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LONG WAVES OVER WAVY BOTTOMS

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Department of the Geophysical Sciences
The University of Chicago

January 1968

Technical Report No. 1

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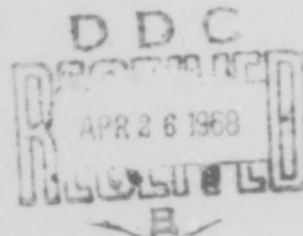
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A B S T R A C T

The propagation of long waves over bottoms having sinusoidal undulations is investigated here within the confines of linearized shallow water theory. It is found that the presence of this irregularity in most cases impedes the propagation of the wave in keeping with the proper application of the Green-DuBoys formula. For wavelengths for which this formula is not valid, or those of the same order as the bottom wavelengths, it is found that there is a region for which the propagation is not retarded, and the travel time is less than that based upon the mean depth. Furthermore, the presence of regular undulations of the bottom of any amplitude prohibits the propagation of an infinite sequence of wavelengths on the surface, the most significant of which are those of the same order as the bottom. These waves are unstable, and through resonance with the bottom will grow without bound as they progress, or at least until the linearized theory is invalidated. An electrical analog is presented which exhibits the same instability, a subharmonic resonance, and can be used to determine the free surface profiles.

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1. Introduction

The laws of propagation of long waves in canals of gradually varying rectangular cross section were investigated by G. Green (1837), leading to the well known Green-DuBoys formula for the travel time of a long wave based on the "local phase speed" which is proportional to the square root of the local depth. The physical conditions for which Green's approximation is valid are well detailed in Defant (1960), and need not be repeated here. Assuming these conditions to be met, then it is easy to determine the progress of a train of long waves over a spatially non-uniform bottom for which over a sufficiently large horizontal extent there is a well defined average depth, say h_0 . Katz (1963) considered the propagation of shallow water waves over a random bottom and showed in some detail that the non-uniformities of the bottom effected a decrease in the average phase speed of the surface waves, of which more later. That is, the actual speed is smaller than the speed based on the average depth, $C_0 = (gh_0)^{1/2}$, by an amount proportional to the mean square of the departure of the depth from its average. The non-uniform bottom in this case retards the surface waves.

What shall be considered here is the somewhat simpler case, conceptually, in which the canal or ocean bottom is sinusoidal. When the wave is above a trough of the bottom, its "local speed" is greater than C_0 and when it is over a crest, its speed is less than C_0 . However, the wave loses more time

progressing the shallow part than it gains over the deeper part, and it is easy to show that the net effect of the bottom is again to retard the wave. This assumes, of course, that the length scale of the bottom is considerably longer than the surface wavelength, in keeping with Green's approximation. The retardation will be calculated in detail in § 7 below.

It is of interest to inquire into the propagation of shallow water waves of wavelengths of the same order as the length of the bottom corrugations, for which Green's results are not really valid. Hidaka (1965) investigated the particular case for which the wavelength of the long wave train is identical with that of the bottom and concluded (correctly) that the Green-DuBoys formula is in error, but his quantitative results appear to be incorrect, as will be pointed out in § 8 below.

In the analysis to follow, no restriction will be made on the wavelength of the waves or on the amplitude and wavelength of the bottom corrugations other than the hydrostatic assumption of long wave theory, that is, the usual neglect of vertical accelerations. In fact, the crests of the bottom corrugations may come relatively close to the free surface in most cases. Standing wave solutions corresponding to eigenfrequencies are given in § 3 for this case. In § 6, the normal modes are combined to give progressive waves, and an average phase speed is defined and calculated. In § 7 the results of a Green's treatment are compared with the more general results given here, and

it is shown in what sense they agree.

If the bottom protrudes above the free surface, it is meaningless to speak of progressive waves with wavelengths of the same order as that of the bottom. The lowest natural frequencies of oscillation in such basins are seiches then. Chrystal (1905), (1906) investigated the seiche periods of a number of basins including in particular that in the shape of a concave quartic. Chrystal's inverted quartic shares a common feature with the cosine-shaped basins of § 4 here. That is, if the water depth and the slope of the bottom are simultaneously zero, then no solutions exist for the lake surface profile which are everywhere finite, but there is a limiting value of the frequency to which all of the natural modes coalesce, independent of the number of nodes. This was spoken of by Chrystal as the "anomalous seiche." In § 4 it is shown that the frequency of the anomalous seiche is independent of the entire shape of the basin, and depends solely on the curvature of the basin at the singular point.

What is most surprising, however, in the ensuing analysis is the result that there exists a range of wavelengths close to that of the bottom for which the waves are not retarded, but in fact travel faster than the phase speed based on the average depth. This is related to a subharmonic resonance between the waves and the bottom. In addition, it will be shown that there exists an infinite sequence of wavelengths for which periodic

propagation, in a sense to be defined, is impossible for any given amplitude of the wavy bottom.

2. The equations for the free surface

The equations to be used here are those of the linearized shallow water theory, and the limitations of the physical variables attendant with this approximation should be kept in mind. Some of the results to be obtained (particularly the asymptotic results) will certainly violate the physical approximations; this will be pointed out from time to time as appropriate. With $y = \eta(x, t)$ the equation for the free surface and $y = -h(x)$ the equation for the bottom contour, then

$$g[h(x)\eta_x(x, t)]_x = \eta_{tt}(x, t) \quad (2.1)$$

Looking for those solutions having harmonic time dependence, with

$$\eta(x, t) = \xi(x) e^{i\sigma t}, \quad (2.1) \text{ is}$$

$$\frac{d}{dx} \left[h(x) \frac{d\xi}{dx} \right] + \frac{\sigma^2}{g} \xi = 0 \quad (2.2)$$

Choose a bottom configuration having sinusoidal corrugations, and write $h(x) = h_0(1 + \alpha \cos px)$ where h_0 is the average depth and α the relative amplitude of the wavy undulations of wavelengths $2\pi/p = L_B$. Then with $z = px/2$, (2.2) becomes

$$(1 + \alpha \cos 2z) \frac{d^2 \xi}{dz^2} - 2\alpha \sin 2z \frac{d\xi}{dz} + \lambda^2 \xi = 0 \quad (2.3)$$

(where $\lambda^2 = 4\sigma^2/p^2 g h_0$), a linear second-order differential

equation with periodic coefficients.

As a preliminary investigation, make a transformation to normal form by the substitution $w(z) = h^{\frac{1}{2}} \xi$, yielding

$$\frac{d^2 w}{dz^2} + \frac{\lambda^2}{1 + \alpha \cos 2z} \left\{ 1 + \frac{\alpha}{\lambda^2} \left(\frac{2 \cos 2z + \alpha + \alpha \cos^2 2z}{1 + \alpha \cos 2z} \right) \right\} w = 0. \quad (2.4)$$

As an approximation for small α / λ^2 , and for small α , this becomes

$$w'' + (\lambda^2 - \lambda^2 \alpha \cos 2z) w = 0$$

which is easily recognized as Mathieu's equation. The preferred standard form of Mathieu's equation is

$$w'' + (a - 2q \cos 2z) w = 0$$

and in the present case $q = \alpha a / 2$. In the a - q plane, then, the familiar regions of stability and instability are shown in the usual crescent shaped figure, with the characteristic line separating alternate regions of stability and instability. Figure 1 shows the present case, for which the admissible values of the parameters a and q lie only on a line including with the axis of a an angle whose tangent is $\alpha/2$, or approximately an angle of $\alpha/2$, α supposed small. On the characteristic lines, of course, the solutions are well known to be periodic with period π or 2π , as the case may be. It may be expected that (2.3) exhibits a sim-

ilar behaviour, at least for $|\alpha| < 1$. Now for α not particularly small, the coefficient of ψ in (2.4) can be expanded as a Fourier series in $\cos 2n\tau$, yielding a Hill's equation. Relatively little is known about the existence of periodic solutions of Hill's equation in all but a few isolated special cases, so it is better to investigate (2.3) itself.

A fundamental theorem regarding linear ordinary differential equations with periodic coefficients, Floquet's Theorem,* states that (2.3) always has at least one solution $\xi(\tau)$ such that $\xi(\tau + \pi) = S \xi(\tau)$, where S is a constant (possibly complex) depending on the values of λ and α . As a corollary, (2.3) always has one solution of the form $e^{\mu\tau}\phi(\tau)$ where ϕ is a periodic function of period π . The periodicity factor S and the characteristic exponent μ are related by $S = e^{\pi\mu}$. Now solutions to (2.3) can be defined such that ξ_1 is even and ξ_2 is odd, and $\xi_1(0) = \xi_1'(0) = 1$, $\xi_1'(0) = \xi_2(0) = 0$. Then the periodicity factor S is given by the roots of the equation $S^2 - 2\xi_1(\pi)S + 1 = 0$. The product of the two periodicity factors S, S_1 is equal to unity. We can determine the nature of the solutions to (2.3) by determining S (or equivalently, μ). The following properties are easily shown:†

$$(1) S_1 S_2 = -1 = e^{\pi(\mu_1 + \mu_2)}, \text{ so } \mu_1 + \mu_2 = 2ni$$

*Ince, pg 381

†See for instance Arscott (1964), where these properties are shown for Mathieu's equation.

$$(ii) \cosh \mu \pi = \xi_1(\pi) = \xi_1'(\pi)$$

(iii) If (2.3) has a solution $\xi(z)$ with periodicity factor $s \neq \pm 1$, then there exists another independent solution with periodicity factor s^{-1} , given by $\xi(-z)$.

(iv) $s_1 = s_2$ if and only if $s_1 = \pm 1$.

(v) (2.3) has a basically periodic solution⁺ if and only if the roots of the periodicity equation are equal ($= \pm 1$).

(vi) If α and λ^2 are real (and z is real variable), and μ is any periodicity exponent, then either $\operatorname{Re}(\mu) = 0$ or $\operatorname{Im}(\mu)$ is an integer.

By (iii) then, if $s \neq \pm 1$, the general solution of (2.3) is $A\xi(z) + B\xi(-z)$, or $Ae^{\mu z}\phi(z) + Be^{-\mu z}\phi(-z)$. Leaving aside the case $s = \pm 1$, then (vi) presents two distinct cases. Suppose $\mu = \mu_r + i\mu_i$. Then $\cosh \mu \pi$ is real, whence $\sinh \mu_r \pi \sin \mu_i \pi = 0$ (from which (vi) follows). In the first case, with $\mu_r \neq 0$, then as $z \rightarrow \pm\infty$, either $e^{\mu z}\phi(z)$ or $e^{-\mu z}\phi(-z)$ is unbounded, and hence the general solution is unbounded also; such solutions are called unstable. In the second case with $\mu_r = 0$, then $|e^{\mu z}| = 1$ and since $\phi(z)$ is periodic, the general solution remains finite

⁺A basically periodic function is one for which $f(x) = \pm f(x+\pi)$.

Taking the upper sign, the period is π ; taking the lower sign, the period is 2π .

as $y \rightarrow \pm \infty$; these are the stable solutions. Two sub-cases present themselves: (a) if $\mu_i = c/d$, a rational fraction with $d \geq 2$, then $e^{\pm 2d\pi\mu} = 1$ and the solution has period $2d\pi$; (b) if μ_i is irrational, the solution is bounded but not periodic.

According to this, then, knowledge of the periodicity factor s or the periodicity exponent μ is sufficient to resolve the question of stability of the solutions. According to (ii) above, μ can be determined (as a function of α and λ) by numerical integration of the differential equation (2.3). Since we do not wish to precisely determine the growth (or decay) rates, but only to delineate the regions of stability and instability or more precisely, to determine their boundaries in the $\lambda - \alpha$ plane, this procedure will be abandoned in favor of a more direct method.

For the more interesting case where $s = \pm 1$, $\xi(z)$ and $\xi(-z)$ are not necessarily independent solutions, and further, both signs must be treated separately. It should be here added that all of the preceding discussion is directly applicable to Mathieu's equation as well as equation (2.3), but the basically periodic solutions present a somewhat different situation. The next section deals with just this case.

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3. The basically periodic solutions

For the case $\mathfrak{S} = \pm 1$, Floquet's theorem assures at least one solution of period π (upper sign) or 2π (lower sign). Furthermore, the solutions are either odd or even, and accordingly we can write the solutions in each of the four cases as trigonometric series. Then by analogy with Mathieu's equation, write the following Ince type solutions:

$$cu_{2n}^{\pm}(z; \alpha) = \sum_{r=0}^{\infty} A_{2r}^{(\pm, 2n)} \cos 2rz \quad (3.1 a)$$

$$cu_{2n+1}^{\pm}(z; \alpha) = \sum_{r=0}^{\infty} A_{2r+1}^{(\pm, 2n+1)} \cos (2r+1)z \quad (3.1 b)$$

$$su_{2n+1}^{\pm}(z; \alpha) = \sum_{r=0}^{\infty} B_{2r+1}^{(\pm, 2n+1)} \sin (2r+1)z \quad (3.1 c)$$

$$su_{2n+2}^{\pm}(z; \alpha) = \sum_{r=0}^{\infty} B_{2r+2}^{(\pm, 2n+2)} \sin (2r+2)z \quad (3.1 d)$$

The solutions cu_{2n}^{\pm} and su_{2n+2}^{\pm} are the even and odd solutions of period π (belonging to $\mathfrak{S} = +1$) and are to reduce to $\cos 2nz$ and $\sin (2n+2)z$ respectively for $\alpha = 0$; the remaining two belong to $\mathfrak{S} = -1$ and are of half-period π .

On substituting these series into (2.3), omitting the superscripts for brevity, the expressions for the coefficients A and B become in order:

$$\left. \begin{aligned} A_0 = 0 \quad ; \quad (\lambda^2 - 4)A_2 - 4\alpha A_4 = 0 \\ 2\alpha r(r+1)A_{2r+2} + (4r^2 - \lambda^2)A_{2r} + 2\alpha r(r-1)A_{2r-2} = 0 \\ r \geq 1 \end{aligned} \right\} (3.2 a)$$

$$\left. \begin{aligned} (\lambda^2 - 1 - \frac{1}{2})A_1 - \frac{3}{2}\alpha A_3 = 0 \quad , \\ \frac{1}{2}\alpha(2r+3)(2r+1)A_{2r+3} + [(2r+1)^2 - \lambda^2]A_{2r+1} + \\ + \frac{1}{2}\alpha(2r+1)(2r-1)A_{2r-1} = 0 \quad , \quad r \geq 1 \end{aligned} \right\} (3.2 b)$$

$$\left. \begin{aligned} (\lambda^2 - 1 - \frac{1}{2})B_1 - \frac{3}{2}\alpha B_3 = 0 \quad , \\ \frac{1}{2}\alpha(2r+3)(2r+1)B_{2r+3} + [(2r+1)^2 - \lambda^2]B_{2r+1} + \\ + \frac{1}{2}\alpha(2r+1)(2r-1)B_{2r-1} = 0 \quad , \quad r \geq 1 \\ \text{and } (\lambda^2 - 4)B_2 - 4\alpha B_4 = 0 \quad ; \end{aligned} \right\} (3.2 c)$$

$$\left. \begin{aligned} 2\alpha r(r+1)B_{2r+2} + (4r^2 - \lambda^2)B_{2r} + 2\alpha r(r-1)B_{2r-2} = 0 \quad , \end{aligned} \right\} (3.2 d)$$

Now, in order that the series (3.1) be uniformly convergent, we must have $\lim_{r \rightarrow \infty} |A_{2r+2}/A_{2r}| < 1$ and $\lim_{r \rightarrow \infty} |B_{2r+2}/B_{2r}| < 1$ which can easily be seen to be satisfied for $|\alpha| < 1$, and all finite λ . Given an α , then, it remains to determine those values of λ such that each of the relations (3.2) can be satisfied for all r . Looking at (3.2a) for the moment, we can either construct an infinite determinant of Hill's type for which the elements are the coefficients of the A 's and determine its roots λ^2 (infinitely many of them) as a function of α , or, what is easier, construct an infinite continued fraction. The latter method is chosen here, and for the details,

one is referred to the standard works on difference equations.*

From (3.2) then four characteristic equations arise:

$$1 - \frac{\lambda^2}{4} - \frac{\alpha^2/4}{1 - \lambda^2/4^2} - \frac{\alpha^2/4}{1 - \lambda^2/6^2} - \frac{\alpha^2/4}{1 - \lambda^2/8^2} - \dots = 0 \quad (3.3 a)$$

$$1 - \frac{\lambda^2}{2} - \frac{\alpha^2/4}{1 - \lambda^2/3^2} - \frac{\alpha^2/4}{1 - \lambda^2/5^2} - \frac{\alpha^2/4}{1 - \lambda^2/7^2} - \dots = 0 \quad (3.3 b)$$

$$1 + \frac{\lambda^2}{2} - \frac{\alpha^2/4}{1 - \lambda^2/3^2} - \frac{\alpha^2/4}{1 - \lambda^2/5^2} - \frac{\alpha^2/4}{1 - \lambda^2/7^2} - \dots = 0 \quad (3.3 c)$$

$$1 - \frac{\lambda^2}{4} - \frac{\alpha^2/4}{1 - \lambda^2/4^2} - \frac{\alpha^2/4}{1 - \lambda^2/6^2} - \frac{\alpha^2/4}{1 - \lambda^2/8^2} - \dots = 0 \quad (3.3 d)$$

for the four types of solutions, respectively. Each of the continued fractions on the left hand side converges for $|\alpha| < 1$ when λ^2 is positive. For λ^2 negative, they converge for $|\alpha| \leq 1$, but these cases are of no physical interest, always representing unstable solutions.

Each of (3.3) represents a transcendental equation for $\lambda = \lambda(\alpha)$, with infinitely many roots. For instance, if α be fixed, the roots of (3.3a) can be arranged in an increasing sequence, $\{\lambda^{(2)}, \lambda^{(4)}, \lambda^{(6)}, \dots, \lambda^{(2n)}, \dots\}$ corresponding to the solution $C\omega_{2n}^r(3, \alpha)$. With $\alpha = 0$, the roots reduce to $\lambda^{(2n)} = 2n$, which corresponds to

$$C\omega_{2n}^r(3, i\alpha) \rightarrow \cos 2n\pi \quad \text{as } \alpha \rightarrow 0.$$

Notice now that (3.2a) and 3.2d) are identical in form (so (3.3a) and (3.3d)), which means that for given α , the ei-

* See for instance Nörlund (1924), chapter XV.

genvalues belonging to the even and odd solutions of period π are identical; further, the two solutions are independent, so the general solution in this case is $\xi = C_1 \text{cu}_{2n}^{\pm}(z; \alpha) + C_2 \text{su}_{2n}^{\pm}(z; \alpha)$. That is, two solutions of period π can coexist, of which more later. The solutions of period 2π , however, cannot coexist. Floquet theory gives the form of the second (non-periodic) solution as $\xi_2 = \gamma \text{cu}_{2n+1}^{\pm}(z; \alpha) + \phi_1(z)$ where ϕ_1 is periodic with period π , which exhibits a so-called secular instability. The complete solution in this case is $\xi = C_1 \text{cu}_{2n+1}^{\pm} + C_2 \{ \gamma \text{cu}_{2n+1}^{\pm} + \phi_1 \}$ or $\xi = C_1 \text{su}_{2n+1}^{\pm} + C_2 \{ \gamma \text{su}_{2n+1}^{\pm} + \phi_2 \}$, say depending on which eigenvalue is chosen. For Mathieu's equation, coexistence is impossible for both $s = +1$ and $s = -1$ (Ince, 1922). The proof that solutions of (2.3) of period 2π cannot coexist is exactly the same as that for Mathieu's equation (Ince (1926) pg 177),[†] and need not be displayed here.

Thus we find that as anticipated in section 2, the solutions of (2.3) are not at all dissimilar to Mathieu functions, but unlike Mathieu's equation, the even intervals of instability disappear, with the exception of the zeroth ($-\infty < \lambda \leq 0$),

[†]Winkler and Magnus (1958) prove in a similar manner a more general theorem concerning coexistence for the generalized Ince equation $(1 + a \cos 2x)y'' + b \sin 2x y' + (c + d \cos 2x)y = 0$ of which (2.3) is a special case. The question of the number of intervals of instability is answered as a corollary.

while there are an infinite number of odd intervals of instability. What remains, then is the determination of the characteristic curves in the $\lambda - \alpha$ plane, which is done by calculating the roots of λ of the characteristic equations containing the continued fractions (3.3a-c). This has been done numerically, and the first eight even and odd modes are tabulated in Table 1, for $0 < \alpha \leq 0.999$. For constant α (neither zero nor one)

then, the roots can be arranged in an increasing sequence

$\{0 < \lambda_e^{(1)} < \lambda_o^{(1)} < \lambda_e^{(2)} < \lambda_o^{(2)} < \lambda_e^{(3)} < \lambda_o^{(3)} < \lambda_e^{(4)} < \lambda_o^{(4)} < \dots\}$ the odd numbered superscripts indicating period 2π solutions, both even and odd, and the even numbered superscripts indicating period π solutions (coexisting). For $\alpha = 0$, $\lambda_e^{(n)} = \lambda_o^{(n)} = n$.

The eigenvalues are plotted in figure 2, which identifies type of solution, with the regions of instability indicated by shading. Note that the intervals of instability decrease rapidly in width as n increases. For negative α , it is obvious on inspection of the characteristic equations that only the odd numbered modes will be affected. That is, for negative α , $\lambda_o^{(2n+1)} < \lambda_e^{(2n+1)}$, $n = 0, 1, 2, \dots$. This affects only the characteristic curves separating the stable from unstable regions, but the overall picture is symmetric. Those for the even numbered modes, however, are symmetric about $\alpha = 0$. Exactly the same phenomenon occurs in the stability diagram for Mathieu's equation. Furthermore, all of the characteristic curves

save that of the first mode have zero slope at the origin (which will be demonstrated shortly), as do those of Mathieu's equation with the same exception.

Now it is not immediately obvious from the form of the characteristic equations (3.3) that the eigenvalues corresponding to the flat bottom $\alpha = 0$ are just the integers 1, 3, 5, ... for the period 2π -modes and 2, 4, 6, ... for the period π -modes. (setting $\alpha = 0$ gives only the lowest eigenvalue). By straightforward algebraic manipulation, it can easily be shown that all four characteristic equations (3.3a-d) can be written in one general form, which is

$$g_n = B_{n-2} + F_{n+2}, \quad n = 1, 2, \dots \quad (3.4)$$

where $g_n = 1 - \lambda^2/n^2$, $n = 2, 3, \dots$
and $g_1 = 1 - \lambda^2 \pm \sqrt{2}$, where the upper sign belongs with sine-type odd-numbered modes and the lower sign with the cosine-type odd numbered modes, each being of period 2π .

The quantities on the right-hand side of (3.4) are each continued fractions, of the form

$$B_{n-2} = \frac{\alpha^2/4}{g_{n-2}} - \frac{\alpha^2/4}{g_{n-4}} - \dots - \frac{\alpha^2/4}{g_i}$$

where $i = 1$ or 2 as n is odd or even, and $B_0 = B_{-1} = B_{-2} \dots = 0$,

$$\text{and } F_{n+2} = \frac{\alpha^2/4}{g_{n+2}} - \frac{\alpha^2/4}{g_{n+4}} - \frac{\alpha^2/4}{g_{n+6}} - \dots$$

B_{n-1} is a terminating continued fraction containing either $(n-1)/2$ or $(n-1)/2$ terms, according to whether n is even or odd. F_{n+2} is an infinite continued fraction (actually part of the original). The right hand side of (3.4) vanishes identically for $\alpha = 0$, easily yielding $\lambda^{(n)}(0) = n$ for all positive n .

In addition to facilitating computation of $\lambda^{(n)}(\alpha)$, the form (3.4) is suitable for approximating the eigenvalues for small α . That is, for small α , the eigenvalues may be written in a Taylor series as

$$\lambda^{(n)}(\alpha) = \lambda^{(n)}(0) + \alpha \left(\frac{d\lambda^{(n)}}{d\alpha} \right)_0 + \frac{\alpha^2}{2!} \left(\frac{d^2\lambda^{(n)}}{d\alpha^2} \right)_0 + \dots \quad (3.5)$$

Carrying out the indicated differentiation of (3.4) and then setting $\alpha = 0$, it is easily shown that $\left(\frac{d\lambda^{(n)}}{d\alpha} \right)_0 = 0$ for all n excepting $n = 1$, for which $\left(\frac{d\lambda^{(1)}}{d\alpha} \right)_0 = \pm 1/4$, where again the upper sign belongs with the SW_1 mode and the lower with SW_2 . This, of course, is consistent with Figure 2, which indicates that the characteristic curves for all of the modes have zero slopes at $\alpha = 0$ with the sole exception of the first.

A second differentiation then yields for the coefficient of α^2 in (3.5)

$$\left(\frac{d^2\lambda^{(n)}}{d\alpha^2} \right)_0 = - \frac{3n}{8} \left(\frac{n^2 - 4/3}{n^2 - 1} \right) \quad (3.6)$$

for all $n \geq 2$, while for $n = 1$, separate calculation (ab initio) yields

$$\left(\frac{d^2 \lambda^{(1)}}{d\alpha^2}\right)_0 = -11/32.$$

The difficulty with the lowest mode, of course, comes about from the form of g_1 .

Continuing on to the third derivative then, it is easy to show that

$$\left(\frac{d^3 \lambda^{(n)}}{d\alpha^3}\right)_0 = 0 \quad \text{for all } n \text{ save } 1 \text{ \& } 3,$$

and

$$\left(\frac{d^3 \lambda^{(n)}}{d\alpha^3}\right)_0 = \pm 105/512 \quad ; \quad \left(\frac{d^3 \lambda^{(n)}}{d\alpha^3}\right)_0 = \pm 9/512$$

where as usual, the \pm signs belong with the 5ω and 4ω modes. For the fourth derivative, the algebra involved is somewhat complicated, but

$$\left(\frac{d^4 \lambda^{(n)}}{d\alpha^4}\right)_0 = -\frac{3 \cdot 69}{128} n \left[\frac{n^6 - 74/23 n^4 + 268/69 n^2 - 32/23}{(n^2 - 1)^3} \right] \quad (3.7)$$

for $n \neq 1, 3$, which must be calculated separately again, and shall not be reproduced here. Calculations of higher derivatives proves to be a fearsome task, and, fortunately, is not necessary in order to arrive at some general features of the characteristic

curves, which can be inferred from the previous steps.

Considering first the even numbered (period π) modes, it becomes apparent upon examining the form which the successive differentiations of (3.4) take that all odd-numbered derivatives are identically zero for $\alpha = 0$. That is, the Taylor series (3.5) involves all even powers of α and no odd powers. This is consistent with a previous statement that the characteristic curves for the even-numbered modes are symmetric about the line $\alpha = 0$ in the λ - α plane, which was deduced from the form of (3.3a and d).

The odd-numbered modes, on the other hand, present a somewhat different behaviour. Each contains all even powers of α . For $n = 1$, the first odd power of α appearing is α^1 preceded by a (\pm) sign, and it is easy to see that thereafter, all higher odd powers will appear (that is, have non-zero coefficients), again with the (\pm) signs. For the third mode, $\left(\frac{d\lambda^{(3)}}{d\alpha}\right)_0 = 0$ and the first odd power of α appearing is α^3 , with the inevitable (\pm) , and all higher odd powers will appear. In general, then, it becomes clear that for n odd, the only odd powers of α involved are those of order n and higher. In other words, for the non-coexisting modes, the series for the $S\omega_n$ and $C\omega_n$ modes are identical up to and including the $(n-1)$ st power of α . Then by subtraction,

$$\lambda_{S\omega_n}^{(n)} - \lambda_{C\omega_n}^{(n)} = 2 \left(\frac{d^n \lambda^{(n)}}{d\alpha^n} \right)_0 \frac{\alpha^n}{n!} + O\left(\frac{\alpha^{n+2}}{(n+1)!}\right) \quad (3.8)$$

This is the reason for the rapid decrease in the widths of the intervals of instability. Note in addition that the appearance of the (\pm) signs agrees with the earlier statement that while the characteristic curves for the odd modes are not symmetric about $\alpha = 0$, the picture (Fig. 2) is.

Figure 3 presents the characteristic curves plotted with $(+\alpha)^2$ as abscissa. The slope of these curves at $\alpha = 0$ is (save for $n = 1$) proportional to the curvature in Fig. 2 and is indicated by the light lines tangent to the characteristic curves. The equations of the light lines are, in virtue of (3.5) and (3.6)

$$\lambda^{(n)}_{(\alpha)} = n \left\{ 1 - \frac{3}{16} \left(\frac{n^2 - 4/3}{n^2 - 1} \right) \alpha^2 \right\} \quad (3.7)$$

Given these eigenvalues, it is a simple task to return them to the suitable recurrence formulas (3.2) to determine in turn the successive values of the Fourier coefficients A_r or B_r and thence determine to any desired accuracy the form of the free surface from (3.1). It is not necessary to the purpose of this paper to present these detailed calculations, but it is enlightening to present a qualitative description of the physical situation, in view of a certain peculiarity which arises.

The completed solutions (3.1) represent standing wave modes in basins of finite length, that being the wavelength of the sinuous bottom. For the even numbered modes, vertical barriers may be erected at $y = \pm (2n+1) \pi/2$, which corres-

ponds to the peaks of the bottom, placing the deepest part of the water (the valleys) in the center of the basin. The appropriate solution here is (3.1a), or $Cu_{2n}(z)$, and the (dimensionless) frequencies are $\lambda_{(n)}^{(2n)}$. Then again, vertical barriers may be erected at $z = \pm n\pi$ (i.e., at the valleys) which places the top of the hill in the center of the basin. (These two cases may be picturesquely called the "hollow" modes and the "hummock" modes, depending on the bottom features at the center of the basin. The difference, really, is only a change in sign of α .) The solution here is (3.1d) and the appropriate frequency is again $\lambda_{(2n)}^{(2n)}$. That is to say, the even hollow modes and the even hummock modes have identical frequencies. Furthermore, the change in frequency relative to the flat-bottom case is always a decrease, and is initially (that is, for small α) of second order in the relative height of the bottom undulations, in view of (3.9) or Fig. 3, for small α .

The form of the free surface is symmetric about the center of the basins in both cases. Qualitatively, the locations of the nodes ($2n$ of them by Sturm's Oscillation Theorem) are displaced toward the shallow water relative to their corresponding position in the undisturbed (flat) bottom case. Finally, as a consequence of the Sonine-Polya Theorem,⁺ the maximum amplitudes of the oscillations (at the anti-nodes, say) are greater

⁺Szego, G. (1939) pg 161 footnote 43.

the water is shallower. That is, for the hollow modes and hummock modes, the maximum vertical displacement of the free surface occurs at the walls and at the center, respectively.

For the odd-numbered modes, the situation is somewhat different. Again, vertical barriers may be erected at the peaks or at the valleys, corresponding to the hollow and hummock modes, with the solutions being J_{2n+1} and Y_{2n+1} , respectively. (Note the reversal from the even modes.) The frequencies are not identical in this case, in virtue of the non-coexistence of these modes. Indeed, the frequencies of the hollow modes are always greater than those of the corresponding hummock mode, the difference being given by (3.8). Again, for all odd modes, save the first, the frequency change relative to the flat-bottom frequency is initially a decrease of second order via (3.9). The fundamental mode is the coriolis mentioned above: the hollow fundamental mode exhibits an increase in frequency for $0 < \alpha < .966$ and a decrease for larger values of α , while the hummock fundamental always exhibits a relative decrease. In both cases the frequency shift is initially of first order since $\left(\frac{d\lambda''}{d\alpha}\right)_0 = \pm \frac{1}{4}$.

The qualitative features of the free surface displacement are similar to those described above for the even-numbered modes with the important exception that the displacement is asymmetric about the center of the basin, which always corresponds to a node.

It is notable that there is a marked decrease in the eigen-frequencies for values of α close to unity. Indeed, the frequencies of all of the modes appear from Fig. 2 and Fig. 3 to be approaching each other, with the sole exception of the hummock fundamental. The case $\alpha = 1$ will be handled in part 5, but before going into this limit, it is well to investigate the case $|\alpha| > 1$ in a somewhat different manner.

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4. Standing waves (seiches) in a cosine shaped basin

When $|\alpha| > 1$, which corresponds physically to the case where the bottom emerges above the mean water level, equation (2.3) is unsuitable because of the nature of the singularity in $h(x)$. A slight reformulation of the problem will prove to be of some advantage. Consider, then, a sinusoidally shaped basin of total depth a containing liquid which is of undisturbed maximum depth $d < a$ at the lateral center of the basin. Then the depth of the fluid measured vertically downwards from the undisturbed free surface is easily shown to be $h(x) = d - a \sin^2 px/2$, where $2\pi/p$ is the overall length of the basin, measured horizontally from peak to peak. Using this in (2.2) and transforming the resulting equation to algebraic form by the substitutions $w = \gamma^{1/2} \sin px/2$, where $\gamma = d/a$, the ratio of the maximum depth of the fluid to the overall height of the basin, then

$$(1-w^2)(1-\gamma w^2) \frac{d^2 \xi}{dw^2} - w(2+\gamma-3\gamma w^2) \frac{d\xi}{dw} + q^2 \xi = 0 \quad (4.1)$$

In (4.1), q is a normalized frequency given by $q^2 = 4\sigma^2/gap^2$. The parameter γ plays the same role in (4.1) that does α in (2.3); that is, it is expected that the eigenfrequencies q shall depend solely on the values of γ .

To determine the frequencies, a series solution of the form

$$\xi(w) = \sum_{n=0}^{\infty} e_n w^n \quad (4.2)$$

is sought. On substitution into (4.1) it is easily seen that two independent solutions arise, one involving only even powers of w and the other, only odd. Again a three-term recurrence relation involving alternate coefficients e appears of the general form

$$\begin{aligned} 2e_2 + q^2 e_0 &= 0, \\ 6e_3 + (q^2 - 2 - \gamma)e_1 &= 0, \end{aligned}$$

and thenceforward for $n \geq 2$,

$$(n+1)(n+2)e_{n+2} + [q^2 - \gamma n^2 - n(n+1)]e_n + \gamma n(n-2)e_{n-2} = 0 \quad (4.3)$$

In order that each of (4.3) be satisfied and that the series (4.2) be convergent, it is a simple matter to construct a characteristic equation again in the form of an infinite continued fraction as in § 3 above. The required manipulations give the result that

$$1 + \frac{\gamma}{2} - \frac{q^2}{2} = \frac{3\gamma/4}{1 + \frac{3\gamma}{4} - \frac{q^2}{3 \cdot 4}} = \frac{5\gamma/6}{1 + \frac{5\gamma}{6} - \frac{q^2}{5 \cdot 6}} = \frac{7\gamma/8}{1 + \frac{7\gamma}{8} - \frac{q^2}{7 \cdot 8}} = \dots \quad (4.4)$$

and

$$1 + \frac{2\gamma}{3} - \frac{q^2}{2 \cdot 3} = \frac{4\gamma/5}{1 + \frac{4\gamma}{5} - \frac{q^2}{4 \cdot 5}} = \frac{6\gamma/7}{1 + \frac{6\gamma}{7} - \frac{q^2}{6 \cdot 7}} = \frac{8\gamma/9}{1 + \frac{8\gamma}{9} - \frac{q^2}{8 \cdot 9}} = \dots \quad (4.5)$$

must be satisfied identically, for the odd-numbered and even-numbered modes, respectively. In order to ensure convergence of the continued fractions (and the solution (4.2) itself) it is easily seen that for $q^2 > 0$, the values of γ must be restricted so that $0 \leq \gamma < 1$.

Each of the characteristic equations (4.4) and (4.5) may be written in a general form, and they may be combined into one formula similar to (3.4). It is then obvious that for $\gamma \rightarrow 0$, the eigenfrequencies reduce to $q^{(n)} \rightarrow [n(n+1)]^{1/2}$. This corresponds physically to the situation of very shallow water lying in the bottom of the basin, which itself may be thought of in this limit as a concave parabolic basin. The eigenfrequencies in this case are well known,⁺ and it should be no surprise that the present results indeed approach this limit.

To find a first approximation for the eigenvalues, say for small ratios of the depths, differentiation of the characteristic equations as in § 3 yields simply the result that

$$\left(\frac{dq^{(n)}}{d\gamma} \right)_0 = - \frac{1}{4} [n(n+1)]^{1/2} \left[\frac{n^2 + n - 1}{n^2 + n - 3/4} \right].$$

Writing $q_0^{(n)} = [n(n+1)]^{1/2}$, then

$$q^{(n)}(\gamma) = q_0^{(n)} \left\{ 1 - \frac{1}{4} \left(\frac{q_0^{(n)2} - 1}{q_0^{(n)2} - 3/4} \right) \gamma + O(\gamma^2) \right\} \quad (4.6)$$

⁺ See, for instance, Lamb (1932) pg 277, or Chrystal (1906).

The eigenvalues for the first eight modes have been computed numerically, and are presented graphically in Fig. 4, and in Table 2. For comparison, the approximation given in (4.6) is indicated on this figure as the light lines tangent to the characteristic curves at $\gamma = 0$. Note that in this case, too, the frequencies decrease markedly and appear to approach one another when the water level approaches the top of the basin. In the limit, as $\gamma \rightarrow 1$, the physical situation corresponds precisely to the case examined in §3 as $\alpha \rightarrow 1$. It is worthwhile now to examine this limiting configuration from several points of view.

5. The degenerate cases $\alpha, \gamma \rightarrow 1$.

It is apparent from the results in Fig. 2-4 and the tables that as the undisturbed water level approaches the crests of the corrugations either from above ($\alpha \rightarrow 1$) or from below ($\gamma \rightarrow 1$), the frequencies of the respective modes decrease rapidly and appear to approach each other. The numerical calculations become increasingly more difficult, requiring hundreds of terms of the appropriate continued fractions for the desired precision.* Physically, the rapid decrease in frequency is to be associated with the appearance of a very gradually shelving bottom: both the depth and the slope approach zero as α and $\gamma \rightarrow 1$.

In the limiting case, then, equations (2.3) and (4.1) become

$$(1 + \cos 2\zeta)\xi'' - 2 \sin 2\zeta \xi' + \lambda^2 \xi = 0 \quad (5.1a, b)$$

$$\text{and } (1 - \omega^2)^2 \xi'' - 3\omega(1 - \omega^2)\xi' + q^2 \xi = 0,$$

respectively. The second of (5.1) is, of course, equivalent to the first with the substitution indicated in §4, and will henceforth be ignored, after noting for later reference that in this case, $q^2 = \lambda^2/2$.

*In contrast, for small α and γ , say .1, six figure precision is attainable with 15 (at the most) terms, provided the remaining terms are suitably approximated. It is of no particular advantage to discuss further the numerical method here.

Now, rather than examining the solutions of (5.1) in detail, which are already known to be unbounded at $\zeta = (2n+1)\pi/2$ (and at $\omega = \pm 1$) because of the failure of the trigonometric series (3.1) to converge, an auxiliary problem containing the pertinent physical properties will be examined. Consider then the localized problem near the singularity. The depth in the neighborhood of the singularity is given by $h(\zeta)/h_0 = 1 + \cos 2(\zeta - \pi/2)$, $\zeta = \zeta + \pi/2$, or $h(\zeta)/h_0 \approx 2\zeta^2$ to the lowest order terms, and (5.1) to this order is then

$$\zeta^2 \zeta'' + 2\zeta \zeta' + (\lambda^2/2)\zeta = 0 \quad (5.2)$$

To avoid unbounded ζ at $\zeta = 0$, rigid vertical walls may be erected at $\zeta = \epsilon$, say, and at $\zeta = 1$, forming a closed basin of length $1 - \epsilon$. The shape of the bottom to this approximation is that of an inverted parabola tangent to the undisturbed water level at $\zeta = 0$. (Introduction of the boundary at $\zeta = 1$ is really quite artificial, but will turn out to be irrelevant to the details of the solution anyway. Its sole purpose is to restrict the eigenvalues to a denumerable set rather than a continuous spectrum.) The oscillatory solutions of (5.2) satisfying the condition of no flow through the vertical walls are simply

$$\zeta = C_1 \zeta^{-1/2} \cos(s \log \zeta) + C_2 \zeta^{-1/2} \sin(s \log \zeta) \quad (5.3)$$

where $S = (2\lambda^2 - 1)^{1/2}$ is real; or $\lambda^2 \geq 1/2$. For $\lambda^2 < 1/2$ there are no oscillatory solutions. Use of the boundary conditions at the two ends then requires that $S \log \epsilon = n\pi$, or simply

$$\lambda = \left\{ \frac{1}{2} + \frac{n^2 \pi^2}{2(\log \epsilon)^2} \right\}^{1/2} \quad (5.4)$$

The behaviour of these characteristic curves $\lambda = \lambda(\epsilon)$ as ϵ becomes vanishingly small is identical to that in the two cases previously considered for $\alpha \rightarrow 1$ and $\gamma \rightarrow 1$. That is, the frequencies approach each other rapidly, and even though the surface displacement becomes unbounded at the singular point, the eigenvalues approach a non-zero limit. That limit ($\lambda = 1/\sqrt{2}$) to which the frequencies all coalesce may be called the "anomalous frequency," since it is only a mathematical limit to which there corresponds no finite oscillation of the free surface.

It is interesting to point out that Chrystal (1905), (1906) observed this same phenomenon of coalescence of modes in the investigation of seiches on a concave quartic lake. The law of depth is given by $h(x) = h_0 (1 - x^2/a^2)^2$. With walls erected at $|x/a| = p < 1$, the eigenvalues are of the form (in dimensional variables, now)

$$\sigma = \frac{\sqrt{gh_0}}{a} \left\{ 1 + \frac{n^2 \pi^2}{\log^2 \left(\frac{1+p}{1-p} \right)} \right\}^{1/2}$$

which exhibit the same anomalous behaviour as in the previous

cases. Chrystal in fact refers to the limit ($p \rightarrow 1$) in the form $T = \frac{2\pi}{\sigma} = 2\pi a / \sqrt{g h_0}$ as the "period of the anomalous seich." His limit is, of course, different from that being investigated here as must be expected, but the phenomenon involved is exactly the same: the modes coalesce to an anomalous frequency, independent of N , whenever the depth and slope of the bottom are simultaneously zero.

Consider, then, the general case of a basin of arbitrary depth $h(x)$ and limited horizontal extent such that at equilibrium, the depth of the fluid and the slope of the bottom are simultaneously zero, say at $x = 0$. By the simple artifice of inserting a barrier at $x = \epsilon$ as above, it is not too difficult to show that for sufficiently small ϵ , the frequencies of all the modes are of the limiting form

$$\sigma^2 = \sigma_a^2 \{ 1 + a_1 n^2 (\log a_2 \epsilon)^{-2} \} \quad (5.5)$$

In this expression, a_1 and a_2 are constants which must be determined from the entire figure of the basin. On the other hand, σ_a (the anomalous frequency) is given by

$$\sigma_a^2 = g h''(0) / 8 \quad (5.6)$$

exactly. It is remarkable that the limit point to which the frequencies of all modes coalesce depends solely on the curvature of basin at the point of singularity, and is otherwise independent

of the details of the basin profile. It is important to keep in mind, however, that as pointed out above, this limit cannot correspond to physically realizable oscillations, and must be considered as a lower limit to the natural frequencies within the framework of the linearized long-wave (or tidal) theory, which is (presumably) invalid at this point.

Returning to Figure 4, then, it must be considered highly likely that the characteristic curves all approach the point $q^2 = 1/4$ as $\gamma \rightarrow 1$ in a manner given by (5.5). In Figs. 2 and 3, the curves all approach $\lambda^2 = 1/2^*$ as $\alpha \rightarrow 1$ save the lowest hummock mode which will be handled below. The proof of this coalescence conjecture has been of course indirect; a direct proof must involve the nature of the characteristic equations in the continued fraction (or equivalent) form at $\alpha, \gamma = 1$, which has been elusive.

For the lowest hummock mode (ω_1), it is easy to show directly from the characteristic equation (3.3b) that $\lambda_e^{(1)}(1) = 0$ by setting $\lambda = 0$ and $\alpha = 1$, thence summing (exactly) the resulting periodic continued fraction. That this single frequency does not approach the anomalous frequency ($1/\sqrt{2}$) should be of no real concern, because on a moments

*The points $(\lambda, \alpha) = (1/\sqrt{2}, 1)$ and $(q, \gamma) = (1/2, 1)$ are really the same points in virtue of the statement made earlier in this section, that $q^2 = \lambda^2/2$ for $\alpha = \gamma = 1$.

reflection it is apparent that this situation requires that a node of the free surface occur at the "shoreline" which is physically implausible.

6. Progressive waves

In the preceding several sections, the analysis has been confined to the case of standing waves, or normal modes. Now, in virtue of the coexistence of solutions on the even numbered characteristic lines, the Fourier coefficients of (3.1a) and (3.1d) are identical, and since $A_0^{(2n)} \equiv 0$, a linear combination of the two independent solutions can be written as

$$\begin{aligned} \eta(z, t, \alpha) &= a [c w_{2n}(z, \alpha) \cos \sigma t + s w_{2n}(z, \alpha) \sin \sigma t] \\ &= a \sum_{r=1}^{\infty} \frac{A_{2r}^{(2n)}}{2r} \cos(2rz - \sigma t) \end{aligned} \quad (6.1)$$

where a is some arbitrary small amplitude. If the Fourier coefficients be suitably normalized, or $\sum [A_{2r}]^2 = 1$, then in the limit as $\alpha \rightarrow 0$, (6.1) becomes

$$\eta(z, t) = a \cos(2nz - \sigma t),$$

since $A_{2r}^{(2n)} \rightarrow \delta_{nr}$, where δ_{nr} is Kronecker's symbol.

That is, if the bottom were flat, the solution (6.1) simply reduces to a progressive wave of wavelength $L_w = 2\pi/n_p = L_0/n$ where L_0 would be the length of the bottom corrugations, were they there. The phase speed of this wave is then given by

$$c_0 = \frac{\sigma}{2n} L_w = \frac{\lambda^{(2n)}(0)}{2n} \sqrt{g h_0} = \sqrt{g h_0}$$

since $\lambda^{(2n)}(\alpha) \rightarrow 2n$ as $\alpha \rightarrow 0$. So the phase speed is constant and equal, of course, to the expected result.

Now (6.1) is really a sum of infinitely many progressive waves, each travelling with a different speed, and the shape of the free surface is continually changing. The "speed of the r^{th} component" is

$$c_r = \frac{\lambda^{(2n)}(\alpha)}{2r} c_0.$$

However, in a time $T = 2\pi/\sigma$, the r^{th} component has advanced exactly one of its wavelengths, so every nT seconds the picture of the free surface is exactly the same: in this sense, the propagating wave is periodic. That is, to an observer moving with a speed $c = \frac{\lambda^{(2n)}(\alpha)}{2n} c_0$, the shape of the free surface repeats itself every nT seconds. It is in this sense, then, that an average phase speed can be defined as

$$c(\alpha) = c_0 \lambda^{(2n)}(\alpha) / \lambda^{(2n)}(0) \quad (6.2)$$

Figure 3 then indicates that progressive waves corresponding to the fundamental, or characteristic, coexisting solutions are always retarded by the presence of the uneven bottom, the retardation becoming greater with increase of corrugation amplitude.

So far, the analysis has been confined to values of the parameters λ and α lying on the characteristic curves, where the solutions are basically periodic. What then of the

solutions of (2.3) for non-characteristic values, but for which λ and α are specified to lie inside a stable region? The question has been partially answered in §2 above, and will now be investigated further, though with less detail.

If the (specified) point (λ, α) lies within a stable region, then the periodicity exponent μ is wholly imaginary. It is easy to construct solutions to (2.3) of the form

$$\left. \begin{aligned} C u_{2n+\beta}^r(z, \alpha) &= \sum_{r=-\infty}^{+\infty} A_{2r}^{(2n+\beta)}(\alpha) \cos(2r+\beta)z \\ S u_{2n+\beta}^r(z, \alpha) &= \sum_{r=-\infty}^{+\infty} B_{2r}^{(2n+\beta)}(\alpha) \sin(2r+\beta)z \end{aligned} \right\} (6.3)$$

and

$$\left. \begin{aligned} C u_{2n+\beta+1}^r(z, \alpha) &= \sum_{r=-\infty}^{+\infty} A_{2r+1}^{(2n+1+\beta)}(\alpha) \cos(2r+1+\beta)z \\ S u_{2n+\beta+1}^r(z, \alpha) &= \sum_{r=-\infty}^{+\infty} B_{2r+1}^{(2n+1+\beta)}(\alpha) \sin(2r+1+\beta)z \end{aligned} \right\} (6.4)$$

with $0 < \beta < 1$. The solutions (6.3) are to be used if the given point (λ, α) lies in a stable region bounded below by an even numbered characteristic curve and above by an odd numbered one; (6.4) are to be used in the alternate stable regions. These solutions then can be called wavy-bottom functions of fractional order by direct analogy with Mathieu functions of fractional order.⁺

⁺See Arscott (1964) pg 133 or McLachlan (1947) § 4.71.

The solutions (3.1a-d) are, of course, wavy bottom functions of integral order then.

Substitution of (6.3) and (6.4) into (2.3) then results in three-term recurrence relations among the coefficients A and among coefficients B somewhat similar to (3.2), but now involving the constant β as well as the parameters λ and α . The relations for A's and B's turn out to be identical; and therefore

$$A_{2r}^{(2n+\beta)} / B_{2r}^{(2n+\beta)} = \text{constant},$$

$$A_{2r+1}^{(2n+\beta+1)} / B_{2r+1}^{(2n+\beta+1)} = \text{constant}$$

for fixed $0 < \beta < 1$, and the arbitrary constants may be chosen to be unity for what follows. Continuing as in § 3, characteristic equations in the form of continued fractions may be constructed, now involving β . For fixed values of β , the roots λ of the characteristic equations may be determined numerically (or otherwise approximated) as a function of α , resulting in the so-called iso- β lines, or fractional characteristics.

It is not really necessary (or desirable) to use the computer for determination of the iso- β lines $\lambda = \lambda(\alpha)$ since their equations may be determined to any desired accuracy from a Taylor series development about $\alpha = 0$ by the method of § 3. The result is

$$\frac{\lambda(\alpha)}{\nu} = 1 - \frac{3}{16} \left[\frac{\nu^2 - 4/3}{\nu^2 - 1} \right] \alpha^2 \quad (6.5)$$

$$- \frac{69}{1024} \left[\frac{\nu^6 - \frac{79}{23} \nu^4 + \frac{269}{69} \nu^2 - \frac{32}{23}}{(\nu^2 - 1)^3} \right] \alpha^4 + O(\alpha^6)$$

where $\nu = n + \beta$, $n = 0, 1, 2, \dots$, and $0 < \beta < 1$.

This expansion is valid for the iso- β curves belonging to both (6.3) and (6.4). Note that the iso- β curves are symmetric about $\alpha = 0$ in the λ - α plane for all allowable values of β . Note further that the coefficients of α in (6.5) are identical with (3.6) and (3.7) if n is replaced by $\nu = n + \beta$.

Since for any $0 < \beta < 1$, there are two independent solutions with identical Fourier coefficients, a superposition similar to (6.1) can be written, now involving the fractional solutions (6.3) or (6.4), as, for example,

$$\eta(z, t; \alpha) = a \left[C W_{2n+\beta}^{(3-\alpha)} \cos \sigma t + S W_{2n+\beta}^{(3-\alpha)} \sin \sigma t \right].$$

This again represents a progressive wave with continually changing wave form. If, however, β is a rational fraction of the form β_1/β_2 , with $\beta_2 \geq 2$, then the instantaneous solution is spatially periodic with least period $2\pi\beta_2$. So as before, the wave form is identical after each component travels a distance of $2\beta_2$ wavelengths of the bottom corrugations L_B . In this sense again, the propagation is periodic, and an "average

phase speed" can be defined as before to be

$$c(\omega) = c_0 \lambda(\omega) / V \quad (6.6)$$

The value of $\lambda(\omega)$ to be used here is that obtained from construction of the iso- β lines, or (6.5), and $\lambda(0)$ is written simply as V .

If β is not a rational fraction, the solutions are not periodic with any finite period, but are bounded, nevertheless. It is not too difficult to see that (6.6) still applies, but now only in a limiting, or asymptotic sense, for large times, or distances of propagation.

It is now proper to consider $V = \lambda(0)$ a continuous variable, and inquire into its physical meaning. It is obvious from the definitions of § 2 that $\lambda(0)$ is simply the ratio of twice the wavelength of the bottom corrugations to the wavelength L_W that the surface wave would have in water of uniform depth h_0 , or $\lambda(0) = V = 2 L_B / L_W$.

For sufficiently small amplitude corrugations, the average phase speed (6.6) can be written as

$$\frac{c(\omega)}{c_0} = 1 - \frac{3}{16} \left[\frac{V^2 - 4/3}{V^2 - 1} \right] \omega^2 + O(\omega^4) \quad (6.7)$$

This indicates a rather surprising result: there is a range of wavelengths for which the propagation is not retarded. That is, if the wavelength of the surface wave lies in the region

$\sqrt{3}L_0 < L_W < 2L_0$, the average phase speed is greater than the corresponding value for a flat bottom, at least for sufficiently small corrugations.

If the iso- β lines were to be drawn on Fig. 3, then (6.5) indicates that their slopes at $\alpha = 0$ are always negative, indicating retardation, except in the region $1 < \lambda < \frac{2}{3}\sqrt{3}$, where the slopes are positive, and the phase speed is augmented by an amount proportional to α^2 . For $\lambda = 1$, that is, on the SW_1 characteristic, no progressive waves are possible because of the secular instability of the second solution as pointed out in §3 above. For $\lambda = \frac{2}{3}\sqrt{3}$, the dotted line in Fig. 3, the average phase speed (obtained from (6.5)) is

$$\frac{c(\alpha)}{c_0} = 1 - \frac{3}{32}\alpha^4 + O(\alpha^6) \quad (6.8)$$

which indicates retardation, but for this wavelength (only) the retardation is initially of fourth order in the corrugation amplitude.

For amplitudes sufficiently large that (6.5) is no longer a good approximation, it is necessary to determine these iso- β characteristics numerically, which has not been done. In the region of augmented propagation, however, some qualitative statements may be made. For augmented propagation, it is only necessary that $\lambda(\alpha) > \lambda(0)$. Now, on the SW_1 characteristic, $\lambda(\alpha) = \lambda(0) = 1$ for $\alpha \approx 0.965$. So for wavelengths just smaller than twice the bottom wavelength, $\alpha = .96$ (say)

is an upper limit to the relative amplitude of the corrugations for which augmentation is possible. At the other end of this region, for wavelengths just longer than $\sqrt{3} L_B$, the maximum amplitude for which augmentation is possible decreases, ultimately vanishing when $L_W = \sqrt{3} L_B$, in accordance with (6.8).

Finally, because of the secular instability of the solutions on all of the odd numbered fundamental characteristics, progressive wave type solutions cannot be formed for wavelength ratios $L_W / L_B = 2, 2/3, 2/5, 2/7, \dots$. So for frequencies lying within the unstable regions and on their boundaries, there can be no stable progressive waves.

7. Some asymptotic results

It was shown by Green (1837) that with suitable restrictions* for a channel of constant width but varying depth, an approximate solution of (2.1) for the surface elevation can be written as

$$\eta(x,t) \sim h^{-1/4} \exp \left\{ i\sigma \left(t \mp \int dx / c(x) \right) \right\} \quad (7.1)$$

where $c(x)$ is the "local" velocity, $\{g h(x)\}^{1/2}$. If the bottom is corrugated, then an average phase speed can be written as the quotient of the length of the undulations by the Green-DuBoys travel time, or

$$\bar{c} = L_B \left\{ \int_0^{L_B} dx / c(x) \right\}^{-1}. \quad (7.2)$$

It is the purpose of this section to elaborate on (7.2) and to show precisely in what sense it agrees with the results of the preceding section.

Using $h(x) = h_0 (1 + \alpha \cos px)$ in (7.2), obtain

$$\bar{c} = \frac{2\pi}{p} \left\{ \int_0^{2\pi/p} [g h_0 (1 + \alpha \cos px)]^{-1/2} dx \right\}^{-1}$$

or $\frac{\bar{c}}{c_0} = \pi (1 + \alpha)^{1/2} / 2K(k) \quad (7.3)$

* See Lamb (1932) §185. Also Jeffries (1962), §3.5.

It is here not necessary to belabor the restrictions for what follows.

where K is the complete elliptic integral of the first kind of modulus $k = (2\alpha/(1+\alpha))^{1/2}$. Now using the series expansion for $K(k)$ and the binomial theorem repeatedly, a little algebra will show that (7.3) can be written as

$$\frac{\bar{c}}{c_0} = 1 - \frac{3}{16} \alpha^2 - \frac{69}{1024} \alpha^4 + O(\alpha^6). \quad (7.4)$$

This result is to be compared with (6.5). Notice that the only difference is that in the exact result, there appears a coefficient written in square brackets which is a quotient of two polynomials in $\sqrt{}$, both of the same order. Rewriting (6.5) again, then

$$\frac{c(\alpha)}{c_0} = 1 - \frac{3}{16} P_2(\sqrt{}) \alpha^2 - \frac{69}{1024} P_4(\sqrt{}) \alpha^4 + O(\alpha^6) \quad (7.5)$$

Now each of the polynomials P_2 , P_4 , approaches unity as $\sqrt{}$ increases. That is, then, the Green-DuBoys formula gives correct results asymptotically, or (physically) for surface waves whose wavelength is small in comparison with the length of the bottom corrugations, but still large in comparison with the mean depth h_0 . For waves of the same scale as the wavy bottoms, then, the Green-DuBoys formula is in error, and the more exact result for small $\sqrt{}$ must be used.

Jeffries (1962) points out that Green's result is indeed an asymptotic result obtained by considering the frequency to be a large parameter. It is interesting to pursue the asymp-

totic result further with a view to improving upon the result of Green, at least for the case at hand. Rather than using (2.2), it is easier (formally) to consider a related equation. If u is the horizontal velocity, then consistent with the long wave approximation, the quantity $f = uh$ (the total flux) satisfies $gh f_{xx} = f_{tt}$. With harmonic time dependence $f = \tilde{f} e^{i\omega t}$ and $h(x) = h_0(1 + \alpha \cos px)$, and with $z = px/2$ as before, then

$$\frac{d^2 \tilde{f}}{dz^2} + \frac{\lambda^2}{1 + \alpha \cos 2z} \tilde{f} = 0 ; \lambda^2 = 4\omega^2 / gh_0 p^2 \quad (7.6)$$

The eigenvalues $\lambda(\alpha)$ of the flux equation (7.6) are identical to those of the surface elevation equation (2.2). That is, if Ince type solutions are substituted into (7.6), the characteristic equations are identical to (3.3a-d). Furthermore, a single differentiation of (7.6) results in (2.2) with $\xi = \tilde{f}'$.

Following the asymptotic methods of Olver (1954), first make a Langer transformation of the dependent and independent variables according to

$$\tilde{f}(z) = (1 + \alpha \cos 2z)^{1/4} W(\zeta)$$

where $\zeta(z)$ satisfies

$$\zeta(z) = \int_0^z dz / (1 + \alpha \cos 2z)^{1/2} = (1 + \alpha)^{-1/2} F(z, k) ,$$

$$k^2 = 2\alpha / (1 + \alpha) ,$$

and $F(z, k)$ is the incomplete elliptic integral of the first kind. Then (7.6) becomes

$$\frac{d^2 W}{ds^2} = \{-\lambda^2 + f(s)\} W \quad (7.7)$$

where $f(s) = \alpha \cos 2z + \frac{3}{4} \alpha^2 \frac{\sin^2 2z}{1 + \alpha \cos 2z}$

Formally, then, solutions of (7.7) can be written as the asymptotic series

$$W(s) \sim \left\{ 1 - \frac{A_2(s)}{\lambda^2} + \dots \right\} \cos \lambda s + \left\{ \frac{A_1(s)}{\lambda} - \frac{A_3(s)}{\lambda^3} + \dots \right\} \sin \lambda s$$

where the A_i 's are given by

$$A_1(s) = \frac{1}{2} \int_0^s f(s) ds,$$

$$A_2(s) = -\frac{1}{4} f(s) + \frac{1}{2} \{A_1(s)\}^2,$$

$$A_3(s) = \frac{1}{8} f'(s) - \frac{1}{4} f(s) A_1(s) + \frac{1}{6} \{A_1(s)\}^3 - \frac{1}{8} \int_0^s f^2(s) ds,$$

etc. Evaluating $A_1(s)$ directly (which is the only term to be used here), have

$$A_1(s) = \frac{1}{8} (1-k^2)^{1/2} \left\{ (2-k^2) F(z, k) - 2E(z, k) + 3 \frac{k^2 \sin 2z \cos 2z}{(1-k^2 \sin^2 z)^{1/2}} \right\}.$$

For present purposes and without loss of generality, it is only necessary to determine periodic solutions of period π , namely those corresponding to the even numbered characteristics of § 3.

The condition of periodicity then requires that $W(\zeta(z+\pi)) = W(\zeta(z))$. Now to the lowest order of approximation, that is neglecting A_1, A_2, \dots , the periodicity condition requires that $\lambda \zeta(\pi) \sim 2n\pi$, $n = 1, 2, \dots$. For the next approximation, involving A_1 but nothing higher, try $\lambda \zeta(\pi) \sim 2n\pi + \lambda_1$. In order to satisfy the periodicity conditions to this order, then $\lambda_1 = \zeta(\pi) A_1(\zeta(\pi)) / 2n\pi$. Finally, the eigenvalues are of the form

$$\lambda_n(\omega) \sim \frac{2n\pi}{\zeta(\pi)} + \frac{A_1(\zeta(\pi))}{2n\pi}$$

or explicitly

$$\lambda_n(\omega) \sim \frac{(1+\omega)^{1/2}}{K(k)} n\pi + \frac{1}{8}(1+\omega)^{1/2} \left\{ (2-k^2)K(k) \cdot 2E(k) \right\} \frac{1}{n\pi} + \dots \quad (7.8)$$

Notice that the leading term in the asymptotic expansion for the eigenvalues is identical in form to the Green-DuBoys result (7.3), recalling that $\lambda_n(0) = 2n$. The second term in (7.8) estimates quantitatively the error involved in using Green's results for this particular configuration. Furthermore, the asymptotic results always indicate that the wavy bottom retards the phase speed of progressive waves, which is of course in accordance with (6.5) for large values of ω .

Finally, a curious result related to the coalescence phenomenon of § 5 should be pointed out. Considering only the Green-DuBoys term of (7.8), then as $\omega \rightarrow 1$, the eigenvalues all ap-

proach zero, regardless of α . However, if the asymptotic series (7.8) is squared, then

$$\lambda_n^2(\alpha) \sim \frac{1-\alpha}{\{K(k)\}^2} n^2 \pi^2 + \frac{(1-\alpha)}{4} \left\{ (1-k^4) - 2 \frac{E(k)}{K(k)} \right\} + O(n^{-2})$$

The second term in this relation is the only term independent of α . Considering just the first two terms, then it is easily seen that as $\alpha \rightarrow 1$, the term involving n^2 vanishes and $\lambda_n^2(\alpha) \rightarrow 1/2$,* which is the (frequency)² of the anomalous seiche obtained in § 5 above, and is as before independent of α . That this is true of the asymptotic result, however, must be regarded as coincidental, since Olver's asymptotic method requires that $f(\zeta)$ in (7.7) be regular, which is not the case here as $\alpha \rightarrow 1$.

*The approach is inverse squared logarithmic, in agreement with (5.5).

8. Discussion and comments

It is now worthwhile to review briefly some of the physical conditions for which the above results may apply. First, the wave amplitude must be everywhere small enough that the linearized equations be a good approximation. This has already been assumed to be the case here, but should be kept in mind. Second, the surface wavelength must always be much larger than the maximum depth so that they are effectively in shallow water, or (loosely) (a) $L_w \gg h_0$. These conditions (at least) must hold for linearized long wave theory (2.1) to be applicable.

In order that Green's results be applicable, it is well known that the depth must not vary by more than a small fraction of itself within the limits of a wavelength of the surface wave. For arbitrary $0 < \alpha < 1$, this then requires that the surface wavelength be a small fraction of the bottom wavelength, or $L_B \gg L_w$. This is of course consistent with the asymptotic results of § 7, which require that $\sqrt{\alpha}$ (or λ) be large. The relative error in using the Green-DuBoys formula is then of order $(L_w/L_B)^2$ and is given explicitly by (7.8). If for somewhat longer surface waves the error should become intolerable, then the first two terms of the asymptotic series may be used to advantage. In any event, however, the wave is always retarded. This is the region that Katz (1963) investigated in a different context.

For waves of sufficient length that the asymptotic re-

sults are not valid, two cases arise. First, suppose the surface and bottom wavelengths are of the same order, $L_B = O(L_W)$. In order that the vertical accelerations be negligible, the maximum slope of the bottom must be small, or $h_0 \ll L_B$. This condition together with (a) above places no restriction on α . That is, the amplitude of the ridges may be quite large as long as the slope is gradual. For this case, if α is moderate, the series (6.5) may be used. Notice that this region contains the only ratio of wavelengths for which augmentation of the phase speed is possible.

If, secondly, the surface wave is much longer than the undulations of the bottom, $L_W \gg L_B$ or $\nu \ll 1$, then to be consistent with (a), the amplitude of the ridges must be a small fraction of the average depth, in addition to the requirement of small bottom slopes. For this region the first several terms of (6.5) should prove adequate. Again, the propagation is always retarded. Geophysically, this region is most applicable to phenomena of tidal magnitudes.

Hidaka (1965) investigated only the equal wavelength case for which $\lambda = 2$, and two independent solutions coexist. He calculated (numerically) several points on the characteristic curve for $\alpha \leq 0.5$ by the continued fraction method correctly. He then incorrectly determined an "average phase speed" numerically on this characteristic to be greater than \sqrt{ghe} .

But this cannot be so, since the propagation is always retarded on this characteristic. Proper calculation of the average phase speed using his numerical values, however, yields a result that is identical with that here, namely $\epsilon_0 \lambda(\omega) / \lambda(0)$.

The odd regions of instability deserve more comment. Hidaka did not find them since he was on an even characteristic. That neither Green's type analysis nor the few-term asymptotic analysis of § 7 will turn them up is clear upon examination of (3.8), which shows that the widths of the intervals of instability are proportional to $(n!)^{-1}$. That is, they vanish faster than the first few terms in the asymptotic series. Moreover, it is to be expected that only the first few intervals of instability will be of any physical significance, in virtue of the rapidity with which they vanish. This is intended to be a further caveat against the use of Green-DuBoys results in unwarranted physical situations, when a more detailed analysis is called for.

In § 3 it was mentioned that the calculation of the Fourier coefficients, while not too difficult, has not been done. The only reason for determining them would be to thence determine the profiles of the surface waves. This indeed can be done for any α and λ by numerical integration of (2.3), but there is an even simpler way to determine the form of the solutions. Consider an analog computer consisting of a capacitor (C) and a motor driven variable inductor of inductance $L_0(1 + \alpha \cos \omega t)$, where ω is the radian frequency of the motor. The current in

a series combination of these elements satisfies

$$\frac{d}{dt} \left[(1 + \alpha \cos \omega t) \frac{di}{dt} \right] + \frac{i}{L_0 C} = 0.$$

With $\omega t = 2\pi$, this is identical with (2.3), and λ is $2/\omega \sqrt{L_0 C}$, or twice the ratio of the natural frequency $\omega_0 = (L_0 C)^{-1/2}$ to the driving frequency ω . The waveforms can now be determined by varying the inductance (α) and the speed of the motor (λ). It is interesting to note that if the driving frequency is exactly twice the natural resonant frequency of the series circuit, or $\lambda = 1$, the solutions are unstable, and the oscillating current will grow with time exponentially in amplitude, until it is limited by the inevitable series resistance in the circuit.

This simple example of a subharmonic resonance is directly analogous to the physical situation that exists if it is attempted to pass surface waves over wavy bottoms for combinations of the parameters lying within unstable regions. By resonance with the wavy bottom, the amplitude of the wave will grow exponentially as the wave progresses, at least until either the shallow water approximation or the linearized equations are no longer reasonable approximations. The wavy bottom, then, is in the linear theory the analog of a band stop filter. The ultimate fate of these waves, however, is a matter most easily settled by experiment.

It is a pleasure to acknowledge the assistance of D. Wedel in the numerical calculation of the characteristic curves. The asymptotic methods of Olver were pointed out by W. H. Reid. The second term of the asymptotic series is essentially his.

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Figure 1. Stability for Mathieu's equation. The approximate solutions of (2.4) for small α/λ^2 lie along the sloping straight line.

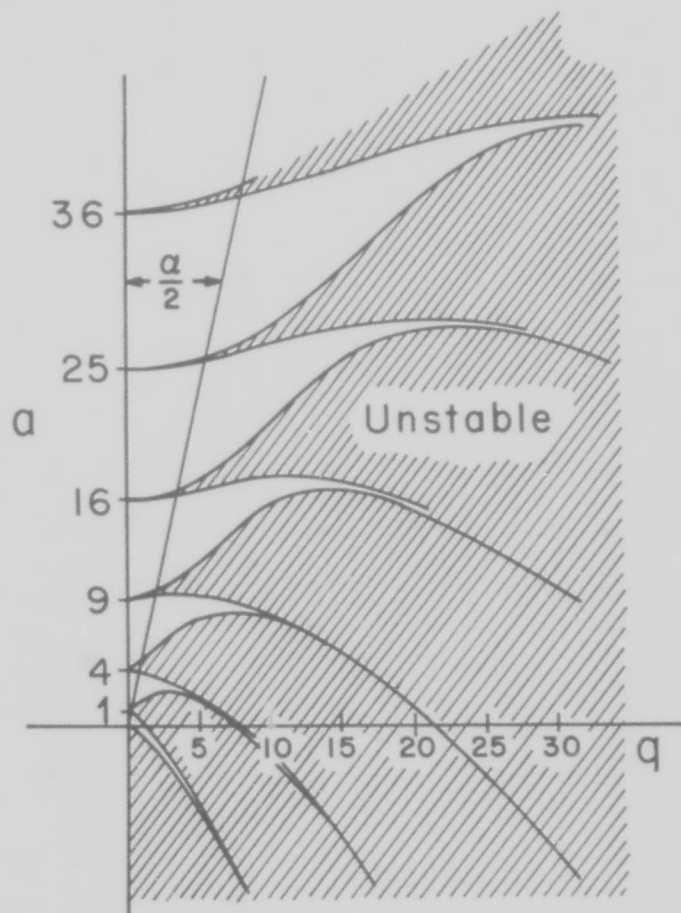


Figure 2. Characteristic curves for the wavy bottom equation.

$\lambda = 2\sigma / \rho \sqrt{g h}$ and α is the relative height of the undulations.

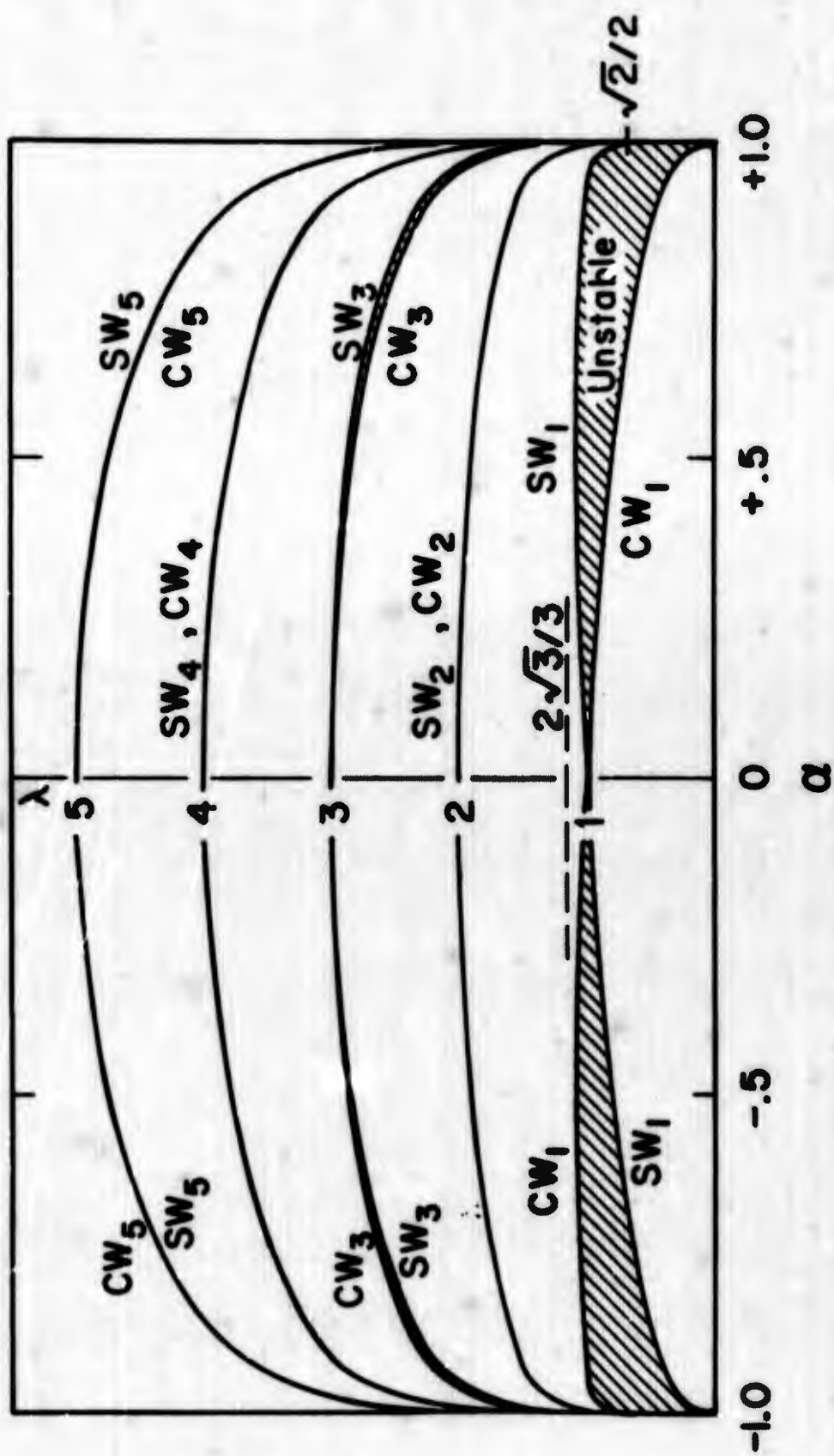


Figure 3. The characteristic values λ as a function of $(\tau_{\text{rel}})^2$.
The light straight lines are the approximations given
by (3.9).

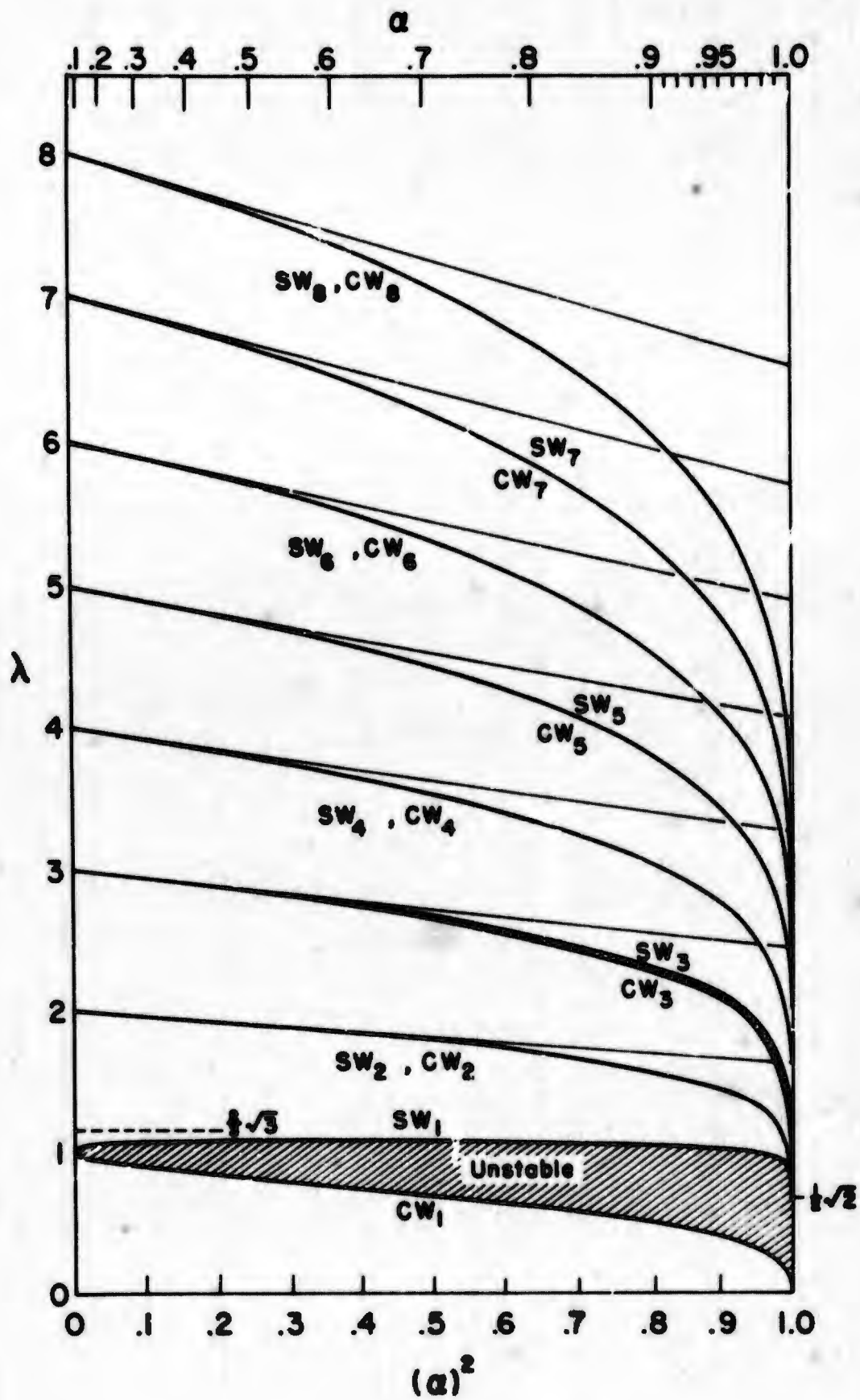
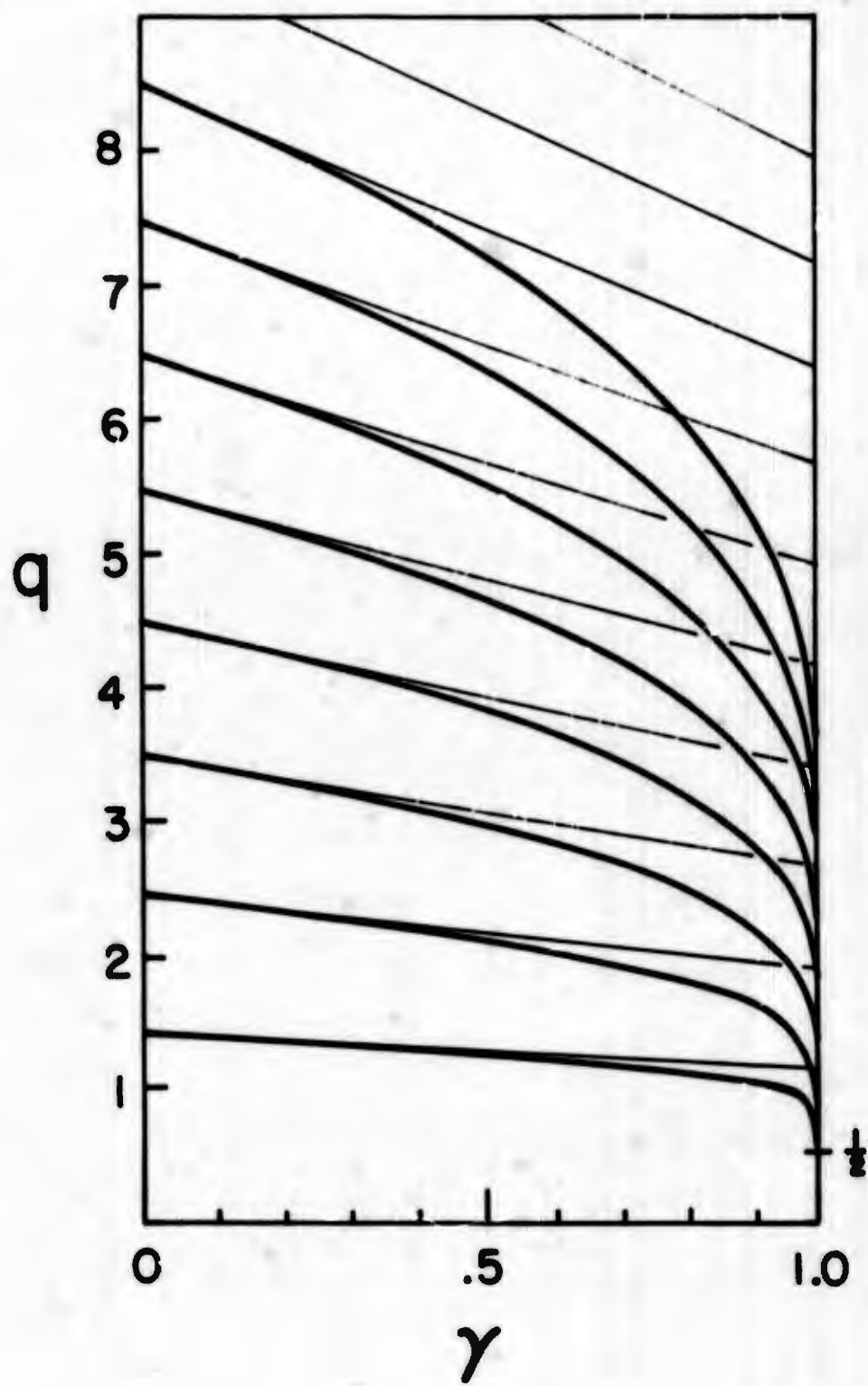


Figure 4. Characteristic curves for seiches in a cosine shaped basin. q is a dimensionless frequency, $q = \sigma l / \pi \sqrt{g a}$ and γ is the ratio of the maximum depth of the water d to the peak-to-peak depth of the basin, a . l is the overall length of the basin, from peak to peak. The light lines are the first approximations to the eigenvalues.



α	1st MODE		2nd MODE		3rd MODE		4th MODE		5th MODE		6th MODE		7th MODE		8th MODE	
	even $\lambda_0^{(1)}$	odd $\lambda_1^{(1)}$	even $\lambda_0^{(2)}$	odd $\lambda_1^{(2)}$	even $\lambda_0^{(3)}$	odd $\lambda_1^{(3)}$	even $\lambda_0^{(4)}$	odd $\lambda_1^{(4)}$	even $\lambda_0^{(5)}$	odd $\lambda_1^{(5)}$	even $\lambda_0^{(6)}$	odd $\lambda_1^{(6)}$	even $\lambda_0^{(7)}$	odd $\lambda_1^{(7)}$	even $\lambda_0^{(8)}$	odd $\lambda_1^{(8)}$
.100	.9732	1.0233	1.9967	2.9966	2.9966	2.9966	3.9926	4.9907	4.9907	4.9907	5.9908	6.9909	6.9909	6.9909	7.9950	.100
.200	.9827	1.0433	1.9965	2.9781	2.9781	2.9781	3.9702	4.9625	4.9625	4.9625	5.9549	6.9471	6.9471	6.9471	7.9396	.200
.300	.9901	1.0600	1.9960	2.9498	2.9498	2.9498	3.9318	4.9140	4.9140	4.9140	5.8963	6.8787	6.8787	6.8787	7.8611	.300
.400	.9966	1.0732	1.9956	2.9081	2.9081	2.9081	3.8753	4.8427	4.8427	4.8427	5.8104	6.7783	6.7783	6.7783	7.7462	.400
.500	.9999	1.0825	1.9952	2.8508	2.8508	2.8508	3.7975	4.7466	4.7466	4.7466	5.6922	6.6399	6.6399	6.6399	7.5977	.500
.600	.9999	1.0872	1.9948	2.7735	2.7735	2.7735	3.6928	4.6125	4.6125	4.6125	5.5329	6.4535	6.4535	6.4535	7.3763	.600
.700	.9999	1.0859	1.9942	2.6687	2.6687	2.6687	3.5599	4.4326	4.4326	4.4326	5.3169	6.2008	6.2008	6.2008	7.0850	.700
.800	.9999	1.0734	1.9934	2.5286	2.5286	2.5286	3.3563	4.1802	4.1802	4.1802	5.0113	5.8431	5.8431	5.8431	6.6754	.800
.900	.9999	1.0470	1.9927	2.3537	2.3537	2.3537	3.0899	3.7748	3.7748	3.7748	4.5222	5.2705	5.2705	5.2705	6.0195	.900
.910	.9999	1.0422	1.9924	2.3033	2.3033	2.3033	2.9848	3.7176	3.7176	3.7176	4.4331	5.1896	5.1896	5.1896	5.9367	.910
.920	.9999	1.0366	1.9918	2.2130	2.2130	2.2130	2.8954	3.5661	3.5661	3.5661	4.3376	5.0112	5.0112	5.0112	5.8255	.920
.930	.9999	1.0306	1.9910	2.1285	2.1285	2.1285	2.8010	3.4067	3.4067	3.4067	4.2043	4.8945	4.8945	4.8945	5.7137	.930
.940	.9999	1.0235	1.9902	2.0774	2.0774	2.0774	2.7111	3.2490	3.2490	3.2490	4.0952	4.7702	4.7702	4.7702	5.5636	.940
.950	.9999	1.0152	1.9894	2.0177	2.0177	2.0177	2.6204	3.1087	3.1087	3.1087	3.9713	4.6251	4.6251	4.6251	5.4442	.950
.960	.9999	1.0052	1.9884	1.9453	1.9453	1.9453	2.5276	2.9643	2.9643	2.9643	3.8210	4.4688	4.4688	4.4688	5.2796	.960
.970	.9999	.9987	1.9870	1.8516	1.8516	1.8516	2.4460	2.8329	2.8329	2.8329	3.6750	4.3200	4.3200	4.3200	5.0777	.970
.980	.9999	.9959	1.9859	1.7714	1.7714	1.7714	2.3664	2.7095	2.7095	2.7095	3.5337	4.1832	4.1832	4.1832	4.8152	.980
.990	.9999	.9942	1.9842	1.6921	1.6921	1.6921	2.2903	2.5770	2.5770	2.5770	3.3922	3.9976	3.9976	3.9976	4.6004	.990
.991	.9999	.9934	1.9834	1.6711	1.6711	1.6711	2.2618	2.5403	2.5403	2.5403	3.3480	3.9300	3.9300	3.9300	4.5458	.991
.992	.9999	.9926	1.9826	1.6480	1.6480	1.6480	2.2356	2.5061	2.5061	2.5061	3.2922	3.8680	3.8680	3.8680	4.4902	.992
.993	.9999	.9918	1.9818	1.6222	1.6222	1.6222	2.2105	2.4682	2.4682	2.4682	3.2480	3.8060	3.8060	3.8060	4.4346	.993
.994	.9999	.9910	1.9810	1.5929	1.5929	1.5929	2.1856	2.4333	2.4333	2.4333	3.2040	3.7440	3.7440	3.7440	4.3790	.994
.995	.9999	.9902	1.9802	1.5607	1.5607	1.5607	2.1605	2.3981	2.3981	2.3981	3.1600	3.6820	3.6820	3.6820	4.3234	.995
.996	.9999	.9894	1.9794	1.5372	1.5372	1.5372	2.1356	2.3556	2.3556	2.3556	3.1160	3.6200	3.6200	3.6200	4.2678	.996
.997	.9999	.9886	1.9786	1.5131	1.5131	1.5131	2.1105	2.3131	2.3131	2.3131	3.0720	3.5580	3.5580	3.5580	4.2122	.997
.998	.9999	.9878	1.9778	1.4886	1.4886	1.4886	2.0856	2.2706	2.2706	2.2706	3.0280	3.4960	3.4960	3.4960	4.1566	.998
.999	.9999	.9870	1.9770	1.4640	1.4640	1.4640	2.0605	2.2281	2.2281	2.2281	2.9840	3.4340	3.4340	3.4340	4.1010	.999

TABLE 1. Numerical values for the characteristic curves for the first eight modes of the upper bottom equation. For the odd numbered modes, the even eigenvalues belong to the even, or $C_{2n}^{(1)}$ solutions and the odd eigenvalues belong to the $S_{2n}^{(1)}$ solutions. For the even numbered modes, the $C_{2n}^{(1)}$ and $S_{2n}^{(1)}$ solutions consist.

γ	1 st mode	2 nd	3 rd	4 th	5 th	6 th	7 th	8 th
0.0	$2\frac{1}{2}$	$6\frac{1}{2}$	$12\frac{1}{2}$	$20\frac{1}{2}$	$30\frac{1}{2}$	$42\frac{1}{2}$	$56\frac{1}{2}$	$72\frac{1}{2}$
.1	1.3850	2.3892	3.3766	4.2827	5.3370	6.3144	7.2909	8.2668
.2	1.3538	2.3246	3.2828	4.2359	5.1866	6.1360	7.0846	8.0327
.3	1.3200	2.2546	3.1813	4.1036	5.0238	5.9429	6.8614	7.7793
.4	1.2831	2.1779	3.0702	3.9588	4.8457	5.7316	6.6170	7.5019
.5	1.2422	2.0926	2.9466	3.7978	4.6476	5.4966	6.3452	7.1934
.6	1.1959	1.9955	2.8061	3.6146	4.4223	5.2293	6.0361	6.8426
.7	1.1417	1.8813	2.6409	3.3992	4.1572	4.9149	5.6724	6.4297
.8	1.0746	1.7387	2.4348	3.1304	3.8264	4.5224	5.2183	5.9143
.9	0.9808	1.5366	2.1431	2.7491	3.3567	3.9651	4.5737	5.1825
.91	0.9685	1.5099	2.1044	2.6984	3.2945	3.8911	4.4880	5.0852
.92	0.9552	1.4810	2.0628	2.6438	3.2269	3.8112	4.3956	4.9806
.93	0.9409	1.4496	2.0174	2.5838	3.1537	3.7243	4.2950	4.8660
.94	0.9251	1.4150	1.9675	2.5178	3.0732	3.6282	4.1840	4.7399
.95	0.9075	1.3762	1.9114	2.4435	2.9820	3.5205	4.0593	4.5984
.96	0.8874	1.3318	1.8472	2.3581	2.8776	3.3966	3.9159	4.4355
.97	0.8636	1.2788	1.7706	2.2597	2.7538	3.2486	3.7446	4.2410
.98	0.8337	—	1.6733	2.1313	2.5953	3.0599	3.5263	3.9930
.99	0.7904	—	1.5213	1.8135	2.3606	2.8580	—	—

TABLE 2. Characteristic values for the first eight seiche modes of a cosine shaped basin.

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13. ABSTRACT The propagation of long waves over bottoms having sinusoidal undulations is investigated here within the confines of linearized shallow water theory. It is found that the presence of this irregularity in most cases impedes the propagation of the wave in keeping with the proper application of the Green-DuBoys formula. For wavelengths for which this formula is not valid, or those of the same order as the bottom wavelengths, it is found that there is a region for which the propagation is <u>not</u> retarded, and the travel time is less than that based upon the mean depth. Furthermore, the presence of regular undulations of the bottom of any amplitude prohibits the propagation of an infinite sequence of wavelengths on the surface, the most significant of which are those of the same order as the bottom. These waves are unstable, and through resonance with the bottom will grow without bound as they progress, or at least until the linearized theory is invalidated. An electrical analog is presented which exhibits the same instability, a subharmonic resonance, and can be used to determine the free surface profiles. ()		

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