Convair Astronautics

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A Note on Rational Function Approximation

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INTRODUCTION

The purpose of this note is two-fold. The first objective is to describe a weighted least squares algorithm for rational function approximation. The second objective is to exhibit some examples of functions approximated by "rational functions in one or more variables".

In obtaining these approximations both the above mentioned algorithm and the linear inequality method are employed. The weighted least squares method is used to obtain a first approximation; the linear inequality method is then used to obtain the "best" or "Chebyshev" approximation. The inequalities, which are encountered in the linear inequality algorithm, are solved using the method outlined in [1].

In describing the weighted least squares procedure only "rational functions in one variable" are considered. This restriction results in a simplification of notation. It will become obvious during the discussion how the algorithm can be used for a more general class of approximating functions. In fact for both algorithms the prime restrictions on the approximation are the following. The numerator of the approximation is a linear function of the unknown coefficients, $(a_0, \ldots, a_n)$; the denominator of the approximation is a linear function of the unknown coefficients, $(b_1, \ldots, b_m)$.

WEIGHTED LEAST SQUARES ALGORITHM

Problem: Approximate f(t) over a set of points, $\{t_1, \ldots, t_v\}$, by a rational function of the form, $\dfrac{\sum_{i=0}^{n} a_i t^i}{1+\sum_{j=1}^{m} b_j t^j}$, where $v > m+n$.

Let:

$$R_k(a,b) = f(t_k) - \frac{\sum_{i=0}^{n} a_i t_k^i}{1+\sum_{j=1}^{m} b_j t_k^j} \quad (k=1, \ldots, v) . \tag{1}$$

Ideally we would like to find values of the vectors, $a = (a_0, \ldots, a_n)$ and $b = (b_1, \ldots, b_m)$, which minimize $\sum_{k=1}^{v} R_k^2(a,b)$. However, this is a difficult non-linear problem. Let us instead consider the following linear problem. Find the values of the vectors a and b which minimize

$$\sum_{k=1}^{v} \left[1+\sum_{j=1}^{m} b_j t_k^j\right]^2 R_k^2(a,b) \equiv \sum_{k=1}^{v} \left[f(t_v)\left(1+\sum_{j=1}^{m} b_j t_k^j\right) - \sum_{i=0}^{n} a_i t_k^i \right]^2 . \tag{2}$$

The above problem can be solved by standard methods. It should be noted that the trivial conditions, $1+\sum_{j=1}^{m} b_j t_k^j = 0$ $(k=1, \ldots, v)$, $a_i = 0$ $(i=0, \ldots, n)$, cannot occur since $v>m$.

The fact that we are minimizing in (2), $\sum_{k=1}^{v} \left[1+\sum_{j=1}^{m} b_j t_k^j\right]^2 R_k^2(a,b)$ rather than $\sum_{k=1}^{v} R_k^2(a,b)$ suggests a sequence of weighted problems be done. On iteration s (s=1, ...), the values of the vectors $a=(a_0, \ldots, a_n)$ and $b=(b_1, \ldots, b_m)$ would be found which minimize

$$\sum_{k=1}^{v} \frac{\left[1+\sum_{j=1}^{m} b_j t_k^j\right]^2}{\left[1+\sum_{j=1}^{m} b_j^{(s-1)} t_k^j\right]^2} R_k^2(a,b) \quad . \tag{3}$$

Here, $b^{(s-1)}=(b_1^{(s-1)}, \ldots, b_m^{(s-1)})$ is the denominator portion of the minimizing form on the iteration s-1. To start the iteration we would set $b_j^{(0)}=0$ $(j=1, \ldots, m)$. This iterative scheme is then known as the weighted least squares method.

No proof of convergence has been established for the weighted least squares method. However, it can be shown that if the sequence $(a^{(s)}, b^{(s)})(s=1, \ldots,)$ converges, the limit point of the sequence will satisfy a system of equations which closely resemble the so-called normal equations,

$$\begin{aligned}
&\frac{\partial \sum_{k=1}^{v} R_k^2(a,b)}{\partial a_0} = 0 \\
&\qquad \vdots \\
&\frac{\partial \sum_{k=1}^{v} R_k^2(a,b)}{\partial a_n} = 0 \\
&\frac{\partial \sum_{k=1}^{v} R_k^2(a,b)}{\partial b_1} = 0 \\
&\qquad \vdots \\
&\frac{\partial \sum_{k=1}^{v} R_k^2(a,b)}{\partial b_m} = 0
\end{aligned} \tag{4}$$

NUMERICAL EXAMPLES

A "Chebyshev" approximation is a couple, say (a^t, b^t), which minimizes

$$M(a,b) = \max_{1 \le i \le v} |R_i(a,b)| . \tag{5}$$

The linear inequality method generates a sequence of couples $\{(a^{(i)}, b^{(i)})\}$ $(i=1, \ldots)$, such that $\lim_{i \to \infty} M(a^{(i)}, b^{(i)}) = M(a^t, b^t)$. A description of this method can be found in [1] and [3].

Let us consider the problem of approximating $\tanh t = \frac{\sinh t}{\cosh t}$ by a rational function. The form of the approximation is suggested by the Taylor's Series expansions for sinh t and cosh t. The form is

$$\frac{a_5 x^5 + a_3 x^3 + a_1 x}{b_4 x^4 + b_2 x^2 + 1} . \tag{6}$$

Forty equally spaced points covering the range from $t=0$ to $t=1$ were chosen. Over this set of points the weighted least squares approximation to tanh t was obtained. The solution yielded the coefficients,

$a_1 = .99999999$	$b_2 = .44393819$
$a_3 = .11060497$	$b_4 = .15672388 \times 10^{-1}$
$a_5 = .10259267 \times 10^{-2}$	

Over this set of points the maximum residual in absolute value was 5×10^{-9}. Because the existing linear inequality code is written in single precision, the "best" approximation could not be obtained. However the excellent results obtained using the weighted least squares method probably make the "best" approximation only of academic interest.

To illustrate the fact that "rational functions in more than one variable" can be used, consider the following example. Let $(x,y)I = \int_0^y e^{-t} t^x dt$, the familiar incomplete gamma function. With this notation, $(x, \infty)!$ designates the complete gamma function. The problem is then to approximate

$$\frac{(x,y)!}{(x,\infty)!} - \frac{y^{x+1}}{(x+1,\infty)!} \tag{7}$$

by a rational function of the form,

$$\frac{y^{x+1}}{(x+1,\infty)!}\left(\frac{a_1x^2y + a_2xy + a_3y + a_4x^2y^2 + a_5xy^2 + a_6y^2}{b_0 + b_1x + b_2x^2}\right). \tag{8}$$

This form was suggested by the series expansions for $(x,y)!$, [4].

$$\frac{(x,y)!}{(x,\infty)!} = \frac{y^{x+1}}{(x+1,\infty)!}\left(1 + \frac{y}{1!}\,\frac{x+1}{x+2} + \frac{y^2}{2!}\,\frac{x+1}{x+3} + \ldots\right). \tag{9}$$

Forty points were chosen to form a grid over the rectangle, $0 \le x \le 1$ and $0 \le y \le .5$. Over this set of points, $\frac{(x,y)!}{(x,\infty)!} - \frac{y^{x+1}}{(x+1,\infty)!}$ was approximated by a function of the form designated in (8). The weighted least squares solution for this set of points yields a maximum residual in absolute value of .000069. The "best" approximation yields a maximum residual in absolute value of .000055. The coefficients for the "best" approximation are

$a_1 = -10002.779$	$b_0 = 1.0000000$
$a_2 = -9657.0846$	$b_1 = 19545.865$
$a_3 = -.49468537$	$b_2 = 10617.318$.
$a_4 = 3119.1040$	
$a_5 = 2643.8523$	
$a_6 = .13716296$	

It should be noted that the coefficients, a_3 and a_6, could both be set to zero without losing too much accuracy in the approximation.

The weighted least squares method has been used for the last three years at Convair Astronautics for various curve fitting problems. Codes for this algorithm exist on both the I. B. M. 704 and E. R. A. 1103. A code for the linear inequality method has been constructed for the I. B. M. 704, [2].

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REFERENCES

[1] E. W. Cheney and A. A. Goldstein, Newton's Method for Convex Programming and Tchebycheff Approximation, to appear in Numerische Mathematik.

[2] H. L. Loeb, Algorithms for Chebyshev Approximations Using the Ratio of Linear Forms, submitted for publication.

[3] E. W. Cheney, A. A. Goldstein and H. L. Loeb, Some Simple Remarks on Rational Approximations, paper given at the Symposium on Methods for Solving Linear Systems at the International Conference on Information Processing, Paris, June 15-20, 1959.

[4] E. Jahnke and F. Emde, Tables of Functions With Formulae and Curves, Dover Publications, New York, 1945, p.p. 22.

[5] K. Pearson, Table of Incomplete Gamma Function, London, 1922, XXXI + 164 p.p.

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