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INSTITUTE OF TECHNOLOGY

AIR UNIVERSITY

UNITED STATES AIR FORCE

SCHOOL OF ENGINEERING

THESIS

WRIGHT-PATTERSON AIR FORCE BASE, OHIO

AF-WP-B-SEP 60 5M

A STUDY OF
PARAMETRIC AMPLIFICATION

THESIS

Presented to the Faculty of the School of Engineering of
the Institute of Technology
Air University
in Partial Fulfillment of the
Requirements for the
Master of Science Degree
in Electrical Engineering

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Graduate Electronics
30 August 1961

AF-WP-O-NOV 61 20

Preface

About a year ago, Major Everette Garrett suggested this topic to the Class of GE-61. I had casually noticed parametric amplifiers mentioned in different articles and was just slightly interested in the subject. However, before I undertook to write a thesis on Parametric Amplifiers, I felt it might be wise to locate an elementary article on the subject and become familiar with some of the basic aspects of parametric amplification. I combed the library for an article that explained the subject and I must admit that I am still looking for one that gives a good clear, elementary approach. The more I searched, the more convinced I became of the need for such a work; there was plenty of information on the subject, but most of the articles assumed the reader had a solid background in the principles of parametric excitation and negative resistance amplification. I sincerely hope that this thesis will partially fill that void and that it will provide an aid in attacking the many advanced articles on this subject.

I can't begin to thank Major Garrett, my faculty advisor, for all his assistance and encouragement. He certainly made this thesis a great deal less painful than it would have been without his aid. Thanks, too, to my ASD sponsor, Mr. William Eppers, and to Captain Frank Brown for their helpful discussions.

My wife deserves a special thanks for her patience throughout the past year and a half . . . and for proof-reading this thesis.

Matt Quinn

Abstract

The parametric amplifier is studied beginning with several mechanical models and progressing to several parallel resonant electrical circuits. Two regenerative parametric amplifiers, the degenerate and the non-degenerate case, are analyzed by linear circuit theory and compared to the negative resistance amplifier. The Manley-Rowe energy equations are then derived by a simplified method. Finally, the degenerate case of the parametric amplifier is analyzed by the Mathieu Equation and on the analog computer.

Contents

A STUDY OF PARAMETRIC AMPLIFICATION

I. Introduction

This report contains a basic explanation and an analysis of parametric amplification. It should provide the background necessary for a student's understanding of the current literature in this field. The aim of the thesis is to give the reader an aid in approaching and understanding advanced articles on this subject; these articles are presently appearing in ever-increasing numbers as the parametric amplifier is improved. Because the field of parametric amplification is literally "exploding", the student should have a basic understanding of the principles of this phenomenon.

Parametric amplifiers are also called reactance amplifiers and MAVARS (Modulator Amplifier by Variable Reactance); however, only the first name will be used in this report.

Basically, a parametric amplifier is an electronic circuit that uses the nonlinear characteristics of a storage element of a resonant circuit to amplify an input signal. This nonlinear storage element is the variable parameter of the amplifier circuit and, hence, the name "parametric" amplifier is used. The underlying theory is quite simple: if two circuits which are resonant at frequencies, ω_1 and ω_2, respectively are coupled through a nonlinear reactance which varies sinusoidally in time, then power of one of the frequencies may be converted to power of the other frequency. The nonlinear capacitor or inductor is the heart of the parametric amplifier because it is responsible for this power conversion from one frequency to the other. Without this element, there would be no amplification.

The principles of parametric excitation are certainly not new; they have been explored and studied for well over one hundred years. Yet in the past decade there has been a tremendous surge of interest in this phenomenon. Actually there are two reasons for this interest: first, parametric amplifiers offer exceptionally low noise qualities; and second, the nonlinear storage elements that are required for this type of amplifier have only recently reached a stage of development that makes parametric amplification feasible.

The noise generated in an ordinary amplifier is caused by the random motion of electric charges passing through the passive circuit elements, the tubes, and the transistors. There is an inherent nonuniformity in these so-called "streams" of charges, and this causes an erratic current which is called noise. The parametric amplifier overcomes this trouble to a great extent because there is no charge transportation through a high-impedance substance. Negligible noise is generated in a nonlinear capacitor because its carriers move such a minute distance. For example, the carriers in a junction diode move about 10^{-4} millimeters during parametric amplification, and such a slight movement generates very little noise. It can be stated somewhat loosely in this way: the mechanism of amplification depends upon reactance rather than resistance for energy conversion. Reactance is a noiseless quantity while resistance is inherently noisy.

One important difference to keep in mind about parametric amplifiers is that radio frequency energy is used as a source of power in amplification, whereas in klystrons, for example, direct current energy is used for the source of energy.

With this in mind, the parallel resonant form of the parametric amplifier is analyzed. The results of this analysis are compared with the known results of a normal regenerative amplifier. In addition, the gain and bandwidth are studied in some detail.

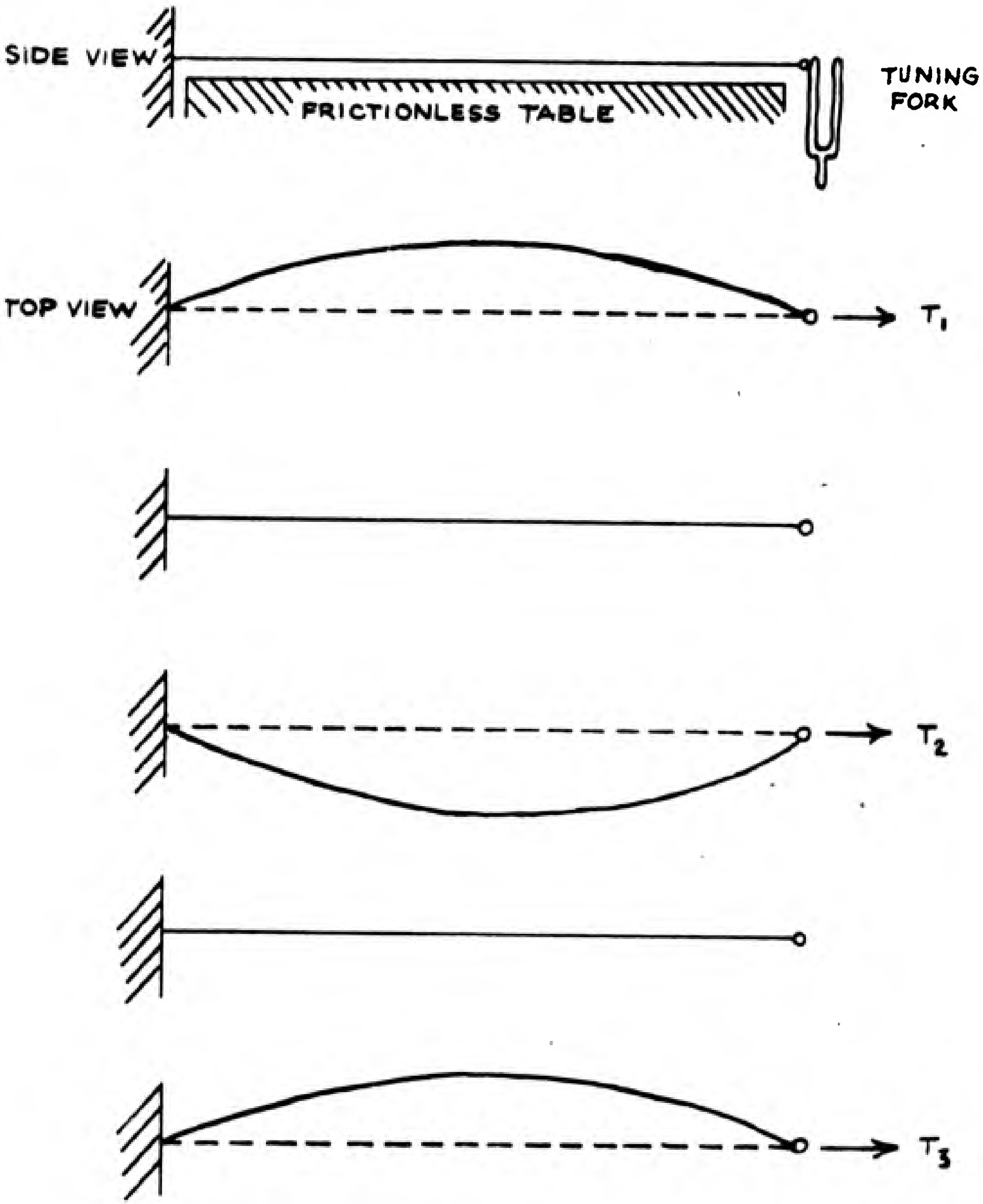


FIGURE 2-1

II. Historical Background

The study of parametric excitation is certainly not new. As early as 1831, Faraday presented a paper on this subject to the Royal Society; his experiment dealt with the vibrations of the particles on the surface of water in a large wine glass. His results would be quite difficult to verify in such an experiment. (Ref 1) However, in 1859 Melde demonstrated these same principles in an experiment that vividly shows parametric oscillation. Figure 2-1 shows a set-up similar to Melde's vibrating string experiment. Melde used a string attached to the vibrating prong of a tuning fork on one end and to a fixed body on the other end. (In this example the string is supported by a frictionless table in order to neglect the weight of the string.) The tuning fork places tension, T, on the string periodically; this tension is applied on the string at the natural frequency of the fork. If the string is initially displaced, then the tension, T_1, will tend to stabilize the string. However, momentum will force the string to continue moving past its undisturbed position. Again when the string has moved to a maximum position in the other direction, the tension, T_2, will tend to stabilize it. As before, its momentum will carry it through until the tension, T_3, has a chance to exert a force on the string. In this manner the

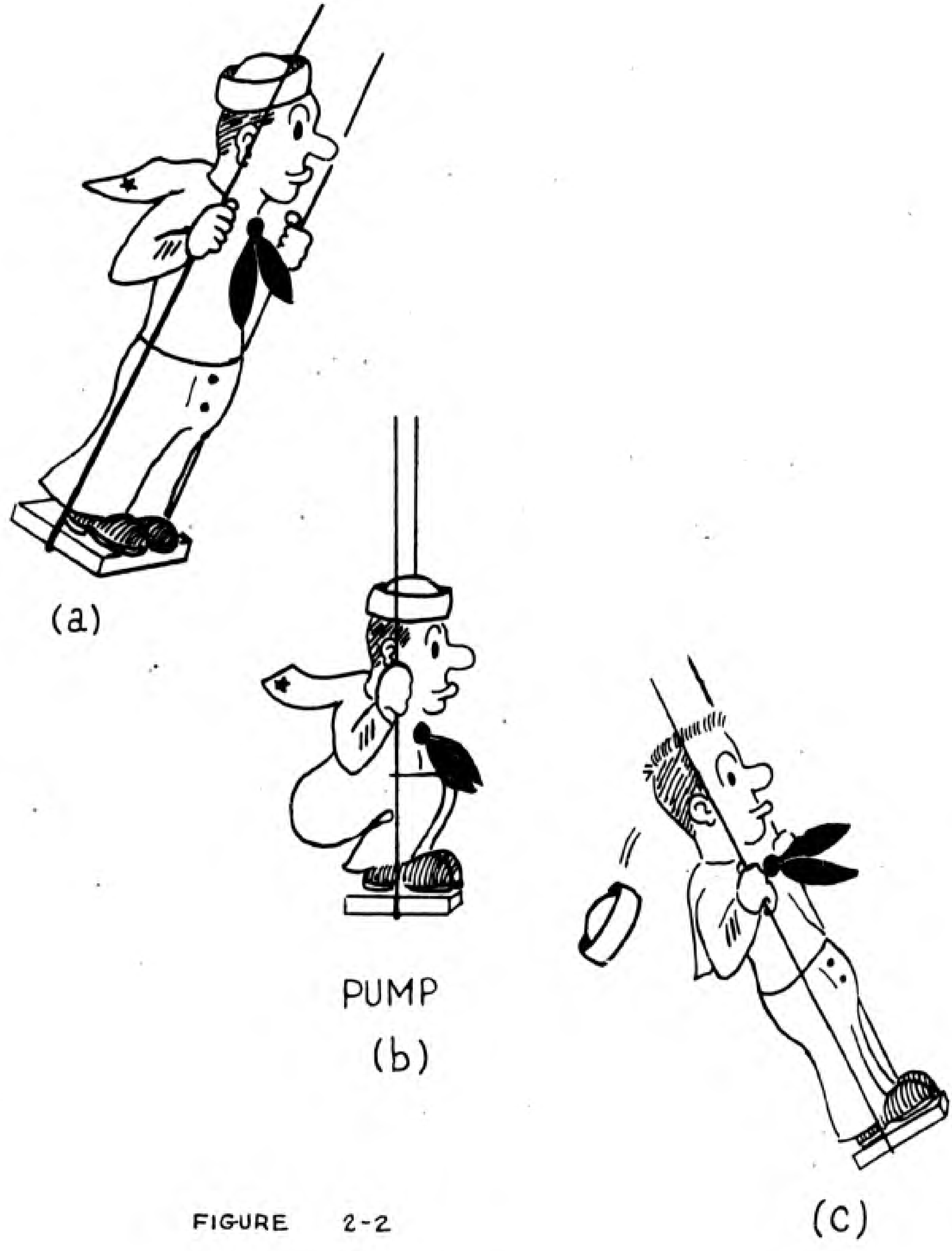


FIGURE 2-2

string maintains oscillation by parametric excitation. It should be noted that the string oscillates at a frequency that is half the frequency of the tuning fork. Stated in other words: for every cycle of motion of the string, the prong of the tuning fork completes two cycles of motion. One other important thing to note is that the oscillation of the string is entirely dependent on the tuning fork. If the tuning fork should be removed, the motion of the string would completely die out due to the losses present in any actual system. The tuning fork adds energy to the system to overcome these losses and, hence, oscillation will continue as long as the tuning fork adds this energy. (Ref 2)

It might prove helpful at this time to look at another example of parametric excitation. The ordinary child's swing operates on these principles. After receiving an initial push a swing can continue to oscillate by a child merely changing his center of gravity with respect to the swing. The child acts much like the tuning fork in the last example because the child adds energy to the system by raising his body; when he raises his body, he is, in effect, simply raising the center of gravity of the swing. Figure 2-2 shows a child "pumping" a swing. When the child stands erect at either end of the arc, the system has more potential energy than if the child was hunched. At the extreme end of the arc, the swing has no kinetic energy; all of its energy is potential energy. Consequently, if the child stands erect and, so to speak, places energy into the system, the swing can oscillate. The child actually overcomes the losses in the system by standing erect. On the other hand, the acceleration of the swing, as with any compound pendulum, is directly proportional to the length of the pendulum. When the child hunches during the pumping of the swing, the length of the pendulum and, in turn, the acceleration of the pendulum is increased. The child's pumping is analogous to the prong of the tuning fork in the previous example.

In a parametric amplifier, the portion of the circuit that supplies the energy is commonly called the "pump". The name is quite logical when the parametric amplifier is compared to the swing, because the child's pumping is responsible for the parametric action. What would happen if the child did not pump; in other words, what would happen if the child remained rigid throughout the swinging? The swing would merely damp out to rest due to the losses in the system. Now to return to the historical background.

Beginning in the 1880's, Lord Rayleigh published several works on parametric excitation. His contribution to

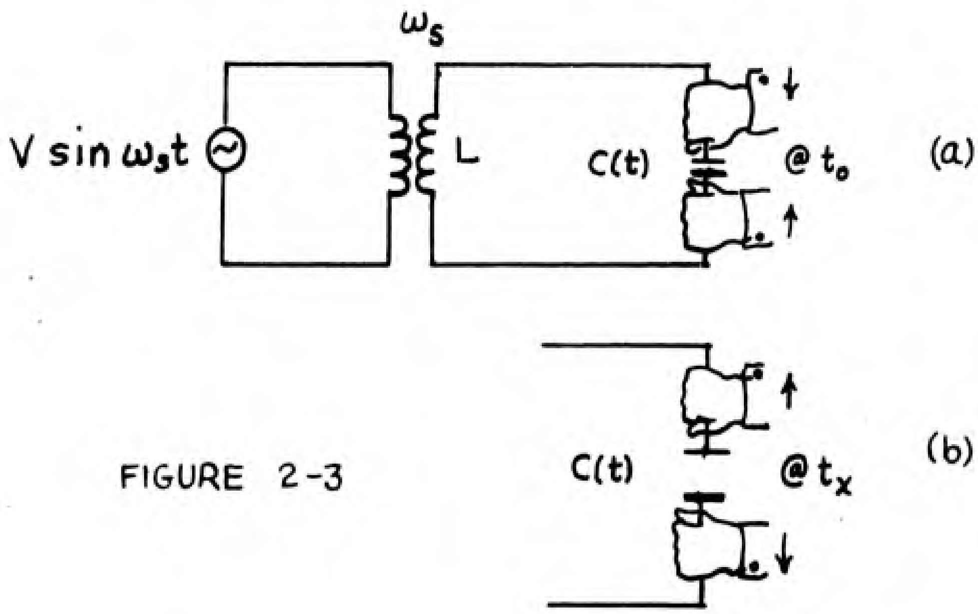


FIGURE 2-3

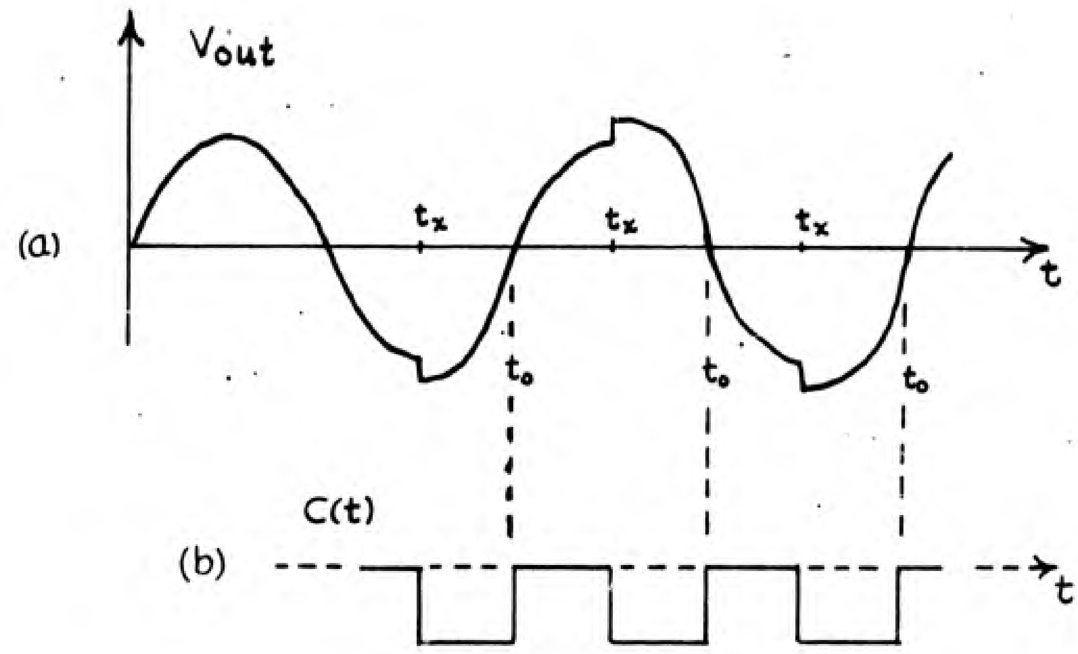


FIGURE 2-4

this field was merely a more detailed analysis of the works of Faraday and Melde. (Ref 3) The first parametric amplifier that closely resembles the amplifiers of today was not made until 1936. At that time Hartley proposed a resonant circuit with a capacitor having moveable plates. A picture of this is in figure 2-3. By moving the plates of the capacitor in and out at twice the frequency of resonance of the tank circuit, a signal can be amplified. (Ref 4)

In detail, the system operates in this manner: the hands pull apart the plates of the capacitor and thereby change the value of the capacitance.

$$C = \frac{\epsilon A}{d} \qquad (2\text{-}1)$$

where ϵ : dielectric constant in farads/meter

A : area of the capacitor plates in meters^2

d : distance between the plates in meters

The only element in this equation that varies is the distance between the plates. Therefore, the capacitence varies inversely as the distance between the plates varies; the capacitance decreases as the plates are pulled apart and it increases as the plates are pushed together.

The charge on the plates, Q, cannot change instantaneously, so that when the capacitance changes, the voltage, V, must change immediately.

$$V = \frac{Q}{C} \tag{2-2}$$

Referring to figure 2-3 (a), the plates are brought back to their closed position at time, $t = t_0$. In figure 2-3 (b) the plates are separated at time, $t = t_x$. The resonant circuit has a sinusoidal output when the plates are left in their normal position. Then, at $t = t_x$ the plates are separated and work is done on the circuit overcoming the attraction of the plates. The hands in this illustration are basically the pump, and the hands supply the energy to the electrical circuit. At $t = t_0$, the plates are returned to their closed position and there is no work done at this time because the voltage across the plates is zero. This process is continued as shown in figure 2-4.

If the output of a device is greater than its input, that device can be considered an amplifier regardless of its use. In this respect, the circuit is an amplifier. However, since this output in figure 2-4 (a) can be amplified by merely increasing the amplitude of the pumping,

PUMPING IN PHASE
W GREATER AMPLITUDE

-- AMPLIFICATION

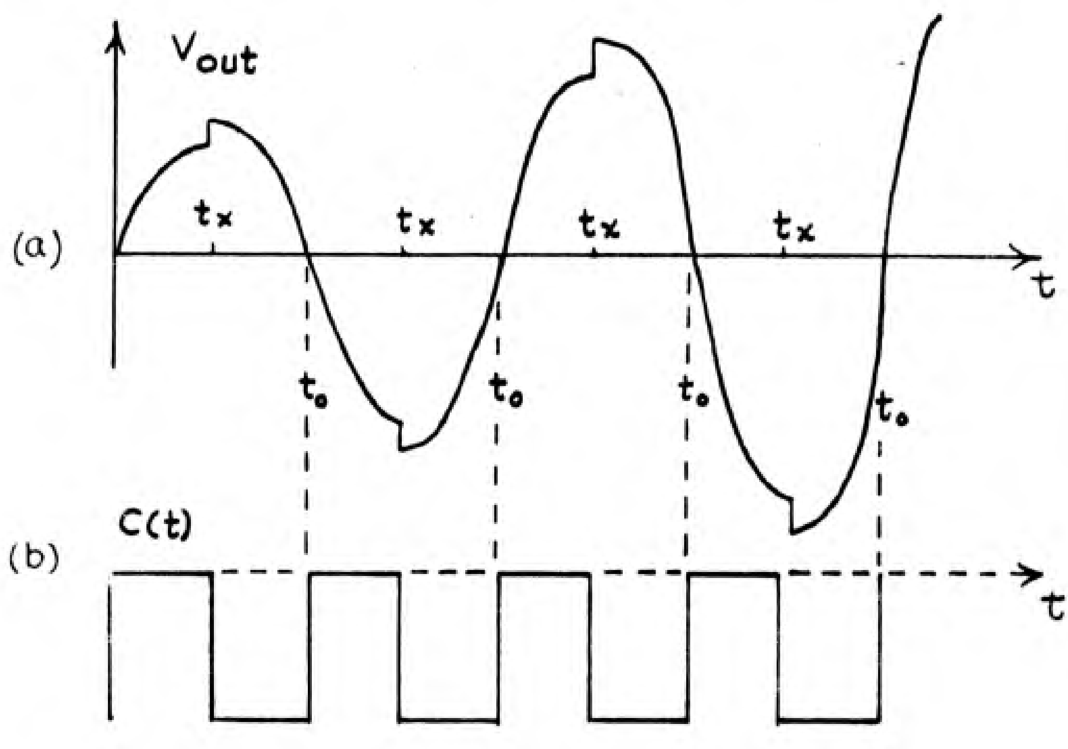


FIGURE 2-5

consider this output to be simply oscillation. Therefore, figure 2-4 (a) illustrates parametric oscillation, and, of course, there is a good deal of distortion present.

Amplification can be obtained from the same circuit by merely increasing the displacement of the two plates. By pulling out and pushing in the plates farther, the output will continuously increase. Theoretically the voltage would increase without bound, but in practical cases the circuit would break down before this would happen. Actually, the entire pumping sequence is just a repetition of the case in figure 2-4. In the case of amplification in figure 2-5, the amplitude of C(t) is greater than in the previous case.

In the case of attenuation of the output, there is a noticeable change. This is illustrated in figure 2-6. The amplitude of C(t) is not important in this case, but rather the phase of the pumping is the critical factor. In this case, the plates are brought together when the charge is a maximum. Therefore, no work is done on the system; in fact, work is done by the attraction of the charges in bringing the plates together. The plates are brought together at time, $t = t_x'$, which is the same as t_o in figure 2-3. On the other hand, the plates are separated at time, $t = t_o'$, when there is no voltage on the plates and, hence, there is no work done.

PUMPING OUT OF PHASE
-- ATTENUATION

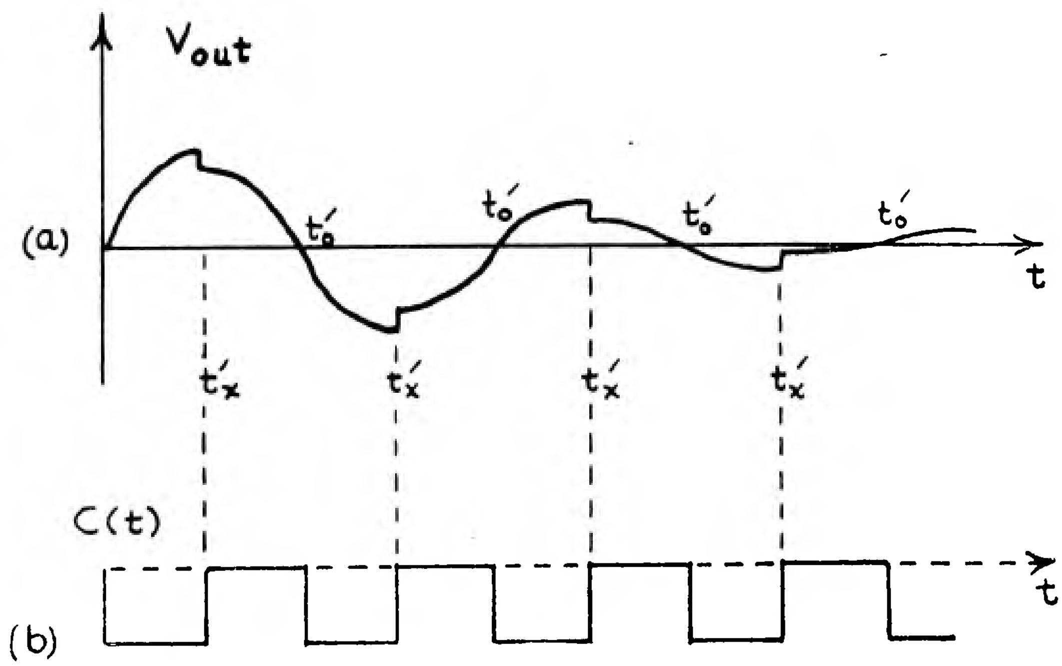
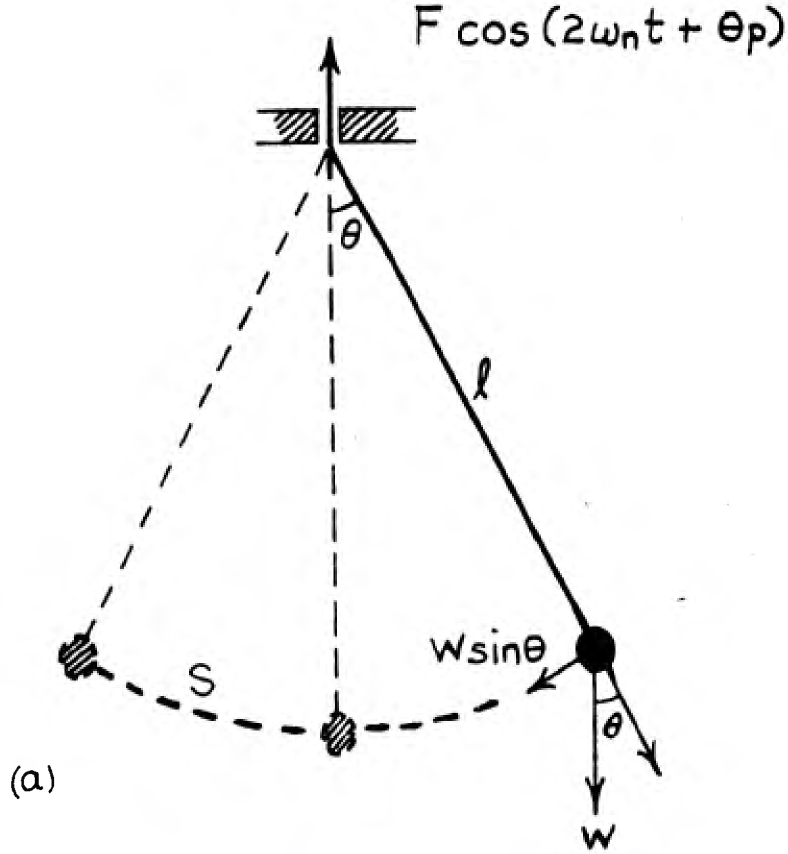


FIGURE 2-6

Since the charge does some internal work on the system while no external work is applied to the system, the voltage decreases. This shows that attenuation is produced by merely changing the phase of the pump. For this reason, the phase is quite critical in this type of parametric amplifier.



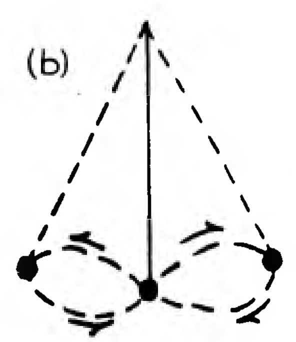


FIGURE 3-1

III A. An Example of Parametric Oscillation

A pendulum bob of weight, $W=mg$ is attached to a weightless string which is movable at a frequency that is twice the frequency of oscillation of the pendulum. The equation of motion for a simple pendulum is

$$F = ma \tag{3-1}$$

where the force: $F_\theta = -W\sin\theta$

the mass: m

the acceleration: a

where $S \cong \ell\theta$; ℓ is the length of the pendulum.

$$m\frac{d^2S}{dt^2} = -W\sin\theta \tag{3-2}$$

$$m\ell\frac{d^2\theta}{dt^2} = -mg\sin\theta \tag{3-3}$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0 \tag{3-4}$$

When the angular displacement is restricted to small values of θ , the sine of the angle can be replaced by the angle.

$$\sin\theta \cong \theta \quad \text{for small } \theta \tag{3-5}$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0 \tag{3-6}$$

$$\frac{d^2\theta}{dt^2} + \omega_n^2\theta = 0 \tag{3-7}$$

Since the natural frequency of the simple pendulum, $\omega_n = \sqrt{\frac{g}{\ell}}$, is independent of the mass of the bob, the frequency depends on the length alone. The acceleration due to gravity is considered to remain constant.

However, the pendulum depicted in figure 3-1 (a) is not a simple pendulum because its length is continuously changed by a force which is acting at twice the natural frequency of oscillation of the pendulum. The motion of the pendulum bob is shown in figure 3-1 (b). Now equation (3-7) no longer describes the motion of this pendulum system. The new equation of motion is

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell(t)}\theta = 0 \qquad (3\text{-}8)$$

where $\ell(t) = \ell_o + \Delta\ell \cos(2\omega_n t + \theta_p)$

and $\ell_o \gg \Delta\ell$

This can be shown to be of the form

$$\frac{d^2\theta}{dt^2} + \omega_n^2\left(1 - 2\frac{\Delta\omega}{\omega_n}\cos 2\omega_n t\right)\theta = 0 \qquad (3\text{-}9)$$

The entire derivation is illustrated in Chapter XII using an electrical analog of the pendulum.

NOTE: The approximation that $S \cong \ell\theta$ is not changed to $S \cong \ell(t)\cdot\theta$ in this case because the variation in ℓ is very small. In other words, $\ell_o \gg \Delta\ell$.

In order to understand how parametric oscillations are produced, it is best to inspect the tension of the string throughout the pendulum swing. When θ is a maximum value, i.e. at either the right or left extreme position of the swing, the tension is the weight of the bob multiplied by $\cos\theta$, which is somewhat less than unity.

$$T\Big|_{\theta = \theta_{MAX}} = W\cos\theta \tag{3-10}$$

On the other hand, when θ is zero, i.e. when the bob goes through the center, the tension is the weight of the bob and the centrifugal force of the bob moving in its curved path. Therefore tension is somewhat greater at this point than at the extreme positions.

$$T\Big|_{\theta = 0} = W + \frac{Wv^2}{g\ell} \tag{3-11}$$

The above analysis is the same as for a simple pendulum. If the force that changes the length of the string was not present, the analysis would be complete. If the pendulum bob was given an initial displacement, the pendulum would swing until the oscillations died down to zero meaning that the bob would come to rest. The motion of the pendulum would be damped by the frictional or resistive forces present in the system.

However, because a force is present that changes the length of the string, the pendulum can continue oscillating indefinitely. The force that acts on the string supplies the energy to overcome the frictional or resistive forces. One important point to observe is that the force is applied at a frequency that is twice the natural frequency of oscillation of the pendulum. The importance of this can not be overemphasized, yet the reason for it is rather basic. The force acts on the string and pulls up on the string when the bob is at the center. Conversely, the string is let down when the bob is at the extreme position. The force, therefore, pulls up against a large tension and lets the string down when the tension is smaller. In this way work is put into the system and this work is converted into energy to overcome the resistive forces that are present. If there is more energy supplied to the system than is needed to overcome the resistive forces, then the pendulum will absorb the energy by increasing its swing. If the pump, as the force is often called, supplies more energy than necessary, then the bob will increase its swing until the system stabilizes. When this occurs the system can be considered a parametric amplifier.

III B. A Comparison with The Electrical Analog

In chapter II, Hartley's experiment with a variable capacitor was explained in detail. The mechanical pendulum system is a mechanical analog of that case. If the pendulum was "pumped" with a rectangular wave instead of a sinusoidal wave, then the three illustrations could be used to portray the angular displacement instead of the voltage. The variation called C(t) in figures 2-4, 2-5, and 2-6 would then be called F(t) in the pendulum system.

For the first case, figure 2-4, the pendulum is pumped with a force of the proper magnitude and phase in order to sustain the oscillations of the pendulum. The frequency of F(t) is exactly twice the frequency of the oscillations, which is expected. This case is parametric oscillation.

In the second case, figure 2-5, the same phase is retained, but the magnitude of F(t) is increased. The bob will increase its swing until the system stabilizes. In this case there is parametric amplification.

Finally in the third case, figure 2-6, the phase is changed by 90° and attenuation results. When this happens, energy is being removed from the system by the pump. The force drops the string when the bob is at the center and pulls on the string when the bob is at an extreme position of the swing. The force lets the string down when the tension is greatest and pulls up on the string when the tension is somewhat smaller. In this way the oscillations are reduced as pictured in figure 2-6 (b).

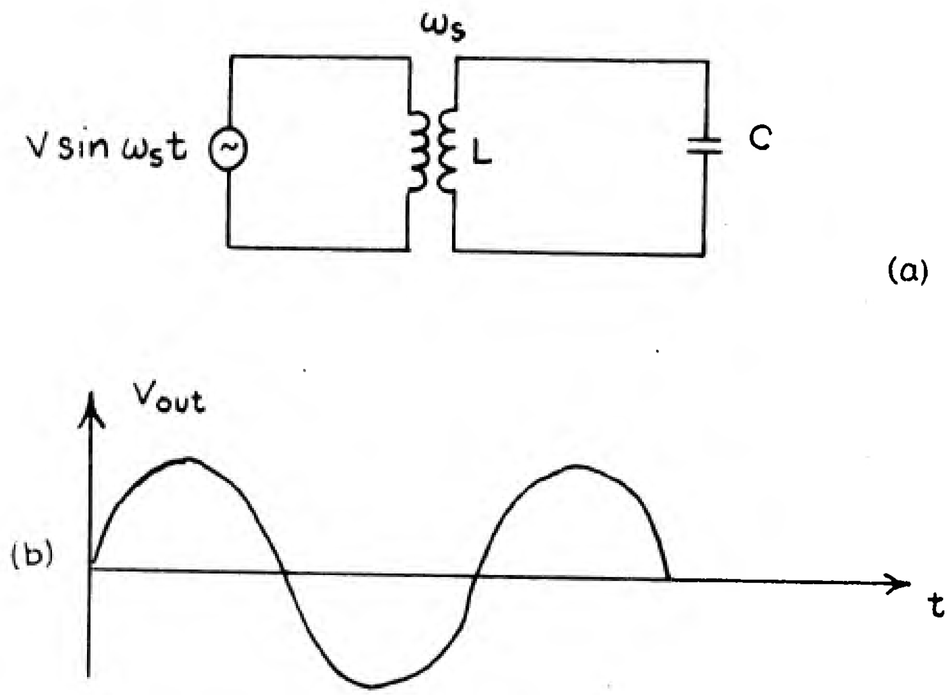


FIGURE 4-1

IV. A Review of Oscillation

Because there are several resonant tank circuits in most parametric amplifiers, a short review of tuned circuit oscillation is presented. The response of a normal tank circuit is included in this thesis so that it can be compared to the response of a tank circuit with a time-varying storage element.

An ideal resonant tank circuit is shown in figure 4-1. When a voltage is applied to the circuit, the tank will resonate at $\omega_S = \frac{1}{\sqrt{LC}}$. The differential equations can be written with either the charge, the voltage, or the current as the dependent variable. The charge is used in this review so that no integro-differential equations result. The driving functions of all of the differential equations will be a constant current placed in the circuit at time, t = 0.

The differential equation of the L-C circuit in figure 4-2 (a) is

$$L \frac{di}{dt} + \frac{1}{C}\int_0^t i \, dt = 0 \qquad (4\text{-}1)$$

and since $i = \frac{dQ}{dt}$ and $V = \frac{Q}{C}$, the expression can be changed to

$$L \frac{d}{dt}\left(\frac{dQ}{dt}\right) + \frac{1}{C}\int_0^t \frac{dQ}{dt} dt = 0 \qquad (4\text{-}2)$$

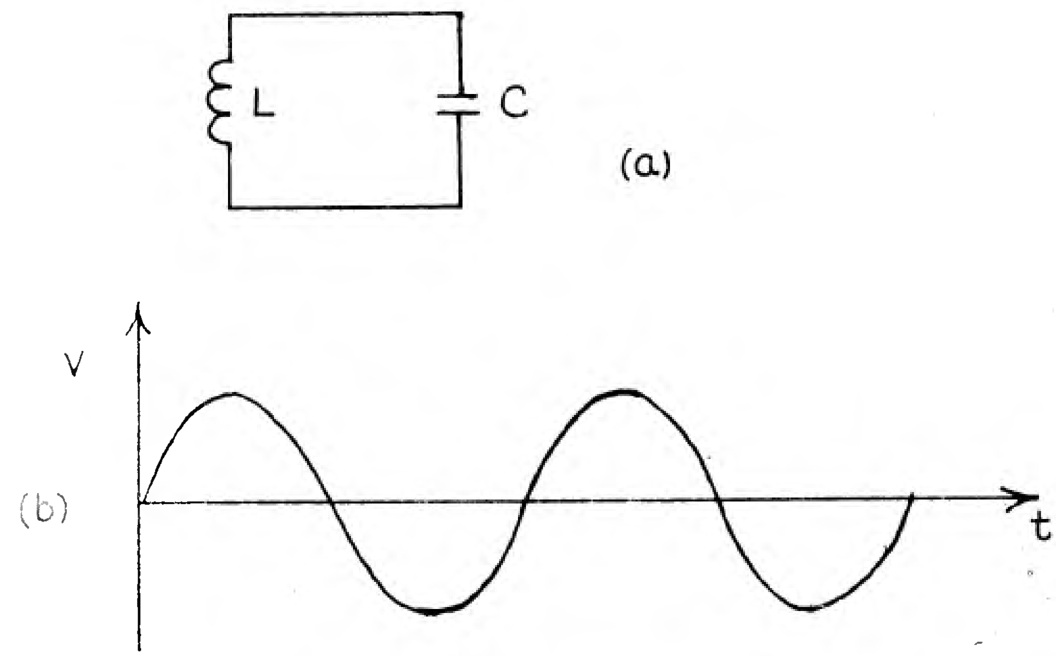


FIGURE 4-2

$$L \frac{d^2Q}{dt^2} + \frac{Q}{C} - \frac{Q(0)}{C} = 0 \qquad (4\text{-}3)$$

$$\frac{d^2Q}{dt^2} + \frac{Q}{LC} = \frac{Q(0)}{LC} = \frac{V(0)}{L} \qquad (4\text{-}4)$$

The homogeneous equation is

$$\frac{d^2Q}{dt^2} + \frac{Q}{LC} = 0 \qquad (4\text{-}5)$$

In this case, sustained oscillations will result since there is no resistance in the ideal circuit of figure 4-2 (a). A graph of the solution is shown in figure 4-2 (b). The solution is of the form

$$v(t) = \frac{q(t)}{C} = V_m \sin(\omega_n t + \theta_d) \qquad (4\text{-}6)$$

where $\omega_n = \frac{1}{\sqrt{LC}}$ and $\theta_d = 0$

In any actual tank circuit there is resistance present; the previous case depicted an ideal case. Figure 4-3 (a) shows an R-L-C circuit and the homogeneous differential equation of such a circuit is

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = 0 \qquad (4\text{-}7)'$$

In order to investigate the solution of this equation, the differential equation should be compared to the normal form of the second order differential equation.

$$\frac{d^2Q}{dt^2} + 2\zeta\omega_n\frac{dQ}{dt} + \omega_n^2 Q = 0 \qquad (4\text{-}8)$$

The coefficients of the two equations, (4-7) and (4-8), can be equated.

$$\omega_n^2 = \frac{1}{LC} \qquad \omega_n = \frac{1}{\sqrt{LC}}$$

$$2\zeta\omega_n = \frac{R}{L} \qquad \zeta = \frac{R}{2\omega_n L} = \frac{1}{2Q}$$

since the quality factor, $Q = \dfrac{\omega_n L}{R}$

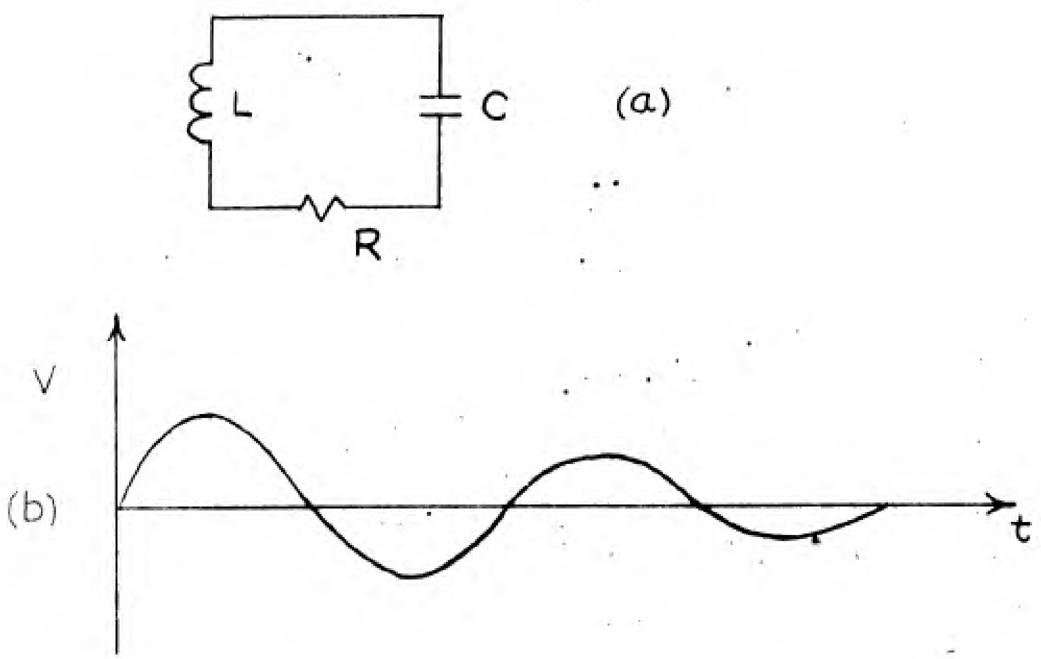


FIGURE 4-3

The general solution of this equation is

$$v(t) = \frac{q(t)}{C} = V_m e^{-\zeta\omega_n t} \sin(\omega_d t + \theta_d) \qquad (4\text{-}9)$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\theta_d = 0$

and this, in turn, can be written

$$v(t) = V_m e^{-\frac{R}{2L}t} \sin(\omega_d t + \theta_d) \qquad (4\text{-}10)$$

In this case, the solution is a damped sinusoid as pictured in figure 4-3 (b). The voltage will "die" down due to the exponential damping of the resistance, R. If R = 0, as in figure 4-2, then sustained oscillations result, and this was found to be true in that case.

If by some means a negative resistance could be introduced in the tank circuit as shown in figure 4-4 (a), then a still different type of solution would result. The homogeneous differential equation of the R-L-C circuit with the negative resistance introduced in the circuit is

$$\frac{d^2Q}{dt^2} - \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = 0 \qquad (4\text{-}11)$$

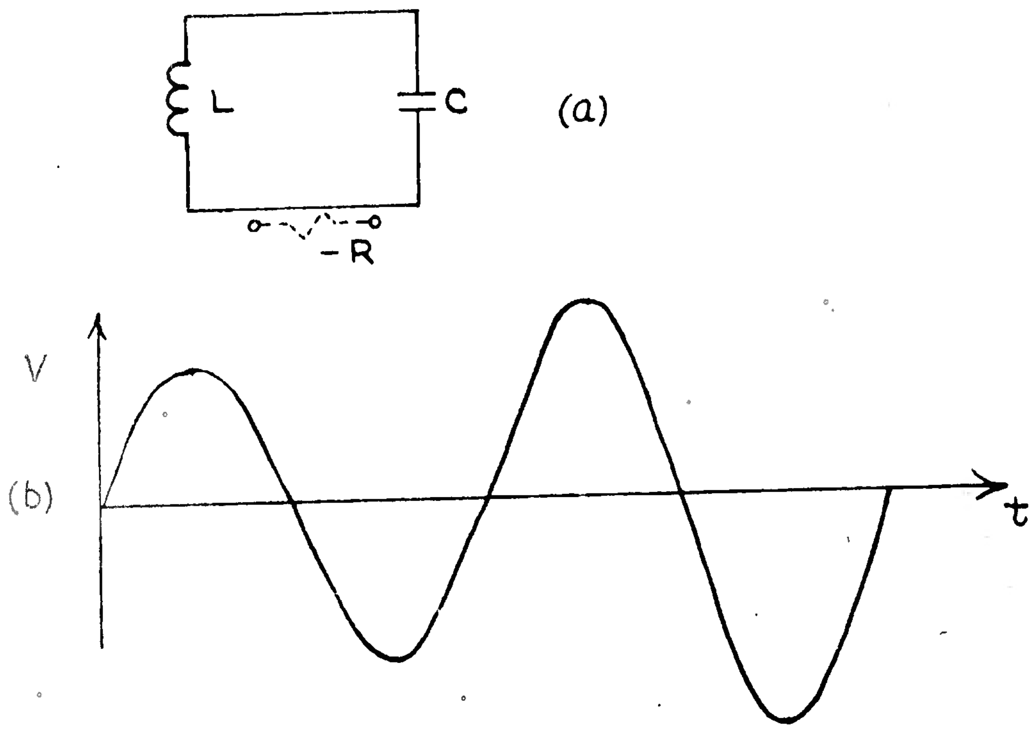


FIGURE 4-4

The general solution of this equation is similar to equation (4-10).

$$v(t) = V_m e^{+\frac{R}{2L}t} \sin(\omega_d t + \theta_d) \qquad (4\text{-}12)$$

In this case, however, the solution is an increasing sinusoid as shown in figure 4-4 (b). The voltage will continue increasing exponentially until it eventually becomes infinite.

The first two solutions, those depicted in figures 4-2 and 4-3, are both stable solutions, while the last solution in figure 4-4 is unstable. These concepts are quite important in understanding parametric amplification because the regenerative parametric amplifier uses negative resistance properties to obtain amplification. However, there is a method to obtain conditionally stable amplification from a circuit with negative resistance. This method is discussed in the following chapter.

NEGATIVE CONDUCTANCE

FIGURE 5-1

V. An Analysis of Parallel and Series Resonant Regenerative Amplifiers

The tank circuit of figure 5-1 (a) can be considered a regenerative or negative resistance amplifier. It contains a negative conductance, g_n. The tank circuit can be assumed to be operating at frequency, ω_1, so that the two reactance elements, L_1 and C_1, have no effect on the circuit. Furthermore, the ideal filter, $\boxed{\omega_1}$, is included in this circuit so that the circuit will have the same form as the parametric amplifier circuit; the two will be analyzed in a similar method.

The circuit of figure 5-1 (b) is equivalent to that of figure 5-1 (a) when the circuit is operating at frequency, ω_1.

$$\omega_1 = \frac{1}{\sqrt{L_1 C_1}} \tag{5-1}$$

The H matrix for the circuit of figure 5-1 (b) is

$$H = \begin{Vmatrix} 1 & 0 \\ G_{g_1} + G_1 + g_n & 1 \end{Vmatrix} \tag{5-2}$$

The power gain forward, ${}_pG_f$, of this circuit is

$$pG_f = \left| \frac{1}{CZ_L + D} \right|^2 \frac{r_L}{r_{in}} \tag{5-3}$$

where

$$Z_{in} = r_{in} = \frac{AZ_L + B}{CZ_L + D} = \frac{1}{G_{g_1} + G_1 + g_n + G_L} \tag{5-4}$$

$$Z_L = r_L = \frac{1}{G_L} \tag{5-5}$$

$$pG_f = \frac{G_L}{G_{g_1} + G_1 + g_n + G_L} \tag{5-6}$$

When the total conductance of the circuit is zero, oscillation should result. That means that

$$G_{g_1} + G_1 + g_n + G_L = 0 \tag{5-7}$$

When oscillation occurs, the power gain should be infinite. Equation (5-6) shows that this condition is met, because the denominator is zero at oscillation; thus the power gain is infinite. When the total conductance of the circuit is negative, there will be amplification, but the circuit will be unstable. This case was mentioned in Chapter IV.

$$G_{g_1} + G_1 + g_n + G_L < 0 \qquad (5\text{-}8)$$

However, stable amplification with a power gain greater than unity will result when

$$G_L > G_{g_1} + G_1 + g_n + G_L > 0 \qquad (5\text{-}9)$$

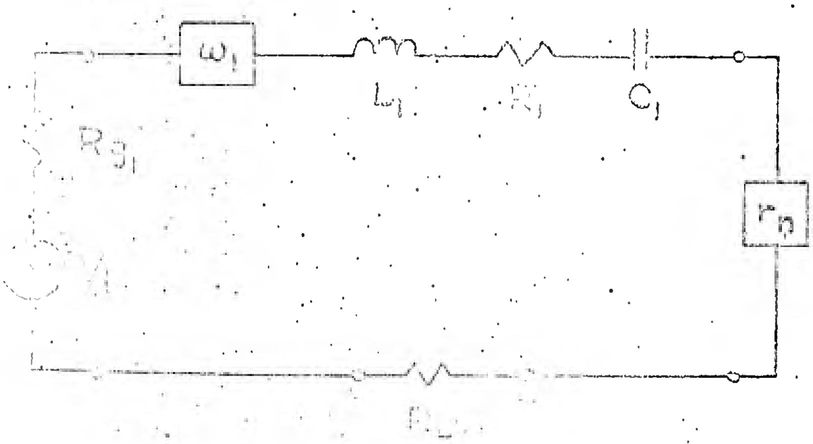
Thus

$$pG_f = \frac{G_L}{G_{g_1} + G_1 + g_n + G_L} > 1 \qquad (5\text{-}10)$$

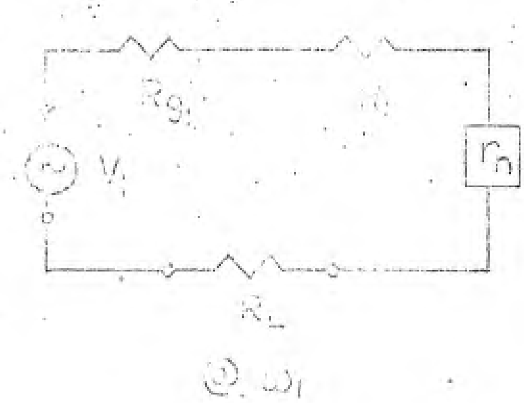
It is important to notice that this circuit is potentially unstable for large gain. This means that in order to realize large gains from such an amplifier, it must be operated in the oscillation threshold. This is a major disadvantage of the negative resistance amplifier. Nevertheless, this is one of the methods that is used in parametric amplifiers to achieve large gains.

Figure 5-2 (a) is the series resonant regenerative amplifier and is a dual of figure 5-1 (a). The power gain of

NEGATIVE RESISTANCE



(a)



(b)

FIGURE 5-2

this configuration can be found in a similar manner to be

$$pG_f = \frac{R_L}{R_{g_1} + R_1 + r_n + R_L} \tag{5-11}$$

The transducer gain of the amplifier, of figure 5-1, is found in order that it can be compared with the transducer gain of the parametric amplifier.

$$pG_t = \frac{P_{out}}{P_{in} \text{ available}} \tag{5-12}$$

$$P_{out} = V_1^2 G_L = \left[\frac{i_1}{G_{g_1} + G_1 + g_n + G_L}\right]^2 G_L \tag{5-13}$$

$$P_{in \text{ available}} = \frac{i_1^2}{4G_{g_1}} \tag{5-14}$$

$$pG_t = \frac{4G_{g_1} G_L}{(G_{g_1} + G_1 + g_n + G_L)^2} \tag{5-15}$$

$$pG_t = \frac{4G_{g_1} G_L}{(G_{T_1} + g_n)^2} \tag{5-16}$$

where

$$G_{T_1} = G_{g_1} + G_1 + G_L \qquad (5-17)$$

The similarity between the regenerative amplifier and the parametric amplifier will become apparent when the latter is analyzed. (Ref 5:104-6)

PARALLEL RESONANT CASE

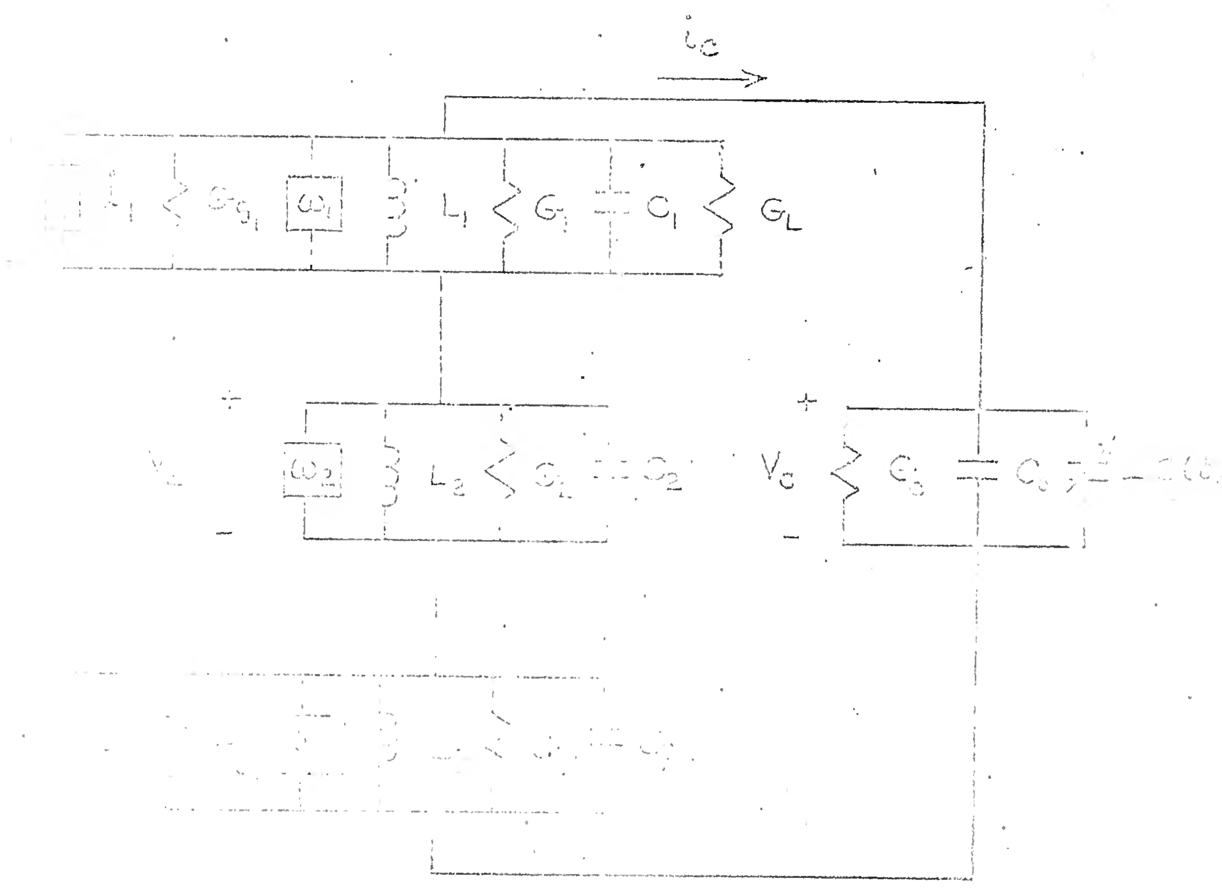


FIGURE B-1

VI. An Analysis of the Parallel Resonant Form of Parametric Amplifiers

A parallel resonant parametric amplifier is pictured in figure 6-1. There are three tuned circuits in the amplifier:

ω_1 : the signal frequency

ω_2 : the idler frequency

ω_p : the pump frequency

These are the names that are commonly used in the literature. The three frequencies are related by:

$$\omega_p = \omega_1 + \omega_2 \qquad (6\text{-}1)$$

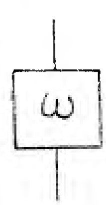
where

$$\omega_1 = \frac{1}{\sqrt{L_1(C_1+C_0)}} \qquad (6\text{-}2)$$

$$\omega_2 = \frac{1}{\sqrt{L_2(C_2+C_0)}} \qquad (6\text{-}3)$$

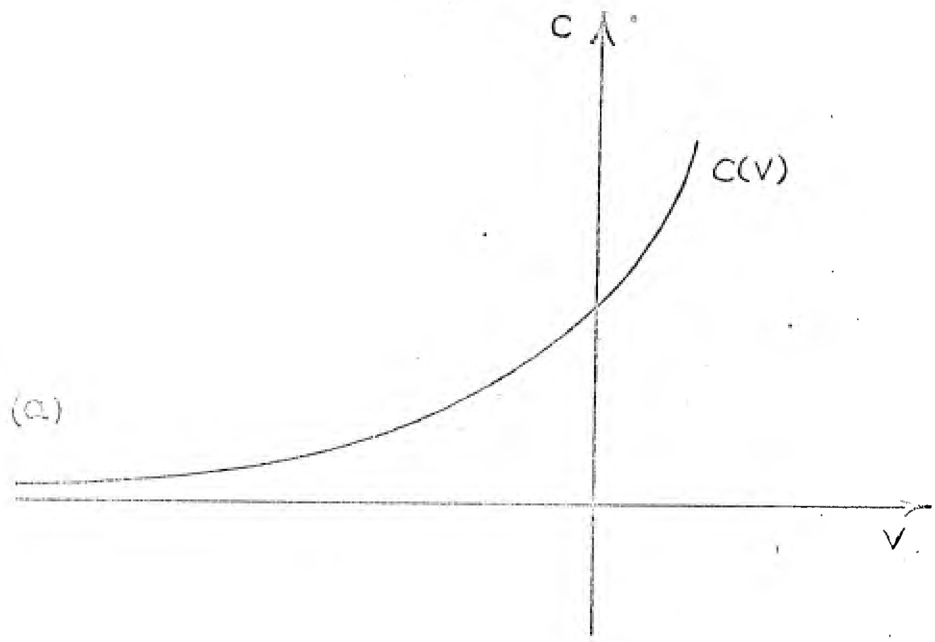
$$\omega_p = \frac{1}{\sqrt{L_p(C_p+C_0)}} \qquad (6\text{-}4)$$

The load conductance, G_L, is placed in the signal circuit, and in each of the three branches there is an ideal filter. The properties of the filter are:



$$Y = 0 \quad @ \quad \Omega = \omega$$

$$Y = \infty \qquad \Omega \neq \omega$$



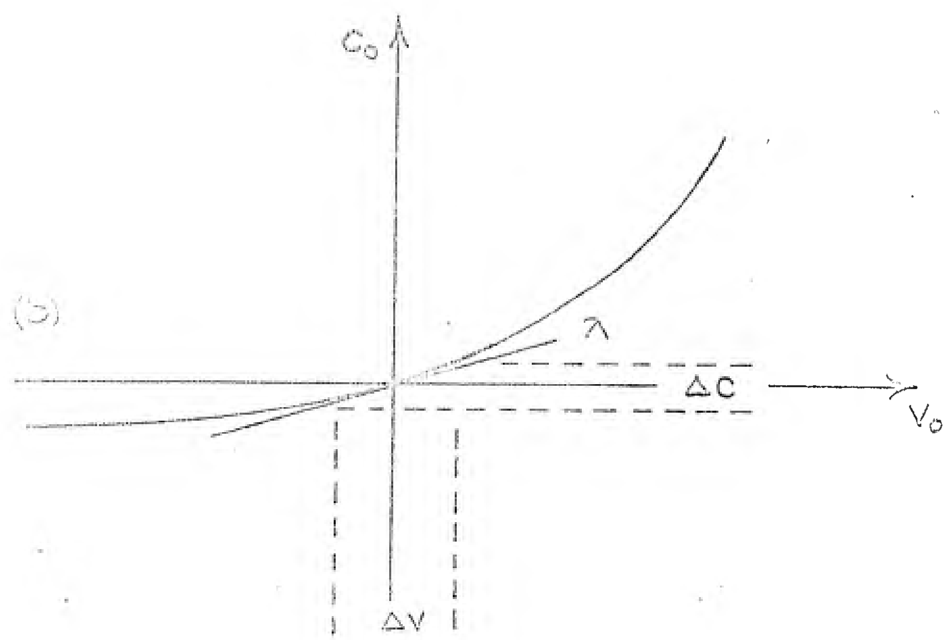


FIGURE 6-2

The current through the nonlinear capacitor, i_c, is found by differentiating the charge.

$$i_c = \frac{dQ}{dt} \tag{6-5}$$

where

$$Q = C \cdot v \tag{6-6}$$

$$i_c = \frac{dQ}{dv} \cdot \frac{dv}{dt} \tag{6-7}$$

$$\frac{dQ}{dv} = C_0 + \lambda v \tag{6-8}$$

$$\lambda = \frac{dC}{dv} \tag{6-9}$$

so that

$$i_c = (C_0 + \lambda v) \frac{dv}{dt} \tag{6-10}$$

By a transformation of axis, C_0 can be deleted from the equation. This is done to simplify the derivation; the C_0 term will be reintroduced later in this chapter.

$$i_c' = \lambda v \frac{dv}{dt} \tag{6-11}$$

The voltage across the nonlinear capacitor, V_c, is made up of many frequencies due to the mixing qualities of such a nonlinear element. However, there are only three frequencies of interest and these are the only ones considered; all the other frequencies have negligible effect on the circuit.

$$v_c(t) = V_1 \cos(\omega_1 t + \theta_1) + V_2 \cos(\omega_2 t + \theta_2) + V_p \cos(\omega_p t + \theta_p) \qquad (6\text{-}12)$$

The voltage is somewhat difficult to work with in this form. Therefore, the voltage will be converted into a contra-rotating vector form.

$$v_c(t) = \frac{V_1}{2} e^{j\theta_1} e^{j\omega_1 t} + \frac{V_1}{2} e^{-j\theta_1} e^{-j\omega_1 t} + \frac{V_2}{2} e^{j\theta_2} e^{j\omega_2 t} + \frac{V_2}{2} e^{-j\theta_2} e^{-j\omega_2 t} + \frac{V_p}{2} e^{j\theta_p} e^{j\omega_p t} + \frac{V_p}{2} e^{-j\theta_p} e^{-j\omega_p t} \qquad (6\text{-}13)$$

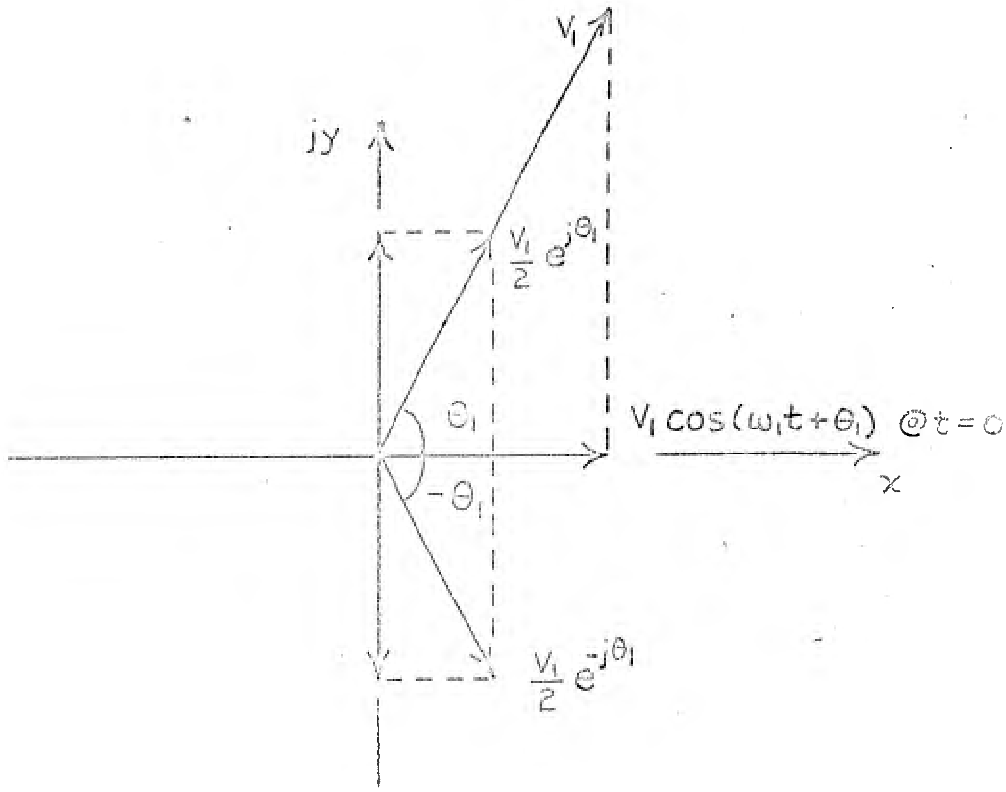

V_1
jy
$\frac{V_1}{2} e^{j\theta_1}$
θ_1
$V_1 \cos(\omega_1 t + \theta_1)$ @ $t = 0$
x
$-\theta_1$
$\frac{V_1}{2} e^{-j\theta_1}$

FIGURE 6-3

The two equations, (6-12) and (6-13), are equivalent. Figures 6-3 and 6-4 are included in order to clarify this equivalence. There are two simple methods of arriving at the contra-rotating vector form: first, directly from the exponential definition of the cosine;

$$\cos \omega_1 t = \frac{e^{j\omega_1 t}}{2} + \frac{e^{-j\omega_1 t}}{2} \qquad (6\text{-}14)$$

second, from the Euler Equations.

$$e^{j\omega_1 t} = \cos \omega_1 t + j \sin \omega_1 t \qquad (6\text{-}15)$$

$$e^{-j\omega_1 t} = \cos \omega_1 t - j \sin \omega_1 t \qquad (6\text{-}16)$$

Adding equation (6-15) to equation (6-16) gives

$$e^{j\omega_1 t} + e^{-j\omega_1 t} = 2 \cos \omega_1 t \qquad (6\text{-}17)$$

In figure 6-3, the real parts of $\frac{V_1}{2} e^{j\theta_1}$ and $\frac{V_1}{2} e^{-j\theta_1}$ add together to form a quantity equal to $V_1 \cos \Theta_1$. The imaginary parts of these two quantities cancel out. This

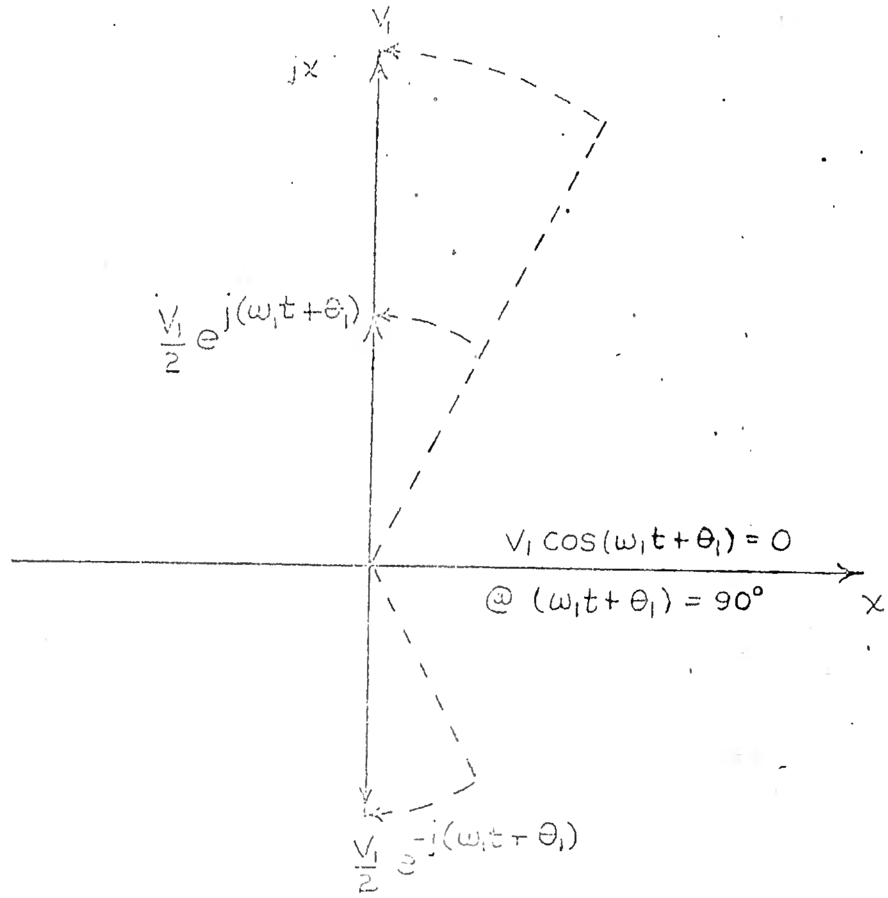

V_1
jx
$\frac{V_1}{2} e^{j(\omega_1 t+\theta_1)}$
$V_1 \cos(\omega_1 t+\theta_1) = 0$
@ $(\omega_1 t+\theta_1) = 90°$
x
$\frac{V_1}{2} e^{-j(\omega_1 t+\theta_1)}$

FIGURE 6-4

last statement is always true in contra-rotating form; the imaginary part of the two quantities always cancels.

In figure 6-4, the time variable has made $\frac{V_1}{2} e^{j(\omega_1 t + \theta_1)}$ and $\frac{V_1}{2} e^{-j(\omega_1 t + \theta_1)}$ such that both are purely imaginary quantities. Since the imaginary portions of the contra-rotating vectors always cancel, then there is nothing left. Since $\omega_1 t + \theta_1 = 90^\circ$ and $V_1 \cos 90^\circ = 0$, the two are equivalent.

Furthermore, these quantities are written in vector form such that,

$$\tilde{V}_1 = \frac{V_1}{2} e^{j\theta_1}$$

$$\tilde{V}_{-1} = \frac{V_1}{2} e^{-j\theta_1}$$

$$\tilde{V}_2 = \frac{V_2}{2} e^{j\theta_2}$$

$$\tilde{V}_{-2} = \frac{V_2}{2} e^{-j\theta_2}$$

$$\tilde{V}_p = \frac{V_p}{2} e^{j\theta_p}$$

$$\tilde{V}_{-p} = \frac{V_p}{2} e^{-j\theta_p} \qquad (6\text{-}18)$$

so that

$$v_c(t) = \tilde{V}_1 e^{j\omega_1 t} + \tilde{V}_{-1} e^{-j\omega_1 t}$$

$$+ \tilde{V}_2 e^{j\omega_2 t} + \tilde{V}_{-2} e^{-j\omega_2 t}$$

$$+ \tilde{V}_p e^{j\omega_p t} + \tilde{V}_{-p} e^{-j\omega_p t} \qquad (6\text{-}19)$$

The derivative of this quantity with respect to time is

$$\frac{dv_c}{dt} = j\omega_1 \left[\tilde{V}_1 e^{j\omega_1 t} - \tilde{V}_{-1} e^{-j\omega_1 t} \right]$$

$$+ j\omega_2 \left[\tilde{V}_2 e^{j\omega_2 t} - \tilde{V}_{-2} e^{-j\omega_2 t} \right]$$

$$+ j\omega_p \left[\tilde{V}_p e^{j\omega_p t} - \tilde{V}_{-p} e^{-j\omega_p t} \right] \qquad (6\text{-}20)$$

The current, i_c', is completely defined in Appendix A. Appendix A contains equation (6-21) which is the product of equations (6-19) and (6-20). Now only the frequencies, ω_1, ω_2, and ω_p, will be considered since the other frequencies will have no effect on this ideal circuit.

The frequencies that are found in equation (6-21) are

$$2\omega_1 \qquad \omega_1 \pm \omega_2 \qquad \omega_1 \pm \omega_p$$

$$2\omega_2 \qquad \omega_2 \pm \omega_1 \qquad \omega_2 \pm \omega_p$$

$$2\omega_p \qquad \omega_p \pm \omega_1 \qquad \omega_p \pm \omega_1 \tag{6-21a}$$

In an actual circuit, due to the high Q of the tank circuits, the frequencies other than

$$\omega_1 = \omega_p - \omega_2$$

$$\omega_2 = \omega_p - \omega_1$$

$$\omega_p = \omega_1 + \omega_2$$

$$\omega_p = \omega_2 + \omega_1 \tag{6-21b}$$

would have negligible effect on the circuit. The d. c. terms cancel out.

$2\omega_1$ and $2\omega_2$ might have some effect on the circuit if the parametric amplifier was quasi-degenerate, i.e. $\omega_1 \cong \omega_2$. However, this amplifier is non-degenerate; the degenerate amplifier is discussed later.

$$i_c' = j\lambda \Big\{ \omega_p \tilde{V}_p \tilde{V}_{-2} e^{j\omega_1 t} - \omega_p \tilde{V}_{-p} \tilde{V}_2 e^{-j\omega_1 t}$$

$$- \omega_2 \tilde{V}_{-2} \tilde{V}_p e^{j\omega_1 t} + \omega_2 \tilde{V}_2 \tilde{V}_{-p} e^{-j\omega_1 t}$$

$$+ \omega_p \tilde{V}_p \tilde{V}_{-1} e^{j\omega_2 t} - \omega_p \tilde{V}_{-p} \tilde{V}_1 e^{-j\omega_2 t}$$

$$- \omega_1 \tilde{V}_{-1} \tilde{V}_p e^{j\omega_2 t} + \omega_1 \tilde{V}_1 \tilde{V}_{-p} e^{-j\omega_2 t}$$

$$+ \omega_1 \tilde{V}_1 \tilde{V}_{+2} e^{j\omega_p t} - \omega_1 \tilde{V}_{-1} \tilde{V}_{-2} e^{-j\omega_p t}$$

$$+ \omega_2 \tilde{V}_2 \tilde{V}_1 e^{j\omega_p t} - \omega_2 \tilde{V}_{-2} \tilde{V}_{-1} e^{-j\omega_p t} \Big\} \qquad (6\text{-}22)$$

This reduces to

$$i_c' = j\lambda \Big\{ \omega_1 \tilde{V}_p \tilde{V}_{-2} e^{j\omega_1 t} - \omega_1 \tilde{V}_2 \tilde{V}_{-p} e^{-j\omega_1 t}$$

$$+ \omega_2 \tilde{V}_p \tilde{V}_{-1} e^{j\omega_2 t} - \omega_2 \tilde{V}_1 \tilde{V}_{-p} e^{-j\omega_2 t}$$

$$+ \omega_p \tilde{V}_1 \tilde{V}_2 e^{j\omega_p t} - \omega_p \tilde{V}_{-1} \tilde{V}_{-2} e^{-j\omega_p t} \Big\} \qquad (6\text{-}23)$$

There are three different components of i_c':

$$i_c'(\omega_1) = j\lambda\omega_1 \Big[\frac{V_2 V_p}{4} e^{j(\theta_p - \theta_2)} e^{j\omega_1 t}$$

$$- \frac{V_2 V_p}{4} e^{-j(\theta_p - \theta_2)} e^{-j\omega_1 t} \Big] \qquad (6\text{-}24)$$

$$i_c'(\omega_2) = j\lambda\omega_2\left[\frac{V_1 V_p}{4} e^{j(\theta_p-\theta_1)} e^{j\omega_2 t} - \frac{V_1 V_p}{4} e^{-j(\theta_p-\theta_1)} e^{-j\omega_2 t}\right] \quad (6\text{-}25)$$

$$i_c'(\omega_p) = j\lambda\omega_p\left[\frac{V_1 V_p}{4} e^{j(\theta_1+\theta_2)} e^{j\omega_p t} - \frac{V_1 V_2}{4} e^{-j(\theta_1+\theta_2)} e^{-j\omega_p t}\right] \quad (6\text{-}26)$$

The admittance of the nonlinear capacitance and the other two tanks as seen from the respective tank is

$$Y'(\omega_1) = \frac{i_c'(\omega_1)}{V_1 e^{j\theta_1} e^{j\omega_1 t}} = \frac{-j\lambda\omega_1 V_2 V_p e^{j(\theta_p-\theta_2)} e^{j\omega_1 t}}{2 V_1 e^{j\theta_1} e^{j\omega_1 t}}$$

$$= -j\lambda\omega_1 \frac{V_2 V_p}{2 V_1} e^{j(\theta_p-\theta_2-\theta_1)} \quad (6\text{-}27)$$

$$Y'(\omega_2) = \frac{i_c'(\omega_2)}{V_2 e^{j\theta_2} e^{j\omega_2 t}} = \frac{-j\lambda\omega_2 V_1 V_p e^{j(\theta_p-\theta_1)} e^{j\omega_2 t}}{2 V_2 e^{j\theta_2} e^{j\omega_2 t}}$$

$$= -j\lambda\omega_2 \frac{V_1 V_p}{2 V_2} e^{j(\theta_p-\theta_2-\theta_1)} \quad (6\text{-}28)$$

$$Y'(\omega_p) = \frac{i_c'(\omega_p)}{V_p e^{j\theta_p} e^{j\omega_p t}} = \frac{-j\lambda\omega_p V_1 V_2 e^{j(\theta_1+\theta_2)} e^{j\omega_p t}}{2 V_p e^{j\theta_p} e^{j\omega_p t}}$$

$$= -j\lambda\omega_p \frac{V_1 V_2}{2 V_p} e^{j(\theta_1+\theta_2-\theta_p)} \qquad (6\text{-}29)$$

In each of these equations the real part of the quantity is used.

The current in each resonant circuit is equal to the voltage of that circuit times the admittance of that resonant tank and the admittance of the remainder of the circuit.

$$i_1 = V_1 Y_{T_1} + V_1 Y'(\omega_1) \qquad (6\text{-}30)$$

$$0 = V_2 Y_{T_2} + V_2 Y'(\omega_2) \qquad (6\text{-}31)$$

$$i_p = V_p Y_{T_p} + V_p Y'(\omega_p) \qquad (6\text{-}32)$$

If the conductance of the diode, G_c, is neglected, then

$$Y_{T_1} = G_{T_1} + j\omega_1(C_1 + C_o) + \frac{1}{j\omega_1 L_1} \qquad (6\text{-}33)$$

where

$$G_{T_1} = G_{g_1} + G_1 + G_L \qquad (6\text{-}34)$$

$$Y_{T_2} = G_{T_2} + j\omega_2 (C_2 + C_0) + \frac{1}{j\omega_2 L_2} \qquad (6\text{-}35)$$

where

$$G_{T_2} = G_2 \qquad (6\text{-}36)$$

$$Y_{T_p} = G_{T_p} + j\omega_p (C_p + C_0) + \frac{1}{j\omega_p L_p} \qquad (6\text{-}37)$$

where

$$G_{T_p} = G_{g_p} + G_p \qquad (6\text{-}38)$$

Then, three equations can be written for this amplifier similar to equations (6-30), (6-31), and (6-32).

$$i_1 = V_1 Y_{T_1} - j\lambda\omega_1 \frac{V_2 V_p}{2} e^{j(\theta_p - \theta_1 - \theta_2)} \qquad (6\text{-}39)$$

$$0 = V_2 Y_{T_2} - j\lambda\omega_2 \frac{V_1 V_p}{2} e^{j(\theta_p - \theta_1 - \theta_2)} \qquad (6\text{-}40)$$

$$i_p = V_p Y_{T_p} - j\lambda\omega_p \frac{V_1 V_2}{2} e^{j(\theta_1 + \theta_2 - \theta_p)} \qquad (6\text{-}41)$$

Equation (6-40) can be reduced in order to find an expression for voltage, V_2.

$$V_2 = j\lambda\omega_2 \frac{V_1 V_p}{2 Y_{T_2}} e^{j(\theta_p - \theta_1 - \theta_2)} \tag{6-42}$$

Taking the conjugate of both the numerator and the denominator does not change the voltage, V_2.

$$V_2 = -j\lambda\omega_2 \frac{V_1 V_p}{2 Y_{T_2}^*} e^{-j(\theta_p - \theta_1 - \theta_2)} \tag{6-43}$$

Substituting these values of V_2 into equations (6-39) and (6-41) gives

$$\dot{i}_1 = V_1 Y_{T_1} - \lambda^2 \frac{V_1 V_p^2}{4 Y_{T_2}^*} \omega_1 \omega_2 \tag{6-44}$$

$$\dot{i}_p = V_p Y_{T_p} + \lambda^2 \frac{V_1^2 V_p}{4 Y_{T_2}} \omega_p \omega_2 \tag{6-45}$$

The total admittance at each frequency can now be found.

$$Y(\omega_1) = Y_{T_1} - \frac{\omega_1 \omega_2 (\lambda V_p)^2}{4 Y_{T_2}^*} \tag{6-46}$$

$$Y(\omega_p) = Y_{T_p} + \frac{\omega_p \omega_2 (\lambda V_1)^2}{4 Y_{T_2}} \tag{6-47}$$

Equation (6-46) shows that the mixing properties of the nonlinear capacitance have made an equivalent negative conductance appear in the signal resonant circuit. Equation (6-47) shows that a normal positive conductance appears in the pump circuit. Equation (6-46) can also be rewritten as:

$$Y(\omega_1) = Y_{T_1} + g_n \tag{6-48}$$

where

$$g_n = - \frac{\omega_1 \omega_2 (\lambda V_p)^2}{4 Y_{T_2}^*} \tag{6-49}$$

By properly choosing the variables in equation (6-49), amplification can be achieved. This circuit can be handled in a similar manner to the regenerative amplifier circuits of Chapter V. (Ref 6:12-14)

PARALLEL RESONANT CASE
DEGENERATE

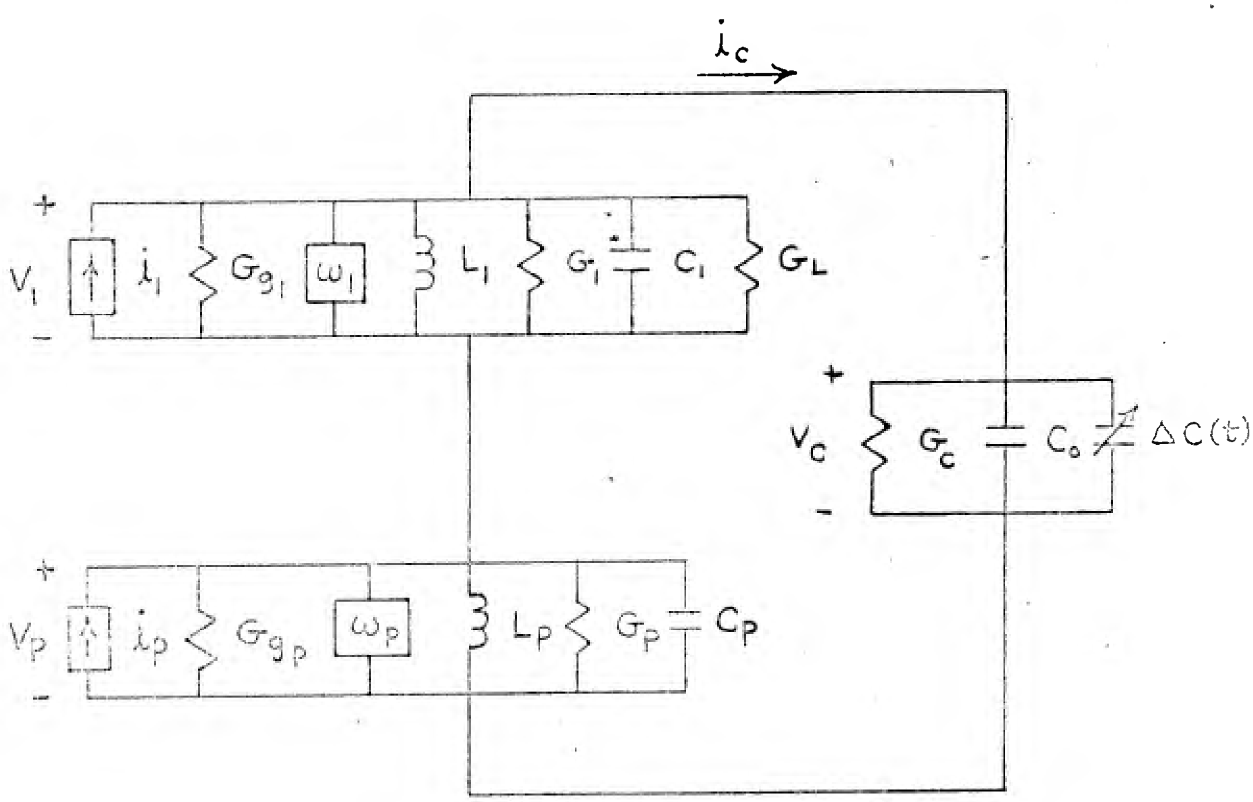


FIGURE 7-1

VII. An Analysis of the Degenerate Parametric Amplifier

A parallel resonant degenerate parametric amplifier is pictured in figure 7-1. This is merely a "degenerate" case of the normal parametric amplifier that has just been analyzed. The idler in this amplifier is tuned to the same frequency as the signal; for this reason, the idler circuit can be deleted.

The frequency relationship in the degenerate case is the same as for the normal amplifier.

$$\omega_p = \omega_1 + \omega_2 \tag{7-1}$$

However, because $\omega_1 = \omega_2$, this relationship can also be expressed as

$$\omega_p = 2\omega_1 \tag{7-2}$$

The degenerate case is analyzed because it brings out several important features of its phase relationship. Although the degenerate case is simpler than the normal parametric amplifier, its phase relationship is a definite disadvantage that outweighs its simpler construction. This analysis shows the importance of the phase relationship.

The frequencies are

$$\omega_1 = \omega_2 = \frac{1}{\sqrt{L_1 (C_1 + C_o)}} \tag{7-3}$$

and

$$\omega_p = \frac{1}{\sqrt{L_p (C_p + C_o)}} \tag{7-4}$$

This derivation is similar to the previous derivation, so that much of the detail is left out. Chapter VI should be consulted for a more detailed explanation of the steps.

$$i_c' = \lambda v \frac{dv}{dt} \tag{7-5}$$

$$v_c = V_1 \cos(\omega_1 t + \theta_1) + V_p \cos(\omega_p t + \theta_p) \tag{7-6}$$

$$v_c = \tilde{V}_1 e^{j\omega_1 t} + \tilde{V}_{-1} e^{-j\omega_1 t} \tag{7-7}$$

$$+ \tilde{V}_p e^{j\omega_p t} + \tilde{V}_{-p} e^{-j\omega_p t} \tag{7-7}$$

$$\frac{dv_c}{dt} = j\omega_1 \left[\tilde{V}_1 e^{j\omega_1 t} - \tilde{V}_{-1} e^{-j\omega_1 t} \right]$$

$$+ j\omega_p \left[\tilde{V}_p e^{j\omega_p t} - \tilde{V}_{-p} e^{-j\omega_p t} \right] \tag{7-8}$$

Appendix B contains equation (7-9) which is the product of equations (7-7) and (7-8). Only frequencies, ω_1 and ω_p will be considered since the other frequencies will have little effect on this ideal circuit.

$$i_c' = j\lambda\omega_1\left[\frac{V_p}{2}\frac{V_1}{2}e^{j(\theta_p-\theta_1)}e^{j\omega_1 t} - \frac{V_p}{2}\frac{V_1}{2}e^{-j(\theta_p-\theta_1)}e^{-j\omega_1 t}\right] + j\lambda\omega_1\left[\frac{V_1}{2}\frac{V_1}{2}e^{j2\theta_1}e^{j\omega_p t} - \frac{V_1}{2}\frac{V_1}{2}e^{-j2\theta_1}e^{-j\omega_p t}\right] \tag{7-10}$$

The current, i_c', consists of two parts

$$i_c'(\omega_1) = j\lambda\omega_1\left[\frac{V_pV_1}{4}e^{j(\theta_p-\theta_1)}e^{j\omega_1 t} - \frac{V_pV_1}{4}e^{-j(\theta_p-\theta_1)}e^{-j\omega_1 t}\right] \tag{7-11}$$

$$i_c'(\omega_p) = j\lambda\omega_1\left[\frac{V_1^2}{4}e^{j2\theta_1}e^{j\omega_p t} - \frac{V_1^2}{4}e^{-j2\theta_1}e^{-j\omega_p t}\right] \tag{7-12}$$

There are also two admittances

$$Y'(\omega_1) = \frac{i_c'(\omega_1)}{V_1 e^{j\theta_1} e^{j\omega_1 t}} = \frac{-j\lambda\,\omega_1 V_p V_1 e^{j(\theta_p-\theta_1)} e^{j\omega_1 t}}{2 V_1 e^{j\theta_1} e^{j\omega_1 t}}$$

$$= -j\lambda\,\omega_1 \frac{V_p V_1}{2 V_1} e^{j(\theta_p - 2\theta_1)} \qquad (7\text{-}13)$$

$$Y'(\omega_p) = \frac{i_c'(\omega_p)}{V_p e^{j\theta_p} e^{j\omega_p t}} = \frac{-j\lambda\,\omega_1 V_1 V_1 e^{j2\theta_1} e^{j\omega_p t}}{2 V_p e^{j\theta_p} e^{j\omega_p t}}$$

$$= -j\lambda\,\omega_1 \frac{V_1^2}{2 V_p} e^{j(2\theta_1 - \theta_p)} \qquad (7\text{-}14)$$

In each of these equations the real part of the quantity is used. Using an attack similar to equations (6-39) through (6-41).

$$i_1 = V_1 Y_{T_1} - j\lambda\,\omega_1 \frac{V_1 V_p}{2} e^{j(\theta_p - 2\theta_1)} \qquad (7\text{-}15)$$

$$i_p = V_p Y_{T_p} - j\lambda\,\omega_1 \frac{V_1^2}{2} e^{j(2\theta_1 - \theta_p)} \qquad (7\text{-}16)$$

where Y_{T_1} is the same as equation (6-33)

and Y_{T_p} is the same as equation (6-37)

$$Y(\omega_1) = Y_{T_1} - j\lambda\omega_1 \frac{V_p}{2} e^{j(\theta_p - 2\theta_1)} \tag{7-17}$$

$$Y(\omega_p) = Y_{T_p} - j\lambda\omega_1 \frac{V_1^2}{2V_p} e^{j(2\theta_1 - \theta_p)} \tag{7-18}$$

The phase of the degenerate case is extremely important.

If $$(2\theta_1 - \theta_p) = \pi/2 \tag{7-19}$$

then $$e^{j(2\theta_1 - \theta_p)} = e^{j\pi/2} \tag{7-20}$$

and $$e^{j\pi/2} = \cos\pi/2 + j\sin\pi/2 = j \tag{7-21}$$

Similarly $$e^{j(\theta_p - 2\theta_1)} = -j \tag{7-22}$$

So that equations (7-17) and (7-18) become

$$Y(\omega_1) = Y_{T_1} - \lambda\omega_1 \frac{V_p}{2} \tag{7-23}$$

$$Y(\omega_p) = Y_{T_p} + \lambda\omega_1 \frac{V_1^2}{2V_p} \tag{7-24}$$

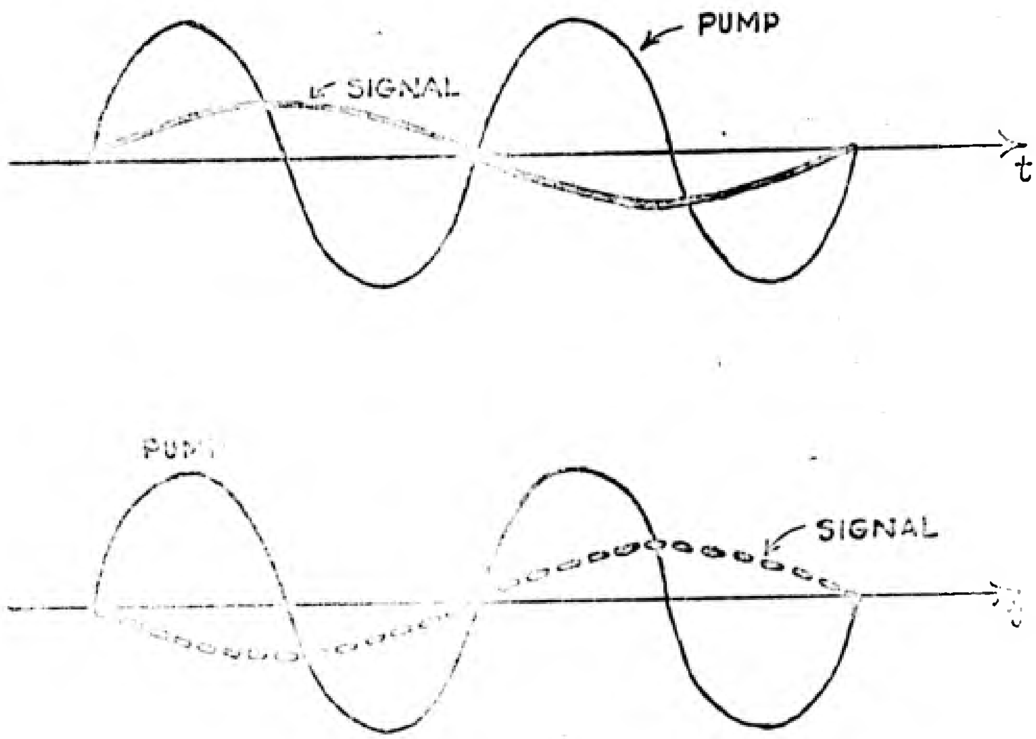


FIGURE 7-2

For this phase requirement, the nonlinear capacitance makes a negative conductance appear in the signal circuit. Equation (7-24) shows that a normal positive conductance appears in the pump circuit.

Equation (7-23) could also be rewritten

$$Y(\omega_1) = Y_{T_1} + g_n \tag{7-25}$$

where $$g_n = - \lambda \omega_1 \frac{V_p}{2} \tag{7-26}$$

By properly choosing the variables in equation (7-26), amplification can be achieved. This circuit could be handled similarly to the regenerative amplifier circuit of Chapter V.

The signal phase angle, θ_1 , can shift 180° and the previous discussion is still valid. Figure 7-2 shows that the 180° shift has no effect on the system.

$$\theta_1 + \pi = \theta_1' \tag{7-27}$$

This quantity can be placed in equation (7-20)

$$e^{j(2\theta_1' - \theta_p)} = e^{j\pi/2} \tag{7-28}$$

$$e^{j[2(\theta_1 + \pi) - \theta_p]} = e^{j\pi/2} \quad (7\text{-}29)$$

$$e^{j2\pi} \cdot e^{j(2\theta_1 - \theta_p)} = 1 \cdot j = j \quad (7\text{-}30)$$

However, if the signal phase angle should shift 90°, there would be an amplification. Instead there would be attenuation.

$$\theta_1 + \pi/2 = \theta_1'' \quad (7\text{-}31)$$

Again this quantity can be placed in equation (7-21)

$$e^{j(2\theta_1'' - \theta_p)} \neq e^{j\pi/2} \quad (7\text{-}32)$$

$$e^{j[2(\theta_1 + \pi/2) - \theta_p]} = e^{-j\pi/2} \quad (7\text{-}33)$$

$$e^{j\pi} \cdot e^{j(2\theta_1 - \theta_p)} = -1 \cdot j = -j \quad (7\text{-}34)$$

Consequently, equation (7-23) becomes

$$Y(\omega_1) = Y_{T_1} + \lambda \, \omega_1 \frac{V_p}{2} \quad (7\text{-}35)$$

With a positive conductance in the signal circuit, attenuation results.

VIII. The Gain and Bandwidth of the Non-Degenerate Parametric Amplifier

The gain of a parametric amplifier can be found in a manner similar to that for the negative resistance amplifier of Chapter V. Consequently, the power gain (transducer), ${}_pG_t$, for on-resonance operation is

$$ {}_pG_t = \frac{4G_{g_1}G_L}{(G_{T_1} + g_n)^2} \tag{8-1} $$

However, the power gain for off-resonance operation is somewhat more difficult to obtain and it is necessary in order to find the bandwidth of the amplifier. The numerator of the power gain is assumed to remain constant off-resonance; the denominator varies as the fractional detuning varies.

$$ Y(\omega_1) = Y_{T_1} - \frac{\omega_1\omega_2(\lambda V_p)^2}{4Y_{T_2}^*} \tag{8-2} $$

The only two quantities that vary off-resonance are Y_{T_1} and Y_{T_2}.

$$ Y_{T_1} = G_{T_1}(1 - j2\delta Q_1) \tag{8-3} $$

$$Y_{T_2} = G_{T_2}\left(1 - j2\delta\frac{\Omega_1}{\Omega_2}Q_2\right) \qquad (8\text{-}4)$$

where

Ω_1 : center frequency of the signal

Ω_2: center frequency of the idler

$$\Omega_p = \omega_p = \Omega_1 + \Omega_2 \qquad (8\text{-}5)$$

$$\delta = \frac{\omega_1 - \Omega_1}{\Omega_1} = \frac{\Omega_2 - \omega_2}{\Omega_1} \qquad (8\text{-}6)$$

ω_p is assumed to remain constant throughout the entire derivation.

Q_1 : quality factor of the signal tank

Q_2 : quality factor of the idler tank

Substituting these expressions into equation (8-2) yields

$$Y(\omega_1) = G_{T_1}(1 - j2\delta Q_1) + \frac{g_n}{\left(1 + j2\delta\frac{\Omega_1}{\Omega_2}Q_2\right)} \qquad (8\text{-}7)$$

Then, in turn, substituting this expression into the power gain of equation (8-1) gives

$$pG_t = \frac{4G_{g_1}G_L}{\left[G_{T_1}(1-j2\delta Q_1) + \dfrac{g_n\left(1-j2\delta\frac{\Omega_1}{\Omega_2}Q_2\right)}{1+\left(2\delta\frac{\Omega_1}{\Omega_2}Q_2\right)^2}\right]^2} \tag{8-8}$$

Defining four quantities, a, b, c, and d, in order to simplify the algebra

$$a = G_{T_1} \tag{8-9}$$

$$b = 2\delta Q_1 G_{T_1} \tag{8-10}$$

$$c = \frac{g_n}{1+\left(2\delta\frac{\Omega_1}{\Omega_2}Q_2\right)^2} \tag{8-11}$$

$$d = \frac{2\delta\frac{\Omega_1}{\Omega_2}Q_2 g_n}{1+\left(2\delta\frac{\Omega_1}{\Omega_2}Q_2\right)^2} \tag{8-12}$$

$$pG_t = \frac{4G_{g_1}G_L}{\left[(a-jb)+(c-jd)\right]^2} \tag{8-13}$$

$$pG_t = \frac{4G_{g_1}G_L}{\text{DENOMINATOR}} \tag{8-14}$$

$$\text{DENOMINATOR} = \Big[(a^2 - j2ab - b^2) + (c^2 - j2cd - d^2)$$

$$+ (2ac - 2bd - j2bc - j2ad)\Big] \qquad (8\text{-}15)$$

$$= \Big[(a^2 + 2ac + c^2) - (b^2 + 2bd + d^2)$$

$$-j(2ab + 2cd + 2bc + 2ad)\Big] \qquad (8\text{-}16)$$

In order to get the real part of the power gain, the numerator and the denominator must be multiplied by the conjugate of the denominator.

$$\text{DENOMINATOR} = \Big[(a+c)^2 - (b+d)^2$$

$$-j2(ab + cd + bc + ad)\Big] \qquad (8\text{-}17)$$

$$\text{DEN} \times \text{DEN}^* = \big[(a+c)^2 - (b+d)^2\big]^2$$

$$+ 4\big[ab + cd + bc + ad\big]^2 \qquad (8\text{-}18)$$

$$\text{DENOMINATOR} = \left[(a+c)^4 + (b+d)^4 + 2(ab+cd+bc+ad)^2 \right] \tag{8-19}$$

$$= \left[(a+c)^2 + (b+d)^2 \right]^2 \tag{8-20}$$

Since the imaginary part of the power gain has no effect on the bandwidth and the magnitude of the power gain, only the real part is considered.

$$pG_t = \frac{4G_{g_1}G_L\left[(a+c)^2 - (b+d)^2\right]}{\left[(a+c)^2 + (b+d)^2\right]^2} \tag{8-21}$$

$$pG_t = \frac{4G_{g_1}G_L\left[1 - \frac{(a+c)^2}{(b+d)^2}\right]}{\left[1 + \frac{(a+c)^2}{(b+d)^2}\right]\left[(a+c)^2 + (b+d)^2\right]} \tag{8-22}$$

$$pG_t = \frac{4G_{g_1}G_L}{\left[(a+c)^2 + (b+d)^2\right]} \tag{8-23}$$

When the gain is large, the quantity

$$\frac{(a+c)^2}{(b+d)^2} \longrightarrow 0 \tag{8-24}$$

Substituting the values for a, b, c, and d into the gain equation (8-23) gives the final power gain expression.

$$pG_t = \frac{4\, G_{g_1} G_1}{\left[\left(G_{T_1} + \frac{g_n}{A}\right)^2 + 4\delta^2 Q_1^2 \left(G_{T_1} + \frac{B g_n}{A}\right)^2\right]} \tag{8-25}$$

where

$$A = 1 + \left(2\delta \frac{\Omega_1}{\Omega_2} Q_2\right)^2$$

$$B = \frac{\Omega_1 Q_2}{\Omega_2 Q_1}$$

This is the power gain for off-resonance operation. The bandwidth can be derived from this expression by equating the power gain off-resonance to one-half of the power gain on-resonance. The equality will give the fractional detuning at the half-power point, and the bandwidth is normally defined as twice the fractional detuning at the half-power point.

$$\frac{4 G_{g_1} G_L}{(G_{T_1} + g_n)^2 + 4\delta^2 Q_1^2 \left(G_{T_1} + \frac{\Omega_1 Q_2}{\Omega_2 Q_1} g_n\right)^2} = \frac{4 G_{g_1} G_L}{2(G_{T_1} + g_n)^2} \qquad (8\text{-}26)$$

Equate the denominators

$$4\delta^2 Q_1^2 \left(G_{T_1} + \frac{\Omega_1 Q_2}{\Omega_2 Q_1} g_n\right)^2 = (G_{T_1} + g_n)^2 \qquad (8\text{-}27)$$

$$\text{Bandwidth} \triangleq 2\delta \qquad (8\text{-}28)$$

$$2\delta \cong \frac{(G_{T_1} + g_n)}{Q_1 \left(G_{T_1} + \frac{\Omega_1 Q_2}{\Omega_2 Q_1} g_n\right)} \qquad (8\text{-}29)$$

This approximation for the bandwidth was checked on the IBM 1620 digital computer using the experimental parametric amplifier of Buckley and Hupert as a model. The formula is correct to about two per cent when Q_1 is in the eighty to one hundred range and is correct to about ten per cent when Q_1 is in the forty to fifty range. (Ref 7:313)

The gain bandwidth product of the parametric amplifier is

$$\sqrt{pG_t} \times \text{BANDWIDTH} =$$

$$\frac{2\sqrt{G_{g_1} G_L}}{(G_{T_1} + g_n)} \cdot \frac{(G_{T_1} + g_n)\,\Omega_2}{\Omega_2 Q_1 G_{T_1} + \Omega_1 Q_2 g_n} \cong \tag{8-30}$$

$$\frac{2\,\Omega_2 (G_{T_1} + g_n)}{\Omega_2 Q_1 G_{T_1} + \Omega_1 Q_2 g_n} \tag{8-31}$$

(Ref 8:1325)

IX. A Derivation of the Manley-Rowe Equations

An understanding of the Manley-Rowe energy equations is necessary for a complete comprehension of parametric amplification. These equations relate the average power at different frequencies in the nonlinear storage element of the parametric amplifier. The only assumptions in the derivation are that the nonlinear storage element is lossless and that the characteristics of this nonlinear element are single-valued. The original derivation is quite sophisticated and quite difficult to follow. (Ref 9:906-8) Consequently the following derivation is based on a simpler analysis of these equations by Salzberg. (Ref 10:1544) An interesting thing to notice is that the Manley-Rowe equations are independent of the shape of the characteristics of the nonlinear element and are also independent of the external circuit to which the nonlinear element is connected.

In this thesis a nonlinear capacitor is used in most examples, and for this reason, a nonlinear capacitor is used in this derivation. However, a nonlinear inductor would give the same results.

Since the capacitor is lossless, the sum of the powers, P_1, P_2, and P_p, should be zero according to the law of conservation of energy.

$$P_1 + P_2 + P_p = 0 \tag{9-1}$$

where

P_1 : power at f_1

P_2 : power at f_2

P_p : pump power at f_p

Since all the voltages and currents are sinusoidal, the average power is

$$P = \frac{VI}{2} \cos\theta \tag{9-2}$$

θ : the angle between voltage, V and current, I.

The impedance, Z, across a capacitor is

$$Z = X_C = \frac{1}{2\pi f C} = \frac{V}{I} \tag{9-3}$$

$$I = 2\pi f C V \tag{9-4}$$

Since

$$Q = CV \tag{9-5}$$

$$I = 2\pi f C V = 2\pi f Q \qquad (9\text{-}6)$$

$$P = \frac{V(2\pi f Q)}{2} \cos\theta \qquad (9\text{-}7)$$

$$P = f(\pi Q V \cos\theta) \qquad (9\text{-}8)$$

$$W = \pi Q V \cos\theta \quad \text{in energy per cycle} \qquad (9\text{-}9)$$

Consequently, equation (9-1) becomes

$$f_1 W_1 + f_2 W_2 + f_p W_p = 0 \qquad (9\text{-}10)$$

There are many different frequency relationships that can be found using the three frequencies, f_1, f_2, and f_p. Nearly every article in the literature refers to sum and difference frequencies. There is no uniformity on this subject because sum equations can be changed to difference equations by merely subtracting a frequency from both sides of the equation. On the other hand, difference equations can be changed to sum equations by adding a frequency to both sides of the equation.

Theoretically there are nine different frequency relationships for parametric amplifiers.

Sum Equations

$$f_1 = f_2 + f_p \quad (9\text{-}11)$$

$$f_2 = f_1 + f_p \quad (9\text{-}12)$$

$$f_p = f_1 + f_2 \quad (9\text{-}13)$$

Difference Equations

$$f_1 = f_2 - f_p \quad (9\text{-}14)$$

$$f_1 = f_p - f_2 \quad (9\text{-}15)$$

$$f_2 = f_1 - f_p \quad (9\text{-}16)$$

$$f_2 = f_p - f_1 \quad (9\text{-}17)$$

$$f_p = f_1 - f_2 \quad (9\text{-}18)$$

$$f_p = f_2 - f_1 \quad (9\text{-}19)$$

However, there are only three distinct frequency relationships, because all are repeated three times in different forms.

There are two classes of parametric amplifiers: (1) Degenerative and (2) Regenerative.

Degenerative

Down Converter $$f_p = f_1 - f_2 \qquad (9\text{-}20)$$

Up Converter $$f_p = f_2 - f_1 \qquad (9\text{-}21)$$

Regenerative

Negative Resistance $$f_p = f_1 + f_2 \qquad (9\text{-}22)$$

First, to analyze the energy relations in a Down Converter:

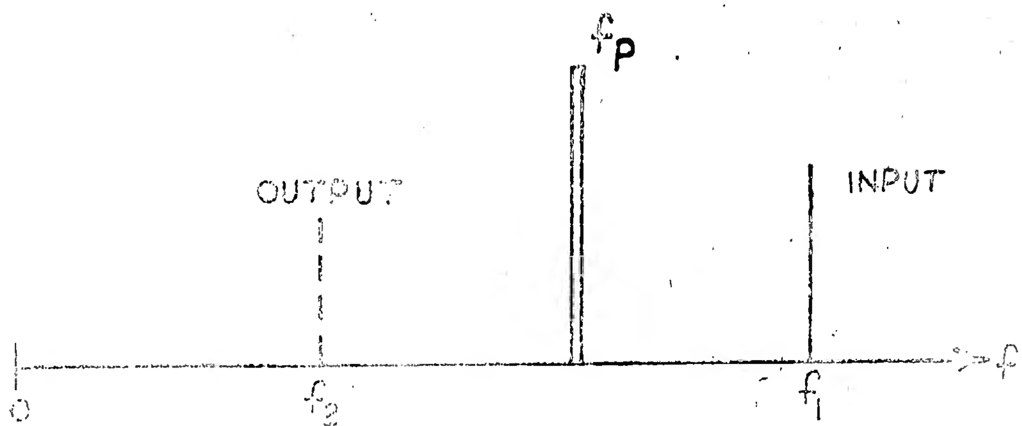
Substitute the frequency relationship into equation (9-10).

$$f_1 W_1 + f_2 W_2 + (f_1 - f_2) W_p = 0 \qquad (9\text{-}23)$$

$$(W_1 + W_p) f_1 + (W_2 - W_p) f_2 = 0 \qquad (9\text{-}24)$$

Since the frequencies are not zero, then

$$W_1 + W_p = 0 \qquad W_2 - W_p = 0 \qquad (9\text{-}25)$$



DOWN CONVERTER

$$f_p = f_1 - f_2$$

FIGURE 9-1

$$\frac{P_1}{f_1} = -\frac{P_p}{f_p} \qquad \frac{P_2}{f_2} = \frac{P_p}{f_p} \tag{9-26}$$

$$P_2 = \frac{f_2}{f_1}(-P_1) \tag{9-27}$$

The frequency spectrum for this case is pictured in figure 9-1.

$$P_2 = \frac{f_1 - f_p}{f_1}(-P_1) = \left[1 - \frac{f_p}{f_1}\right](-P_1) \tag{9-28}$$

The negative power, P_1, shows that power is being supplied to the output from the input. The power absorbed by the output, P_2, is a fraction of the power supplied. Consequently the minimum loss of the down converter is $\frac{f_2}{f_1}$. The down converter is stable.

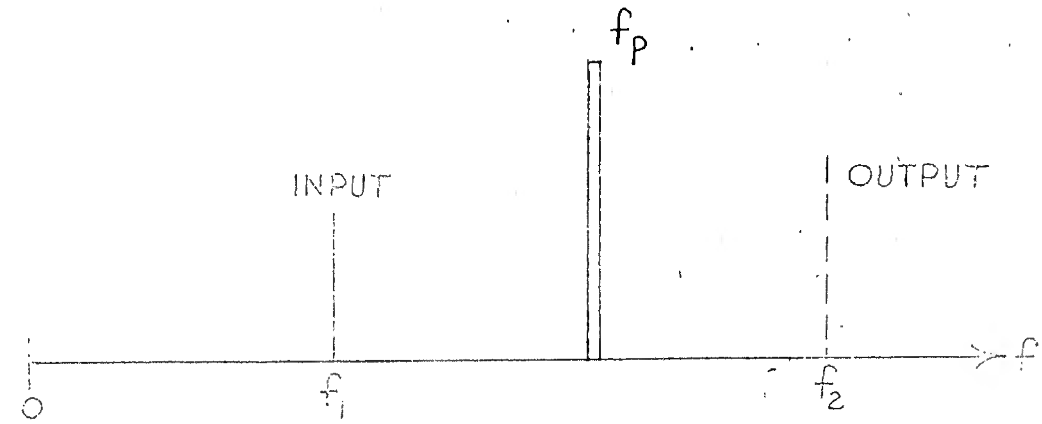
Second, to analyze the energy relations in an up converter:

Substitute the frequency relationship into equation (9-10).

$$f_1 W_1 + f_2 W_2 + (f_2 - f_1) W_p = 0 \tag{9-29}$$

$$\frac{P_1}{f_1} = \frac{P_p}{f_p} \qquad \frac{P_2}{f_2} = -\frac{P_p}{f_p} \tag{9-30}$$

$$P_2 = \frac{f_2}{f_1}(-P_1) \tag{9-31}$$



UP CONVERTER

$$f_p = f_2 - f_1$$

FIGURE 9-2

The frequency spectrum for this case is pictured in figure 9-2.

$$P_2 = \frac{f_1 + f_p}{f_1}(-P_1) = \left[1 + \frac{f_p}{f_1}\right](-P_1) \qquad (9\text{-}32)$$

The negative power, P_1, shows that power is being supplied to the output from the input. The power absorbed by the output, P_2, is a multiple of the power supplied. Consequently the maximum gain of the up converter is $\frac{f_2}{f_1}$.
The up converter is stable.

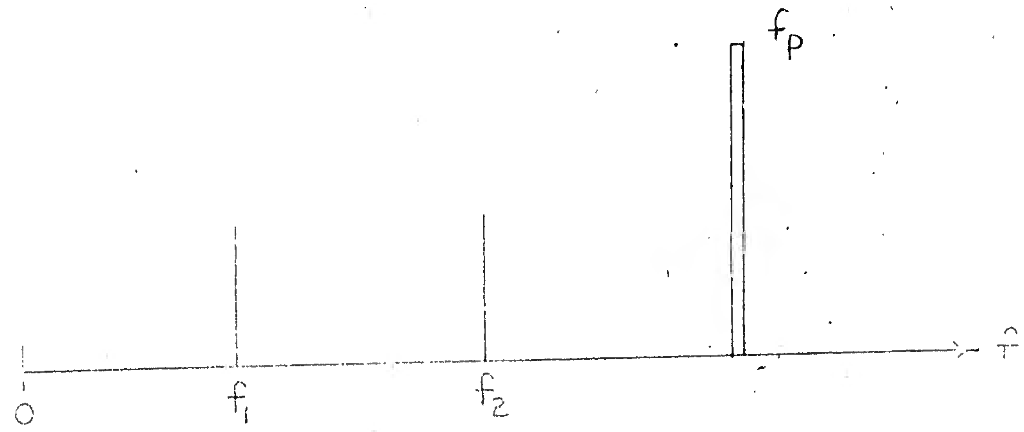
Third, to analyze the energy relations in a Negative Resistance case:

Substitute the frequency relationship into equation (9-10).

$$f_1 W_1 + f_2 W_2 + (f_1 + f_2) W_2 = 0 \qquad (9\text{-}33)$$

$$\frac{P_1}{f_1} = -\frac{P_p}{f_p} \qquad \frac{P_2}{f_2} = -\frac{P_p}{f_p} \qquad (9\text{-}34)$$

$$P_1 = \frac{f_1}{f_p}(-P_p) \qquad P_2 = \frac{f_2}{f_p}(-P_p) \qquad (9\text{-}35)$$



NEGATIVE RESISTANCE

$$f_p = f_1 + f_2$$

FIGURE 9-3

The frequency spectrum for this case is pictured in figure 9-3.

$$P_2 = \frac{f_2}{f_1} P_1 \qquad\qquad f_2 > f_1 \tag{9-36}$$

In the case pictured in figure 9-3, there would be a conversion gain since f_2 is greater than f_1. More important than this gain is the fact that the pump supplies power to both circuits. The pump will supply more power to the higher frequency circuit; in this case circuit 2 would receive more power. The maximum gain in a negative resistance amplifier is unlimited, but the amplifier operates in a potentially unstable region in order to realize this high gain.

The reader should recall that there are two cases of regenerative parametric amplifiers: the degenerate case and the non-degenerate case. Both have been analyzed in earlier chapters.

UP CONVERTER

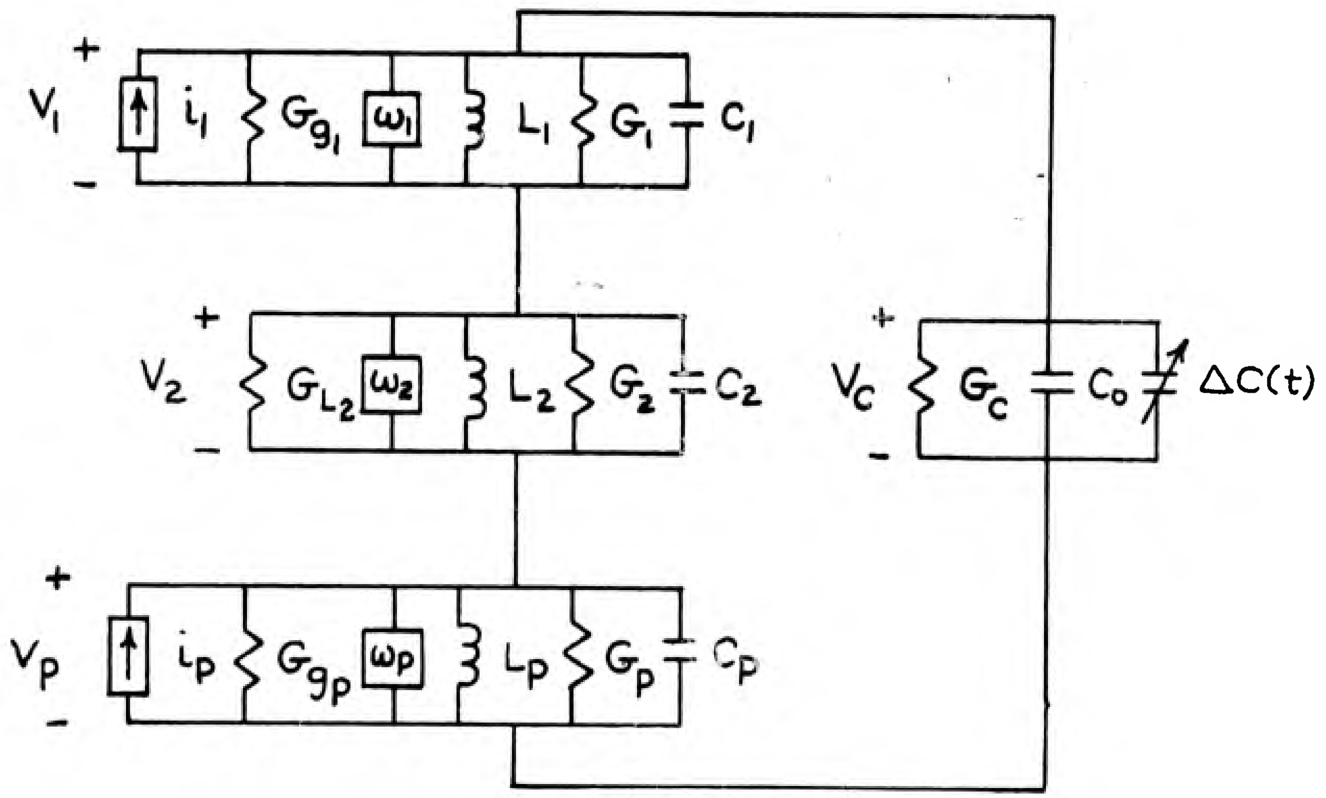


FIGURE 10-1

X. The Gain and Bandwidth of the Parallel Resonant Form of the Parametric Up-Converter

The parallel resonant form of the parametric up-converter is pictured in figure 10-1. The only difference between the parametric up-converter and the parametric amplifier of figure 6-1 is that the load, G_L, has been removed from the signal branch and placed in the idler branch to become G_{L_2}. It should be noted that this is the regenerative parametric up-converter and its basic frequency relationship is

$$\omega_p = \omega_1 + \omega_2 \tag{10-1}$$

A number of steps will be left out of this derivation due to the similarity between this derivation and the one for the normal parametric amplifier in Chapter VIII. In addition the conversion gain will be compared to the value obtained from the Manley-Rowe equations of the previous chapter. The conversion gain from the Manley-Rowe equation is

$$pG_c = \frac{f_2}{f_1} P_1 = \frac{\omega_2}{\omega_1} P_1 \tag{10-2}$$

Two quantities are changed due to the load being relocated in the idler circuit

$$G_{T_1}' = G_{g_1} + G_1 \qquad (10\text{-}3)$$

$$G_{T_2}' = G_2 + G_{L_2} \qquad (10\text{-}4)$$

The conversion gain, ${}_pG_c$, is

$$pG_c = \frac{V_2^2 \cdot G_{L_2}}{\frac{i^2}{4G_{g_1}}} \qquad (10\text{-}5)$$

$$pG_c = \frac{V_2^2}{i^2} 4G_{g_1} G_{L_2} \qquad (10\text{-}6)$$

$$pG_c = \frac{V_2^2}{V_1^2} \frac{4G_{g_1} G_{L_2}}{(G_{T_1}' + g_n)^2} \qquad (10\text{-}7)$$

$$V_2 = \frac{\wedge \omega_2 V_1 V_p}{2Y_{T_2}'} \qquad (10\text{-}8)$$

$$V_1 = V_1 \qquad (10\text{-}9)$$

$$pG_c = \frac{\omega_2}{Y_{T_2}'} \cdot \frac{\omega_1}{\omega_1} \cdot \frac{\omega_2 (\lambda V_p)^2}{4 Y_{T_2}'} \cdot \frac{4 G_{g_1} G_{L_2}}{(G_{T_1}' + g_n)^2} \qquad (10\text{-}10)$$

$$pG_c = \frac{\omega_2}{Y_{T_2}'} \quad \frac{g_n}{\omega_1} \quad \frac{4 G_{g_1} G_{L_2}}{(G_{T_1}' + g_n)^2} \qquad (10\text{-}11)$$

Where conversion gain on-resonance is

$$pG_c = \frac{\omega_2}{\omega_1} \quad \frac{g_n}{G_{T_2}'} \quad \frac{4 G_{g_1} G_{L_2}}{(G_{T_1}' + g_n)^2} \qquad (10\text{-}12)$$

Where conversion gain off-resonance is

$$pG_c = \frac{\omega_2}{\omega_1} \quad \frac{g_n}{G_{T_2}' \cdot A} \cdot$$

$$\frac{4 G_{g_1} G_{L_2}}{\left[(G_{T_1}' + \frac{g_n}{A})^2 + 4\delta^2 Q_1^2 (G_{T_1}' + \frac{g_n B}{A})^2\right]}$$

(10-13)

where

$$A = 1 + \left(2\delta \frac{\Omega_1}{\Omega_2} Q_2\right)^2$$

$$B = \frac{\Omega_1 Q_2}{\Omega_2 Q_1}$$

The bandwidth for the up-converter is actually the same expression that was found in equation (8-29) for the bandwidth of the normal parametric amplifier.

$$2\delta \cong \frac{G_{T_1}' + g_n}{Q_1\left(G_{T_1}' + \frac{\Omega_1 Q_2}{\Omega_2 Q_1} g_n\right)} \qquad (10\text{-}14)$$

(Ref 8:1327)

XI. Parametric Amplifier Simulation on the Analog Computer

In order to better understand the parametric amplifier, its operation was simulated on the analog computer. The degenerate case was chosen because its frequency relationship is well defined and its operation is the basis for all preliminary study of the parametric amplifier. In addition the phase of the pump, which is quite critical in the degenerate mode, can be accurately controlled on the analog computer.

The computer results show that the output of the parametric amplifier is dependent on both the magnitude and the phase of the pump. Furthermore, the loading of the signal branch was investigated in order to see the effect of different loads on the amplifier.

The tank circuit with losses is merely

$$\ddot{y} + \beta \dot{y} + \omega_s^2 y = 0 \qquad (11\text{-}1)$$

where $\ddot{y} = \dfrac{d^2y}{dt^2}$

and $\dot{y} = \dfrac{dy}{dt}$

However, this differential equation must have a driving function, and Ax(t) is chosen to represent the time-varying driving function.

$$\ddot{y} + \beta \dot{y} + \omega_s^2 y = A x(t) \tag{11-2}$$

In this case, the action of the nonlinear capacitor is incorporated into the circuit by merely multiplying the signal and the pump together.

$$\ddot{y} + \beta \dot{y} + \omega_s^2 y + \frac{u}{25} \omega_s^2 y = A x(t) \tag{11-3}$$

This reduces to

$$\ddot{y} + \beta \dot{y} + \omega_s^2 \left(1 + \frac{u}{25}\right) y = A x(t) \tag{11-4}$$

where

ω_s : signal frequency

ω_p : pump frequency

β : accounts for the losses

ϕ : the angle of the pump voltage in relation to the pump current

$u = P \cos(\omega_p t - \phi)$: the pump

$A x(t) = A \cos \omega_s t$: the driving function

The final equation that was used on the computer is

$$\ddot{y} + \beta \dot{y} + \omega_s^2\left[1 + \frac{P}{25}\cos(\omega_p t - \phi)\right]y = 0.5 \cos \omega_s t \qquad (11\text{-}5)$$

There are four series of computer runs:

Series 1 : $\beta = .4$ @ $P = 45$

Series 2 : $\beta = .4$ @ $P = 30$

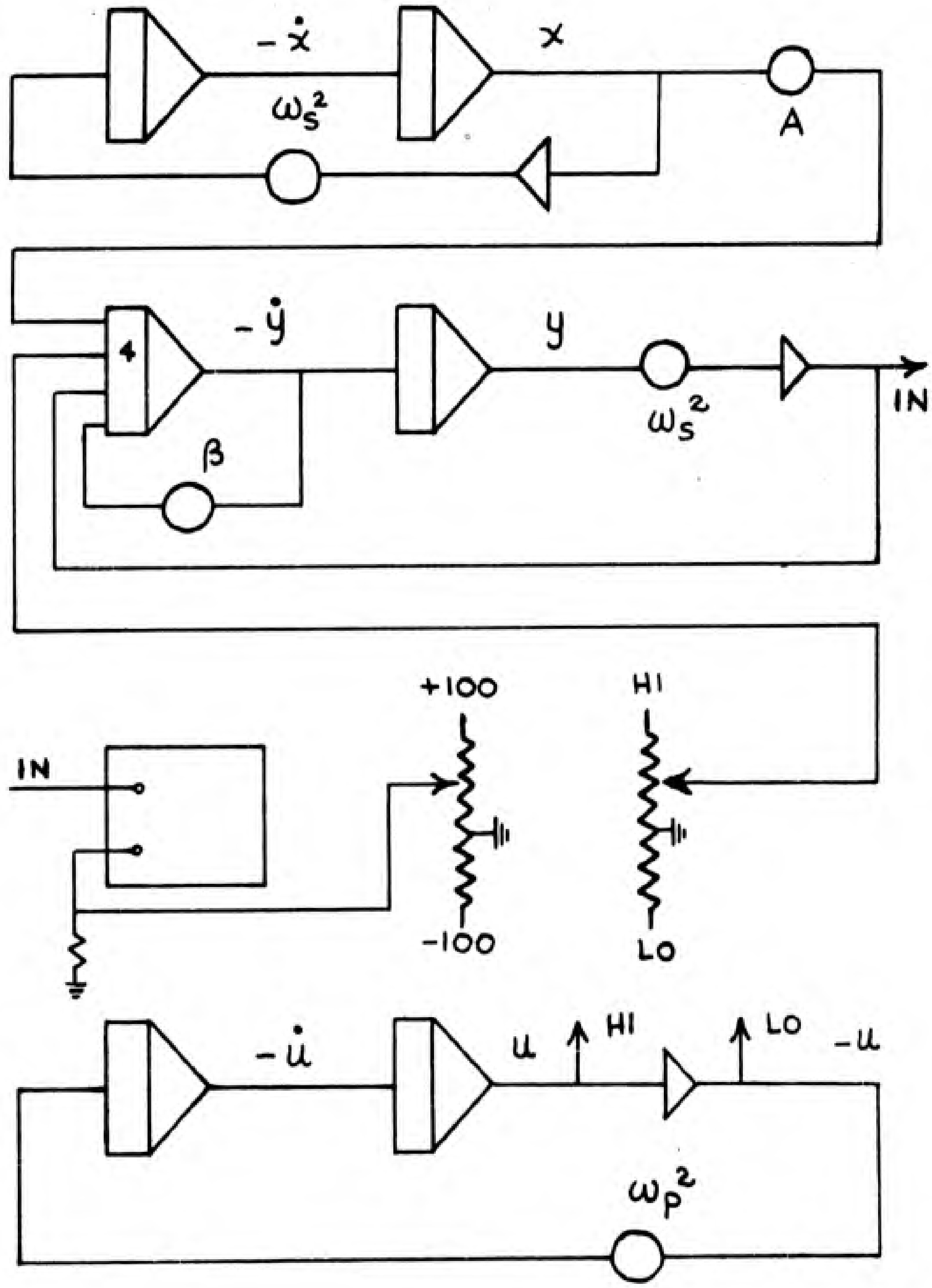
Series 3 : $\beta = .5$ @ $P = 30$

Series 4 : $\beta = .3$ @ $P = 30$

Series (1) and (2) show the effect of the magnitude of pumping. Several runs were made with the magnitude of the pump, P = 60, but the output was unstable. Series (3) and (4) can be compared to Series (2) in order to see the effects of loading on the output.

The runs were all made with no pumping on the first run in each series to compare the effects of pumping. In the second run, the pump was made a cosine function while in the third run, the pump was shifted 90° so that it was a sine function.

In the second run of each series, amplification should be noted; in the third run, attenuation should be noted. Each can be compared with the first run of that series to see the amount of amplification or attenuation present.



$-\dot{x}$
x
ω_s^2
A
4
$-\dot{y}$
y
ω_s^2
IN
β
+100
HI
IN
-100
LO
$-\dot{u}$
u
HI
LO
$-u$
ω_p^2

FIGURE II-1

CONSTANT POTENTIOMETER SETTINGS

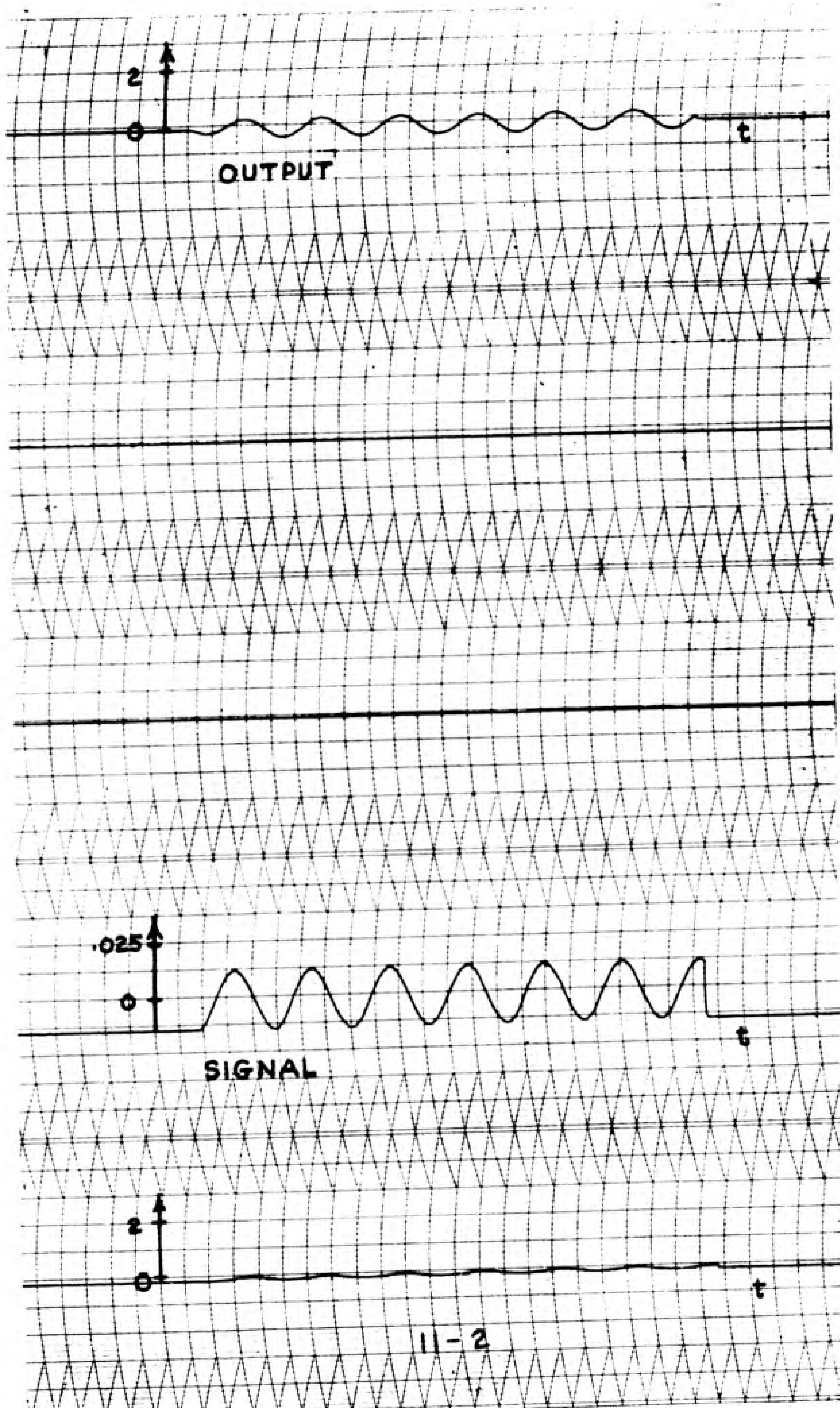
$\omega_s^2 = .16$ $\omega_s = .4$

$\omega_p^2 = .64$ $\omega_p = .8$

$A = .01$

INITIAL CONDITIONS

$x(0) = 1$



2
0
t
OUTPUT
.025
0
t
SIGNAL
2
0
t
11-2

CASE ONE

$\beta = .4$

(11-2)

$y(0) = 0$

$\dot{y}(0) = 0$

(11-3)

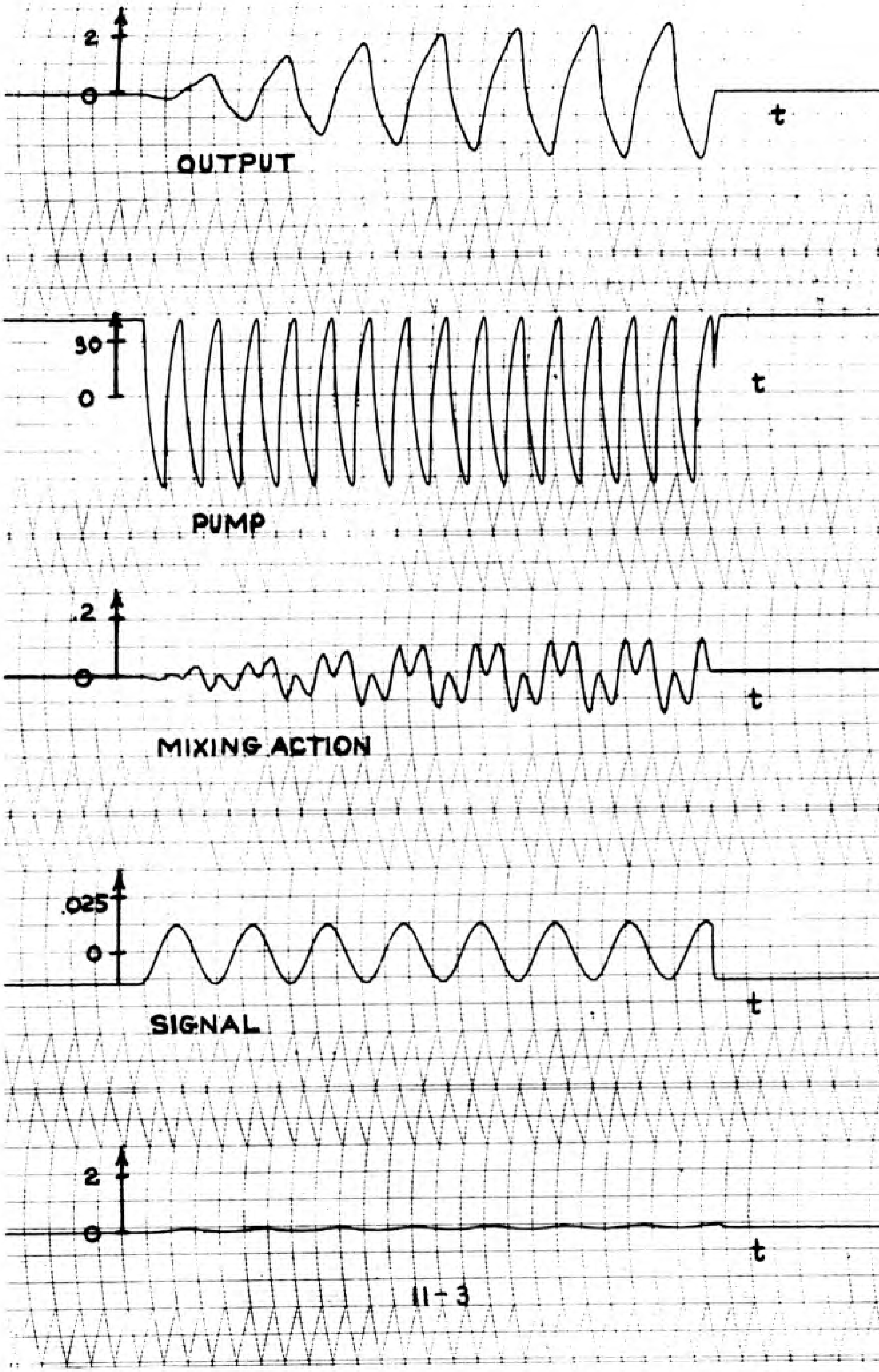
$y(0) = 45$

$\dot{y}(0) = 0$

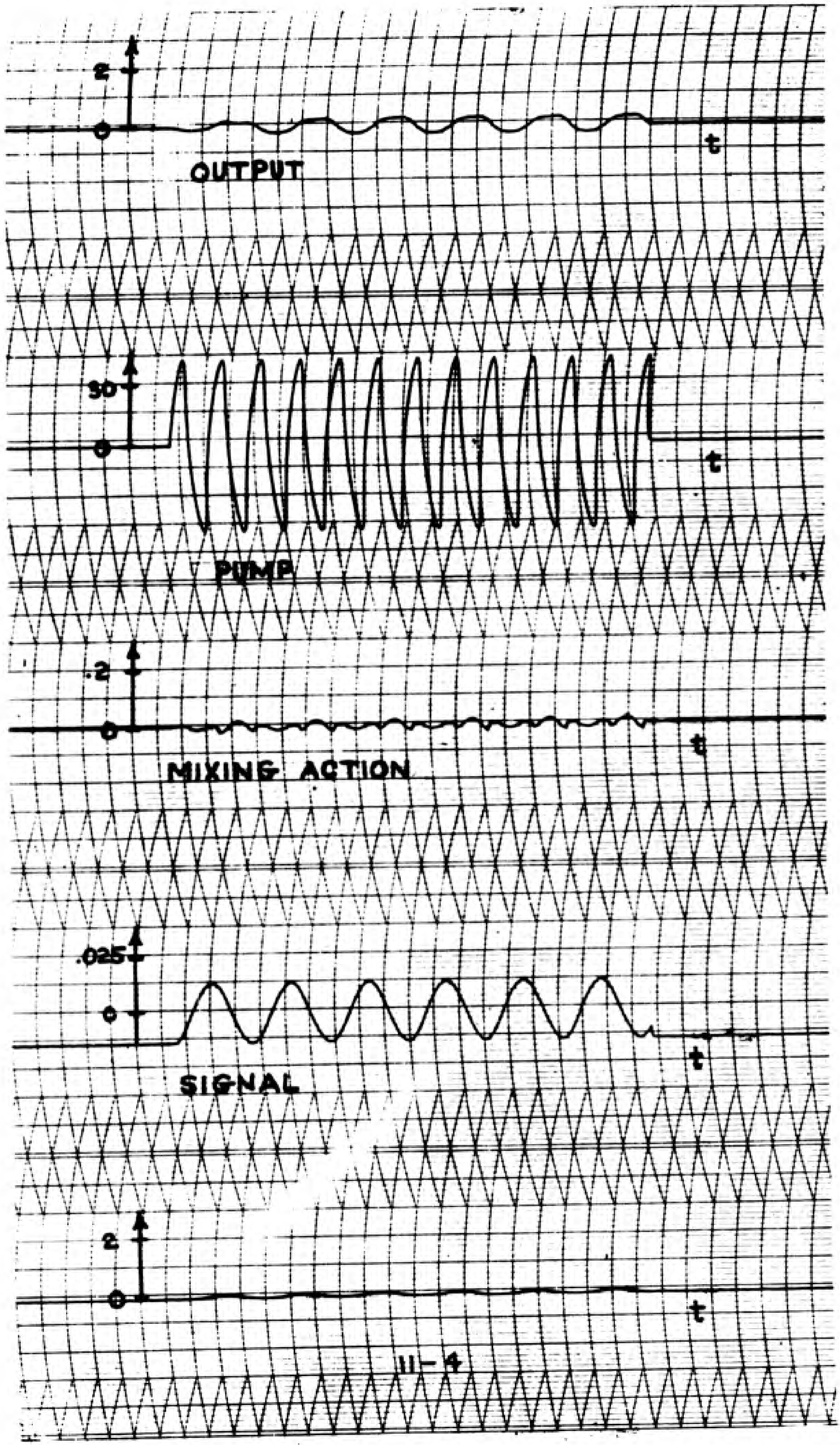
(11-4)

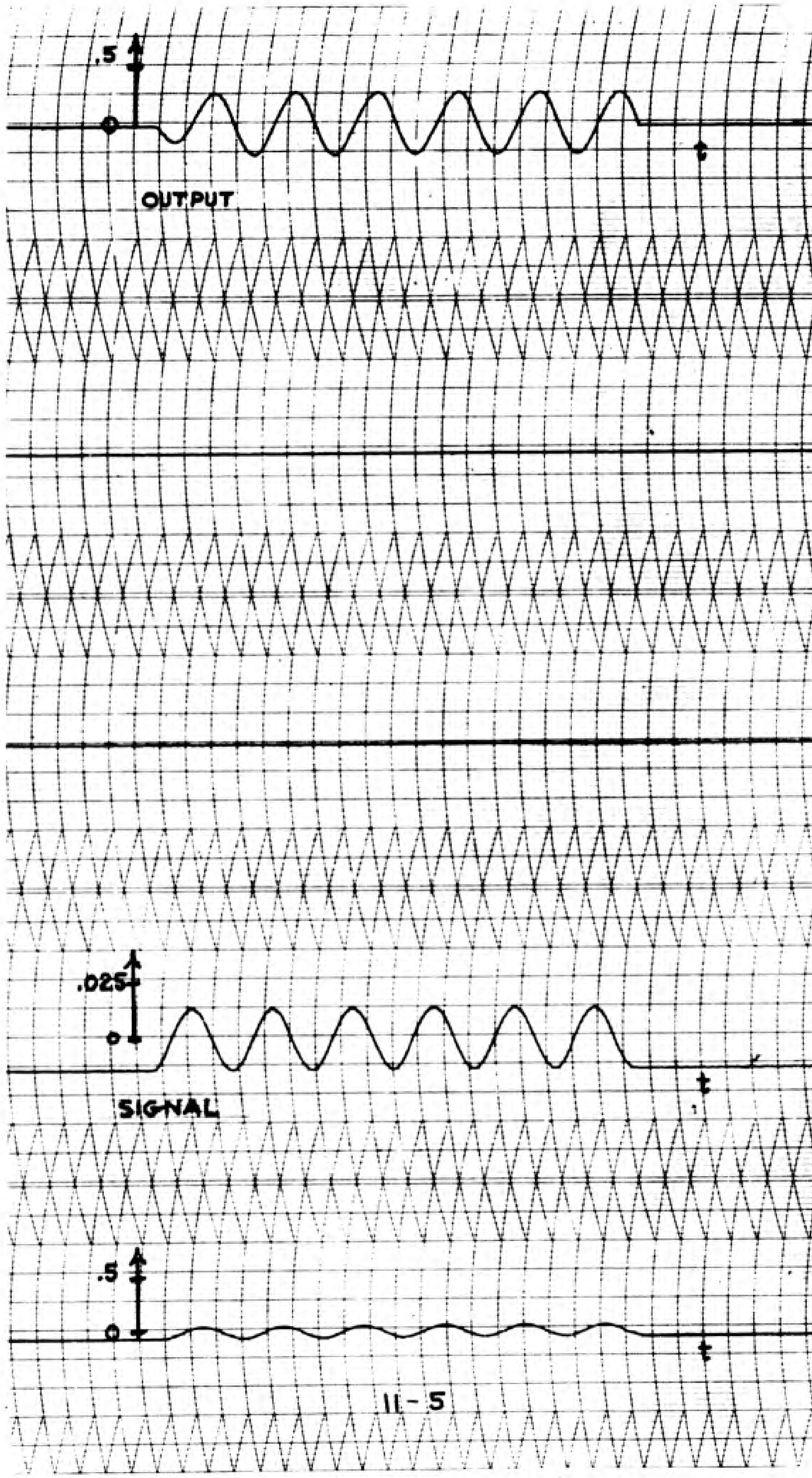
$y(0) = 0$

$\dot{y}(0) = 36$



11-3


2
0
t
OUTPUT
30
0
t
PUMP
.2
0
t
MIXING ACTION
.025
0
t
SIGNAL
2
0
t
11-4


.5
0
t
OUTPUT
.025
0
t
SIGNAL
.5
0
t
II-5

CASE TWO

$\beta = .4$

(11-5)

$y(0) = 0$

$\dot{y}(0) = 0$

(11-6)

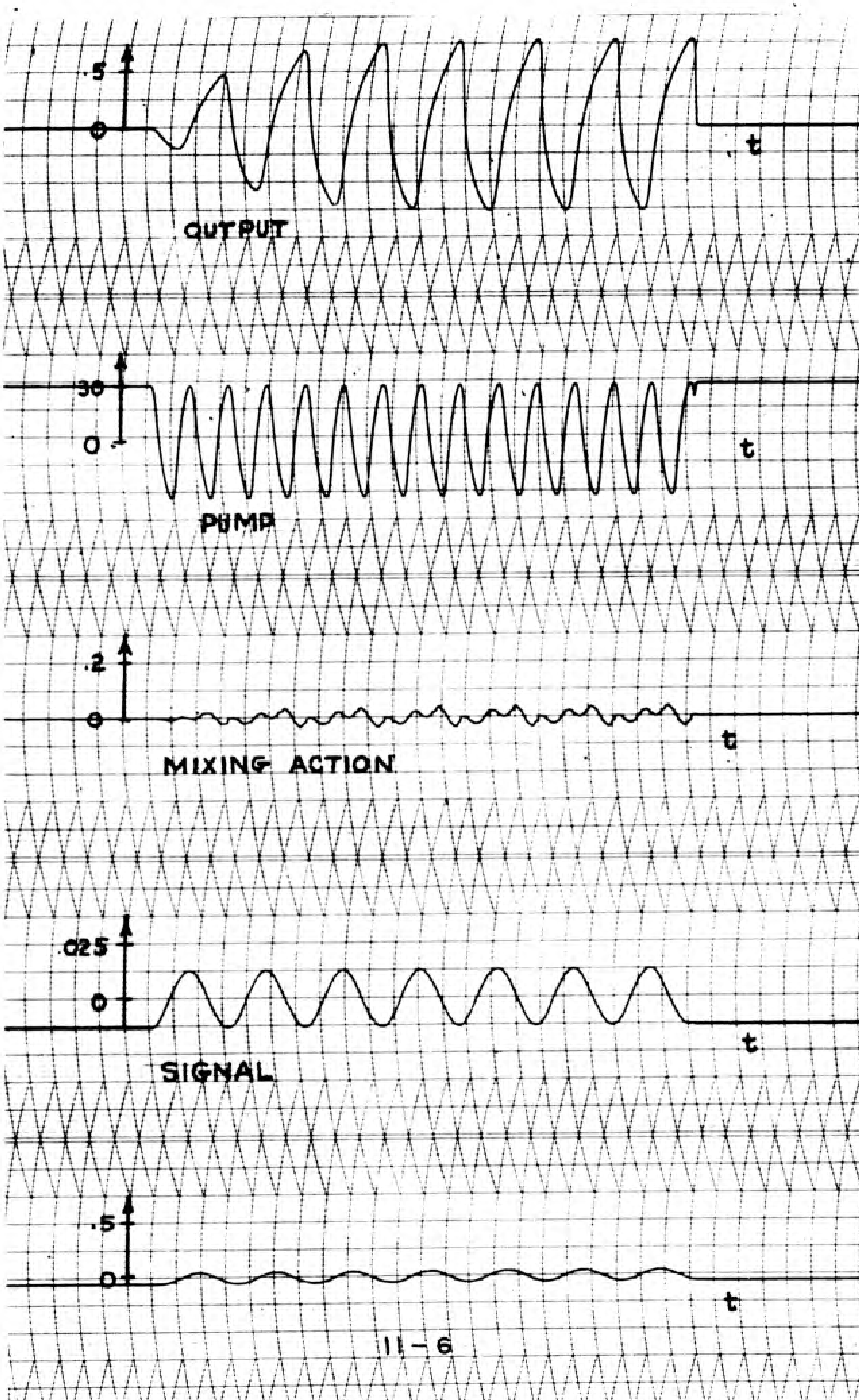
$y(0) = 30$

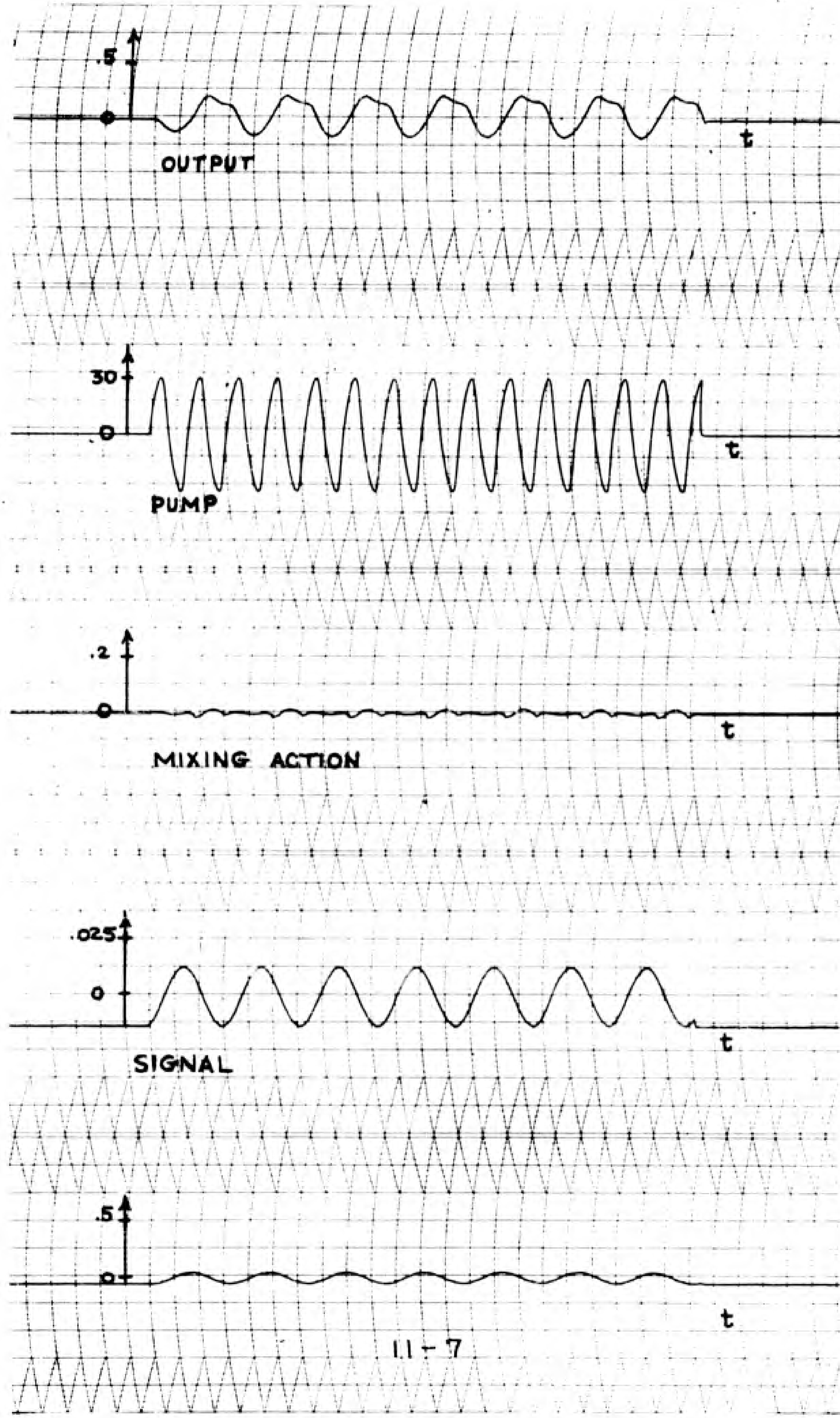
$\dot{y}(0) = 0$

(11-7)

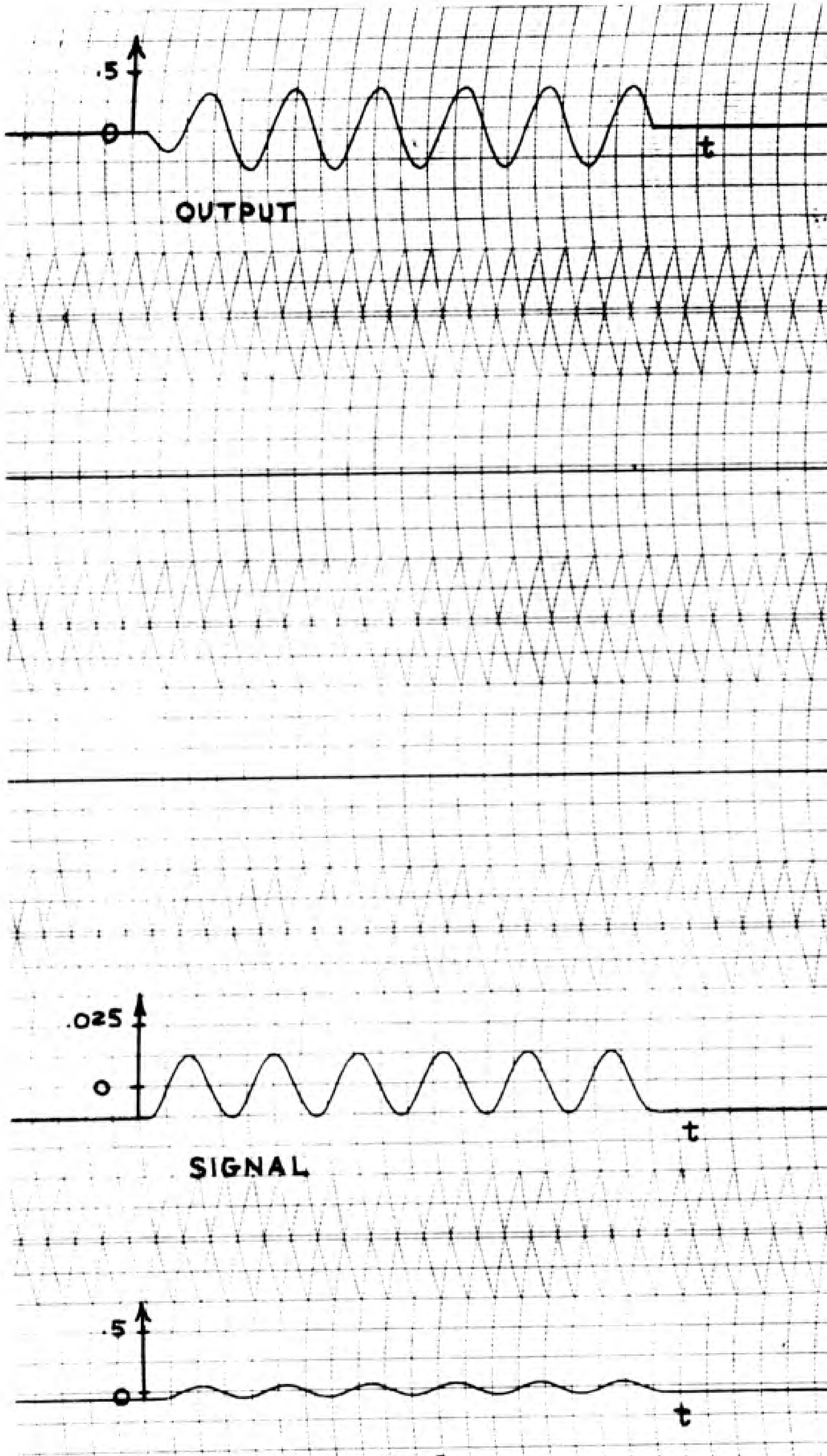
$y(0) = 0$

$\dot{y}(0) = 24$


.5
0
t
OUTPUT
30
0
t
PUMP
.2
0
t
MIXING ACTION
.025
0
t
SIGNAL
.5
0
t
II-6



II - 7



11-8

CASE THREE

$\beta = .5$

(11-8)

$y(0) = 0$

$\dot{y}(0) = 0$

(11-9)

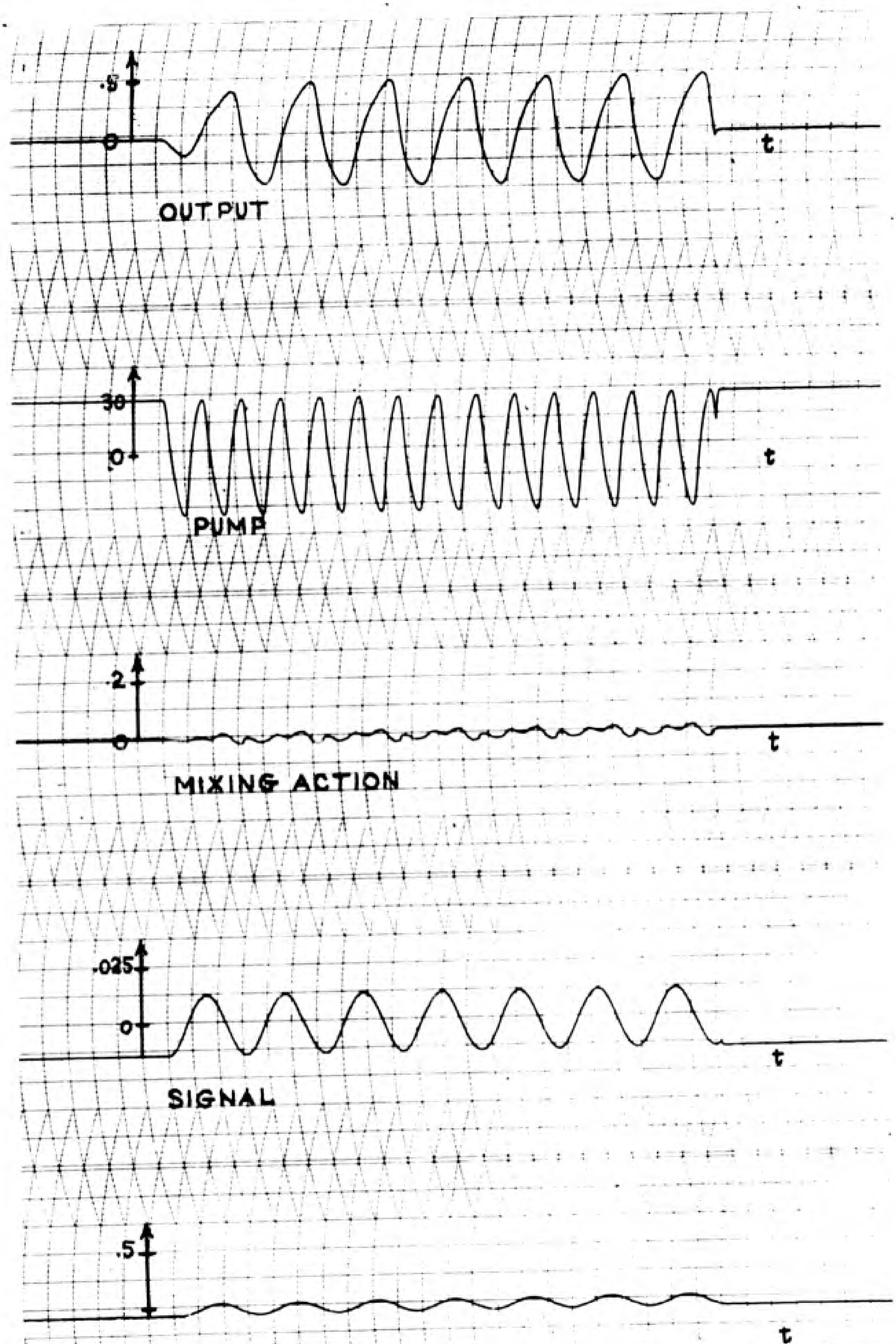
$y(0) = 30$

$\dot{y}(0) = 0$

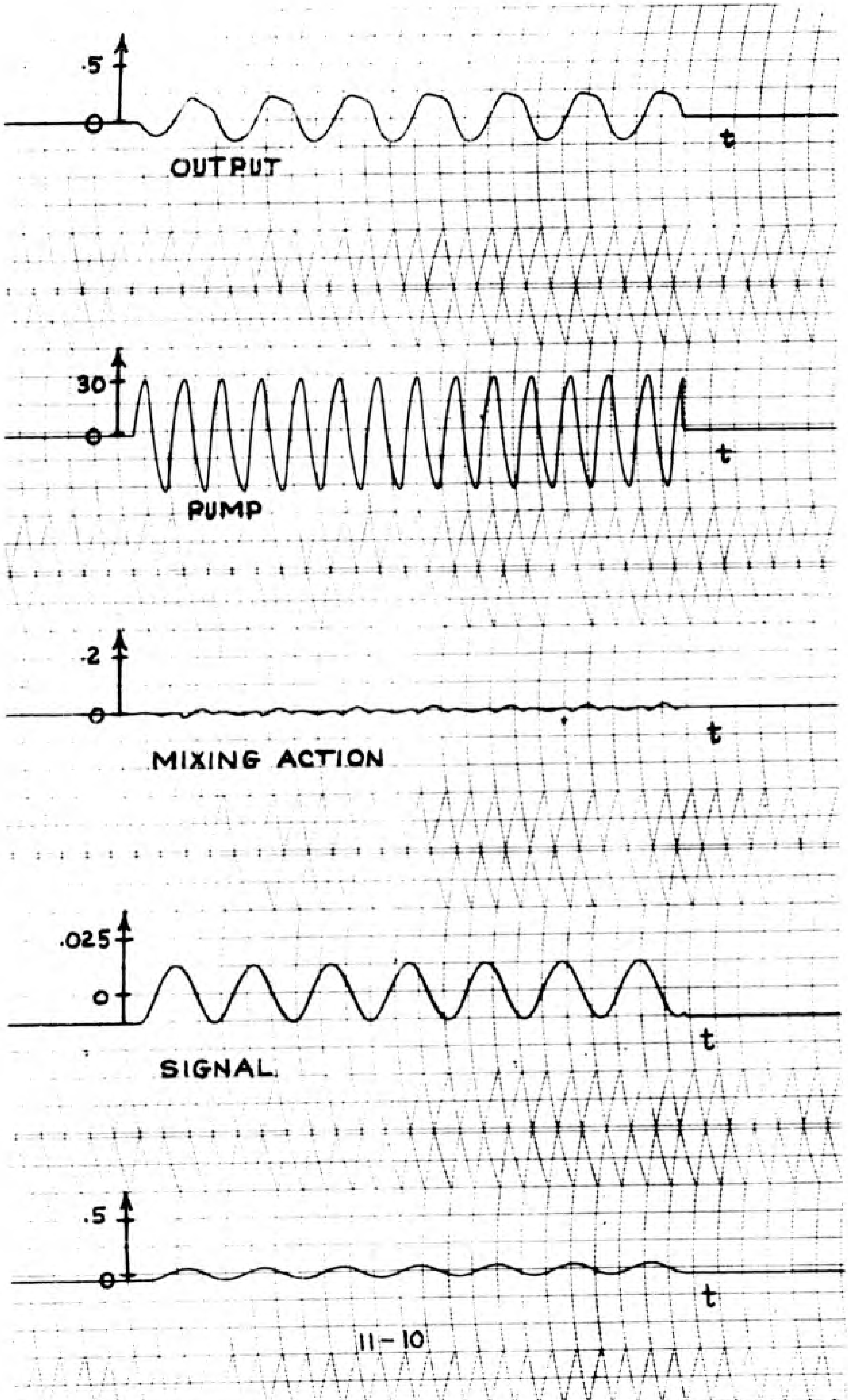
(11-10)

$y(0) = 0$

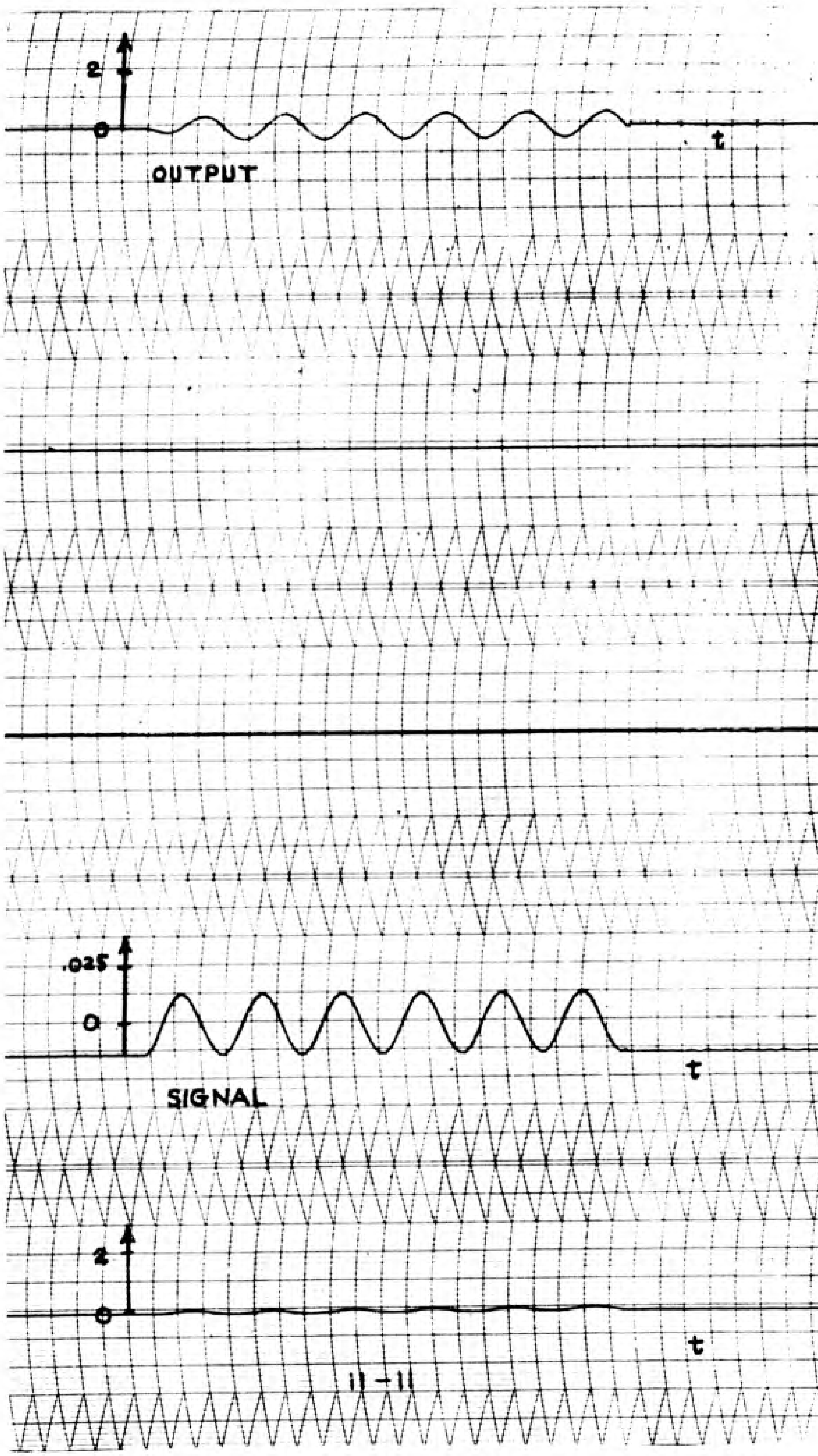
$\dot{y}(0) = 24$



11 - 9



11-10


2
0
t
OUTPUT
.025
0
t
SIGNAL
2
0
t
11-11

CASE FOUR

$\beta = .3$

(11-11)

$y(0) = 0$

$\dot{y}(0) = 0$

(11-12)

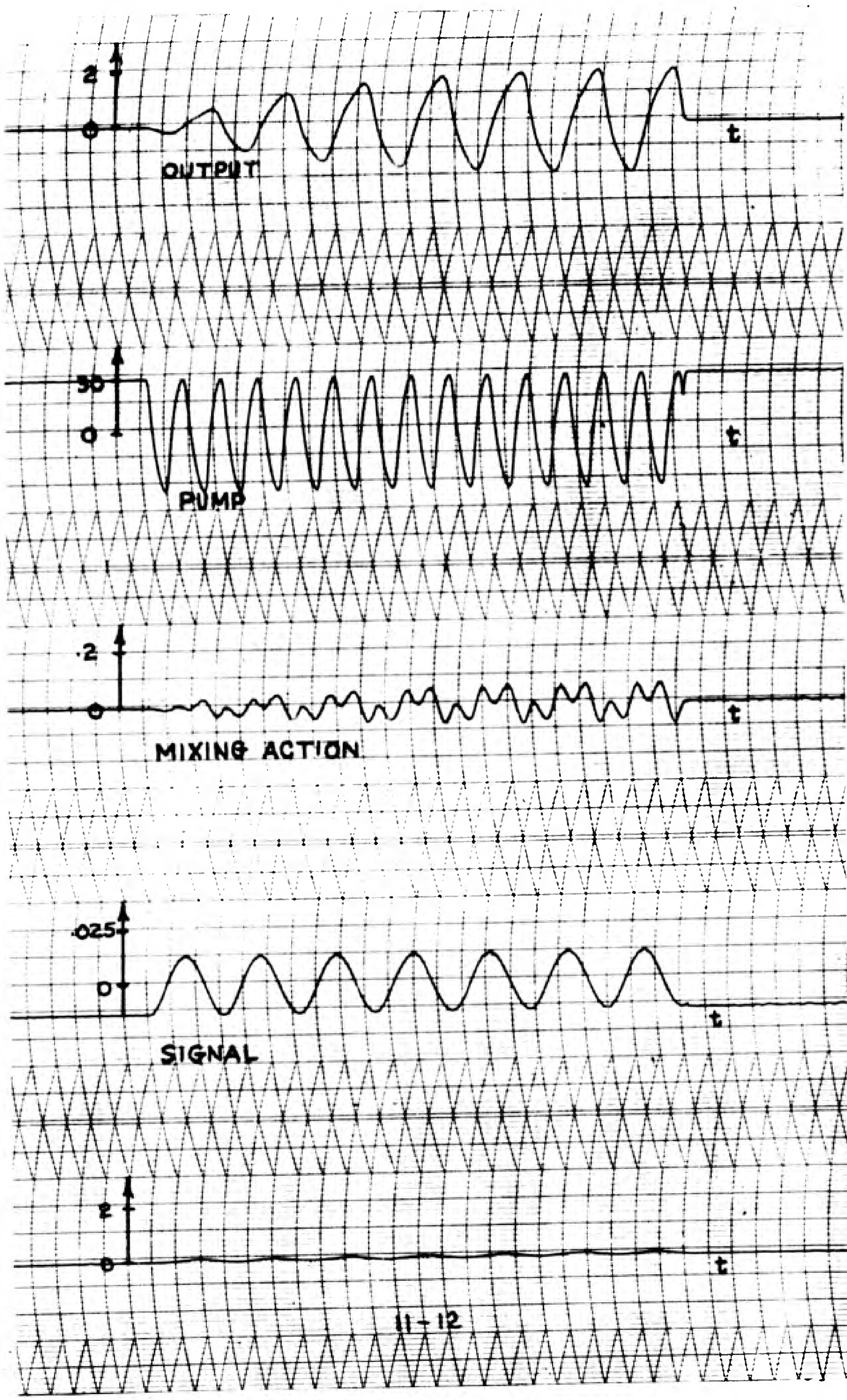
$y(0) = 30$

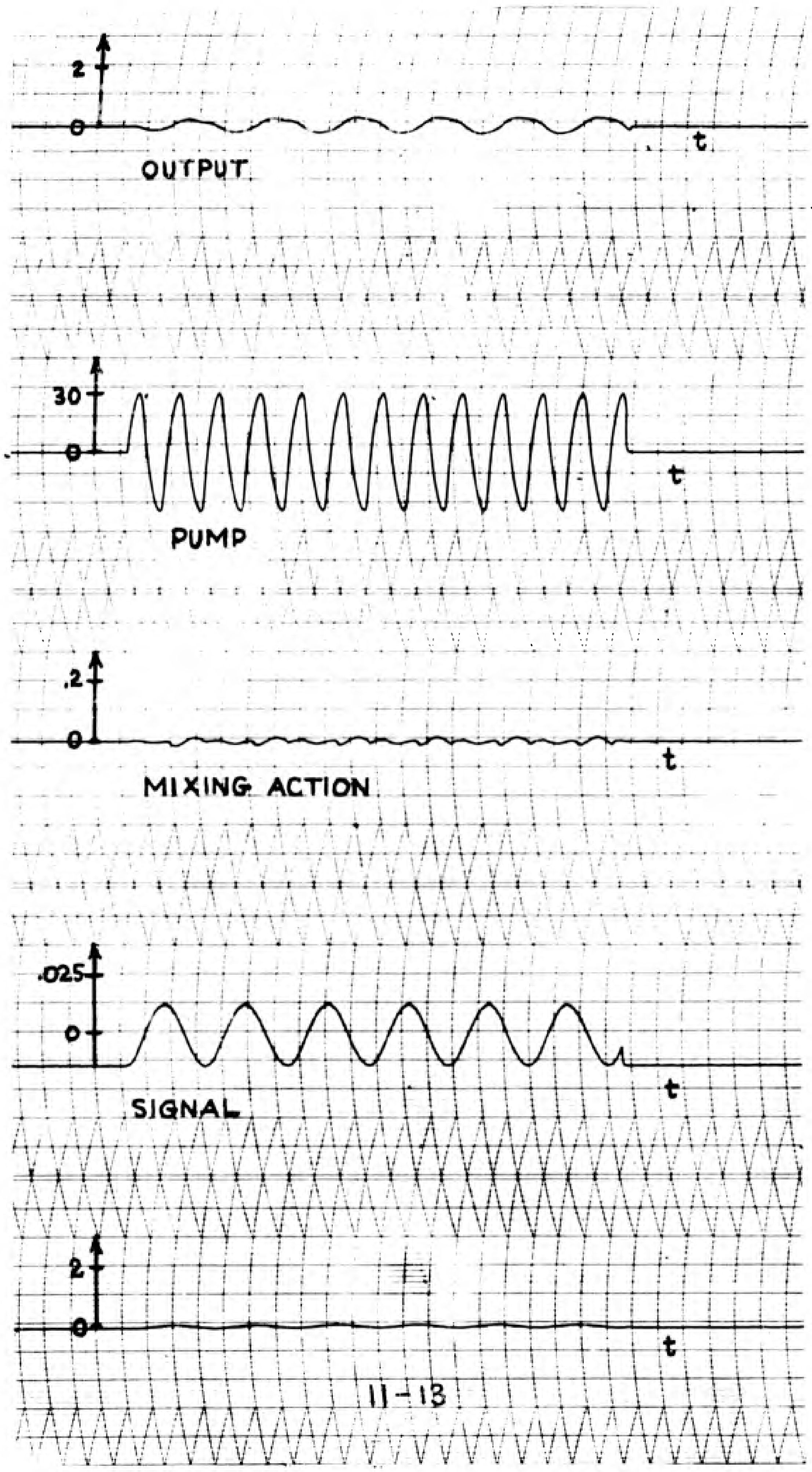
$\dot{y}(0) = 0$

(11-13)

$y(0) = 0$

$\dot{y}(0) = 24$


2
0
t
OUTPUT
30
0
t
PUMP
.2
0
t
MIXING ACTION
.025
0
t
SIGNAL
2
0
t
11-12


2
0
t
OUTPUT
30
0
t
PUMP
.2
0
t
MIXING ACTION
.025
0
t
SIGNAL
2
0
t
11-13

XII. A Derivation of the Mathieu Equation

No analysis of parametric amplifiers would be complete without some reference to the Mathieu Equation. Classically, the Mathieu Equation has been the tool used to explain parametric excitation. Furthermore, the model of the parametric amplifier on the analog computer can be supported by a derivation of the Mathieu Equation.

There is one short-coming to this derivation; the model of the nonlinear capacitor is not as realistic as the model used in the circuit theory analysis of Chapter VI. Actually the capacitance is a function of voltage, and voltage, in turn, is a function of time. The derivation of Chapter VI used C(v) and included all the voltages that appeared in the circuit, while in this derivation C(t) is used and it is considered to be varied by the dominant voltage. This derivation, then, assumes that the "pump" is much greater than the signal. The assumption is quite valid for a first-order approximation; a more detailed derivation might include the second-order effect of the signal. Although the results are very nearly the same, this basic difference should be kept in mind throughout the analysis. The Mathieu Equation does two important things; it links the model simulated on the analog computer to the model of Chapter VI and it shows

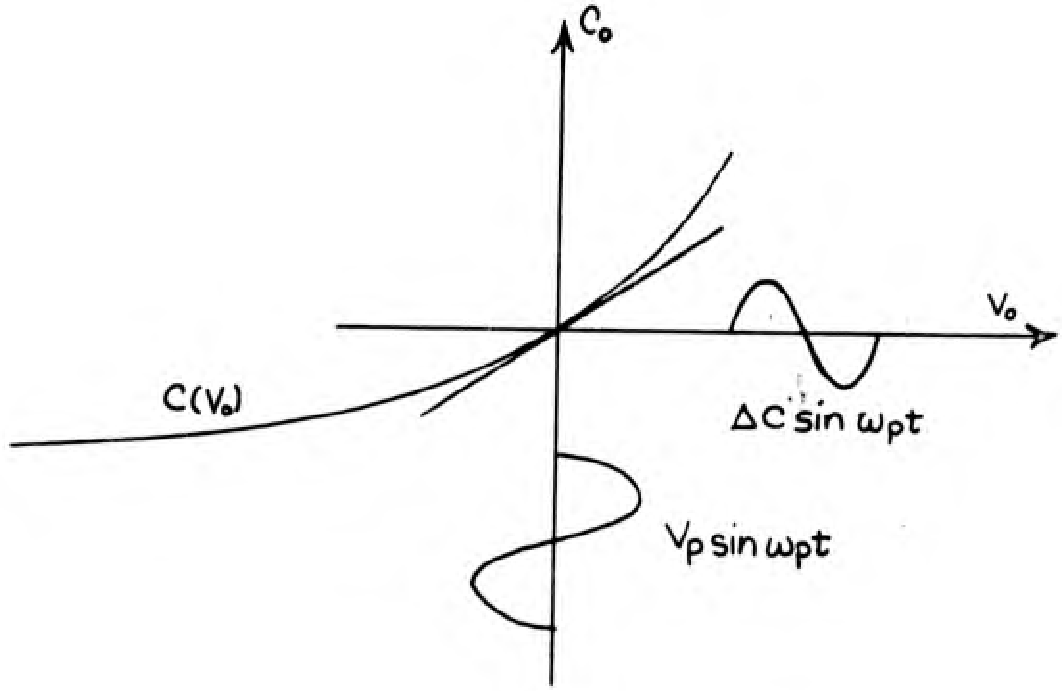


FIGURE 12-1

the classical attack to the parametric excitation problem.

The capacitance of the L-C tank circuit of figure 12-2 is shown in figure 12-1.

$$C(t) = C_o + \Delta C \sin(\omega_p t + \theta_p) \tag{12-1}$$

where C_o is constant

and $C_o >> \Delta C$

If $\Delta C = 0$, then the discussion in Chapter IV is applicable because the tank circuit is a normal resonant L-C tank. However, ΔC does have a finite value in a parametric amplifier and its magnitude is dependent on the amount of non-linearity of the variable capacitor. (This subject is discussed in more detail in Appendix C.)

The homogeneous differential equation of the circuit in figure 12-2 is

$$\frac{d^2Q}{dt^2} + \frac{Q}{L\,C(t)} = 0 \tag{12-2}$$

The equation can be rewritten

$$\frac{d^2Q}{dt^2} + \frac{1}{LC_o} \frac{Q}{\left[1 + \frac{\Delta C}{C_o} \cos \omega_p t\right]} = 0 \tag{12-3}$$

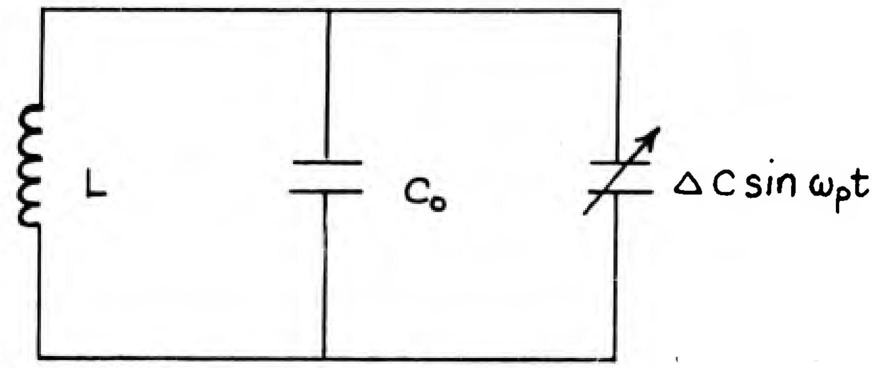


FIGURE 12-2

where $\theta_p = 90^\circ$

Using the binomial expansion and since

$$|\cos \omega_p t| \leq 1$$

and $\dfrac{\Delta C}{C_o} \ll 1$

then
$$\frac{1}{\left[1 + \frac{\Delta C}{C_o} \cos \omega_p t\right]} \cong \left[1 - \frac{\Delta C}{C_o} \cos \omega_p t\right] \qquad (12\text{-}4)$$

The extreme values of C(t) are found in order to get the equation into the desired form

$$\omega^2_{MAX} = \frac{1}{LC_o}\left[1 + \frac{\Delta C}{C_o}\right] \qquad (12\text{-}5)$$

$$\omega^2_{MIN} = \frac{1}{LC_o}\left[1 + \frac{\Delta C}{C_o}\right] \qquad (12\text{-}6)$$

$$\Delta\omega \triangleq \frac{1}{2}\left(\omega_{MAX} - \omega_{MIN}\right) \qquad (12\text{-}7)$$

Using the approximation that $\sqrt{1+x} \cong 1 + \dfrac{x}{2}$

for small values of x

$$\Delta\omega = \frac{1}{2}\left[\left(\omega_0 + \frac{\omega_0}{2}\frac{\Delta C}{C_0}\right) - \left(\omega_0 - \frac{\omega_0}{2}\frac{\Delta C}{C_0}\right)\right] \tag{12-8}$$

$$\Delta\omega = \frac{1}{2}\omega_0\frac{\Delta C}{C_0} \tag{12-9}$$

$$\frac{\Delta C}{C_0} = 2\,\frac{\Delta\omega}{\omega_0} \tag{12-10}$$

This can be substituted into the original equation

$$\frac{d^2Q}{dt^2} + \omega_0^2\left[1 - \frac{\Delta C}{C_0}\cos\omega_p t\right]Q = 0 \tag{12-11}$$

The result is

$$\frac{d^2Q}{dt^2} + \omega_0^2\left[1 - 2\,\frac{\Delta\omega}{\omega_0}\cos\omega_p t\right]Q = 0 \tag{12-12}$$

This compares very closely with the normal form of the Mathieu Equation

$$\frac{d^2y}{dz^2} + (a - 2q\cos 2z)y = 0 \tag{12-13}$$

where

$$z = \frac{\omega_p t}{2}$$

$$Q = y$$

$$a = \frac{\omega_0^2 \omega_p^2}{4} = \left(\frac{\omega_0 \omega_p}{2}\right)^2$$

$$q = \frac{\omega_p^2}{4} \omega_0 \Delta\omega$$

EXAMPLE 1: The degenerate case requires that $\omega_p = 2\omega_0$

then if $\omega_0 = 1$

$\omega_p = 2$

then $z = t$ $a = 1$ $q = \Delta\omega$

The solution to this differential equation is

$$Q = A\, Ce_1(t, \Delta\omega) + B\, Se_1(t, \Delta\omega) \quad (12\text{-}14)$$

where Ce_1 : the even Mathieu function of degree one

Se_1 : the odd Mathieu function of degree one

A and B are arbitrary constants which can be found when the initial conditions of the differential equation are known.

EXAMPLE 2: The non-degenerate case requires that $\omega_p > \omega_0$

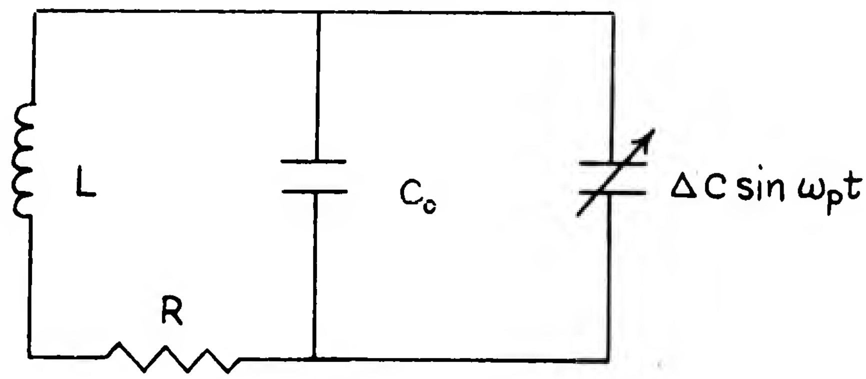


FIGURE 12-3

then if $\omega_o = 1$

$\omega_p = 4$

then $z = 2t$ $a = 4$ $q = 4\Delta\omega$

The solution to this differential equation is

$$Q = C\,Ce_2(2t, 4\Delta\omega) + D\,Se_2(2t, 4\Delta\omega) \tag{12-15}$$

where Ce_2 : the even Mathieu function of degree two

Se_2 : the odd Mathieu function of degree two

C and D are arbitrary constants which can be found when the initial conditions of the differential equation are known.

Again resistance in the circuit must be considered, so that the homogeneous differential equation for the circuit of figure 12-3 is

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC(t)} = 0 \tag{12-16}$$

It is somewhat more difficult to change this equation to the form of the Mathieu Equation, but it is advantageous to do this. Using a change of variables

$$Q = u e^{-Kt} \tag{12-17}$$

where $2K = \frac{R}{L}$

The new differential equation is

$$\ddot{Q} = \ddot{u}e^{-Kt} - 2K\dot{u}e^{-Kt} + K^2 u e^{-Kt}$$

$$2K\dot{Q} = + 2K\dot{u}e^{-Kt} - 2K^2 u e^{-Kt}$$

$$\frac{Q}{LC(t)} = + \frac{1}{LC(t)} u e^{-Kt}$$

$$\ddot{u}e^{-Kt} + \left[\frac{1}{LC(t)} - K^2\right] u e^{-Kt} = 0 \qquad (12\text{-}18)$$

and since $e^{-Kt} \neq 0$ for finite values of K and t, then

$$\ddot{u} + \left[-K^2 + \frac{1}{LC(t)}\right] u = 0 \qquad (12\text{-}19)$$

$$\ddot{u} + \left[-K^2 + \frac{1}{LC_0}\left(1 - \frac{\Delta C}{C_0}\cos\omega_p t\right)\right] u = 0 \qquad (12\text{-}20)$$

$$\ddot{u} + \omega_0^2\left[\left(1 - \frac{K^2}{\omega_0^2}\right) - 2\frac{\Delta\omega}{\omega_0}\cos\omega_p t\right] u = 0 \qquad (12\text{-}21)$$

This, too, compares very closely with the normal form of the Mathieu Equation

$$\frac{d^2y}{dz^2} + (a - 2q\cos 2z)\,y = 0 \qquad (12\text{-}22)$$

where $z = \frac{\omega_p t}{2}$

$$u = y \quad \text{and} \quad u = Qe^{Kt}$$

$$a = \frac{\omega_p^2 \omega_0^2}{4}\left(1 - \frac{K^2}{\omega_0^2}\right) = (\omega_0^2 - K^2)\frac{\omega_p^2}{4}$$

$$q = \frac{\omega_p^2}{4}\omega_0 \Delta\omega$$

The result of the change of variable is quite logical; the exponential can be considered the damping term due to the resistance in the circuit. The two solutions closely resemble the solutions found in Chapter IV for a normal L-C tank and a normal R-L-C tank circuit. The even Mathieu function can be replaced by the cosine, while the odd Mathieu function can be replaced by the sine. If the quantity, $\Delta\omega$, was zero, the solution to the ordinary differential equation could be expressed in terms of cosines and sines. The limits of the Mathieu functions as that quantity, $\Delta\omega$, approaches zero are

$$Ce_m(t, \Delta\omega)\Big|_{\Delta\omega \to 0} = \cos mt \qquad (12\text{-}23)$$

$$Se_m(t, \Delta\omega)\Big|_{\Delta\omega \to 0} = \sin mt \qquad (12\text{-}24)$$

(Ref 11:274-6)

XIII. Conclusions

In the past decade outstanding results have been obtained from parametric amplifiers. These amplifiers have overcome many of their original disadvantages and appear to be exceeding many early operational predictions. The chief asset of parametric amplifiers is their low noise figure. Typical noise figures for these amplifiers range from 2 to 6 db, while comparable noise figures for conventional tube amplifiers range from 12 to 20 db. (Ref 7:311) On the other hand, parametric amplifiers do not have as low noise figures as masers; masers are considered the ultimate in low noise amplification at this time. However, masers cannot be operated at room temperature while parametric amplifiers are designed for such operation. Masers must be operated in the liquid helium temperature region, and the disadvantages of this low temperature operation are apparent. Heffner claims that materials may be discovered that allow maser operation in the liquid hydrogen region and perhaps, even in the liquid nitrogen region, but he doubts if masers can ever be used at room temperature. (Ref 12:3) Incidentally, parametric amplifiers that have been designed to operate at these reduced temperatures have noise figures that compare favorably with masers, so that parametric amplifiers may even displace masers in some operations.

There are several short-comings of the parametric amplifier. One important disadvantage of the negative resistance parametric amplifier is that its operation is only conditionally stable. In order to obtain high gains, this amplifier, like all regenerative amplifiers, must be operated near the threshold of oscillation. In addition the bandwidth of this amplifier is quite narrow. To overcome both these defects to some extent, distributed or traveling wave parametric amplifiers have been proposed; the traveling wave amplifiers are more stable and have increased bandwidths.

Another disadvantage of the regenerative parametric amplifier is the high frequency of pumping. Since pumping must be performed at the sum of the idler and the signal frequencies, the pump must have a frequency greater than the signal. A higher frequency pump source is often not desirable--particularly when the signal frequency is in the high microwave range. To remedy this fundamental drawback, a parametric amplifier using lower frequency pumping has been suggested in which two or more pump sources at frequencies lower than the signal frequency are used. Chang and Bloom constructed such an amplifier using these frequencies:

pump $f_p = 300$ mc

signal $f_s = 380$ mc

idle $f_i = 220$ mc (Ref 13:23)

A third disadvantage is that the degenerate amplifier is sensitive to the phase of the pumping. This can be easily overcome by merely changing the frequency relationship so that an idler is used. In effect the degenerate amplifier is changed to a non-degenerate amplifier; after such a change, the phase no longer affects the operation of the amplifier.

There are relatively few conclusions that can be drawn from this type of work. One conclusion is that the analog computer can be used to simulate a parametric amplifier.

XIV. Recommendations

There are still several things that should be done before a parametric amplifier is built. First, the non-degenerate amplifier should be investigated thoroughly on the analog computer. In addition, second-order effects should be taken into account on both the degenerate and the non-degenerate case. The computer provides an excellent check on the stability of the different amplifier configurations. The stability can be theoretically calculated by the Mathieu Equation and then, verified on the analog computer.

Second, the nonlinear differential equation could be analyzed on the digital computer. Such an attack could include both small signal and large signal investigations. The analog computer approach used in this study and the Mathieu Equation are only valid for small signals. The Mathieu Equation is a linear differential equation and can only be used for portions of the C(v) curve that approximate a straight line. By working with the digital computer or the analog computer using a function generator, large signal investigations can be made.

After this analysis is complete, a non-degenerate parametric amplifier should be built. The range of frequencies should be in the low radio frequency range. To the author's knowledge, this area has not yet been investigated. Semiconductor diodes that are specifically designed for this application should be used in the amplifier. Also, piezoelectric crystals with high Q and narrow pass-band properties should be considered for use in the circuit. Finally the gain and bandwidth equations of this thesis should be verified by an experimental amplifier and an experimental up-converter. Of course, the final step could be miniaturizing the entire set-up.

For any further reading on the subject, three comprehensive bibliographies are recommended:

1. Louisell has over 200 references. (Ref 14)
2. Mount and Begg have an annotated bibliography. (Ref 15)
3. Mumford has 200 references and a history of parametric amplification. (Ref 16)

Appendix A

Equation 6-21

$$
\begin{aligned}
i_c' = j\lambda \Big\{ & \omega_1 \tilde{V}_1 \tilde{V}_1 e^{j2\omega_1 t} + \omega_1 \tilde{V}_1 \tilde{V}_{-1} \\
& + \omega_1 \tilde{V}_1 \tilde{V}_2 e^{j(\omega_1+\omega_2)t} + \omega_1 \tilde{V}_1 \tilde{V}_{-2} e^{j(\omega_1-\omega_2)t} \\
& + \omega_1 \tilde{V}_1 \tilde{V}_p e^{j(\omega_1+\omega_p)t} + \omega_1 \tilde{V}_1 \tilde{V}_{-p} e^{j(\omega_1-\omega_p)t} \\
& - \omega_1 \tilde{V}_{-1} \tilde{V}_1 + - \omega_1 \tilde{V}_{-1} \tilde{V}_{-1} e^{-j2\omega_1 t} \\
& - \omega_1 \tilde{V}_{-1} \tilde{V}_2 e^{j(\omega_2-\omega_1)t} - \omega_1 \tilde{V}_{-1} \tilde{V}_{-2} e^{-j(\omega_1+\omega_2)t} \\
& - \omega_1 \tilde{V}_{-1} \tilde{V}_p e^{j(\omega_p-\omega_1)t} - \omega_1 \tilde{V}_{-1} \tilde{V}_{-p} e^{-j(\omega_1+\omega_p)t} \\
& + \omega_2 \tilde{V}_2 \tilde{V}_1 e^{j(\omega_1+\omega_2)t} + \omega_2 \tilde{V}_2 \tilde{V}_{-1} e^{j(\omega_2-\omega_1)t} \\
& + \omega_2 \tilde{V}_2 \tilde{V}_2 e^{j2\omega_2 t} + \omega_2 \tilde{V}_2 \tilde{V}_{-2} \\
& + \omega_2 \tilde{V}_2 \tilde{V}_p e^{j(\omega_2+\omega_p)t} + \omega_2 \tilde{V}_2 \tilde{V}_{-p} e^{j(\omega_2-\omega_p)t}
\end{aligned}
$$

$$
\begin{aligned}
&- \omega_2 \tilde{V}_{-2} \tilde{V}_1 e^{j(\omega_1-\omega_2)t} && - \omega_2 \tilde{V}_{-2} \tilde{V}_{-1} e^{-j(\omega_1+\omega_2)t} \\
&- \omega_2 \tilde{V}_{-2} \tilde{V}_2 && - \omega_2 \tilde{V}_{-2} \tilde{V}_{-2} e^{-j2\omega_2 t} \\
&- \omega_2 \tilde{V}_{-2} \tilde{V}_p e^{j(\omega_p-\omega_2)t} && - \omega_2 \tilde{V}_{-2} \tilde{V}_{-p} e^{-j(\omega_2+\omega_p)t} \\
&+ \omega_p \tilde{V}_p \tilde{V}_1 e^{j(\omega_1+\omega_p)t} && + \omega_p \tilde{V}_p \tilde{V}_{-1} e^{j(\omega_p-\omega_1)t} \\
&+ \omega_p \tilde{V}_p \tilde{V}_p e^{j2\omega_p t} && + \omega_p \tilde{V}_p \tilde{V}_{-p} \\
&+ \omega_p \tilde{V}_p \tilde{V}_2 e^{j(\omega_2+\omega_p)t} && + \omega_p \tilde{V}_p \tilde{V}_{-2} e^{j(\omega_p-\omega_2)t} \\
&- \omega_p \tilde{V}_{-p} \tilde{V}_1 e^{j(\omega_1-\omega_p)t} && - \omega_p \tilde{V}_{-p} \tilde{V}_{-1} e^{-j(\omega_1+\omega_p)t} \\
&- \omega_p \tilde{V}_{-p} \tilde{V}_2 e^{j(\omega_2-\omega_p)t} && - \omega_p \tilde{V}_{-p} \tilde{V}_{-2} e^{-j(\omega_2+\omega_p)} \\
&- \omega_p \tilde{V}_{-p} \tilde{V}_p && - \omega_p \tilde{V}_{-p} \tilde{V}_{-p} e^{-j2\omega_p t} \Big\}
\end{aligned}
$$

Appendix B

Equation 7-9

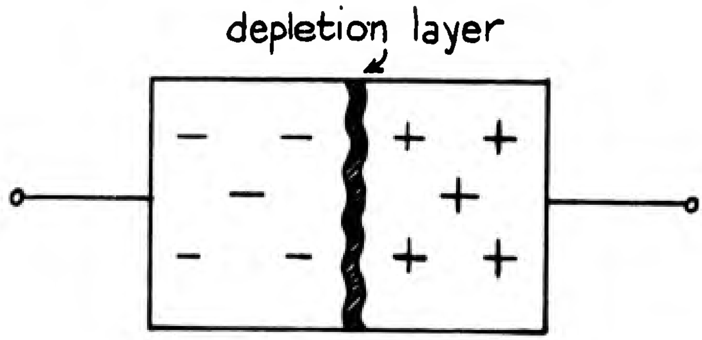
$$
\begin{aligned}
i_c' = j\lambda \Big\{ & \omega_1 \tilde{V}_1 \tilde{V}_1 e^{j2\omega_1 t} && + \omega_1 \tilde{V}_1 \tilde{V}_{-1} \\
& + \omega_1 \tilde{V}_1 \tilde{V}_p e^{j(\omega_1+\omega_p)t} && + \omega_1 \tilde{V}_1 \tilde{V}_{-p} e^{j(\omega_1-\omega_p)t} \\
& - \omega_1 \tilde{V}_{-1} V_1 && - \omega_1 \tilde{V}_{-1} \tilde{V}_{-1} e^{-j2\omega_1 t} \\
& - \omega_1 \tilde{V}_{-1} \tilde{V}_p e^{j(\omega_p-\omega_1)t} && - \omega_1 \tilde{V}_{-1} \tilde{V}_{-p} e^{-j(\omega_1+\omega_p)t} \\
& + \omega_p \tilde{V}_p \tilde{V}_1 e^{j(\omega_1+\omega_p)t} && + \omega_p \tilde{V}_p \tilde{V}_{-1} e^{j(\omega_p-\omega_1)t} \\
& + \omega_p \tilde{V}_p \tilde{V}_p e^{j2\omega_p t} && + \omega_p \tilde{V}_p \tilde{V}_{-p} \\
& - \omega_p \tilde{V}_{-p} \tilde{V}_1 e^{j(\omega_1-\omega_p)t} && - \omega_p \tilde{V}_{-p} \tilde{V}_{-1} e^{-j(\omega_1+\omega_p)t} \\
& - \omega_p \tilde{V}_{-p} \tilde{V}_p && - \omega_p \tilde{V}_{-p} \tilde{V}_{-p} e^{-j2\omega_p t} \Big\}
\end{aligned}
$$

Appendix C

Properties of a Semiconductor Diode

Although the voltage sensitivity of a semiconductor junction's capacitance has been known for some time, these diodes have only recently been used in parametric amplifiers. One reason for this is that the diodes have just now been perfected to a degree worthy of good parametric amplification. In fact, the losses in the diode are constantly being reduced by new methods of manufacture. The present parametric amplifiers use the properties of bound electrons in diodes rather than free electrons which are used in klystrons and traveling wave amplifiers. By using the electrons in a solid rather than using free electrons boiled off a hot cathode, low noise devices can be realized because the temperature of the working substance is responsible to a large degree for the amount of noise generated. Also because the movement of the carriers is so slight, there is little noise generated. This factor was mentioned in more detail in the Introduction.

One other important factor in favor of parametric amplifiers over their conventional counter-parts is that parametric amplifiers can be used at much higher frequencies. A typical electron beam contains about 10^8 electrons per



P·N JUNCTION

C-1

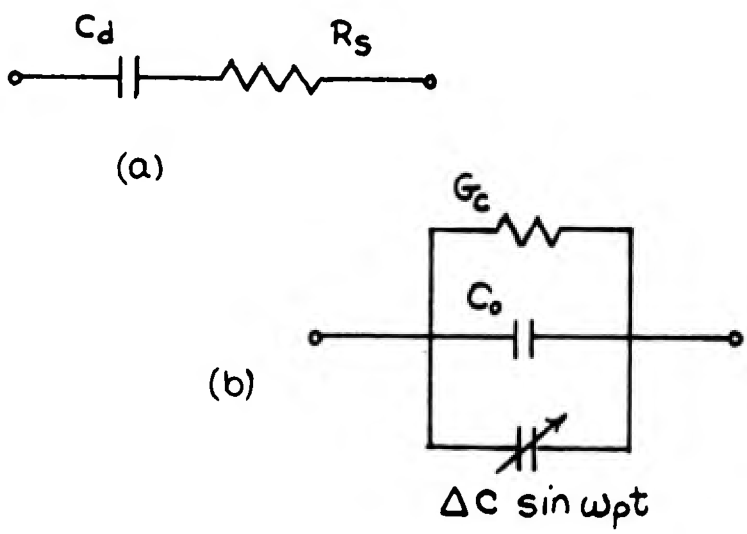


FIGURE C-2

cubic centimeter while the useable electrons in a solid may be between 10^{19} to 10^{23} electrons per cubic centimeter. This great difference in the amount of electrons offers great possibilities for high frequency amplification. (Ref 12:3)

McMahon and Straube sum up the other important advantages very well: "Voltage-variable capacitors have opened new opportunities in miniature circuitry, and have given rise to a burst of activity in the parametric amplification field. Prime advantages relative to mechanical capacitors are size, weight, electronic rather than mechanical control, speed of response, and stability under shock and vibration. Advantages relative to reactance tubes are in size, weight, life, total power, heat generation, frequency range, and stability under shock and vibration." (Ref 17:74)

Nearly every study on parametric excitation contains an account of the potentials in the semiconductor diode; consequently, this is omitted and a basic discussion of the operation of the junction is included in its place. When a reverse bias is placed across a semiconductor junction, a region which is depleted of mobile carriers is supported in the junction. This is the depletion layer that is pictured in figure C-1. As an applied voltage is varied across the junction, the width of the depletion layer changes. This depletion layer acts as an insulator, and the result is a capacitance which varies inversely as frequency.

Although the equivalent circuit of a semiconductor junction is quite complex, a simplified model can be used that is accurate in the normal operating range. This is the model of figure C-2 (a). The capacitance, C_d, is a variable capacitance that depends on the voltage. The equivalent series circuit of figure C-2 (a) can be converted into the parallel circuit of figure C-2 (b).

Since the figure of merit of the circuit is

$$Q = \frac{1}{2\pi f R_S C_d} \qquad \text{(C-1)}$$

then

$$G_c = \frac{1}{R_S(1 + Q^2)} \cong \frac{1}{R_S Q^2} \qquad \text{(C-2)}$$

and

$$C_o = \frac{C_d}{1 + \frac{1}{Q^2}} \cong C_d \qquad \text{(C-3)}$$

The approximations are valid when Q is large, and this is normally true.

A more generally useful quality figure is the $C_d R_s$ product, although the cut-off frequency, f_c, is now quite widely

C-3

C-4

C-5

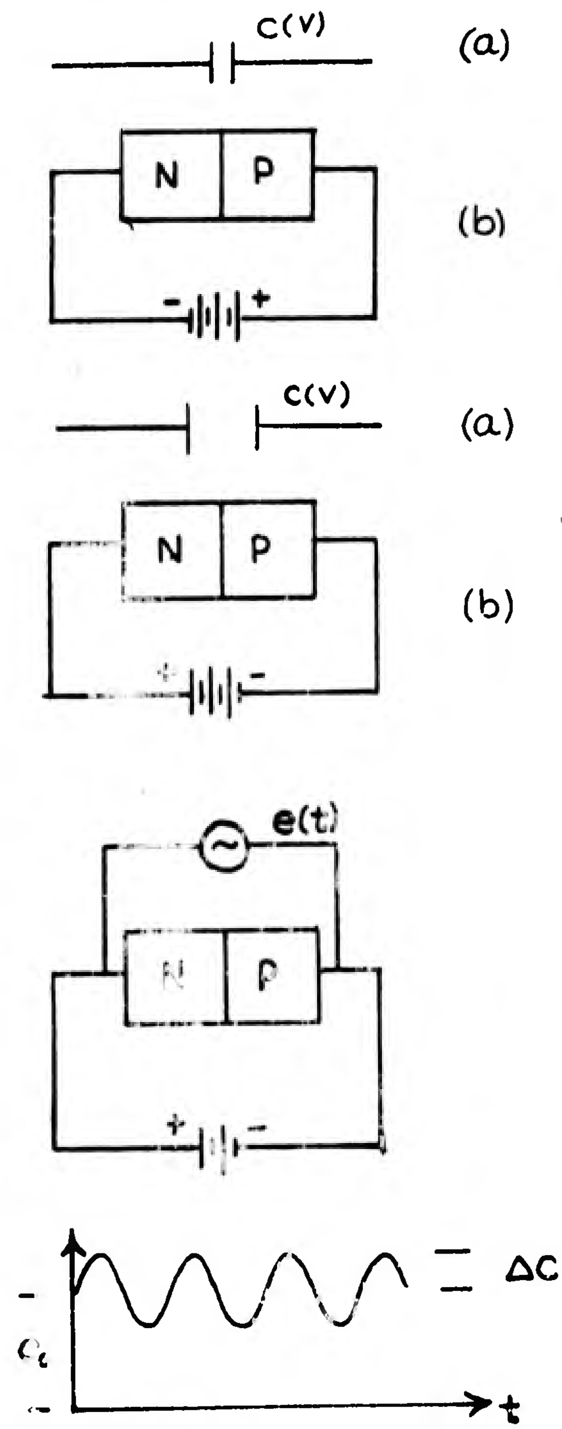


FIGURE C-6

used to show the quality of a semiconductor junction.

$$f_c = \frac{1}{2\pi R_s C_d} \qquad \text{(C-4)}$$

This semiconductor junction is similar to the capacitor that is described in Chapter II; the semiconductor junction's capacitance is changed by electronic means while the capacitor of Chapter II was varied by hand. In figure 2-3 the plates of the capacitor are periodically pulled apart and pushed back together by hand. If a junction is forward biased as in figure C-3 (b), its capacitance is similar to a capacitor with its plates nearly together: figure C-3 (a). However, if a junction is reverse biased as in figure C-4 (b), its capacitance is similar to a capacitor with its plates pulled apart: figure C-4 (a). These two cases are analogous to figure 2-4 (a) and (b); that is when the hands pulled the plates apart and pushed them back together.

In actual practice, the junction is reverse biased as shown in figure C-5 with a sinusoidal generator placed across the junction. Then the capacitance varies sinusoidally about a constant capacitance, C_o, as shown in figure C-6. This is the reason for the equivalent circuit of figure C-2 (b) where there is a constant capacitance, C_o, and a sinusoidally varying capacitance in parallel with C_o.

The capacitance, C(v), is a nonlinear function of voltage. A typical curve of the capacitance versus voltage is in figure 6-2 (a) and (b). The equation of C(v) is

$$C(v) = C\left(1 + \frac{v_o}{V_d}\right)^{-\frac{1}{m}} \tag{C-5}$$

where n = 3 for graded junction

n = 2 for abrupt junction

n $<$ 2 for hyper-abrupt junction

$$v_o = V_o \cos(\omega_o t + \theta_o)$$

V_d = direct current bias voltage (Ref 17:77)

DUAL

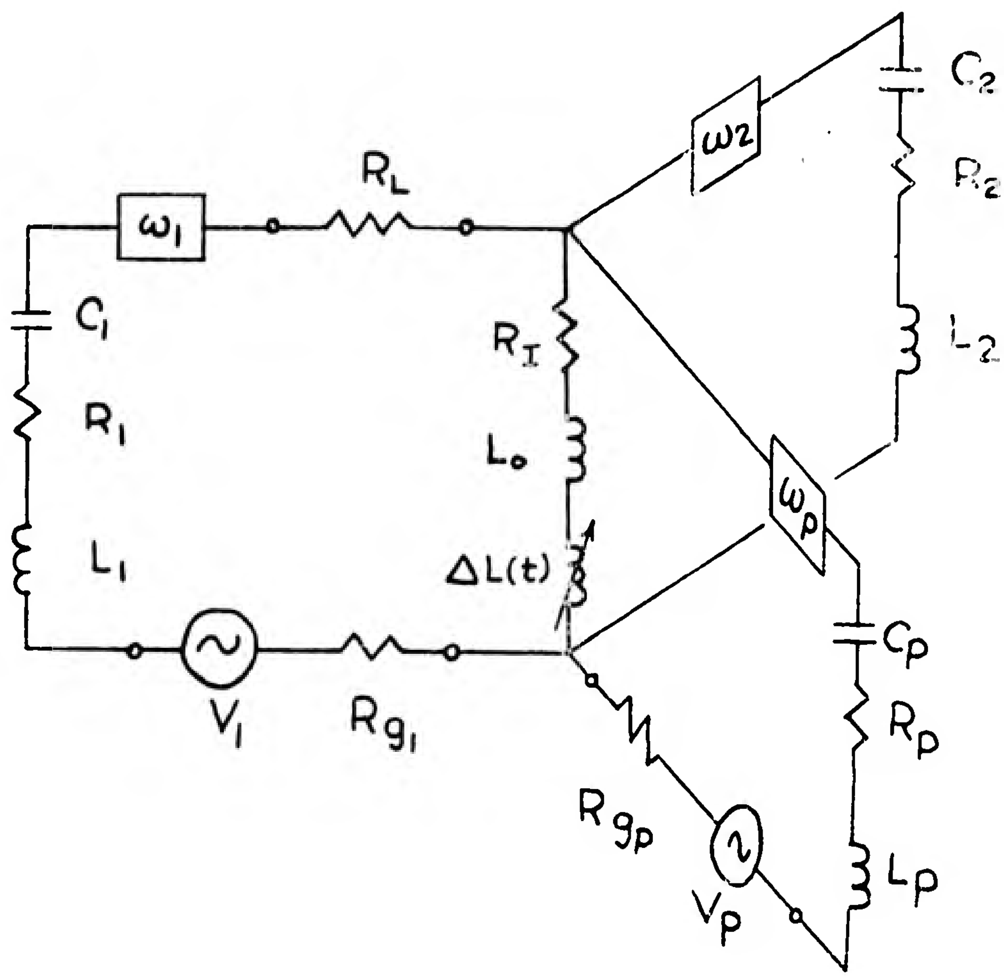


FIGURE D-1

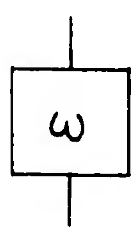
Appendix D

Some Dual Circuits

Appendix D contains several dual circuits. The derivations of the gain and bandwidth are not given because they can easily be derived by a dual approach to the expressions that are given in the text.

Figure D-1 is the series resonant form of the parametric amplifier. It is the dual of the amplifier in figure 6-1. This circuit is discussed in detail by Bloom and Chang. (Ref 18)

The properties of the ideal filter are:



$$z = 0 \quad @ \quad \Omega = \omega$$

$$z = \infty \quad @ \quad \Omega \neq \omega$$

DUAL

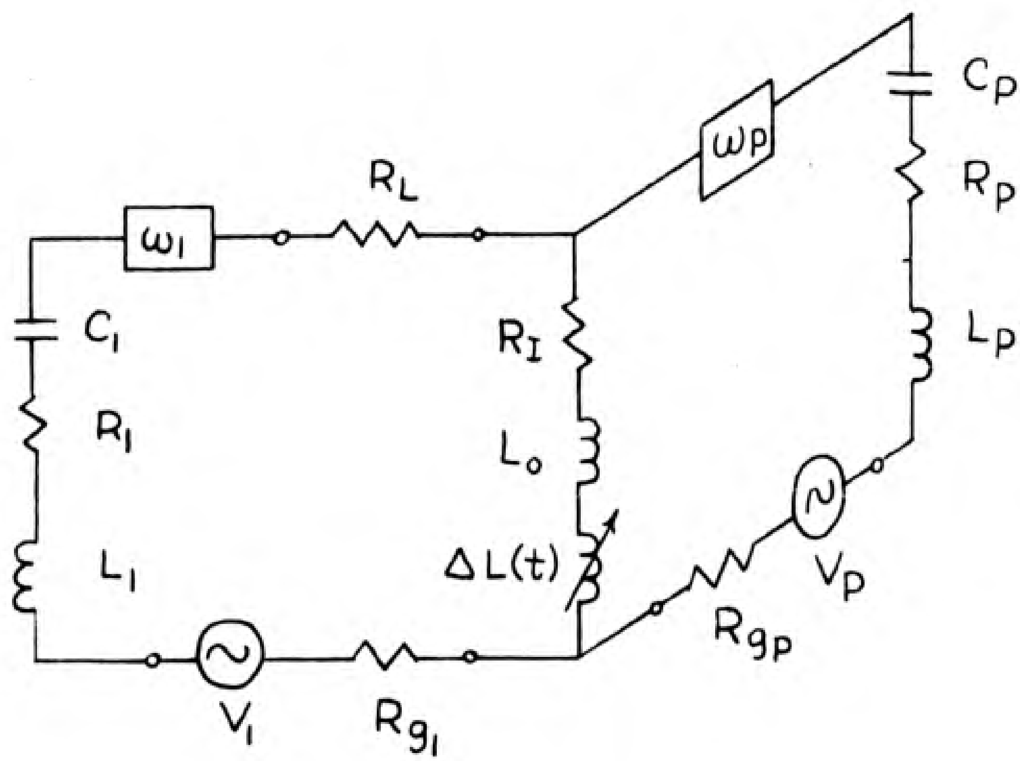


FIGURE D-2

Figure D-2 is the series resonant form of the degenerate parametric amplifier. It is the dual of the degenerate amplifier of figure 7-1.

DUAL

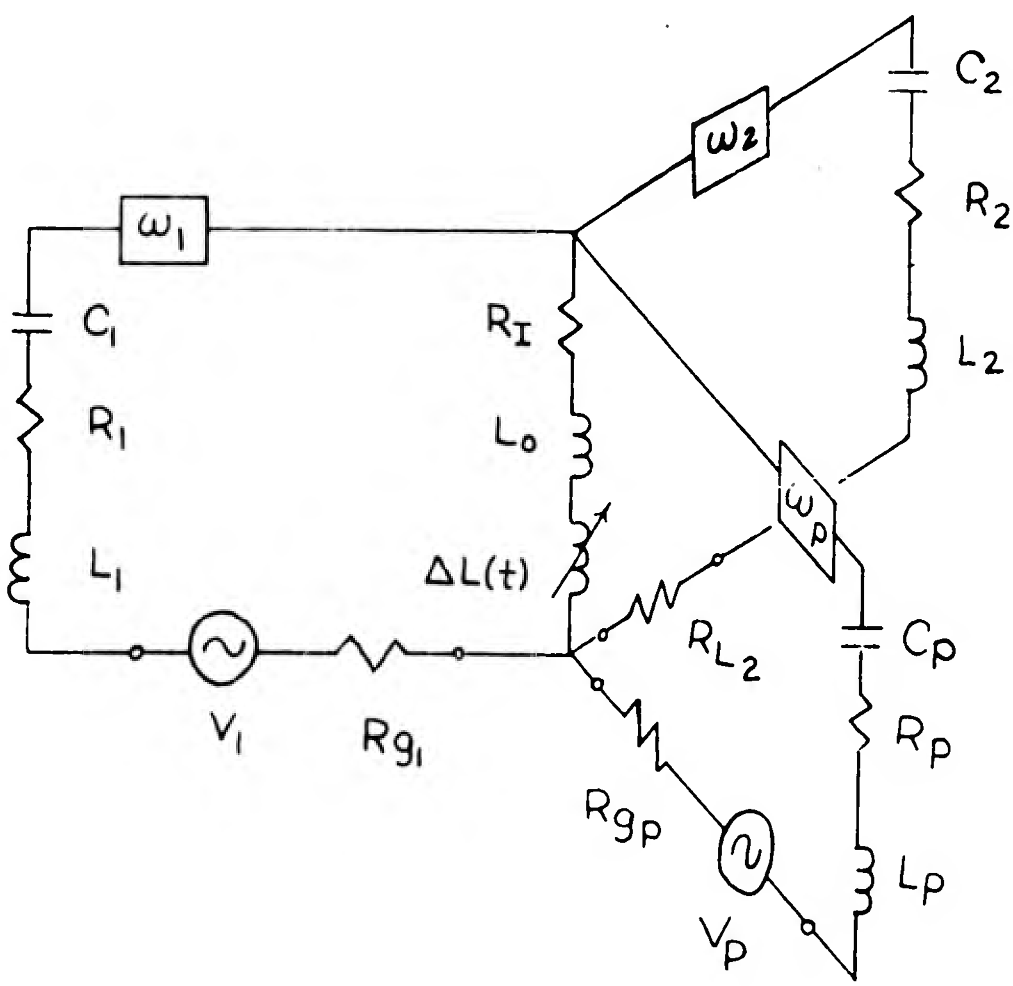


FIGURE D-3

Figure D-3 is the series resonant form of the parametric up converter. It is the dual of the amplifier of figure 10-1.

SERIES RESONANT CASE

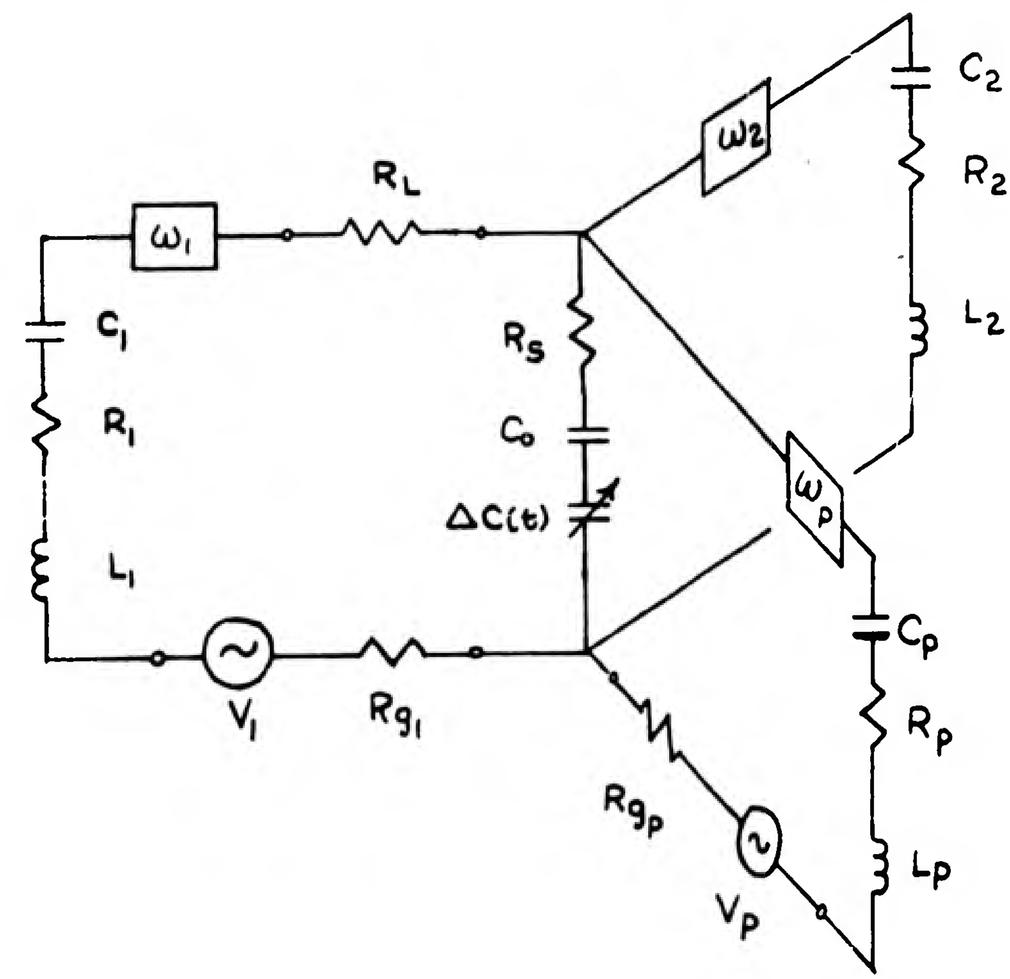
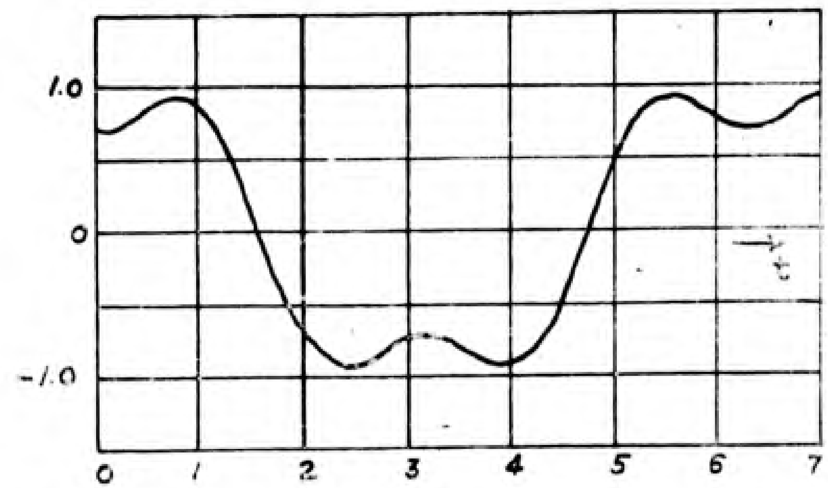


FIGURE D-4

Figure D-4 is a series resonant form of the parametric amplifier with a variable capacitor. The author has never seen such a circuit discussed; it is included in order to show the many different combinations of elements that are theoretically possible in a parametric amplifier. There is no other circuit in this thesis that is a dual of this circuit.

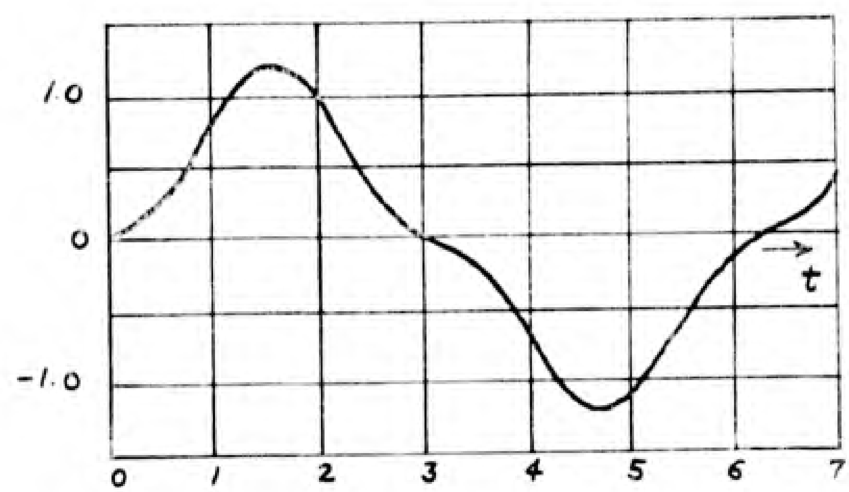
Appendix E

Mathieu Functions of Degree One



$Ce_1(t, 1)$

$\Delta\omega = 1$



$Se_1(t, 1)$

(Ref 9:14)

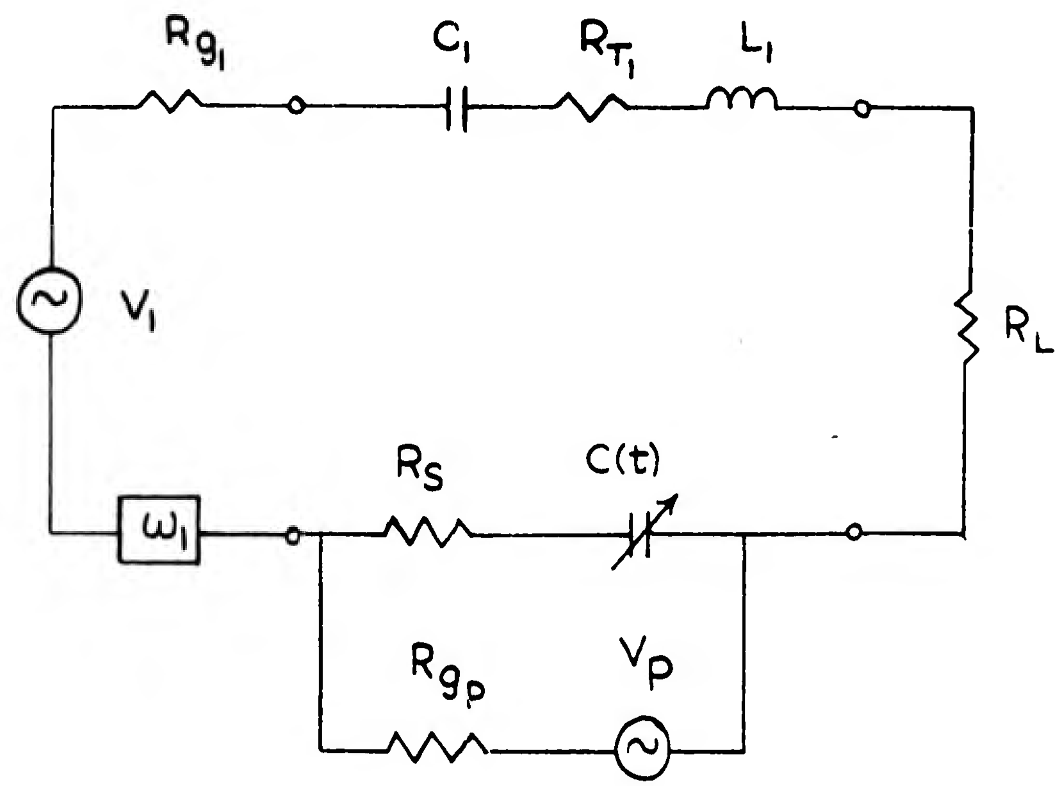


FIGURE F-1

Appendix F

A Derivation of the Mathieu Equation From the Linear Model

After this thesis was completed, it was felt that the derivation of the Mathieu Equation was open to criticism because a simplified model was used in the derivation. The following derivation is included to support the previous simplified work. In other words, there was no direct connection between the parametric amplifier of the linear analysis and the parametric amplifier of the Mathieu Equation derivation. Consequently the parametric amplifier of figure F-1 is used to derive the Mathieu Equation.

$$C(t) = C_o + \Delta C \cos \omega_p t \qquad \text{(F-1)}$$

$$C_o \gg \Delta C$$

$$C_{11} = \frac{C_1 C(t)}{C_1 + C(t)} \qquad \text{(F-2)}$$

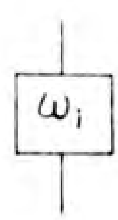
$$C_{11} = \frac{C_1 C_o + C_1 \cdot \Delta C \cos \omega_p t}{C_1 + C_o + \Delta C \cos \omega_p t} \qquad \text{(F-3)}$$

$$C_{11} \cong \frac{C_1 C_0}{C_1 + C_0} + \frac{C_1 \cdot \Delta C \cos \omega_p t}{C_1 + C_0} \qquad \text{(F-4)}$$

$$\omega_1 = \frac{1}{\sqrt{L_1 C_{11}}} \qquad \text{(F-5)}$$

$$R_{11} = R_{g_1} + R_{T_1} + R_L + R_S \qquad \text{(F-6)}$$

The properties of an ideal filter are



$$Z = 0 \quad @ \quad \Omega = \omega_1$$

$$Z = \infty \qquad \Omega \neq \omega_1$$

The differential equation of the parametric amplifier of figure F-1 is

$$V_1 \sin \omega_1 t = L_1 \frac{di}{dt} + R_{11} i + \frac{1}{C_{11}} \int_0^t i \, dt \qquad \text{(F-7)}$$

This can be changed to

$$\frac{V_1}{L_1} \sin \omega_1 t = \frac{d^2 Q}{dt^2} + \frac{R_{11}}{L_1} \frac{dQ}{dt} + \frac{1}{L_1 C_{11}(t)} Q \qquad \text{(F-8)}$$

In Chapter XII an equation similar to equation (F-8) was shown to be of the form of the Mathieu Equation.

In this case, the pump voltage is considered to be much greater than the signal voltage. For this reason, the variable capacitor is assumed to be a function of the pump frequency alone. If the magnitude of the pump and signal voltages were of the same order, then this assumption would not be true.

This parametric amplifier is the model used to simulate parametric amplifier operation on the analog computer.

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Vita

Matthew J[redacted] Quinn, Jr. was born on [redacted] in [redacted], the son of Matthew James and Mary [redacted] Quinn. After completing his work at [redacted] in 1951, he enrolled at Saint Thomas College in Saint Paul, Minnesota. In 1952 he enrolled at the United States Military Academy at West Point, New York, and graduated from that school in June 1956 with a Bachelor of Science Degree in Military Art and Engineering. He won pilot wings in 1957 and for two years flight checked navigational aids with AACS at Randolph AFB, Texas. In 1960 he entered the Air Force Institute of Technology.

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This thesis was typed by Mrs. Frances L. Athey

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