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THE THEORY OF OPTIMAL CONTROL AND THE CALCULUS OF VARIATIONS

by
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Abstract:

~~Using Carathéodory's framework,~~ the hamiltonian theory of the calculus of variations is formulated for a wide variety of problems in the theory of control. The hamiltonian function is constructed with the aid of the Minimum Principle, which is the counterpart of the same principle due to Pontryagin. The canonical differential equations of Hamilton are shown to imply Pontryagin's theorem.

A number of concrete examples are ~~also~~ included.

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1. Background.

We call today "system theory" a loose collection of problems and methods which are held together by a central theme: to understand better the complex systems created by modern technology. Aside from certain combinatorial questions, most of present system theory is concerned with problems in automatic control and in statistical estimation and prediction, with emphasis on solutions which are optimal in some sense. At present, a large variety of *ad hoc* methods are employed in system theory.

Recent research has shown how to formulate and resolve these problems in the spirit of the classical calculus of variations. This provides a unifying point of view. Eventually it should be possible to organize system theory as a rigorous and well-defined discipline. One example of this trend is the author's Duality Principle [1-3] relating control and estimation. Conversely, problems in system theory are stimulating further research in the calculus of variations.

Let us sketch here briefly the historical background of the hamiltonian formulation of the calculus of variations. There is a long stream of scientific thought concerned with wave propagation and variational principles in Nature. It begins with Huygens, continues with the work of John Bernoulli and receives maturity at the hands of the great masters of the nineteenth century: Hamilton, Jacobi, Lie. The most articulate representative of this tradition in recent times was C. Carathéodory (1873-1950). Beginning with his famous dissertation of 1904, Carathéodory insisted on the hamiltonian point of view in the calculus of variations throughout his lifetime. The evolution of his thinking on this subject is carefully integrated in his last major work [4] -- a book which is hard to obtain and difficult to digest.

The theory of optimal control (under the assumption that the equations of motion are known exactly and the state can be instantaneously measured) may be regarded as a generalization of the problem of Lagrange in the calculus of variations: minimization of an integral subject to side conditions which may be ordinary or differential equations. Carathéodory's work on the lagrange problem is incomplete, consisting of only two papers

[5-6] (which are sketched in Chapter 18 of [14]). The problem is one of extreme difficulty, and has received very little attention until very recently.

In [7] the present writer gave a new formulation of the problem of optimal control from the hamiltonian point of view. The purpose of this paper is to extend this approach. We shall see that this formulation -- which differs from Carathéodory's in essential details -- explains a number of recent results in the theory of control and provides a very general framework for further research. In particular, the so-called "maximum principle" of Pontryagin [8] will arise in a simple and natural way as part of the definition of the hamiltonian function of the problem.

We hope to give a deeper and more detailed treatment of the subject in the near future.

2. The Variational Problem in the Theory of Control.

We assume that the control object is a dynamical system governed by the differential equation

$$(2.1) \qquad dx/dt \equiv \dot{x} = f(x, u(t), t).$$

Here x is a real n-vector, called the state of the system; $u(t)$ is a real m-vector for each t; f is a real n-vector which is continuously differentiable in all arguments.

To avoid the cumbersome phrase "the state x at time t", we shall refer to the couple (x, t) as a phase. The phase space is thus the cartesian product of the state space $X(= R^n)$ with the set $T(= R^1)$ of all values of the time.

We call the function $u(t)$ in (2.1) an admissible control if (i) it is piecewise continuous in t; (ii) for each t, its values belong to a given closed subset $U(t)$ of R^m.

For any admissible control u and any initial phase (x_0, t_0) there exists a unique absolutely continuous function ϕ of t, denoted by

$$\phi(t) \equiv \phi_u(t; x_0, t_0)$$

which satisfies (2.1) identically almost everywhere* and which has the property

$$\phi(t_0) = \phi_u(t_0; x_0, t_0) = x_0.$$

We call $\phi_u(t; x_0, t_0)$ the motion of (2.1) passing through x_0 at time t_0 under the action of the control u. Sometimes we shall write $x(t) = \phi(t)$ to emphasize the fact that the value of ϕ at some fixed t is the state of the system at that time.

We call x^* an equilibrium state if there is some control u such that $\phi_u(t; x^*, t_0) = x^*$ for all t, t_0, or, equivalently, $f(x^*, u(t), t) \equiv 0$.

* In some region G of the phase space containing (x_0, t_0).

To state the control problem in its simplest form, it is assumed further that physical measurements are available which provide the exact numerical value of the state at every instant of time. (Of course, this is a gross idealization from the engineering point of view.) We want to determine $u(t)$ as a function of $x(t)$ so that motions of (2.1) have certain extremal properties. To express $u(t)$ as a function of $x(t)$ is commonly called feedback in engineering. We denote this functional relationship by

$$(2.2) \qquad u(t) = k(x(t), t),$$

and refer to the function k as the control law. A control law is admissible if $k(x, t) \in U(t)$ for all t.

Let (x_0, t_0) be an arbitrary phase and let S be a surface in the phase space. Consider the following scalar functional of motions of (2.1):

$$(2.3) \qquad V(x_0, t_0, S; u) = \lambda(\phi_u(t_1; x_0, t_0), t_1) + \int_{t_0}^{t_1} L(\phi_u(t; x_0, t_0), u(t), t)dt,$$

where L, λ are scalar functions and t_1 is the first instant of time after t_0 when the motion enters the set S. Thus λ need be defined only on S. We call t_1 the terminal time. We assume that L, λ are continuously differentiable in all arguments.

In terms of these notations, we can now state the

(2.4) OPTIMAL CONTROL PROBLEM. Given any initial phase (x_0, t_0), find a corresponding admissible control u defined in the interval $[t_0, t_1]$ at which the functional (2.3) assumes its infimum (or supremum) with regard to the set of all admissible controls.

Actually, for technological reasons one usually sets a slightly stronger objective.

(2.5) OPTIMAL FEEDBACK CONTROL PROBLEM. Find a control law such that (when (2.2) is substituted in (2.1)) the functional (2.3) assumes its infimum (or supremum) with regard to the set of all admissible control laws.

Bellman's Principle of Optimality shows that we can always define an optimal control law along every optimal motion. Hence (2.4) and (2.5) are abstractly equivalent.

If equation (2.1) depends on stochastic factors, however, then the infimum of (2.3) with respect to all admissible control laws will be usually lower than with respect to all admissible controls which are uniquely determined by the initial phase. This is because the control law takes into account not only the initial state but successive states as well; the added information so obtained may result in a better optimum.

Before embarking on a detailed analysis of the control problem, let us mention a number of typical examples which may be put into this formulation.

(2.6) TERMINAL CONTROL. The problem is to bring the state of the system as close as possible to a given terminal state x_1 at a given terminal time t_1. Then $L = 0$, $\lambda(x)$ is the distance of x from x_1, and $S = X \times \{t_1\}$.

(2.7) MINIMAL-TIME CONTROL. Suppose we want to reach a state x_1 from (x_0, t_0) in the shortest possible time. We then set $L = 1$, $\lambda = 0$, and $S = \{x_1\} \times T$. This problem has a solution as a rule only if $U(t)$ is a bounded set for all $t \geq t_0$.

(2.8) REGULATOR PROBLEM. We assume that the system is in some initial phase (x_0, t_0) and we wish to return to some equilibrium state x^* in such a way that some integral of the motion is minimized. We then usually take L and λ as nonnegative. The dependence of L on u is needed because otherwise the problem may not have a solution. The set S is again $X \times \{t_1\}$.

(2.9) PURSUIT PROBLEM. We are given a moving target $\xi(t)$. The problem is to bring the motion to phase $(\xi(t), t)$ as soon as possible.

This is a generalization of the minimal-time problem; we take $S = \{(\xi(t), t);\ t \in T\}$.

(2.10) SERVOMECHANISM PROBLEM. This is a generalization of the regulator problem. We are given a desired state $\xi(t)$, $t \in T$. The problem is to cause the phase of the controlled motion to be as close as possible to $(\xi(t), t)$ on the interval $[t_o, t_1]$. The instantaneous distance between $(x(t), t)$ and $(\xi(t), t)$ is measured by L. The set S is again as in (2.8).

(2.11) MINIMUM ENERGY CONTROL. We wish to transfer from an initial phase (x_o, t_o) to a final phase (x_1, t_1) with the expenditure of a minimal amount of control energy. In this case we take L to be a nonnegative function of u, independent of ϕ; S is the set consisting of the single point (x_1, t_1); λ is immaterial.

(2.12) ISOPERIMETRIC PROBLEMS. Suppose that the optimal motions must satisfy also the so-called isoperimetric constraints

(2.13) $$\int_{t_o}^{t_1} f_{n+k}(\phi_u(t;\ x_o,\ t_o),\ u(t),\ t)dt \leq \alpha_k, \qquad k = 1, \ldots, N - n.$$

These problems reduce immediately to the preceding ones, by replacing the n-vector x by an N-vector whose last N - n components satisfy the differential equations

(2.14) $$dx_{n+k}/dt = f_{n+k}(x,\ u(t),\ t), \qquad k = 1, \ldots, N - n;$$

the initial values are $x_{n+k}(t_o) = 0$ and the final values $x_{n+k}(t_1)$ are to lie on a surface S where $x_{n+k} \leq \alpha_k$.

3. Relations with the Calculus of Variations.

The classical problem of Lagrange in the calculus of variations is concerned with the minimization of the integral

$$\int L(x(t), \dot{x}(t), t)dt \tag{3.1}$$

with respect to any smooth curve $x(t)$ which (i) connects a given point (x_0, t_0) with a point (x_1, t_1) lying on a given surface S, and (ii) satisfies the constraints

$$g_i(x(t), \dot{x}(t), t) = 0, \qquad i = 1, \ldots, n - m. \tag{3.2}$$

There are two ways in which the optimal control problem discussed above differs from the lagrange problem. First, the function L depends on u rather than on $\dot{x}$. Second, the constraints are of a mixed type:

$$\dot{x} - f(x, u(t), t) = 0 \quad \text{and} \quad u(t) \in U(t). \tag{3.3}$$

Neither of these differences is essential. Inequality constraints such as $\alpha \geq 0$ can be replaced by equality constraints such as $\beta(\alpha) = 0$ where β is a smooth function which is zero if $\alpha \geq 0$ and positive otherwise. Similarly, one can always express $u(t)$ from (3.3) as a function of $x, \dot{x}, t$, introducing, if necessary, additional equality type constraints. Hence the optimal control problem is formally identical with the lagrange problem. However, the transformations necessary to establish the equivalence will be usually rather complicated. Moreover, because of difficulties arising from an explicit treatment of the constraints (3.2), the theory of the lagrange problem today is far from adequate.

We therefore prefer to treat directly the problem of minimizing (2.3), subject to the constraints (3.3). This treatment includes of course the ordinary problem of the calculus of variations (upon setting $f(x, u, t) = u$ and $U(t) = R^n$), as well as the lagrange problem (after suitable transformations of the type just discussed).

Using the hamiltonian point of view, we need not transform the constraints (3.3) but can treat them directly. The principal idea is the following. We define a hamiltonian function not with the aid of the legrendre transformation (as is usual), but by a more general procedure, the so-called Minimum Principle. In this way the optimum control problem can be reduced to the solution of the hamilton-jacobi partial differential equation. The existence of a solution of the hamilton-jacobi equation is a <u>sufficient</u> condition for the solution of the optimal control problem. If the function $V^o(x, t)$ is smooth, this condition is also necessary.

Unfortunately, quite often $V^o(x, t)$ does not have continuous partial derivatives with respect to x. In that case one cannot state necessary and sufficient conditions <u>solely in terms of differential equations</u>. But this is not the issue. The main objective is always to develop methods by which we can eventually discover a complete solution of the problem. These methods usually take the form of sufficient conditions. As a matter of fact, early in his career, Carathéodory took the position that:

"The distinction between necessary and sufficient conditions seems, however, always a little artificial; explicit proof that certain conditions are necessary is of interest only in cases where one cannot resolve a problem at once, and it serves, above all, to limit the scope of future investigations. When, on the other hand, one has a solution possessing all the properties required by the theorem, it suffices to show that this solution is unique in order to have at the same time the proof that <u>all the conditions which serve to determine the solution are necessary</u>."*

It has unfortunately become very common in physical and engineering applications to regard the extremals supplied by the euler equations as the "solution" of a variational problem. There are two long-standing objections to this: (i) the euler equations may not exist (as when L is not sufficiently smooth); (ii) the solutions of the euler equations may cease to define a minimum or a maximum after a certain interval of time (as

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*Writer's translation from French; writer's italics. See [9, Introduction].

when the extremal contains conjugate points). The hamiltonian point of view, which aims to obtain sufficient conditions, avoids such difficulties at the outset by considering only these initial phases which can be connected by optimal motion with a phase on S, and by regarding the function $V^0 = \min V$ as abstractly defined in advance.

The dynamic programming method of Bellman proceeds from the same fundamental idea, differing only in detail from the hamiltonian methods. For a nonrigorous but highly enlightening discussion of the relations between the two, see the recent paper of Dreyfus [10].

4. The Hamilton-Jacobi Equation; Minimum Principle.

Let us first obtain the sufficient condition. The starting point is the following trivial, well-known, but important observation:

(4.1) CARATHEODORY LEMMA [4, p. 198]. Consider Problem (2.4). Suppose there is a function $k(x, t)$, continuously differentiable in both arguments, and such that for all (x, t) in some region G of the phase space

(i) $k(x, t) \in U(t)$,

(ii) $L(x, k, t) = 0$,

(iii) $L(x, u, t) > 0$ if $u \neq k(x, t)$.

Consider motions of (2.1) with control law defined by (2.2), i.e.,

$$dx/dt = f(x, k(x, t), t). \tag{4.2}$$

Let the initial phase (x_0, t_0) belong to G. Let $\lambda(x, t)$ be identically zero on some surface $S \subset G$ of the phase space. Then for any motion ϕ^0 of (4.2) which connects (x_0, t_0) with a phase on S and remains entirely in G

(a) the value of the integral (2.3) is zero;

(b) the motion ϕ^0 provides the absolute minimum of (2.3) with respect to any other motion of (2.1) which connects (x_0, t_0) with S and remains entirely in G.

In short, the hypotheses of the lemma mean that at every point in G the integrand L has a unique, absolute minimum $u^0 = k(x, t)$ with respect to all u satisfying the constraint (3.3). Then k is the unique optimal control law, and Problem (2.5) is also solved.

Proof. Conclusion (a) is immediate, since for any motion ϕ^0 of (4.2) the integral (2.4) is zero by hypothesis (ii). Now let ϕ^1 be any other motion of (2.1) which connects (x_0, t_0) with S without leaving G, and for which $V = 0$. Then by hypothesis (iii) and the continuity of

L, it is clear that along ϕ^1 we must have $u^1(t) = k(\phi^1(t; x_o, t_o), t)$ at every continuity point of $u^1(t)$, since otherwise $V > 0$. We would obtain the same motion if we let $u^1(t)$ be always defined by this relation; in other words, $\phi_{u^1}(t; x_o, t_o) = \phi^1(t; x_o, t_o)$. But since k is continuously differentiable in x, (4.2) defines a unique motion, and the proof of (b) is complete.

It should be noted that there may be phases in G such that the motion defined by (4.2) going through these phases is not optimal -- this is due to the possibility that a motion may leave G prior to reaching S.

Now we try to construct a lagrange function L* and a corresponding function k which satisfy the requirements of the lemma.

Suppose $V^o(x, t)$ is a scalar function which is twice continuously differentiable in both arguments. Then

$$(4.3) \qquad \int_{t_o}^{t_1} [V_t^o(x,t) + f(x,u(t),t)\cdot V_x^o(x,t)]dt = V^o(x_1,t_o) - V^o(x_o,t_o) \quad *$$

along any motion of (2.1) which connects the phase (x_o, t_o) with the phase (x_1, t_1) on S. If we let

$$(4.4) \qquad V^o(x, t) = \lambda(x, t)$$

on S, then the variational problem (2.4) obtained by replacing λ with $\lambda^* = 0$ and L with

$$(4.5) \qquad L^*(x, u, t) = L(x, u, t) + V_t^o(x, t) + f(x, u, t)\cdot V_x^o(x, t)$$

will be equivalent to the original problem, because the values of V and V* will differ only by $V^o(x_o, t_o)$ which does not depend on the control u.

Let p be a real n-vector, called the costate.

We define a scalar function H by

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* The dot denotes the inner product; $V_t = \partial V/\partial t$, $V_x = \text{grad}_x V$.

$$H(x, p, t, u) = L(x, u, t) + f(x, u, t)\cdot p. \tag{4.6}$$

We assume that H has a unique absolute minimum for each t with respect to $u(t) \in U(t)$ at the point

$$u^o(t) = c(k, p, t); \tag{4.7}$$

moreover, c is continuously differentiable in all arguments.

The scalar function H^o defined by

$$\begin{aligned} H^o(x, p, t) &= \min_{u(t) \in U(t)} H(x, p, t, u) \\ &= L(x, c(x, p, t), t) + f(x, c(x, p, t), t)\cdot p \end{aligned} \tag{4.8}$$

is the hamiltonian of the problem.

Finally, we assume that $V^o(x, t)$ satisfies the hamilton-jacobi partial differential equation

$$V^o_t + H^o(x, V^o_x, t) = 0 \tag{4.9}$$

with the boundary condition (4.4).

If these assumptions hold, we let the costate be defined by

$$p = V^o_x(x, t). \tag{4.10}$$

Then

$$L^*(x, u, t) = V^o_t(x, t) + H^o(x, V^o_x(x, t), t) \tag{4.11}$$

will clearly satisfy the hypotheses of the carathéodory lemma, with k defined by

$$k(x, t) = c(x, V^o_x(x, t), t). \tag{4.12}$$

Moreover, by the lemma, we have also that

$$(4.13) \qquad V^o(x_o, t_o) = \lambda(\phi^o(t_1), t_1) + \int_{t_o}^{t_1} L(\phi^o(t), k(\phi^o(t), t), t)dt$$

along motions ϕ^o satisfying (4.2). In other words,

$$(4.14) \qquad V^o(x_o, t_o) = \min_u V(x_o, t_o, S; u)$$

is the absolute minimum of the integral (2.3) with respect to admissible controls; (4.12) is the optimal control law, and we have also solved Problem (2.5).

Hence we have arrived at the following

(4.15) SUFFICIENT CONDITION. Let H^o be the absolute minimum of $H = L + f \cdot p$ with respect to $u(t) \epsilon U(t)$. Suppose that the preceding continuity and differentiability hypotheses hold, and that V^o satisfies the hamilton-jacobi partial differential equation $V^o_t + H^o(x, V^o_x, t) = 0$ in a region $G \subset S$, and that furthermore $V = \lambda$ on S. Then

(a) $V^o(x_o, t_o)$ is the absolute minimum of (2.3) with respect to all motions which connect (x_o, t_o) with a phase on S without leaving G;

(b) the optimal control law is given by (4.12); with this control law any motion which eventually reaches S without leaving G is optimal.

The introduction of the hamiltonian function H^o reduces the problem to one of ordinary minimization, which defines the optimal value of $u(t)$ at each moment through (4.7). In order to achieve this, we brought in the auxiliary variable p. To make sure that the point-by-point optimization based on p is consistent, we eliminate p by (4.10). The construction succeeds whenever V^o is a solution of the hamilton-jacobi equation, provided V^o is a sufficiently smooth function of x.

(4.16) NECESSARY CONDITION. Let G be a region in the phase space which possesses the following properties:

(i) There is an optimal motion from every phase in G to a phase on S which never leaves G;

(ii) the minimum value of (2.3), denoted by $V^o(x, t)$, is twice continuously differentiable in both arguments;

(iii) every point in G which is not also on S has a neighborhood lying entirely in G;

(iv)* for every phase in G, $H(x, V^o_x, t, u)$ given by (4.6) has an absolute minimum $H^o(x, V^o_x, t)$ at $u^o = k(x, t)$ with respect to $u(t) \in U(t)$;

(v)* the function k defining the minimum is differentiable in x and continuous in t.

Then the function $V^o(x, t)$ satisfies the hamilton-jacobi equation $V^o_t + H^o(x, V^o_x, t) = 0$ in the region G.

Proof. Let (x_o, t_o) be a phase in G where the theorem is false. There are then two possibilities. We consider first

$$V^o_t(x_o, t_o) + H^o(x_o, V^o_x(x_o, t_o), t_o) > 0. \tag{4.17}$$

Let $N \subset G$ be an open neighborhood of (x_o, t_o) which is small enough so that the inequality (4.17) remains true everywhere in N. It is clear that N exists because of (iii) and because the left-hand side of (4.17) is continuous in x and t. Let $\phi^o(t)$ be an optimal motion originating at (x_o, t_o), and let $u^o(t)$ be the corresponding optimal control. Then, because of the definition of H^o, we have for all t such that $(\phi^o(t),t) \in N$

$$H(\phi^o(t), V^o_x(\phi^o(t),t),t, u^o(t)) \geq H^o(\phi^o(t), V^o_x(\phi^o(t),t),t). \tag{4.18}$$

Combining (4.17 - 18), we have

$$-V^o_t(\phi^o(t),t) - V^o_x(\phi^o(t),t)\cdot f(\phi^o(t),u^o(t),t) = L(\phi^o(t),u^o(t),t) - \epsilon(t) \tag{4.19}$$

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* These conditions may be checked from the given form of L, f, and U -- i.e., without solving the variational problem.

with $\epsilon > 0$ as long as $(\phi^o(t), t)$ remains in N. Let $t_1 > t_o$ such that $(\phi^o(t), t) \in N$ for all $t \in [t_o, t_1]$. Integrating both sides of (4.19) we have

$$V^o(x_o,t_o) - V^o(x_1,t_1) = \int_{t_o}^{t_1}[L(\phi^o(t),u^o(t),t) - \epsilon(t)]dt < \int_{t_o}^{t_1} L(\phi^o(t),u^o(t),t)dt$$

or, using the definition of $V^o(x_1, t_1)$ and letting (x_2, t_2) be the phase on S reached by an optimal motion ϕ^o starting at (x_1, t_1),

$$V^o(x_o, t_o) < \lambda(x_2, t_2) + \int_{t_o}^{t_2} L(\phi^o(t), u^o(t), t)dt$$

which contradicts the assumption that ϕ^o is optimal.

Now we suppose that N is an open neighborhood of (x_o, t_o) throughout which the inequality (4.17) holds in the opposite sense. Therefore, by definition of H^o

$$-V^o_t(x,t) - V^o_t(x,t)\cdot f(x,k(x,t),t) = L(x,k(x,t),t) + \epsilon(x,t)$$

where $\epsilon > 0$ throughout N.

Hence, integrating along the unique motion $\phi_k(t; x_o, t_o)$ defined by (4.2) we have

$$V^o(x_o,t_o) - V^o(x_1,t_1) = \int_{t_o}^{t_1}[L(x,k(x,t),t) + \epsilon(x,t)]dt > \int_{t_o}^{t_1} L(x,k(x,t),t)dt$$

provided $(\phi_k(t; x_o, t_o), t) \in N$ for all $t \in [t_o, t_1]$. This contradicts the definition of V^o, by the same argument as above. Q. E. D.

The essence of the arguments in this section is replacing the hamiltonian H by the hamiltonian H^o by eliminating u with the aid of the minimum operation (4.8). We shall call this the Minimum Principle.

5. Canonical Differential Equations; Pontryagin's Theorem.

At this stage, the optimal control problem is reduced to the problem of solving the hamilton-jacobi partial differential equation. Following Carathéodory's program, one can go a step further and show that the optimal motions must be solutions of the characteristics of the hamilton-jacobi equation, which are a set of 2n-th order ordinary differential equations. They are the euler equations in canonical form -- or simply the canonical equations -- of the problem.

In this way, the determination of optimal motions reduces to the solution of the canonical equations. But in order to show that a given motion is really optimal, one must still construct -- abstractly or explicitly -- a solution of the hamilton-jacobi partial differential equation or, what is the same thing in view of (4.16), the function $V^o(x, t)$. Moreover, the solution of the canonical equations does not provide the optimal control law for which -- see (4.12) -- knowledge of V^o is essential.

Let G be a region in the phase space satisfying the hypotheses (i-v) of Theorem (4.16). Let $\phi^o(t)$ be an optimal motion which starts at some phase in G and eventually reaches a phase on S without leaving G. We define

$$\psi^o(t) = V^o_x(\phi^o(t), t). \tag{5.1}$$

Differentiating $\psi^o(t)$ with respect to t we have

$$d\psi^o(t)/dt = V^o_{xt}(\phi^o(t), t) + V^o_{xx}(\phi^o(t), t)\cdot f(\phi^o(t), u^o(t), t). \tag{5.2}$$

Differentiating the hamilton-jacobi equation (regarded as an identity) with respect to x yields

$$V^o_{xt}(x, t) + H^o_x(x, V^o_x(x, t), t) = 0 \tag{5.3}$$

throughout G. Recalling the definition of H and combining (5.2-3), we see that the $\phi^o(t)$, $\psi^o(t)$ must be solutions of the equations

$$(5.4)\qquad \begin{cases} dx/dt = H_p(x,\ c(x,\ p,\ t),\ t), \\ dp/dt = -\ H_x^o(x,\ p,\ t). \end{cases}$$

These are not yet the canonical equations because they are not stated in terms of a single hamiltonian. To remove this difficulty, we assume that the boundary of $U(t)$ is smooth in the space $R^m \times T$.

Consider a point $(x_o,\ p_o,\ t_o)$ and the corresponding $u_o^o = c(x_o,\ p_o,\ t_o) \in U(t_o)$ at which H assumes its unique absolute minimum. Recall that $U(t_o)$ is closed. The following possibilities arise:

(1) u_o^o is an interior point of $U(t_o)$. Then the first derivative of H with respect to u must vanish at u_o^o:

$$(5.5)\qquad L_u(x_o,\ u_o^o,\ t_o) + p_o \cdot f_u(x_o,\ u_o^o,\ t_o) = 0.$$

(2) u_o^o is on the boundary of $U(t_o)$. There are now two subcases.

(2-i) There is at least one point in every neighborhood of $(x_o,\ p_o,\ t_o)$ such that the corresponding u^o is an interior point of $U(t)$. Then (5.5) holds also at $(x_o,\ p_o,\ t_o)$ since L_u, f_u, c are continuous in all arguments.

(2-ii) There is a neighborhood N of $(x_o,\ p_o,\ t_o)$ such that every u^o corresponding to points in N lies on the boundary of $U(t)$. In this case we must have throughout N

$$g^i(c(x,\ p,\ t),\ t) = 0, \qquad i = 1,\ \ldots,\ q \leq m.$$

Since the boundary of $U(t)$ is to be smooth, we assume that the functions g^i are differentiable in both arguments and also that the determinant

$$\left| \frac{\partial g^i(u,\ t)}{\partial u_j} \right|$$

has rank q at the point $(u_o^o,\ t_o)$. Then the well-known lagrange multiplier rule [4: p. 166] implies that

$$(5.6)\qquad L_u(x_o, u_o^o, t_o) + p_o \cdot f_u(x_o, u_o^o, t_o) + \sum_{i=1}^{q} \nu_i g_u^i(u_o^o, t_o) = 0.$$

with $v_i \neq 0$. On the other hand, differentiating $g^i(c(x, p, t), t) = 0$ with respect to x and p shows that

$$\left.\begin{array}{l} g^i_u(c(x, p, t), t)\cdot c_x(x, p, t) = 0 \\ g^i_u(c(x, p, t), t)\cdot c_p(x, p, t) = 0 \end{array}\right\} \quad i = 1, \ldots, q.$$

Combining the foregoing two equations, we have

$$(5.7) \quad \begin{cases} [L_u(x_o, u^o_o, t_o) + p_o\cdot f_u(x_o, u^o_o, t_o)]\cdot c_x(x_o, p_o, t_o) = 0, \\ [L_u(x_o, u^o_o, t_o) + p_o\cdot f_u(x_o, u^o_o, t_o)]\cdot c_p(x_o, p_o, t_o) = 0. \end{cases}$$

Hence we conclude that

$$(5.8) \quad \begin{cases} H_x(x, p, t, \; c(x, p, t)) = H^o_x(x, p, t) \\ H_p(x, p, t, \; c(x, p, t)) = H^o_p(x, p, t) \end{cases}$$

which follows immediately from (5.7) in case (2-ii) and from (5.6) in the other cases.

In view of (5.8), the canonical equations (5.4) take on their usual form

$$(5.9) \quad \begin{cases} dx/dt = H^o_p(x, p, t) = H_p(x, p, t, \; c(x, p, t)), \\ dp/dt = - H^o_x(x, p, t) = - H_x(x, p, t, \; c(x, p, t)), \end{cases}$$

which could also be written as the identities

$$(5.10) \quad \begin{cases} d\phi^o(t)/dt = H_p(\phi^o(t), \; \psi^o(t), t, u^o(t)), \\ d\psi^o(t)/dt = - H_x(\phi^o(t), \psi^o(t), t, u^o(t)). \end{cases}$$

The last equations constitute in effect

(5.11) PONTRYAGIN'S THEOREM [8]. <u>If the motion</u> $\phi^o(t)$ <u>is optimal with control</u> $u^o(t)$, <u>then there must exist a function</u> $\psi^o(t)$ <u>such that</u> (5.10) <u>is satisfied, and in addition the relation</u>

$$(5.12) \qquad H(\phi^o(t), \psi^o(t), t, u) > H(\phi^o(t), \psi^o(t), t, u^o(t))$$

<u>must hold for all</u> $u \in U(t)$ <u>not equal to</u> $u^o(t)$.

Equation (5.12) is Pontryagin's form of the Minimum Principle. It is proved [8, 11] by constructing a special first variation of the function $u^o(t)$. (In Pontryagin's paper, the standard convention of defining H is followed, which is equal to minus the quantity (4.6). For this reason, Pontryagin speaks of the "maximum" principle. We feel the present choice of sign, which is motivated by the dynamic programming approach to the definition of V^o, is more natural.)

Actually, Pontryagin's theorem can be proved nowadays [11] without the strong smoothness assumptions concerning V^o. But in that case one cannot identify $\psi^o(t)$ with $V^o_x(\phi^o(t), t)$, and there remains a gap between the necessary condition represented by Pontryagin's form (5.10) of the euler equations and the hamilton-jacobi-carathéodory theory which we have sketched above.

Nevertheless, our theory can still be used for the effective solution of problems where $V^o(x, t)$ does not have continuous second derivatives throughout the phase space. This is illustrated in the next section.

6. Solution of a Minimal-Time Problem.

Consider the linear system (harmonic oscillator)

$$(6.1) \qquad \begin{cases} dx_1/dt = x_2, \\ dx_2/dt = -x_1 + u_1(t), \end{cases}$$

with

$$(6.2) \qquad |u_1(t)| \leq 1.$$

Determine a control law which takes the state of the system to the origin in the shortest possible time. See Problem (2.7).

This celebrated problem seems to have been first mentioned by Doll [12] in 1943 in a U. S. Patent. The first rigorous solution of the problem appeared in 1952 in the doctoral dissertation of Bushaw [13]. Bushaw states that the problem does not fall within the framework of the classical calculus of variations, and he solves it by elementary but highly intricate direct geometric arguments.

The hamiltonian theory developed above can be applied quite simply to give a rigorous proof of Bushaw's theorem.

We rewrite equations (6.1) in matrix form as

$$(6.3) \qquad dx/dt = Fx + Gu(t)$$

where

$$F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad g = G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The Minimum Principle shows that the optimal control must satisfy the relation

$$u^o(t) = -\operatorname{sgn}[G'\psi^o(t)] \tag{6.4}$$

where sgn is a scalar function of the scalar $G'p$, which takes on the value 1 when $G'p > 0$, -1 when $G'p < 0$, and is undetermined when $G'p = 0$.

Since the problem is invariant under translation in time, we shall drop the arguments referring to the initial time, which can be taken as 0 for convenience. Instead of considering motions in the phase space (x_1, x_2, t), we need to consider them only in the state space (x_1, x_2).

First we consider all possible optimal motions which pass through the origin. There are three of these: either $\phi^o(t)$ is identically zero (which is trivial), or $\phi^o(t)$ is a solution of (6.1) with $u^o(t) = +1$ or -1.

Let $u^o(t) = 1$. Then the motion of (6.1) passing through the origin is a circular arc γ^+ about the point $(1, 0)$. See Fig. 1. To check whether this motion is really optimal, we must verify first of all that $G'\psi^o(t) < 0$ along the entire arc. Now (5.9b) in this case is

$$dp/dt = -F'p \tag{6.5}$$

which is independent of x and has the solution

$$\psi^o(t) = \begin{bmatrix} \cos(t - t_o) & \sin(t - t_o) \\ -\sin(t - t_o) & \cos(t - t_o) \end{bmatrix} \psi^o(t_o). \tag{6.6}$$

It is clear that $\psi^o(t)$ is periodic with period 2π; therefore the largest interval over which $G'\psi^o(t) < 0$ is at most of length $< \pi$. This is actually achieved by choosing $\psi^o(0) = (\epsilon, 1)$ so that

$$G'\psi^o(t) < 0 \qquad \text{for all} \qquad 0 \leq t < \pi - \epsilon. \tag{6.7}$$

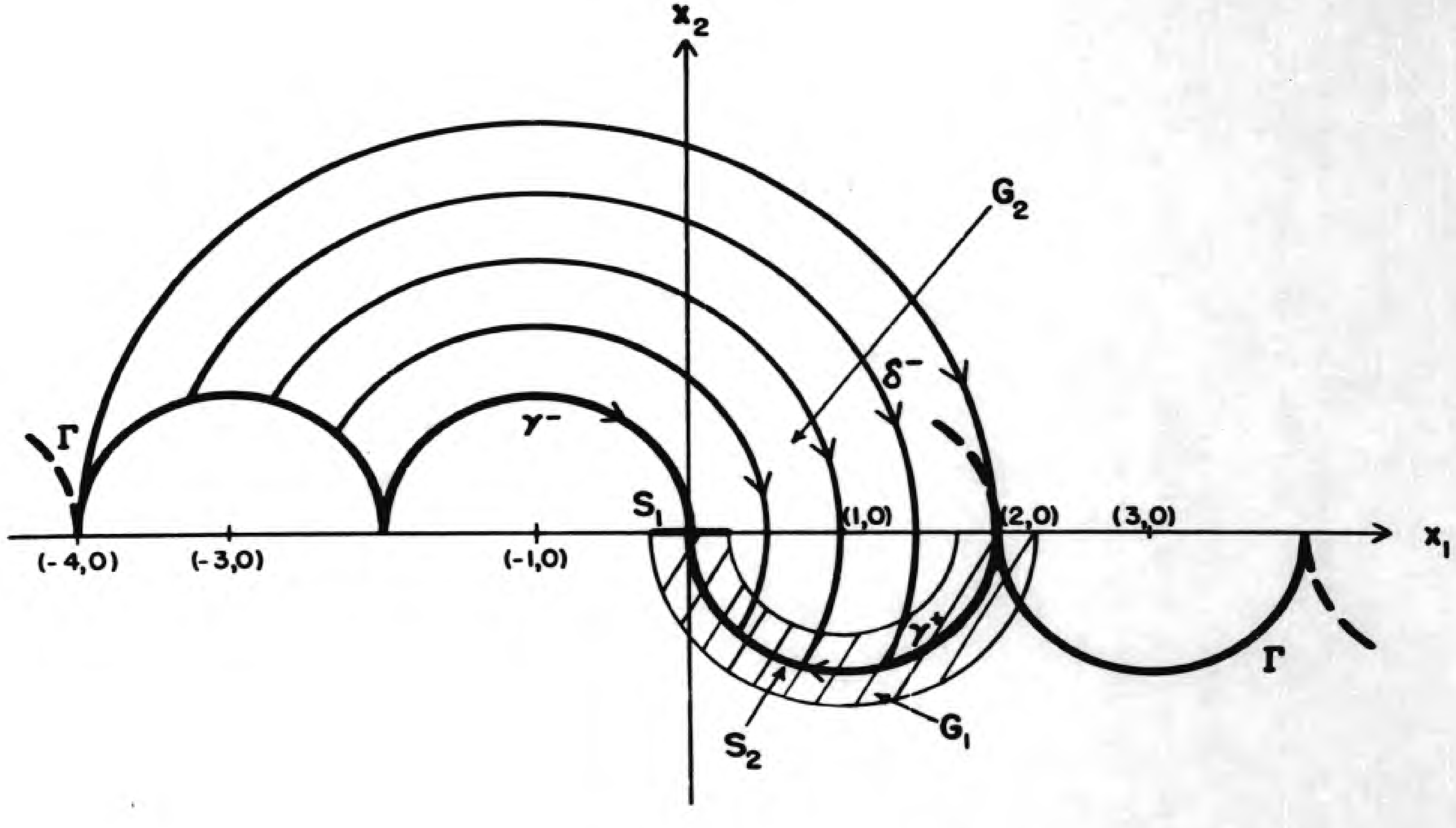

Fig. 1

Thus the necessary condition provided by the euler equations (6.10) is satisfied along the arc γ^+ up to the state (2, 0). (The remaining portion of the arc γ^+ is shown by a dashed curve in Fig. 1.)

Now we must establish a sufficiency condition; in other words, we must show that the arc γ^+ is indeed an optimal motion between the states (0, 0) and (2, 0).

Let us define the set S_1 by

$$S_1 = \{x;\ -0.1 < x_1 < 0.1,\ \ x_2 = 0\},$$

as shown in Fig. 1. We consider the problem of reaching S_1 in minimal time. The hamilton-jacobi equation for this problem is $(V_t^o = 0)$

$$(6.8) \qquad V_x^o \cdot (Fx + g) = -1,$$

which has the solution

$$V_1^o(x_1, x_2) = \pi/2 + \arctan\,[(1 - x_1)/x_2].$$

The value of V_1^o for $x_2 = 0$ is defined by its limit as $x_2 \to 0$ from <u>negative</u> values. Then $V_1^o = 0$ on S_1 as required. Moreover, the region G_1 where V_1^o is to satisfy (6.8) is taken as the semicircular band indicated by the cross-hatching in Fig. 1.

It follows by Theorems (4.15) and (4.16) that <u>if we connect any state on</u> γ^+ <u>with</u> S_1 <u>by means of a motion of</u> (6.1) <u>which is distinct from</u> γ^+ <u>and remains entirely in</u> G_1, <u>then the value of</u> V <u>in</u> (2.3) <u>is greater than</u> V^o.

But if it is not possible to reach S_1 from γ^+ faster than by proceeding along γ^+ itself, the same is true a forteriori as concerns reaching the state (0, 0) on S_1. Hence we have proved:

(6.8) <u>The motion</u> γ^+ <u>is optimal relative to the region</u> G_1.

The same construction establishes the local optimality of the motion γ^- (see Fig. 1).

Now we let $S_2 = \gamma^+$, $\lambda(x) = V_1^o(x)$ and we consider the minimal time problem relative to S_2. All optimal motions (denoted by δ^- in the figure) necessarily correspond to $u^o = -1$. They are circular arcs of radius 1 about the point $(-1, 0)$. Applying the euler equations the same way as before, we find that all optimal arcs δ^- must terminate on the semicircle of radius 1 centered at $(-3, 0)$, which is part of the curve Γ in Fig. 1. The arcs δ^- therefore fill up a region G_2 bounded by Γ and the semicircle centered at $(-1, 0)$ which connects $(2, 0)$ with $(-4, 0)$. If we calculate the time needed to reach γ^+ starting from a point in G_2 and proceeding along δ^-, we get a smooth function $V_2^o(x)$ which satisfies the hamilton-jacobi equation (6.8). Details are left to the reader. This proves that all motions consisting of an arc of δ^- and an arc of δ^+ are optimal.

The construction can be continued in a similar fashion until it covers any point in the plane. The optimal control law will be

$$(6.9) \qquad u_1^o(x) = k(x) = \begin{cases} +1 & \text{below the curve } \Gamma \text{ composed of semicircles of radius 1, and on } \gamma^+; \\ -1 & \text{above the curve } \Gamma \text{ and on } \gamma^-. \end{cases}$$

On $\Gamma - (\gamma^+ \cup \gamma^-)$ the value of u^o is not determined by the Minimum Principle; it is easily verified that the choice of u^o on $\Gamma - (\gamma^+ \cup \gamma^-)$ is immaterial as long as $|u_1| \leq 1$.

The control law (6.9) is <u>Bushaw's theorem</u>.

It should be noted that the function V^o, which is determined piecewise as V_1^o, V_2^o, etc., is <u>not</u> continuously differentiable at a point P on γ^+. The limit of V_x^o is infinite if we approach P from below γ^+ along points which lie on the continuation of δ^-; and the same limit is finite if we approach P from above γ^+ along δ^-.

As a result, the euler equations (5.9) do not have continuous solutions along optimal motions; the conjugate vector p receives an "impulse" on passing through Γ. But the more general proof [11] of Pontryagin's theorem shows that relations (5.10) remain true so that

(6.10) $$u^{o}(t) = -\operatorname{sgn}[G'\psi^{o}(t; p_{o}, t_{o})]$$

where the initial condition p_o for the adjoint equation (6.5) may be defined by

$$p_o = k(x_o)$$

where k is the optimal control law (6.9). Then $\psi^{o}(t)$ vanishes on Γ, which shows that ψ_o cannot be interpreted as V_x^o.

One may use Pontryagin's theorem (in the form just mentioned) as a necessary condition to determine all possible optimal motions. But one must then still carry out the explicit construction given above. For (6.10) can be interpreted as the optimal control only if the corresponding motion actually reaches the origin.

7. General Solution of the Linear Optimal Regulator Problem.

A class of problems which can be completely solved by the hamiltonian theory is represented by the functional

$$(7.1) \qquad \tfrac{1}{2}[\|\phi_u(t_1;x_o,t_o)\|_A^2 + \int_{t_o}^{t_1}\{\|H(t)\phi_u(t;x_o,t_o)\|_{Q(t)}^2 + \|u(t)\|_{R(t)}^2\}dt]^*$$

where the motions ϕ_u are defined by the linear differential equation

$$(7.2) \qquad dx/dt = F(t)x + G(t)u(t);$$

there are no constraints on u.

This is a slight generalization of Problem (2.8).

The matrices $Q(t)$, $R(t)$ are to be positive definite for all t. This assumption on R implies that

$$(7.3) \qquad 2H(x,p,t,u) = \|H(t)x\|_{Q(t)}^2 + \|u\|_{R(t)}^2 + 2p\cdot[F(t)x + G(t)u(t)]$$

has a unique absolute minimum for every (x, p, t) at

$$c(x, p, t) = - R^{-1}(t)G'(t)p,$$

so that

$$(7.4) \qquad 2H^o(x,p,t) = \|H(t)x\|_{Q(t)}^2 + 2p\cdot F(t)x + \|G'(t)p\|_{R^{-1}(t)}^2.$$

The hamilton-jacobi equation corresponding to (7.4) has a unique solution given any nonnegative definite matrix A and any $t_1 > t_o$. We assume that this solution has the form

$$(7.5) \qquad 2V^o(x, t) = \|x\|_{P(t)}^2,$$

which implies the linear control law

---

*We use the notation $\|x\|_A^2$ for a quadratic form defined by a symmetric, non-negative definite matrix A.

$$k(x, t) = -R^{-1}(t)G'(t)P(t)x. \tag{7.6}$$

It is easily checked that (4.9) with the hamiltonian defined by (7.4) has a solution of the type (7.5) if and only if the symmetric matrix $P(t)$ is a solution of the riccati equation

$$-dP/dt = F'(t)P + PF(t) - PG(t)R^{-1}(t)G'(t)P + H'(t)Q(t)H(t). \tag{7.7}$$

Moreover, the boundary condition

$$V^{o}(x_1, t_1) = \|x\|_A^2$$

(which is the concrete form of (4.4)) implies that the solution of (7.7) must satisfy the initial condition

$$P(t_1) = A. \tag{7.8}$$

Since (7.7) is nonlinear, it is not clear at once that $P(t)$ exists outside of a small neighborhood of t_1. However, the integral (7.1) may be bounded from above by the free motions of (7.2) (i.e. by setting $u(t) = 0$), which in view of (7.5) is equivalent to a bound on $\|P(t)\|$. Utilizing the a priori bound so obtained in the standard existence theorem for differential equations shows that solutions of (7.7) exist for all $t \leq t_1$. This conclusion is in general no longer valid if A has negative eigenvalues or if $t > t_1$.

The existence of solutions of (7.7) (and therefore of the hamilton-jacobi equation) being assured, they can be expressed [2, 3, 7] with the aid of solutions of the canonical differential equations

$$\begin{bmatrix} dx/dt \\ dp/dt \end{bmatrix} = \begin{bmatrix} F(t) & -G(t)R^{-1}(t)G'(t) \\ -H'(t)Q(t)H(t) & -F'(t) \end{bmatrix} \cdot \begin{bmatrix} x \\ p \end{bmatrix}. \tag{7.9}$$

Further difficulties arise, however, in studying the stability of (7.7) as well as the stability of the optimal motions defined by (7.6). Additional details on these problems may be found particularly in [7].

8. General Solution of the Linear Optimal Servomechanism Problem.

The problem considered in the previous section can be generalized in several ways. We consider here simultaneously two such generalizations.

First, we assume that the motions, in addition to control, are subject to "disturbances" represented by the term $w(t)$ in the equations

$$dx/dt = F(t)x + G(t)u(t) + w(t). \tag{8.1}$$

Second, we assume that the functional to be minimized is

$$\tfrac{1}{2}\{\|\eta(t_1) - H(t_1)\phi_u(t_1)\|_A^2 + \int_{t_0}^{t_1}[\|\eta(t) - H(t)\phi_u(t)\|_{Q(t)}^2 + \|u(t)\|_{R(t)}^2]dt\}. \tag{8.2}$$

We call the p-vector

$$y(t) = H(t)x(t) \tag{8.3}$$

the output of the system (8.1); by analogy, the vector function $\eta(t)$ is the desired output.

This setup is a slight generalization of Problem (2.10). A number of formal solutions have appeared in the engineering literature [14-15]. The hamiltonian theory provides a simple rigorous proof of the known formulas.

Proceeding exactly as in Section 7, we find that the hamiltonian of the problem is:

$$2H^0(x, p, t) = \|\eta(t) - H(t)x\|_{Q(t)}^2 + 2p\cdot[F(t)x + w(t)] - \|G'(t)p\|_{R^{-1}(t)}^2. \tag{8.4}$$

To solve the corresponding hamilton-jacobi equation (4.9), we assume that

$$2V^0(x, t) = \|x\|_{P(t)} - 2z(t)\cdot x + v(t). \tag{8.5}$$

Substituting, we find that

(8.6) $V^0(x, t)$ given by (8.5) satisfies the hamilton-jacobi equation defined by (8.4), with $V^0(x, t_1) = \|\eta(t_1) - H(t_1)x\|_A$, if and only if the matrix $P(t)$, the vector $z(t)$, and the scalar $v(t)$ satisfy the following ordinary differential equations:

(a) $P(t)$ is the solution of the riccati equation (7.7) with $P(t_1) = A$;

(b) $z(t)$ is the solution of

$$(8.7) \quad dz/dt = -[F(t) - G(t)R^{-1}(t)G(t)P(t)]'\cdot z + P(t)w(t) - H'(t)Q(t)\eta(t)$$

with

$$(8.8) \quad H'(t_1)A\eta(t_1) = z(t_1);$$

(c) $v(t)$ is the solution of

$$(8.9) \quad -dv/dt = \tfrac{1}{2}[\|\eta(t)\|^2_{Q(t)} - \|G'(t)z(t)\|^2_{R^{-1}(t)}] - z(t)\cdot w(t)$$

with

$$v(t_1) = \|\eta(t_1)\|^2_A.$$

The control law is linear, for it is given by

$$(8.10) \quad u^o(t) = -R^{-1}(t)G'(t)p(t) = R^{-1}(t)G'(t)[z(t) - P(t)x(t)].$$

The control law (8.10) is unrealizable, because it involves $z(t)$ which, according to (8.7 - 8), must be computed backwards in time and requires the knowledge of $\eta(t)$ and $w(t)$ in the interval $[t_o, t_1]$ -- this is usually not known at time t_o in practical applications.

It should be noted that the differential equation for $z(t)$ (minus the forcing terms) is the adjoint of the differential equation of optimal motions of Sect. 7.

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