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ANALYSIS OF INERTIAL GUIDANCE POSITION ERRORS CAUSED BY ERRORS IN REFERENCE VELOCITY

THESIS

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Preface

The inception of the idea for this report occurred on a familiarization visit to the Martin Aircraft Company, builders of the Titan ICBM. This was an introduction to the critical procedure of aligning the missile guidance system, which is a limiting factor on the missile accuracy. A missile, however sophisticated, is only as accurate as the initial alignment.

The investigation was given further impetus during a discussion of the GAM-77 air-launched missile. Since only the equipment aboard the carrier aircraft is available for alignment, the guidance system is more susceptible to inaccuracies due to initial alignment errors. At this point, I decided to investigate the magnitude and effect of these errors when a Doppler velocity is used to align the guidance system.

This report represents a qualitative determination of the sources of these initial alignment errors. Then the propagation of the errors due to reference velocity error and their effect on final missile position are investigated. The results apply to no specific missile, but give some idea of the relationships between the missile system gain constants, and various reference velocity noise error frequencies.

My sincere thanks to my sponsor, Captain Harold J. Shirley, for his invaluable guidance and advice in spite of considerable personal inconvenience.

Joseph P. Crocco

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A, B	correlation constants
$\bar{A}$	missile acceleration vector with respect to inertial space
a(t)	measured acceleration with respect to inertial space
e	rms value of reference velocity noise error
$e_{\Delta\dot{x}\Delta\dot{x}}$	rms value of system velocity noise error
$e_{\phi_y\phi_y}$	rms value of platform tilt noise error
E	center of the earth and assumed origin of inertial reference system
F(s),G(s),H(s)	system transfer functions
G	universal gravitation constant
g	gravitational acceleration
K_1,K_2	inertial system gains (K_1 = sec^{-1})
m_e	mass of the earth
p	operator notation for differentation
$\bar{R}$	position vector with respect to inertial space
R_e	earth radius
s	La Place variable
$\widehat{sec}$	seconds of arc
t	time of flight from launch (min)
V_x,V_y	missile velocity in missile system coordinates
x, y, z	coordinates of right-handed missile coordinate system, with z upward along local vertical, x along missile longitudinal axis and origin located at CG of missile
X_E, Y_E, Z_E	earth reference coordinates

List of Symbols

X_I, Y_I, Z_I	inertial reference coordinates
α	platform wander angle
β	half-power frequency of reference velocity noise
$\Delta\dot{x}$, $\Delta\dot{y}$	missile velocity errors measured in missile system coordinates
Δx, Δy	missile position errors measured in missile system coordinates
δV	reference velocity error due to Doppler
$\delta(0)$	unit impulse (sec^{-1})
$\bar{\epsilon}$	total gyro drift rate vector
$\overline{\delta\theta}$	vector angular error between computer and true axes
$\Phi(s)$	spectral density function of reference velocity noise
$\Phi_s(s)$	system spectral density function
$\bar{\phi}$	vector angular error between platform and true axes
$\bar{\psi}$	vector angular error between platform and computer axes
ω_s	Schuler frequency
(n)	superscript number in parenthesis indicates bibliography reference number

Abstract

Doppler velocity can be used to perform the in-flight alignment of inertial guidance systems. Reference velocity errors, both noise and bias, cause initial guidance system velocity errors and initial platform tilt errors, which in turn cause position errors. The values of these initial errors must be found to determine their effect on the position errors.

In this report, the position error propagation equation is developed and the position error and CEP of the missile due to reference velocity errors are found. Also, the dependence of the position error on the half-power frequency of the reference velocity noise is investigated.

ANALYSIS OF INERTIAL GUIDANCE POSITION ERRORS CAUSED BY ERRORS IN REFERENCE VELOCITY

I. Introduction

Inertial Guidance

An inertial guidance system depends on dynamical measurements by accelerometers. These measured accelerations are integrated twice to produce positional information. Unfortunately, the accelerometers are unable to distinguish between gravitational and non-gravitational accelerations, and the total acceleration measured is the sum of these two quantities. This can be written analytically as

$$\frac{d^2(\bar{R})}{dt^2} = \bar{a}(t) - \bar{1}_R \frac{Gm_e}{R_e^2} .$$

In this expression, a(t) is the measured acceleration and the second term is the acceleration due to specific gravity force, where G is the universal gravitation constant, m_e is the mass of the earth and R_e is the radius of the earth.

Schuler Tuning. If the accelerometers are mounted on a truly horizontal platform, the gravitational acceleration and the resultant false acceleration measurements are eliminated. A system of this type was devised theoretically by Schuler in the form of a pendulum. The pendulum has a natural period in which the angular acceleration of the pendulum bob due to its inertia reaction torque is equal to the angular acceleration of the true vertical at the horizontally accelerating pivot.

A Schuler-tuned pendulum may not be realized physically, but it is possible to design a closed-loop servo system with these dynamic characteristics. The entire system is a second-order system with a natural frequency of 84.4 minutes and no damping. The equation governing the system is of the form,

$$\left(\frac{d^2}{dt^2} + \omega_s{}^2\right)\bar{R} = \bar{a}(t)$$

where $\omega_s = \sqrt{\frac{g}{R_e}}$, is the Schuler frequency.

Since there is no damping, initial errors in position and velocity, and distrubances which occur in flight will initiate the 84-minute oscillations. Once begun, the oscillations will continue indefinitely and undiminished in amplitude. The resultant maximum errors are prohibitively large, especially those due to initial position and velocity errors. It is desirable to damp these oscillations, and a common solution is the introduction of velocity measurements to the system. However, the damping is accomplished at the expense of a forced dynamic error proportional to the error in velocity measurement. This dynamic error increases the total error in the system caused by accelerometer inaccuracies, gyro and integrator drift, and other system imperfections. This solution is always a compromise between the desired damping of the transient and the forced error introduced.

Effect of Vertical Misalignment. Therefore, especially in an undamped system, precise initial alignment is necessary. In long-range surface-to-surface missiles, great effort is made to align the inertial guidance system before launch with elaborate, permanently emplaced equipment at the launch site. For air to surface missiles, the problem is complicated because the initial alignment must be made with the navigational equipment of the carrier aircraft. The final accuracy of the missile, therefore, will be a function of the error in initial alignment and initial velocity supplied by the bombing-navigation system of the aircraft. The effect of these errors in velocity and alignment on the final position of the missile will be investigated in this report.

Reference Velocity Errors. The missile system used as a model is hypothetical, but is based on an air-to-surface missile of the GAM-77 type. It is a high-altitude, short-range, air-breathing missile. The guidance system used is an undamped inertial guidance system during the free-flight phase. Before the launch, however, the Doppler radar of the aircraft navigational system provides damping for the guidance system.

During the initial alignment period, position fixes and Doppler velocity provide initial conditions for the missile guidance. After the missile guidance system is in operation, the Doppler velocity and missile velocity are compared, and any error is interpreted as a missile platform misalignment. The platform is then corrected to eliminate the error. Therefore, at launch, any error in the Doppler reference velocity will be reflected in the missile platform alignment and in the missile velocity. The initial errors in platform alignment and velocity will, therefore, be a function of the reference velocity error.

Since the reference velocity consists of both bias and noise errors, both will have an effect on the missile velocity and platform tilt. In the case of bias error sources, the errors in tilt and velocity are correlated. The bias error is constant and may be treated as a step input. The final value theorem is then used to determine the steady-state value of the error in missile velocity and platform tilt after the transient has ceased.

The noise error, however, produces errors in tilt and velocity that are both correlated and uncorrelated. Therefore, the power density spectrum of the noise is used to determine the correlated and uncorrelated errors. The correlated errors are combined, but the uncorrelated errors must be treated independently in order to find the total rms value of the error.

Scope of the Report

In this report, the position error propagation equation as a function of the initial guidance system velocity error and the initial platform tilt error is derived. The initial velocity error and the initial platform tilt error are both functions of the reference velocity error. Since the reference velocity error is composed of both noise and bias, statistical methods are used to determine the effect of the noise component. Finally, the effect of various noise half-power frequencies on the missile position is investigated.

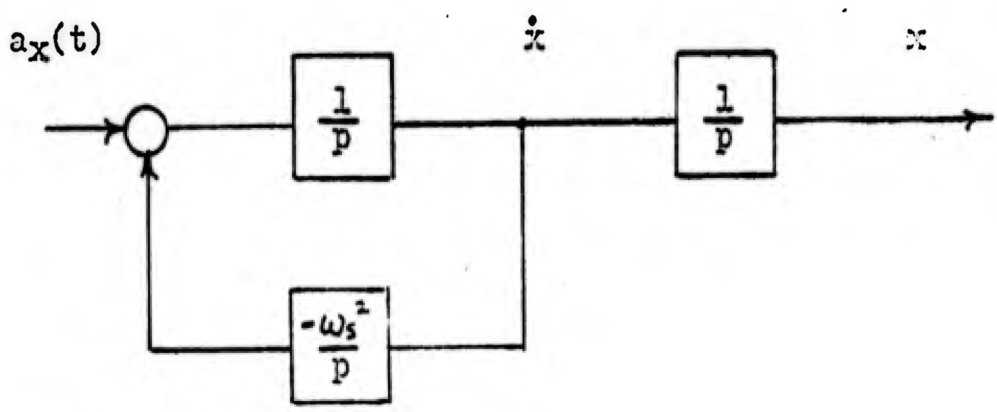
Error Propagation. The position error propagation equation for an undamped inertial guidance system is derived to determine the effect of initial system velocity error and initial platform tilt on the final missile position, since, in this case, the missile guidance system is undamped after being launched from the carrier aircraft. This equation is derived primarily for one channel, the X channel, since the errors propagate similarly in both dimensions.

From a block diagram for a velocity damped inertial system, since the missile guidance system is velocity damped before launch, the system transfer functions are derived. These transfer functions relate missile system velocity error to reference velocity error, and relate platform tilt to reference velocity error. After the system equations have been derived, the nature of the two errors, system velocity and platform tilt, is determined using spectral density analysis.

Spectral Density Analysis. Spectral density must be used because of the noise component of the reference velocity error. The transfer functions relating platform tilt and missile velocity error to reference velocity error are used with the power density spectrum of the noise to determine the correlation and cross-correlation functions of missile velocity and tilt error. Then, the effect of varying the half-power frequency of the reference velocity noise on the correlated and uncorrelated errors is investigated for a representative range of frequencies.

Finally, the position error caused by these initial errors in velocity and tilt is found using the previously described error propagation equation. Again, the effect of a variable half-power frequency is investigated. For an assumed time of flight, the rms value of position error is then determined, and with the rms value of y position error, the Circular Error Probability is calculated.

Coordinate Systems. There is a description of the reference coordinate systems in Appendix J.



$$(p^2 + \omega_s^2)x = a_x(t)$$

p = operator notation for differentiation

ω_s = Schuler frequency

Figure 1 Block Diagram for X Channel of Undamped Inertial Guidance System

II. System Equations

Although the missile guidance system is Doppler velocity damped before launch, it operates as a completely undamped guidance system after launch. Therefore, the initial system velocity and platform tilt errors will be propagated according to the equation governing undamped guidance systems. The propagation equation will be derived using one form of a typical undamped inertial guidance system.

Error Propagation Equations

In a simple undamped inertial guidance system, a straight-forward, double integration of acceleration gives position. In the x dimension, this may be represented as shown in Figure (1). The block diagram is a mechanization of the second order equation

$$(p^2 + \omega_s^2)x = a_x(t), \qquad (1)$$

for the x coordinate of the position vector R. In this equation, p is the operator notation for differentiation, and ω_s is the Schuler frequency. This equation assumes that the centripetal acceleration and Coriolis acceleration are negligible or have been compensated for. Since the accelerometer is measuring both gravitational and non-gravitational accelerations, there is an error in the position, Δx, caused by any platform misaligmment. The error propagation equation is shown to be

$$(p^2 + \omega_s^2)\Delta x = 0 \qquad (2)$$

in Appendix A.

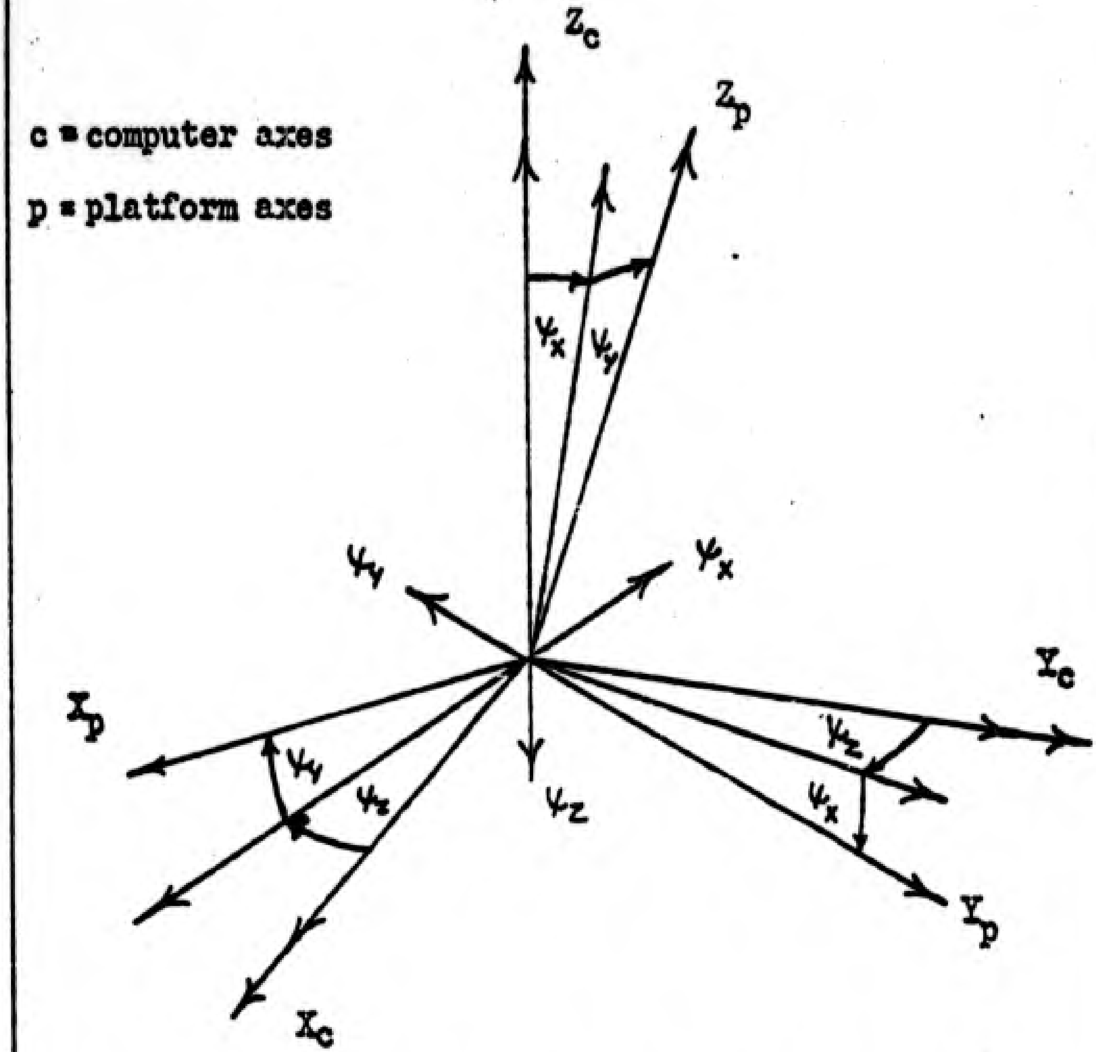


Figure 2 Components of the Vector Angular Error Between Computer and Platform Axes, $\bar{\psi}$

Initial Condition Errors. When the La Place transformation of equation (2) is made, the transient effects caused by initial errors in position and velocity are introduced. The La Place transform of equation (2) is

$$(s^2 + \omega s^2)\Delta x(s) = s\Delta x(0) + \Delta\dot{x}(0). \tag{3}$$

The solution of this equation in the time domain is

$$x(t) = \Delta x_o \cos \omega_s t + \frac{\Delta\dot{x}_o \sin\omega_s t}{\omega s}, \tag{4}$$

showing that the transient effects are functions of the sine and cosine of the Schuler frequency. These oscillations are excited by initial errors in position and velocity.

Error Driving Functions. The above solution assumes no external error driving functions. However, when the missile is accelerated at launch, an error driving function is introduced by any platform misalignment. If the misalignment is small, as it must be in any acceptable system, the error driving function is expressed as

$$[\bar{A} \times \bar{\psi}]$$, as shown in Appendix B.

In the above expression, $\bar{\psi}$ = vector angular error between platform and computer axes. There is a brief discussion of the vector representation of angles in Appendix C, and the angle $\bar{\psi}$ is illustrated in Figure (2). The acceleration vector of the missile is

$$\bar{A} = \bar{1}_x (V_x - V_{xo})\,\delta(0) + \bar{1}_y (V_y - V_{yo})\,\delta(0) + \bar{1}_z g. \tag{5}$$

Since the acceleration time is relatively short compared with the total time of flight, the acceleration is expressed as the difference in missile velocity before and after acceleration, multiplied by the unit impulse $\delta(0)$. The x component of the driving function,

$$[\bar{A} X \bar{\Psi}]_x = A_y \Psi_z - A_z \Psi_y \, , \tag{6}$$

is then included on the right-hand side of equation (2).

The effect of the gyro drift rate vector, $\bar{\epsilon}$, as a function of time is also a driving function of equation (2), and the x component,

$$\epsilon_y t \quad ,$$

of gyro drift is added to the right-hand side. Equation (2) now takes the form

$$(p^2 + \omega_s^2)\, \Delta x = [\bar{A} X \bar{\Psi}]_x + g \epsilon_y t = A_y \Psi_z - A_z \Psi_y + g \epsilon_y t \tag{7}$$

Taking the LaPlace transform of equation (7) results in

$$(s^2 + \omega_s^2)\, \Delta x(s) = (V_y - V_{yo}) \Psi_z - \frac{g \Psi_y}{s} + \frac{g \epsilon_y}{s^2} + s \Delta x(0) + \Delta \dot{x}(0) \tag{8}$$

Substitutions for the components of acceleration, A_y and A_z, were made from equation (5).

The vector angle $\bar{\Psi}$, the angular error between the platform and computer axes, can be defined further in terms of the true axes reference. It is made up of two components, the computer axes error and the platform axes error. Therefore the angle $\bar{\Psi}$ is function of the vector angles

Only X components of angles are used for clarity.

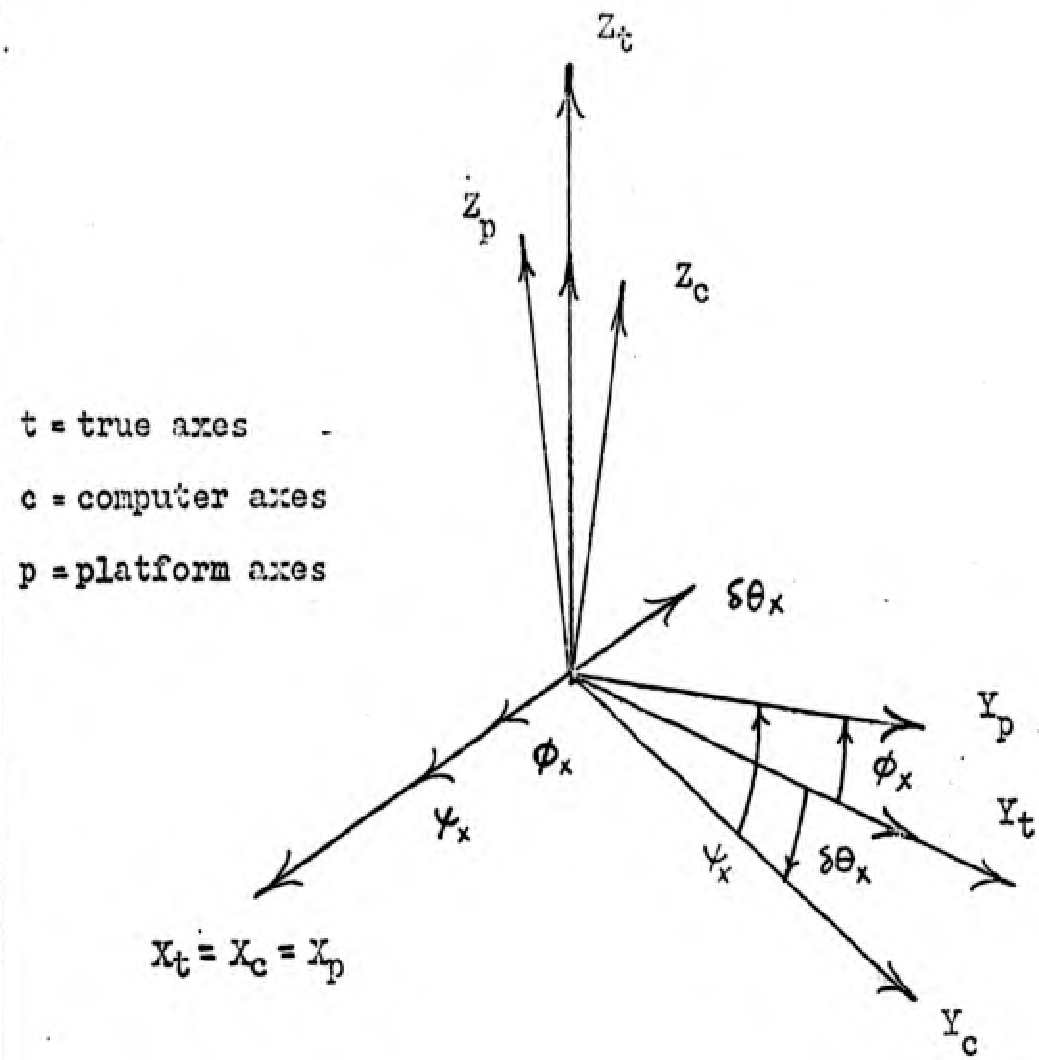


Figure 3 Relationship Between Vector Angles ϕ, $\delta\theta$, and ψ.

$\bar{\phi}$ and $\overline{\delta\theta}$ as shown in Figure (3) or

$$\bar{\psi} = \bar{\phi} - \overline{\delta\theta} \tag{9}$$

where

$\bar{\phi}$ = the vector angular error between platform and true axes and

$\overline{\delta\theta}$ = the vector angular error between computer and true axes. The computer error angle, $\overline{\delta\theta}$, is a function of the position error as shown in Appendix D. Therefore, the error between computer and true axes is expressed vectorially in component form as

$$\overline{\delta\theta} = -\bar{1}_x \frac{\Delta y}{R_e} + \bar{1}_y \frac{\Delta x}{R_e} \tag{10}$$

where Δx and Δy are the components of the position error and R_e is the earth radius. The components of the initial vector angle at launch, $\bar{\psi}(0)$, are then written

$$\begin{aligned} \psi_x(0) &= \phi_x(0) - \delta\theta_x(0) = \phi_x(0) + \frac{\Delta Y(0)}{R_e} \\ \psi_y(0) &= \phi_y(0) - \delta\theta_y(0) = \phi_y(0) - \frac{\Delta X(0)}{R_e} \\ \psi_z(0) &= \phi_z(0) + \alpha(0) \approx \phi_z(0) \end{aligned} \tag{11}$$

In the expression for the z component, α is the angle of platform wander, and is small enough to be neglected.[(5)] Also, since the Schuler frequency

$$\omega_s^2 = \frac{g}{R_e},$$

where g is the gravitational acceleration, and R_e is the earth radius, then

$$R_e = \frac{g}{\omega_s^2}. \tag{12}$$

Substituting the expression in equation (12), and the expressions in equations (11) for the appropriate components of $\bar{\Psi}$, into equation (8), produces the final form of the error propagation equation. When solved in the time domain, this equation becomes

$$x(t) = (V_y - V_{yo})\,\phi_{z_o}\frac{\sin\omega_s t}{\omega_s} - R_e\left(\phi_{yo} - \frac{\Delta x_o}{R_e}\right)(1-\cos\omega_s t)$$

$$+\frac{R_e\epsilon_y}{\omega_s}(\omega_s t - \sin\omega_s t) + \Delta x_0\cos\omega_s t + \Delta\dot{x}_0\frac{\sin\ {}_s t}{\omega_s}\,. \qquad (13)$$

In equation (13), the position error is a function of the azimuth misalignment, platform tilt, initial velocity error, gyro drift and initial position error.

The position error propagation due only to initial errors in missile velocity and platform tilt, $\Delta\dot{x}$ and ϕ_y, is written

$$x(t) = \Delta\dot{x}_0\frac{\sin\omega_s t}{\omega_s} - R_e\phi_{y_0}(1 - \cos\omega_s t) \qquad (14)$$

This is the position error propagation equation for an undamped inertial guidance system, since the system being investigated operates as an undamped system after launch. This error propagation equation due to initial errors in system velocity and platform tilt will be used throughout the remainder of this report.

The transfer functions which relate the initial missile system velocity error and platform tilt to the reference velocity, which is an input to the system before launch, must now be found.

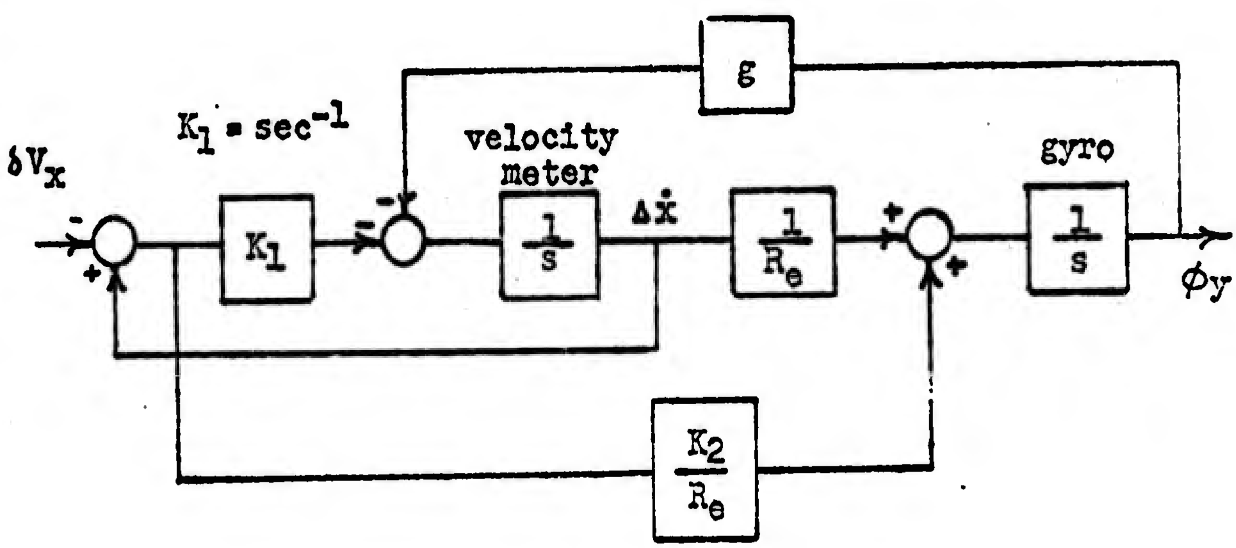


Figure 4 Error Block Diagram for X Channel[3] of Model Inertial Guidance System

System Transfer Functions

The transfer functions relating missile system velocity error to reference velocity error, and relating platform tilt error to reference velocity error must be derived in order to find the individual errors resulting from a particular reference velocity error. Using the block diagram in Figure (4) for a model velocity-damped inertial guidance system, the required transfer functions are found. This block diagram describes the system before the missile is launched. The transfer functions are:

$$\frac{\Delta \dot{x}(s)}{\delta V_x(s)} = \frac{K_1 s + K_2 \omega_s^2}{s^2 + K_1 s + (1 + K_2)\omega_s^2} = G(s) \qquad (15)$$

and
$$\frac{\phi y(s)}{\delta V_x(s)} = \frac{K_1 - K_2 s}{R_e\left[s^2 + K_1 s + (1 + K_2)\,\omega_s^2 \right]} = H(s) \qquad (16)$$

or
$$\Delta \dot{x}(s) = G(s)\ \delta V_x(s) \qquad (17)$$

$$\phi y(s) = H(s)\ \delta V_x(s) \qquad (18)$$

These equations show that both the system velocity error and platform tilt are functions of the reference velocity error. Since the reference velocity has a noise error component, the velocity and tilt errors will be composed of both correlated and uncorrelated errors. The correlation between the two errors, velocity and tilt, must be found using the power spectral density of the noise and subsequent error propagation must account for this correlation.

III. Spectral Density Analysis

Now that the position error propagation equation and the transfer functions that describe the effect of the reference velocity error on the initial system velocity and platform tilt errors, have been derived, the reference velocity must be analyzed. The reference velocity error is composed of a noise and bias component. The noise error, being a random function must be analyzed using a statistical approach.

Spectral Density of the System Noise

To analyze the errors in the system velocity, $\Delta\dot{x}$, and platform tilt, ϕ_y, caused by random noise in the reference velocity, δV_x, a statistical analysis is used. The noise is approximated by the exponential decay function,

$$R(t) = e^{-(2\beta t)} \quad (4) \tag{19}$$

and the spectral density of the noise is expressed as

$$\phi(\omega) = \frac{2e^2\beta}{(\omega^2 + \beta^2)} \tag{20}$$

where

$\phi(\omega)$ = spectral density of reference velocity noise

e = rms value of reference velocity noise (power/rad/sec)

β = half power frequency of reference noise (rad/sec)

The quantity $\frac{1}{\beta}$ is called the correlation time constant. Then the spectral density function of the resultant system velocity noise is written

$$\phi_s(\omega) = \phi(\omega)F(j\omega)F(-j\omega) \tag{21}$$

or in terms of the La Place variable, s,

$$\Phi_s(s) = \Phi(s)F(s)F(-s) \tag{22}$$

The expression F(s) is any transfer function between the reference velocity error and some system error, or

$$F(s) = \frac{\Delta V(s)}{\delta V(s)}$$

The value of the mean squared system error, e_s^2, is found by performing the indicated integration

$$e_s^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi s(\omega)d\omega \; . \tag{23}$$

Using the transfer functions derived in Section (II), the correlated and cross-correlated functions due to initial values of the velocity and platform tilt errors are,

(5)

$$e^2_{\Delta\dot{x}\Delta\dot{x}}(0) = \frac{1}{2j\pi}\int_{-j\infty}^{j\infty} \Phi(s)G(s)G(-s)$$

$$e^2_{\phi_y\phi_y}(0) = \frac{1}{2j\pi}\int_{-j\infty}^{j\infty} \Phi(s)H(s)H(-s) \tag{24}$$

$$e^2_{\Delta\dot{x}\phi_y}(0) = \frac{1}{2j\pi}\int_{-j\infty}^{j\infty} \Phi(s)G(s)H(-s).$$

In equations (24), $e^2_{\Delta\dot{x}\Delta\dot{x}}$ (0) is the mean squared value of the velocity error, $e^2_{\phi_y\phi_y}$ (0) is the mean squared value of the platform tilt error, and $e^2_{\Delta\dot{x}\phi_y}$ (0) is the mean squared value of the cross-correlation between the two errors. These integrals are evaluated using the table of integrals shown in Appendix F.

Error Correlation

A measure of the correspondence between the correlated errors due to noise, is the relationship of $e^2_{\Delta\dot{x}\phi_Y}(0)$ to $e^2_{\Delta\dot{x}\Delta\dot{x}}(0)$, and the relationship of $e^2_{\Delta\dot{x}\phi_Y}(0)$ to $e^2_{\phi_Y\phi_Y}(0)$. Complete correlation exists if

(5)

$$e^2_{\Delta\dot{x}\phi_Y} = \sqrt{e^2_{\Delta\dot{x}\Delta\dot{x}}(0)\, e^2_{\phi_Y\phi_Y}(0)} \tag{25}$$

When the individual errors in $\Delta\dot{x}$ and ϕ_y are multiplied by

$$A = \left[\frac{e^2_{\Delta\dot{x}\phi_Y}(0)}{\sqrt{e^2_{\Delta\dot{x}\Delta\dot{x}}(0)\, e^2_{\phi_Y\phi_Y}(0)}}\right]^{\frac{1}{2}} \tag{26}$$

the products are correlated. When the errors are multiplied by

$$B = \left[1 - \frac{e^2_{\Delta\dot{x}\phi_Y}(0)}{\sqrt{e^2_{\Delta\dot{x}\Delta\dot{x}}(0)\, e^2_{\phi_Y\phi_Y}(0)}}\right]^{\frac{1}{2}} \tag{27}$$

the products are uncorrelated and treated as independent random errors. The correlated and uncorrelated errors are expressed as

$$\begin{aligned} \Delta\dot{x}(cor) &= Ae_{\Delta\dot{x}\Delta\dot{x}}(0) \\ \Delta\dot{x}(unc) &= Be_{\Delta\dot{x}\Delta\dot{x}}(0) \\ \phi_Y(cor) &= Ae_{\phi_Y\phi_Y}(0) \\ \phi_Y(unc) &= Be_{\phi_Y\phi_Y}(0) \end{aligned} \tag{28}$$

Although there is a possible reduction of the final error by proper correlation of $\Delta\dot{x}$ and ϕ_y, it is also possible for the opposite to occur if the errors are added algebraically due to the sign of $e^2_{\Delta\dot{x}\phi_Y}(0)$ as explained below.

Position Errors

The two correlated and two uncorrelated errors found thus far, when substituted in the error propagation equation derived in Section (II), result in the position error contribution of each source. The error propagation equation is repeated below for reference.

$$\Delta x = \frac{\Delta \dot{x}_o \sin \omega_s t}{\omega_s} - R_e \phi_{yo}(1 - \cos \omega_s t) \tag{29}$$

The correlated position errors due to system velocity and platform tilt are then combined algebraically. The sign of $e^2_{\Delta \dot{x} \phi_Y}(o)$ gives the relation of the signs of the two errors. If $e^2_{\Delta \dot{x} \phi_Y}(o)$ is positive, the signs of the two errors are the same, and if the sign of $e^2_{\Delta \dot{x} \phi_Y}(o)$ is negative, the signs of the two errors are opposite. The combined correlated errors and the independent uncorrelated errors are then used to compute the rms value of the total position error by the equation

$$\Delta x_{rms} = \sqrt{\left[\Delta x_{\Delta \dot{x}}\ (cor) + \Delta x_{\phi_Y}\ (cor)\right]^2 + \Delta x_{\Delta \dot{x}}\ (unc)^2 + \Delta x_{\phi_Y} (unc)^2} \tag{30}$$

If the rms value of the error in the y direction is found, using a similar procedure with the characteristics of the y channel, the Circular Error Probability is computed using

$$CEP = .589 \left| \Delta x_{rms} + \Delta y_{rms} \right| \quad (5) \tag{31}$$

Bias Error

The bias error in the reference velocity is treated as a steady state error since it has a constant magnitude. The bias error is approximated by a step function and in the La Place notation is

$$\delta V_x(t) = \frac{\delta V_x}{s} \tag{32}$$

Therefore, the error in system velocity caused by the bias error is

$$\Delta \dot{x}(s) = \frac{\delta V_x(s) G(s)}{s} = \frac{\delta V_x}{s} \cdot \frac{K_1 s + K_2 \omega_s^2}{s^2 + K_1 s + (1 + K_2)\, \omega_s^2} \tag{33}$$

where G(s) is the transfer function derived in Section (II). Applying the final value theorem,

$$\lim_{s \to 0} sF(s) = \lim_{t \to 0} f(t) \tag{34}$$

to the system velocity error to find the steady-state error yields

$$\Delta \dot{x}_{ss} \underset{s \to 0}{=} sG(s) \frac{\delta V_x}{s} = \delta V_x \frac{K_2}{(1 + K_2)} \tag{35}$$

Similarly, the steady-state error in platform tilt caused by bias error in reference velocity is

$$\phi_{yss} \underset{s \to 0}{=} sH(s) \frac{\delta V_x}{s} = \delta V_x \frac{K_1}{R_e[\omega_s^2 \;\; (1 + K_2)]} \tag{36}$$

IV. Results

Effect of Variation of the Half Power Frequency β

The effect of varying the half-power frequency of the noise, β, on the rms value of the system velocity error, $e_{\Delta\dot{x}\Delta\dot{x}}(0)$, and the rms value of the platform tilt error $e_{\phi_Y\phi_Y}(0)$, and on the correlation constants, A and B, is investigated first. The frequency is varied through a representative range of values and the following values are assumed for the system constants:

$$K_1 = 10^{-3} \text{ sec}^{-1}$$

$$K_2 = 1$$

$$\delta V_x = 1 \text{ fps}$$

$$\omega_s^2 = 1.5\text{x}10^{-6} \text{ sec}^{-2}$$

$$R_e = 20.9\text{x}10^{6} \text{ feet}$$

$$\beta = 10^{-6} \text{ to } 10^{-1} \text{ sec}^{-1}$$

Computations similar to those in Appendix G, give the results shown in Table I, and the curves in Figures 5, 6, and 7. The values are plotted for a unit reference velocity noise error.

The further effect of a variable β on the correlated and uncorrelated velocity and tilt errors, and on the rms value of the x position error is shown in Tables II and III and, in Figures 8, 9, 10, and 11. Again, these errors are due to a unit Doppler reference velocity noise error.

Table I

Variation of Velocity and Tilt Error and Correlation Constants, A and B, with β for Unit Reference Velocity Error

	β	$e_{\dot{\Delta x}\dot{\Delta x}}$ (fps)	$e_{\phi_y \phi_y}$ $[x10^{-5}$ rad$]$	(sec)	A	B
1	10^{-6}	.500	1.62	3.4	.986	.165
2	10^{-5}	.500	1.62	3.4	.975	.224
3	10^{-4}	.545	1.85	3.9	.822	.567
4	$5.0x10^{-4}$	.685	2.48	5.2	.430	.896
5	10^{-3}	.707	2.77	5.8	0	1.000
6	$1.5x10^{-3}$	.707	2.82	5.9	.298	.955
7	$2.0x10^{-3}$	.686	2.76	5.7	.374	.930
8	$4.0x10^{-3}$	.580	2.38	4.9	.475	.880
9	10^{-2}	.400	1.67	3.5	.530	.846
10	10^{-1}	.130	0.55	1.1	.570	.820

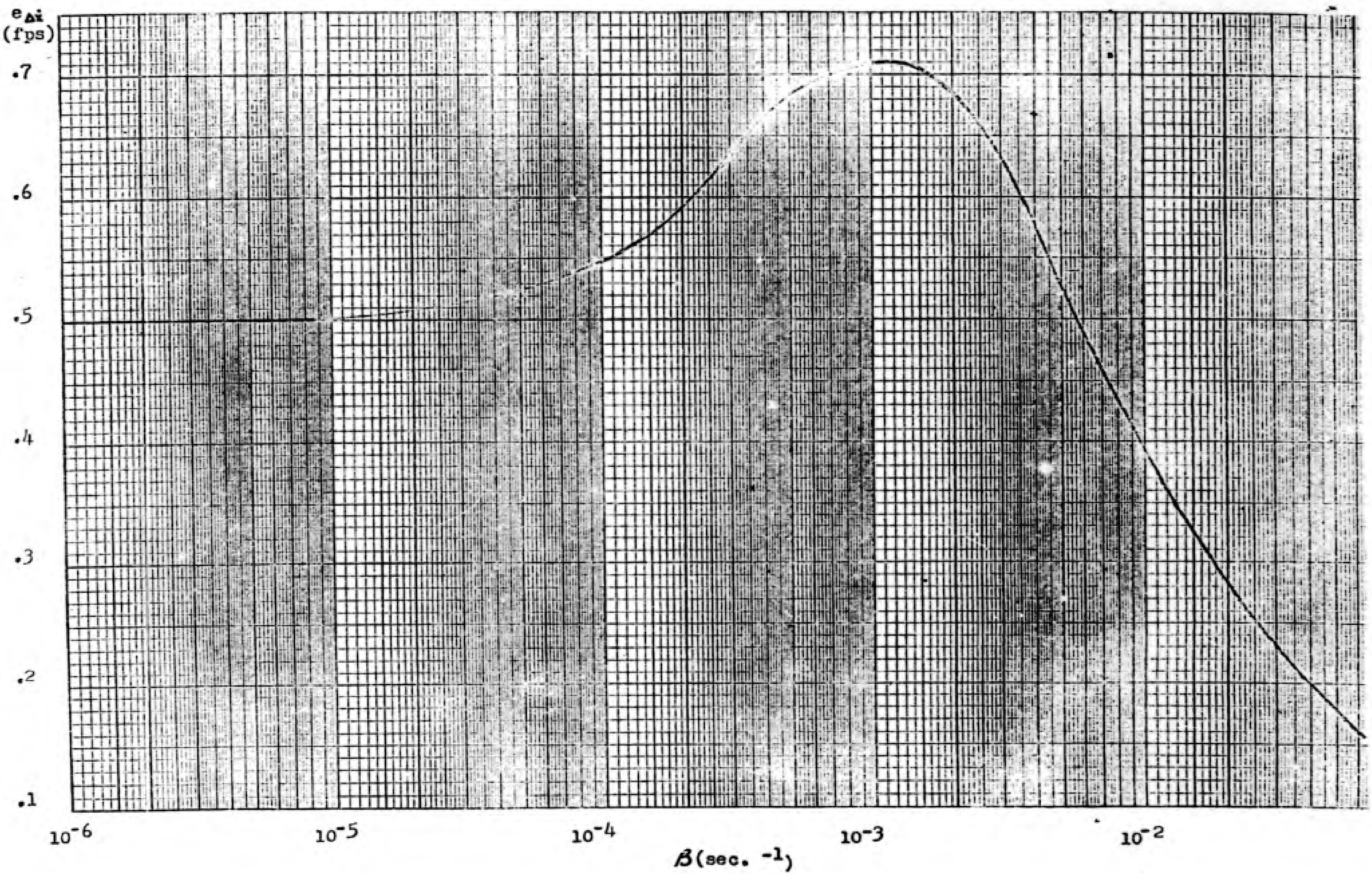


Figure 5 Variation of $e_{\Delta\dot{x}}$ with β for Unit Reference Velocity Error

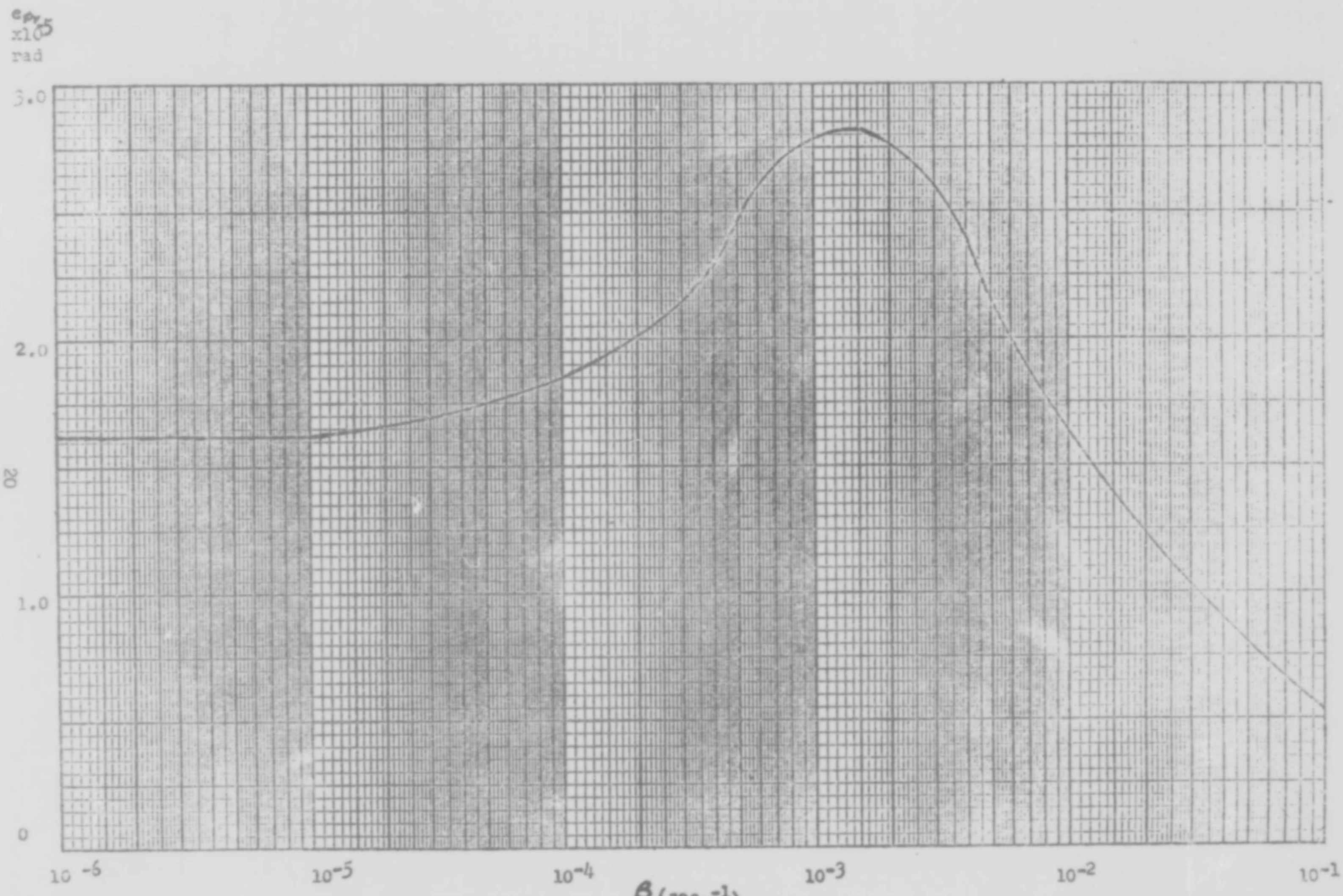


Figure 6 Variation of e_{ϕ_r} with β for Unit Reference Velocity Error

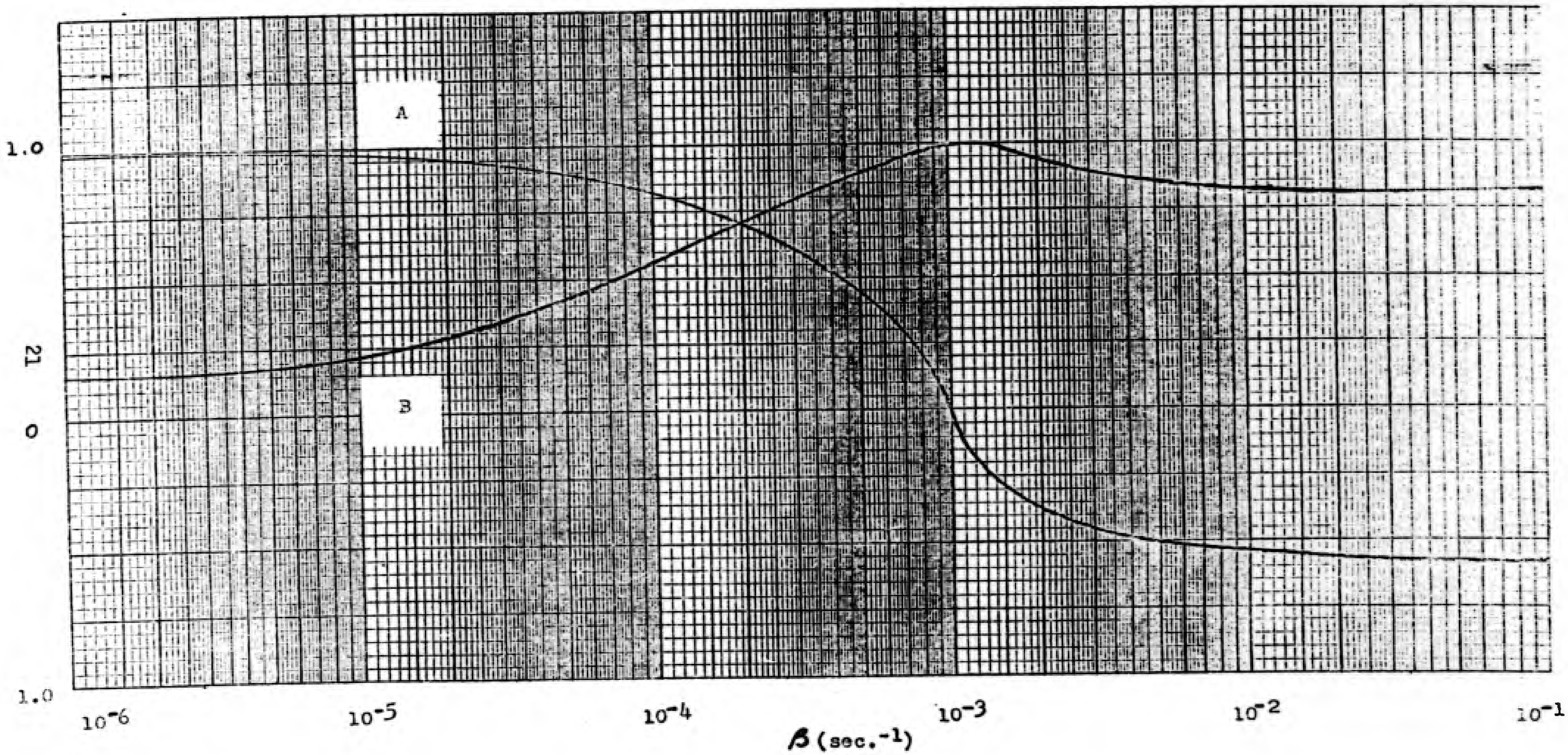


Figure 7 Variation of Correlation Constants A and B with β

Table II

Variation of Correlated and Uncorrelated Position Errors with β for Unit Reference Velocity Error

	$\Delta\dot{x}$ (fps)		ϕ_y ($\times 10^{-5}$ rad)		$\Delta x_{\Delta\dot{x}}$ (ft)*		Δx_{ϕ_y} (ft)*	
	cor	unc	cor	unc	cor	unc	cor	unc
1	.496	.083	1.60	0.27	405	68	300	50
2	.487	.112	1.58	0.36	396	91	297	68
3	.448	.309	1.52	1.05	365	252	286	197
4	.294	.615	1.07	2.22	240	500	201	415
5	0	.707	0	2.77	0	575	0	520
6	.211	.675	0.84	2.70	172	550	158	508
7	.257	.638	1.03	2.56	209	520	194	480
8	.276	.510	1.13	2.10	233	415	212	395
9	.212	.338	0.89	1.42	173	275	167	267
10	.074	.107	0.31	0.45	60	87	58	85

* for t equals 20 min.

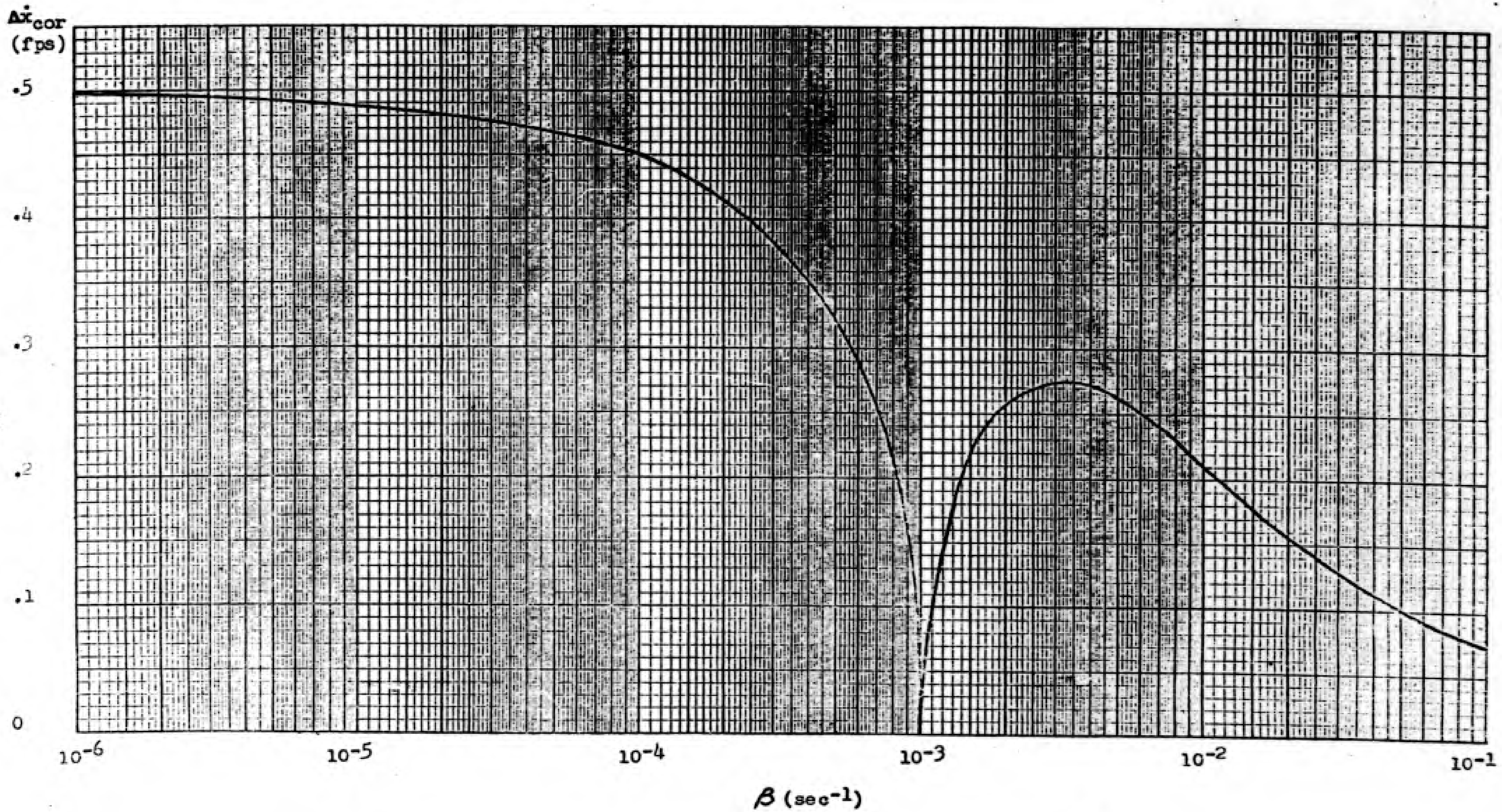


Figure 8 Variation of Correlated Velocity Error with β for Unit Reference Velocity Error

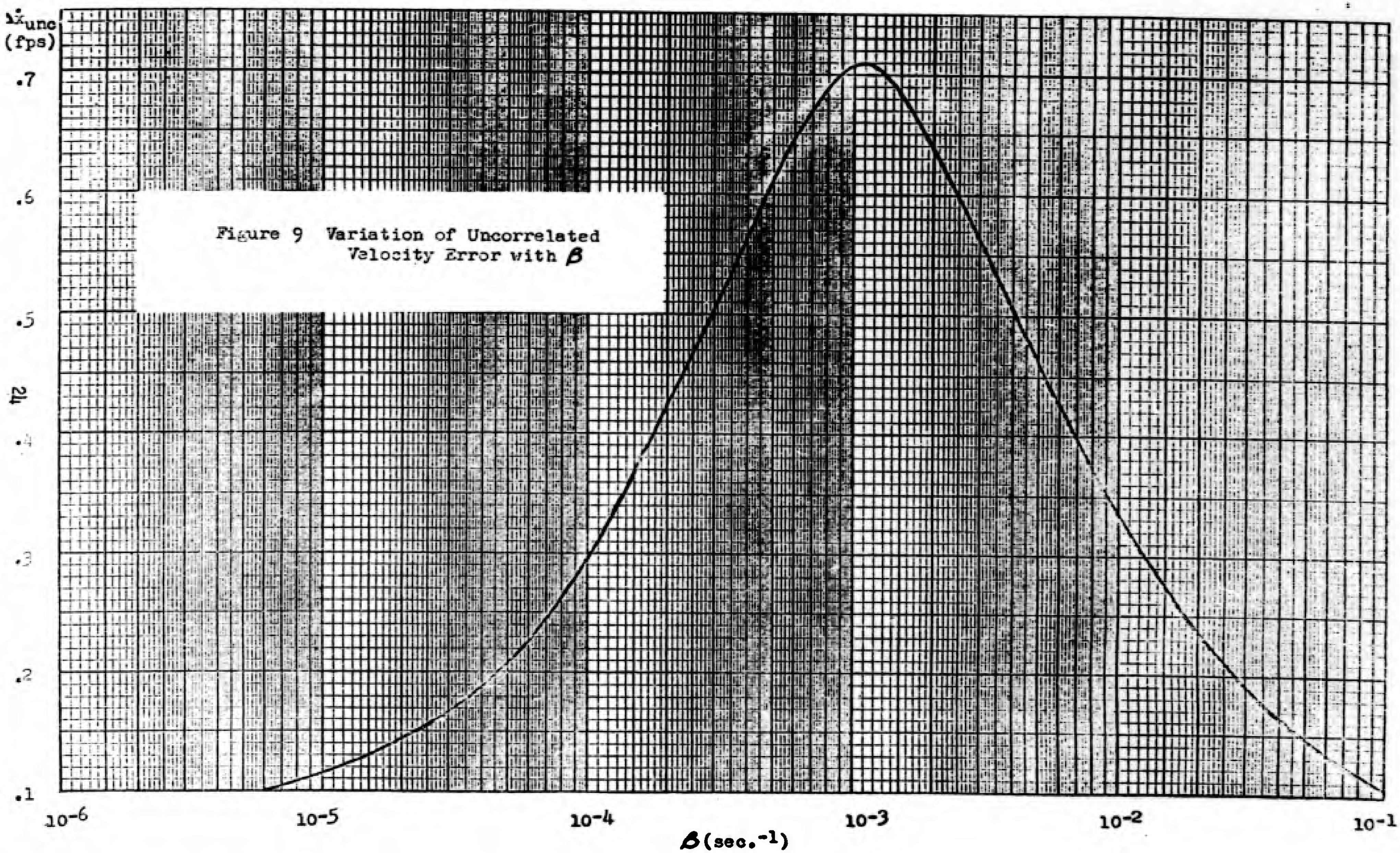


Figure 9 Variation of Uncorrelated Velocity Error with β

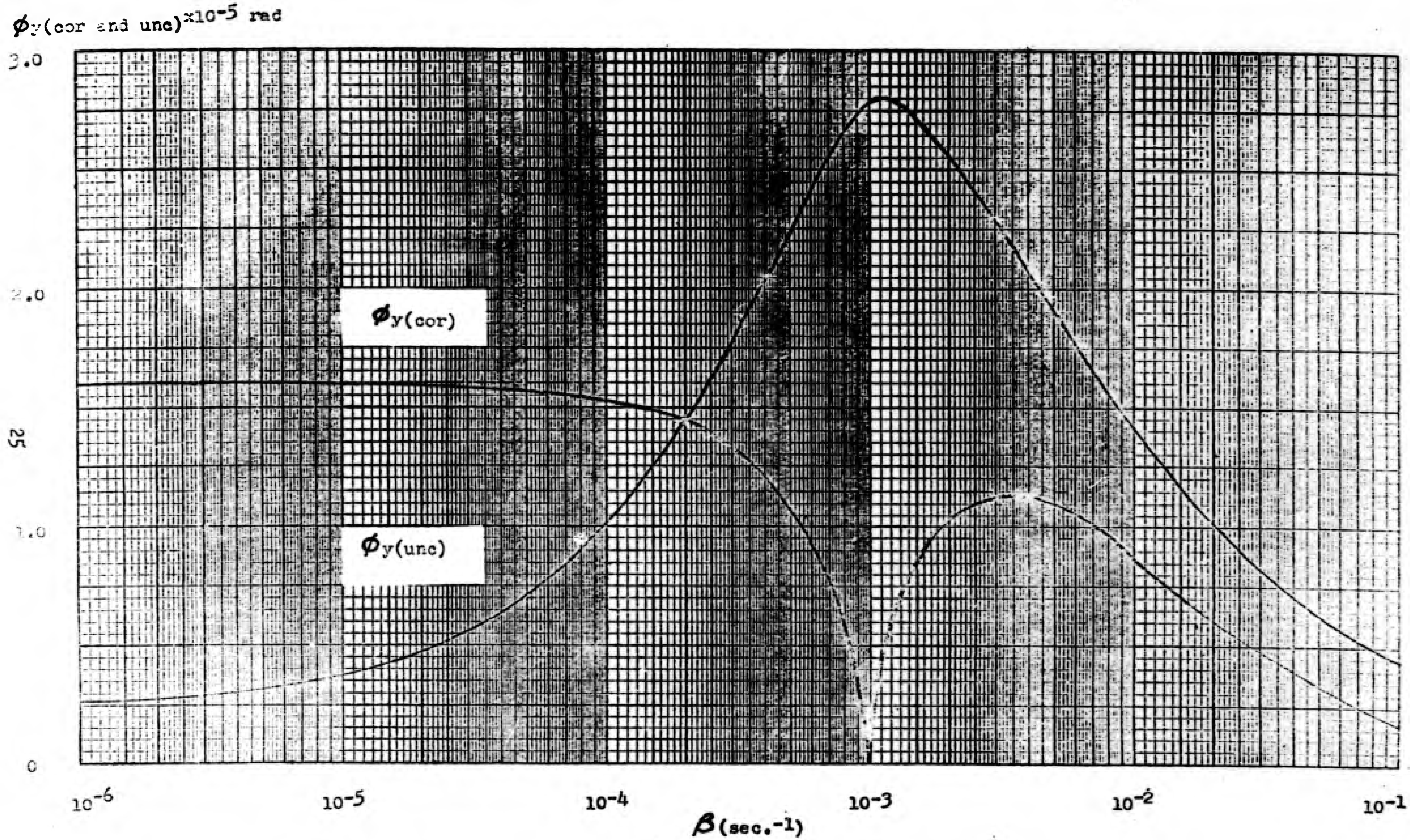


Figure 10 Variation of Correlated and Uncorrelated Tilt Error for Unit Reference Velocity Error

Table III

Variation of RMS Value of Position Error with β

	Δx_{cor} (ft)	$\Delta x_{\Delta\dot{x}}$ unc (ft)	Δx_{ϕ_y} unc (ft)	Δx_{rms} (ft)
1	105	68	50	135
2	99	91	68	151
3	79	252	197	332
4	39	500	416	652
5	0	575	520	775
6	330	550	508	820
7	403	520	480	815
8	445	415	395	725
9	340	275	267	512
10	118	87	58	170

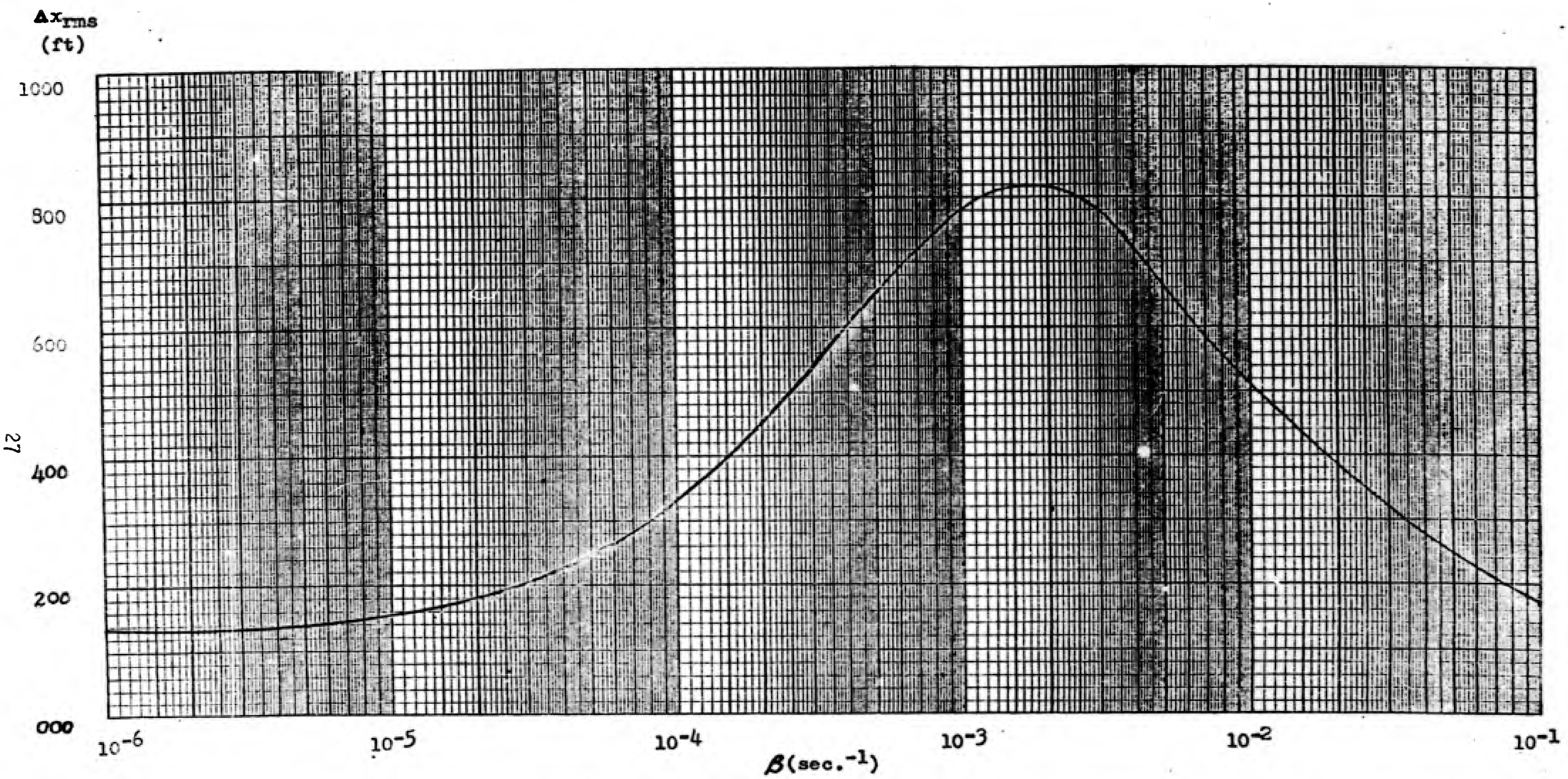


Figure 11 Variation of RMS Value of Position Error with β for Unit Reference Velocity Error

Circular Error Probability

X Position Errors. In practice, the half-power frequency of the noise in a particular system, in this case the Doppler velocity, is fixed. Therefore, to determine the rms value of the x position error for a specific reference velocity noise and bias error, a representative value of β will be assumed. System constants and a reasonable flight time for this type of missile are also assumed. The values used are:

$$K_1 = 10^{-3} \text{ sec}^{-1}$$

$$K_2 = 1$$

$$\beta = 2\text{x}10^{-3}$$

$$\delta V_x(\text{noise}) = 5 \text{ fps}$$

$$\delta V_x(\text{bias}) = 2 \text{ fps}$$

$$t = 20 \text{ min.}$$

The sample computations in Appendix G indicate the method for computing the total rms value of the x position error, 4180 feet, as shown in Table IV.

Since the curves of Figures 5 through 11 are plotted for a unit reference velocity noise error, they may be used to compute the errors for any larger value of reference velocity noise error. The appropriate half-power frequency is used as the entering argument, and the extracted values of the errors are multiplied by five, in this case. To find the position error, the system velocity and platform tilt errors thus obtained are substituted in the error propagation equations. The values of $\frac{\sin \omega_s t}{\omega_s}$ and $(1 - \cos \omega_s t)$, for the given time of flight are extracted from the curves in Figures 12 and 13. These position errors, computed for both the correlated and uncorrelated velocity and tilt errors, and the correlated error due to reference velocity bias, are then combined, in accordance with equation (30), to give the rms value of the x position error.

Y Position Error. Using a parallel derivation to that in Section (II), the error propagation in the y direction is shown to be

$$\Delta y = \frac{\Delta \dot{y} \sin \omega_s t}{\omega_s} - R_e \phi_x (1 - \cos \omega_s t).$$

Therefore, to determine the rms value of the y position error, identical computations to those in Appendix G are made, or the x position error curves may be used as described above. To determine the y_{rms} position error in this case, the following values are assumed:

$$K_1 = 10^{-3}$$

$$K_2 = 1$$

$$\beta = 4 \times 10^{-3}$$

$$\delta V_y(\text{noise}) = 4 \text{ fps}$$

$$\delta V_y(\text{bias}) = 2 \text{ fps}$$

$$t = 20 \text{ min.}$$

The results are shown in Table IV for rms value of the y position error, which is 3020 feet.

To determine the circular error probability due to the rms value of x and y position error, the relationship

$$CEP = .589 \left| \Delta x_{rms} + \Delta y_{rms} \right|$$

is used. In this case, the CEP as shown in Table IV and in Appendix G, is 4240 feet. This is the error probability caused only by Doppler reference velocity noise and bias error.

Table IV

Circular Error Probability Caused by Reference Velocity

Noise and Bias Error

		x	y
β ($x10^{-3}$ sec.$^{-1}$)		2	4
Reference Velocity Error (fps)	noise	5	4
	bias	2	2
System Velocity Error (ft)	cor	1050	932
	unc	2610	1660
Platform Tilt Error (ft)	cor	955	848
	unc	2390	1580
Total Bias Error (ft)	cor	230	230
RMS Position Error (ft)		4180	3059
CEP (ft)	4250		

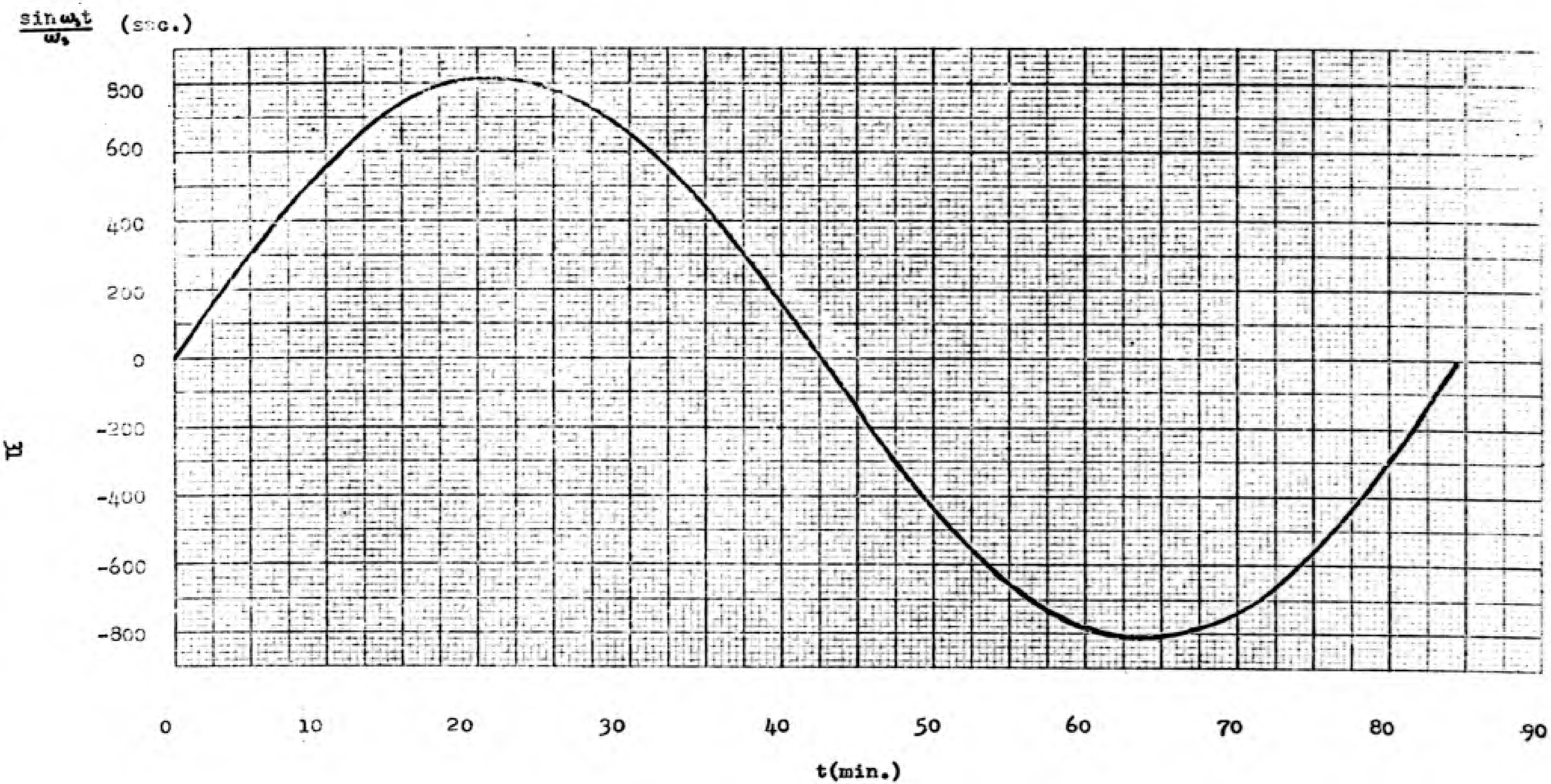


Figure 12 $\frac{\sin\omega_s t}{\omega_s}$ for One Schuler Period (84.4 min.)

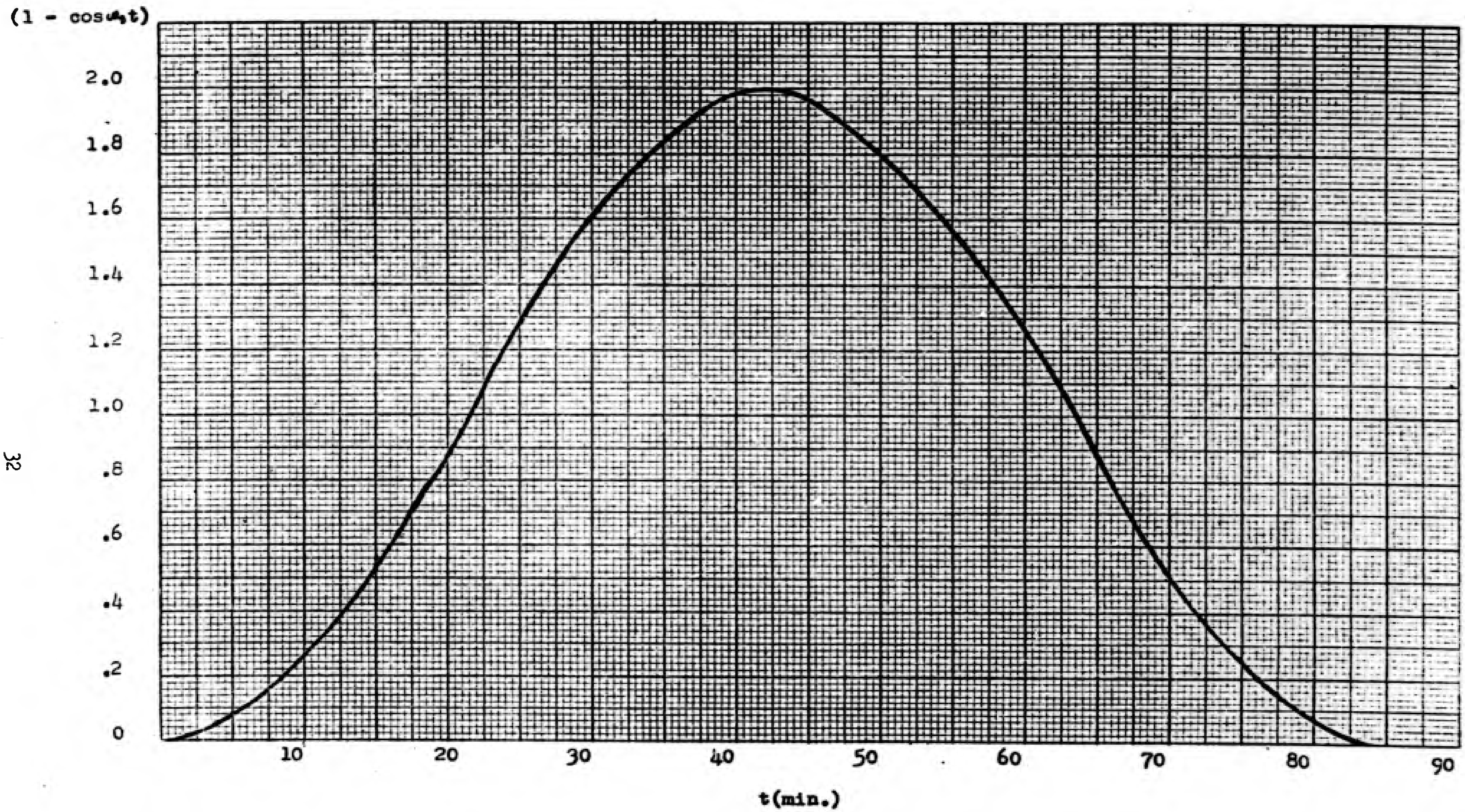


Figure 13 (1 - cosω_st) for One Schuler Period (84.4 min.)

Conclusions

1. The mean squared value of the initial system velocity error, $e^2_{\Delta\dot{x}\Delta\dot{x}}(0)$, is shown to approach limiting values as the noise half-power frequency, β, approaches zero or as it becomes infinite. See Appendix H and Figure 5. As β approaches zero, $e^2_{\Delta\dot{x}\Delta\dot{x}}(0)$ approaches a constant value dependent only on the value of the system gain, K_2. In this case, since the value of K_2 is one, $e^2_{\Delta\dot{x}\Delta\dot{x}}(0)$ approaches $.25e^2\,(fps)^2$, where e^2 is the reference velocity error squared. As β becomes infinite, $e^2_{\Delta\dot{x}\Delta\dot{x}}(0)$ approaches zero.

2. The mean squared value of the platform tilt error also approaches a constant value as the noise half-power frequency goes to its limits. As β goes to zero, $e^2_{\phi_y\phi_y}(0)$ approaches a constant dependent upon the gain constants, K_1 and K_2, earth radius and the reference velocity error. In this case with $K_1 = 10^{-3}\ sec^{-1}$, and $K_2 = 1$, $e^2_{\phi_y\phi_y}(0)$ approaches $2.55 \times 10^{-10} e^2\ (rad)^2$. As β becomes infinite, $e^2_{\phi_y\phi_y}(0)$ approaches zero.

3. The correlated initial system velocity and tilt errors become zero when the correlation constant, A, is zero. In Appendix H, equation (26) is used to show that A and the correlated error become zero when $\beta = K_1/K_2$. Since in this case, $K_2 = 1$, the correlated error goes to zero when $\beta = K_1$ or $10^{-3}\ sec^{-1}$. The uncorrelated initial velocity and tilt errors depend upon the correlation constant B, which is expressed in terms of A as

$$B = (1 - A^2)^{\frac{1}{2}}$$

Hence, the uncorrelated error is a maximum, or equal to one, when the correlated error is zero. This relationship is shown in Figure (7).

4. Figure (11) shows that the position error, Δ_x, is a maximum in the vicinity of $\beta = K_1$. Therefore, in this particular system configuration, the maximum final position error occurs when the half-power frequency of the noise is approximately equal to the system gain constant K_1.

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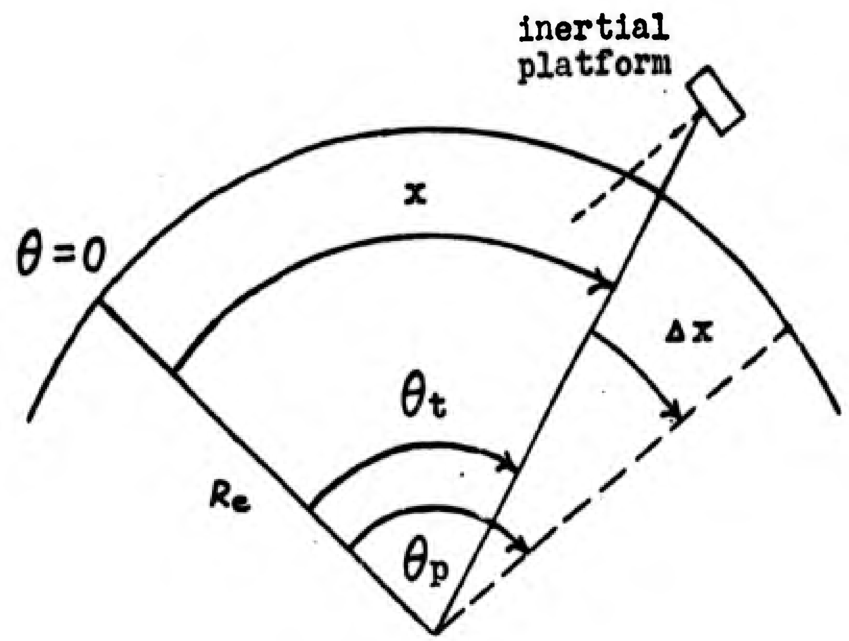
Appendix A

Derivation of Position Error Propagation Caused by Inertial Platform Tilt [1]



$$\Delta\theta = \theta_p - \theta_t$$

Figure 1A Position Error Caused by Tilted Inertial Platform

An accelerometer on the tilted inertial platform will measure the apparent acceleration,

$$\begin{aligned} a_{meas} &= a_{true} - (\theta_{plat} - \theta_{true}) \\ &= a_{true} + g\left(\frac{x}{R_e} - \frac{x + \Delta x}{R_e}\right) \end{aligned} \qquad (1A)$$

Substituting, $p^2 x = a$ and $\omega_s^{\,2} = \dfrac{g}{R_e}$

into equation (1A) gives

$$a_{meas} = (p^2 + \omega_s^{\,2})x - \omega_s^{\,2}(x + \Delta x) \qquad (2A)$$

observed horizontal acceleration	=	inherent accelerometer measurement	-	platform position feedback

But $a_{meas} = p^2(x + \Delta x)$

Therefore,

$$p^2(x + \Delta x) = (p^2 + \omega_s^{\,2})x - \omega_s^{\,2}(x + \Delta x) \qquad (3A)$$

or

$$(p^2 + \omega_s^{\,2})\Delta x = 0 \qquad (4A)$$

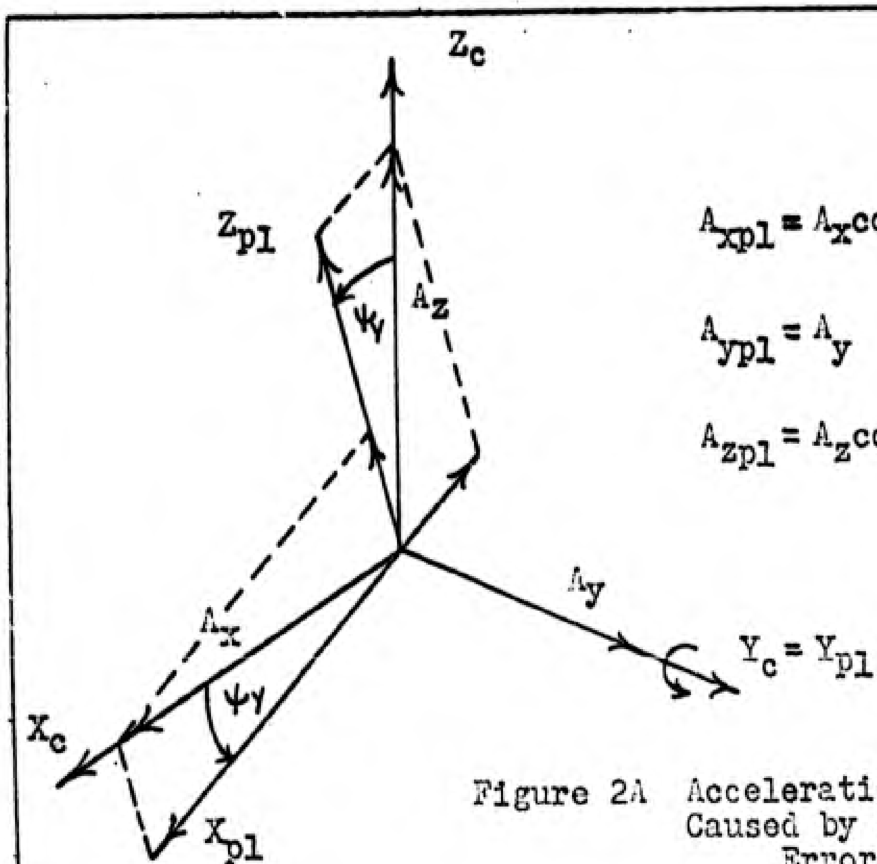
Appendix B

Derivation of the Error Driving Function Due to Acceleration and Platform Misalignment $\bar{\Psi}$.

In order to determine the acceleration error driving function due to the angular error between computer and platform axes, the error components of the acceleration vector,

$$\bar{A} = \bar{1}_x A_x + \bar{1}_y A_y + \bar{1}_z A_z \tag{5A}$$

must be derived. A right-handed coordinate system is used, with the X axis in the direction of the velocity vector and the Z axis up. For the purposes of this derivation, the computer axes and true axes are assumed to be aligned. Misalignment of the computer axes is treated in a later appendix. The components of the vector angle $\bar{\Psi}$, are treated separately.



$$A_{xpl} = A_x \cos \Psi_y - A_z \sin \Psi_y$$

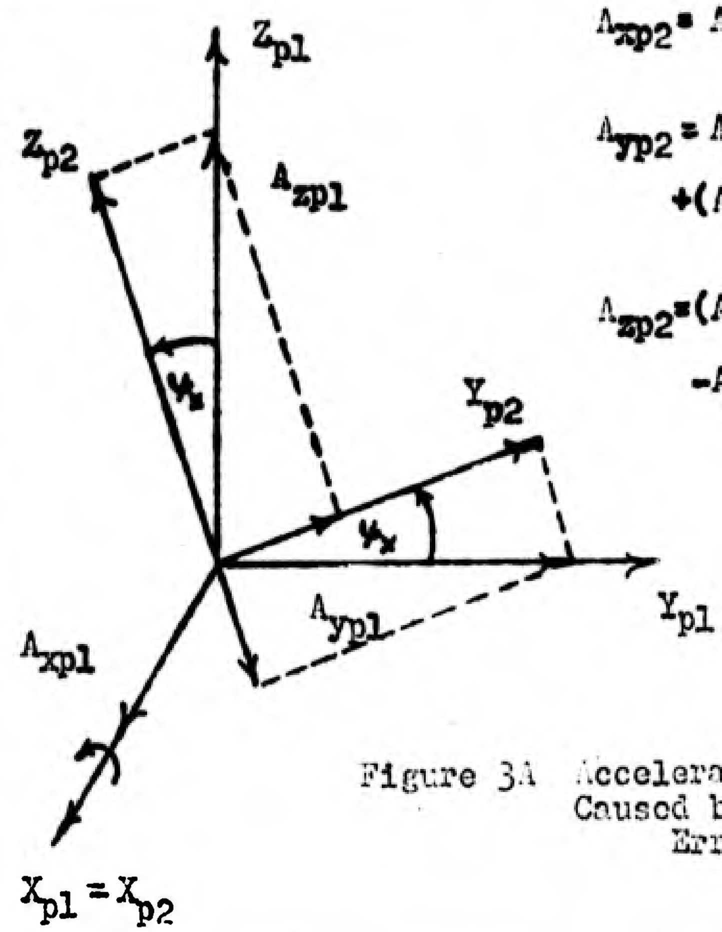
$$A_{ypl} = A_y$$

$$A_{zpl} = A_z \cos \Psi_y + A_x \sin \Psi_y$$

Figure 2A Acceleration Error Components Caused by Platform Alignment Error Angle Ψ_y

Introducing a component of platform angular error, ψ_y, and resolving the acceleration components to platform axes, produces acceleration error components in the (-)X and (+)Z directions, as shown in Figure 2A. In the figures, the subscript c indicates computer axes, t indicates true axes, and p1, p2, and p3 indicate successive positions of the platform axes.

Introducing a component of platform angular error, ψ_x , and resolving the acceleration components to the new platform axes produces acceleration error components in the (+)Y and (-)Z directions as shown in Figure 3A.



$$A_{xp2} = A_x \cos \psi_y - A_z \sin \psi_y$$

$$A_{yp2} = A_y \cos \psi_x + (A_z \cos \psi_y + A_x \sin \psi_y) \sin \psi_x$$

$$A_{zp2} = (A_z \cos \psi_y + A_x \sin \psi_y) \cos \psi_x - A_y \sin \psi_x$$

Figure 3A Acceleration Error Components Caused by Platform Alignment Error Angle ψ_x

A component of platform angular error, ψ_z, produces acceleration error components in the (+)X and (-)Y directions as shown in Figure 4A.

$$A_{xp3} = (A_x \cos\psi_y - A_z \sin\psi_y)\cos\psi_z + (A_y \cos\psi_x + A_z \cos\psi_y \sin\psi_x + A_x \sin\psi_y \sin\psi_x)\sin\psi_z$$

$$A_{yp3} = (A_y \cos\psi_x + A_z \cos\psi_y \sin\psi_x + A_x \sin\psi_y \sin\psi_x)\cos\psi_z - (A_x \cos\psi_y - A_z \sin\psi_y)\sin\psi_z$$

$$A_{zp3} = A_z \cos\psi_y \cos\psi_x + A_x \sin\psi_y \cos\psi_x - A_y \sin\psi_x$$

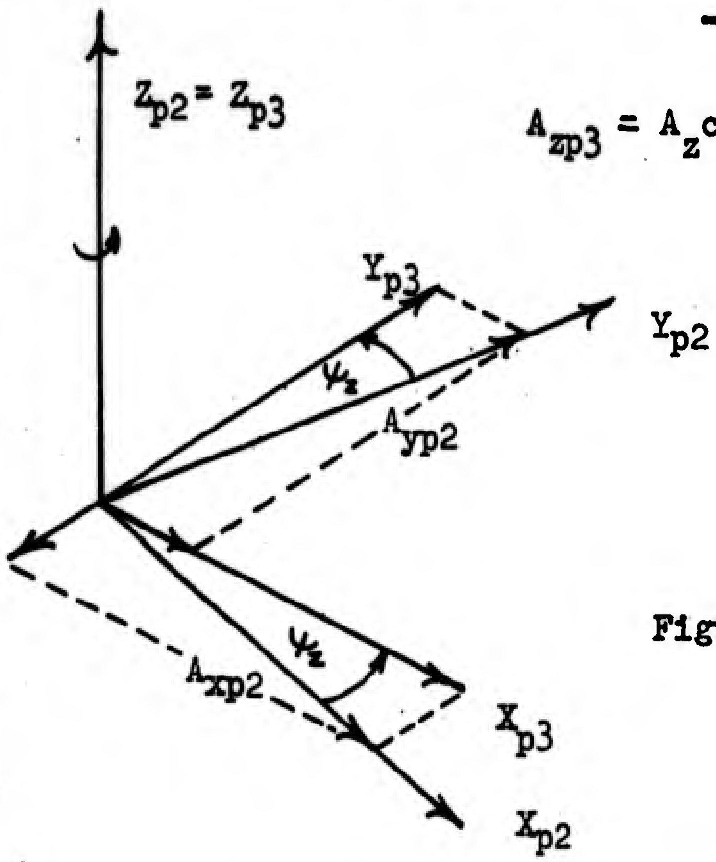


Figure 4A Acceleration Error Components Caused by Platform Alignment Error Angle ψ_z.

Therefore, the acceleration components in platform coordinates are:

$$A_{xp} = A_x \cos\psi_y \cos\psi_z - A_z \sin\psi_y \cos\psi_z + A_y \cos\psi_x \sin\psi_z + A_z \cos\psi_y \sin\psi_x \sin\psi_z + A_x \sin\psi_y \sin\psi_x \sin\psi_z \qquad (6A)$$

$$A_{yp} = A_y \cos\psi_x \cos\psi_z + A_z \cos\psi_y \sin\psi_x \cos\psi_z + A_x \sin\psi_y \sin\psi_x \cos\psi_z - A_x \cos\psi_y \sin\psi_z + A_z \sin\psi_y \sin\psi_z \qquad (7A)$$

$$A_{zp} = A_z \cos\psi_y \cos\psi_x + A_x \sin\psi_y \cos\psi_x - A_y \sin\psi_x \qquad (8A)$$

The error components of acceleration caused by platform misalignment are therefore:

$$A_{x(error)} = - A_z \sin\Psi_y \cos\Psi_z + A_y \cos\Psi_x \sin\Psi_z + A_z \cos\Psi_y \sin\Psi_x \sin\Psi_z \quad (9A)$$

$$+ A_x \sin\Psi_y \sin\Psi_x \sin\Psi_z$$

$$A_{y(error)} = A_z \cos\Psi_y \sin\Psi_x \cos\Psi_z + A_x \sin\Psi_y \sin\Psi_x \cos\Psi_z \quad (10A)$$

$$- A_x \cos\Psi_y \sin\Psi_z + A_z \sin\Psi_y \sin\Psi_z$$

$$A_{z(error)} = A_x \sin\Psi_y \cos\Psi_x - A_y \sin\Psi_x \quad (11A)$$

If the platform misalignment is assumed to be small then,

$$\sin\Psi \approx \Psi \quad , \quad \cos\Psi \approx 1 \quad \text{and} \quad \sin\Psi \sin\Psi \approx 0$$

Then the acceleration error vector is written,

$$\bar{A}_{error} = \bar{1}_x(A_y\Psi_z - A_z\Psi_y) - \bar{1}_y(A_x\Psi_z - A_z\Psi_x) \quad \bar{1}_z(A_x\Psi_y - A_y\Psi_x) \quad (12A)$$

$$= \left[\bar{A} \times \bar{\Psi}\right]$$

Appendix C

Vector Representation of Angles

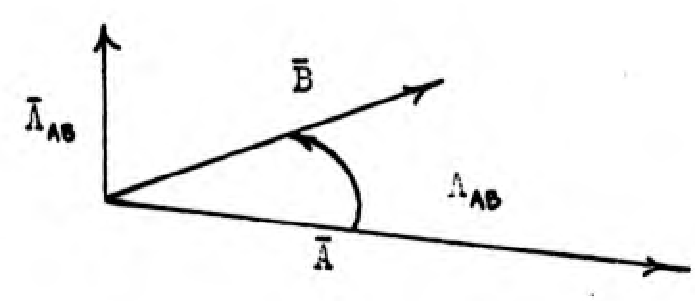
The vector representation of an angle is derived from the definition of a vector cross-product,

$$\bar{A} \times \bar{B} = \bar{1}_{\perp AB}\, AB \sin A_{AB} \tag{13A}$$

where $\bar{1}_{\perp AB}$ is the unit vector perpendicular to the vectors $\bar{A}$ and $\bar{B}$, and A_{AB} is the angle between the two vectors. Dividing through by $AB\sin A_{AB}$ and multiplying both sides of the equation by A_{AB} gives

$$\frac{(\bar{A} \times \bar{B})A_{AB}}{AB\sin A_{AB}} = \bar{1}_{\perp AB}\, A_{AB} \triangleq \bar{A}_{AB} \tag{14A}$$

which is the definition of the vector angle $\bar{A}_{AB}$.



$$\bar{A}_{AB} = \frac{(\bar{A} \times \bar{B})A_{AB}}{AB\sin A_{AB}} \approx \bar{1}_A \times \bar{1}_B$$

Figure 5A Vector Representation of an Angle

For small angles, the quantity

$$\frac{A_{AB}}{\text{Sin } A_{AB}}$$

is approximately unity, therefore, the vector angle is approximated

$$\bar{A}_{AB} \approx \frac{\bar{A} \times \bar{B}}{AB} = \bar{1}_A \times \bar{1}_B \qquad (15A)$$

For example, when $A_{AB} = 30^\circ$,

$$A_{AB} = .523 \approx \sin A_{AB} = .5 \qquad (16A)$$

Appendix D

Computer Axis Angular Error as a Function of Position Error

The vector angular error, $\overline{\delta\theta}$, between the computer axis and true axes may be defined in terms of the components of a position error. In Figure 6A, the X component of the angular error, $\delta\theta_x$, is defined in terms of Y component of the position error. The coordinate system is assumed as shown in the figure and R_e, the earth radius is assumed equal to the position vector $\bar{R}$.

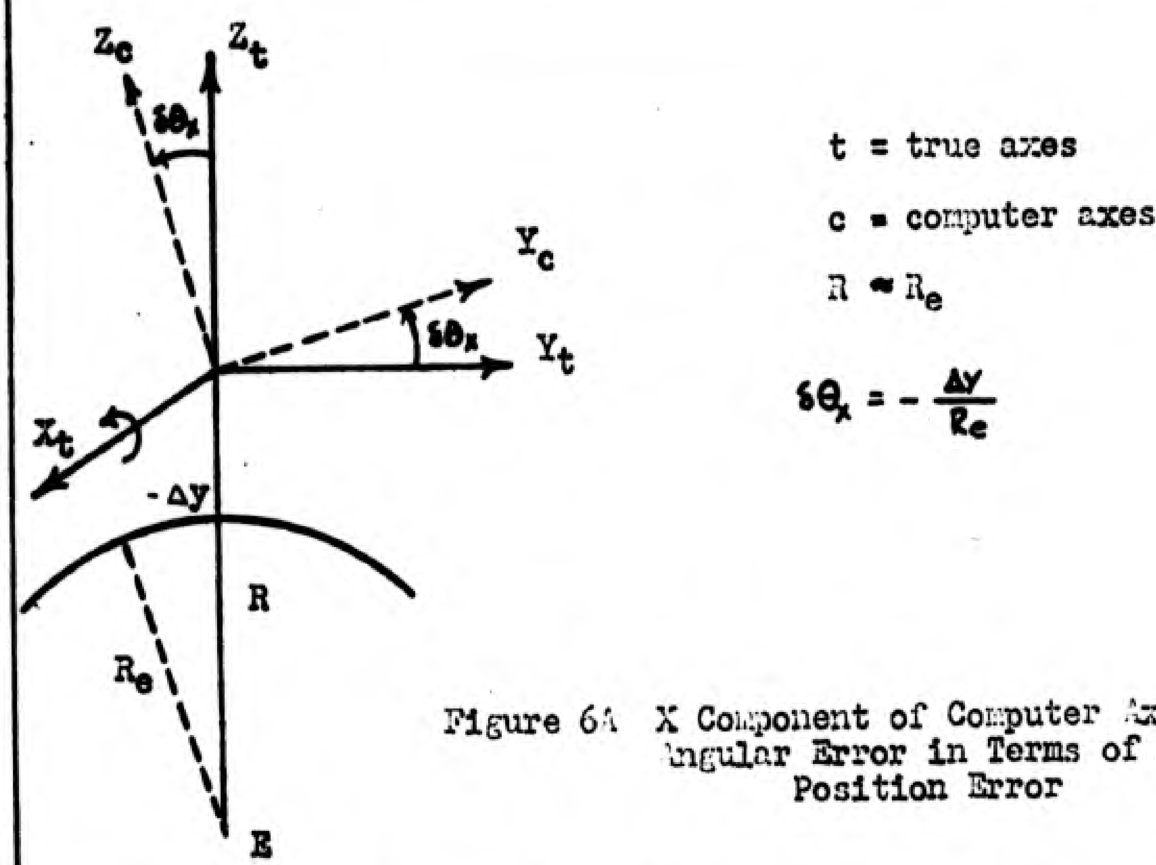


Figure 6A X Component of Computer Axes Angular Error in Terms of Y Position Error

In Figure 7A, the Y component of the computer axes angular error is defined in terms of the X component of the position error.

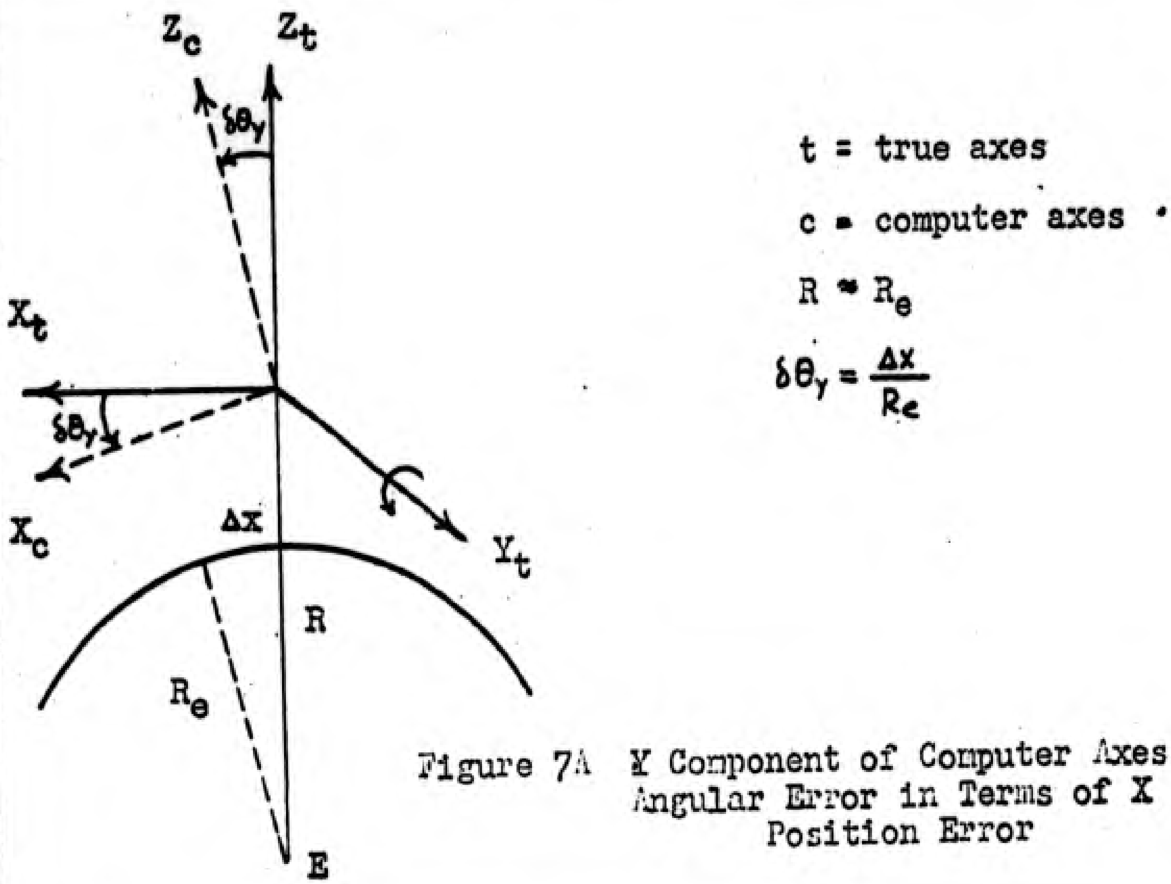


Figure 7A Y Component of Computer Axes Angular Error in Terms of X Position Error

Therefore, the vector angle, $\overline{\delta\theta}$, is written

$$\overline{\delta\theta} = -\bar{i}_x \frac{\Delta y}{R_e} + \bar{i}_y \frac{\Delta x}{R_e} \tag{17A}$$

Appendix B

Derivation of System Velocity Error and Platform Tilt Transfer Functions

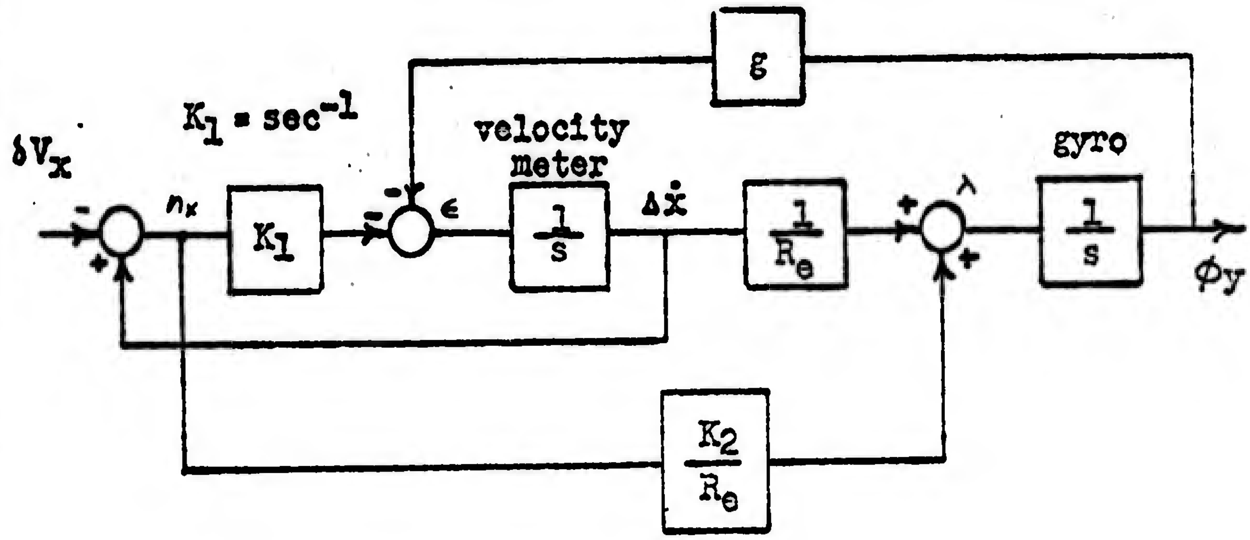


Figure 8A Error Block Diagram for X Channel of Model Inertial Guidance System [5]

$$\epsilon = -(\phi_y g + n_x K_1) \tag{18A}$$

$$\Delta\dot{x} = \frac{\epsilon}{s} = \frac{-(\phi_y g + n_x K_1)}{s} \tag{19A}$$

$$\lambda = \frac{\Delta\dot{x}}{Re} + n_x \frac{K_2}{Re} \tag{20A}$$

$$n_x = -\delta V_x + \Delta\dot{x} \tag{21A}$$

$$\phi_y = \frac{\lambda}{s} \tag{22A}$$

Substituting equation (19A) into (21A),

$$n_x = -\delta V_x - \frac{(\phi_y g + n_x K_1)}{s}$$

$$s\eta_x = -s\delta V_x - \phi_y g - \eta_x k_1$$

$$\eta_x = -\frac{s\delta V_x + \phi_y g}{s + K_1} \qquad (23A)$$

Then substituting (20A) and (21A) into (22A),

$$\phi_y = \frac{\Delta\dot{x}}{R_e s} + \frac{\eta_x k_2}{R_e s} = \frac{\Delta\dot{x}}{R_e s} - \frac{\delta V_x K_2}{R_e s} + \frac{\Delta\dot{x} K_2}{R_e s}$$

$$= \frac{\Delta\dot{x}}{R_e s}(1 + K_2) - \frac{\delta V_x K_2}{R_e s} \qquad (24A)$$

Substituting the value of $\Delta\dot{x}$ from equation (19A), the value of η_x from (23A).

$$\phi_y = \frac{-(\phi_y g + \eta_x K_1)(1 + K_2)}{R_e s^2} - \frac{\delta V_x K_2}{R_e s}$$

$$\phi_y = -\frac{\phi_y g(1 + K_2)}{R_e s^2} + \frac{K_1(1 + K_2)(s\delta V_x + \phi_y g)}{R_e s^2(s + K_1)} - \frac{\delta V_x K_2}{R_e s}$$

Clearing and simplifying,

$$\phi_y R_e s^2(s + K_1) + \phi_y g(1 + K_2)(s + K_1) - \phi_y g K_1(1 + K_2)$$

$$= K_1(1 + K_2)s\delta V_x - K_2(s + K_1)s\delta V_x$$

$$\phi_y(R_e s^3 + R_e s^2 K_1 + gs + gsK_2) = (K_1 - K_2 s)s\delta V_x$$

$$s\phi_y R_e\left[s^2 + sK_1 + \frac{g}{R_e}(1 + K_2)\right] = (K_1 - K_2 s)s\delta V_x$$

Finally, substituting ω_s^2 for $\frac{g}{R_e}$ gives the required function

$$\frac{\phi_y}{\delta V_x} = \frac{(K_1 - K_2 s)}{R_e\left[s^2 + K_1 s + \omega_s^2(1 + K_2)\right]}$$

From (24A), $\phi_y = \frac{\Delta\dot{x}}{R_e S} + n_x \frac{K_2}{R_e S}$

From (19A), $\phi_y = \frac{-n_x K_1 - S\Delta\dot{x}}{g}$

$$-R_e s(n_x K_1 + s\Delta\dot{x}) = g\Delta\dot{x} + n_x g K_2$$

$$\Delta\dot{x}(R_e s^2 + g) = n_x(-K_2 g - R_e K_1 s)$$

Substituting the value of n_x from equation (21A),

$$\Delta\dot{x}(R_e s^2 + g) = (-\delta V_x + \Delta\dot{x})(-K_2 g - R_e K_1 s)$$

$$\Delta\dot{x}(R_e s^2 + g + K_2 g + R_e s K_1) = \delta V_x(K_2 g + R_e K_1 s)$$

$$\Delta\dot{x} R_e\left[s^2 + sK_1 + \frac{g}{R_e}(1+K_2)\right] = \delta V_x R_e\left(K_1 s + \frac{g}{R_e}K_2\right)$$

Substituting ω_s^2 for $\frac{g}{R_e}$,

$$\frac{\Delta\dot{x}}{\delta V_x} = \frac{K_1 s + \omega_s^2 K_2}{S^2 + SK_1 + \omega_s^2(1+K_2)}$$

Appendix F

Table of Integrals [4]

The followin is a table of integrals of the type

$$I_n = \frac{1}{2\pi j}\int_{-\infty}^{\infty} \frac{g_n(x)}{h_n(x)h_n(-x)}$$

where

$$h_n(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$$

$$g_n(x) = b_0x^{2n-2} + b_1x^{2n-4} + \dots + b_{n-1} ,$$

and the roots of $h_n(x)$ all lie in the upper half plane. The table lists the integrals I_n for values of n from 1 to 3 inclusive.

$$I_1 = \frac{b_0}{2a_0a_1}$$

$$I_2 = \frac{-b_0 + \frac{a_0b_1}{a_2}}{2a_0a_1}$$

$$I_3 = \frac{-a_2b_0 + a_0b_1 - \frac{a_0a_1b_2}{a_3}}{2a_0(a_0a_3 - a_1a_2)}$$

Appendix G

Sample Computations

From equation (24)

$$e^2_{\Delta x \Delta x}(0) = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \Phi_s(s)\, G(s)\, G(-s)\, ds$$

$$= \frac{1}{2\pi j}\int \frac{2\beta e^2(-K_1^2 s^2 + K_2^2 \omega_s^4)}{\left|(s+\beta)\left[s^2+K_1 s+\omega_s^2(1+K_2)\right]\right|^2} ds = \frac{1}{2\pi j}\int \frac{g_n(x)}{h_n(x)\, h_n(-x)} dx$$

where

$$g_n(s) = b_0 x^{2n-2} + b_1 x^{2n-4} + \cdots + b_{n-1} = -K_1^2 s^2 + K_2^2 \omega_s^4$$

$$h_n(s) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$$

$$= s^3 + (K_1+\beta)s^2 + \left[\omega_s^2(1+K_2) + K_1\beta\right]s + \beta\omega_s^2(1+K_2)$$

$$n = 3 \qquad 2n-2 = 4 \qquad 2n-4 = 2$$

From integral table, Appendix F,

$$I_3 = \frac{-a_2 b_0 + a_0 b_1 - \dfrac{a_0 a_1 b_2}{a_3}}{2a_0(a_0 a_3 - a_1 a_2)}$$

where

$a_0 = 1$ $\qquad b_0 = 0$

$a_1 = (K_1 + \beta)$ $\qquad b_1 = -K_1^2$

$a_2 = \omega_s^2(1+K_2) + K_1\beta$ $\qquad b_2 = K_2^2\omega_s^4$

$a_3 = \beta\omega_s^2(1+K_2)$

Appendix G

Sample Computations

From equation (24)

$$e^2_{\Delta x \Delta x}(0) = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \Phi_s(s)\, G(s)\, G(-s)\, ds$$

$$= \frac{1}{2\pi j}\int \frac{2\beta c^2(-K_1^2 s^2 + K_2^2 \omega_s^4)}{\left|(s+\beta)\left[s^2+K_1 s+\omega_s^2(1+K_2)\right]\right|^2} ds = \frac{1}{2\pi j}\int \frac{g_n(x)}{h_n(x)\, h_n(-x)} dx$$

where

$$g_n(s) = b_0 x^{2n-2} + b_1 x^{2n-4} + \quad + b_{n-1} = -K_1^2 s^2 + K_2^2 \omega_s^4$$

$$h_n(s) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$$

$$= s^3 + (K_1+\beta)s^2 + \left[\omega_s^2(1+K_2) + K_1\beta\right]s + \beta\omega_s^2(1+K_2)$$

$$n = 3 \qquad 2n-2 = 4 \qquad 2n-4 = 2$$

From integral table, Appendix F,

$$I_3 = \frac{-a_2 b_0 + a_0 b_1 - \dfrac{a_0 a_1 b_2}{a_3}}{2a_0(a_0 a_3 - a_1 a_2)}$$

where

$a_0 = 1$ $\qquad$ $b_0 = 0$

$a_1 = (K_1 + \beta)$ $\qquad$ $b_1 = -K_1^2$

$a_2 = \omega_s^2(1+K_2) + K_1\beta$ $\qquad$ $b_2 = K_2^2\omega_s^4$

$a_3 = \beta\omega_s^2(1+K_2)$

So, if

$$K_1 = 10^{-3} \qquad \beta = 2\times10^{-5}$$

$$K_2 = 1 \qquad \omega_s^2 = 1.5\times10^{-6}$$

then

$$a_0 = 1 \qquad b_0 = 0$$

$$a_1 = 3\times10^{-3} \qquad b_1 = 10^{-6}$$

$$a_2 = 5\times10^{-6} \qquad b_2 = 2.25\times10^{-12}$$

$$a_3 = 6\times10^{-9}$$

and

$$I_5 = \frac{-10^{-6} - \dfrac{3\times10^{-3}\times2.25\times10^{-12}}{6\times10^{-9}}}{2\left[6\times10^{-9} - 15\times10^{-9}\right]}$$

$$= \frac{-10^{-6} - 1.13\times10^{-6}}{-2\,(9\times10^{-9})} = .118\times10^{3}$$

If the reference velocity error,

$$e = 5 \text{ fps}$$

then

$$e^2_{\Delta\dot{x}\Delta\dot{x}}(0) = 2\beta e^2 I_5 = 2(2\times10^{-3})(25)(118)$$

$$= 11.8$$

$$e_{\Delta\dot{x}\Delta\dot{x}}(0) = 3.44 \text{ fps}$$

Similarly, from equation (24)

$$e^2_{\phi_y\phi_y}(0) = \frac{1}{2\pi J}\int \frac{2\beta e^2(-K_2^2 s^2 + K_1^2)}{R_e^2\left|(s+\beta)\left[s^2 + K_1 s + \omega_s^2(1+K_2)\right]\right|^2}\,ds$$

where

$$a_0 = 1 \qquad b_0 = 0$$
$$a_1 = 3\times 10^{-3} \qquad b_1 = -K_2^2 = -1$$
$$a_2 = 5\times 10^{-6} \qquad b_2 = K_1^2 = 10^{-6}$$
$$a_3 = 6\times 10^{-9}$$

Then

$$I_3 = \frac{-1 - \dfrac{3\times 10^{-3}\times 10^{-6}}{6\times 10^{-9}}}{2\left[6\times 10^{-9} - 15\times 10^{-9}\right]}$$

$$= \frac{-1.5}{-18\times 10^{-9}} = .0832\times 10^{9}$$

and

$$e^2_{\phi_y\phi_y}(0) = \frac{2\beta e^2}{Re^2} = \frac{2(2\times 10^{-3})(25)(.0832\times 10^{9})}{435\times 10^{12}}$$

$$= .0192\times 10^{-6}$$

$$e_{\phi_y\phi_y}(0) = .1385\times 10^{-3}\ \text{rad}$$

$$= \frac{.1385\times 10^{-3}\ \text{rad}}{4.84\times 10^{-6}\ \text{rad/sec}} = 28.3\ \widehat{\text{sec}}$$

Also,

$$e^2_{\Delta x\phi_y}(0) = \frac{1}{2\pi j}\int \frac{2\beta e^2\left[K_1K_2s^2 + (K_1^2 + K_1K_2\omega_s^2)s + K_1K_2\omega_s^2\right]}{Re\left|(s+\beta)\left[s^2 + K_1 s + \omega_s^2(1+K_2)\right]\right|^2}\,ds$$

$$a_0 = 1 \qquad b_0 = 0$$
$$a_1 = 3\times 10^{-3} \qquad b_1 = K_1K_2 = 10^{-3}$$
$$a_2 = 5\times 10^{-6} \qquad b_2 = K_1K_2\omega_s^2 = 1.5\times 10^{-9}$$
$$a_3 = 6\times 10^{-9}$$

$$I_3 = \frac{10^{-3} - \frac{3\times10^{-3} \times 1.5\times10^{-9}}{6\times10^{-4}}}{2\left[6\times10^{-9} - 15\times10^{-9}\right]}$$

$$= \frac{.25\times10^{-3}}{-18\times10^{-9}} = -.0139\times10^{6}$$

$$e^2{}_{\Delta\dot{x}\phi y}(0) = \frac{2\beta e^2}{Re} I_3 = -\frac{2(2\times10^{-3})(25)}{20.4\times10^{6}}\, 1.39\times10^{4}$$

$$= -\ 6.65\times10^{-5}$$

Therefore,

$$A = \left\{ \frac{e^2{}_{\Delta\dot{x}\phi y}(0)}{\sqrt{e^2{}_{\Delta\dot{x}\Delta\dot{x}}(0)\, e^2{}_{\phi_y\phi_y}(0)}} \right\}^{1/2}$$

$$= \left\{ \frac{6.65\times10^{-5}}{\sqrt{11.8 \times .0192\times10^{-6}}} \right\}^{1/2}$$

$$= \sqrt{14\times10^{-2}} = .374$$

and

$$B = \left\{ 1 - A^2 \right\}^{1/2}$$

$$= \left\{ 1 - .14 \right\}^{1/2} = .93$$

Then, the correlated and uncorrelated system velocity and platform tilt errors are;

$$\Delta\dot{x}_{cor} = .374\,(3.44) = 1.29 \text{ fps}$$

$$\Delta\dot{x}_{unc} = .930\,(3.44) = 3.20 \text{ fps}$$

$$\Phi_{ycor} = .374\,(.1385 \times 10^{-3}) = .0518 \times 10^{-3} \text{ rad/sec.}$$

$$\Phi_{yunc} = .930\,(.1385 \times 10^{-3}) = .130 \times 10^{-3} \text{ rad/sec.}$$

Using a flight time, t = 20 min., the x propagation equation, and remembering that since $e^2_{\Delta\dot{x}\Phi_y}$ is negative, the signs of the correlated errors are opposite, then;

$$\Delta x = \Delta\dot{x}\,\frac{\sin \omega_s t}{\omega_s} - R_e\Phi_y\,(1 - \cos \omega_s t)$$

$$\Delta x_{cor} = (1.29)815 - \left[-(20.9 \times 10^6)(.0518 \times 10^{-3})(.88)\right]$$

$$= 1050 + 955 = 2005 \text{ ft}$$

$$\Delta x_{unc} = (3.2)(815) - (20.9 \times 10^6)(.130 \times 10^{-3})(.88)$$

$$= 2610 - 2390$$

Bias Error

From equations (35) and (36)

$$\Delta x_{ss}\Big|_{s\to 0} = s\,G(s)\,\frac{\delta V_x(s)}{s} = \delta V_x\,\frac{K_2}{1+K_2}$$

$$\phi_{Y_{SS}} = S\,H(s)\,\frac{\delta V_X(s)}{S} = \delta V_X(s)\,\frac{K_1}{a\left[\omega_S^2(1+K_2)\right]}$$

$$\delta V_{X_{bias}} = 2 \text{ fps}$$

$$\Delta X_{SS} = \frac{(2)(1)}{2} = 1 \text{ fps}$$

$$\phi_{Y_{SS}} = \frac{2\times10^{-3}}{20.9\times10^{6}\,(1.5\times10^{-6})(2)} = \frac{10^{-3}}{31.4}$$

$$= .0318\times10^{-3} \text{ rad.}$$

$$= \frac{31.8\times10^{-6}}{4.84\times10^{-6}} = 6.5 \text{ sec}$$

$$\Delta X_{bias} = 1(815) - (.0318\times10^{-3})(20.9\times10^{6})(.88)$$

$$= 815 - 585 \quad 230 \text{ ft}$$

Then,

$$\Delta X_{rms} = \sqrt{(1050+955+230)^2 + (2610)^2 + (2390)^2}$$

$$= 4180 \text{ ft}$$

Similarly,

$$\Delta Y_{rms} = \sqrt{(1780+230)^2 + (1660)^2 + (1580)^2}$$

$$= 3050 \text{ ft}$$

And, finally,

$$CEP = .589\,|4180 + 3050|$$

$$= 4250 \text{ ft}$$

Appendix H

Limiting Values of System Velocity Error and Platform Tilt

In Appendix G, it is shown that

$$e^2_{\Delta\dot{x}\Delta\dot{x}}(0) = 2\beta e^2 I_3$$

where

$$I_3 = \frac{-a_2 b_0 + a_0 b_1 - \dfrac{a_0 a_1 b_2}{a_3}}{2a_0(a_0 a_3 - a_1 a_2)}$$

and

$$a_0 = 1 \qquad b_0 = 0$$

$$a_1 = (K_1 + \beta) \qquad b_1 = -K_1^2$$

$$a_2 = \omega_s^2(1+K_2) + K_1\beta \qquad b_2 = K_2^2\omega_s^4$$

$$a_3 = \beta\omega_s^2(1+K_2)$$

Therefore,

$$e^2_{\Delta\dot{x}\Delta\dot{x}}(0) = \cancel{2}\beta e^2 \left[\frac{-K_1^2 - \dfrac{(K_1+\beta)\omega_s^4 K_2^2}{\beta\omega_s^2(1+K_2)}}{\cancel{2}\left\{\beta\omega_s^2(1+K_2) - (K_1+\beta)\left[\omega_s^2(1+K_2) + K_1\beta\right]\right\}}\right]$$

$$= \cancel{\beta} e^2 \left[\frac{-K_1\beta(1+K_2) - (K_1+\beta)\omega_s^2 K_2^2}{\cancel{\beta}(1+K_2)\left[-K_1K_2\omega_s^2 - K_1\omega_s^2 - K_1\beta - K_1\beta^2\right]}\right]$$

$$\lim_{\beta\to 0} e^2_{\Delta\dot{x}\Delta\dot{x}}(0) = e^2\left[\frac{-K_1K_2^2\omega_s^2}{(1+K_2)(-K_1K_2\omega_s^2 - K_1\omega_s^2)}\right]$$

$$= e^2\left[\frac{K_1K_2^2\omega_s^2}{K_1\omega_s^2(1+2K_2+K_2^2)}\right]$$

$$= e^2\left[\frac{K_2^2}{(1+2K_2+K_2^2)}\right]$$

Since $K_2 = 1$

$$\lim_{\beta \to 0} e^2_{\Delta\dot{x}\Delta\dot{x}}(0) = .25 e^2$$

$$\lim_{\beta \to 0} e_{\Delta\dot{x}\Delta\dot{x}}(0) = .50 e \text{ fps}$$

To find limit as $\beta \to \infty$, divide through by highest power of β in denominator.

$$e^2_{\Delta\dot{x}\Delta\dot{x}}(0) = e^2 \left[\frac{-\frac{K_1^2 \beta}{\beta^2}(1+K_2) - (K_1+\beta)\frac{\omega_s^2 K_2^2}{\beta^2}}{\frac{(1+K_2)}{\beta^2}(-K_1 K_2 \omega_s^2 - K_1 \omega_s^2 - K_1 \beta - K_1 \beta^2)} \right]$$

$$\lim_{\beta \to \infty} e^2_{\Delta\dot{x}\Delta\dot{x}}(0) = 0$$

Similarly,

$$e^2_{\phi_y\phi_y}(0) = \frac{2\beta e^2}{R_e^2} I_3$$

$a_0 = 1$

$a_1 = (K_1 + \beta)$

$a_2 = \omega_s^2(1+K_2) + K_1\beta$

$a_3 = \beta\omega_s^2(1+K_2)$

$b_0 = 0$

$b_1 = -K_2^2$

$b_2 = K_1^2$

$$e^2_{\phi_y\phi_y}(0) = \frac{2\beta e^2}{R_e^2} \left[\frac{-K_2^2 - \frac{(K_1+\beta)K_1^2}{\beta\omega_s^2(1+K_2)}}{2\left\{\beta\omega_s^2(1+K_2) - (K_1+\beta)\left[\omega_s^2(1+K_2)+K_1\beta\right]\right\}} \right]$$

$$= \frac{\beta e^2}{R_e^2}\left[\frac{-K_1\beta\omega_s^2(1+K_2)-(K_1+\beta)K_1^2}{\beta\omega_s^2(1+K_2)(-\omega_s^2K_1-\omega_s^2K_1K_2-K_1^2\beta-K_1\beta^2)}\right]$$

$$\lim_{\beta\to 0} e^2_{\phi_y\phi_y}(0) = \frac{e^2}{R_e^2}\left[\frac{-K_1^3}{\omega_s^2(1+K_2)(-\omega_s^2K_1-\omega_s^2K_1K_2)}\right]$$

$$= \frac{e^2}{R_e^2}\left[\frac{-K_1^3}{-K_1\omega_s^4(1+2K_2+K_2^2)}\right]$$

$$= \frac{e^2}{R_e^2}\left[\frac{K_1^2}{\omega_s^4(1+2K_2+K_2^2)}\right]$$

Since $K_1 = 10^{-3}\ \text{sec}^{-1}$ and $K_2 = 1$,

$$\lim_{\beta\to 0} e^2_{\phi_y\phi_y}(0) = \frac{e^2}{(20.9\times10^6)^2}\left[\frac{10^{-6}}{(1.5\times10^{-6})^2(4)}\right]$$

$$= 2.55\times10^{-10}e^2$$

$$\lim_{\beta\to 0} e_{\phi_y\phi_y}(0) = 1.6\times10^{-5}e \text{ rad.}$$

To find the limit as $\beta\to\infty$, divide through by the highest power of β in the denominator.

$$e^2_{\phi_y\phi_y}(0) = \frac{\beta e^2}{R_e^2}\left[\frac{-\frac{K_1\beta\omega_s^2(1+K_2)}{\beta^2}-\frac{(K_1+\beta)K_1^2}{\beta^2}}{\frac{\beta\omega_s^2(1+K_2)}{\beta^2}(-\omega_s^2K_1-\omega_s^2K_1K_2-K_1^2\beta-K_1\beta^2)}\right]$$

$$\lim_{\beta\to\infty} e^2_{\phi_y\phi_y}(0) = 0$$

For the correlated initial system velocity and platform tilt error to equal zero,

$$A = \left\{ \frac{e^2_{\Delta\dot{x}\phi_Y}}{\sqrt{e^2_{\Delta\dot{x}\Delta x}(0)\, e^2_{\phi_Y \phi_Y}(0)}} \right\}^{1/2} = 0$$

or

$$e^2_{\Delta\dot{x}\phi_y}(0) = \frac{2\beta e^2}{Re} I_3 = \frac{2\beta e^2}{Re} \left[\frac{-a_2 b_0 + a_0 b_1 - \frac{a_0 a_1 b_2}{a_3}}{2a_0(a_0 a_3 - a_1 a_2)} \right] = 0$$

where

$a_0 = 1$ $\qquad b_0 = 0$

$a_1 = (K_1 + \beta)$ $\qquad b_1 = K_1 K_2$

$a_2 = \omega_s^2 (1 + K_2) + K_1 \beta$ $\qquad b_2 = K_1 K_2 \omega_s^2$

$a_3 = \beta \omega_s^2 (1 + K_2)$

$$K_1 K_2 - \frac{(K_1 + \beta) K_1 K_2 \omega_s^2}{\beta \omega_s^2 (1 + K_2)} = 0$$

$$K_1 K_2 \beta \omega_s^2 (1 + K_2) = (K_1 + \beta) K_1 K_2 \omega_s^2$$

$$K_1 K_2 \beta (1 + K_2) - \beta K_1 K_2 = K_1^2 K_2$$

$$\beta = \frac{K_1^2 K_2}{K_1 K_2 (1 + K_2) - K_1 K_2}$$

$$= \frac{K_1^2 K_2}{K_1 (K_2 + K_2^2 \quad K_2)}$$

$$= \frac{K_1}{K_2}$$

Since $K_1 = 10^{-3}$ and $K_2 = 1$,

$$\beta = 10^{-3} \text{ sec}^{-1}$$

Appendix J

Coordinate Systems

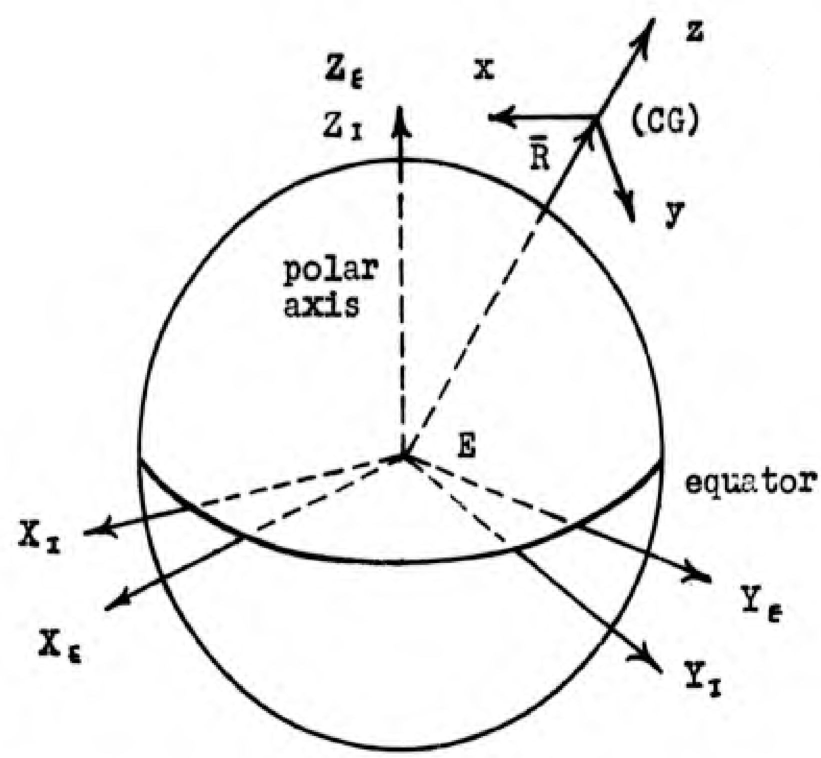


Figure 9A Reference Coordinate Systems

The inertial reference coordinate system has its origin assumed at the center of the earth and is non-rotating with respect to inertial space. The Z_I axis is northward along the polar axis and the X_I and Y_I axes are arbitrarily located in the equatorial plane, forming an orthogonal, right-handed system.

The earth reference coordinate system has its origin and Z_E axis coincident with the inertial frame, but the axes are fixed in the earth and are, therefore, rotating with respect to the inertial reference frame with an angular velocity equal to earth rate, ω_{IE}. The X_E and Y_E axes are also arbitrarily oriented in the equatorial plane to make the system orthogonally right-handed.

The missile coordinate system is also an orthogonal, right-handed system with the origin located at the center of gravity (CG) of the missile. The z axis is directed upward along the local vertical and the x axis along the longitudinal axis of the missile. The y axis is located in the horizontal plane to make the system right-handed.

Vita

Joseph [redacted] Crocco [redacted] After completing primary and secondary school he was admitted to the United States Military Academy in July 1947. He was graduated in June 1951 with a Bachelor of Science degree and entered the United States Air Force. His military assignments included a tour of combat duty in Korea and navigation instructor in the Air Training Command. In August 1960 he was assigned to the Institute of Technology.

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This thesis was typed by Marjorie M. Keppler.

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