. Suppose first that $\mathfrak{C}$ and $\mathfrak{J}$ are not left orthogonal. Let $\mathfrak{D} = \mathfrak{R} \cap \mathfrak{M}$ and $\mathfrak{E} = \mathfrak{S} \cap \mathfrak{L}$. Note that by Lemma 7.4, $\mathfrak{D} = \{0\}$ if and only if $\mathfrak{E} = \{0\}$. We shall show that the assumption $\mathfrak{E} = \{0\}$ leads to a contradiction. Choose $C \in \mathfrak{C} \subseteq \mathfrak{L}$ and $S \in \mathfrak{S}$. Then $SS^*C \in \mathfrak{S} \cap \mathfrak{L} = \mathfrak{E} = \{0\}$, so that $\mathfrak{C}$ is left orthogonal to $\mathfrak{S}$ and is therefore left orthogonal to $\mathfrak{J}$, a contradiction. Hence, $\mathfrak{C}, \mathfrak{D}, \mathfrak{E}$ and $\mathfrak{J}$ are all non-zero, and by Lemma 12.3 all have the same dimension.

A similar proof applies in the case when $\mathfrak{C}$ and $\mathfrak{J}$ are not right orthogonal.

Finally, suppose that $\mathfrak{C}$ is orthogonal to $\mathfrak{J}$. If $\mathfrak{R}$ is not left orthogonal to $\mathfrak{S}$, there exist non-zero elements $R \in \mathfrak{R}$ and $S \in \mathfrak{S}$ such that $RR^*S \neq 0$. If T and U are any two non-zero elements of $\mathfrak{R}$ and $\mathfrak{S}$ respectively, such that $TT^*U = 0$, then $T'T^*U' = 0$ and

$$0 = RR^*T'T^*U'U^*S = R(T'T^*R)^*(U'U^*S) = RR^*S ,$$

a contradiction. Hence, $TT^*U \neq 0$ for any non-zero elements $T \in \mathfrak{R}$ and $U \in \mathfrak{S}$. In particular, if $T \in \mathfrak{R} \cap \mathfrak{L} = \mathfrak{C}$ and $U \in \mathfrak{S} \cap \mathfrak{M} = \mathfrak{J}$ and $T \neq 0$ and $U \neq 0$, then $TT^*U \neq 0$, contradicting the fact that $\mathfrak{C}$ and $\mathfrak{J}$ are orthogonal. Therefore, $\mathfrak{R}$ is left orthogonal to $\mathfrak{S}$. Similarly, $\mathfrak{L}$ is right orthogonal to $\mathfrak{M}$. Let $\mathfrak{R}_1 = \mathfrak{R}$, $\mathfrak{R}_2 = \mathfrak{S}$, $\mathfrak{L}_1 = \mathfrak{L}$, $\mathfrak{L}_2 = \mathfrak{M}$. Then as in the proof of Theorem 7.1,

$$\mathfrak{A} = \mathfrak{R}_1 \oplus \mathfrak{R}_2 \oplus (\mathfrak{R}_1^{\lrcorner} \cap \mathfrak{R}_2^{\lrcorner})$$

and

$$\mathfrak{A} = \mathfrak{L}_1 \oplus \mathfrak{L}_2 \oplus (\mathfrak{L}_1^{\llcorner} \cap \mathfrak{L}_2^{\llcorner}) .$$

Continuing the procedure as before,

$$\mathcal{A} = \mathcal{R}_1 \oplus \cdots \oplus \mathcal{R}_m$$

and

$$\mathcal{A} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n ,$$

Where $\mathcal{R}_1,\ldots,\mathcal{R}_m$ are mutually left orthogonal minimal right ideals and $\mathcal{L}_1,\ldots,\mathcal{L}_n$ are mutually right orthogonal minimal left ideals. Since $\mathcal{A}$ is simple, $\mathcal{R}_i \cap \mathcal{L}_j \neq (0)$ for $i = 1,\ldots,m$ and $j = 1,\ldots,n$. In particular, $\mathcal{R}_1 \cap \mathcal{L}_2 = \mathcal{R} \cap \mathcal{M} = \mathcal{D} \neq (0)$ and $\mathcal{R}_2 \cap \mathcal{L}_1 = \mathcal{S} \cap \mathcal{L} = \mathcal{E} \neq (0)$. Lemma 12.3 again implies that $\mathcal{C},\mathcal{D},\mathcal{E}$ and $\mathcal{F}$ all have the same dimension. This completes the proof.

Suppose now that $\mathcal{B}$ is a real or complex ternary algebra containing a positive unit U such that $AU^*U = UU^*A = A$ for every element A in $\mathcal{B}$. Let a new law of composition be defined in $\mathcal{B}$ as follows:

$$A \circ B = AU^*B .$$

With this operation $\mathcal{B}$ is a binary algebra. The <u>adjoint</u> of A relative to U is defined to be

$$A^+ = UA^*U .$$

If A,B,C is any triple of elements in $\mathcal{B}$, then

$$A \circ B^+ \circ C = AU^*B^+U^*C = AU^*UB^*UU^*C = AB^*C ,$$

and the original ternary composition is regained. Each central ideal $\mathcal{C}_{ij}$ in the algebra $\mathcal{A} = \oplus_{i=1} \oplus_{j=1}^{n} \mathcal{C}_{ij}$ contains such a positive unit and is therefore a binary subalgebra of $\mathcal{A}$.

Theorems 12.1 and 12.2 combine to give us that if $\mathcal{A}$ is simple, then each C_{ij} has either dimension one, two or four. If $\dim(C_{ij}) = 1$, then C_{ij} is isomorphic to the field of real numbers. In the case when C_{ij} is of dimension two, Lemma 12.2 ensures the existence of linearly independent elements U and V in C_{ij} satisfying

$$UU^*U = U \ , \ VV^*V = V \ ,$$
$$VU^*V = -U \ \text{ and } \ UV^*U = -V \ .$$

The element U also has the property that $AU^*U = UU^*A = A$ for each $A \in C_{ij}$. As above, define $A \circ B = AU^*B$ and for convenience, $A^2 = A \circ A$. Then $U^2 = U$, $V^2 = -U$ and $V^+ = -V$. If A and B are elements of C_{ij}, there exist real numbers α, β, γ and δ such that

$$A = \alpha U + \beta V \quad \text{and} \quad B = \gamma U + \delta V \ .$$

Hence,

$$\begin{aligned} A \circ B &= (\alpha U + \beta V)U^*(\gamma U + \delta V) \\ &= \alpha\gamma\, UU^*U + \alpha\delta UU^*V + \beta\gamma VU^*U + \beta\delta VU^*V \\ &= \alpha\gamma U + \alpha\delta V + \beta\gamma V - \beta\delta U \\ &= (\alpha\gamma - \beta\delta)U + (\alpha\delta + \beta\gamma)V \ , \end{aligned}$$

and

$$A^+ = UA^*U = U(\alpha U + \beta V)^*U = \alpha U - \beta V \ .$$

Therefore, C_{ij} is isomorphic to the field of complex numbers. Finally, if C_{ij} has dimension four, then Theorem 12.1 gives the following characterization of C_{ij}. Let U, I, J, K denote the basis elements of C_{ij} as described in the aforementioned theorem. If A and B belong to C_{ij}, again define $A \circ B = AU^*B$ and $A^2 = A \circ A$. Then the elements U, I, J and K satisfy

$$I^2 = J^2 = K^2 = -U ,$$

$$I \circ J = -J \circ I = K , \; J \circ K = -K \circ J = I ,$$

$$K \circ I = -I \circ K = J .$$

C_{ij} is thus isomorphic to the division ring of real quaternions.

These facts, along with Theorem 12.3, prove the following result:

THEOREM 12.4. <u>Let $\mathfrak{a}$ be a simple, real finite-dimensional ternary algebra. Then there exist integers m and n such that either:</u>

(i) $\mathfrak{a}$ <u>is isomorphic to the ternary algebra of all real $m \times n$ matrices, or:</u>

(ii) $\mathfrak{a}$ <u>is the direct sum of mn minimal central ideals C_{ij}, each of which is isomorphic to the complex numbers and such that C_{ij} is left orthogonal to C_{kj} if $i \neq k$ and C_{ij} is right orthogonal to $C_{i\ell}$ if $j \neq \ell$, or:</u>

(iii) $\mathfrak{A}$ <u>is the direct sum of</u> mn <u>minimal central ideals</u> C_{ij}, <u>each of which is isomorphic to the division ring of real quaternions and such that</u> C_{ij} <u>is left orthogonal to</u> C_{kj} <u>if</u> $i \neq k$ <u>and</u> C_{ij} <u>is right orthogonal to</u> $C_{i\ell}$ <u>if</u> $j \neq \ell$.

In the case when $\mathfrak{A}$ is not simple, we decompose $\mathfrak{A}$ into mutually orthogonal simple subalgebras $\mathfrak{B}_1, \ldots, \mathfrak{B}_p$:

$$\mathfrak{A} = \mathfrak{B}_1 \oplus \cdots \oplus \mathfrak{B}_p \ .$$

Theorem 12.4 may then be applied to each $\mathfrak{B}_i$ to obtain a representation of $\mathfrak{A}$.

13. APPLICATIONS TO MATRICES

Our attention will now be restricted to the study of ternary algebras of matrices. The applications will include a characterization of an orthonormal basis of such an algebra.

The $*$-operation will as usual denote conjugate transpose. All vectors will be understood to be column vectors. If x is a vector then the norm of x is $\|x\| = \sqrt{x^*x}$.

The following well-known result [2] will be of importance.

LEMMA 13.1. *Let* A *be an* $m \times n$ *matrix of rank* $r > 0$. *Then there exists an orthonormal set of m-dimensional vectors* $\{x_1,\ldots,x_r\}$ *and an orthonormal set of n-dimensional vectors* $\{y_1,\ldots,y_r\}$ *with a set of* r *positive numbers* $\{\lambda_1,\ldots,\lambda_r\}$ *such that*

$$A = \sum_{i=1}^{r} \lambda_i x_i y_i^* .$$

Furthermore, $Ay_i = \lambda_i x_i$, $A^*x_i = \lambda_i y_i$ *and* $x_1,\ldots,x_r$ *are eigenvectors of* AA^* *while* $y_1,\ldots,y_r$ *are eigenvectors of* A^*A.

Since A is of rank r, A^*A is of rank r also. Thus, let $\lambda_1^2,\ldots,\lambda_r^2$ be the non-zero eigenvalues of A^*A with associated orthonormal eigenvectors $y_1,\ldots,y_r$. We may assume that $\lambda_i > 0$ $(i = 1,\ldots,r)$.

Let $y_{r+1},\ldots,y_n$ be an orthonormal basis for the null-space of A^*A such that $y_1,\ldots,y_r,y_{r+1},\ldots,y_n$ is an orthonormal basis of E^n, Euclidean n-dimensional space.

Let

$$x_i = \frac{1}{\|Ay_i\|} Ay_i$$

for $i = 1,\ldots,r$. Then

$$\|Ay_i\|^2 = (Ay_i)^*(Ay_i) = y_i^*(A^*Ay_i)$$
$$= \lambda_i^2 y_i^* y_i = \lambda_i^2 ,$$

so that $\lambda_i = \|Ay_i\|$ $(i = 1,\ldots,r)$. Also,

$$AA^*x_i = \frac{1}{\|Ay_i\|} A(A^*Ay_i)$$
$$= \lambda_i^2 \frac{1}{\|Ay_i\|} Ay_i = \lambda_i^2 x_i$$

for $i = 1,\ldots,r$ and so $x_1,\ldots,x_r$ are eigenvectors of AA^*. To see that $x_1,\ldots,x_r$ are orthonormal, note that

$$x_i^*x_j = \frac{1}{\lambda_i\lambda_j}(Ay_i)^*(Ay_j) = \frac{1}{\lambda_i\lambda_j} y_i^*(A^*Ay_j)$$
$$= \frac{\lambda_j^2}{\lambda_i\lambda_j} y_i^*y_j = \frac{\lambda_j}{\lambda_i}\delta_{ij} = \delta_{ij} .$$

In addition, if $1 \le i \le r$,

$$Ay_i = \|Ay_i\|x_i = \lambda_i x_i$$

and

$$A^*x_i = \frac{1}{\|Ay_i\|} A^*Ay_i = \frac{1}{\|Ay_i\|} \lambda_i^2 y_i = \lambda_i y_i \ .$$

Now let $D = A - \Sigma_{i=1}^r \lambda_i x_i y_i^*$. Then if $1 \leq j \leq r$,

$$Dy_j = Ay_j - \sum_{i=1}^r \lambda_i x_i y_i^* y_j$$

$$= \lambda_j x_j - \lambda_j x_j$$

$$= 0 \ .$$

If $r + 1 \leq j \leq n$, then

$$Dy_j = Ay_j - \sum_{i=1}^r \lambda_i x_i y_i^* y_j$$

$$= 0 \ ,$$

since y_j is in the null-space of A^*A and therefore in the null-space of A. Hence, $D = 0$ on E^n so that

$$A = \sum_{i=1}^r \lambda_i x_i y_i^*$$

as desired.

The following theorem is the converse to a result mentioned in Section 4. $\mathfrak{M}_{mn}$ will denote as before the ternary algebra of all $m \times n$ complex matrices.

THEOREM 13.1. _If_ $\mathfrak{R}$ _is a minimal right ideal in_ $\mathfrak{M}_{mn}$, _or in a subalgebra of_ $\mathfrak{M}_{mn}$, _then every element_ $A \in \mathfrak{R}$ _is of the form_

$$A = uv^* ,$$

where u _is an m-dimensional vector independent of_ A _and_ v _is an n-dimensional vector._

Suppose that $A \in \mathfrak{R}$ and assume that A has rank $r > 0$. It will be shown that $r = 1$. If $r > 1$, we may write

$$A = \sum_{i=1}^{r} \lambda_i x_i y_i^*$$

where $(x_1,\ldots,x_r)$ and $(y_1,\ldots,y_r)$ are orthonormal sets. Let $E_i = x_i y_i^*$ for $i = 1,\ldots,r$. Then

$$E_i^* E_j = y_i x_i^* x_j y_j^* = \delta_{ij} y_i y_j^*$$

and

$$E_i E_j^* = x_i y_i^* y_j x_j^* = \delta_{ij} x_i x_j^* ,$$

so that E_i is orthogonal to E_j if $i \neq j$. Repeated use of Lemma 4.1 gives us that $E_1,\ldots,E_r$ are all elements of $\mathfrak{R}$. But the fact that E_i is left orthogonal to E_j for $i \neq j$ contradicts Theorem 4.1. Hence, $r = 1$, so that for some integer k,

$$A = \lambda_k x_k y_k^* .$$

Let $u = \lambda_k x_k$ and $v = y_k$. Then

$$A = uv^* .$$

To show that u is independent of A, let $B \in \mathfrak{R}$. Then as above, we can show the existence of an m-dimensional vector w and an n-dimensional vector x such that

$$B = wx^* .$$

Since $\mathfrak{R}$ is a minimal right ideal and $A \neq 0$, $A'A^*B = B$. Also, it is easy to check that $A' = (1/\alpha)A$, where $\alpha = \|u\|^2\|v\|^2$. Therefore,

$$B = A'A^*B = \alpha^{-1}AA^*B = \alpha^{-1}(uv^*)(vu^*)(wx^*)$$

$$= \alpha^{-1}(v^*v)(u^*w)ux^* = uy^* ,$$

where $y = \alpha^{-1}(v^*v)(w^*u)x$. This completes the proof.

The corresponding result for left ideals is as follows:

THEOREM 13.2. <u>If</u> $\mathfrak{L}$ <u>is a minimal left ideal in</u> $\mathfrak{M}_{mn}$ <u>or in a subalgebra of</u> $\mathfrak{M}_{mn}$, <u>then every</u> $A \in \mathfrak{L}$ <u>is of the form</u>

$$A = uv^* ,$$

<u>where</u> u <u>is an m-dimensional vector and</u> v <u>is an n-dimensional vector independent of</u> A.

Within the proper coordinate system, minimal right ideals $\mathfrak{R}$ in $\mathfrak{M}_{mn}$ may be regarded as <u>rows</u> of an $m \times n$ matrix. Each element $R \in \mathfrak{R}$ has the form $R = uv^*$ where u is some fixed m-dimensional vector. If it happens that u is the vector with a 1 in the i-th coordinate and zeroes elsewhere, then R has the form

$$R = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

where $\alpha_1,\ldots,\alpha_n$ (occurring in row i) are the components of the vector v dependent upon R. In a suitable coordinate system, minimal left ideals in $\mathfrak{M}_{mn}$ may be regarded as columns of an $m \times n$ matrix. Also, every minimal central ideal C in $\mathfrak{M}_{mn}$ is of dimension one, and so each element $C \in \mathbf{C}$ may be regarded as an individual entry of an $m \times n$ matrix.

Every finite-dimensional ternary algebra $\mathfrak{A}$ is either isomorphic to $\mathfrak{M}_{mn}$ or to a subalgebra thereof. The above remarks indicate that in the proper coordinate system, the decomposition of $\mathfrak{A}$ into left orthogonal minimal right ideals may be regarded as the sum of the rows of the matrix associated with an element $A \in \mathfrak{A}$. The decomposition into right orthogonal minimal left ideals can be interpreted as the sum of the columns of the matrix. Finally, when $\mathfrak{A}$ is decomposed into minimal central ideals, we may regard this as the sum of mn matrices, each containing a single component of the associated matrix and zeroes elsewhere.

The following theorem provides a characterization of an orthonormal basis of $\mathfrak{M}_{mn}$.

THEOREM 13.3. A set of non-zero matrices E_{ij} $(i = 1,\ldots,m\ ;\ j = 1,\ldots,n)$ forms an orthonormal basis of $\mathfrak{M}_{mn}$ if and only if there exists an orthonormal set of m-dimensional vectors $\{e_1,\ldots,e_m\}$ and an orthonormal set of n-dimensional vectors $\{f_1,\ldots,f_n\}$ such that

$$E_{ij} = e_i f_j^*$$

for $i = 1,\ldots,m$ and $j = 1,\ldots,n$.

It is easy to see that this condition is sufficient. For, we have that

$$\begin{aligned} E_{gh}E_{ij}^*E_{k\ell} &= e_g(f_h^*f_j)(e_i^*e_k)f_\ell^* \\ &= \delta_{hj}\delta_{ik}e_g f_\ell^* \\ &= \delta_{hj}\delta_{ik}E_{g\ell}\ . \end{aligned}$$

To prove the necessity, for each $i = 1,\ldots,m$, let $\mathfrak{R}_i$ denote the set of all elements $A \in \mathfrak{M}_{mn}$ which can be written in the form

$$A = \sum_{j=1}^{n} \alpha_j E_{ij}\ ,$$

where $\alpha_1,\ldots,\alpha_n$ are scalars. Using exactly the same argument as in the proof of Lemma 10.1, we get that $\mathcal{R}_1,\ldots,\mathcal{R}_m$ are minimal right ideals in $\mathfrak{G}$. Also, since E_{ij} is right orthogonal to $E_{i\ell}$ if $j \neq \ell$, repeated use of Lemma 4.1 shows that $E_{ij} \in \mathcal{R}_i$ for $j = 1,\ldots,n$. Therefore, by Theorem 13.1 there exists an m-dimensional vector a_i and an n-dimensional vector x_{ij} such that

$$E_{ij} = a_i x_{ij}^* .$$

If we now denote by $\mathcal{L}_j$ all elements B in $\mathfrak{G}$ which are of the form

$$B = \sum_{i=1}^{m} \beta_i E_{ij} ,$$

then it is not difficult to show that $\mathcal{L}_1,\ldots,\mathcal{L}_n$ are minimal left ideals in $\mathfrak{G}$ and that $E_{ij} \in \mathcal{L}_j$ for $i = 1,\ldots,m$. Hence, by Theorem 13.2,

$$E_{ij} = y_{ij} b_j^* ,$$

where y_{ij} is an m-dimensional vector and b_j is n-dimensional.

Thus,

$$a_i x_{ij}^* = y_{ij} b_j^* ,$$

so that $y_{ij} = \lambda_{ij} a_i$, where $\lambda_{ij} = (x_{ij}^* x_{ij})(b_j^* x_{ij})^{-1}$. Note that if $b_j^* x_{ij} = 0$, then $a_i \|x_{ij}\|^2 = 0$ and therefore $E_{ij} = 0$, a contradiction. Hence,

$$E_{ij} = \lambda_{ij} a_i b_j^* \ .$$

Assume without loss of generality that $\|a_i\| = \|b_j\| = 1$, since otherwise we may absorb the normalizing constants into the scalar λ_{ij}. Now note that

$$\begin{aligned}
\delta_{hj}\delta_{ik}(\lambda_{g\ell} a_g b_\ell^*) &= \delta_{hj}\delta_{ik} E_{g\ell} \\
&= E_{gh} E_{ij}^* E_{k\ell} \\
&= \lambda_{gh}\bar{\lambda}_{ij}\lambda_{k\ell} a_g b_h^* b_j a_i^* a_k b_\ell^* \\
&= \lambda_{gh}\bar{\lambda}_{ij}\lambda_{k\ell}(a_i^* a_k)(b_h^* b_j) a_g b_\ell^* \ .
\end{aligned}$$

Let $h = j$. Then

$$\delta_{ik}(\lambda_{g\ell} a_g b_\ell^*) = \lambda_{gj}\bar{\lambda}_{ij}\lambda_{k\ell}(a_i^* a_k) a_g b_\ell^* \ ,$$

so that

$$a_i^* a_k = \lambda_{g\ell}(\lambda_{gj}\bar{\lambda}_{ij}\lambda_{k\ell})^{-1}\delta_{ik} \ , \tag{13.1}$$

and therefore a_i is orthogonal to a_k if $i \neq k$. Also, if $i = k$, then

$$\delta_{hj}(\lambda_{g\ell} a_g b_\ell^*) = \lambda_{gh}\bar{\lambda}_{kj}\lambda_{k\ell}(b_h^* b_j) a_g b_\ell^* \ ,$$

and b_h is orthogonal to b_j if $h \neq j$.

Let $i = k$ in equation (13.1). Then

$$\lambda_{g\ell} = \lambda_{gj}\bar{\lambda}_{ij}\lambda_{k\ell} .$$

Since the left-hand side of this last equation is independent of i, j and k, we may let $i = j = k = 1$ to get that

$$\lambda_{g\ell} = \lambda_{g1}\bar{\lambda}_{11}\lambda_{1\ell} .$$

Therefore,

$$E_{ij} = \lambda_{ij}a_i b_j^* = \lambda_{i1}\bar{\lambda}_{11}\lambda_{1j}a_i b_j^* = c_i d_j^* ,$$

where $c_i = \lambda_{i1}\bar{\lambda}_{11}a_i$ and $d_j = \bar{\lambda}_{1j}b_j$. This representation of E_{ij} yields

$$\begin{aligned} c_i d_\ell^* &= E_{i\ell} \\ &= E_{ij}E_{ij}^*E_{i\ell} \\ &= c_i d_j^* d_j c_i^* c_i d_\ell^* \\ &= \|c_i\|^2\|d_j\|^2 c_i d_\ell^* , \end{aligned}$$

and so $\|c_i\| \cdot \|d_j\| = 1$. Also, since c_i is a multiple of a_i and d_j is a multiple of b_j, we have that $c_1,\ldots,c_m$ are mutually orthogonal and that the same is true of $d_1,\ldots,d_n$. Thus, letting

$$e_i = \frac{1}{\|c_i\|}\, c_i \quad \text{and} \quad f_j = \frac{1}{\|d_j\|}\, d_j$$

we have that

$$e_i f_j^* = \frac{1}{\|c_i\| \cdot \|d_j\|} c_i d_j^* = E_{ij} ,$$

where $e_1,\ldots,e_m$ are orthonormal and $f_1,\ldots,f_n$ are orthonormal.

14. CIRCULANTS

A special class of matrices will now be studied within the framework of a ternary algebra. Attention will at first be restricted to the case when the matrices are square. A generalization will then be given for the rectangular case. This, along with previous results, will allow a decomposition of any finite-dimensional ternary algebra into commutative binary subalgebras.

An $n \times n$ matrix of the form

$$A = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\ \alpha_n & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ \alpha_{n-1} & \alpha_n & \alpha_1 & \cdots & \alpha_{n-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_1 \end{bmatrix}$$

where $\alpha_1, \ldots, \alpha_n$ are complex numbers, is called a <u>circulant</u>. Let I_k denote the $k \times k$ identity matrix and let

$$J = \begin{bmatrix} 0 & I_{n-1} \\ 1 & 0 \end{bmatrix},$$

where the elements 0 are blocks of zeroes of appropriate sizes. Then

$$J^2 = \begin{bmatrix} 0 & I_{n-2} \\ I_2 & 0 \end{bmatrix} , \quad J^3 = \begin{bmatrix} 0 & I_{n-3} \\ I_3 & 0 \end{bmatrix}, \text{ etc.}$$

Any circulant A may therefore be expressed in the form

$$A = \alpha_1 I_n + \alpha_2 J + \alpha_3 J^2 + \cdots + \alpha_n J^{n-1} .$$

Note also that since $J^n = I_n$, we have that $J^{-1} = J^{n-1} = J^*$, where $*$ as usual denotes conjugate transpose. From this last equation it is seen that

$$(J^k)^* = (J^*)^k = (J^{-1})^k = J^{-k} .$$

As a convention, set $J^0 = J^n = I_n$. The above expansion of A now becomes

$$A = \sum_{i=1}^{n} \alpha_i J^{i-1} .$$

A ternary algebra $\mathfrak{a}$ is called <u>commutative</u> if $AB^*C = CB^*A$ for all elements A, B and C in $\mathfrak{a}$.

THEOREM 14.1. <u>The class</u> C <u>of all</u> $n \times n$ <u>complex circulants is a commutative binary algebra of dimension</u> n.

C is clearly a linear space over the complex numbers. Let A and B be any two elements of C. Then there exist scalars

α_i and β_i , $i = 1,\ldots,n$) such that

$$A = \sum_{i=1}^{n} \alpha_i J^{i-1} \quad \text{and} \quad B = \sum_{i=1}^{n} \beta_i J^{i-1} .$$

Hence,

$$AB = \sum_{i,j=1}^{n} \alpha_i \beta_j J^{i+j-2}$$

$$= \sum_{k=1}^{n} \left(\sum_{i=1}^{k} \alpha_i \beta_{k+1-i} \right) J^{k-1} ,$$

which is a circulant. Obviously, $AB=BA$. The remaining axioms for a binary algebra are readily verified.

Finally, since $I_n, J, \ldots, J^{n-1}$ form a basis for C, the dimension of C is n.

If B is a circulant, then B^* is also. Thus, the preceding theorem implies that C is closed under the triple product AB^*C and $AB^*C = CB^*A$ for every triple A,B,C in C. This proves the following

COROLLARY. C <u>is a commutative ternary algebra of dimension</u> n.

The following theorem characterizes minimal ideals in C, regarded as a ternary algebra.

THEOREM 14.2. *Let* $\mathfrak{B}$ *be either a right or a left ideal in* C. *Then* $\mathfrak{B}$ *is minimal if and only if the dimension of* $\mathfrak{B}$ *is one*.

If $\mathfrak{B}$ has dimension one, then $\mathfrak{B}$ is clearly minimal.

For the converse, suppose that $\mathfrak{B}$ is a minimal right ideal in C. Let

$$R = \begin{bmatrix} \rho_1 & \rho_2 & \rho_3 & \cdots & \rho_n \\ \rho_n & \rho_1 & \rho_2 & \cdots & \rho_{n-1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \rho_2 & \rho_3 & \rho_4 & \cdots & \rho_1 \end{bmatrix},$$

be a non-zero element of $\mathfrak{B}$. Theorem 13.1 implies that R has rank one. Thus, there is one column of R of which every other is a multiple. Since all columns of R contain exactly the same elements, there exists a scalar α such that

$$\begin{bmatrix} \rho_2 \\ \rho_1 \\ \rho_n \\ \rho_{n-1} \\ \vdots \\ \rho_4 \\ \rho_3 \end{bmatrix} = \alpha \cdot \begin{bmatrix} \rho_1 \\ \rho_n \\ \rho_{n-1} \\ \rho_{n-2} \\ \vdots \\ \rho_3 \\ \rho_2 \end{bmatrix}.$$

Therefore, $\rho_i = \alpha\rho_{i-1}$ for $i = 2,3,\ldots,n$ and $\rho_1 = \alpha\rho_n$. This yields $\rho_n = \alpha^{n-1}\rho_1 = \alpha^n\rho_n$. If $\rho_n = 0$, then $\rho_i = 0$ for $i = 1,2,\ldots,n-1$, and so $R = 0$, a contradiction. Hence, $\rho_n \neq 0$, so that $\alpha^n = 1$. Since $\rho_i = \alpha^{i-1}\rho_1$ for $i = 1,2,\ldots,n$, the form of R is now

$$R = \begin{bmatrix} \rho_1 & \alpha\rho_1 & \alpha^2\rho_1 & \cdots & \alpha^{n-1}\rho_1 \\ \alpha^{n-1}\rho_1 & \rho_1 & \alpha\rho_1 & \cdots & \alpha^{n-2}\rho_1 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \alpha\rho_1 & \alpha^2\rho_1 & \alpha^3\rho_1 & \cdots & \rho_1 \end{bmatrix}$$

$$= \rho_1 \begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{n-1} \\ \alpha^{n-1} & 1 & \alpha & \cdots & \alpha^{n-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \alpha & \alpha^2 & \alpha^3 & \cdots & 1 \end{bmatrix}.$$

$$= \lambda\tilde{\alpha a}a^* ,$$

where $\lambda = \rho_1$,

$$a = \begin{bmatrix} 1 \\ \alpha \\ \alpha^2 \\ \vdots \\ \alpha^{n-2} \\ \alpha^{n-1} \end{bmatrix} \quad \text{and} \quad \tilde{a} = \begin{bmatrix} \alpha^{n-1} \\ \alpha^{n-2} \\ \vdots \\ \alpha^2 \\ \alpha \\ 1 \end{bmatrix}.$$

Now let B be any other non-zero element of $\mathfrak{R}$. Then $B = \mu\beta\tilde{b}b^*$, where μ is a scalar, $\beta^n = 1$,

$$b = \begin{bmatrix} 1 \\ \beta \\ \beta^2 \\ \vdots \\ \beta^{n-2} \\ \beta^{n-1} \end{bmatrix} \quad \text{and} \quad \tilde{b} = \begin{bmatrix} \beta^{n-1} \\ \beta^{n-2} \\ \vdots \\ \beta^2 \\ \beta \\ 1 \end{bmatrix}.$$

If $a \neq b$, then $\tilde{a} \neq \tilde{b}$ and $\alpha \neq \beta$. Therefore,

$$\begin{aligned} A^*B &= (\overline{\lambda\alpha})(\mu\beta)a\tilde{a}^*\tilde{b}b^* \\ &= (\overline{\lambda\alpha})(\mu\beta)(\tilde{a}^*\tilde{b})ab^* . \end{aligned}$$

Now

$$\begin{aligned} \tilde{a}^*\tilde{b} &= 1 + \bar{\alpha}\beta + \bar{\alpha}^2\beta^2 + \cdots + \bar{\alpha}^{n-1}\beta^{n-1} \\ &= \frac{1 - \bar{\alpha}^n\beta^n}{1 - \bar{\alpha}\beta} \\ &= 0 , \end{aligned}$$

so that A and B are two non-zero left orthogonal elements of the minimal right ideal $\mathfrak{R}$, a contradiction to Theorem 4.1. Hence, $a = b$, $\tilde{a} = \tilde{b}$ and $\alpha = \beta$. Therefore,

$$B = \mu\alpha\tilde{a}a^* = \mu\lambda^{-1}A ,$$

and the proof is completed.

Let $\omega_1,\ldots,\omega_n$ denote the n distinct n-th roots of unity and let

$$a_i = \begin{bmatrix} 1 \\ \omega_i \\ \omega_i^2 \\ \vdots \\ \omega_i^{n-1} \end{bmatrix} \quad \text{and} \quad \tilde{a}_i = \begin{bmatrix} \omega_i^{n-1} \\ \omega_i^{n-2} \\ \vdots \\ \omega_i \\ 1 \end{bmatrix}$$

for $i = 1,\ldots,n$. Finally, put

$$A_i = \omega_i \tilde{a}_i a_i^* \qquad (i = 1,\ldots,n) .$$

Lemma 14.1. $A_1,\ldots,A_n$ <u>are</u> n <u>mutually orthogonal elements which form a basis for</u> C.

It was shown in the proof of the preceding theorem that if $i \neq j$, then $A_i^* A_j = 0$. Also, if $i \neq j$, then

$$A_i A_j^* = \omega_i \bar{\omega}_j \tilde{a}_i a_i^* a_j \tilde{a}_j$$

and

$$\begin{aligned} a_i^* a_j &= 1 + \bar{\omega}_i \omega_j + \bar{\omega}_i^2 \omega_j^2 + \cdots + \bar{\omega}_i^{n-1} \omega_j^{n-1} \\ &= \frac{1 - \bar{\omega}_i^n \omega_j^n}{1 - \bar{\omega}_i \omega_j} \\ &= 0 . \end{aligned}$$

Hence, $A_1,\dots,A_n$ are mutually orthogonal and are therefore n linearly independent elements of the n-dimensional algebra C.

THEOREM 14.3. C <u>is the direct sum of</u> n <u>mutually orthogonal one-dimensional subalgebras, each of which is a minimal two-sided ideal in</u> C.

Let $A_1,\dots,A_n$ again denote the n orthogonal elements of the preceding lemma. For each $i = 1,\dots,n$, let C_i be the set of all elements $A \in C$ which are scalar multiples of A_i. Then C_i is orthogonal to C_j if $i \neq j$. Also, since $A_1,\dots,A_n$ form a basis for C, the one-dimensional subalgebras $C_1,\dots,C_n$ generate C. Therefore,

$$C = C_1 \oplus \cdots \oplus C_n \, .$$

By Lemma 8.1, $C_1,\dots,C_n$ are all two-sided ideals in C.

The concept of a circulant will now be generalized to include rectangular matrices. An $m \times n$ matrix $A = (\alpha_{ij})$, where $1 < m \leq n$, will be called a circulant if the elements α_{ij} satisfy

$$\alpha_{i+1,j} = \alpha_{i,j-1} \quad (i = 1,\dots,m-1 \; ; \; j = 2,\dots,n)$$

and

$$\alpha_{i+1,1} = \alpha_{i,n} \qquad (i = 1,\dots,m-1) \, .$$

We shall make the convention that any $1 \times n$ matrix forms a circulant. If $m > n$, then A is called a circulant if A^* is. For example, a 2×3 circulant has the form

$$\begin{bmatrix} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \end{bmatrix},$$

and a 4×3 circulant would appear as

$$\begin{bmatrix} \alpha & \beta & \gamma \\ \delta & \alpha & \beta \\ \gamma & \delta & \alpha \\ \beta & \gamma & \delta \end{bmatrix}$$

Rectangular circulants do not, in general, form ternary algebras. In the class $\mathbf{C}$ of all 2×3 circulants, let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then

$$AB^*C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is not an element of C. Such circulants, however, are useful with regard to the study of finite-dimensional ternary algebras.

We begin by considering a simple example. Let $\mathcal{A}$ be the ternary algebra of all 2×3 complex matrices. If $A = (\alpha_{ij})$ is an element of $\mathcal{A}$, write $A = A_1 + A_2 + A_3$, where

$$A_1 = \begin{bmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{22} & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & \alpha_{12} & 0 \\ 0 & 0 & \alpha_{23} \end{bmatrix}$$

and

$$A_3 = \begin{bmatrix} 0 & 0 & \alpha_{13} \\ \alpha_{21} & 0 & 0 \end{bmatrix}.$$

The form of each of A_1, A_2 and A_3 is suggestive of a 2×3 circulant. Let $\mathcal{A}_i$ be the class of all matrices having the same form as A_i for $i = 1,2,3$. Let

$$U_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$U_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then it is easy to check that each $\mathfrak{A}_i$ is a ternary subalgebra of $\mathfrak{A}$ with the positive unit U_i satisfying $U_iU_i^*A_i = A_iU_i^*U_i = A_i$ for each element A_i in $\mathfrak{A}_i$. Therefore, by the remarks made in Section 12, $\mathfrak{A}_i$ is a binary algebra with the composition $A_i \circ B_i = A_iU_i^*B_i$. Also, we have that $A_i \circ B_i = B_i \circ A_i$. Thus, $\mathfrak{A}$ is decomposed into a direct sum of three commutative binary algebras.

The following theorem generalizes this result.

THEOREM 14.4. Every simple ternary algebra $\mathfrak{A}$ of finite dimension is the direct sum of a finite number of commutative binary subalgebras, all having the same dimension.

Let E_{ij} $(i = 1,\ldots,m\ ;\ j = 1,\ldots,n)$ be an orthonormal basis for $\mathfrak{A}$. Assume for the present that $m \leq n$. If A is any element of $\mathfrak{A}$, then there exist scalars α_{ij} such that

$$A = \sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_{ij}E_{ij}\ .$$

Let

$$A_j = \sum_{i=1}^{m} \alpha_{i,i+j-1}E_{i,i+j-1} \tag{14.1}$$

for $j = 1,\ldots,n - m + 1$, and

$$A_j = \sum_{i=1}^{n-j+1} \alpha_{i,i+j-1}E_{i,i+j-1}$$

(14.2)

$$+ \sum_{i=n-j+2}^{m} \alpha_{i,i+j-(n+1)}E_{i,i+j-(n+1)}$$

for $j = n - m + 2,\ldots,n$. It should be noted that the above decomposition of the matrix (α_{ij}) is the same as that of the preceding example. Also, we have that

$$\sum_{j=1}^{n} A_j = A .$$

For each $j = 1,\ldots,n$, let G_j denote the class of all elements A_j in G having the form (14.1) if $1 \leq j \leq n - m + 1$ and the form (14.2) if $n - m + 2 \leq j \leq n$. Suppose that $1 \leq j \leq n - m + 1$ and let

$$B_j = \sum_{i=1}^{m} \beta_{i,i+j-1}E_{i,i+j-1}$$

and

$$C_j = \sum_{i=1}^{m} \gamma_{i,i+j-1} E_{i,i+j-1}$$

Then

$A_j B_j^* C_j =$

$$= \sum_{i,k,\ell=1}^{m} \alpha_{i,i+j-1} \bar{\beta}_{k,k+j-1} \gamma_{\ell,\ell+j-1} E_{i,i+j-1} E^*_{k,k+j-1} E_{\ell,\ell+j-1}$$

$$= \sum_{i,k,\ell=1}^{m} \alpha_{i,i+j-1} \bar{\beta}_{k,k+j-1} \gamma_{\ell,\ell+j-1} \delta_{i+j-1,k+j-1} \delta_{k\ell} E_{i,\ell+j-1}$$

$$= \sum_{i=1}^{m} \alpha_{i,i+j-1} \bar{\beta}_{i,i+j-1} \gamma_{i,i+j-1} E_{i,i+j-1} .$$

By symmetry, $A_j B_j^* C_j = C_j B_j^* A_j$. If we put

$$U_j = \sum_{i=1}^{m} E_{i,i+j-1} ,$$

then clearly $A_j U_j^* U_j = U_j U_j^* A_j = A_j$ for each A_j in $\mathfrak{A}_j$. Therefore, each $\mathfrak{A}_j$ $(j = 1,\ldots,n - m + 1)$ is an m-dimensional commutative binary subalgebra of $\mathfrak{A}$.

Similarly, if $n - m + 2 \leq j \leq n$, and

$$B_j = \sum_{i=1}^{n-j+1} \beta_{i,i+j-1}E_{i,i+j-1}$$

$$+ \sum_{i=n-j+2}^{m} \beta_{i,i+j-(n+1)}E_{i,i+j-(n+1)}$$

and

$$C_j = \sum_{i=1}^{n-j+1} \gamma_{i,i+j-1}E_{i,i+j-1}$$

$$+ \sum_{i=n-j+2}^{m} \gamma_{i,i+j-(n+1)}E_{i,i+j-(n+1)} ,$$

then a straightforward computation yields

$$A_jB_j^*C_j = \sum_{i=1}^{n-j+1} \alpha_{i,i+j-1}\bar{\beta}_{i,i+j-1}\gamma_{i,i+j-1}E_{i,i+j-1}$$

$$+ \sum_{i=n-j+2}^{m} \alpha_{i,i+j-(n+1)}\bar{\beta}_{i,i+j-(n+1)}\gamma_{i,i+j-(n+1)}E_{i,i+j-(n+1)} .$$

Symmetry again gives us that $A_jB_j^*C_j = C_jB_j^*A_j$. Now if we let

$$U_j = \sum_{i=1}^{n-j+1} E_{i,i+j-1} + \sum_{i=n-j+2}^{m} E_{i,i+j-(n+1)} ,$$

then $A_jU_j^*U_j = U_jU_j^*A_j = A_j$ for every A_j in $\mathfrak{A}_j$. Hence, if $n - m + 2 \leq j \leq n$, then $\mathfrak{A}_j$ is an m-dimensional commutative binary subalgebra of $\mathfrak{A}$.

The linear independence of the elements E_{ij} proves that the subalgebras $\mathfrak{A}_1,\ldots,\mathfrak{A}_n$ are independent, so that

$$\mathfrak{A} = \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_n .$$

A similar argument may be used to prove the theorem in the case when $m > n$.

If $\mathfrak{A}$ is not simple, then $\mathfrak{A}$ has a unique decomposition into mutually orthogonal simple subalgebras $\mathfrak{B}_1,\ldots,\mathfrak{B}_p$:

$$\mathfrak{A} = \mathfrak{B}_1 \oplus \cdots \oplus \mathfrak{B}_p .$$

By Theorem 14.4, each $\mathfrak{B}_i$ is the direct sum of a finite number of commutative binary subalgebras, all of the same dimension:

$$\mathfrak{B}_i = \mathfrak{A}_{i1} \oplus \cdots \oplus \mathfrak{A}_{i,j_i} ,$$

so that

$$\mathfrak{a} = \bigoplus_{i=1}^{p} \mathfrak{a}_{i,j_i} .$$

This proves

THEOREM 14.5. <u>Every finite-dimensional ternary algebra is the direct sum of a finite number of commutative binary subalgebras</u>.

BIBLIOGRAPHY

[1] HESTENES, M. R., "A Ternary Algebra with Applications to Matrices and Linear Transformations", Archive for Rational Mechanics and Analysis, 11, 138-194 (1962).

[2] ______________, "Relative Hermitian Matrices", Pacific Journal of Mathematics, 11, 225-245 (1961).

[3] MOORE, E. H., Bull. Amer. Math. Soc., 26 (1920), 394-395.

[4] __________, General Analysis, Part I, Mem. Amer. Philos. Soc., I, 1935.

[5] PENROSE, R., "A Generalized Inverse for Matrices", Proceedings of the Cambridge Philosophical Society, 51, 406-413 (1955).

[6] WEDDERBURN, J. H. M., "On Hypercomplex Numbers", Proceedings of the London Mathematical Society, 6, 77-118 (1908).

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13. Abstract

A ternary algebra is a linear space $\mathfrak{A}$ over the complex numbers such that for any three elements A, B and C in $\mathfrak{A}$ there exists a product AB*C in $\mathfrak{A}$ satisfying certain axioms which reduce to ordinary properties of matrix multiplication when the elements of $\mathfrak{A}$ are matrices and * denotes conjugate transpose. The present paper is devoted to a study of the algebraic structure of ternary algebras. In particular it is shown that an arbitrary finite-dimensional ternary algebra has a representation as a ternary algebra of matrices. U

The aforementioned characterization is obtained by making use of an ideal theory in a ternary algebra. Right, left and central ideals are introduced and studied. Minimal ideals are then characterized as inner-product spaces. U

Using decompositions of a finite-dimensional ternary algebra $\mathfrak{A}$ into direct sums of minimal right, left and central ideals, a generalized orthonormal basis is obtained. Every element in $\mathfrak{A}$ then has a matrix representation relative to this basis, and an additive and multiplicative isomorphism is established between $\mathfrak{A}$ and a ternary algebra of matrices. U

The implications of the absence of the axiom called the positivity condition are then given for a finite-dimensional ternary algebra.

A characterization is found for a finite-dimensional ternary algebra when the scalars are restricted to be real. The paper concludes with applications of several of the results to ternary algebras of matrices. U

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