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APPLICATIONS OF THEORIES OF GENERALIZED COSSERAT CONTINUA TO THE DYNAMICS OF COMPOSITE MATERIALS

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ABSTRACT

It is shown that the Cosserat continuum and the theory of elasticity with micro-structure can be interpreted as analytical models describing the dynamic behavior of a composite material. The nonclassical material constants are simply functions of the geometry and the classical constants of the two materials constituting the composite. The study of wave propagation in a laminated composite reveals that a more complex micro-structure needs to be introduced in a continuum in order to describe adequately the dispersive character of (essentially) longitudinal waves.

I. Introduction

The 19th century idea that models of physical bodies should consist not merely of an assemblage of points, but should also include effects of directions associated with the points, (oriented bodies), as suggested by VOIGT and DUHEM, sprang from the desire to describe various phenomena on the microscale which ordinary continuum mechanics is not able to accommodate. E. and F. COSSERAT constructed a theory of elasticity corresponding to this idea for a special case, namely, when orientation is specified at each point by a rigid triad, entailing the introduction of the couple per unit area, acting across a surface within a material volume or on its boundary, in addition to the usual force per unit area. A modern derivation of a COSSERAT-type theory and a discussion of typical effects of couple stresses within the framework of a linearized form of the couple-stress theory for perfectly elastic, centrosymmetric-isotropic materials were given by MINDLIN and TIERSTEN [1]. It was mentioned by these authors that in their theory the new material constant ℓ, which has the dimension of length and which embodies all the difference between analogous equations or solutions with and without couple stresses, is presumably small in comparison with bodily dimensions and wave lengths normally encountered, as there appears to be no conclusive experimental evidence of its existence. Various other aspects of COSSERAT-type continua and related theories were discussed by TOUPIN [2,3], KOITER [4], SCHAEFER [5] and MINDLIN and ESHEL [6].

To incorporate in a continuum theory of mechanics further microscale phenomena occurring in a crystal lattice, MINDLIN [7] established a theory of linear elasticity with micro-structure (TEMS) by assuming, in effect, that each leg of the COSSERAT triad can stretch and rotate independently of the other two. This model is equivalent to the inclusion, at each point of the macro-medium, of a unit cell of a micro-medium which deforms homogeneously. For a centrosymmetric-isotropic material there are sixteen additional independent material constants which describe the properties of this continuum. If the cell is made rigid, but is allowed to rotate independently of the macro-rotation, one reverts to the COSSERAT theory (COST). With the further constraint of the cell having the same rotation as the macro-rotation (COSSERAT's "triedre caché"), one is led to the special theory of elasticity with couple stresses (TECS). Alternatively, the theory can be made more complex, for example, by placing into each cell several mass points, and by specifying interaction forces between mass points in the same cell and in neighboring cells, as was discussed by KUNIN [8].

It appears now, as some recent work by the authors indicates, that the concepts and theories of a COSSERAT continuum and its generalizations have broad applicability in describing phenomena which occur on a macro- rather than a microscale. Indeed, if one wishes to describe the dynamic behavior of periodically macro-heterogeneous solids, such as, for example, fiber-reinforced or laminated composites, one can be led to similar mathematical relations. One approach explored by the authors [9,10] consists in using representative elastic moduli for the binder (soft layers) and combining the elastic and geometric properties of the fibers or the sheets (stiff layers) into "effective stiffnesses." Depending upon certain supplementary kinematical assumptions, describing the deformation of reinforcing elements, a continuum theory can be evolved which bears strong resemblance to MINDLIN'S theory of elasticity with micro-structure or, in its simpler version, to the COSSERAT continuum. What is noteworthy, however, is that the nonclassical material constants are now simply functions of the geometric layout and of the classical constants of the two constituent homogeneous materials.

To render the indicated connection specific and precise, the most important concepts and relations of TEMS are set down in Section II. To make the point, it suffices to consider plane deformation and unidirectional structuring, and, for the sake of brevity, boundary conditions are not discussed. Section III presents the fundamental relations of one version of the effective stiffness theory for laminated composites recently proposed by the authors [9,10] and identifies the material coefficients of TEMS in terms of the classical material constants and the geometric layout of the composite. This interpretation is followed by a reduction corresponding to COST and TECS.

In Section IV, the viability of the proposed theories is discussed by a study of the dispersion characteristics of free plane harmonic waves in the direction of lamination. Comparisons with "exact" dispersion curves obtained by solving the appropriate classical elasticity problem [11] reveal that the lowest (predominantly) transverse mode is rather well described in its strong dispersion, by contrast to the lowest (predominantly) longitudinal mode. The reason for strong dispersion in this mode, as revealed by examining the exact solution, is due primarily to the dispersive properties of the soft layers. Authors' more complicated versions of the effective stiffness theory for laminated media [12] are briefly summarized.

II. Uni-Directional Micro-Structure in Linear Plane Elasticity

Let us assume that in MINDLIN'S TEMS [7] the deformation is two-dimensional. With a CARTESIAN frame of reference x_1, x_2 the components of displacement u_i are

$$u_1 = u_1(x_1,x_2,t); \qquad u_2 = u_2(x_1,x_2,t) \tag{1}$$

where t is the time. If micro-structure is introduced in one direction only, say x_2, there will be only two nonvanishing components of micro-deformation ψ_{ij}, namely

$$\psi_{21} = \psi_{21}(x_1,x_2,t); \qquad \psi_{22} = \psi_{22}(x_1,x_2,t) \tag{2}$$

The components of macro-strain are

$$\epsilon_{11} = \partial u_1/\partial x_1; \quad \epsilon_{22} = \partial u_2/\partial x_2; \quad \epsilon_{12} = (\partial u_2/\partial x_1+\partial u_1/\partial x_2)/2 \tag{3}$$

and the components of macro-rotation are

$$\omega_{12} = - \omega_{21} = (\partial u_2/\partial x_1 - \partial u_1/\partial x_2)/2 \tag{4}$$

The relative deformation has only two relevant components, namely

$$\gamma_{21} = \partial u_1/\partial x_2 - \psi_{21}; \qquad \gamma_{22} = \partial u_2/\partial x_2 - \psi_{22} \tag{5}$$

The four nonvanishing components of the micro-deformation gradient are

$$\begin{aligned} \kappa_{121} &= \partial\psi_{21}/\partial x_1; & \kappa_{122} &= \partial\psi_{22}/\partial x_1 \\ \kappa_{221} &= \partial\psi_{21}/\partial x_2; & \kappa_{222} &= \partial\psi_{22}/\partial x_2 \end{aligned} \tag{6}$$

It is assumed, however, that the gradients in the direction of structuring do not contribute to the potential energy and thus will be ignored in the sequel, i.e. $\kappa_{221} \equiv \kappa_{222} \equiv 0$. The reason for this assumption will be discussed in Section III.

The potential energy W is assumed to be a function of the seven variables ϵ_{11}, ϵ_{12}, ϵ_{22}, γ_{21}, γ_{22}, κ_{121} and κ_{122}.

The three nonvanishing components of CAUCHY stress τ_{ij} are defined as

$$\tau_{ij} = \partial W/\partial \epsilon_{ij} = \tau_{ji} \qquad i,j = 1,2 \tag{7}$$

The two nonvanishing components of relative stress σ_{ij} are defined as

$$\sigma_{21} = \partial W/\partial \gamma_{21}; \qquad \sigma_{22} = \partial W/\partial \gamma_{22} \tag{8}$$

and the two nonvanishing components of double stress μ_{ijk} are defined as

$$\mu_{121} = \partial W/\partial \kappa_{121}; \quad \mu_{122} \equiv \partial W/\kappa_{122} \tag{9}$$

The kinetic energy T is taken in the form

$$T = \frac{1}{2} \rho(\dot{u}_1{}^2 + \dot{u}_2{}^2) + \frac{1}{6} \rho' d^2 (\dot{\psi}_{21}{}^2 + \dot{\psi}_{22}{}^2) \tag{10}$$

where ρ is the sum of the masses of macro-material and micro-material per unit macro-volume, ρ' is the mass of the micro-material per unit macro-volume and 2d is the characteristic length of the (presently one-dimensional) micro-medium. The dot indicates differentiation with respect to time.

HAMILTON'S principle for independent variations δu_i and $\delta\psi_{ij}$ leads to the following four stress-equations of motion, in the absence of body forces and body double forces,

$$\begin{aligned}
&\partial\tau_{11}/\partial x_1 + \partial(\tau_{21} + \sigma_{21})/\partial x_2 = \rho\ddot{u}_1 \\
&\partial\tau_{12}/\partial x_1 + \partial(\tau_{22} + \sigma_{22})/\partial x_2 = \rho\ddot{u}_2 \\
&\partial\mu_{121}/\partial x_1 + \sigma_{21} = \frac{1}{3} \rho' d^2 \ddot{\psi}_{21} \\
&\partial\mu_{122}/\partial x_1 + \sigma_{22} = \frac{1}{3} \rho' d^2 \ddot{\psi}_{22}
\end{aligned} \tag{11}$$

For an isotropic material and for the restricted deformations presently considered MINDLIN'S general potential energy density reduces to

$$\begin{aligned}
W = &\frac{1}{2} \lambda(\epsilon_{11}{}^2 + 2\epsilon_{11}\epsilon_{22} + \epsilon_{22}{}^2) + \mu(\epsilon_{11}{}^2 + \epsilon_{22}{}^2 + 2\epsilon_{12}{}^2) \\
&+ \frac{1}{2} b_1\gamma_{22}{}^2 + \frac{1}{2} b_2(\gamma_{21}{}^2 + \gamma_{22}{}^2) + \frac{1}{2} b_3\gamma_{22}{}^2 \\
&+ g_1(\gamma_{22}\epsilon_{11} + \gamma_{22}\epsilon_{22}) + g_2(\gamma_{21}\epsilon_{12} + \gamma_{21}\epsilon_{21} + 2\gamma_{22}\epsilon_{22}) \\
&+ \frac{1}{2} (a_4 + a_{10} + a_{13}) \kappa_{122}{}^2 + \frac{1}{2} (a_8 + a_{10} + a_{15}) \kappa_{121}{}^2
\end{aligned} \tag{12}$$

The substitution of constitutive equations, after replacement of ϵ_{ij}, γ_{ij} and $\varkappa_{ijk}$ by u_i and ψ_{ij}, into the stress-equations of motion results readily in displacement-equations of motion, which will not be displayed for the sake of brevity. The displacement-equations of motion can, alternatively, be derived directly by appropriate reduction of MINDLIN'S displacement-equations of motion, [7], Eqs. (6.1), (6.2).

The equations of motion appropriate to TEMS can be reduced to those of COST and then to TECS in different ways as discussed in detail by MINDLIN [7]. In the following we outline one possible procedure of simplification.

To perform the first step of reduction we let $\sigma_{22} \to 0$, which permits to express γ_{22} in terms of ϵ_{11} and ϵ_{22}. The three first equations of motion (11) will thus involve only the unknown functions u_1, u_2 and ψ_{21} and are those of COST. The fourth equation of motion of (11) is ignored.

To carry out the next step of reduction the deformation associated with σ_{21} is made to vanish, i.e.,

$$\psi_{21} = \frac{g_2}{b_2}\frac{\partial u_2}{\partial x_1} + \left(1 + \frac{g_2}{b_2}\right)\frac{\partial u_1}{\partial x_2} \tag{13}$$

The relative deformation γ_{21} contributing to τ_{12}, on the other hand, is expressed in terms of ϵ_{21} and σ_{21} resulting in

$$\tau_{12} = \tau_{21} = \frac{g_2}{b_2}\sigma_{21} + \left(2\mu - 2\,\frac{g_2^2}{b_2}\right)\epsilon_{21} \tag{14}$$

Only two stress-equations of motion remain, namely

$$\frac{\partial\tau_{11}}{\partial x_1} + \frac{\partial\tau_{21}}{\partial x_2} - \frac{\partial\mu_{121}}{\partial x_1\partial x_2} = \left[\rho - \frac{1}{3}\,\rho'\left(1+\frac{g_2}{b_2}\right)d^2\,\frac{\partial^2}{\partial x_2^2}\right]\ddot{u}_1 - \frac{1}{3}\,\rho' d^2\,\frac{g_2}{b_2}\,\frac{\partial^2\ddot{u}_2}{\partial x_1\partial x_2}$$

and $$\frac{\partial\tau_{12}}{\partial x_1} + \frac{\partial\tau_{22}}{\partial x_2} = \rho\ddot{u}_2 \tag{15}$$

where the third equation of motion of (11) has been employed. With

$$\kappa_{121} = \frac{\partial\psi_{21}}{\partial x_1} = \frac{g_2}{b_2}\frac{\partial^2 u_2}{\partial x_1^2} + \left(1 + \frac{g_2}{b_2}\right)\frac{\partial^2 u_1}{\partial x_1 \partial x_2} \tag{16}$$

the two displacement-equations of motion, with $d^2 = 0$, are those of TECS and can be written down immediately.

III. Interpretation of TEMS as a Continuum Theory for a Laminated Material

In an earlier paper [9] the authors derived an expression for the potential energy density of a uniformly laminated composite according to what was termed the effective single-stiffness theory. In this approximate theory it is assumed that the components of displacement (in plane strain) of the k^{th} reinforcing sheet, whose midplane position is defined by x_2^k, may be expressed in the form

$$\begin{aligned} u_1^{fk} &= u_1^k(x_1,x_2^k,t) + x_2'\psi_{21}^k(x_1,x_2^k,t) \\ u_2^{fk} &= u_2^k(x_1,x_2^k,t) + x_2'\psi_{22}^k(x_1,x_2^k,t) \end{aligned} \tag{17}$$

where x_2' is the coordinate in a local coordinate system, and u_i^k are the displacements in the midplanes. The displacement distributions (17) may be used to compute the potential energy V_f^k per unit surface of the kth reinforcing sheet. If there are n reinforcing sheets per unit length in x_2 direction the potential energy stored in the reinforcing sheets is obtained as a summation of V_f^k over the n discrete points x_2^k. The basic premise of the effective stiffness theory is that the sum may be approximated by a weighted integral (smoothing operation)

$$\sum_{k=1}^{n} V_f^k \simeq \frac{\eta}{h}\int V_f dx_2 \tag{18}$$

where η is the density of the reinforcing sheets

$$\eta = h/(h + H) \tag{19}$$

In Eq. (19) h and H are the thicknesses of the reinforcing sheets and the matrix layers, respectively. By means of the smoothing operation the field variables which were previously defined at discrete points x_2^k, now have become continuously varying functions of x_2 and the superscript k is, henceforth, omitted. The resulting expression for the potential energy density V_f is

$$V_f = \frac{1}{2} D_f\left(\frac{\partial\psi_{21}}{\partial x_1}\right)^2 + \frac{1}{2} G_f\left(\psi_{21} + \frac{\partial u_2}{\partial x_1}\right)^2 + \frac{1}{2} \lambda_f h \left(\frac{\partial u_1}{\partial x_1} + \psi_{22}\right)^2$$

$$+ \mu_f h \left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \psi_{22}{}^2\right] + \frac{1}{24} \mu_f h^3 \left(\frac{\partial\psi_{22}}{\partial x_1}\right)^2 \tag{20}$$

where D_f is the bending stiffness

$$D_f = \mu_f h^3/6(1 - \nu_f), \tag{21}$$

and G_f is the shear stiffness, which is, as an approximation

$$G_f = \mu_f h \tag{22}$$

In Eqs. (20)-(22) μ_f, λ_f and ν_f are Lamé's elastic constants and Poisson's ratio of the reinforcing material, respectively. The first two terms in Eq. (20) represent the strain energy of bending and transverse shear of a single reinforcing sheet, respectively, and the remaining three terms represent the strain energy of extension.

By applying a similar smoothing operation to the matrix layers the contribution to the total potential energy density is obtained as $(1-\eta)V_m$, where for V_m we write (see [9] for greater detail),

$$V_m = \frac{1}{2} \bar{\lambda} \left(\epsilon_{11}{}^2 + 2\epsilon_{11}\epsilon_{22} + \epsilon_{22}{}^2\right) + \bar{\mu} \left(\epsilon_{11}{}^2 + \epsilon_{22}{}^2 + 2\epsilon_{12}{}^2\right) \tag{23}$$

In Eq. (23) $\bar{\lambda}$ and $\bar{\mu}$ could be the elastic constants of the matrix, but more appropriate values can be assigned based on solutions for wave motion at long wave lengths.

The total potential energy density of the laminated medium may thus be written as

$$V = (\eta/h)V_f + (1 - \eta)V_m \tag{24}$$

where V_f and V_m are defined by Eqs. (20) and (23), respectively.

In terms of the kinematic variables γ_{21}, γ_{22}, κ_{121} and κ_{122} defined by Eqs. (5) and (6) and the usual components of strain ϵ_{ij} (3), the strain energy V_f may also be written as

$$\begin{aligned} V_f = {} & \left(\frac{1}{2}\lambda_f + \mu_f\right) h \left(\epsilon_{11}{}^2 + \epsilon_{22}{}^2\right) + \lambda_f h \epsilon_{11}\epsilon_{22} \\ & + 2\mu_f h \epsilon_{12}{}^2 - 2\mu_f h \epsilon_{12}\gamma_{21} - \lambda_f h \epsilon_{11}\gamma_{22} \\ & - \left(\lambda_f + 2\mu_f\right) h \epsilon_{22}\gamma_{22} + \frac{1}{2}\mu_f h \gamma_{21}{}^2 + \left(\frac{1}{2}\lambda_f + \mu_f\right) h \gamma_{22}{}^2 \\ & + \frac{1}{2} D_f \kappa_{121}{}^2 + \frac{1}{2}\mu_f \frac{h^3}{12}\kappa_{122}{}^2 \end{aligned} \tag{25}$$

The micro-deformation gradients κ_{221} and κ_{222} do not appear in the above expression because of the underlying assumption that the differences in rotation ψ_{21} and stretch ψ_{22} of two neighboring reinforcing sheets for like x_1 do not contribute to the potential energy of the laminated continuum.

Comparing now the expression $\eta V_f/h + (1 - \eta)V_m$ with that for W as given by Eq. (12), we recognize that they can be made identical provided the coefficients in one expression are related to the coefficients in the other by the following

$$\begin{aligned} & \lambda = \eta\lambda_f + (1-\eta)\bar{\lambda}; \qquad \mu = \eta\mu_f + (1-\eta)\bar{\mu} \\ & b_1 + b_3 = \eta(\lambda_f + \mu_f); \quad b_2 = \eta\mu_f; \quad g_1 = -\eta\lambda_f; \quad g_2 = -\eta\mu_f \\ & a_8 + a_{10} + a_{15} = \eta D_f/h; \quad a_4 + a_{10} + a_{13} = \eta\mu_f h^2/12 \end{aligned} \tag{26}$$

Thus the elastic coefficients of the theory of elasticity with micro-structure, if interpreted as those of a laminated composite, are seen to be determined in terms of the classical Lamé coefficients of the two constituent materials and the geometric lay-out as described by η and h.

A similar juxtaposition can also be carried out for the kinetic energy. For a laminated composite it was taken by the authors in [9] as

$$T = \frac{\eta}{h} T_f + (1-\eta)T_m \tag{27}$$

where

$$T_f = \frac{1}{2} \rho_f h \left(\dot{u}_1^2 + \dot{u}_2^2\right) + \frac{1}{2} \rho_f \frac{h^3}{12} \left(\dot{\psi}_{21}^2 + \dot{\psi}_{22}^2\right) \tag{28}$$

$$T_m = \frac{1}{2} \rho_m \left(\dot{u}_1^2 + \dot{u}_2^2\right) \tag{29}$$

and ρ_f and ρ_m are the mass densities of the laminate and the matrix material, respectively. Comparing the above with the expression (10) we find

$$\rho = \eta\rho_f + (1-\eta)\rho_m \tag{30}$$

$$\rho' d^2 = \eta\rho_f h^2/4 \tag{31}$$

Since, from the definition of ρ', $\rho' = \eta\rho_f$, it follows that h = 2d is the length characterizing micro-structure.

The components of CAUCHY stress for a composite under consideration then are

$$\begin{aligned}
\tau_{11} &= \left[\eta(\lambda_f+2\mu_f)+(1-\eta)(\bar{\lambda}+2\bar{\mu})\right] \frac{\partial u_1}{\partial x_1} + (1-\eta)\bar{\lambda} \frac{\partial u_2}{\partial x_2} + \eta\lambda_f\psi_{22} \\
\tau_{12} = \tau_{21} &= \eta\mu_f\left(\frac{\partial u_2}{\partial x_1} + \psi_{21}\right) + (1-\eta)\bar{\mu} \left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) \\
\tau_{22} &= \eta \left[(\lambda_f+2\mu_f)\psi_{22}+ \lambda_f \frac{\partial u_1}{\partial x_1}\right] + (1-\eta)(\bar{\lambda}+2\bar{\mu}) \frac{\partial u_2}{\partial x_2} + (1-\eta)\bar{\lambda} \frac{\partial u_1}{\partial x_1}
\end{aligned} \tag{32}$$

and the components of relative stress

$$\begin{aligned}
\sigma_{21} &= - \eta\mu_f \left(\frac{\partial u_2}{\partial x_1} + \psi_{21}\right) \\
\sigma_{22} &= - \eta(\lambda_f+2\mu_f)\psi_{22} - \eta\lambda_f \frac{\partial u_1}{\partial x_1}
\end{aligned} \tag{33}$$

The substitution of these constitutive relations into the stress-equations of motion (11) results in displacement-equations of motion identical to Eqs. (38), (39), (40), (41) of ref. [9] and are not reproduced here.

For a COSSERAT medium (COST) $\sigma_{22} \to 0$, hence $\psi_{22} = -[\nu_f/(1-\nu_f)]\partial u_1/\partial x_1$ and the three remaining displacement-equations of motion are the same as Eqs. (38), (39) and (40) in [9], except that the two terms $(\lambda_f + 2\mu_f) \times \partial^2 u_1/\partial x_1^2 + \lambda_f\, \partial\psi_{22}/\partial x_1$ in (38) combine into a single term $[E_f/(1-\nu_f^2)] \times \partial^2 u_1/\partial x_1^2$ with coefficient appropriate to compression of a plate in plane strain.

For the restricted COSSERAT medium (TECS) τ_{12} should be expressed in terms of σ_{21}, see Eq. (14). Next, σ_{21} is to be made a "reactive stress" through the third equation of motion in (11) and the associated deformation has to vanish, i.e., $\partial u_2/\partial x_1 = -\psi_{21}$. The two remaining equations of motion (15) become in terms of displacements

$$(1-\eta)\left[(\bar\lambda+2\bar\mu)\frac{\partial^2 u_1}{\partial x_1^2} + \bar\mu\frac{\partial^2 u_1}{\partial x_2^2} + (\bar\lambda+\bar\mu)\frac{\partial^2 u_2}{\partial x_1 \partial x_2}\right] + \eta\frac{E_f}{1-\nu_f^2}\frac{\partial^2 u_1}{\partial x_1^2} = \rho\ddot u_1$$

$$(1-\eta)\left[(\bar\lambda+2\bar\mu)\frac{\partial^2 u_2}{\partial x_2^2} + \bar\mu\frac{\partial^2 u_2}{\partial x_1^2} + (\bar\lambda+\bar\mu)\frac{\partial^2 u_1}{\partial x_1 \partial x_2}\right] - \frac{\eta}{h}D_f\frac{\partial^4 u_2}{\partial x_1^4} = \rho\ddot u_2 - \frac{\eta}{12}\rho_f h^2\frac{\partial^2 \ddot u_2}{\partial x_1^2} \tag{34}$$

These equations are those of TECS, as discussed in Section II, except for the inertia term with h^2. The material constant ℓ, of the dimension of length, discussed by MINDLIN and TIERSTEN [1], is here

$$\ell^2 = \frac{\eta}{h}\frac{D_f}{(1-\eta)\bar\mu} = \frac{\eta}{1-\eta}\frac{h^2}{6(1-\nu_f)}\frac{\mu_f}{\bar\mu} \tag{35}$$

and depends again, just as the material coefficients of TEMS, only on the classical material properties of the two constituent materials of the composite and on its geometric properties.

IV. Wave Motion and More Complex Theories

The equations of motion (38) through (41) of ref. [9], which are now referred to as those of the effective single stiffness theory, have been used to study the propagation of plane waves in the direction of the layering, focusing attention on the lowest (predominantly) transverse and the lowest (predominantly) longitudinal mode. For a laminated composite of periodic structure it is possible to calculate "exact" dispersion curves of those modes by solving an appropriate eigenvalue problem of the classical theory of elasticity, as was done in [11]. The comparison revealed that the proposed approximate theory of the TEMS-type describes rather well the strong dispersion of the medium in the transverse mode for wave lengths which are large as compared to the lengths which characterize the lamination of the composite, as indicated in Fig. 1, which shows a plot of dimensionless frequency vs. dimensionless wave number.

By contrast, the dispersion predicted by the single stiffness theory advanced in [9] is not too satisfactory for the predominantly longitudinal mode. A detailed examination of the displacement distribution as determined by the "exact" analysis reveals that in this mode the dispersive behavior of the soft matrix is responsible for the discrepancy, which is not accounted for by the version of ref. [9] of the effective stiffness theory. The authors were thus induced to construct a continuum theory with more complex micro-structure [12], in which the effective stiffnesses of both the laminates and the matrix layers are introduced (effective double stiffness theory) and supplemented by appropriate continuity conditions of perfect bond at the layer interfaces. A first- and a second order theory of this type were developed, where, in the latter, quadratic terms in the displacement expansions were retained. The effective double stiffness theory corresponds to a theory of elasticity with microstructure in which the unit cell is allowed to undergo two different microdeformations, whose weighted sum is proportional to the macro-displacement gradient. Alternatively, the micro-kinematics of this theory can be interpreted as that of two deformable COSSERAT triads being defined at each point which are, however, suitably related to the macro-displacement gradient.

Fig. 2 gives a plot of dimensionless frequency vs. dimensionless wave number, and it is concluded that the second order approximation contributes to an improvement of the dispersion characteristics, particularly for large values of the ratio of the shear moduli of the two materials.

Finally it may be remarked that just as TEMS can be reduced to COST and then to TECS, similarly the effective double stiffness theory may thus be reduced by introducing corresponding constraints in the micro-structure.

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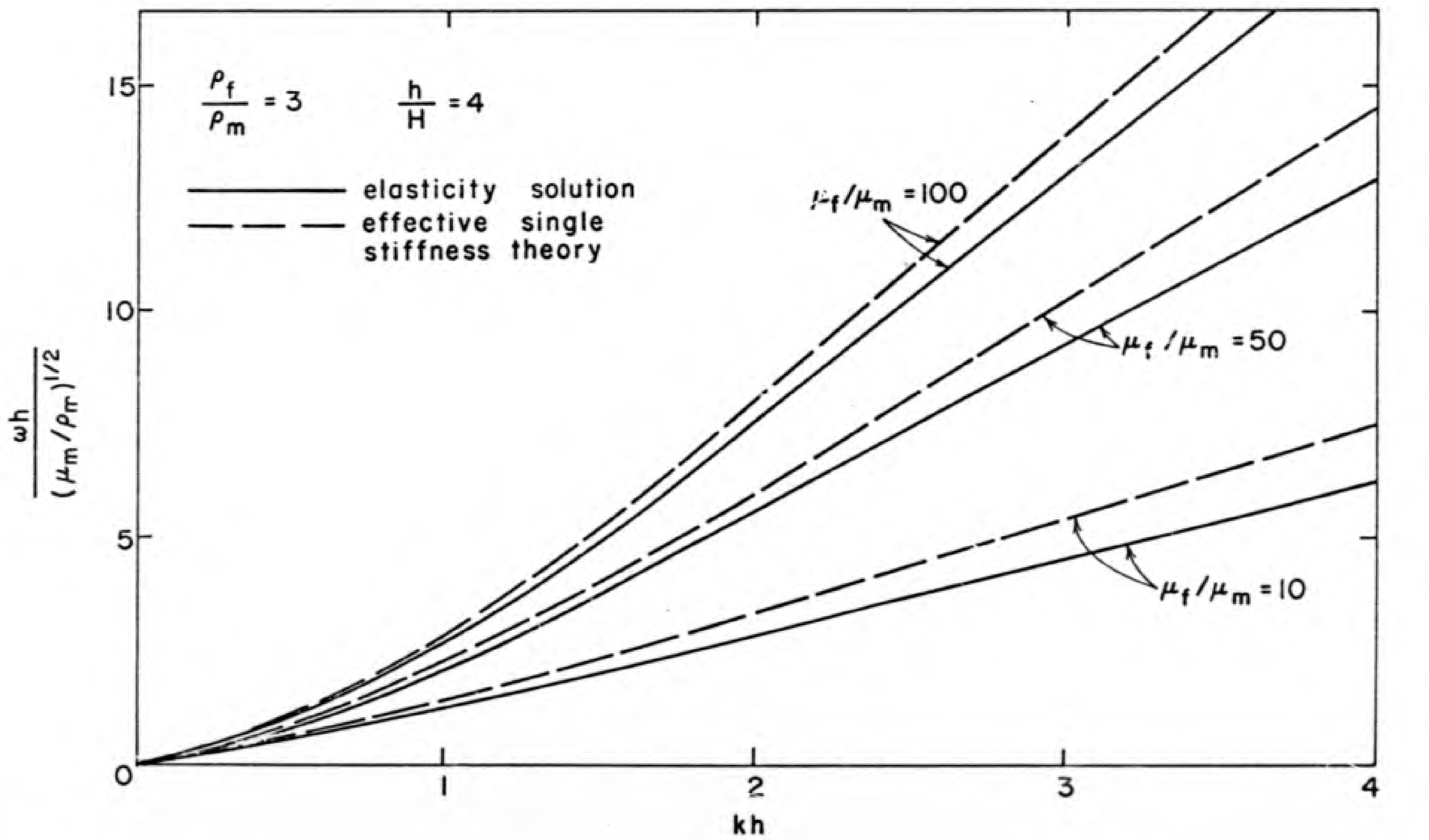

Fig. 1 Lowest transverse mode according to effective single stiffness theory

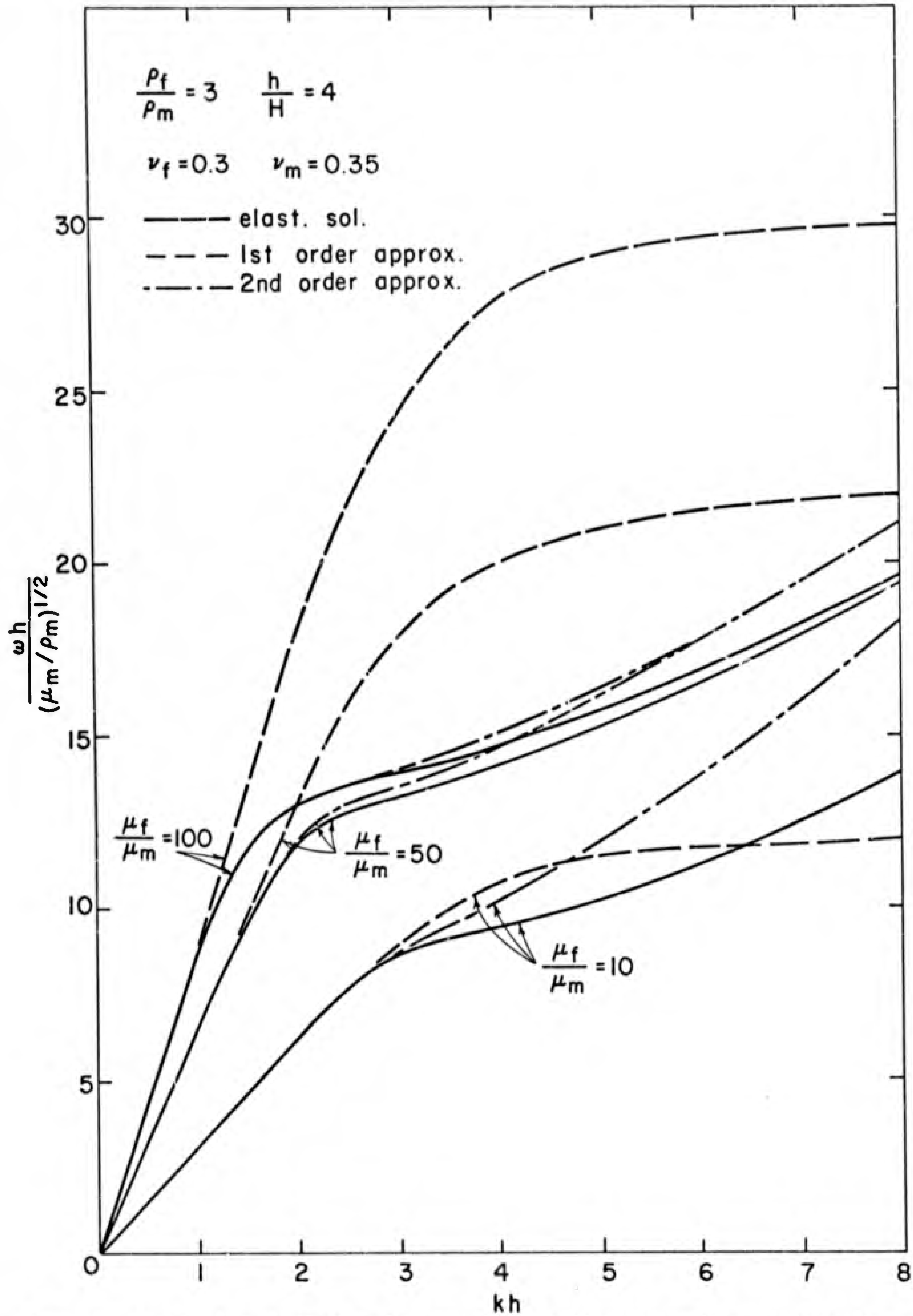

Fig. 2 Lowest longitudinal mode according to effective double stiffness theory

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It is shown that the Cosserat continuum and the theory of elasticity with micro-structure can be interpreted as analytical models describing the dynamic behavior of a composite material. The nonclassical material constants are simply functions of the geometry and the classical constants of the two materials constituting the composite. The study of wave propagation in a laminated composite reveals that a more complex micro-structure needs to be introduced in a continuum in order to describe adequately the dispersive character of (essentially) longitudinal waves.

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14. KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
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