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RESEARCH MEMORANDUM

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RESEARCH MEMORANDUM

LUNAR INSTRUMENT CARRIER -- TRAJECTORY STUDIES

H. A. Lieske

RM-1728

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Assigned to ____________________

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PREFACE

This study complements other RAND studies of the problems of landing a package of scientific instruments and associated radio equipment on the surface of the moon. These other studies have been reported in the following research memorandums:

RM-1720, General Report on the Lunar Instrument Carrier (U) by R. W. Buchheim, May 28, 1956 (Secret).

RM-1725, Lunar Instrument Carrier: Landing Factors by M. A. Lang, June 4, 1956.

RM-1727, Lunar Instrument Carrier: Ascent Flight Mechanics (U) by H. A. Lieske, June 4, 1956 (Secret).

RM-1729, Lunar Instrument Carrier: Powered Flight Guidance (U) by W. E. Frye, June 4, 1956 (Secret).

RM-1730, Lunar Instrument Carrier: Attitude Stabilization by R. W. Buchheim, June 4, 1956.

RM-1731, Lunar Instrument Carrier Tracking and Communication (U) by R. T. Gabler and M. R. O'Mara, June 4, 1956 (Confidential).

RM-1994, Lunar Instrument Carrier: Launch-Time Tolerance (U) by H. A. Lieske, October 4, 1957 (Secret).

The material in this memorandum was first prepared several years ago, and was reported in part in the General Report, RM-1720.

SUMMARY

This research memorandum summarizes the results of a study of free-flight earth-moon trajectories, which describe the two-dimensional ballistic motion of a vehicle from a standard initial altitude above the earth to impact on the surface of the moon. Trajectories originating at various positions relative to the initial position of the moon are studied to cover the complete spectrum, including retrograde launching.

A design transit trajectory is chosen to examine the effect of small variations of initial parameters on the location of the lunar impact point for use in error studies.

A "hit band" of transit trajectories that lead to impact on the moon is computed by varying initial conditions in the vicinity of the design point. Transit trajectories in the vicinity of the design point that miss the moon are studied for use in establishing artificial satellites of the moon.

CONTENTS

LIST OF FIGURES

I. INTRODUCTION

This memorandum presents the results of the earth-moon trajectory study for the Lunar Instrument Carrier Project which is summarized in Ref. 1. The trajectories investigated herein describe the two-dimensional unpowered, or ballistic, motion of a vehicle from a standard initial altitude above the earth to impact on the surface of the moon. Trajectories originating at various positions relative to the initial position of the moon were studied in order to cover the complete spectrum, including "launching" westward (against the rotation of the earth).

A design transit trajectory is rather arbitrarily chosen in order to study the effect of small variations of the initial parameters on the location of the lunar impact point for use in error studies.

Transit trajectories in the neighborhood of this design point which miss the moon are studied for use in establishing an artificial satellite of the moon (see Ref. 2). Subsequent and more general studies, including circumlunar trajectories, have been partially described in Refs. 3 - 5.

The matching of powered-ascent trajectories to the free-flight trajectories, discussed herein, is described in Ref. 6. A representative booster-vehicle combination was used for a numerical example.

II. GENERAL DISCUSSION

If a body of negligible mass moves under the influence of a single gravitational center, its total energy and angular momentum are conserved. The equations of motion can be integrated in closed form to give Kepler's equations.

If the body is moving under the influence of two or more attracting centers, however, its position relative to the attracting bodies cannot, in general, be described analytically. The equations of motion, therefore, must be integrated numerically to describe the trajectory of the body.

An approximate description of the trajectory can be obtained by "patching" together segments of conic sections relative to the appropriate "isolated" gravitational centers. This approach will provide a reasonable approximation to the required initial velocity but is probably not sensitive enough to provide error coefficients required for guidance studies. An example of this method of analysis is given in Ref. 7.

EARTH-MOON MODEL

In the "real world" the motion of the vehicle is governed by the gravitational attractions and relative positions of the earth, moon, and sun. The sun's gravitational attraction, besides determining the motion of the earth-moon system, will exert a net acceleration on the vehicle as it departs from the earth (actually, the earth-moon barycenter). This trajectory perturbation by the sun is, of course, a function of the time of the month. Since the motion of the three bodies is not entirely periodic, each computation would apply to a single specific date.

The three-dimensional character of the earth-moon system -- as defined by the inclination of the moon's orbital plane relative to the earth's equator -- would require a set of six initial conditions as well as the specification of a launch date. The inclination of the moon's orbital plane varies between limits of about 18.5 and 28.5 deg over a period of 18.6 years.

For this preliminary study, the actual earth-moon-sun system is replaced by a simplified model of the earth-moon system. In this model, the earth and moon are assumed to be moving around their common center of mass at constant angular velocity. The bodies are also assumed to be homogeneous spheres with the inverse-square gravitational fields of point masses. The principal dimensions of this simplified model[8] are shown in Fig. 1. The earth and moon, whose radii are 3950 and 1080 st mi, respectively, are separated by a distance of 238,857 st mi. The ratio of the mass of the moon to the total mass of the earth-moon system is assumed to be 1/82.45. The center of mass of the system is, therefore, displaced from the center of the earth by a distance of 2897 mi, which is less than the earth's radius. The angular velocity of the system around its center of mass is about 0.23 rad/day, giving a period of about 27.3 days.

An internally self-consistent set of numerical constants which could be used to describe a hypothetical earth-moon system is derived in Ref. 9.

Some of the factors which have been neglected in this model are

- The gravitational field of the sun
- The eccentricity of the moon's orbit
- The oblateness of the earth
- The inclination of the moon's orbital plane relative to the earth's equator

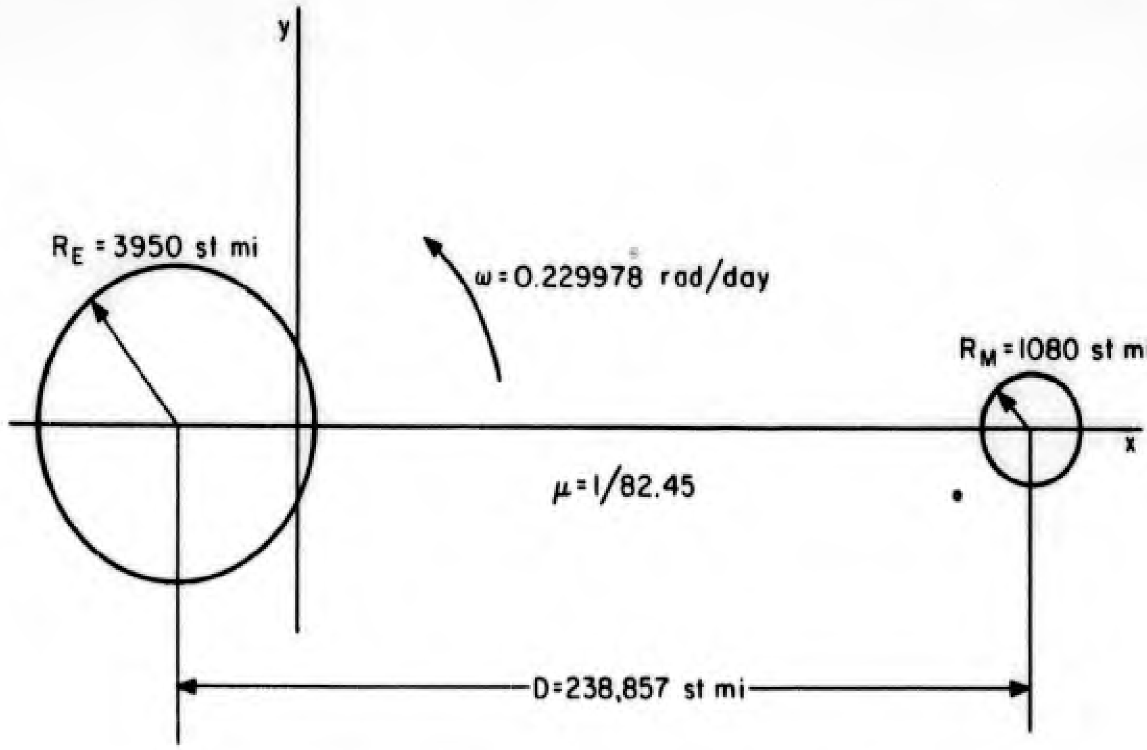

Fig. 1—Earth-moon model

The magnitude of the effective velocity shifts due to the neglect of these factors is estimated in Ref. 9. It is concluded that the general character of the results would not be changed by use of this idealized model. The sun's net gravitational attraction on the vehicle, for example, would be equivalent to an increment of approximately 20 ft/sec in the required velocity for a transit trajectory which has a flight time of about 2.5 days. This velocity shift, which is a function of the vehicle's transit time, would be positive for a launching at full moon and negative at the time of new moon. The sun's attraction would have little effect on the trajectory at first or last quarter.

EQUATIONS OF MOTION

The equations of motion of a body of negligible mass under the gravitational attraction of the earth and moon can be written most readily relative to an inertial coordinate system whose origin is at the center of mass of the system. These equations, however, include the coordinates of the earth and the moon as functions of time. If the equations of motion are referred to a right-handed coordinate system (x, y, z) which is rotating with the earth and moon and is oriented so that the earth and moon are located on the x-axis, the three-dimensional equations become

$$\begin{aligned}
\ddot{x} &= 2\omega\,\dot{y} + \omega^2\left[x - (1-\mu)\frac{(x - x_1)}{r_1^3} - \mu\,\frac{(x - x_2)}{r_2^3}\right] \\
\ddot{y} &= -2\omega\,\dot{x} + \omega^2\left[y - (1-\mu)\frac{y}{r_1^3} - \mu\,\frac{y}{r_2^3}\right] \qquad (1) \\
\ddot{z} &= -\omega^2\left[(1-\mu)\frac{z}{r_1^3} = \mu\,\frac{z}{r_2^3}\right]
\end{aligned}$$

where the subscripts 1 and 2 refer to the earth and moon, respectively. The angular velocity of the system, ω, can be used to define the gravitational constant if the distances are non-dimensionalized by use of D, the distance between the earth and the moon. Also

$$\mu = \frac{M_2}{M_1 + M_2}$$

$$r_1 = \sqrt{(x - x_1)^2 + y^2 + z^2}$$

$$r_2 = \sqrt{(x - x_2)^2 + y^2 + z^2}$$

The terms $2\omega\dot{x}$ and $2\omega\dot{z}$ are components of the Coriolis acceleration, and $\omega^2 x$ and $\omega^2 y$ are the centrifugal acceleration components which are introduced by use of the rotating coordinate system.

The constants in the above equations have been taken as

$$\omega = 0.2299708 \text{ rad/day}$$

$$\mu = 1/82.45$$

The positions of the earth and moon are

$$x_1 = - 0.01212856 \text{ lunar unit}$$

$$x_2 = 0.98787144 \text{ lunar unit}$$

using D = 238,857 st mi.

The equations of motion (Eq. (1)) for this idealized earth-moon model are those of the classical Problem of Three Bodies discussed in Ref. 10.

For this study we will consider the motion of the body to be in the plane of the earth-moon orbit, so that $\ddot{z}_o = \dot{z}_o = z_o = 0$. One check case with an initial value of $\dot{z}_o \neq 0$ was computed to estimate the out-of-plane sensitivity of the impact location (see Appendix B).

Following the derivation given in Ref. 10, the first integral to the equations of motion is given by

$$C = \omega^2 \left[\frac{2(1 - \mu)}{r_1} + \frac{2\mu}{r_2} + (x^2 + y^2) \right] - (\dot{x}^2 + \dot{y}^2) \qquad (2)$$

which is known as Jacobi's Integral.

It is shown in Ref. 9 that Jacobi's Integral expresses the conservation of the difference of the total energy (referred to the rotating coordinate system) and the angular momentum of the vehicle relative to the z-axis. This law of conservation differs from the case of the motion of a body under the influence of a single gravitational attraction, for which the total energy and angular momentum are each conserved.

Jacobi's Integral (Eq. (2)) was used as a check on the numerical integration at each computational step. The Runge-Kutta method of integration was used for machine computations.

Jacobi's Integral can be used to determine some interesting features of the motion in the three-body problem as discussed later in this memorandum and also in Ref. 9.

INITIAL TRAJECTORY PARAMETERS

A set of four initial parameters is required to define the unpowered earth-moon trajectory in two-dimensional study. The four parameters selected for use in this investigation are indicated in Fig. 2. The earth is represented by the circle which is offset from the origin of the rotating x, y coordinate system by a distance, $x = -\mu$, which corresponds to the earth-

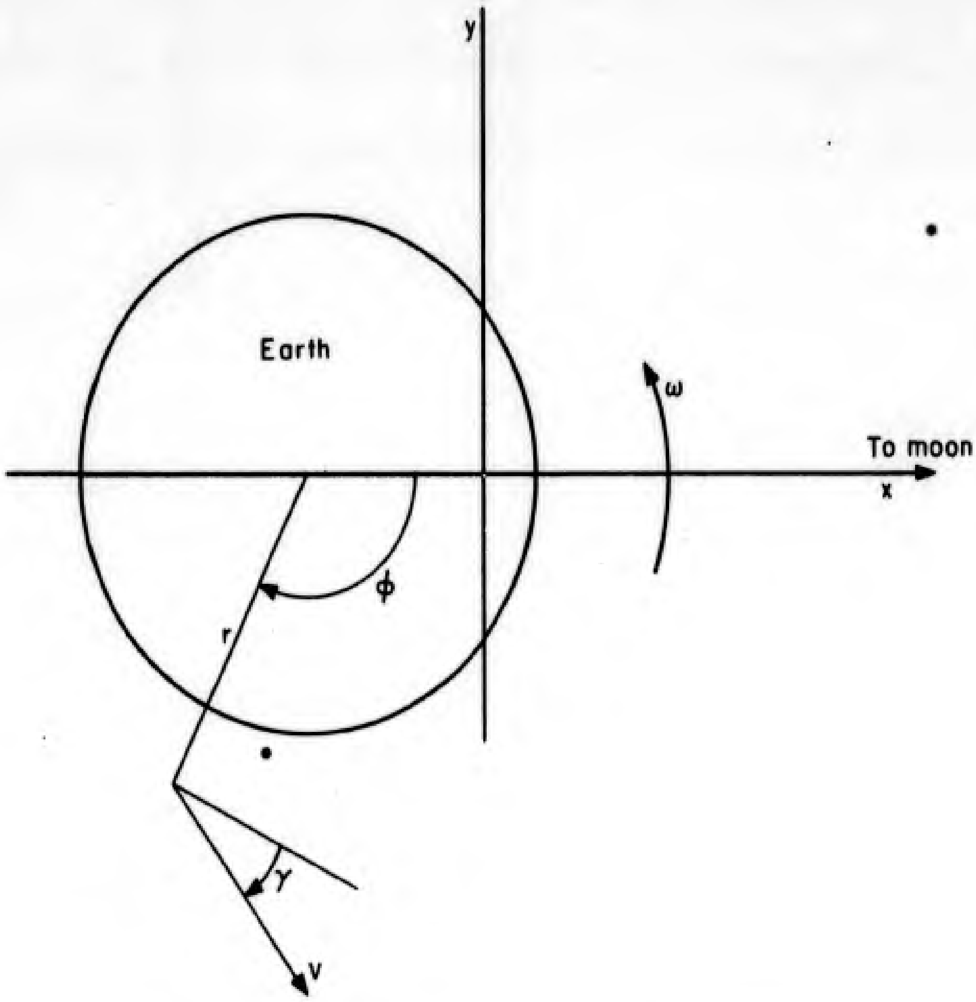

Fig. 2—Initial trajectory parameters

moon mass ratio. The moon is located at a position $(1 - \mu)$ on the positive x-axis.

The initial position of the free-flight trajectory is defined by the polar coordinates r and ϕ, relative to the center of the earth. For this study, the initial radial distance from the center of the earth, r, is held fixed at a value of 4300 st mi, corresponding to an altitude of about 350 st mi. The angle ϕ defines the initial position of the trajectory relative to the moon. This angle has been varied through 360 deg to investigate the complete range of earth-moon transit trajectories which leave the earth in both direct and retrograde motion.

The initial free-flight velocity vector, relative to the rotating coordinate system, is defined by its magnitude V and path angle γ relative to the local horizontal. The initial velocity V is varied between values of about 34,800 ft/sec, which is nearly the minimum initial velocity required to reach the moon on the first pass, and a value of 37,000 ft/sec. The local value of the escape velocity, assuming an isolated earth, is 35,166 ft/sec relative to an earth-fixed inertial coordinate system. This velocity, referred to the rotating coordinate system, will vary between values of about 35,100 and 35,220 ft/sec depending on the initial path angle, as discussed later in this section.

The initial portions of transit trajectories defined by velocities that are less than the local escape velocity will, of course, be elliptical relative to the earth, while initial velocities greater than that value will initially be hyperbolic relative to the earth.

This set of initial trajectory parameters (r, ϕ, V, γ) is equivalent to a set $(x, y, \dot{x}, \dot{y})$ relative to the rotating coordinate system.

The conversion of the initial parameters used in this study to a set (r, ϕ, V_e, γ_e) relative to a non-rotating inertial coordinate system fixed at the center of the earth is discussed later in this section. This set, in the earth-inertial coordinate system, can be used to define an equivalent set of initial osculating elements (a, e, ϕ_{ma}, and direction of rotation). The quantities a and e are the semimajor axis and the eccentricity of the initial conic section. The quantity ϕ_{ma} defines the angle between the major axis of the initial conic section and the initial position of the moon. The fourth initial parameter, which is required in this set, is the type of motion relative to the earth -- direct or retrograde.

The semimajor axis of the equivalent initial conic section -- a for an ellipse, or -a for a hyperbola -- relative to the earth is defined by

$$a = \frac{r_1}{2 - \frac{V_e^2 r_1}{\mu_e}}$$

where

$$\mu_e = g_e R_e^2 = \omega^2(1 - \mu) = 0.05224515 \text{ lunar unit}^3/\text{day}^2$$

The eccentricity, e, is given by

$$e = \sqrt{1 - (2 - \frac{V_e^2 r_1}{\mu_e}) \frac{V_e^2 r_1}{\mu_e} \cos^2 \gamma_e}$$

For the range of initial velocities used in this study (34,800 to 37,000 ft/sec relative to the rotating coordinate system), the semimajor axis varies from a minimum value of a = 0.519 lunar unit for the ellipse to a value of -a = 0.0814 lunar unit for the hyperbola. The eccentricities of the initial osculating ellipses vary from a minimum of about 0.965 to a maximum value of 1.0 for a rectilinear ellipse ($\gamma_e = 90$ deg). The eccentricity of the hyperbolas varies from 1.0 for the rectilinear case to a maximum value of about 1.22.

COORDINATE TRANSFORMATION

The equations of motion of the small body were numerically integrated relative to the x, y coordinate system which has its origin at the center of mass of the earth-moon system and which rotates to keep the earth and moon on the x-axis.

In order to correlate the transit trajectories defined by various initial path angles, it is necessary to refer the trajectory parameters to a non-rotating or inertial coordinate system (x_e, y_e) which is fixed at the center of the earth. This coordinate system is actually accelerating toward the center of mass of the system due to the earth's circular orbital motion, so that it is truly inertial only at $x_e = y_e = 0$.

Given the position and velocity of the body relative to the rotating coordinate system (x, y), the position and velocity relative to the earth-fixed system (x_e, y_e) are[9]

$$x_e = (x + \mu)\cos\omega t - y\sin\omega t$$

$$y_e = (x + \mu)\sin\omega t + y\cos\omega t$$

$$\dot{x}_e = (\dot{x} - \omega y)\cos\omega t - [\dot{y} + \omega(x + \mu)]\sin\omega t$$

$$\dot{y}_e = (\dot{x} - \omega y)\sin\omega t + [\dot{y} + \omega(x + \mu)]\cos\omega t$$

where x and y are non-dimensional and the center of the (x_e, y_e) coordinate system is at ($x = -\mu$, $y = 0$) relative to the rotating coordinate system.

The velocity in the earth-inertial system is given by

$$V_e^2 = \dot{x}_e^2 + \dot{y}_e^2$$

$$= V^2 + \omega^2 r_1^2 + 2\omega \; [\dot{y} \, (x + \mu) - \dot{x} \, y]$$

$$= V^2 + \omega^2 r_1^2 \pm 2\omega \, V \, r_1 \cos \gamma$$

where V is the magnitude of the velocity and γ is the path angle, both relative to the rotating coordinate system, and r_1 is the radial distance from the trajectory point to the center of the earth. The plus and minus signs on the last term correspond to direct and retrograde motion relative to the earth-moon system, respectively. The path angle relative to the local horizontal in the earth inertial system is given by

$$\sin \gamma_e = \frac{V \sin \gamma}{V_e}$$

The increments in the values of the initial velocity and path angle produced by the transfer from the rotating to the earth-fixed coordinate system can be defined as

$$\Delta V_e = V_e - V$$

$$\Delta \gamma_e = \gamma_e - \gamma$$

The velocity increment, which is independent of the initial position of the trajectory relative to the moon, is shown in Fig. 3 for the standard initial-transit-trajectory radius of 4300 mi. This increment, which has a maximum value of about 60 ft/sec at γ = 0 deg, is positive for direct motion and negative for motion which is retrograde relative to the earth.

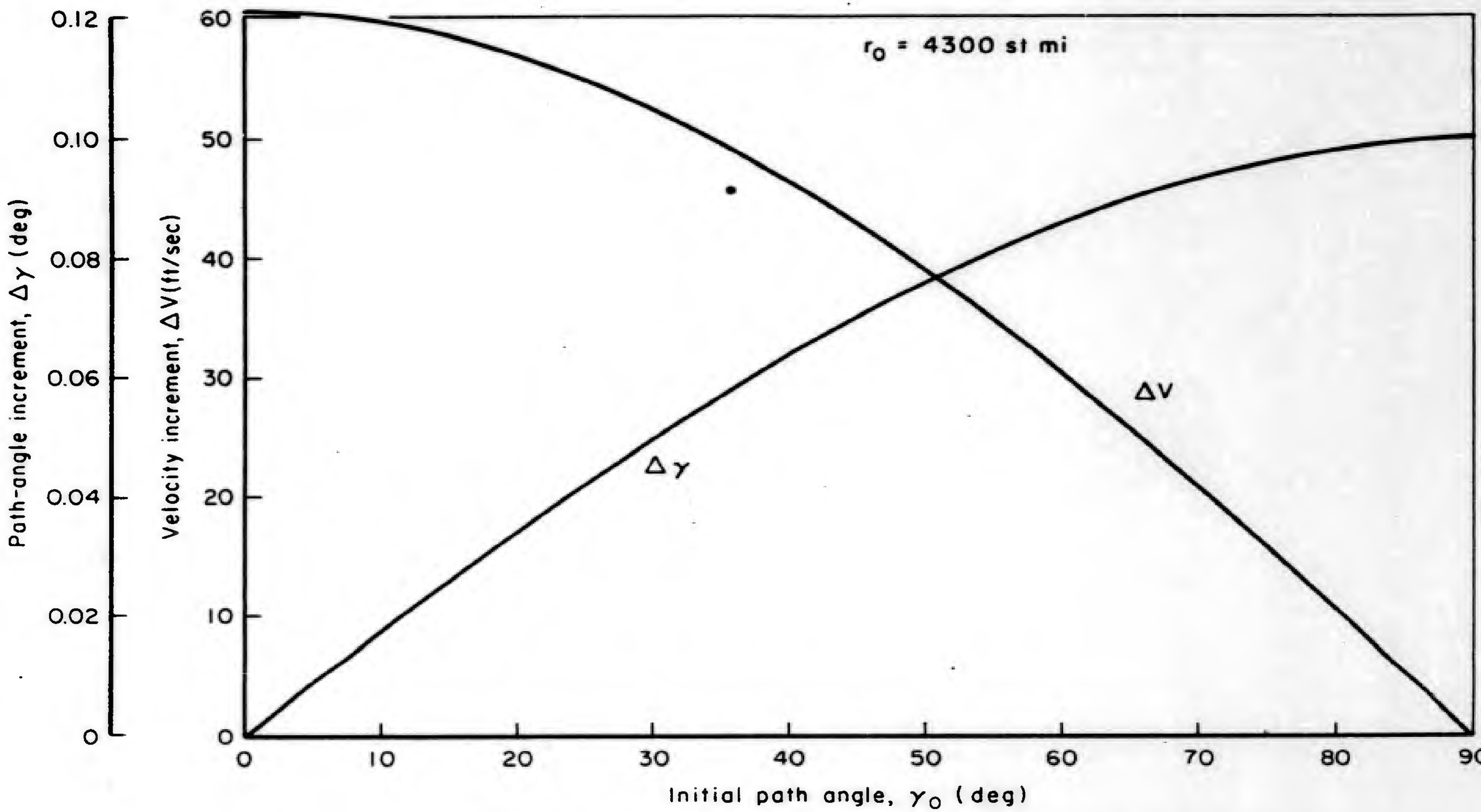

Fig. 3 — Increments for coordinate transfer
(earth)

The value of ΔV is essentially constant for the range of initial velocities in the rotating system from 34,800 to 37,000 ft/sec.

The corresponding change in the initial path angle resulting from the coordinate transfer is also given in Fig. 3. The angular increment, which has a maximum value of about 0.10 deg at $\gamma = 90$ deg, varies by a maximum of about 0.005 deg between the initial velocity extremes of 34,800 and 37,000 ft/sec. The path angle value is decreased for direct motion and increased for retrograde motion by the change from the rotating to the earth-fixed coordinate system.

The corresponding transformation from the CG-centered rotating coordinates to a non-rotating, or inertial, system fixed at the center of the moon gives

$$V_m^2 = V^2 + \omega^2 r_2^2 \pm 2V \omega r_2 \cos \gamma'$$

and

$$\sin \gamma_m = \frac{V \sin \gamma'}{V_m}$$

where r_2 is the radial distance to the center of the moon. At the surface of the moon ($r_2 = r_m$) the velocity increment resulting from the coordinate transformation, which is

$$\Delta V_m = V_m - V$$

has a maximum value of about 15 ft/sec for $\gamma = 0$ deg. The corresponding value of the path-angle increment

$$\Delta \gamma_m = \gamma_m - \gamma'$$

is less than 0.10 deg for γ = 90 deg and an impact speed of 9000 ft/sec.

The variation of these velocity and path-angle increments at the moon with respect to path angle is similar to that for the case of the earth shown in Fig. 3.

III. GENERAL RESULTS

The rotation of the earth-moon system around its center of mass is in the counterclockwise direction, as indicated in Fig. 1. The rotation of the earth (and also of the moon) about its own axis is also in the counterclockwise direction. The resulting tangential velocity of the earth's surface, 1000 to 1500 ft/sec at reasonable launch-site latitudes, makes it desirable to launch the vehicle in a generally eastward direction; that is, in direct motion relative to the earth-moon system. These preferred initial positions will "lag" the moon, and are located generally around the lower portion of the earth as shown in Fig. 2.

The general results of the numerical experimentation on the digital computer can be summarized by a set of curves which present the initial path angle required to hit the moon as a function of the initial velocity from various initial positions relative to the moon.

DIRECT MOTION

Figure 4 shows the variation of the required initial path angle as a function of the initial velocity (referred to the rotating coordinate system) for trajectories that depart from the earth in direct motion. The initial trajectory positions are varied by increments of 22.5 deg, from 135 deg below the x-axis to 22.5 deg above the axis. Initial position angles of 110 to 130 deg below the axis correspond to transit trajectories that start at perigee of the initial conic, while position angles of 10 to 40 deg above the x-axis correspond to free-flight trajectories that leave the earth in a radial direction. These special cases, $\gamma = 0$ and 90 deg, will be discussed in more detail later in this section.

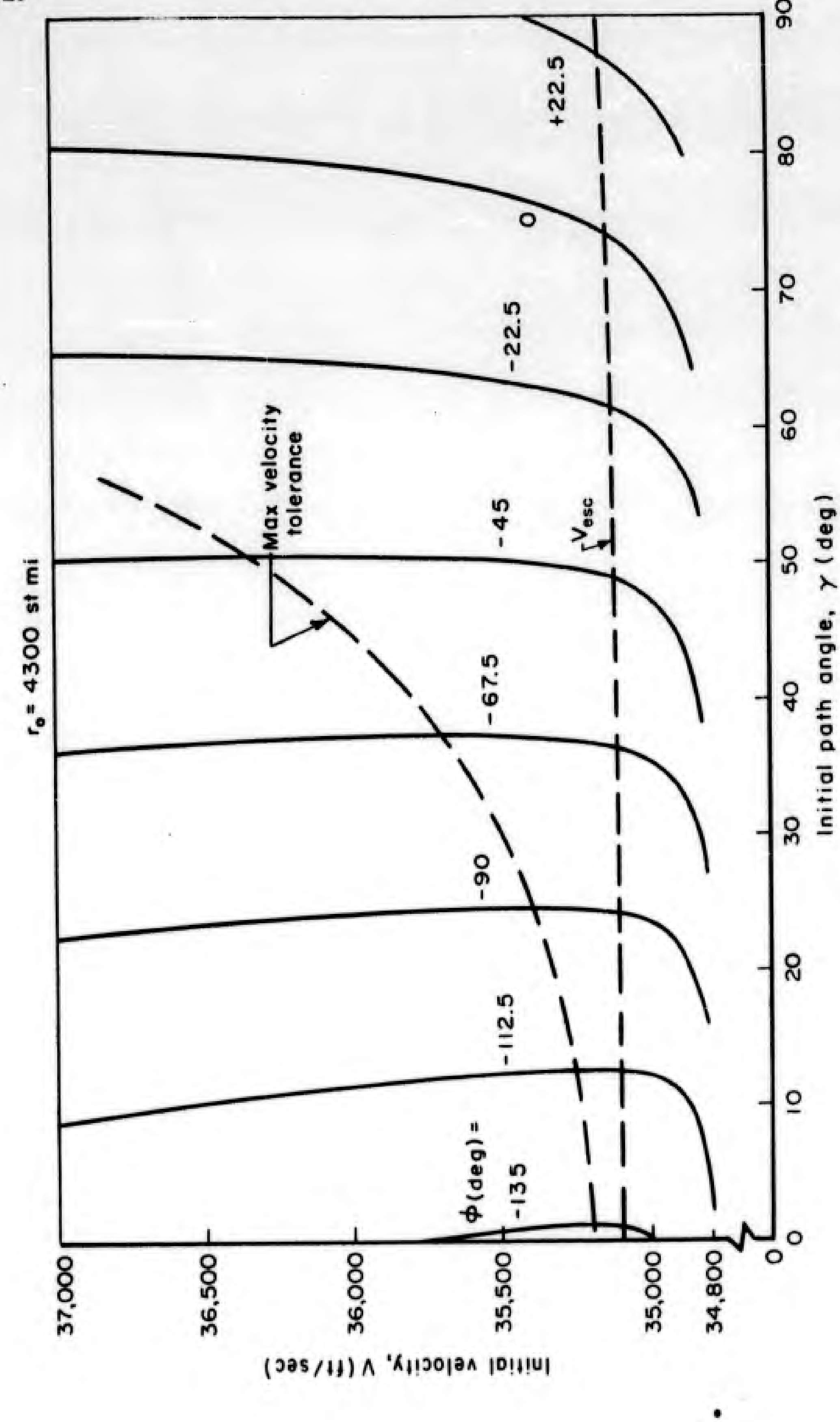

Fig.4 — Initial conditions – direct motion
(rotating coordinates)

The short section of the curve for a position angle of -135 deg could be extended further along the velocity scale, but the required initial path angles would be negative; that is, the vehicle would pass through perigee on its free-flight transit trajectory to the moon.

The local value of the escape velocity from an isolated earth is indicated on the graph. Plotting the curves relative to the rotating coordinate system is seen to skew the escape velocity from a value of about 35,100 ft/sec at $\gamma = 0$ deg to the inertial value of 35,166 ft/sec at $\gamma = 90$ deg. The velocity shift due to the use of this rotating coordinate system was discussed in Section II.

The curves for initial positions which are more than 45 deg from the moon are seen to "bend over" within the velocity range covered in Fig. 4. The locus of the points at which the curves are vertical is indicated by the dashed line. Since each of these curves actually represents a narrow band of initial velocity - path angle combinations, this dashed line defines the locus of the points of maximum tolerance in the magnitude of the initial velocity.

For trajectory-correlation (and also vehicle-performance) studies, the initial velocity - path angle combinations of Fig. 4 should be referred to an earth-fixed, or inertial, coordinate system. The lower portion of Fig. 4 is replotted in Fig. 5 with the initial velocity and path angle referred to this earth-fixed system. The value of escape velocity is again included for reference.

The theoretical minimum initial velocities required to reach the moon's distance (assuming the mass of the moon to be zero) vary between

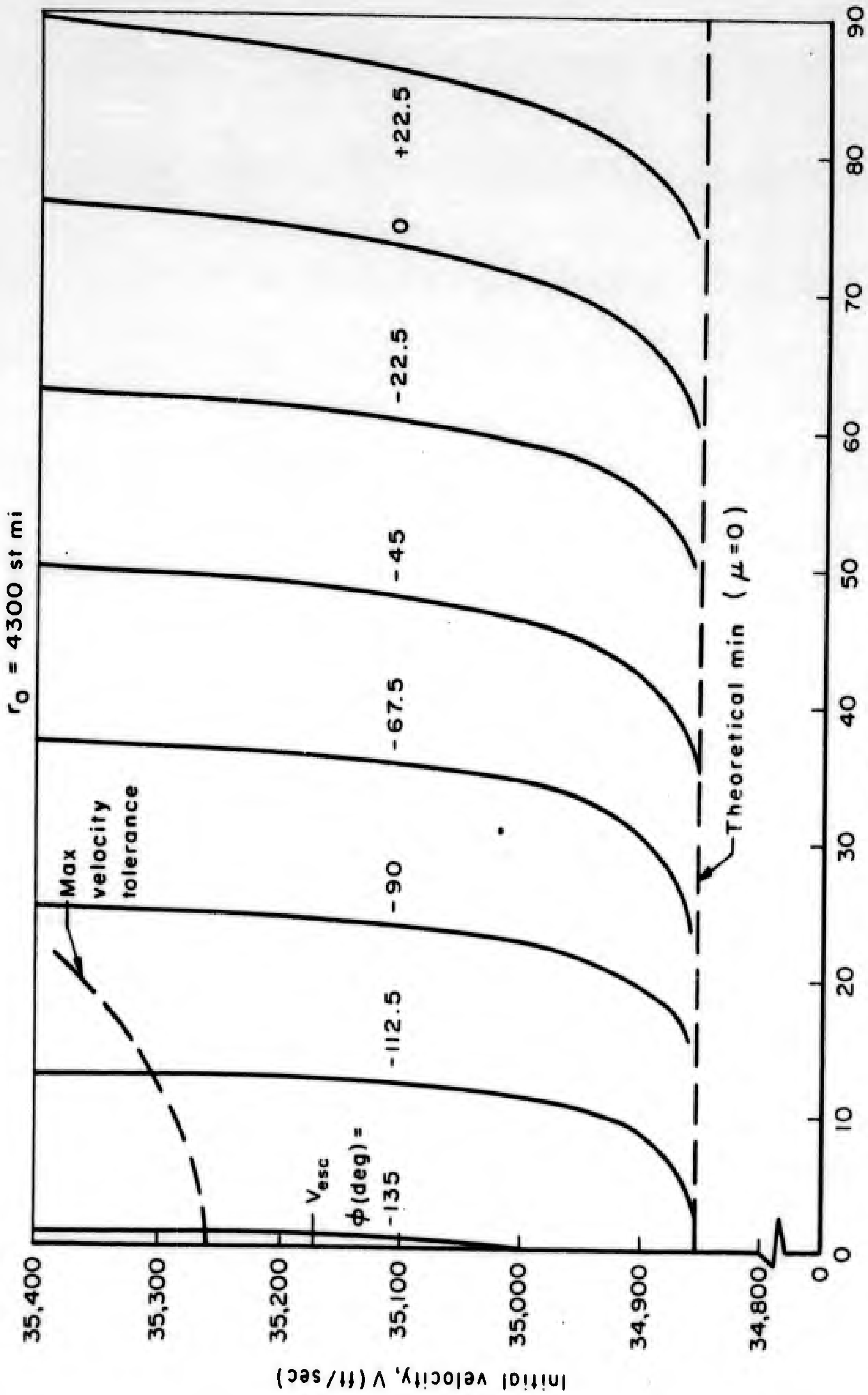

Fig. 5 — Initial conditions — direct motion
(inertial system)

values of 34,854 and 34,849 ft/sec as the initial path angle is varied from zero to 90 deg. These theoretical values form an approximate lower bound to the curves of Fig. 5. The extremely-low-velocity portion of these velocity - path angle curves has not been investigated in great detail since these trajectories would not be of much practical interest.

The initial path angles required for intermediate initial positions can be determined from a cross-plot, such as that shown in Fig. 6, which presents curves for several selected initial velocities relative to the earth-inertial-coordinate system.

The initial conditions shown in Figs. 4 and 5 can also be presented in the form of the parameters of the initial earth-centered conic sections. In order to avoid the difficulty of passing through infinity at escape velocity, the semimajor axis, a, can be replaced by the factor $V_e^2 r_1/\mu_e$, which is the square of the ratio of the initial velocity to the local circular orbital velocity. This factor is plotted against the eccentricity, e, for several values of the initial-position angle in Fig. 7. The initial velocity relative to the rotating system is shown as a parameter of the curves.

The minimum elliptical point is approximately equal to the theoretical value of $V_e^2 r_1/\mu_e = 1.965$, $e = 0.965$, corresponding to perigee launch at this initial radial distance of 4300 st mi.

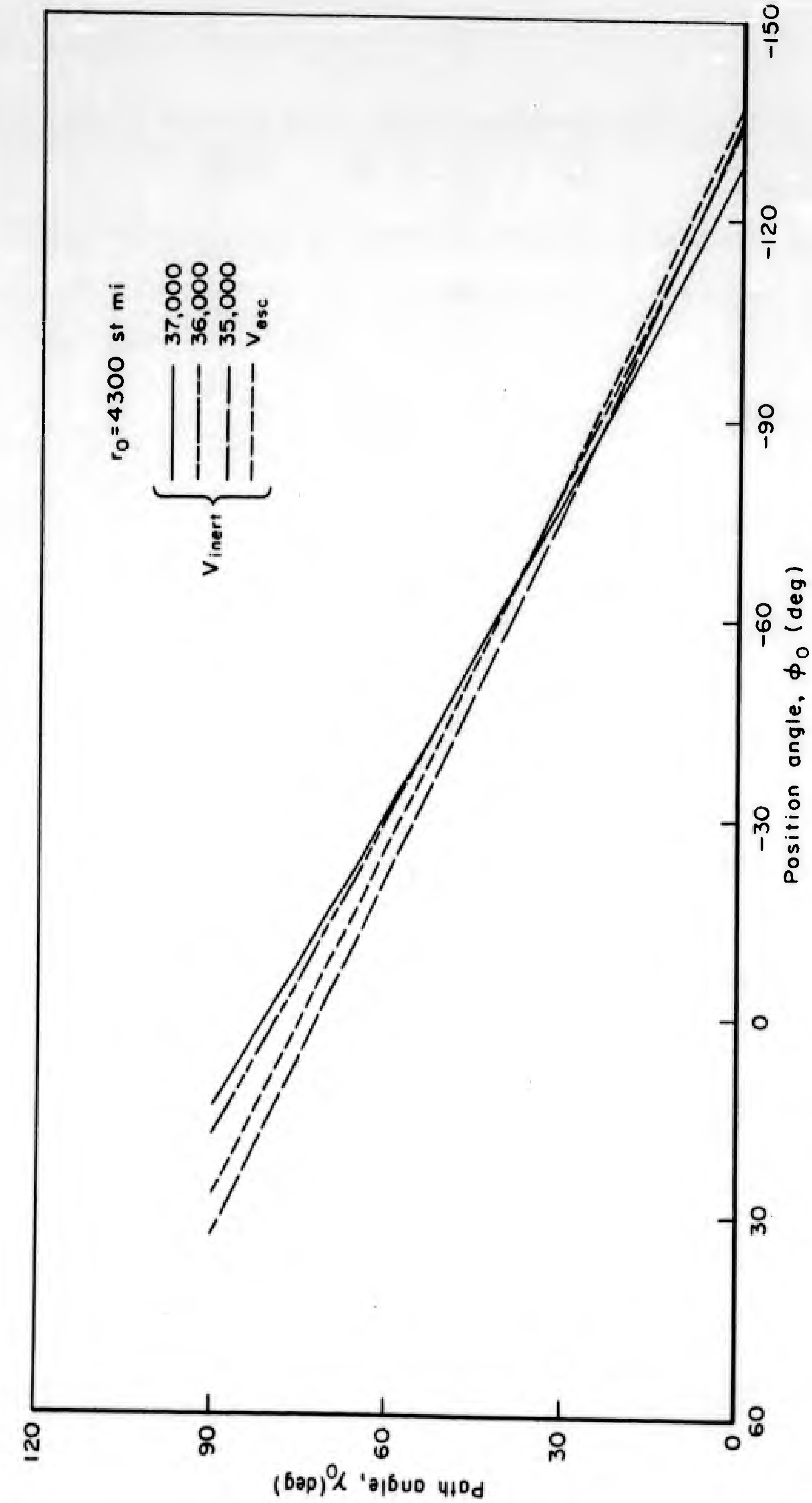

Fig.6—Path angle–position angle
(direct motion)

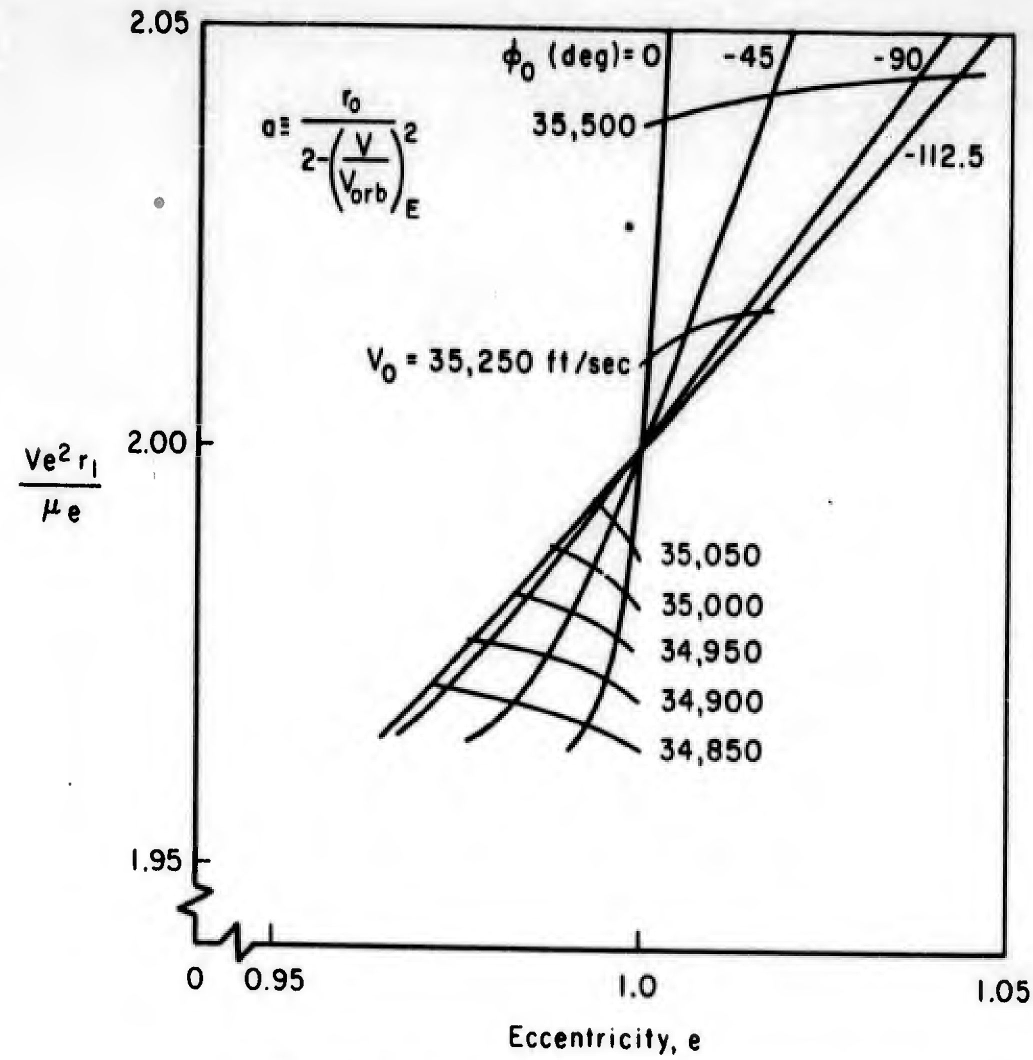

Fig. 7 — Initial conic parameters
(direct motion)

RETROGRADE MOTION

For completeness of the study of general initial conditions required to hit the moon, transit trajectories for vehicles launched in retrograde motion relative to the earth were also investigated. The initial positions for these trajectories (launched westward, or against the rotation of the earth) are located generally around the upper portion of the earth, shown in Fig. 2. The initial velocity - path angle combinations relative to the rotating system (corresponding to Fig. 4, which is for direct motion) are given in Fig. 8. The use of the rotating coordinate system again skews the velocity scale as shown by the dashed line indicating the local escape velocity.

As before, initial position angles from 10 to 40 deg above the x-axis correspond to initially radial trajectories, while initial positions which are 160 to 220 deg ahead of the moon are required for transit trajectories which start at perigee of the initial conic section.

The curves for constant initial position angles are seen to curve very gradually as the velocity is increased. There are no hooks as in the case of direct motion (Fig. 4) so that the initial velocity tolerance for impact on the moon will be monotonically increasing. The variation of the initial path-angle tolerance will be generally similar to the case of direct motion.

PERIGEE AND RADIAL TRAJECTORIES

The form of presentation of the initial trajectory parameters in Figs. 4 and 8 does not allow the initial position angles for trajectories defined by initial path angles of either 0 or 90 deg to be easily determined. Figure 9 presents the initial position angles required for these cases

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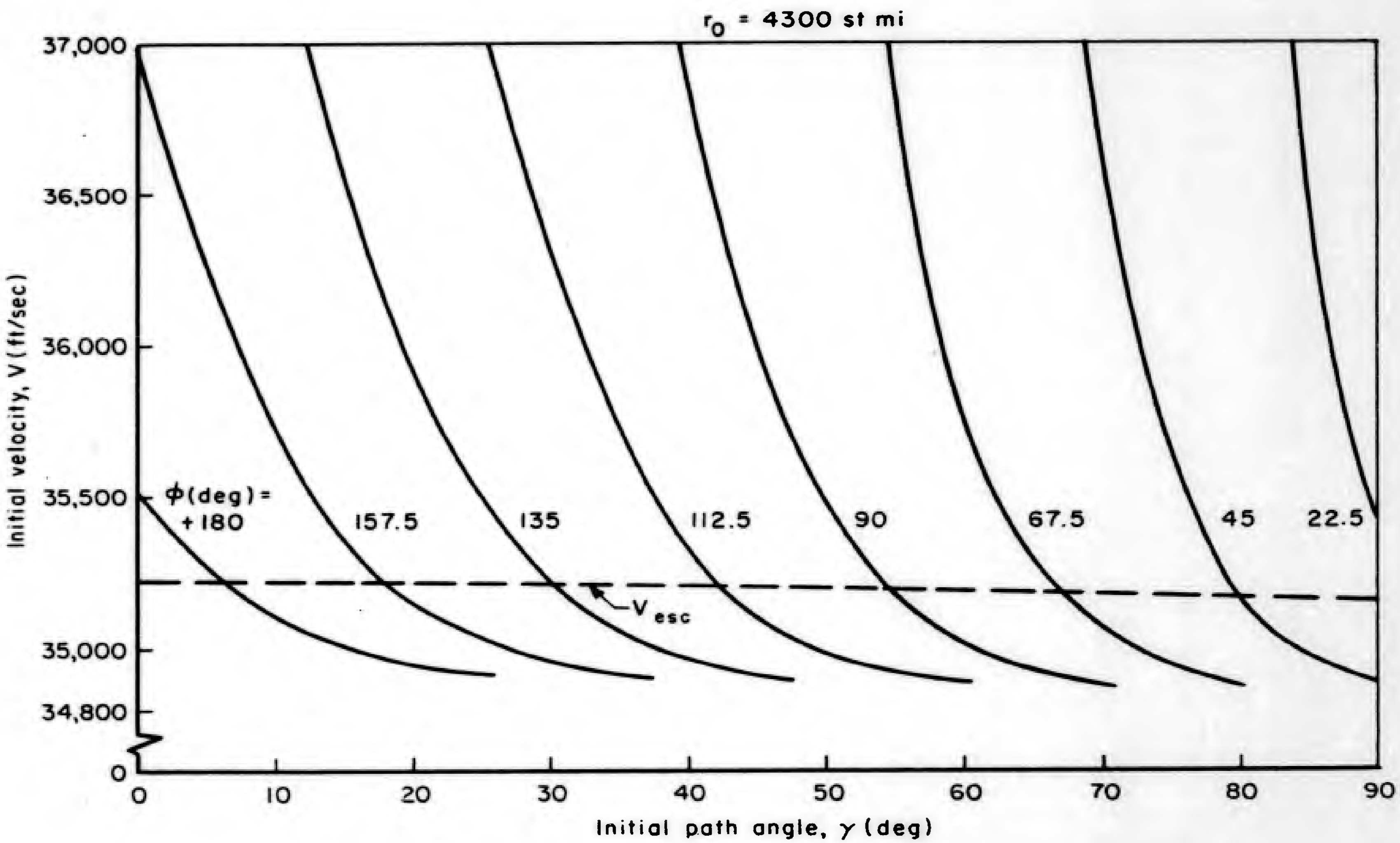

Fig. 8 — Initial conditions — retrograde motion
(rotating system)

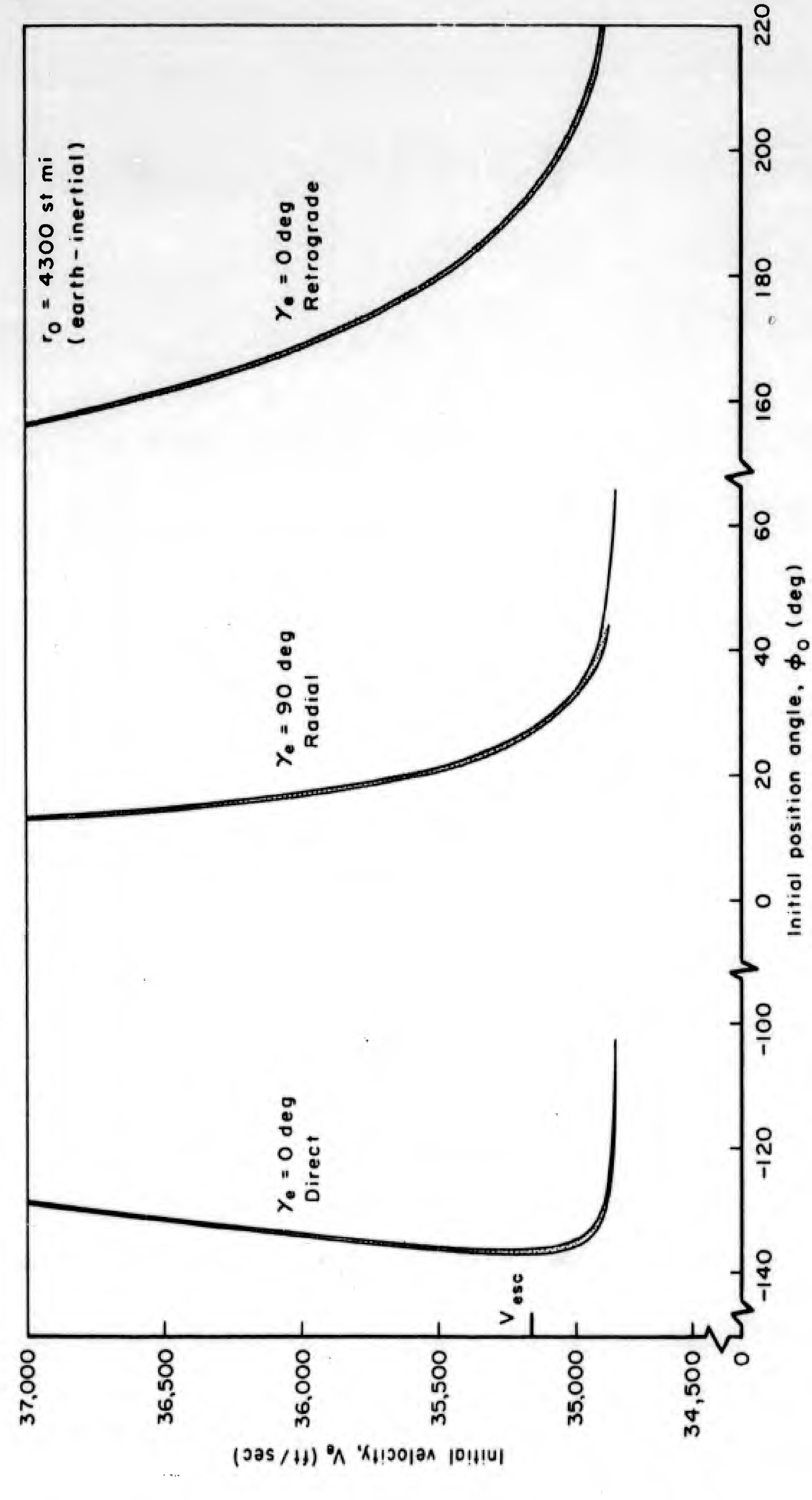

Fig. 9 — Initial position angles for perigee and radial trajectories

as a function of the initial velocity (relative to the earth-inertial coordinate system).

The curve for $\gamma = 0$ deg in direct motion shows a pronounced hook, resulting in large initial velocity tolerances over a range of velocities, as would be expected from the curves in Fig. 4. The initial position angles required for $\gamma = 90$ deg (radial trajectories), however, simply become asymptotic to a value of zero deg as the initial velocity approaches a value of infinity. The curve for $\gamma = 0$ deg in retrograde motion varies even more gradually.

The extremely-low-velocity portions of the curves are relatively horizontal, resulting in a large tolerance in initial position angle of the trajectory relative to the moon. Thus, large tolerances in initial position angle (which is equivalent to a launch-time tolerance for a rotating earth) can be obtained at the expense of the requirement for strict velocity control.

TRANSIT TIME

The time of flight for unpowered trajectories to the moon is extremely sensitive to the value of the initial velocity. The transit time is also dependent on the type of trajectory at the earth (perigee through radial initial path angles), and to some extent on the location of the impact point on the moon. Figure 10 presents the transit time as a function of the initial velocity relative to the earth-inertial coordinate system. The transit time increases very steeply for initial velocities below the local escape velocity indicated on the graph.

The transit-time difference between trajectories defined by initial path angles of 0 and 90 deg is not more than about 0.04 day for hyperbolic

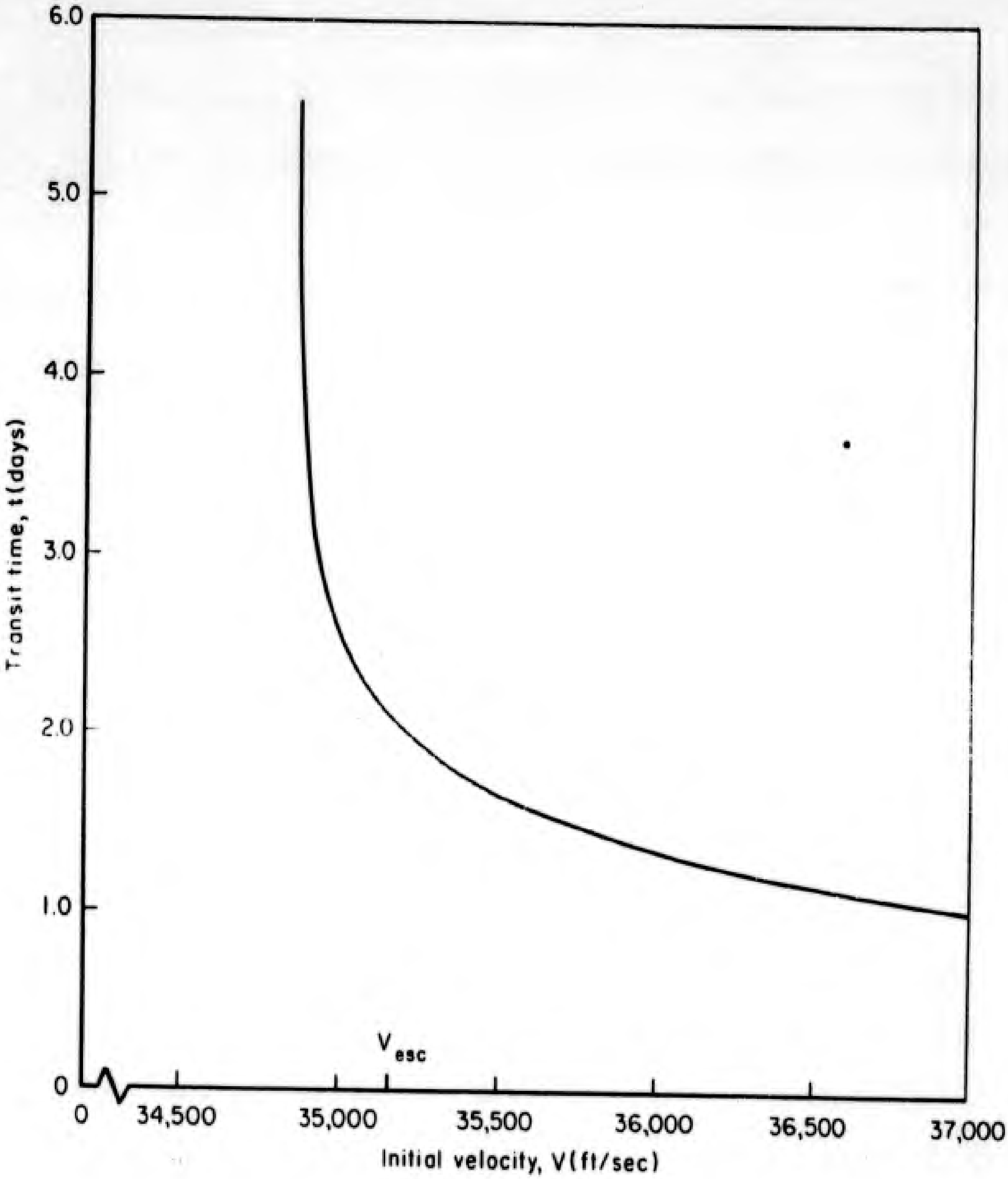

Fig. 10—Transit time

initial velocities. At these initial velocities the location of the lunar impact point will change the transit time by less than 0.02 day. For near-minimum initial velocities, however, the transit time can vary by almost one day, depending on the trajectory relative to the moon.

The theoretical maximum transit times to reach the moon's distance (mass of the moon equal to zero) are about 5.0 days for $\gamma = 0$ deg and 4.85 days for $\gamma = 90$ deg; the shape of the initial trajectory affects transit time by about 0.15 day.

DISCUSSION OF INITIAL TOLERANCES

The curves of Figs. 4 and 8 represent narrow bands of initial velocity - path angle combinations that define trajectories resulting in impact somewhere on the moon, as mentioned previously. The slope of a band, which is a measure of the relative value of the tolerances in initial velocity and path angle, is seen to vary with the magnitude of the initial velocity, and also with the initial trajectory position relative to the moon.

A "hit band," representative of position angles near -100 deg, is sketched in Fig. 11 for use in the following discussion. The left edge of the band, labeled E, gives the limiting initial velocity - path angle combinations defining trajectories which are tangent to the moon's surface beyond its leading, or eastern, limb. The other edge of the hit band, labeled W, defines the limiting initial combinations for trajectories which are tangent to the moon near its western limb. The dashed line in the band is included to indicate the locus of initial conditions for trajectories resulting in impacts that are normal to the surface of the moon.

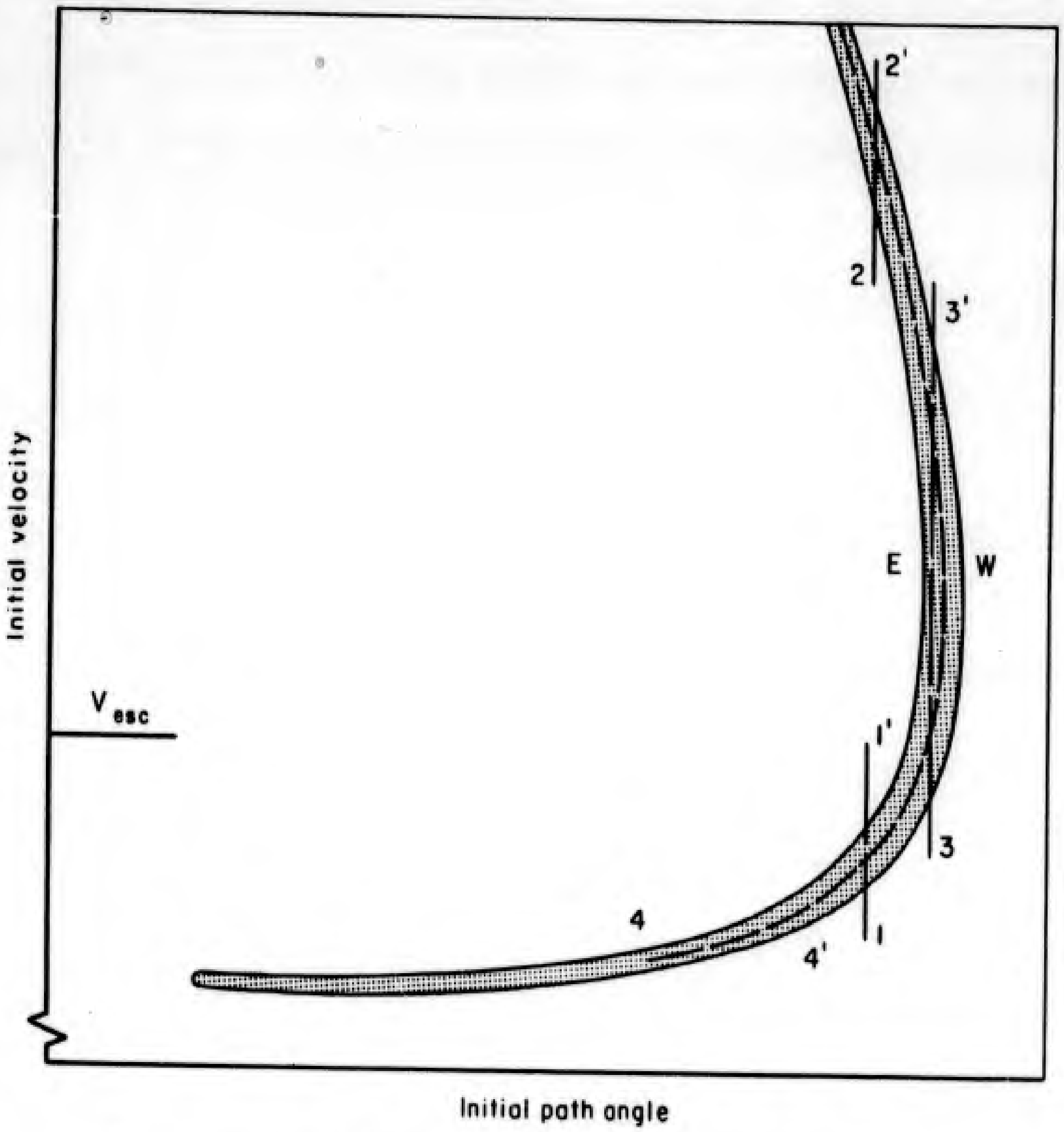

Fig. 11 — Hit band for a typical initial position at the earth

The initial tolerance can be defined as the variation in the parameter which causes the lunar impact point to travel from one limb of the moon to the other. Slices 1 - 1', 2 - 2', and 3 - 3' represent the variations of initial velocity at constant initial path angle. As the initial velocity is increased in slice 1 - 1', the lunar impact position moves from the western to the eastern limb of the moon. For slice 2 - 2', which is above the reflex of the band, the impact locations move from the eastern to the western limb of the moon as the velocity is increased. Slice 3 - 3', which cuts the hit band in the region where it is nearly vertical, results in the impact location moving from the western limb toward the eastern limb and finally back to the western limb of the moon. Slice 4 - 4' is included to show the variation of initial path angle at constant initial velocity. The lunar impact point is seen to move from the eastern to the western limb of the moon as the path angle is increased.

Thus it can be seen that the value of the initial velocity tolerance and the corresponding direction of travel of the impact point around the surface of the moon are functions of the magnitude of the initial velocity. The value of the initial path-angle tolerance is also a function of the magnitude of the initial velocity, but the direction of travel of the lunar impact point remains constant.

Figure 12 presents a sketch of the total initial velocity tolerance around the value of the velocity corresponding to an impact normal to the surface of the moon. At near-minimum initial velocities the tolerance has a value of about 10 ft/sec. As the magnitude of the initial velocity is increased, the value of the velocity tolerance

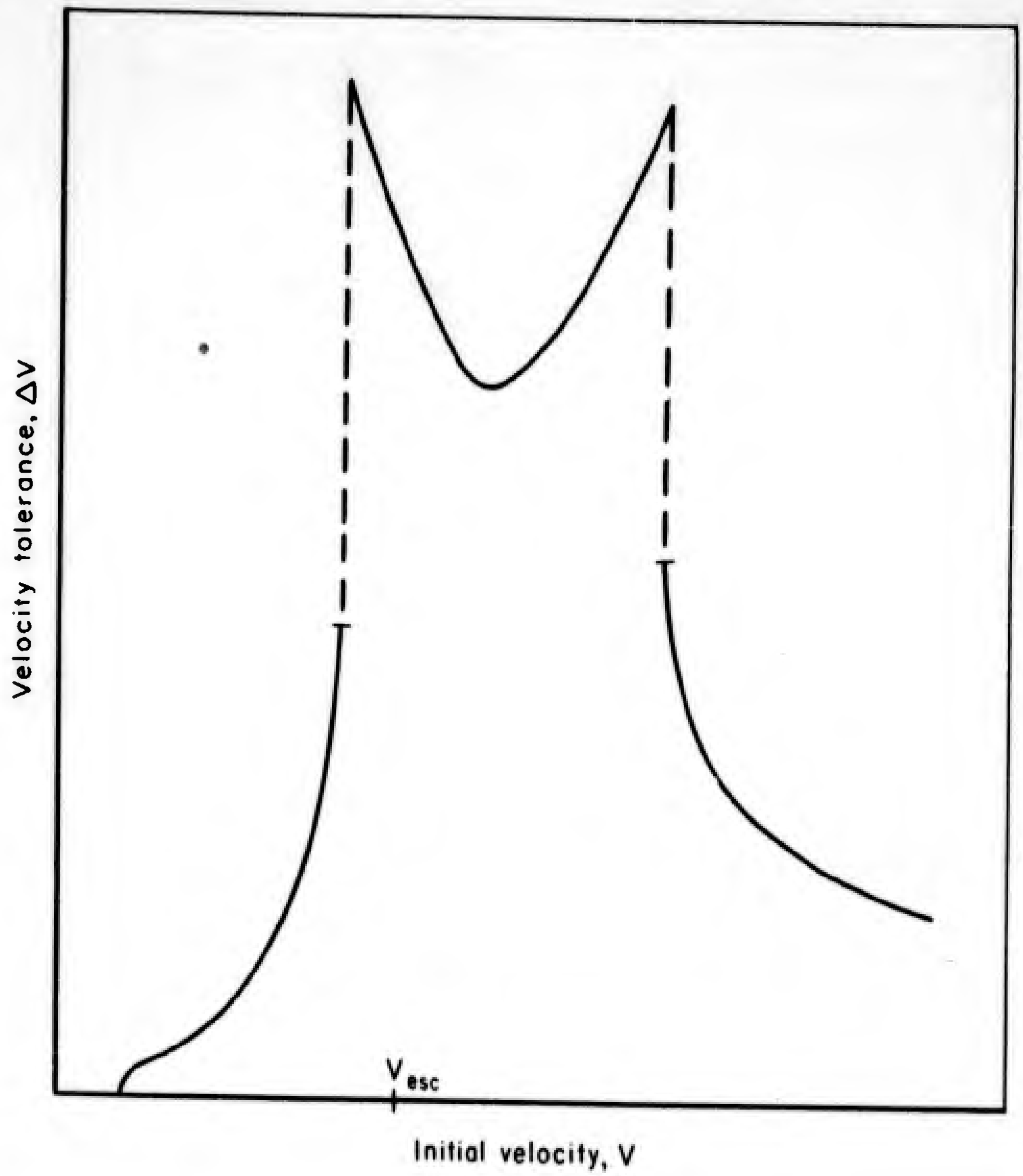

Fig.12 — Initial velocity tolerance as a function of initial velocity

increases rapidly to a value on the order of a few hundred ft/sec as slice 1 - 1' of Fig. 11 becomes tangent to the left edge of the hit band.

The maximum value of the initial velocity tolerance corresponding to slice 2 - 2' has approximately the same value as the maximum for slice 1 - 1'. As the initial velocity is increased further, the tolerance decreases gradually as the hit band continues to bend over.

The velocity tolerance in the portion of the hit band where the impact locations first move eastward and then westward over the surface of the moon (typified by slice 3 - 3' of Fig. 11) is quite large. At the points of tangency, the velocity tolerance is equal to the sum of values for slices 1 - 1' and 2 - 2', a value of approximately 600 ft/sec. The tolerance decreases to a local minimum as the locus for impacts normal to the moon is vertical.

The effect of increasing the initial path angle (with an appropriate shift of the initial trajectory position) is to shift this region of very large tolerances to higher initial velocities (see Fig. 4).

For an initial position of zero deg (on the earth-moon axis), the point of maximum velocity tolerance has moved to a velocity above that covered in Fig. 4 so that the tolerance values are still increasing.

Figure 13 presents a sketch of the corresponding total tolerance in the initial path angle. The tolerance value is seen to increase very rapidly and to peak at about 5 deg, at a value which is about 20 ft/sec greater than the minimum velocity required to reach the moon. As the magnitude of the velocity is increased the path-angle tolerance decreases, reaching a value of approximately 0.5 deg at high velocities.

The initial velocity and path-angle tolerances for trajectories

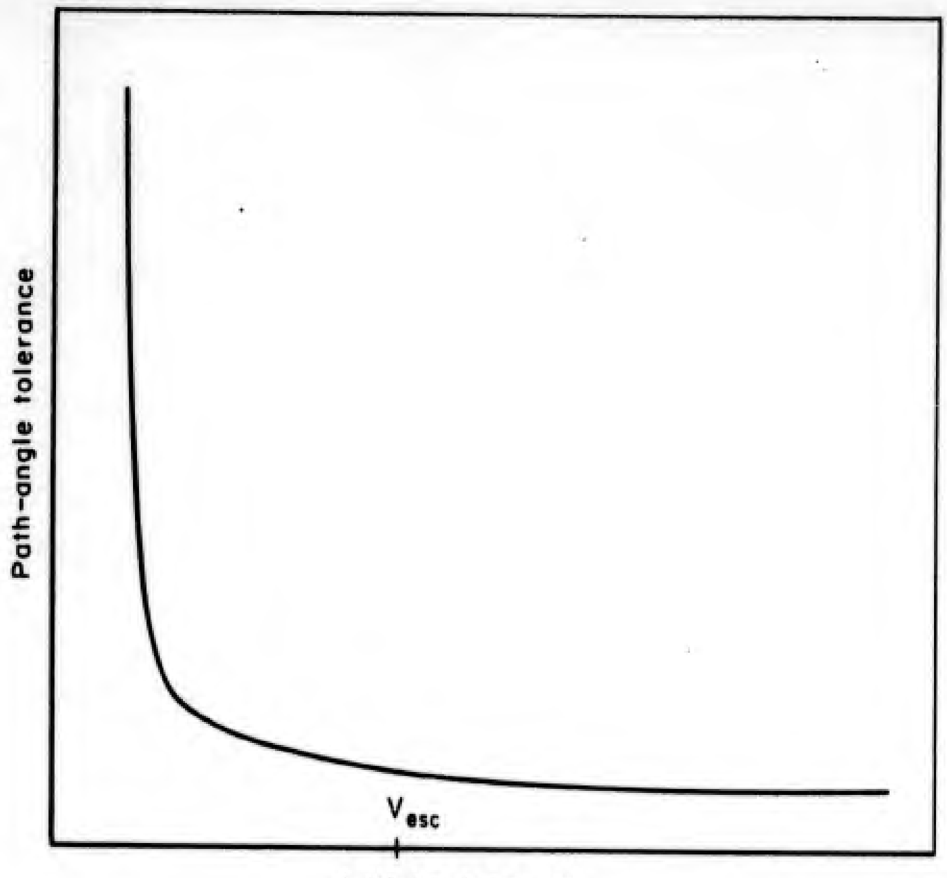

Fig. 13 — Initial path-angle tolerance as a function of initial velocity

which are launched in retrograde motion relative to the earth vary much more gradually, as can be inferred from Fig. 8. The hit bands do not bend over as in the case of direct motion, so that the initial velocity tolerance increases gradually while the path-angle tolerance decreases gradually as the magnitude of the initial velocity is increased.

For this class of trajectories, the location of the lunar impact point travels from the western to the eastern limb as the initial velocity or the initial path angle is increased within the hit band.

IV. TRANSIT-TRAJECTORY DESIGN CASE

Following the general investigation of trajectories resulting in impacts on the moon from various initial positions, a specific set of initial conditions was chosen for a vehicle-design study. The selection of the magnitude of the initial free-flight velocity for a transit trajectory to the moon is governed by two main factors: the first is a desire for a relatively low value of the initial (or burnout) velocity so that the payload weight for a given vehicle will be as large as possible; the second is imposed by the ascent guidance system. For initial velocities which are near the minimum value required to reach the moon, the allowable tolerance on initial path angle is very large, but the tolerance on the magnitude of the initial velocity is quite small. As the magnitude of the initial velocity is increased, the relative size of the tolerances on the magnitude and direction of the velocity are reversed. Therefore the choice of a typical transit trajectory requires an assessment of the relative capabilities of the ascent guidance system in measuring and controlling the magnitude and the direction of the burnout velocity vector.

The other parameters -- the direction of motion, initial position relative to the moon, and the corresponding initial path angle -- are also subject to choice. The selection of the direction of launch is relatively straightforward. The diurnal rotation of the earth about its axis contributes an increment to the burnout velocity, relative to an earth-fixed inertial coordinate system, of as much as about 1500 ft/sec for an eastward launch at the equator. Thus, it would be advantageous to launch the vehicle in a generally eastward direction, in direct motion relative to the earth. From vehicle-payload considerations, it is desirable to have a reasonably low path angle at the end of powered flight in order to decrease the velocity loss due to gravity during the powered-ascent trajectory.*

Since the initial position angle relative to the moon is a function of the initial path angle (Fig. 4), it is specified when the trajectory path angle is chosen.

As a reasonable first approximation to the requirements on the flight path discussed above, a transit-trajectory design point was arbitrarily chosen for vehicle-performance studies.[6] The trajectory will be discussed to show various features of earth-moon transit trajectories. The initial conditions that result in an impact at an angle nearly normal to the surface of the moon are

$$V = 35{,}000 \text{ ft/sec}$$

$$\gamma = 14.20 \text{ deg}$$

$$\phi = -108.0 \text{ deg}$$

$$r_1 = 4300 \text{ st mi}$$

relative to the coordinate system which rotates with the earth and the moon.

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*This statement is applicable to the case of two-dimensional trajectories, where the transit trajectory is in the plane of the moon's orbit. In the general case of three-dimensional motion, however, the initial trajectory azimuth angle must also be considered. As the initial path angle approaches zero deg, the corresponding required initial azimuth approaches zero deg (due north) so that a compromise between the velocity loss due to gravity during powered ascent and the vector-addition of the earth's angular velocity must be made.

This design transit trajectory is plotted in Fig. 14 relative to the coordinate system which is rotating with the earth and the moon. The initial position of the trajectory "lags" the moon by an angle of 108 deg. The vehicle's angular velocity (relative to the earth) is greater than that of the moon so that the vehicle "catches up" to the moon and crosses the x-axis after a time of about 2.6 hr. The vehicle moves to a maximum distance ahead of the earth-moon axis (an angle of about 12 deg ahead of the moon) at 1.12 days. From that time on, the vehicle's lead angle decreases until impact on the moon occurs after a transit time of 2.312 days.

Figure 15 presents this transit trajectory from the earth to the moon plotted in the earth-inertial coordinate system (x_e, y_e) with the x_e-axis directed toward the initial position of the moon. The initial trajectory parameters relative to this coordinate system are

$$V_e = 35{,}058.7 \text{ ft/sec}$$

$$\gamma_e = 14.176 \text{ deg}$$

while the initial position angle and radius from the earth remain the same as before. The velocity and path angle given above can be used to define the initial osculating conic section, assuming an isolated earth. The parameters of this ellipse are

$$a = 1.466945 \text{ lunar units} = 350{,}390 \text{ st mi}$$

$$e = 0.988468$$

giving perigee and apogee radii of

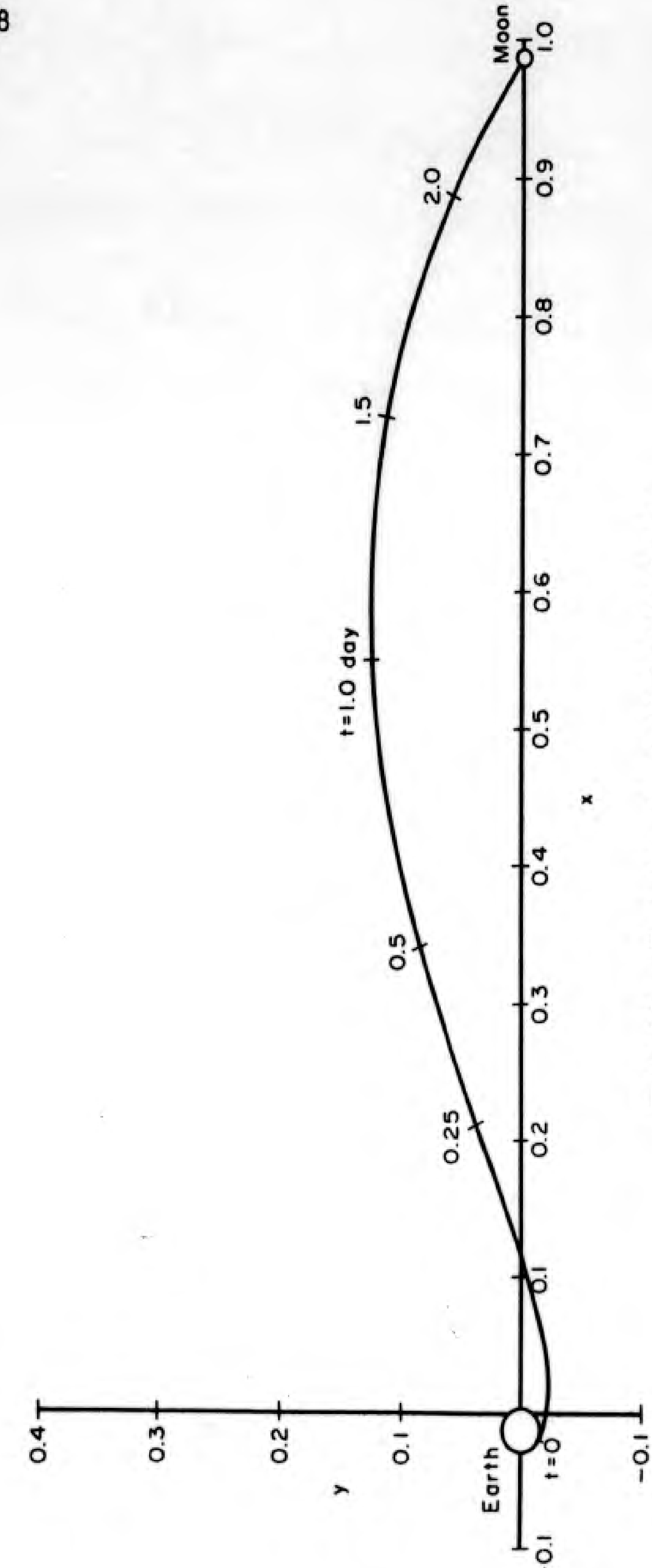

Fig. 14 — Design transit trajectory
(rotating coordinates)

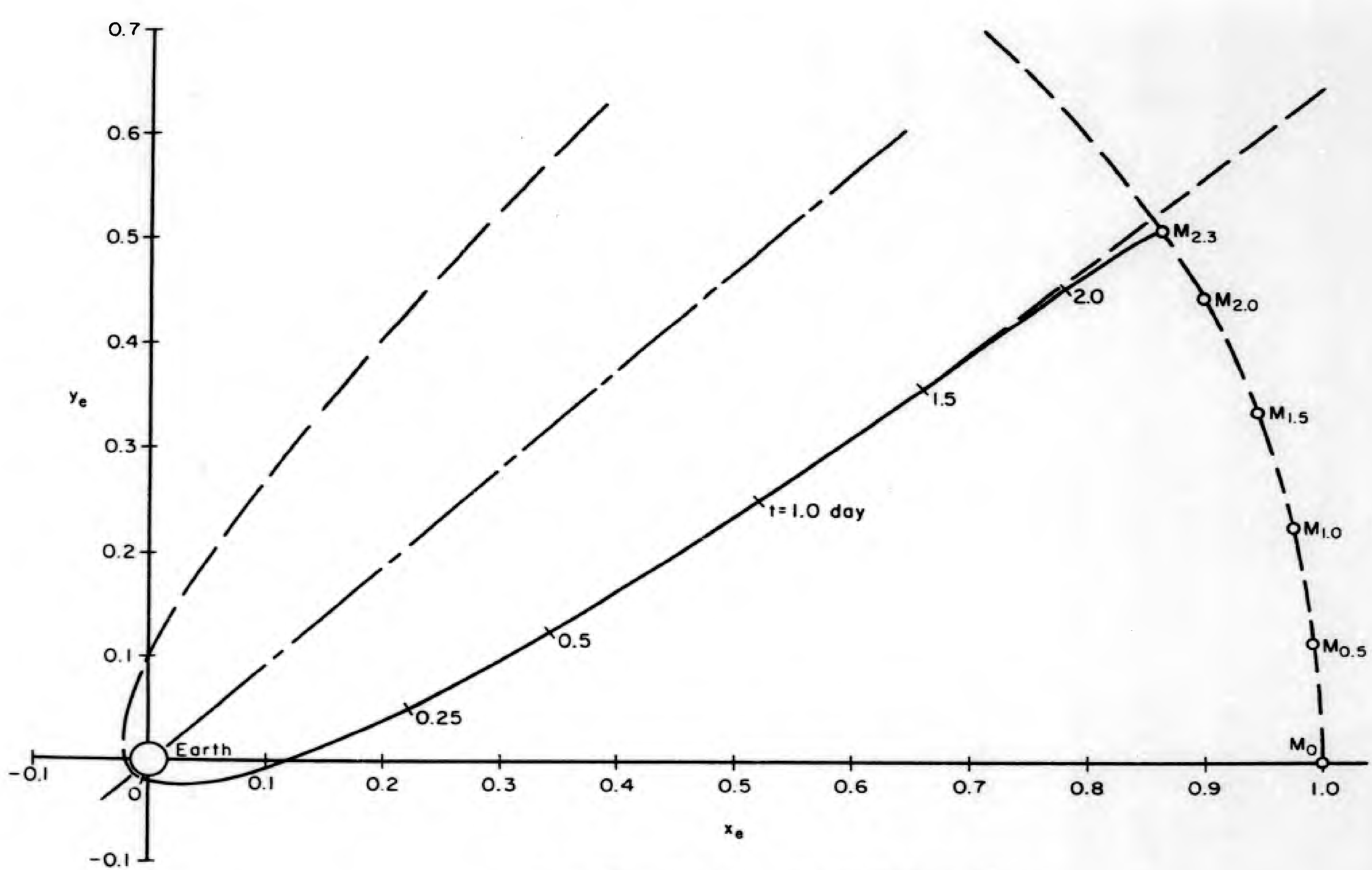

Fig. 15 — Design transit trajectory
(earth-inertial coordinates)

$$r_p = 4060.6 \text{ st mi}$$

$$r_a = 696{,}700 \text{ st mi}$$

The orbital period of the unperturbed ellipse would be 48.84 days, and the time of arrival at the moon's distance (assuming no moon) would be about 2.41 days.

The major axis of the osculating ellipse is at an angle of 43.48 deg to the x_e-axis so that perigee is located at an angle of -136.52 deg relative to the initial position of the moon. The osculating ellipse is indicated in Fig. 15 by the dashed line to show that the moon's attraction substantially affects the vehicle's trajectory -- in this case, only during approximately the last 1/2 day of the transit time.

The time-history of the instantaneous conic-section parameters for this trajectory is given in Section V.

HIT BAND

Transit trajectories defined by initial velocity - path angle combinations in the vicinity of the design point were computed to investigate the sensitivity of the trajectories near the moon to relatively small changes in the initial conditions. Trajectories which both impacted on and missed the moon were investigated. These latter transit trajectories could be used to establish artificial satellites of the moon[2] if the capability of an additional velocity increment is assumed.

Figure 16 shows an enlarged-scale portion of the initial-conditions graph of Fig. 4 in the vicinity of the design case for trajectories

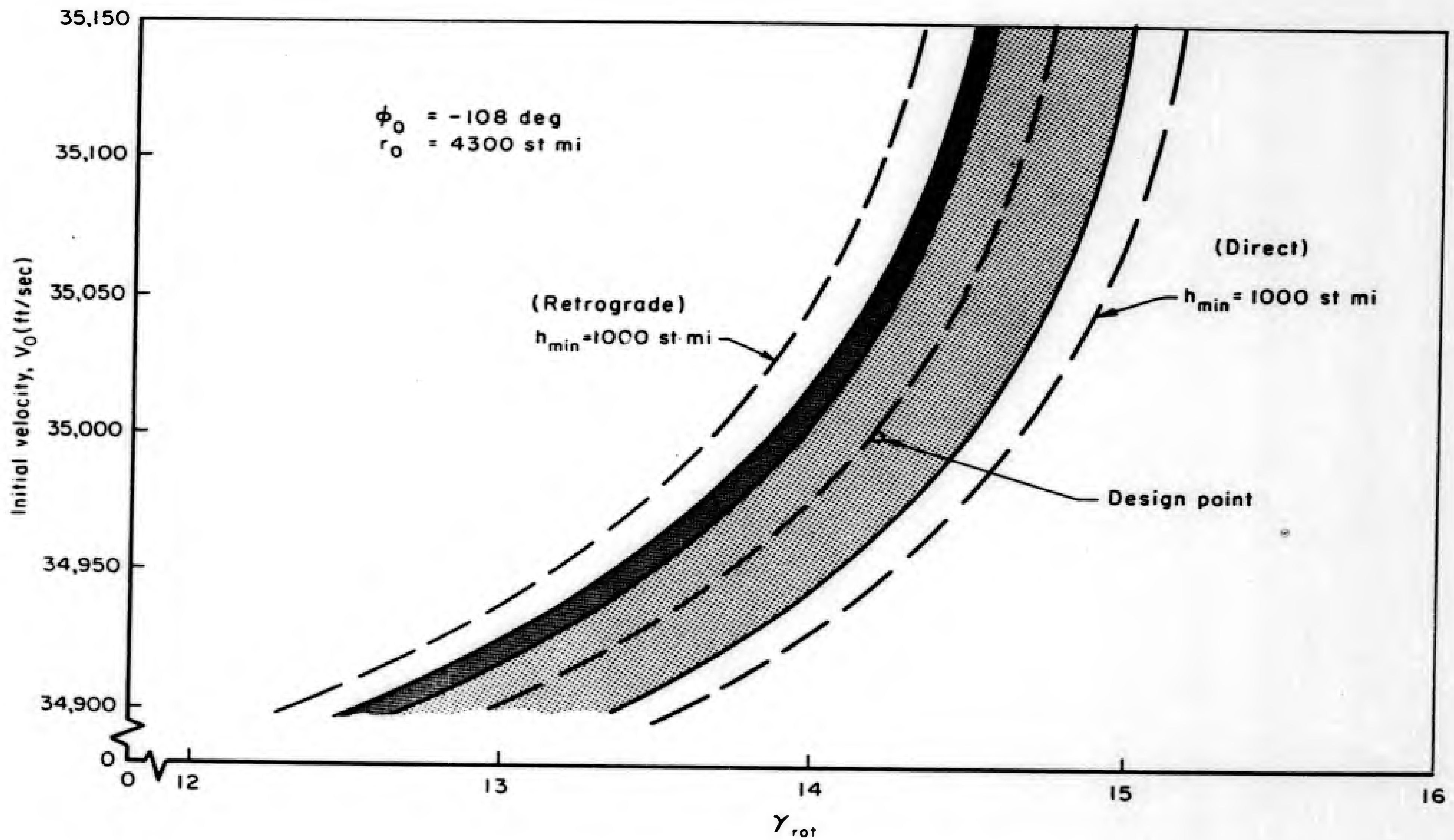

Fig. 16 — Hit band in vicinity of design point
(rotating system)

starting at the same initial position (ϕ = -108 deg, r = 4300 st mi). The initial-velocity and path-angle scales are referred to the rotating coordinate system, as in Fig. 4.

The scales of Fig. 16 can be converted to the earth-inertial coordinate system (see Fig. 5) by use of constant increments of $\Delta V = + 58.7$ ft/sec and $\Delta\gamma = - 0.024$ deg for the region covered in this graph.

The shaded band represents the combinations of initial velocity - path angle combinations for transit trajectories which result in impacts on the surface of the moon. The heavily shaded portion at the left of the band defines trajectories which intersect the lunar surface at points beyond the eastern limb of the moon (therefore not visible from the earth). The larger, lightly shaded area defines the initial conditions for trajectories resulting in impacts somewhere on the face of the moon, which is visible from the earth. The dashed line in this region, passing near the small circle which defines the design case, specifies the locus of initial velocity - path angle combinations for trajectories which approach normal to the surface of the moon.

Dashed lines lying outside the hit band are included to show the loci of initial combinations defining transit trajectories which pass the moon at a minimum altitude of 1000 st mi. A vehicle fired on a trajectory defined by a point on the curve to the left of the shaded band will reach the moon's orbit at a point ahead of the moon and pass it in a clockwise direction so that the vehicle's motion will be retrograde relative to the moon. Conversely, points on the curve to the right of the hit band define trajectories for which the vehicle's motion relative to the moon is direct.

The total, non-simultaneous, allowable initial tolerances around the design point are 75 ft/sec or 0.52 deg for impacts on the visible face of the moon (lightly shaded band). If impacts beyond the eastern limb of the moon are included, the allowable tolerances are increased to values of about 90 ft/sec or 0.63 deg.

Rectangles of various proportions, centered on the design point, could be drawn to give the maximum allowable simultaneous errors in the initial velocity and path angle. A representative set of these tolerances could be approximately 60 ft/sec and 0.2 deg for impacts on the visible face of the moon.

Since the hit band of Fig. 16 is curved, the allowable initial trajectory tolerances will be changed as the nominal design point is shifted along the dashed line (which defines impacts normal to the surface of the moon). Figure 17 presents these initial tolerances as functions of the velocity of the new design point (again, relative to the rotating coordinate system) for impacts both on the front face of the moon and anywhere on its surface. The tolerance in the initial velocity is seen to vary by a factor of approximately 5 within the velocity interval shown, while the initial path-angle tolerance remains almost constant. These curves indicate the possibility of adjusting the allowable initial tolerances by means of shifting the magnitude of the velocity for the design case.

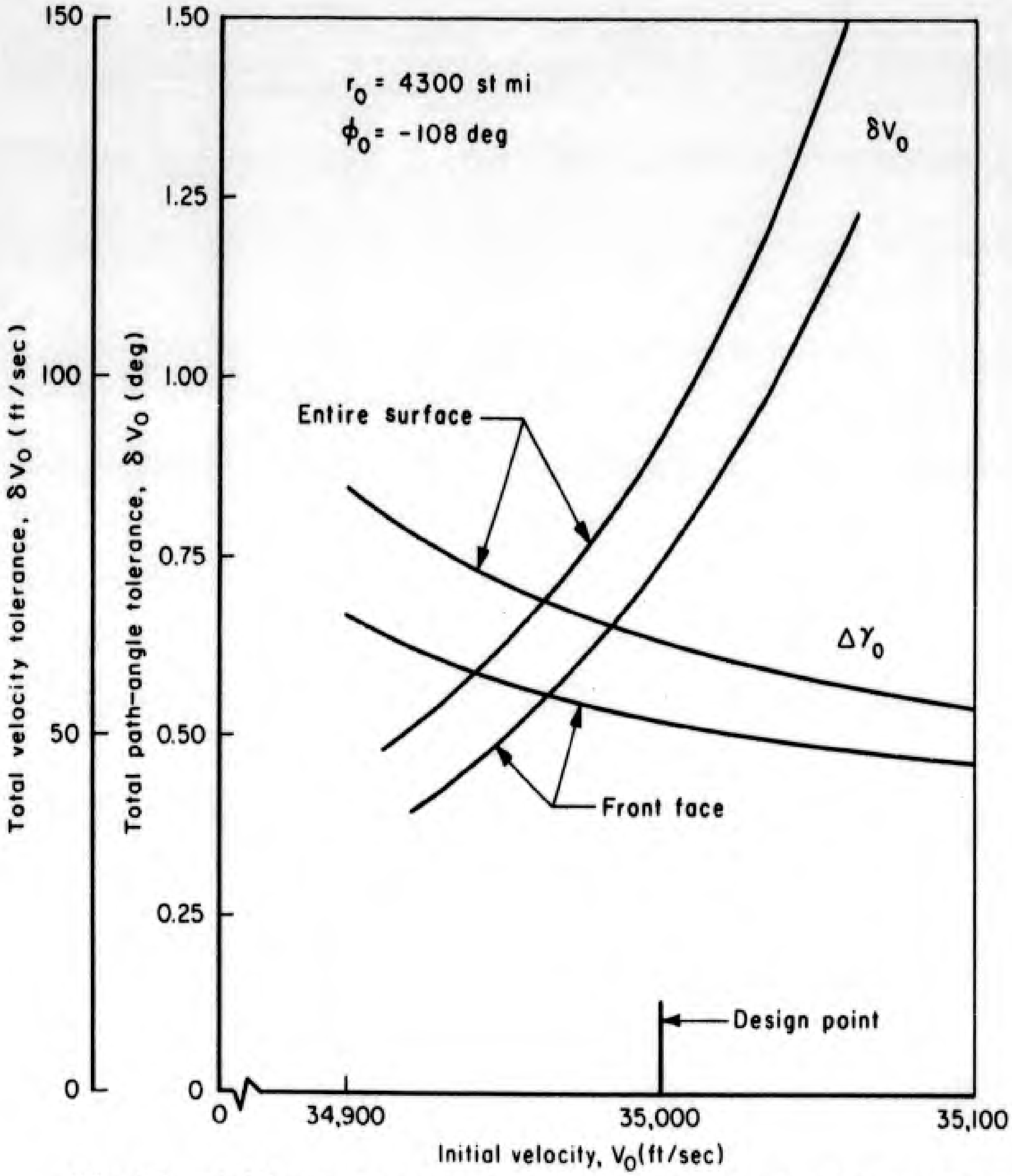

Fig. 17 — Initial tolerances as a function of initial velocity

TRAJECTORIES NEAR THE MOON

The transit trajectories computed to define the hit band in the vicinity of the design case are used to show the general behavior of trajectories in the neighborhood of the moon and the effects of errors in the initial velocity or path angle.

Portions of several transit trajectories in the vicinity of the moon are shown in Fig. 18. These curves are plotted relative to the rotating coordinate system so that the moon's position remains fixed regardless of the flight time, and the various angles, as viewed from the earth, are preserved. This set of trajectories is generated by varying the magnitude of the initial velocity at the earth, corresponding to a vertical slice through the hit band of Fig. 16, and includes cases which miss the moon in retrograde and direct motion. The direction to the earth is indicated for reference.

The trajectory of the nominal design case (V_0 = 35,000 ft/sec) is seen to impact at a path angle which is nearly normal to the surface of the moon at a point which is located approximately 30 deg east of the center of the visible face of the moon.

As the magnitude of the initial velocity is decreased, the lunar-impact position is moved toward the western limb of the moon. The limiting trajectory, with a tangential impact, is given by an initial velocity of approximately 34,965 ft/sec. The position of this impact is very nearly at the western limb of the moon.

The corresponding upper limit on the initial velocity is 35,055 ft/sec. The position for this tangential impact, however, is at an angle of approximately 65 deg beyond the eastern limb of the moon. The initial

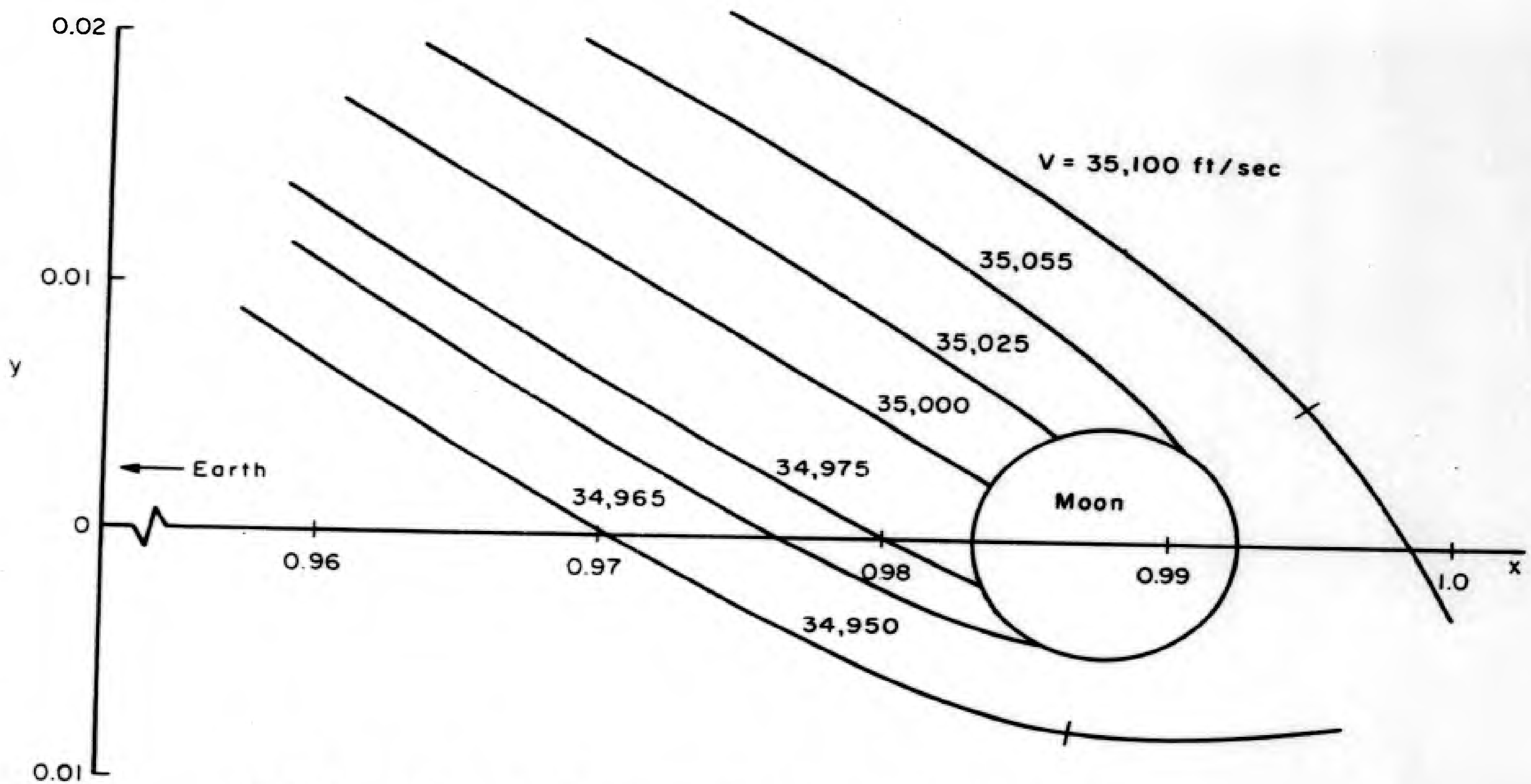

Fig. 18 — Terminal trajectories for variation of initial velocity

velocity for a trajectory which impacts <u>at</u> the eastern limb of the moon is about 35,040 ft/sec, resulting in the total initial velocity tolerances of 75 and 90 ft/sec.

The trajectory given by an initial velocity of 35,100 ft/sec is included as an example of a path which crosses the moon's orbit ahead of the moon. The vehicle, in retrograde motion, passes the lunar surface at a minimum altitude of about 1030 st mi. The tick on the trajectory indicates the point of closest approach, or perilune, which is located at an angle of about 140 deg relative to the line joining the earth and moon.

The other near miss, defined by an initial velocity of 34,950 ft/sec shows a trajectory which reaches the moon's orbit after the moon has passed. A vehicle on this trajectory would miss the moon in direct motion at a minimum altitude of about 720 st mi. The position of perilune, indicated by the tick, is located at an angle of about 80 deg from the earth-moon axis -- that is, over the visible face of the moon.

CONIC SECTIONS RELATIVE TO THE MOON

The set of transit trajectories described above, and a similar set resulting from variation of the initial path angle (at a constant velocity of 35,000 ft/sec), are used to show the variation of the parameters of moon-centered conic sections as a function of the initial conditions at the earth. The specific numerical values quoted apply to only the trajectories being discussed herein, but the relative variations in these values can be considered to be representative of the behavior of sets of trajectories defined in comparable regions of Fig. 4.

The portions of transit trajectories in the vicinity of the moon can be correlated by assuming the vehicle's motion to be under the influence of an isolated moon. The conic-section parameters computed on this basis will actually be slowly varying quantities due to the differential effect of the earth's gravitational attraction. However, for short time intervals, the vehicle's motion can be considered to be relative to an inertial moon.

The following set of graphs shows the variation of these parameters as a function of variations in both the initial velocity and path angle. The values corresponding to the design point (V_0 = 35,000 ft/sec and γ_0 = 14.2 deg) are located at a common point on the abscissa of the graphs.

The absolute value of the hyperbolic semimajor axis, $|a|$, relative to the moon is shown in Fig. 19 as a function of the initial velocity and path angle (relative to the rotating coordinate system). The intersection of the curves corresponds to the design case, and the ticks to either side indicate the limiting values for tangential impacts on the moon. The semimajor axis, which is nondimensionalized by the earth-moon distance (238,857 st mi), is seen to vary principally with the initial velocity at the earth. The slight variation with initial path angle is produced by referring the trajectory position and velocity data to the nonrotating moon-centered coordinate system. (As shown in Appendix A, the magnitude of the velocity at the moon, relative to the rotating coordinate system, is a function only of the initial velocity at the earth.) The difference in semimajor axis between limiting values of initial path angle is equivalent to the velocity increment arising from the change of coordinate-system reference.

The corresponding eccentricity, e, of these moon-centered hyperbolas

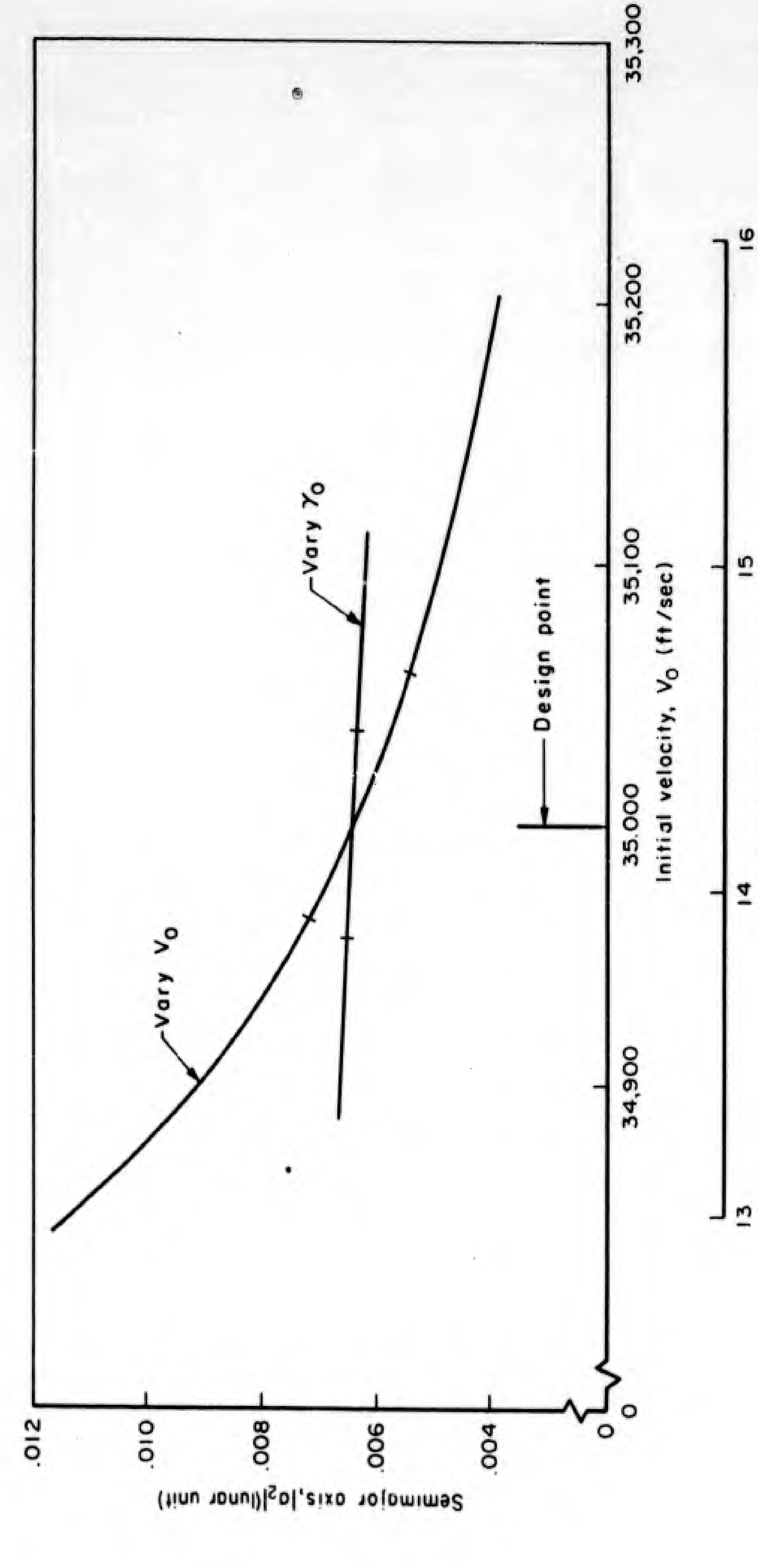

Fig. 19 — Semimajor axis relative to the moon

is shown in Fig. 20, the ticks again indicating the limiting values for lunar impact. The locations of the points of $e = 1.0$ (which corresponds to a radial trajectory) are seen to be displaced slightly from the design point since it does not quite lie on the curve defining radial impacts in Fig. 16.

The perilune radius, given by $r_p = |a|(e - 1)$, is plotted in non-dimensional form in Fig. 21. The moon's radius, as well as several values of lunar altitude, is indicated for reference.

The branches of the curves are labeled to indicate the direction of motion of the trajectory relative to the moon. For very small values (generally less than the moon's radius), the perilune radius varies as a quadratic function of the "error" from the value corresponding to a radial impact ($r_p = 0$) for variations in both the initial velocity and path angle. As the perilune distance is increased, the variation becomes more nearly linear so that the sensitivity can be expressed simply as the slope of the curve at a given radial distance.

The variation of perilune radius as a function of initial path angle is symmetrical around the value corresponding to radial impact out to distances of several moon radii. Thus the sensitivities of perilune altitude for trajectories passing the moon in either direct or retrograde motion will be equal. The perilune sensitivity has a value of $\frac{\partial r_p}{\partial \gamma_o} =$ 4400 st mi/deg at the surface of the moon, increasing to 5300 st mi/deg at a lunar altitude of 2000 st mi.

The curve of perilune radius as a function of initial velocity (at constant path angle), however, is not symmetrical around the velocity for $r_p = 0$ for perilune values which are greater than the moon's radius. For

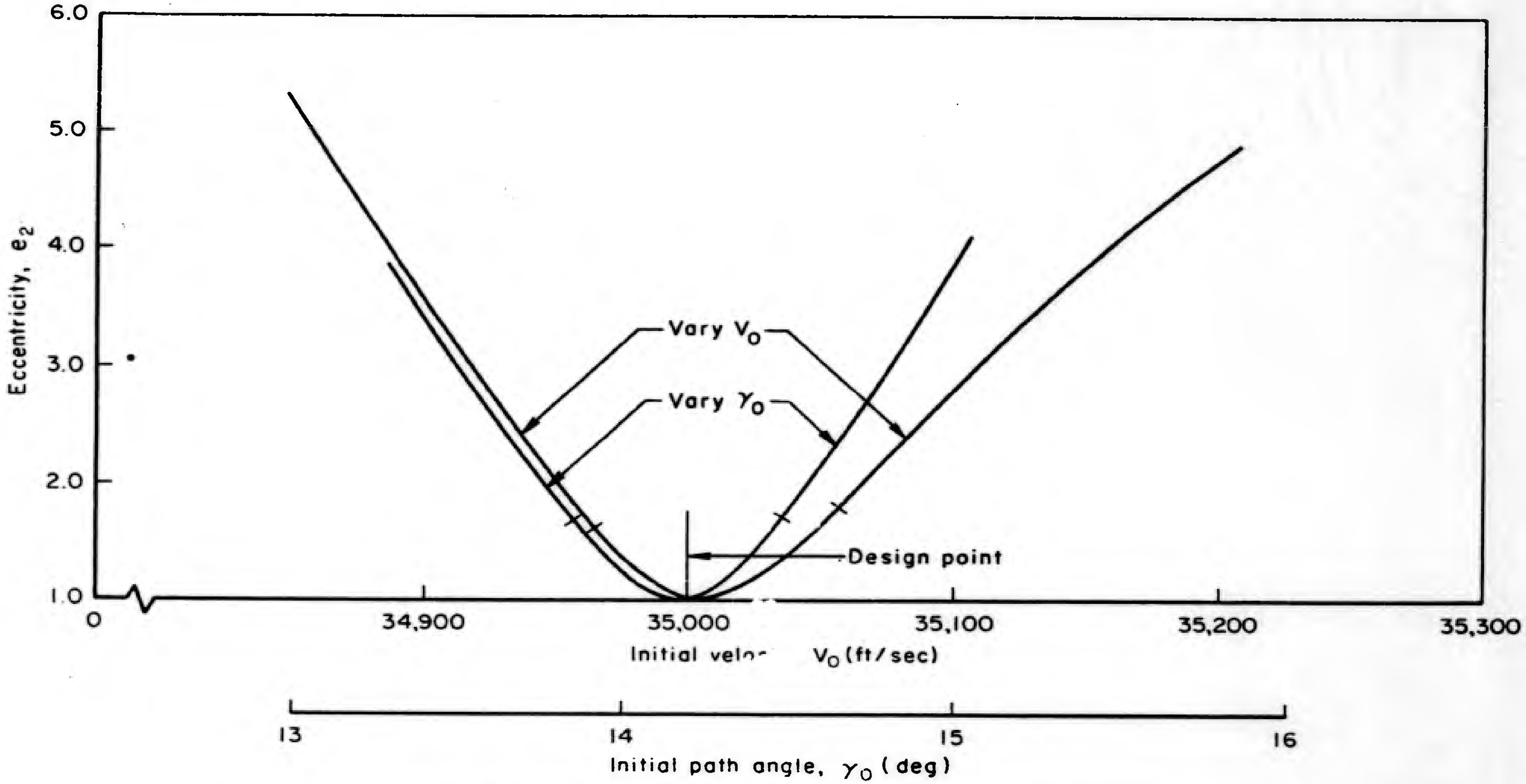

Fig. 20 — Eccentricity relative to the moon

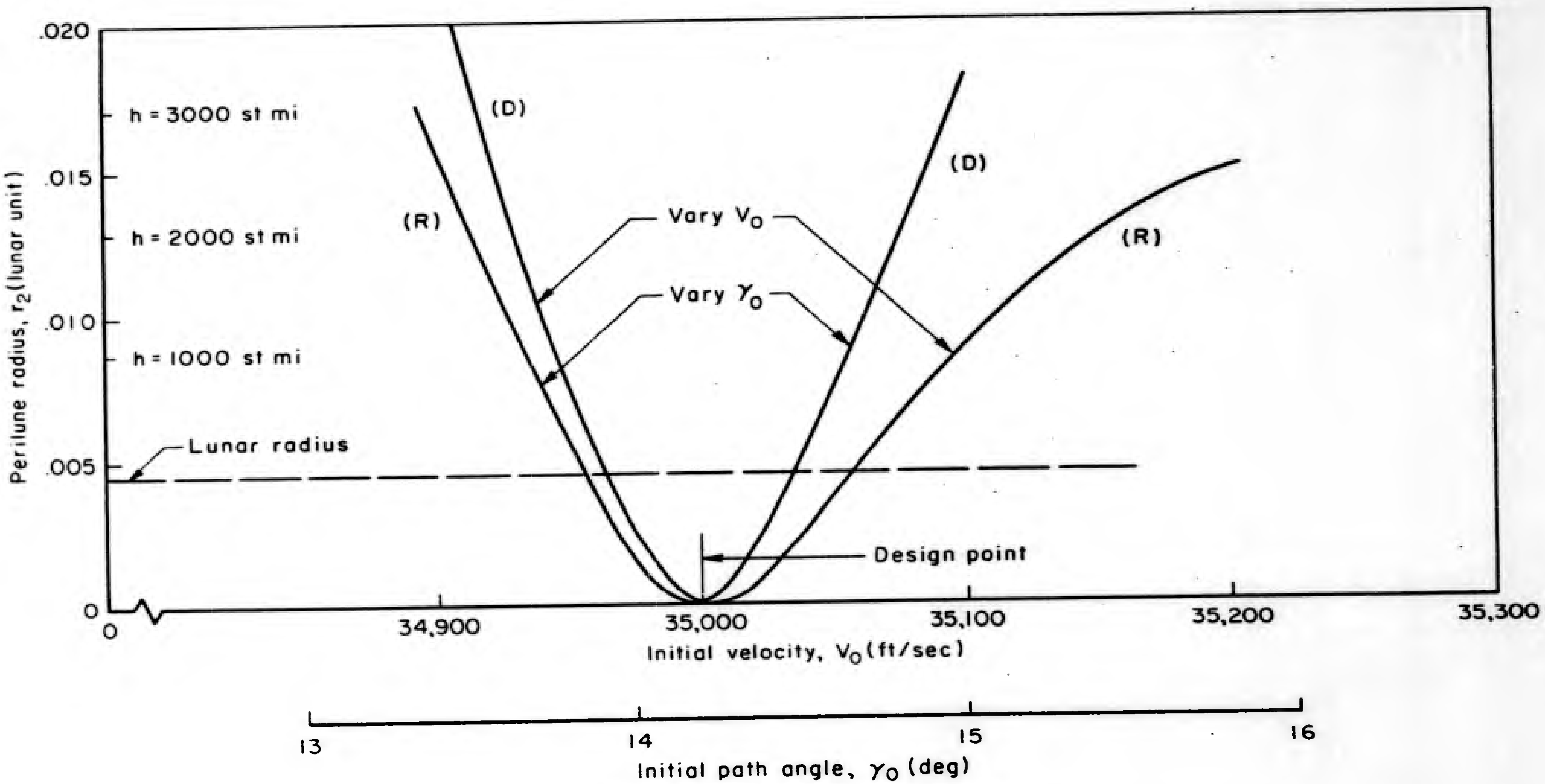

Fig. 21 — Perilune radius

initial velocities less than the design value, the perilune radius increases monotonically with the error in initial velocity. The perilune sensitivity for these trajectories, which are in direct motion relative to the moon, is $\frac{\partial r_p}{\partial V_o} = 48.2$ st mi/ft/sec at the surface of the moon, and increases to a value of about 70.7 st mi/ft/sec at a lunar altitude of 2000 st mi. For trajectories defined by initial velocities greater than the design value, the perilune behavior is typical of the variation around a design point located below the knee of the curves in Fig. 4 (the design initial velocity being less than the velocity corresponding to the maximum velocity tolerance). The corresponding perilune sensitivity for these trajectories (in retrograde motion relative to the moon) is 26.8 st mi/ft/sec at the surface of the moon, decreasing to a value of about 13 st mi/ft/sec at a lunar altitude of 2000 st mi.

As this curve is extended to higher initial velocities (still at $\gamma_o = 14.2$ deg), the perilune radius reaches a maximum value of about 0.0163 lunar unit (3900 st mi) at a velocity of 35,280 ft/sec. At this value of velocity, the hit band for this initial position angle, $\phi = -108$ deg, is parallel to the slice, so that the perilune sensitivity $\frac{\partial r_p}{\partial V_o}$ is equal to zero. Increasing the initial velocity still further results in a second set of lunar-impact trajectories as the constant-path-angle slice intersects the upper branch of the hit band. These lunar impacts occur in the velocity interval between approximately 35,650 and 35,860 ft/sec, with radial impact occurring at an initial velocity of 35,760 ft/sec.

The position of perilune, which determines the orientation of the moon-centered conic section, is shown in Fig. 22 as a function of perilune radius for variations in both initial velocity and path angle. The

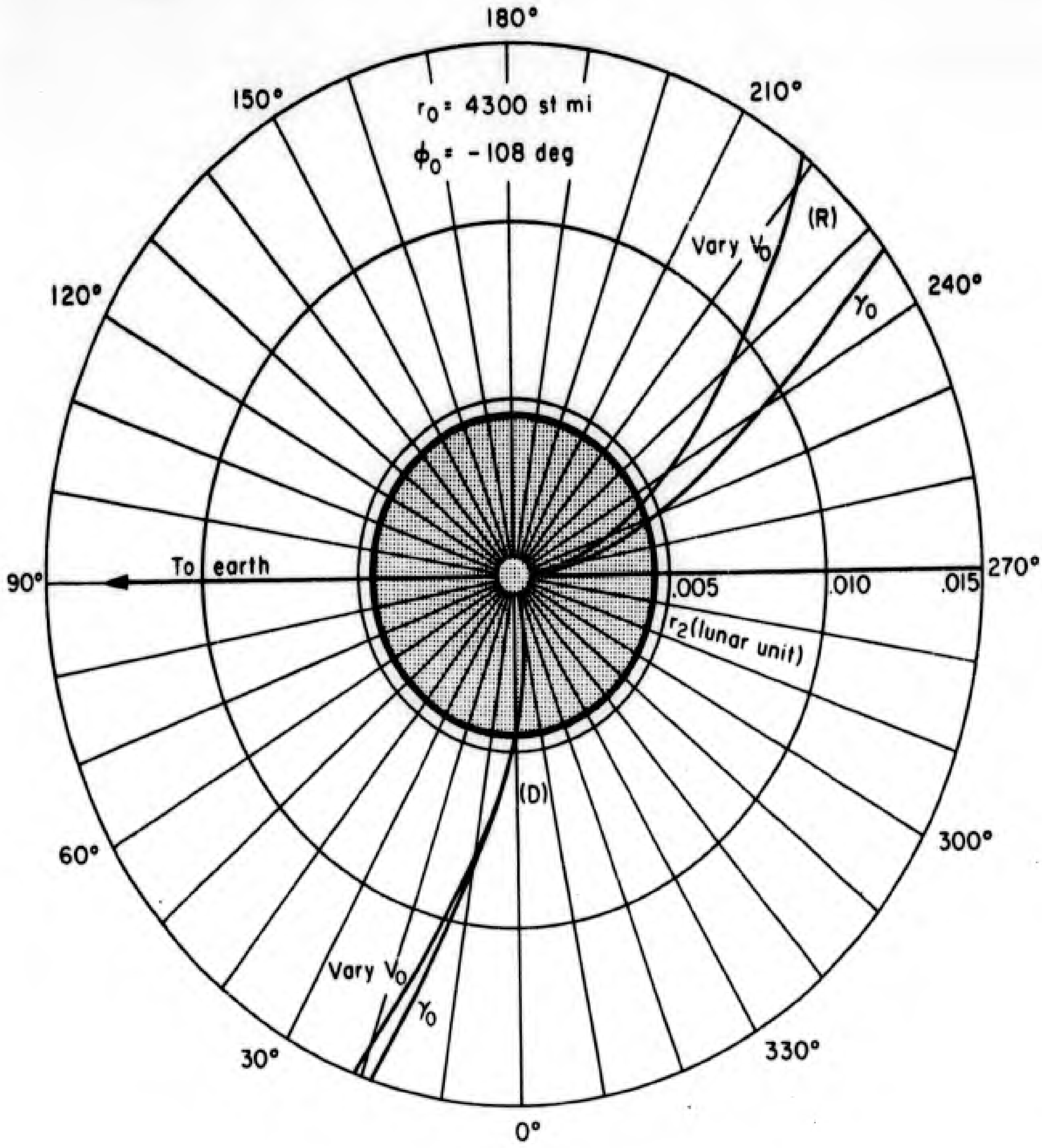

Fig. 22 — Perilune position

direction to the earth is indicated for reference. Transit trajectories which miss the moon in direct motion have perilunes located over the visible face of the moon (see Fig. 18), with perilune positions moving toward the western limb of the moon as the limiting case of tangential impact is approached. For retrograde miss trajectories, which have perilune located in the quadrant beyond the eastern limb of the moon, its position moves further around the far side as the perilune radius is decreased to a value equal to the moon's radius. These limiting positions for tangential impact are 155 and 162 deg from the earth-moon line for variations in initial velocity and path angle, respectively.

The limiting position angles for retrograde trajectories retreat toward the eastern limb of the moon as the initial velocity is increased, while the limiting positions for direct motion remain essentially at the western limb. The fraction of the moon's circumference which can be hit by an unpowered vehicle, therefore, decreases as the magnitude of the initial velocity is increased. This fraction approaches 0.5 as the initial velocity becomes very large. The positions of the corresponding impacts normal to the surface approach the center of the visible face of the moon.

Conversely, as the magnitude of the initial velocity is reduced, the limiting retrograde impact positions move further around the far side of the moon, while the limiting positions for direct motion gradually move away from the western limb onto the visible face of the moon. The fraction of the moon's circumference which can be hit approaches a value of 1.0, and the position for a normal impact approaches the eastern limb of the moon.

For these extremely-low-velocity trajectories (which have their unperturbed apogee located approximately at the moon's orbit) the moon can be considered, effectively, to "run into" the vehicle.

IMPACT POSITION

The position of the lunar impact point, relative to the earth-moon axis, is shown in Fig. 23 as a function of both initial velocity and path angle. The center of the visible face, as well as the eastern and western limbs of the moon, is indicated for reference. As the initial velocity is increased, the impact position is seen to move from the western limb of the moon to a point which is approximately 65 deg beyond the eastern limb (see Fig. 18). The corresponding impact locations for an initial-path-angle variation move from beyond the eastern limb of the moon, over the visible face of the moon, to the western limb of the moon as the path angle is increased.

The intersection of the curves, at approximately 27 deg east of the center, corresponds to the design trajectory which results in an impact nearly normal to the surface. The sensitivity of impact location to initial errors is a minimum in the vicinity of this point. The impact-position sensitivities, expressed as the distance along the surface of the moon, are approximately

$$\frac{\partial s_m}{\partial V_o} = 31 \text{ st mi/ft/sec}$$

and

$$\frac{\partial s_m}{\partial \gamma_o} = 4500 \text{ st mi/deg}$$

at the design point. As seen in Fig. 23, these slopes are reasonably

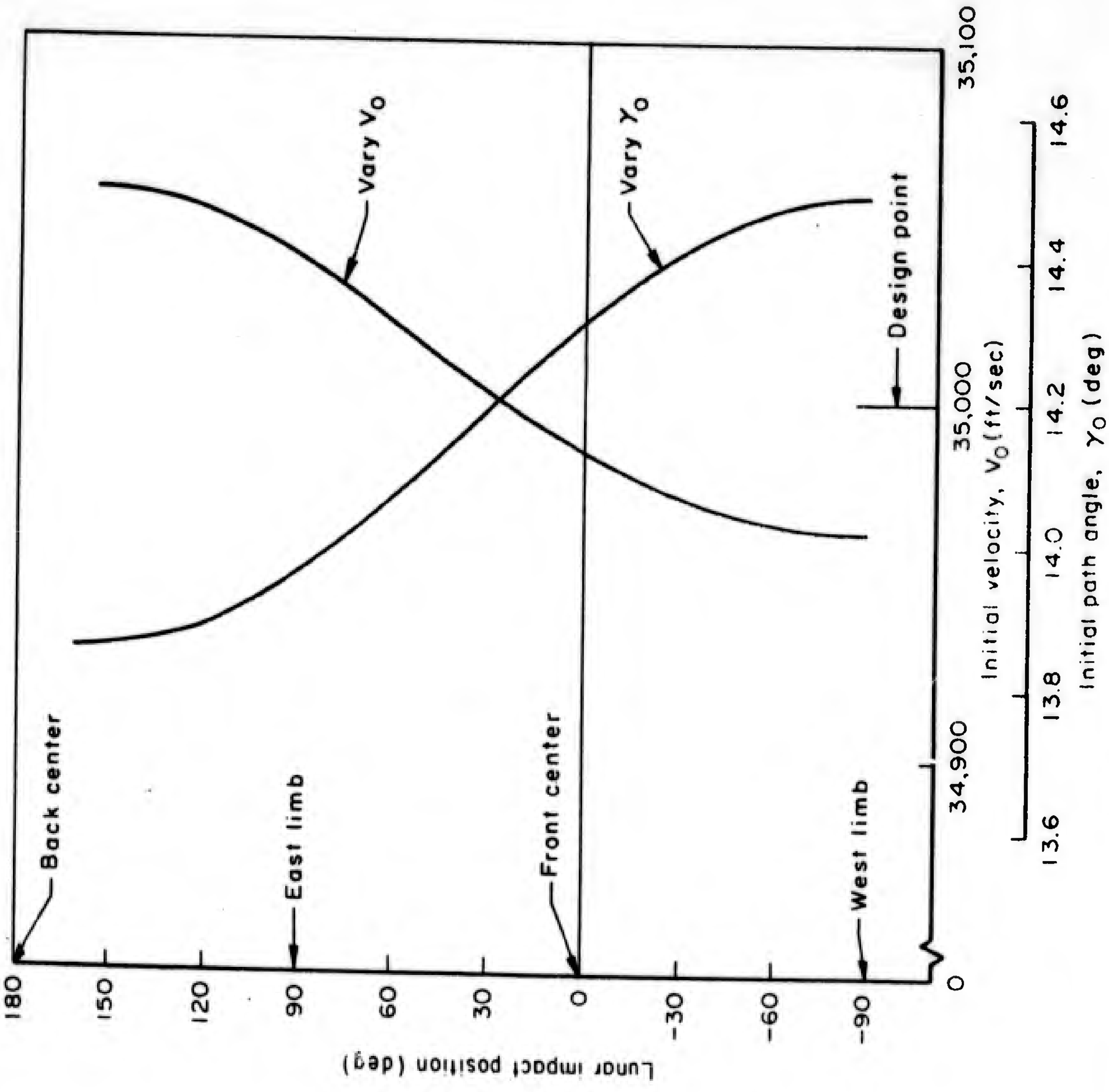

Fig. 23 — Lunar impact position

constant over intervals of ± 10 ft/sec and ± 0.1 deg around the design point.

As the value of either initial parameter departs beyond these intervals, the impact position is moved toward the limb of the moon and the corresponding sensitivity increases very sharply.

Thus, if minimum lunar CEP is desired, the nominal or design trajectory should result in an impact which is normal to the surface of the moon.

On the other hand, if it is desired to maximize the probability of an impact on the visible face of the moon, the nominal trajectory should be shifted so that the impact occurs slightly nearer the center of the visible face (see Fig. 16). The resulting lunar approach trajectory will, however, not be radial. This required biasing of the nominal trajectory becomes more significant as the initial velocity is decreased to near-minimum velocities.

IMPACT VELOCITY AND PATH ANGLE

For a mission involving a soft landing on the moon, the braking rocket's velocity potential must absorb the free-fall lunar-approach velocity plus a small velocity increment which is contributed by the moon's gravitational field during this final powered-flight phase.

The unbraked lunar-approach velocity, which can be computed from the semimajor axis (Fig. 19), is given as a function of the initial trajectory parameters in Fig. 24. The impact velocity for the design case is 9040 ft/sec, which is approximately 1270 ft/sec greater than the escape velocity from an isolated moon. The impact velocity is seen to be mainly a function of the initial velocity at the earth, and varies from about 8900 to 9250 ft/sec between limiting impacts. The variation

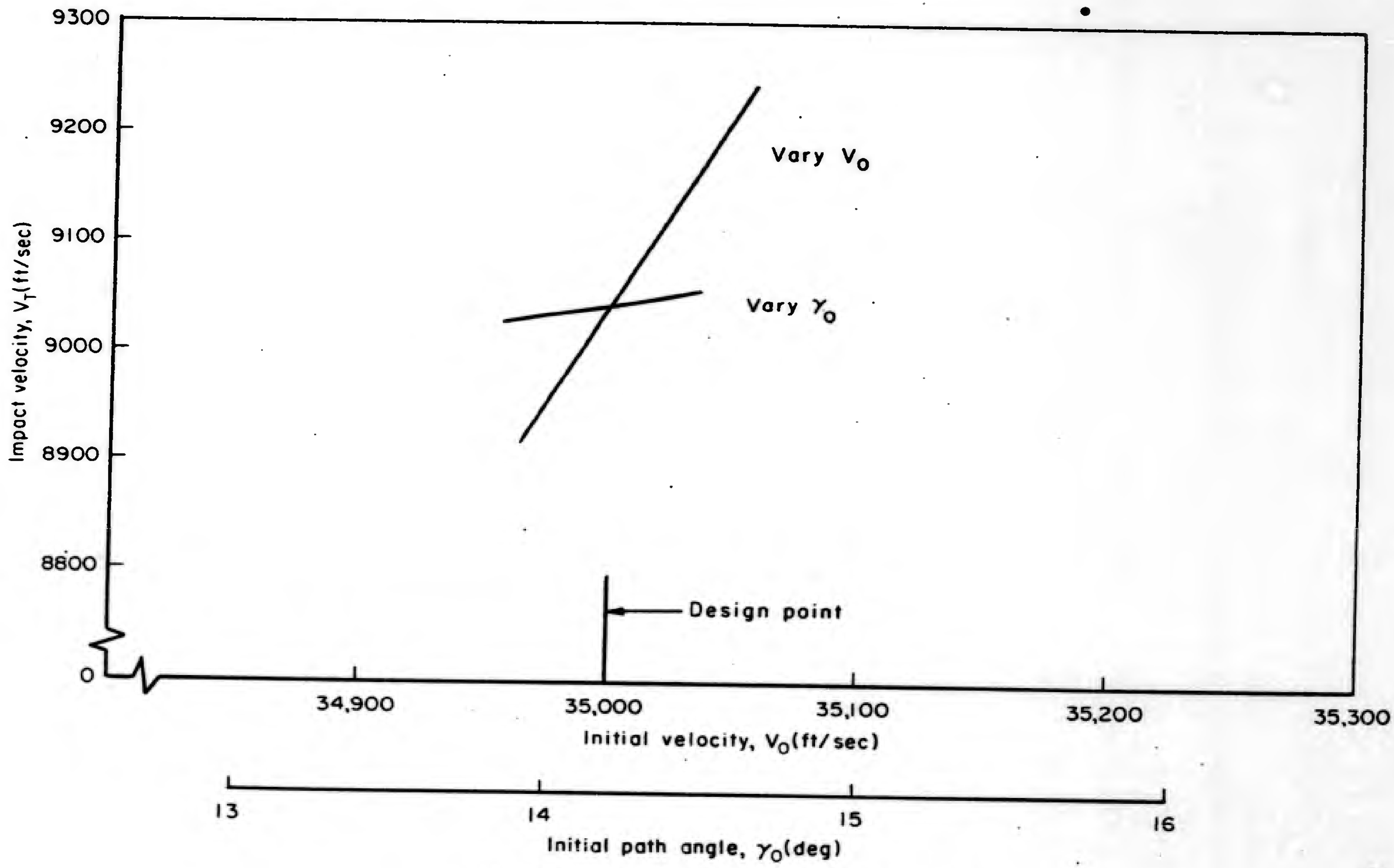

Fig. 24—Lunar impact velocity

of ±15 ft/sec as a function of initial path angle corresponds to the increment resulting from the change to a moon-centered coordinate system.

The corresponding unbraked terminal path angle is given in Fig. 25. The variation is reasonably linear in the vicinity of the design point.

LUNAR-SATELLITE VELOCITY INCREMENT

If the vehicle is assumed to be capable of reducing its velocity in the vicinity of the moon, its path can be changed from the earth-moon transit trajectory to the orbit of a satellite of the moon.[2] The velocity decrements required can be computed from the curves of the semi-major axis (Fig. 19) and perilune radius (Fig. 21). If the retroimpulse occurs at perilune and the resulting satellite orbit is to be circular, the required velocity increments are those shown in Fig. 26 as a function of perilune radius. For a fixed initial transit trajectory velocity of 35,000 ft/sec, the required velocity reduction varies only slightly with lunar orbital altitude, decreasing from a value of 3500 ft/sec near the surface to 3300 ft/sec at an altitude of a few thousand miles. The variation of required velocity increment with initial transit trajectory velocity is greater, but is still between 3000 and 4000 ft/sec.

The increments required to establish lunar satellites in direct motion are seen to be slightly lower than for retrograde satellites.

VEHICLE-AXIS ROTATION REQUIRED

Early vehicles to the moon will probably make use of spin stabilization for attitude control, at least during the period of free flight from the earth to the moon.[11] If the vehicle-roll axis is to be aligned

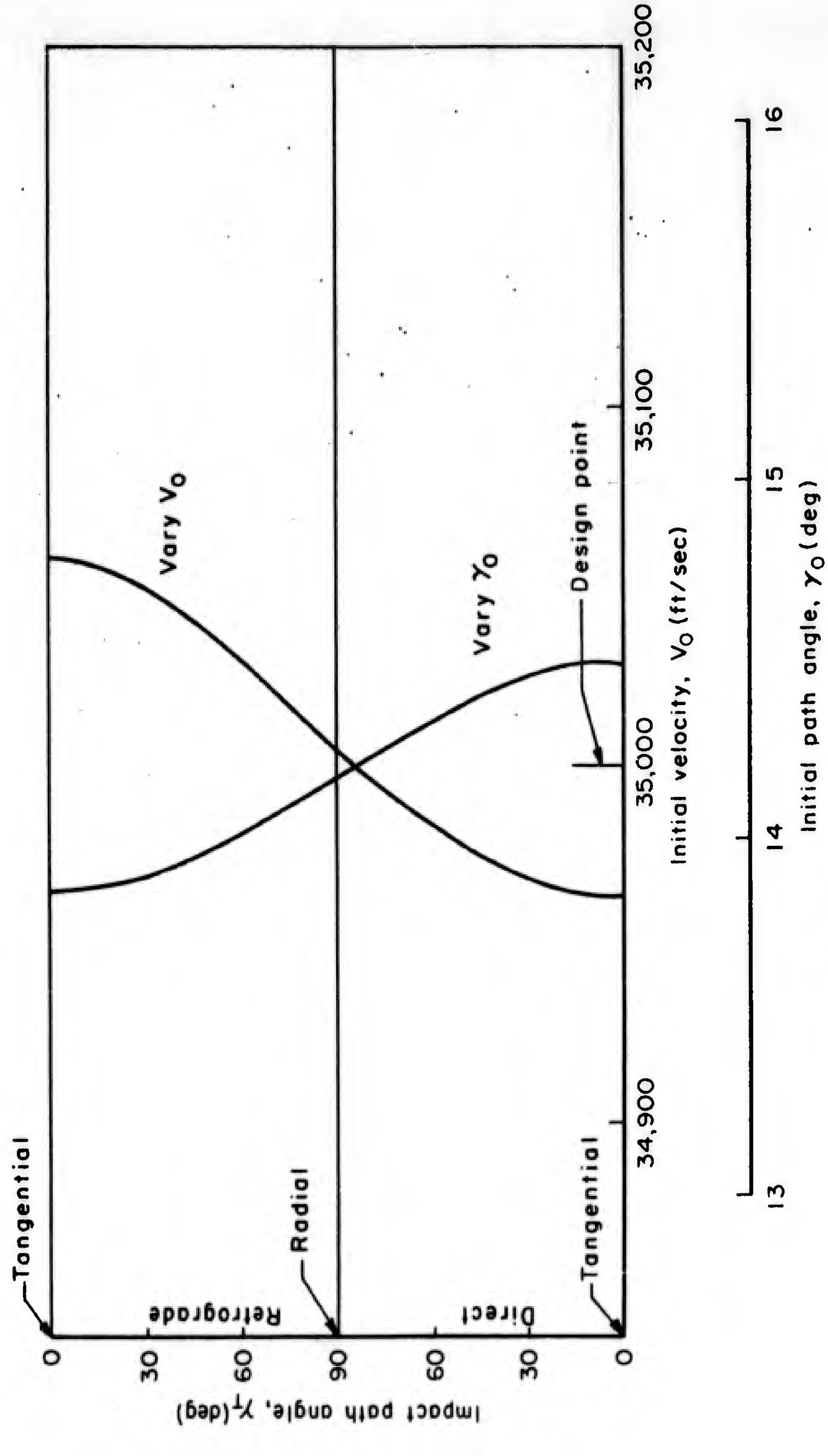

Fig. 25 — Impact path angle

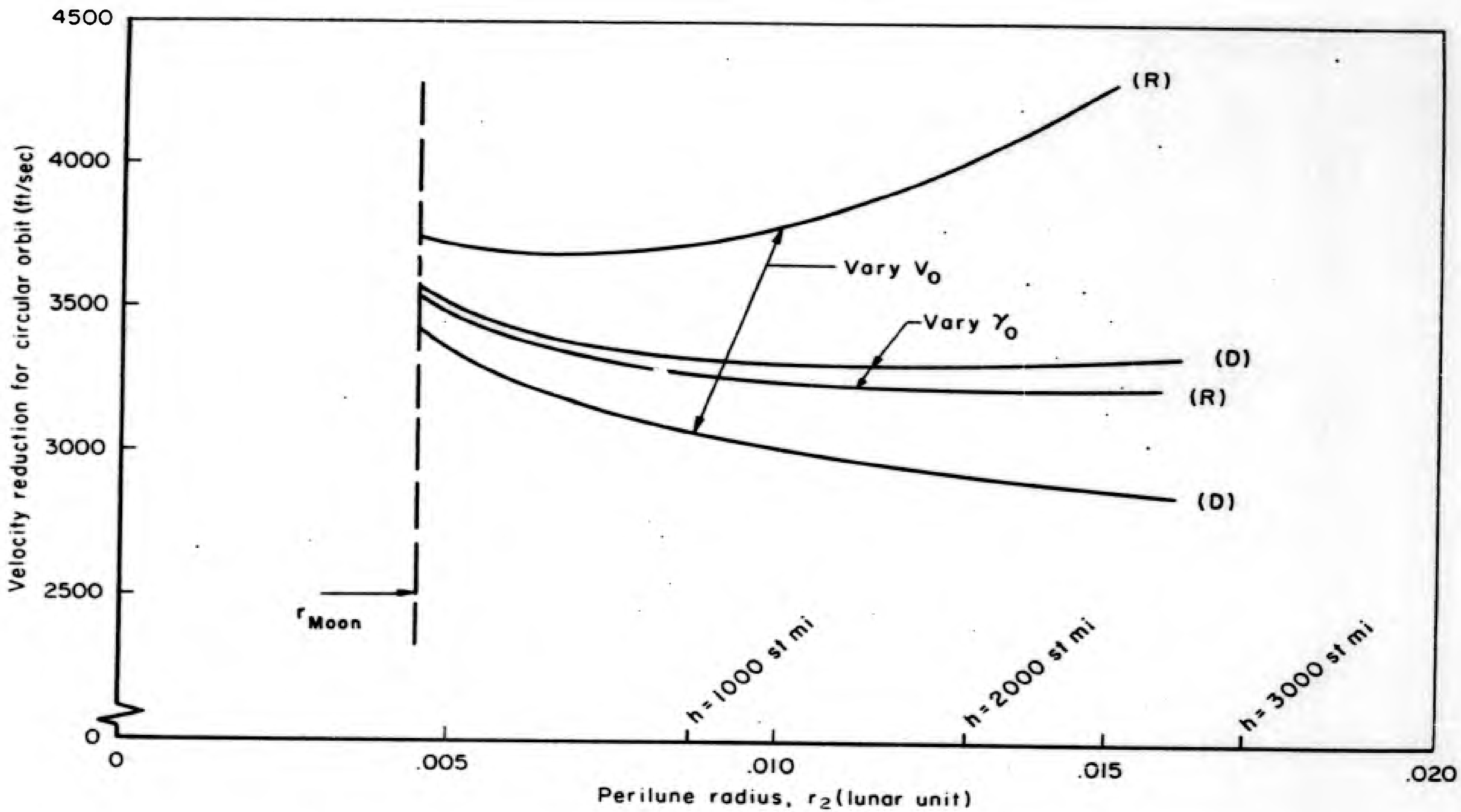

Fig. 26 — Velocity increment required for lunar satellite

with the velocity vector at the moon (for either lunar landing or satellite retrorocket firing), the spatial orientation of the axis will have to be changed at booster burnout. Figure 27 shows the axis-rotation angle required as a function of the initial trajectory parameters at the earth. The design trajectory (impact normal to the moon's surface) is seen to require a rotation of 30 deg counterclockwise, assuming the vehicle-roll axis aligned with the initial velocity vector at $r_o = 4300$ st mi. For off-design impact trajectories this rotation angle changes fairly slowly -- 0.4 deg/ft/sec.

The rotation angle required for lunar satellites shows the influence of the position of perilune relative to the earth-moon line and the transit time. Use of Fig. 21 shows that the rotation angle required for satellites in direct motion approaches 40 deg as the satellite altitude is increased. The comparable rotation angles for retrograde satellite orbits are between 10 and 20 deg.

TRANSIT TIME

The free-flight time from the earth for these transit trajectories is given in Fig. 28 for variations in both initial velocity and path angle. The portions of the curves between the ticks correspond to the time of impact on the lunar surface, while the extensions of the curves show the time of perilune, or closest approach. The transit time for the design impact case is 2.312 days.

The total variation of transit time is about 5.5 hr as the initial velocity is varied between limits for lunar impact. In the vicinity of the design case, the sensitivity of arrival time to an initial velocity

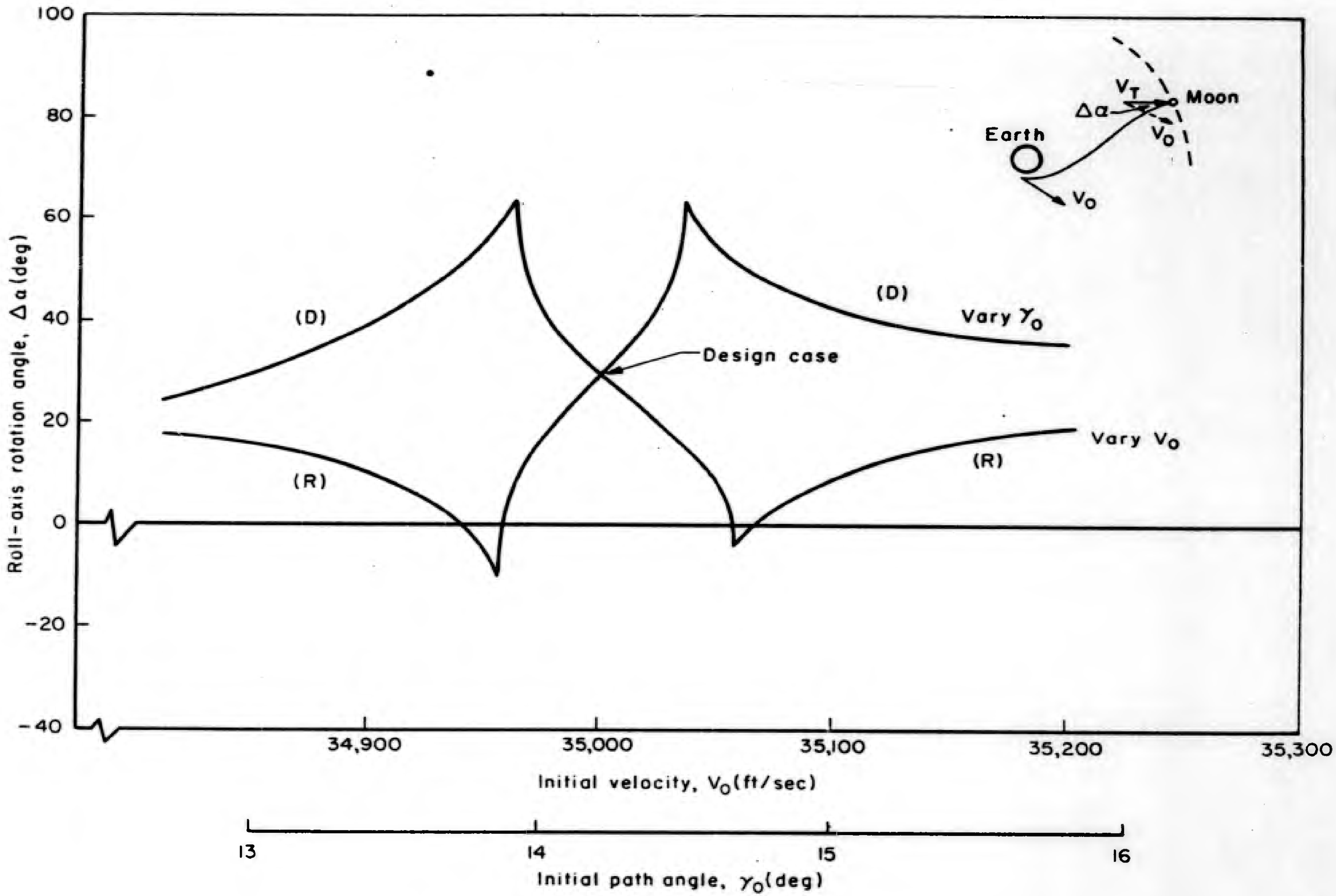

Fig. 27 — Roll-axis rotation angle
(initial to terminal)

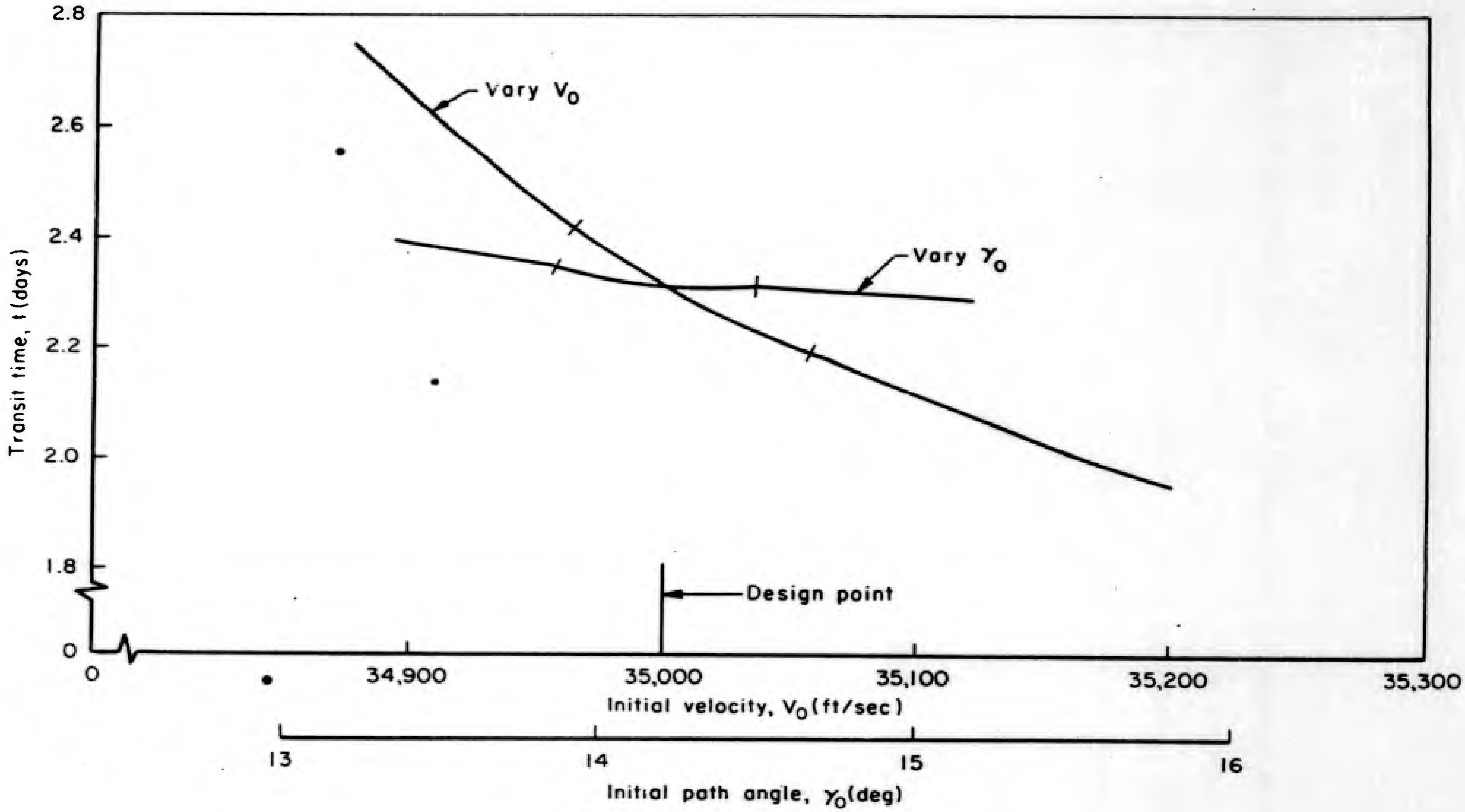

Fig. 28 — Transit time

error is approximately 3.9 min/ft/sec. Thus the use of a preset clock to initiate retrorocket ignition would not be practical, and some form of lunar-altitude-sensing equipment would be preferable.

This variation of 5.5 hr in arrival time would also affect the observation of the lunar approach from the earth. Assuming that the design trajectory is chosen so that the moon is to be over a certain meridian on the earth for observation (in the two-dimensional case), the moon would be at an angle of as much as 40 deg from the zenith for the extreme limits of lunar impact.

V. TRANSIT-TRAJECTORY TIME HISTORY

A body placed in a low-altitude circular satellite orbit around the earth will be weakly perturbed by the moon's gravitational attraction. Similarly, a body initially in a low-altitude circular orbit around the moon will remain, for some time at least, in a lunar satellite orbit which is perturbed by the earth's net gravitational attraction.

As a body departs from the earth on a trajectory toward the moon, the moon's gravitational attraction becomes an increasingly large fraction of the total acceleration which governs the motion of the vehicle. Thus the two-body (earth-vehicle) conic section, defined by the initial velocity and radius at the earth, becomes an increasingly poor approximation to the actual trajectory which the vehicle is following.

For the two-dimensional trajectories discussed herein, the conic-section elements which are affected by the moon's attraction are the semi-major axis, eccentricity, and the orientation of the major axis (line of apsides). Figure 29 presents these instantaneous elements as a function of the radial distance from the earth.

The gravitational attraction of the moon is seen to result in an increase in the instantaneous semimajor axis of the earth-centered ellipse, showing that the total energy of the vehicle, relative to the earth, is increasing. The semimajor axis of the initial ellipse increases until at a radial distance of 0.981 lunar unit the motion of the vehicle becomes parabolic relative to the earth. The transit time to this point is about 2.27 days. The instantaneous eccentricity of the earth-centered ellipse also increases, passing a value of 1.0 at the position and time quoted above.

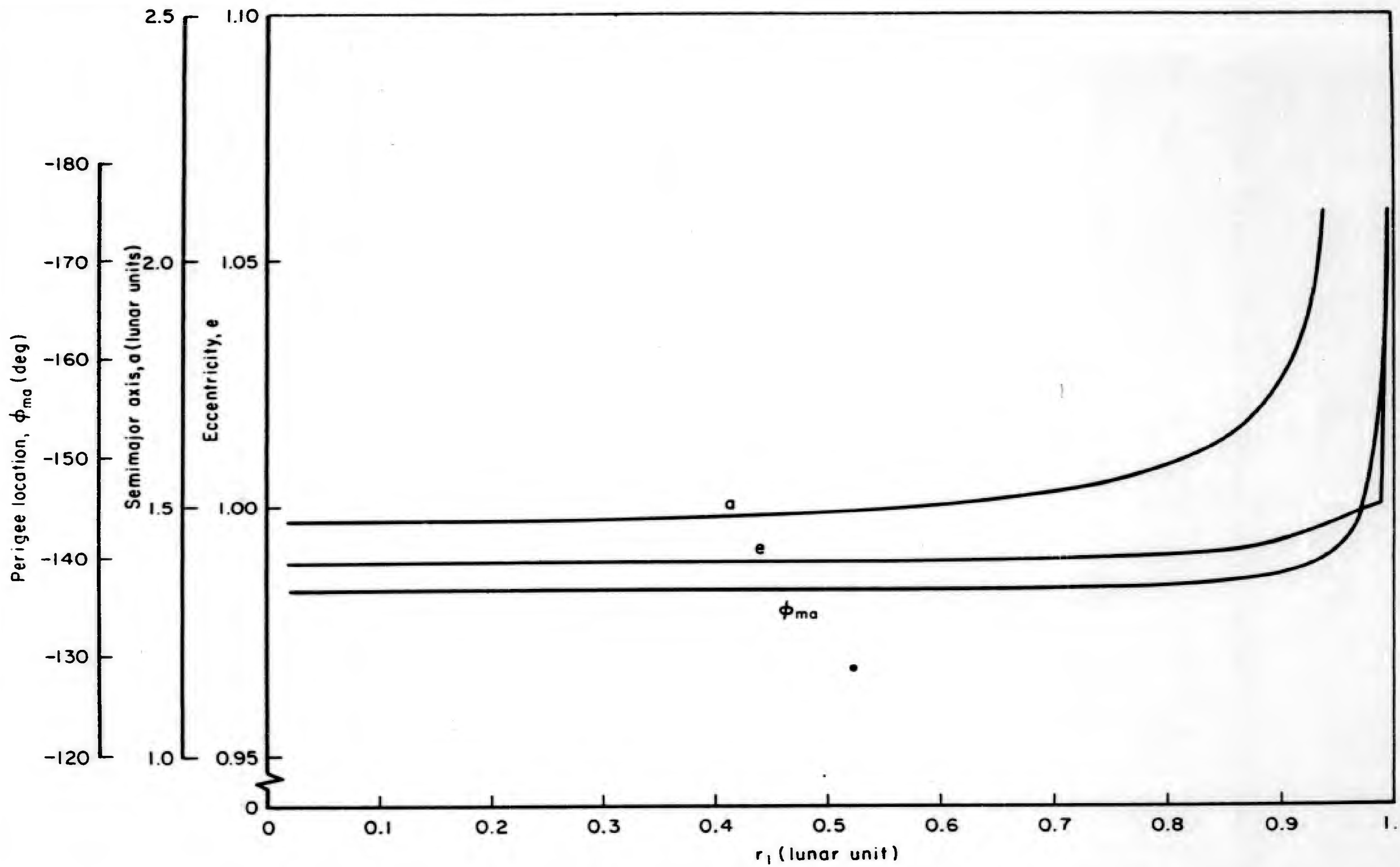

Fig. 29— Instantaneous conic elements relative to the earth (design case)

The moon's attraction also affects the orientation of the major axis of the earth-centered ellipse. This effect is noticeable during the last quarter of the trajectory. The angle ϕ_{MA} defines the angular position of perigee, measured clockwise from the x-axis.

The increase in the instantaneous semimajor axis of the trajectory can be related to an effective-velocity increase at the initial point of the trajectory (r_o = 4300 st mi). This effective-velocity increment, which can be defined by

$$\Delta V_{eff} = V_o - \sqrt{V^2 - \frac{2\mu_E}{r} + \frac{2\mu_E}{r_o}}$$

is shown in Fig. 30 as a function of the x-coordinate between the earth and moon. At x = 0.87 lunar unit, the velocity increment ΔV_{eff} reaches a value of 108 ft/sec which corresponds to the difference between the escape velocity from an isolated earth and the initial velocity of this trajectory.

Also included in Fig. 30 is the comparable effective-velocity increment for the moon-centered hyperbola, measured relative to the terminal velocity at the surface of the moon.

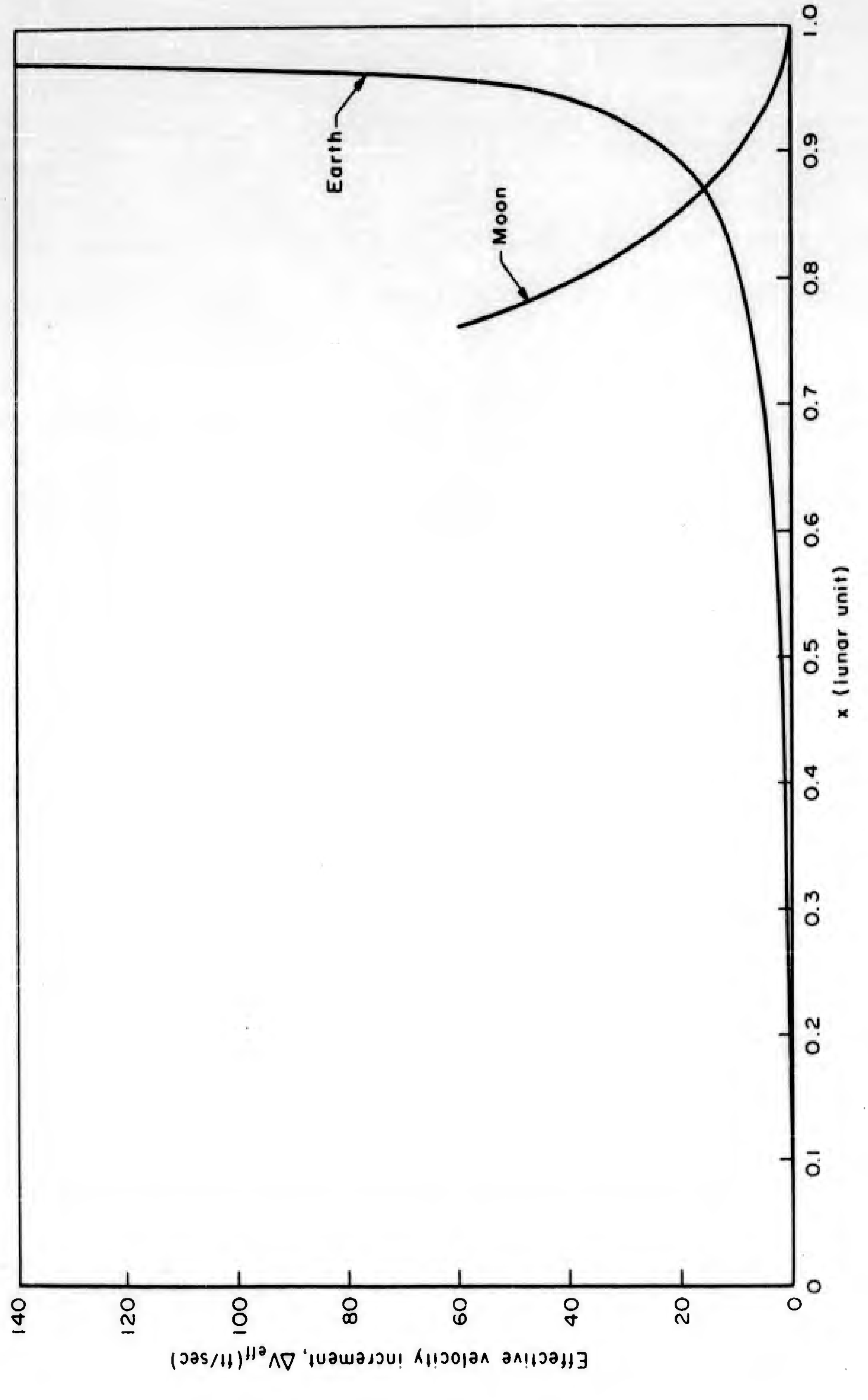

Fig. 30 — Effective — velocity increments

Appendix A

THEORETICAL RELATIONSHIP BETWEEN INITIAL AND FINAL VELOCITIES

Jacobi's integral, which is constant for any free-flight trajectory in the earth-moon system, can be used to determine analytically the lunar-impact velocity as a function of the initial velocity at the earth.

This integral, which can be written as

$$C = \omega^2 \left[\frac{2(1 - \mu)}{r_1} + \frac{2\mu}{r_2} + x^2 + y^2 \right] - V^2 \tag{3}$$

allows computation of the velocity at any point (x, y) in the earth-moon system corresponding to the C-value defined by some other position-velocity combination. A pair of standard positions can be used for the position-dependent terms in the brackets of Eq. (3). Evaluating the equation at a point at an initial radial distance of r_o = 4300 st mi, located on the x-axis in the direction of the moon (x = +0.00587384 lunar unit), we have

$$C_1 = 12.3697196 \times 10^8 - V_1^2 \quad (\text{ft/sec})^2 \tag{4}$$

The other point, at the surface of the moon in the direction of the earth (x = 0.98334991 lunar unit), gives

$$C_2 = 0.9371354 \times 10^8 - V_2^2 \quad (\text{ft/sec})^2 \tag{5}$$

Since C is constant for any given trajectory, $C_1 = C_2$, so that these equations can be combined to give

$$V_2^2 = V_1^2 - 11.432584 \times 10^8 \quad (\text{ft/sec})^2 \tag{6}$$

where V_2 and V_1 are the final and initial velocities relative to the rotating coordinate system. This function is plotted in Fig. 31.

The exact angular positions of the points used in the numerical evaluation of the equation are not significant, since the dominant term in the bracket for C evaluated at the earth is $2(1 - \mu)/r_1$, and for C at the moon the dominant term is $2\mu/r_2$. These terms correspond to the escape velocity from the respective isolated bodies. The lower limit of applicability of this function can be determined by the C-value corresponding to the center of libration lying between the earth and moon which determines the velocity for which motion from the earth to the moon first becomes theoretically possible.(9)

The value of Jacobi's integral for this point is

$$C = 0.3592520 \times 10^8 \quad (\text{ft/sec})^2$$

Substitution of this value in Eqs. (4) and (5) gives a velocity of 34,656 ft/sec at the earth and 7602 ft/sec at the moon.

Two other interesting points for this curve are the local escape velocities at the earth and the moon (assuming isolated bodies). The earth-escape velocity at the initial radial distance of 4,300 st mi is 35,167 ft/sec and the moon-escape velocity at its surface is 7,775 ft/sec relative to an inertial coordinate system fixed at each body.*

---

*As shown in Section II, this inertial value is very nearly equal to the velocity relative to the rotating coordinate system for a radial (γ = 90 deg) direction. For γ = 0, however, the velocities relative to the rotating coordinate systems will differ by ±60 ft/sec at the earth position, and by ±15 ft/sec at the surface of the moon, depending on the direction of motion.

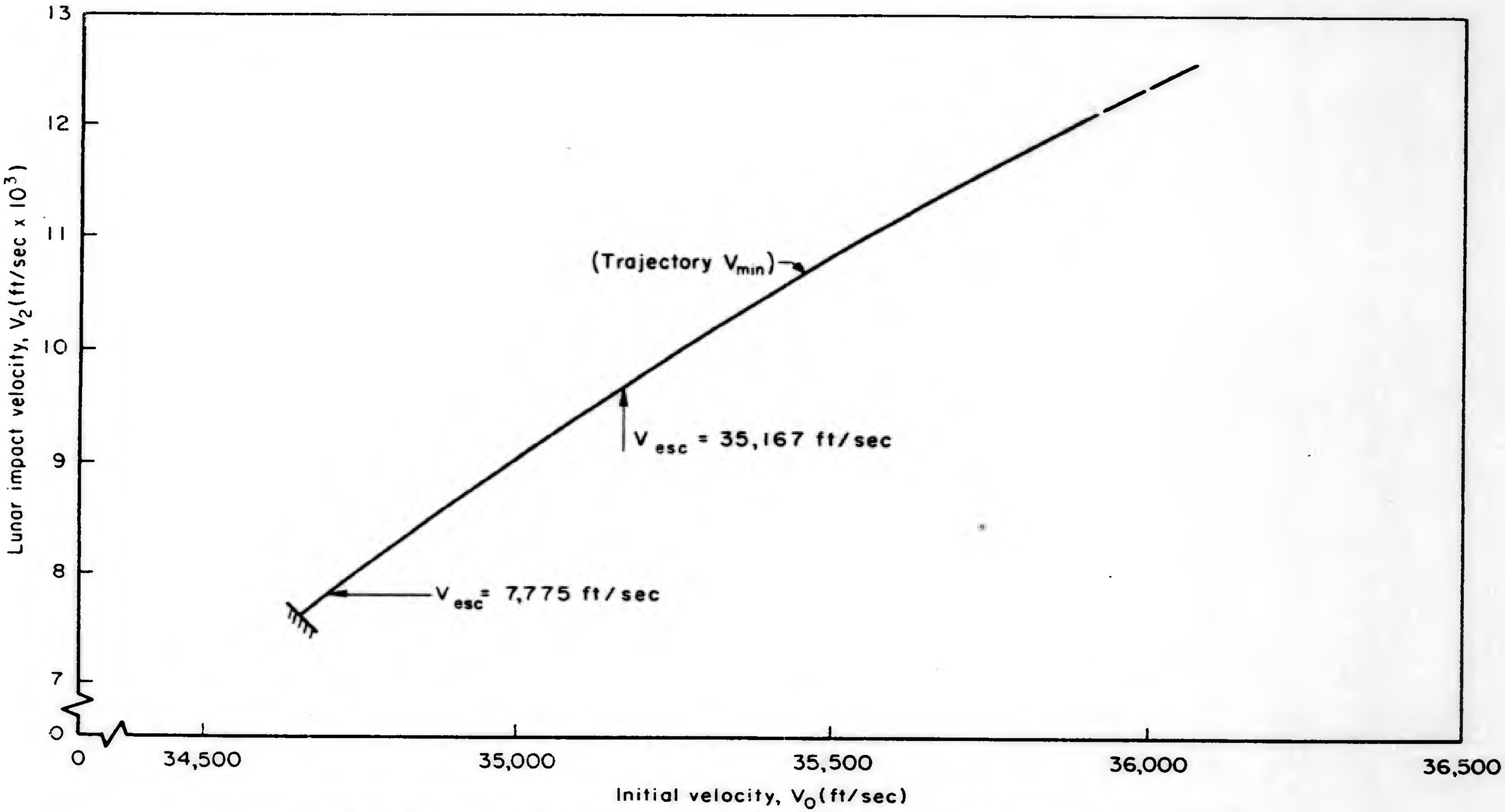

Fig. 31 — Impact velocity as a function of initial velocity

Thus, it is theoretically possible to fire a vehicle from the earth on a trajectory which approaches the moon at less than lunar-escape velocity -- that is, on a trajectory whose energy is elliptical relative to the moon. In this study the lowest-velocity trajectory found which hit the moon on the first pass, however, was defined by an initial velocity of 34,790 ft/sec. The resulting impact velocity on the moon was 8200 ft/sec -- a value 425 ft/sec greater than the escape velocity, and about 600 ft/sec above the theoretical minimum lunar-impact velocity.

Equation (6) can also be used to estimate the sensitivity of the value of lunar-impact velocity as a function of errors in the initial velocity. Differentiation yields

$$\frac{dV_2}{dV_1} = \sqrt{\frac{V_1^2}{V_1^2 - K}} \tag{7}$$

where $K = 11.432584 \times 10^8 \text{ (ft/sec)}^2$. This is plotted in Fig. 32 as a function of the initial velocity. For low initial velocities the sensitivity is seen to be about 4, and to become asymptotic to a value of 1 as the launch velocity approaches infinity.

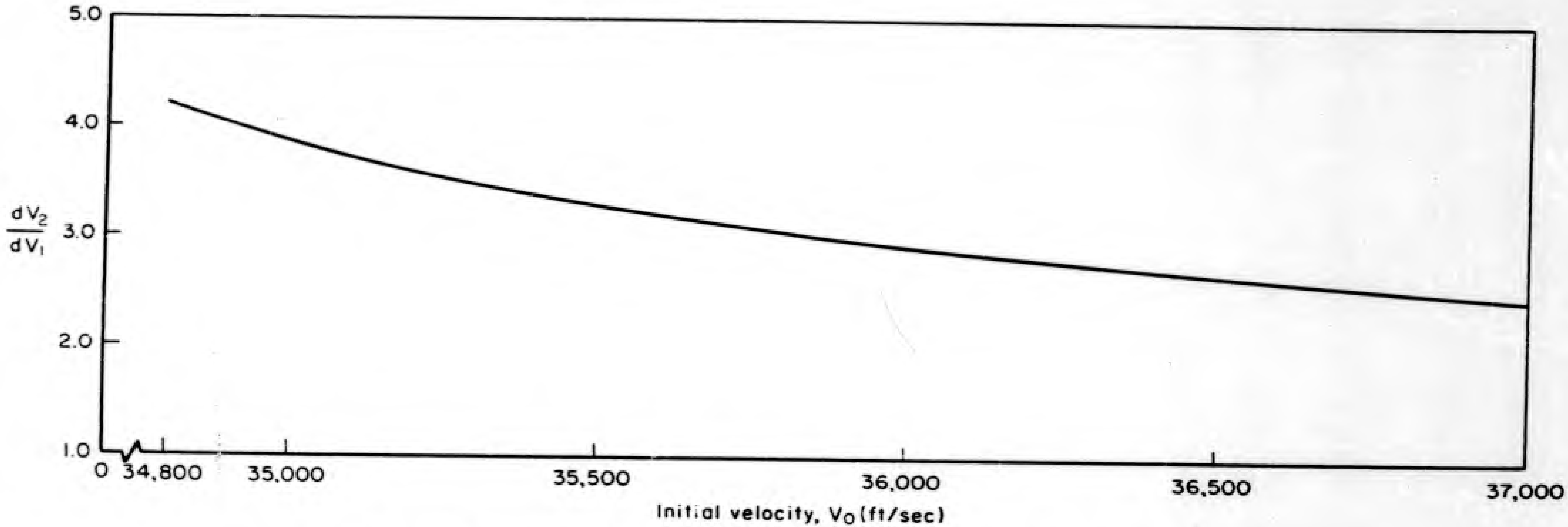

Fig. 32 — Sensitivity of moon impact velocity to initial velocity error

Appendix B

MOTION IN THE z-DIRECTION

As mentioned in Section II, the motion of the vehicle in the general study was restricted to the x, y plane, the plane of motion of the earth and the moon. A first-order calculation, using the design transit trajectory was made to estimate the out-of-plane or z-displacement, resulting from an initial lateral velocity, $\dot{z}_o$.

The equation of motion in the z-direction is

$$\ddot{z} = - \omega^2 \left[\frac{(1 - \mu)}{r_1^3} + \frac{\mu}{r_2^3} \right] z \tag{8}$$

where

$$r_1 = \sqrt{(x - x_1)^2 + y^2 + z^2} \tag{9}$$

and

$$r_2 = \sqrt{(x - x_2) + y^2 + z^2} \tag{10}$$

For small displacements in the z-direction, the radial distances to the earth and moon can be approximated satisfactorily by the two-dimensional values, so that the equation of motion in the z-direction can be written as

$$\ddot{z} = - f(t)\, z \tag{11}$$

where

$$f(t) = \omega^2 \left[\frac{(1 - \mu)}{r_1^3} + \frac{\mu}{r_2^3} \right] \tag{12}$$

can be found from the two-dimensional trajectory time history. The initial values for this check calculation are

$$z(0) = \ddot{z}(0) = 0$$

$$\dot{z}(0) = 10 \text{ ft/sec} = 6.85 \times 10^{-4} \text{ lunar unit/day}$$

After an initial transient in $\ddot{z}$ caused by the starting position being at $z = 0$, the z-acceleration gradually decreases, reaching a minimum at a time of about 1.5 days. At this time, which corresponds to a radial distance of approximately 0.28 lunar unit from the moon, the terms due to the earth and moon in $f(t)$ are equal. The z-acceleration then increases until impact on the moon.

The z-displacement is shown in Fig. 33 as a function of the transit time to the moon. The maximum out-of-plane distance is about 31 st mi, at a transit time of 2.15 days. At impact on the moon, the z-displacement has a value of 20.8 st mi. This corresponds to an out-of-plane angle of about 1.1 deg measured at the center of the moon. The sensitivity to an initial lateral velocity, therefore, can be expressed approximately as

$$\frac{\partial z_m}{\partial \dot{z}_0} = 2.08 \text{ st mi/ft/sec} \qquad (13)$$

for an initial velocity corresponding to the design case.

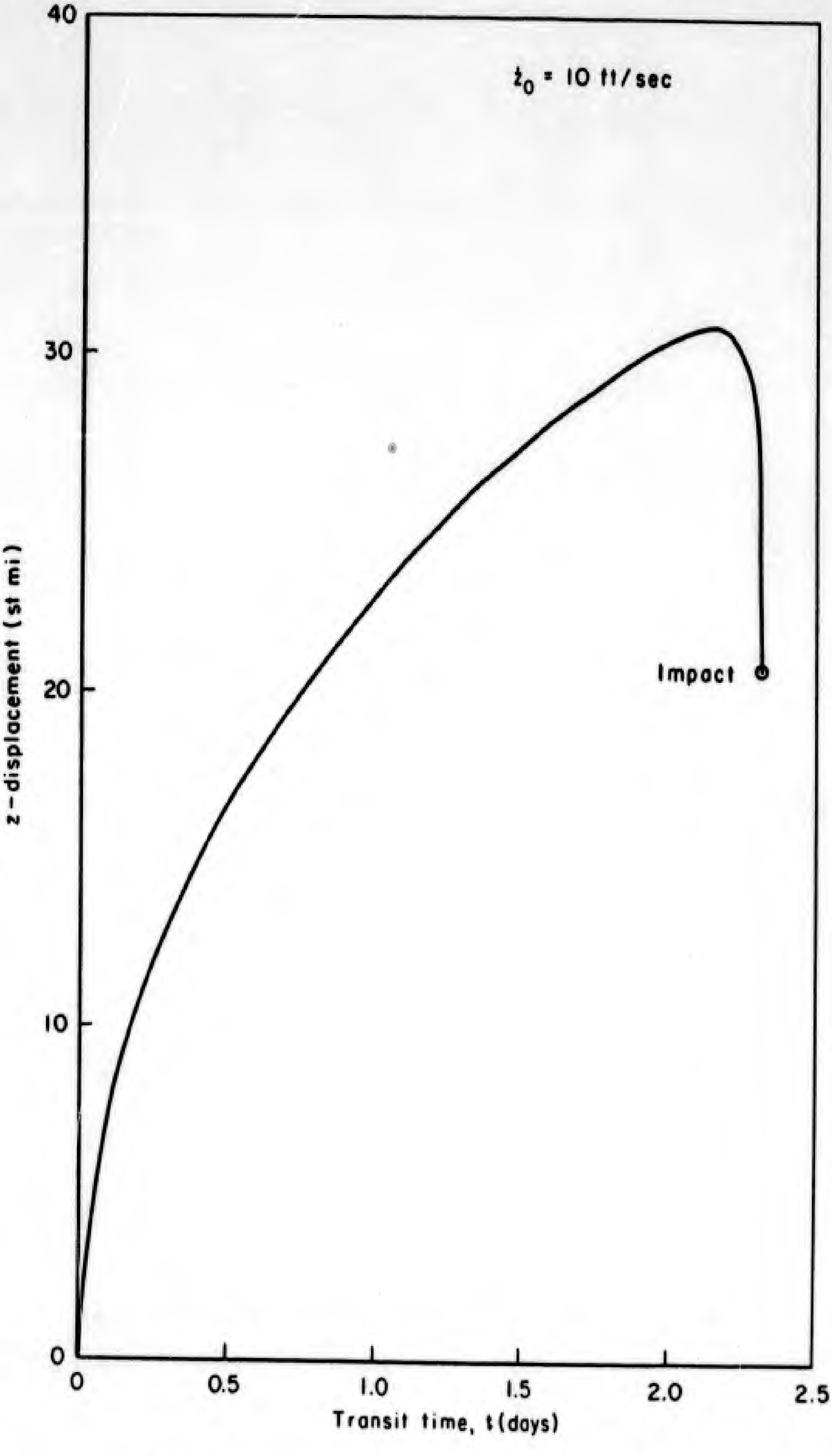

Fig. 33 — Out-of-plane displacement
(design case)

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The RAND Corporation, Santa Monica, Calif.
USAF Project RAND RM-1728

LUNAR INSTRUMENT CARRIER--TRAJECTORY STUDIES by H. A. Lieske; Research Memorandum RM-1728, June 4, 1956, Revised June 25, 1958, UNCLASSIFIED. 93 pp. incl. illus.

This report presents the results of a study of earth-moon "transit" trajectories describing the two-dimensional free-flight motion of a vehicle from a given point above the earth--reached by a matching powered-ascent trajectory--to impact on the surface of the moon. By varying the initial parameters of the flight, the author examines their effect upon the location of the impact point and, for transit trajectories that miss the moon, upon the proper position for placing the vehicle in orbit around the moon. RM-1728 is one of several RAND reports on problems related to landing a package of scientific instruments and associated equipment on the surface of the moon. For a general summary of the results, see RM-1720, General Report on the Lunar Instrument Carrier.

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RAND RESEARCH MEMORANDUM

LUNAR INSTRUMENT CARRIER--TRAJECTORY STUDIES

H. A. Lieske

June 4, 1956
RM-1728 Revised June 25, 1958

in brief

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