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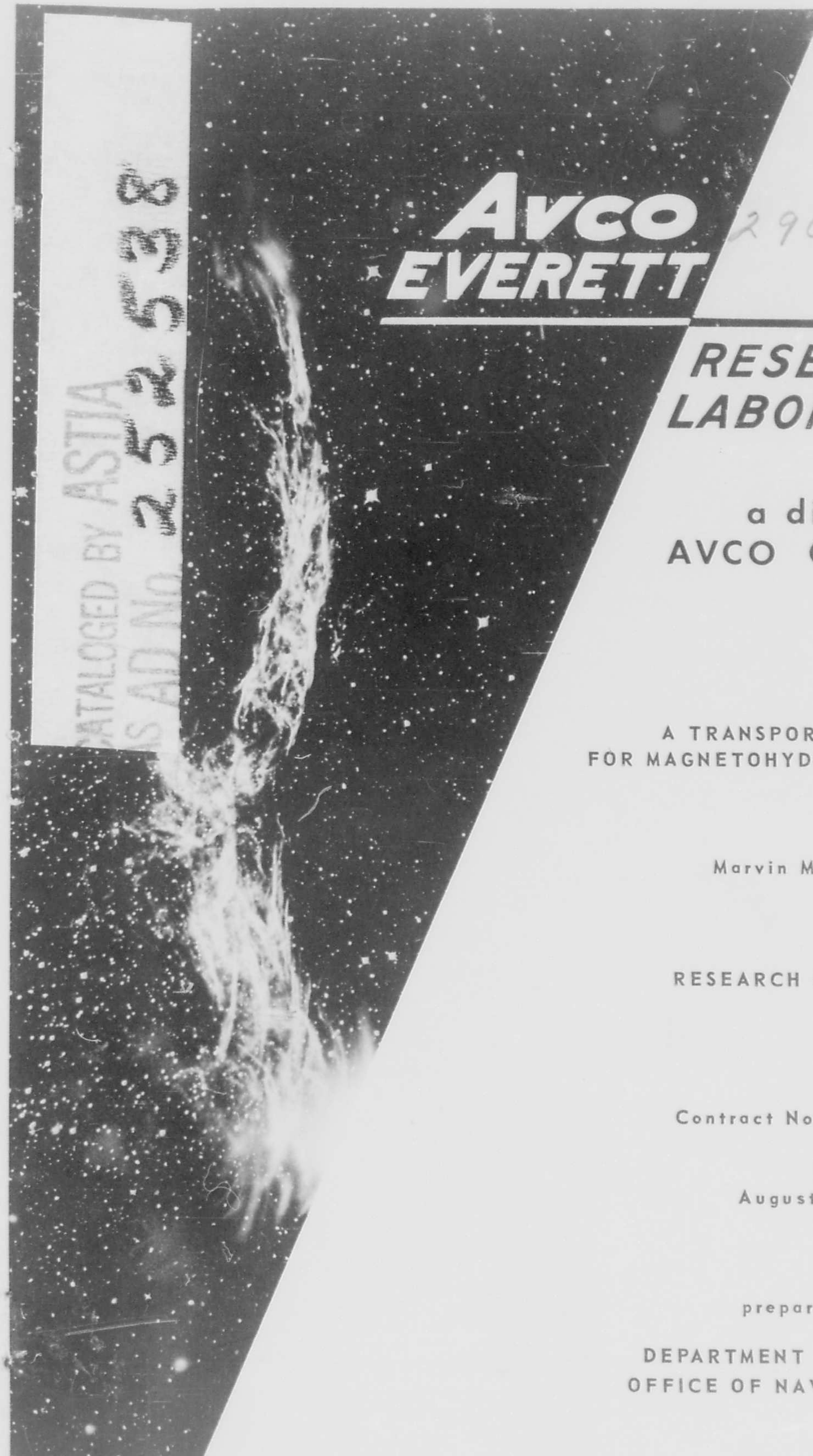

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A TRANSPORT EQUATION FOR MAGNETOHYDRODYNAMIC WAVES

Marvin M. Litvak

RESEARCH REPORT 92

Contract Nonr-2524(00)

August 1960

prepared for

DEPARTMENT OF THE NAVY
OFFICE OF NAVAL RESEARCH

AD 252 538

ERRATA

"A Transport Equation for Magnetohydrodynamic Waves"

Marvin M. Litvak

Avco-Everett Research Laboratory Research Report 92
Avco-Everett Research Laboratory, Everett, Mass.

PAGE	
27	In (3.16), there is no "z" subscript on m_k.
31	(4.11) is: $\hat{E}_T + \frac{1}{c}\left(\hat{u}_T - \frac{m}{e\rho_T}\hat{j}_T\right) \times \hat{B}_T = 0$
41	In Fig. 1 : ω", not W" ; k_1 , not $k_\perp$
49	Line 14: Read "..... and of not too low temperature....."
51	Line 7: Read "..... Λ_k k then is of the order of magnitude of β^{-1}"
	Line 13: Read "..... compared to k^{-1}...."
54	The left-hand side of (5.8) is $\frac{\partial}{\partial t}\lvert A_k(t)\rvert^2$.
55	Footnote: In the definition of H_{1k} replace "M_{kpq}" by "$iS_{kpq}\sqrt{\frac{\omega_p^0\omega_q^0}{\omega_k^0}}$"
56	In (5.11), M_{kpq} is half of the expression given if p, q, k refer to the same mode.
58	In (5.15), M_{kpq} is half the expression given.
60	Footnote, line 3: Sum over k, as well as p and q.
63	In Fig. 2: q, not g

AD 252 538

ERRATA (continued)

Avco-Everett Research Laboratory Research Report 92

Marvin M. Litwak

PAGE

21 Equation (3.4) has " + i " on the right-hand side instead of "-i".

24 Equations (3.14) are written in terms of the following non-dimensional quantities:

$$\hat{B} = \frac{1}{B_o} \text{ x magnetic field}$$

$$\hat{E} = \frac{1}{B_o} \text{ x } \frac{c}{V_A} \text{ x electric field}$$

$$\hat{u} = \frac{1}{V_A} \text{ x fluid velocity}$$

25 Equation (3.18): Right-hand side is " $-\frac{1}{c}\frac{\partial \hat{B}}{\partial t}$ "

27 Equation (3.15), (3.16), (3.17) should be numbered (3.23), (3.24), (3.25). Equation (3.17), renumbered,(3.25),has ω_{c_i} , not ω_{ce_i} .

38 Renumber Equation (4.28) as (4.28').

39 Call the equation which follows line 8, Equation (4.28).

50 Line 11 should read: "... the potential of a collection of simple harmonic..."

65 Line 13 should read: " ω_k is fixed; ω_q is a constant..."
In Equation (5.29), $x = P_z/k_z = \sqrt{\frac{V_A r_i P_z^2}{\omega_k}}$

RESEARCH REPORT 92

A TRANSPORT EQUATION FOR MAGNETOHYDRODYNAMIC WAVES

by

Marvin M. Litvak

AVCO-EVERETT RESEARCH LABORATORY
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Everett, Massachusetts

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DEPARTMENT OF THE NAVY
OFFICE OF NAVAL RESEARCH
Washington, D. C.

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ABSTRACT

We deal with a plasma of electrons and ions at a density of 10^{16} particles per cc and temperature 10^5 to 10^6 degrees K. There is a magnetic field of about 10^4 gauss whose pressure is about 10^1 or 10^2 times larger than the gas pressure.

We limit our attention to physical phenomena occurring over lengths of the order of $r_i = \sqrt{m_i c^2/(4\pi ne^2)}$ which is about one centimeter or about 10^4 times larger than the Debye length. r_i, the largest characteristic length next to the particle mean free path, is the gyro-radius of ions which move at the Alfvén speed, the characteristic speed of magnetohydrodynamic (MHD) flows.

The mean free path for multiple Coulomb scattering, the only particle collision process for these plasma conditions, is large so that diffusion and dissipation processes due to particle collisions are slow. However, experimental evidence exists which indicates high diffusion and dissipation rates. We attribute these processes to the diffusion and randomization of waves which are excited in the plasma.

In particular, the important waves are the fast MHD waves, which are one of the six types of normal waves which we obtain from the Boltzmann equation with a self-consistent Lorentz force and no collision term. The fast waves were derived under the restrictions that the particle thermal velocities are unimportant, that the wavelengths lie between $r_i \sqrt{m_e/m_i}$ and r_i, and that the Maxwell displacement current is small. The fast waves are important because they are not heavily damped, and they have phase and group velocities of the order of the Alfvén speed.

Our main effort is to derive a wave transport equation for the fast MHD waves which describes the motion of a wave in a non-uniform medium, the amplification of a wave because of the pressure of the surrounding medium, and the scattering of a wave by other waves. This equation resembles the transport equation for lattice waves in a crystal.

The equations for the structure of a steady MHD shock are derived from the wave transport equation but not solved. The shock model is based on the assumption that behind the steady shock is a distribution of fast waves whose pressure is much greater than the gas pressure. The waves scatter with other waves and provide the entropy increase for the jump of conditions across the shock. The particles become thermalized only some distance further back of the shock by means of damping of wave motion into particle motion. Estimates of relaxation times for the wave collision processes are made, and an estimate of the shock thickness, as a function of shock velocity, is made assuming that it is a few wave mean free paths.

ACKNOWLEDGMENTS

The author wishes to thank Dr. Harry E. Petschek for his guidance, patience, and considerable technical help concerning the research reported here and the preparation of the manuscript. The author sincerely thanks Prof. Edwin E. Salpeter, for his advice, encouragement, and technical help especially in his role as critic of the research and of the manuscript. The author thanks Dr. Arthur R. Kantrowitz, for his encouragement and for extending to the author the hospitality of the Avco-Everett Research Laboratory.

The author acknowledges the helpful discussions with Dr. Frank Fishman and the computational work done by Miss Petrina Biondo.

The author acknowledges the financial support of the National Science Foundation and the Office of Naval Research, under Contract Nonr-2524(00).

TABLE OF CONTENTS

SECTION I

INTRODUCTION

PLASMA CONDITIONS

The medium that we deal with is an over-all neutral plasma consisting of electrons and ions and a negligible number of neutral particles. The density, n, is about 10^{16} particles per cubic centimeter. The temperature, T, is about 10^1 to 10^2 e.v. (10^5 to 10^6 degrees K.). *

The Debye length (shielding distance) h is then about 10^{-3} to 10^{-4} cm. N(h), the number of particles in a cube of side equal to the Debye length, is about 10^5 particles. The formulae for h and N(h) are the following:

$$h = \sqrt{kT/4\pi ne^2}$$

$$N(h) = nh^3 = \left(\frac{4\pi e^2 n^{1/3}}{kT}\right)^{-\frac{3}{2}} \sim \frac{kT}{e^2/h} \qquad (1.1)$$

where k is Boltzmann's constant.

Because N(h) is large, the Coulomb interaction energy between nearest electrons, the separation distance being approximately $n^{-\frac{1}{3}}$, will be small compared to kT. Under these circumstances the multiple Coulomb scattering through small angles is the only particle collision process. The mean free path for appreciable deflection by this process is of the order of $hN(h)/\ln N(h)$ which is between one and ten centimeters.

The plasma contains a large magnetic field, B_o, equal to about 10^4 gauss. The gas pressure is about 10^{-2} to 10^{-1} times smaller than the magnetic pressure $B_o^2/8\pi$. The ions have a gyro-radius of about 10^{-2} cm. if they move with their thermal speed, about 10^6 cm/sec. The gyro-radius is about 3×10^{-1} cm. for ions moving at the Alfvén speed $V_A = \sqrt{B_o^2/4\pi\rho_o}$, the characteristic speed of small amplitude MHD (magnetohydrodynamic)

* The conditions of density, temperature, and magnetic field need not be restricted to the values indicated here. For the theory to apply, only the non-dimensional parameters $(nkT)/\left(\frac{B_o^2}{8\pi}\right)$, $(nh^3)^{-1}$, $(kT/m_e c^2)^{1/2}$, r_e/L, need be specified as small compared to one, where $r_e = h\left(\frac{m_e c^2}{kT}\right)^{1/2}$ and L is the characteristic size of the apparatus, flow, etc.

signals or waves. The electron gyro-radius based on thermal speed is about 10^{-3} cm., and, based on Alfvén speed, it is about 10^{-2} cm.

We concern ourselves with physical situations in which the average properties of the plasma vary appreciably only over lengths that are smaller than the particle mean free path but comparable to the ion gyro-radius based on Alfvén speed. This gyro-radius, $r_i = \sqrt{m_i c^2/4\pi n e^2}$, is the largest characteristic length next to the particle mean free path. Theoretical interest in this region of plasma behavior arose mainly from the speculation that this gyro-radius is the scale of dissipative effects that are more important for this plasma than particle collisions. Considerable experimental work in this region, particularly by means of MHD shock tubes, lends support to this speculation.[6,25]

BOLTZMANN EQUATION

The starting point of our analysis is the Boltzmann equations for electrons and ions. The equations are as follows:

$$\left(\frac{\partial}{\partial t} + \hat{V}\cdot\frac{\partial}{\partial \hat{r}} + \hat{F}_{e,i}\cdot\frac{\partial}{\partial \hat{V}}\right) f_{e,i}(\hat{r},\hat{V},t) = \left(\frac{\partial f_{e,i}}{\partial t}\right)_{\text{collisions}}$$

where $f(\hat{r},\hat{V},t)\, d^3r\, d^3V$ is the number of particles whose positions lie between $\hat{r}$ and $\hat{r} + d\hat{r}$ and whose velocities lie between $\hat{V}$ and $\hat{V} + d\hat{V}$. f is called the one-particle distribution function. The subscripts denote electron, e, and ion, i. $\hat{F}_{e,i} = \mp \frac{e}{m_{e,i}} (\hat{E} + \frac{\hat{V}}{c} \times \hat{B})$, the Lorentz force per unit mass. $\hat{E}(\hat{r},t)$ is the electric field and $\hat{B}(\hat{r},t)$ is the magnetic field acting on a particle at $\hat{r}$ at time t. $\hat{E}$ and $\hat{B}$ are the self-consistent fields which satisfy the Maxwell equations which contain, as source terms, the charge and current due to the average motion of the particles. The Maxwell equations and the Boltzmann equations must be solved simultaneously so that the distribution functions $f_{e,i}$ give average charges and currents which determine $\hat{E}$ and $\hat{B}$ which fit the Boltzmann equations that $f_{e,i}$ satisfy. The average charge density is $e\int d^3V \left[f_i(\hat{r},\hat{V},t) - f_e(\hat{r},\hat{V},t)\right]$ and the average current density is $e\int d^3V\ \hat{V}\left[f_i(\hat{r},\hat{V},t) - f_e(\hat{r},\hat{V},t)\right]$.

For the density and temperature range of interest, the right-hand side $\left(\frac{\partial f_{e,i}}{\partial t}\right)_{\text{collisions}}$ is the rate of change of the distribution functions due to multiple Coulomb scatterings through the small angles. The mean free path for this collision process has been shown to be large compared to the characteristic lengths of the flow field, the magnetic field, and the electrostatic field. The lengths for the magnetic field, we recall, are the gyro-radii and for the electrostatic field, the Debye length. It can be shown that the relative importance of the terms of the Boltzmann equation is given by the ratios of the corresponding characteristic lengths.[18] From this we conclude that the collision term can be neglected as small compared to the other terms of the Boltzmann equation. The Boltzmann equation with zero collision term is called the Vlasov[24] equation, and we make exclusive use of it hereafter.

NORMAL WAVES

The Vlasov and Maxwell equations may be linearized by supposing that each particle distribution function f is the sum of a time-independent distribution f_o, usually a Maxwell velocity-distribution, plus a small amplitude disturbance f_1, $|f_1| \ll |f_o|$. Terms in the equations which depend upon powers of f_1 higher than the first are omitted. The resulting linear equations for f_1 can then be solved by Fourier or Laplace transform methods. Gross[11] and Bernstein[2] have dealt with waves in the electron distribution which propagate parallel or perpendicular to the uniform magnetic field. Considerable mathematical complication arises when arbitrary direction of propagation of waves, which involve both electrons and ions, is dealt with. We avoid these difficulties by taking moments of the Vlasov equations in velocity space and then use the resulting magnetohydrodynamic equations to obtain the normal waves and the nonlinear interaction of the waves. Oster has already pointed out for several cases the equivalence of the two methods for obtaining the normal waves of the plasma; one method consists in solving for the velocity distribution f_1 from the linearized Vlasov equation and the other method consists in solving the linearized MHD equations in terms of the macroscopic properties like fluid-velocity, particle density, and the self-consistent fields. In the first method the choice of f_o as the Maxwell distribution introduces temperature into the wave properties. Other choices of f_o require careful examination, as discussed by van Kampen[23] and Backus[1]. Oster[15] explains by showing a more general solution why the "transmission gaps" at multiples of the electron gyro-frequency obtained by Gross and Bernstein for electron waves by the distribution-function method do not disprove the equivalence of the two methods despite the absence of such transmission gaps in the MHD moment method.

Considerable discussion by van Kampen[22] and Backus[1] concerning the relationship between different methods of solution of the linearized equations for the above-mentioned simple cases have also touched upon the following topics: the relationship of the Laplace transform method used by Landau[13] to the Fourier transform or superposition of normal waves method used by van Kampen[22] for the solution of initial-value problems; the nature of the singular integral equation which is the dispersion relation for plasma oscillations and the correct choice of the path of integration for this dispersion relation, obtained earlier by Vlasov[24], Bohm and Gross[4], Gordeev[10], and many others.

Properties of normal waves are described by Gershman, Ginzburg, and Denisov[9] mainly by means of solutions of moment equations similar to ones we use. These authors indicate certain discrepancies between the two methods, mentioned before, of obtaining normal waves. These discrepancies arise from the approximate treatment in the moment method of the particle pressure tensors, which introduce the mean thermal velocities into the dispersion relations. These authors describe the effects of particle collisions, which we ignore, and the thermal velocities on wave phase velocity. Details are provided only for the cases of propagation parallel or perpendicular to the magnetic field. MHD waves are discussed for these special propagation direction for low frequencies (less than the ion gyro-frequency) and long wavelengths (greater than the ion gyro-radius). This case has been treated for general directions of propagation and infinite conductivity by Friedrichs[7]. Since we deal with waves of shorter wavelength and higher frequency we cannot use the results of these authors, and we derive from moment equations expressions for the normal waves which make no assumption at first concerning the wavelength or frequency range. We then deal with the case of arbitrary propagation direction of MHD waves with wavelengths greater than r_e and frequencies less than the electron gyro-frequency.

The Vlasov equation enables us to write down the continuity equations for average number-density and particle velocity for electrons and ions. The velocity equation contains the particle pressure tensor $\mathbb{P}_{ij}$ as well as the magnetic pressure tensor. As mentioned earlier, the gas pressure is about 10^{-2} to 10^{-1} times smaller than the magnetic pressure. Hence, we neglect the pressure tensor. Because of this assumption of zero gas pressure, the continuity equations for number-density and particle velocity, together with the Maxwell equations, form a set of simultaneous equations with as many unknowns as equations. Without the assumption of zero pressure, we would have had too many unknowns and the introduction of additional equations from the Vlasov equations by taking higher velocity moments would add still too many unknowns. The Vlasov equations and the Maxwell equations do form a solvable set of equations but complicated compared to the two moment equations and the Maxwell equation.

The equations are solved for the small-amplitude waves of the periodic form $e^{i(\hat{k}\cdot\hat{x}-\omega t)}$, where $\hat{k}$ is the wave-vector and ω is the corresponding frequency. The non-linear terms of the equations are neglected for the while since the wave amplitudes are assumed small. A dispersion relation $\omega = \omega(\hat{k})$ is derived which shows that there are six modes with complicated orthogonal polarization vectors of electric, magnetic and particle velocity fields. In order to simplify the wave properties and in order to describe physical phenomena of order of magnitude equal to the ion gyroradius based on Alfvén speed, $r_i = \sqrt{m_i c^2/4\pi ne^2}$, as we mentioned before was of interest, we restrict ourselves to waves with wavelength of this same order of magnitude. Furthermore, if we restrict ourselves to flows with velocities of the order of the Alfvén speed, which is the main concern of magnetohydrodynamics, we find only two of the wave modes are interesting. The others, for phase velocities of the order of the Alfvén speed, have wavelengths several orders of magnitude less than r_i.

It can be shown that the effects of Landau damping can be discussed qualitatively by noticing for what wavelengths $\omega \pm \omega_{c_{e,i}}$ first becomes comparable to k times the ion or electron thermal velocities, where $\omega_{c_{e,i}}$ is the gyrofrequency for electrons or ions, respectively. The signs $\pm$ are determined by the sense of polarization of the electric vector relative to the gyromotion. Wavelengths less than this will be appreciably damped. The quantitative expressions for Landau damping can be obtained only from the Vlasov equation itself.

The two interesting wave modes differ not only in their dispersion curves but also in their damping behavior. For small values of the wave vector component along the magnetic field direction, compared to r_i^{-1}, one wave moves with the Alfvén speed in all directions and the other moves with the slower Alfvén speed based upon the component of magnetic field parallel to the wave vector. As the wave number increases, the phase velocity of the faster wave increases while that of the slower one decreases. As the wave number increases the slower one is more heavily damped by the ions; its frequency is monotonically approaching the ion cyclotron frequency at the same time. The fast wave is not heavily damped by the ions for $r_i^{-1} < k < r_e^{-1}$ but becomes heavily damped by the electrons as k approaches r_e^{-1}; the frequency monotonically approaches the electron cyclotron frequency. The electric vector of the slow wave rotates predominantly with the gyromotion of the ions, while that of the fast wave is predominantly the opposite. These points are discussed in detail later on.

WAVE TRANSPORT EQUATION

We have learned from the example of electrostatic plasma oscillations that wave-motion may be a more useful description of plasma behavior than the picture of particles in random thermal motion interacting by electromagnetic forces, when we treat effects occurring over distances larger than the appropriate cut-off distance.[5] For the plasma oscillations this cut-off is the Debye length. For the faster of the two MHD waves, which we introduced earlier, the cut-off is $r_e = \sqrt{m_e c^2/4\pi n e^2} = \sqrt{m_e c^2/kT}$ times the Debye length. $\sqrt{m_e c^2/kT}$ is approximately 10^2 for $kT = 10$ e.v. However, the slower MHD wave is cut off at wavelengths less than $r_i = \sqrt{m_i c^2/4\pi n e^2}$. The physical scale-length of the order of r_i and, furthermore, the scale-velocity of the order of V_A, the Alfvén speed, which we are considering here, make the fast MHD wave the one most likely to provide an interesting wave description, since it is not heavily damped and is the only one with the appropriate phase and group velocity range, for the wavelengths in question.

It is shown later that if equilibrium exists between particles and waves, then the energy invested in fast MHD waves is about 10^{-12} times smaller than that invested in the thermal motion of particles.

However, we find that a wave-packet moving in the direction of increasing density of a non-uniform plasma will be compressed and its wave energy will be increased. We also find that the waves scatter with other waves because the medium behaves non-linearly. If the non-uniformity occurs over a distance ℓ and if the relaxation time associated with the change of wave amplitude due to the scattering process is τ such that a velocity ℓ/τ is comparable to the group velocity of the wave-packet, then considerable non-isentropic growth of wave energy can occur at the expense of the energy in the non-uniformity. In the case that the non-uniformity is an MHD shock propagating with a velocity two or three times the Alfvén speed the wave pressure is about one-tenth the large magnetic pressure $B_o^2/8\pi$ of the plasma. This wave pressure is, therefore, about ten times the gas pressure. The gas pressure is then unimportant compared to the wave pressure.

The compression of the wave-packet due to a density gradient in the plasma is examined in detail by describing the propagation of a single fast MHD wave of constant frequency. The non-uniformities are taken as stationary gradients, in a direction perpendicular to the applied magnetic field, of the density, average particle velocity, and magnetic field. The amplitude and wave-vector are functions of position in the plasma. It is found that the amplitude becomes infinitely large at places where the group velocity is zero. If the non-uniformity is a compression front or shock wave, then the description is made in the coordinate system moving with the front or shock.

We consider the plasma as being divided into small cubes whose side-length L is small compared to the distance ℓ over which the plasma properties vary appreciably. In each cube, then, the plasma appears uniform. We apply periodic boundary conditions at the cube faces. The waves in a cube are then the normal waves of a uniform plasma, which is not necessarily at rest. The approximation introduced by the use of normal waves of a uniform plasma is that the space Fourier components of the non-uniformities are small for wavelengths comparable with those of the waves which we consider. We eventually average all effects over the volume of the cube.

Because of the non-linear terms in the equations of motion, the fields of different normal waves in a cube interact. The non-linear terms can be represented as the effect of anharmonic corrections to the potential of a harmonic oscillator, the normal wave being the oscillator.[20] In solid state physics, the scattering of lattice waves has been dealt with in a similar fashion.[17] The rate of change of the amplitude of the wave with wave-vector $\hat{k}$ is the sum over $\hat{p}$ of the product of the amplitudes of the waves with wave vector $\hat{p}$ and $\hat{k}-\hat{p}$ with a coupling coefficient. An approximate expression for the rate of change of amplitude is obtained for times long compared to the periods of the waves but short compared to the time τ in which an appreciable change of the amplitude occurs. This approximate expression is the first term in a perturbation expansion in powers of β the ratio of wave pressure to magnetic pressure $B_o^2/8\pi$. β is about one-tenth for waves behind an MHD shock wave moving with twice the Alfvén speed. The further approximation is made that the complex wave amplitudes are statistically independent, i.e., the phases are randomly distributed. This random phase approximation excludes a term that is at least $\frac{kT}{r_e^3\, B_o^2/8\pi}$ times smaller than the mean wave energy kT if the phases were correlated according to the wave-interaction energy at the equilibrium temperature T. $\frac{kT}{r_e^3\, B_o^2/8\pi}$ is about 10^{-14}. The physical justification for the random phase approximation is that three waves which scatter once do not retain the phase correlations, which they produce in the interaction, by the time the next scattering of these three waves occurs. The effects of wave scattering which are omitted by the random phase approximation are shown in Appendix 3 to be approximately a factor $(\omega\tau)^{-1}$ smaller than the effects which are retained, where ω is the frequency of the wave in question and τ is the relaxation time for wave amplitude changes. It is shown that the phase changes at a faster rate than the amplitude so that the phases of the waves most likely randomize in times short compared to τ. This is the same as saying that, Λ, the mean free path for the wave, which is the group velocity times τ, must be large compared to the wavelength, λ. From the estimate of τ that we make later, we conclude that $\Lambda/\lambda \sim 1/\beta$. As mentioned above, β is about 1/10, so that the mean free path is large compared to the wavelength. If we use the descriptive language of particle collisions, we may say that we apply the expression for the rate of change of amplitude, which is valid for times short compared to τ, anew each time there is a wave collision. We treat the collisions as statistically independent processes. We determine the rates from the amplitudes of the colliding waves at or near the time of the collision and not to follow several successive collisions of a wave.*

In a way similar to treatments of lattice waves in a temperature gradient, we derive a transport equation for the fast MHD waves in a density or magnetic field gradient.[17] The left-hand side of the transport equation is the time-rate of change of the wave amplitude of wave-vector $\hat{k}$ due to the motion of the wave, which transports properties to neighboring positions with the wave group velocity, and due to the force exerted by the medium in compressing the wave and amplifying it. These terms are akin to the convective derivative and force terms of the particle Boltzmann equations.

The right-hand side of the transport equation is the rate of change of the wave amplitude due to wave-wave scattering, which we have already described. The transport equation for steady-state non-equilibrium expresses the balance of the three effects: the convection of properties with the group velocity, the amplification of the wave because of the pressure of

* Prigigione and Henin[28] consider the approach to equilibrium of a distribution of waves by means of a Liouville equation for waves with a cubic interaction Hamiltonian, with initial conditions on the wave distribution which allow the definition of extensive and intensive thermodynamic variables at equilibrium. The authors then show that the phase correlation of three waves approaches the equilibrium correlation in two ways:

1) the initial phase correlations die out with a relaxation time which is $\frac{[\Omega_{0,0}]_1}{[\Omega_{3,3}]_1} < 1$ times that for action.
2) the equilibrium phase correlations build up always proportional to the action.

The operators $[\Omega_{0,0}]_1$ and $[\Omega_{3,3}]_1$ are the contributions of the cycle and the cycle plus three freely propagating lines, respectively. Their eigenvalues are negative, the second operator having more negative eigenvalues.

the surrounding medium, and the scattering into and out of the wave-state because of waves with different wave-vector at a given position in the medium.

COLLISIONLESS MHD SHOCK STRUCTURE

We then apply the transport equation to the problem of a steady MHD shock wave moving perpendicular to the magnetic field lines. A shock wave is usually formed by the steepening of gradual compression fronts. The steepening of a compression front or pulse in a plasma with a magnetic field can be shown to occur for the same reasons that an ordinary aerodynamic pulse steepens. The higher pressure parts of the pulse move with a higher speed than the other parts of the pulse, since the characteristic speed (with respect to the still fluid) of signals depends upon the local properties of the medium including the added effects of the signal itself. For the aerodynamic case, the higher pressure part of the compression pulse moves at a higher sound speed relative to the still fluid than the lower pressure part ahead of it, the temperature and particle velocity being higher at the higher pressure. Therefore, the higher pressure region will catch up with the lower pressure region ahead of it, thereby steepening the front edge of the pulse and flattening the back edge. For the magnetohydrodynamic case, it has been shown that similar effects occur with the Alfvén speed, instead of the sound speed, and the steepening proceeds until some diffusion process becomes sufficiently important to transfer the required momentum and energy and to produce the required entropy by dissipation so that a steady shock profile is attained.[18,•]

Because the particle mean free path is large compared to the ion gyro-radius r_i, a pulse will continue to steepen until its width is narrower than a mean free path. The strong magnetic field has reduced the electron (or ion) viscosity and heat conduction perpendicular to the magnetic field by a factor which is nearly the square of the ratio of electron (or ion) gyro-radius to the mean free path.[21] The reduction of transport properties perpendicular to the magnetic field arises mainly because of the spiraling

• For discussion of the steepening of an ordinary longitudinal magneto-acoustic wave, see L. Davis, R. Lüst, and A. Schlüter.[27]

motion of the electrons (or ions) between collisions which decreases the effective mean free path and diffusion speed, each by a factor of electron (or ion) gyro-radius divided by electron (or ion) mean free path; the product of mean free path and diffusion speed is the important factor in both the viscosity and heat conduction coefficients. Ion-ion collisions seem to dominate viscosity effects because of higher ion momenta, and electron-electron and electron-ion collisions seem to dominate heat conduction effects. So that viscosity is reduced by the square of the factor: ion gyro-radius divided by ion mean free path and heat conduction by the square of the factor: electron gyro-radius divided by electron mean free path.*

When the pulse width becomes comparable to the ion gyro-radius, the wave processes involving the fast MHD waves are estimated to be an important diffusion and dissipation mechanism for inhibiting further steepening of the pulse. Chapter VI deals with the steady state structure of the shock, assuming that the steepening has ended at the ion gyro-radius and that a fast MHD wave distribution exists on the high density side of the shock.

Kahn[12] and Parker[16] have suggested that plasma oscillations provide a dissipative mechanism which predicts shock thicknesses of the order of the Debye length. Kahn considers another but similar physical situation to a shock, that of the interpenetration of two ionized streams. The counterstreaming is stopped by the excitation and amplification of irregular plasma oscillations at the expense of the kinetic energy of the streams. He does not have any magnetic field in the plasma. Since the Debye length is characteristic of wavelengths of electrostatic plasma oscillations which are closely coupled with particle motions, one would expect this dissipation mechanism to produce a shock thickness of that order of magnitude. Parker also considers the interpenetration of two ionized but overall-neutral streams. He then considers two cases: the relative velocity of the streams is greater than or less than the electron thermal velocity. In the first case, the electron components of the two streams interact quickly, the electron streams being stopped by the conversion of their kinetic energy to electron

* In the shock theory of L. Davis, R. Lüst and A. Schlüter,[27] the fundamental dissipative mechanism is still particle collisions. They predict a shock front with a fine structure consisting of a damped wave train having a scale of the order of the geometric mean of the gyro radii of the ions and electrons but the total width of the front is of the order of a mean free path.

plasma oscillations. Then the ions interact with the electron waves, and the ion streams are stopped in an ion plasma period. The thickness of the region of interaction is then of the order of the initial relative velocity multiplied by the ion plasma period. This thickness is somewhat larger than the square-root of the ion-to-electron mass-ratio times the Debye length. In the second case, when the relative velocity of the streams is less than the electron thermal velocity, only the ion clouds interact, again producing dissipative effects in an ion plasma period, and again resulting in a thickness of interacting plasma of the order of the streaming velocity times the ion plasma period. This thickness is of the order of the Debye length. Like Kahn's, Parker's analysis does not include the effects of a magnetic field in the plasma.

Gardner, et al.[8] suggest that a permanent shock structure exists in the form of a series of magnetohydrodynamic pulses whose wavelength is of the order of r_e, the electron gyro-radius based on Alfvén speed. The dissipation has to do with the random phasing of the ion orbits in the series of pulses. These authors conclude that the electrons move adiabatically but the ions do not and are probably heated much more than the electrons.

Both of the shock thicknesses suggested by others are considerably smaller than the ion gyro-radius based on Alfvén speed that we suggest here and we expect that these other shock structures are broadened out by the fast MHD wave dissipative mechanisms.

We derive but do not solve the equations for the steady shock structure. They are the continuity equations for action, magnetic pressor tensor, energy, momentum, and mass assuming a space-varying distribution of waves, and space-varying properties of the plasma, and negligible gas pressure and temperature, as mentioned before. These equations are derived from the wave transport equation. We include the effects of collisions among three fast waves and among two fast waves and one slow wave, which may be important in thermalizing the particles some distance behind the shock and in adjusting the momentum of the fast waves so that they do not escape with their energy ahead of the shock.

Estimated solutions given elsewhere[6] have shown the wave pressure as approximately one-tenth of the magnetic pressure, the number we have quoted many times, for a shock moving with twice the Alfvén speed.

If we consider a distribution of fast MHD waves to be limited to a narrow region of wave-vectors of magnitude approximately r_i^{-1}, and if we say the shock thickness is about two average wave mean free paths, as is the case for aerodynamic shocks and particle mean free path, and if we say the average group velocity of this wave distribution must be equal to the shock velocity in order to keep up with the shock, then the calculated shock thickness fits very well the experimentally determined shock thicknesses, as a function of the shock velocity. The estimate of two mean free paths for the shock thickness is not made from the shock equations, which at present have not been solved, but is inferred from the fact that the mean free path is the distance over which dissipative effects have appreciable influence. Refinements of this estimate by means of the shock equations will be made in a later paper.

SECTION II

EQUATIONS OF MOTION

The Vlasov equations, which are the basis of our analysis, are

$$\left(\frac{\partial}{\partial t} + \hat{V} \cdot \frac{\partial}{\partial \hat{r}} + \hat{F}_{e,i} \cdot \frac{\partial}{\partial \hat{V}}\right) f_{e,i}\,(\hat{r}, \hat{V}, t) = 0 \qquad (2.1)$$

where $\hat{r}$ is the position of the particle, $\hat{V}$ is the particle velocity, and t is the time. $\partial / \partial \hat{r}$ is the gradient operator with respect to position. $\partial / \partial \hat{V}$ is the gradient operator with respect to velocity. $f_{e,i}\,(\hat{r}, \hat{V}, t)$ is the one-particle distribution function. $f_{e,i}\, d^3r d^3V$ is the number of electrons or ions, respectively, having positions in the range $\hat{r}$ to $\hat{r} + d\hat{r}$ and velocities in the range $\hat{V}$ to $\hat{V} + d\hat{V}$, all at time t.

$\hat{r}$ and $\hat{V}$ really refer to the position of a small cell of phase space which contains many more than one particle and over which properties can be smoothed out by averaging. We always deal with large numbers of particles even down to the smallest scale-lengths of our problem.

$\hat{F}_{e,i}$ is the self-consistent Lorentz force acting on the particle.

$$\hat{F}_{e,i} = \mp \frac{e}{m_{e,i}} \left(\hat{E} + \frac{\hat{V}}{c} \times \hat{B}\right) \qquad (2.2)$$

where $\hat{E}$ and $\hat{B}$ are the electric and magnetic fields which satisfy Maxwell's equations:

$$\hat{\nabla} \times \hat{B} = \frac{4\pi}{c}\hat{j} + \frac{1}{c}\frac{\partial \hat{E}}{\partial t} \qquad (2.3)$$

$$\hat{\nabla} \times \hat{E} = -\frac{1}{c}\frac{\partial \hat{B}}{\partial t} \qquad (2.4)$$

$$\hat{\nabla} \cdot \hat{B} = 0 \qquad (2.5)$$

$$\hat{\nabla} \cdot \hat{E} = 4\pi e(n_i - n_e) \qquad (2.6)$$

$n_{e,i} = \int d^3V\, f_{e,i}\,(\hat{r}, \hat{V}, t)$ is the average number-density of particles, electrons and ions, respectively at $\hat{r}$ and t.

$\hat{j} = e \int d^3V\, \hat{V} \left[f_i\,(\hat{r}, \hat{V}, t) - f_e\,(\hat{r}, \hat{V}, t)\right]$ is the average current-density of particles at $\hat{r}$ and t.

$\hat{\nabla} \equiv \frac{\partial}{\partial \hat{r}}$ is the gradient operator on position variables.

If we integrate the Vlasov equations over velocity space we obtain the continuity equation for the number of electrons and ions respectively.

$$\frac{\partial}{\partial t} n_{e,i} + \nabla \cdot n_{e,i} V_{e,i} = 0 \qquad (2.7)$$

$\hat{V}_{e,i} = \int d^3V \hat{V} f_{e,i}$ is the average particle velocity. Often we refer to the fluid velocity, which is $\hat{u} = \frac{m_i \hat{V}_i + m_e \hat{V}_e}{m_i + m_e}$. The fluid density is $\rho = m_i n_i + m_e n_e$.

If we multiply the Vlasov equation by $\hat{V}$ and then integrate over velocity space, we obtain the continuity equation for momentum i.e., Newton's second law.

$$n_{i,e}\, m_{i,e} \left(\frac{\partial}{\partial t} + \hat{V}_{i,e} \cdot \hat{\nabla}\right) \hat{V}_{i,e} + \hat{\nabla} \cdot \hat{\hat{P}}_{i,e} = \pm\, e n_{i,e} \left(\hat{E} + \frac{\hat{V}_{i,e}}{c} \times \hat{B}\right) \qquad (2.8)$$

where $\hat{\hat{\mathbb{P}}}_{i,e} = m_{i,e} \int d^3V f_{i,e} (\hat{V} - \hat{V}_{i,e})(\hat{V} - \hat{V}_{i,e})$ is the particle pressure tensor. $(\mathbb{P}_{i,e})_{ij}$ is the flux of the i component of particle momentum (relative to the average momentum), through a unit area whose normal lies in the j direction. $i, j = x, y, z$, the position coordinates.

If we multiply the Vlasov equation by $\hat{V}\hat{V}$ or any higher rank velocity tensor and then integrate over velocity space, we will obtain partial differential equations which relate the average quantities which we have already introduced with new ones. The averages of the velocity tensors are called moments of the particle distribution. We find that the number of moments always exceeds the number of equations for them so that additional relations among the moments are necessary to terminate the sequence of equations so that we have an equal number of unknowns and equations which system may then be solved simultaneously.

We terminate the sequence of 'moment' equations at equation (2.8), the continuity equation for momentum by neglecting the pressure tensor altogether. The omission of the pressure tensor is not merely a matter of convenience but has the following physical justification. The largest components of the pressure tensor are of order of magnitude $n_{i,e} k T_{i,e}$ where k is Boltzmann's constant and $T_{i,e}$ is the temperature. $n_{i,e} k T_{i,e}$ is about 10^{-2} times smaller than the magnetic pressure $B_o^2/8\pi$ of the magnetic field in the plasma. The magnetic force term in the momentum equation is of the order of magnitude of either $\hat{\nabla} B_o^2/8\pi$ for a magnetic field $\hat{B}_o$ which varies with position in the plasma, or $\hat{j} \times \frac{\hat{B}}{c} \simeq e\frac{n}{c} V_A B_o \simeq \frac{1}{r_i} B_o^2/4\pi$ for average particle velocities around the Alfvén speed $V_A = B_o/\sqrt{4\pi\rho}$. r_i is $\sqrt{m_i c^2/4\pi n e^2}$ and is of the same order as $|\hat{\nabla} \ln B_o|^{-1}$. Both of these estimates based upon the average properties of the plasma, are much larger than $\hat{\nabla} \cdot \hat{\hat{\mathbb{P}}}_{i,e}$.

Later we consider the normal waves which are the eigen solutions of the linearized Maxwell and continuity equations. The omission of the pressure tensor will cause two errors: the wave phase velocity will be missing terms comparable to the particle thermal velocities and the frequency will be missing Landau damping. Since we are interested in wave phenomena and flows with velocities comparable to the Alfvén speed the corrections to the phase velocity will be small. The influence of Landau damping is discussed qualitatively later on. The fast MHD waves are interesting partly because they are not heavily damped by either ions or electrons for wavelengths between $\sqrt{m_e/m_i}\, r_i$ and r_i, which is the range of physical intersst, as mentioned before.

SECTION III

NORMAL WAVES

CLASSES OF WAVES

The normal waves are the small-amplitude periodic solutions of the equations of motion, Equations (2.3) through (2.8). The waves have the space and time dependence $e^{i(\hat{k}\cdot\hat{x}-\omega t)}$ of traveling disturbances of magnetic field, electric field, particle velocity, and number-density. We consider, for the moment, that we have waves in a uniform, quiescent plasma having a magnetic field $\hat{B}_o$ and density n_o. Later, we allow motion of the plasma and non-uniformities of the fields. We then describe the propagation of a single-wave in such a medium. However, for the present we discuss the different classes of waves and the details of the frequencies and polarizations of the waves for limited regions of wavelength and phase velocity, in a uniform quiescent plasma, so that the influence of non-uniformities and plasma motion can be better understood in terms of amplification and Doppler shifting, respectively, of these waves.

It is generally recognized that in a uniform plasma with a magnetic field there are four vaguely-distinct types of waves: transverse electromagnetic waves, magnetohydrodynamic waves, electrostatic charge-separation waves, and acoustic waves. The electrons or ions vary in importance in these waves depending upon the frequency and wavelength. The general normal waves are not simply related to the types listed above but are combinations of these depending upon the direction of propagation relative to the magnetic field, which provides the complication of a "preferred" direction in addition to that of the wave-vector.

The magnetohydrodynamic waves are distinguished from the plasma waves by the fact that the frequencies of the MHD waves go to zero for zero wave-vector, while those of the plasma waves are non-zero, and of the order of the electron plasma frequency. One may say that, for a given wave-vector, there are six modes, three plasma waves and three MHD waves.* If we have a very small magnetic field so that the gyro-frequencies are small compared to the plasma frequencies, we would have two distinct branches for frequency as a function of wave-vector. The upper branch consists of three modes: the longitudinal plasma oscillation and the two degenerate electromagnetic transverse modes, which are slightly split apart in frequency by the magnetic field. One of these, the ordinary wave is unaffected by the transverse component of magnetic field. The other, the extraordinary wave, has some longitudinal components, as the longitudinal wave likewise has some transverse components, when these two propagate at an angle with the magnetic field. The situation with the ordinary and extraordinary waves is similar to that of double refraction in an anisotropic crystal. The magnetic field provides the anisotropic effect in the plasma. This branch, as we stated earlier, is characterized by high frequencies even for low wave-vector. Hence, this branch is predominantly electron oscillation; the ion motion produces a very small correction.

The lower branch in the absence of magnetic field, consists of a longitudinal wave and two trivially degenerate transverse waves, i.e. an ordinary sound wave and two identically zero-frequency waves, respectively. When there is a little magnetic field the transverse modes are non-trivial and are split. If the wave propagates along the magnetic field, there are still longitudinal waves and two degenerate transverse waves. For arbitrary direction of $\hat{k}$ all degeneracy is removed. In the region of low wave-vector such that the wavelength is much larger than the gyro-radii, the charged particles move adiabatically and the conditions for infinite conductivity are effectively in operation. For propagation parallel to the magnetic field the two transverse modes have the Alfvén speed and the longitudinal mode has the sound speed. When we omit the particle pressure tensor from the equations of motion we will not have acoustic waves or the sound speed appearing at all. In this case, the longitudinal mode will correspond to the frequency being identically zero for all wave vectors.

Now let us consider wavelengths that are comparable to the ion gyro-

* The two branches, plasma and MHD, arise from the two species, electrons and ions, as is the case in crystal lattices with two different atoms per unit cell. The high frequency branch corresponds to the relative motions of the two "atoms", and the reduced mass of the two is the mass of the vibrating system. The low frequency branch corresponds to motion of both "atoms" in the same direction. The pass bands may overlap for the different modes.
V. L. Brillouin, Wave Propagation in Periodic Structures, Dover Publications, Inc. (1953). pp. 14-16

radius. The electrons still move adiabatically. In this case we still have the zero frequency longitudinal mode, if we omit the pressure tensor. The transverse waves are the fast and slow waves which we will describe in greater detail later on. They consist of elliptically polarized magnetic field and fluid velocity for arbitrary direction of propagation. The electric field is polarized similarly to the fluid velocity but the electric field is smaller than the magnetic field by the ratio of phase velocity to light speed.

When the magnetic field becomes sufficiently strong so that the electron gyro-frequency is equal to or greater than the electron plasma frequency then the MHD branch and the plasma wave branch do not have easily-distinguishable properties. In this range, the electron Alfvén speed (based on electron mass) is comparable to the light speed; the ratio of electron Alfvén speed to light speed is the ratio of electron gyro-frequency to plasma frequency. For our plasma however, this ratio is still small, about 10^{-2}.

Let us now linearize the equations of motion to obtain the general eigen vector equations for the normal waves. We will then solve the dispersion relation in the approximation that $m_e/m_i \ll 1$, that $r_e k \ll 1$, and that $\omega/kc \ll 1$. We then obtain the fast and slow MHD waves, plus a zero frequency solution corresponding to the absent acoustic-type wave.

We substitute in Equations (2.3) through (2.8) for the magnetic field: $\hat{B}_o + \hat{E}\, e^{i(\hat{k}\cdot\hat{x}-\omega t)}$; for the electric field: $\hat{E}\, e^{i(\hat{k}\cdot\hat{x}-\omega t)}$; for the particle velocities: $\hat{V}_{i,e}\, e^{i(\hat{k}\cdot\hat{x}-\omega t)}$; and for the densities: $n_o + n_{i,e}\, e^{i(\hat{k}\cdot\hat{x}-\omega t)}$. We then omit terms which involve the product of two or more of the periodic fields, thereby linearizing the equations. The eigen equations are then:

$$-i\, m_{i,e}\, \omega\, \hat{V}_{i,e} = \pm\, e\left(\hat{E} + \frac{\hat{V}_{i,e}}{c} \times \hat{B}_o\right) \tag{3.1}$$

$$i\, \hat{k} \times \hat{B} = \frac{4\pi e}{c}\, n_o\, (\hat{V}_i - \hat{V}_e) - i\,\frac{\omega}{c}\, \hat{E} \tag{3.2}$$

$$i\, \hat{k}\cdot \hat{E} = 4\pi\, (n_i - n_e) \tag{3.3}$$

$$i\, \hat{k} \times \hat{E} = -i\, \frac{\omega}{c}\, \hat{B} \tag{3.4}$$

$$i\hat{k}\cdot\hat{B} = 0 \tag{3.5}$$

$$-i\,\omega\, n_{i,e} + n_o\, i\hat{k}\cdot\hat{V}_{i,e} = 0 \tag{3.6}$$

There are four vector equations and four scalar equations, two of which are redundant. The non-redundant scalar equations give $\omega^2=0$. Thus, there are at most, six non-zero eigen values for ω^2. The ω eigen values are real and appear in positive and negative pairs because of the Hermitean nature of the matrix of coefficients of equations (3.1) through (3.6) and the time reversal symmetry of the original equations of motion.

Using $\hat{V} = \hat{V}_i - \hat{V}_e$ in equation (3.1) and eliminating $\hat{B}$ between equations (3.2) and (3.4), we obtain:

$$\omega^2 \frac{m_i m_e}{m_i+m_e}\hat{V} + i\omega\frac{m_i-m_e}{m_i+m_e}\frac{\hat{V}}{c}\times\hat{B}_o + \frac{e^2}{m_i+m_e}\frac{\hat{V}\times\hat{B}_o}{c^2}\times\hat{B}_o = i\omega e\hat{E} \tag{3.7}$$

$$i\hat{k}\times(i\hat{k}\times\hat{E}) - \frac{\omega^2}{c^2}\hat{E} = \frac{i4\pi n_o\omega}{c^2}\hat{V} \tag{3.8}$$

Upon eliminating $\hat{V}$ between equations (3.7) and (3.8), we obtain the final eigen vector equation: $\Lambda\hat{E} = 0$ where

$$\Lambda = \begin{bmatrix} \omega^2+(\omega^2-\omega_{ce}\omega_{ci})r_e^2(k^2-k_x^2-\frac{\omega^2}{c^2}) - i\omega\omega_{ce}r_e^2k_xk_y & -(\omega^2-\omega_{ce}\omega_{ci})r_e^2k_xk_y+i\omega\omega_{ce}r_e^2(k^2-k_y^2-\frac{\omega^2}{c^2}) & -(\omega^2-\omega_{ce}\omega_{ci})r_e^2k_xk_z-i\omega\omega_{ce}r_e^2k_yk_z \\ -(\omega^2-\omega_{ce}\omega_{ci})r_e^2k_yk_z-i\omega\omega_{ce}r_e^2(k^2-k_x^2-\frac{\omega^2}{c^2}) & \omega^2+(\omega^2-\omega_{ce}\omega_{ci})r_e^2(k^2-k_y^2-\frac{\omega^2}{c^2})+i\omega\omega_{ce}r_e^2k_xk_y & -(\omega^2-\omega_{ce}\omega_{ci})r_e^2k_yk_z+i\omega\omega_{ce}r_e^2k_xk_z \\ -\omega^2r_e^2k_zk_x & -\omega^2r_e^2k_zk_y & \omega^2[1+r_e^2(k^2-k_z^2-\frac{\omega^2}{c^2})] \end{bmatrix} \tag{3.9}$$

We have neglected m_e compared to m_i in (3.9).

The roots of det $\Lambda=0$ for ω are real. The imaginary terms must multiply in pairs to give only real terms in det Λ or they must cancel out. We conclude then that det Λ is a polynomial in ω^2, whose highest power is $(\omega^2)^6$, corresponding to the six modes, three being plasma oscillations, and three being MHD waves.

In the approximation that $r_ek \ll 1$ we have the following matrix:

$$
\mathbb{A} \simeq \begin{bmatrix}
\omega^2 + (\omega^2 - \omega_{ci}\omega_{ce}) r_e^2 (k^2 - k_x^2 - \frac{\omega^2}{c^2}) - i\omega\omega_{ci} r_i^2 k_x k_y & \omega_{ci}^2 r_i^2 k_x k_y + i\omega\omega_{ci} r_i^2 (k^2 - k_y^2 - \frac{\omega^2}{c^2}) & \omega_{ci}^2 r_i^2 k_x k_z - i\omega\omega_{ci} r_i^2 k_y k_z \\
\omega_{ci}^2 r_i^2 k_y k_z - i\omega\omega_{ci} r_i^2 (k^2 - k_x^2 - \frac{\omega^2}{c^2}) & \omega^2 + (\omega^2 - \omega_{ci}\omega_{ce}) r_e^2 (k^2 - k_y^2 - \frac{\omega^2}{c^2}) + i\omega\omega_{ci} r_i^2 k_x k_y & \omega_{ci}^2 r_i^2 k_y k_z + i\omega\omega_{ci} r_i^2 k_x k_z \\
0 & 0 & \omega^2 (1 - r_e^2 \frac{\omega^2}{c^2})
\end{bmatrix} \qquad (3.10)
$$

We still have a polynomial of sixth degree in ω^2. Two roots are already evident: $\omega^2 = 0$ and $1 - \frac{r_e{}^2 \omega^2}{c^2} = 0$, the latter being $\omega^2 = \omega^2_{Pe}$, the electron plasma frequency squared. We obtain some idea of the other four roots as follows. When $\hat{k} = 0$

$$
\mathbb{A} \simeq \begin{bmatrix}
\omega^2 (1 - r_e^2 \frac{\omega^2}{c^2} + \frac{\omega_{ci}^2}{c^2} r_i^2) & \omega^2 (-i \frac{\omega_{ci} r_i^2 \omega}{c^2}) & 0 \\
\omega^2 (i \frac{\omega_{ci} r_i^2 \omega}{c^2}) & \omega^2 (1 - r_e^2 \frac{\omega^2}{c^2} + \frac{\omega_{ci}^2}{c^2} r_i^2) & 0 \\
0 & 0 & \omega^2 (1 - r_e^2 \frac{\omega^2}{c^2})
\end{bmatrix}
$$

(3.11)

we obtain $(\omega^2)^2 = 0$ and $\left(1 - r_e^2 \frac{\omega^2}{c^2} + \frac{\omega_{ci}^2}{c^2} r_i^2\right)^2 = \left(\frac{\omega_{ci} r_i^2 \omega}{c^2}\right)^2$ the roots of the latter being $(\omega^2)^2 \simeq (\omega_{Pe}^2)^2$ for $\left(\frac{\omega_{ci} r_i}{c}\right)^2 = \left(\frac{V_A}{c}\right)^2 \ll 1$. So we have three modes whose frequencies are zero when $\hat{k} = 0$: these comprise the MHD branch. And, we obtain three modes whose frequencies are approximately the electron plasma frequency when $\hat{k} = 0$. These comprise the plasma oscillation branch. Furthermore, we obtain two of the zero frequency modes and two of the plasma frequency modes from the 2x2 sub-matrix in the upper left-hand corner of (3.11). Hence, we expect two MHD waves and two plasma oscillations to come out of the corresponding sub-matrix of (3.10) for non-zero $\hat{k}$. The two MHD waves can be obtained approximately by now neglecting ω/kc if $V_A/c \ll 1$. This leaves the determinant of the 2x2 sub-matrix as a polynomial of second degree in ω^2, the two roots of which correspond to the MHD waves, when $V_A/c \ll 1$. This approximation fails when $V_A/c \simeq 1$.

Making the approximation that $(\omega/kc)^2 \ll 1$ we have for the eigen vector equation for the two non-trivial MHD modes:

$$\left(1-\frac{k'^2-k_x'^2}{\omega'^2}-i\,\frac{k_x'k_y'}{\omega'}\right)E_x \quad + \quad \left(\frac{k_x'k_y'}{\omega'^2}+i\,\frac{k'^2-k_y'^2}{\omega'}\right)E_y \qquad = 0$$

$$\left(\frac{k_y'k_x'}{\omega'^2}-i\,\frac{k'^2-k_x'^2}{\omega'}\right)E_x \quad + \quad \left(1-\frac{k'^2-k_y'^2}{\omega'^2}+i\,\frac{k_x'k_y'}{\omega'}\right)E_y \qquad = 0$$

$$E_z = 0 \tag{3.12}$$

where $\quad \omega' = \omega/\omega_{ci} \quad \hat{k}' = \hat{k}\, r_i$

$$(\omega'^2)^2 - (\omega'^2)\,k'^2\left[1+\cos^2\theta + k'^2\cos^2\theta\right] + (k'^2)^2\cos^2\theta = 0 \tag{3.13}$$

is the dispersion relation. $\quad \cos\theta = k_z/k$

If we call $\omega'/k' = V_{ph} = (\omega/k)/V_A$, and set $k_y = 0$, we have for the two non-trivial MHD modes:

$$\begin{aligned}
E_x &= \frac{i}{\omega'\cos\theta}(V_{ph}-1)\,E_y \\
E_z &= 0 \\
B_x &= \frac{1}{V_{ph}}\cos\theta \;\; E_y \\
B_y &= \frac{i}{\omega' V_{ph}\cos\theta}\left(1-V_{ph}^2\right)E_y \\
B_z &= -\frac{1}{V_{ph}}\sin\theta \;\; E_y \\
u_x &= \frac{1}{V_{ph}^2}\,E_y \\
u_y &= \frac{i}{\omega' V_{ph}^2}\left(1-V_{ph}^2\right)E_y \\
u_z &= 0
\end{aligned} \tag{3.14}$$

We now discuss some formal aspects of the eigen vector solutions for the MHD waves.

The approximation that $k\, r_e \ll 1$ and $(\omega/kc)^2 \ll 1$ is equivalent to the use of the following linear equations for the wave fields:

$$\hat{E} + \frac{\hat{V}_e}{c}\times\hat{B}_o = 0 \tag{3.15}$$

$$m_i\frac{\partial\hat{V}_i}{\partial t} = e\left(\hat{E}+\frac{\hat{V}_i}{c}\times\hat{B}_o\right) \tag{3.16}$$

$$\hat{\nabla}\times\hat{B} = 4\pi n_o\frac{e}{c}\left(\hat{V}_i-\hat{V}_e\right) \tag{3.17}$$

$$\hat{\nabla} \times \hat{E} = -\frac{1}{c} \frac{\partial \hat{E}}{\partial t} \tag{3.18}$$

This is easily seen from equations (3.1) to (3.6). By means of (3.15), we can eliminate $\hat{E}$ and obtain the following eqs:

$$\frac{\partial \hat{\eta}}{\partial t} - (\hat{\nabla} \times \hat{\mathcal{E}}) \times \hat{e}_z V_A = 0 \tag{3.19}$$

$$\frac{\partial \hat{\mathcal{E}}}{\partial t} - \hat{\nabla} \times [\hat{\eta} \times \hat{e}_z V_A - (\hat{\nabla} \times \hat{\mathcal{E}}) \times \hat{e}_z V_A r_i] = 0$$

where

$$\left.\begin{array}{l} \hat{\eta} = \sqrt{\frac{\rho}{2}}\, \hat{u} \\ \hat{\mathcal{E}} = \frac{1}{\sqrt{8\pi}} \hat{B} \end{array}\right\} = \mathcal{Y}_{\hat{k}}\, e^{i(\hat{k}\cdot\hat{x} - \omega_{\hat{k}} t)} \tag{3.20}$$

is the six dimensional eigen-vector belonging to the eigen-value $\omega_{\hat{k}}$ for the given $\hat{k}$. The components of $\hat{\eta}_{\hat{k}}$ and $\hat{\mathcal{E}}_{\hat{k}}$ have already been given in equations (3.14), as has the dispersion relation giving $\omega_{\hat{k}}$ in terms of $\hat{k}$ in eq. (3.13).

Because of time-reflection symmetry, the dispersion relation involves integral powers of ω^2, not of ω. Hence the two signs of ω are allowed and each corresponds to a different eigen-vector.

In field theory it is customary to use the following conventions related to the Fourier expansion:

$$\begin{pmatrix} \sqrt{\frac{\rho}{2}}\, \hat{u} \\ \frac{1}{\sqrt{8\pi}} \hat{B} \end{pmatrix} = \int \frac{d^3k}{(2\pi)^{3/2}} \sum_{m_k = \pm 1} a_{\hat{k}, m_k}\, \mathcal{Y}_{m_k}(\hat{k})\, e^{i(\hat{k}\cdot\hat{x} - \omega_{\hat{k}, m_k} t)}$$

$$\begin{aligned} a_{\hat{k}, m_k} &= a^{+}_{-\hat{k}, -m_k} \\ \omega_{\hat{k}, m_k} &= \omega_{-\hat{k}, m_k} = -\omega_{\hat{k}, -m_k} = -\omega_{-\hat{k}, -m_k} \\ \hat{\eta}_{\hat{k}, m_k} &= \hat{\eta}^{*}_{-\hat{k}, -m_k} \\ \hat{\mathcal{E}}_{\hat{k}, m_k} &= \hat{\mathcal{E}}^{*}_{-\hat{k}, -m_k} \end{aligned} \tag{3.21}$$

This makes, as desired, $\hat{u}(\hat{x}, t)$ and $\hat{B}(\hat{x}, t)$ real-valued vectors. According to this convention the complex amplitudes $a_{\hat{k}, m_k}$ and $a_{-\hat{k}, m_k}$ correspond, in field theory, to particles moving in opposite directions, the direction of $\hat{k}$ being the direction of the particle. m_k distinguishes particle and antiparticle fields. $a_{\hat{k},m_k}$ and $a_{\hat{k},-m_k}$ are unrelated. However, $a_{\hat{k},m_k} = a^{+}_{-\hat{k},-m_k}$.

However, it is more convenient to use m_k to distinguish waves moving to the right or to the left along the applied magnetic field, which we take along the z-axis. Therefore, we set

$$
\begin{aligned}
a_{\hat{k},m_k} &= a^{+}_{-\hat{k},m_k} \\
\omega_{\hat{k},m_k} &= -\omega_{-\hat{k},m_k} = -\omega_{\hat{k},-m_k} = \omega_{-\hat{k},-m_k} \\
\hat{\eta}_{\hat{k},m_k} &= \hat{\eta}^{*}_{-\hat{k},m_k} \\
\hat{\varepsilon}_{\hat{k},m_k} &= \hat{\varepsilon}^{*}_{-\hat{k},m_k}
\end{aligned}
\tag{3.22}
$$

The complex amplitudes,for our convention, $a_{\hat{k},m_k}$and $a_{\hat{k},-m_k}$are unrelated and are associated with waves traveling in the two opposite z-directions. The sign of k_z is not the sign of the wave velocity in the z-direction, but that of m_k is.

For the case $k\, r_i \gg 1$, the fast waves are transverse waves of circularly-polarized magnetic field and of negligible fluid velocity. If we view the wave in wave-coordinates, i.e. moving along with the phase-velocity, we will see the applied magnetic field plus the wave's magnetic field as a steady configuration of magnetic lines which are helices wound on elliptical cylinders. If the wave moves parallel to the applied field, the helices are circular and the electrons move along these lines (with negative phase-velocity) because it happens that r_e the electron gyro-radius is small. The ions move straight along the direction of the applied field through the wave with the negative phase-velocity. The negative phase-velocity occurs because we are using the coordinate system moving with the wave. If the wave moves at some angle to the applied field then in this coordinate system the electrons feel an electric field, $\hat{E} = \frac{\hat{V}_{ph}}{c} \times \hat{B}$. The electrons will drift. If we let the whole steady pattern of magnetic lines move with the component of phase velocity perpendicular to the applied field, we will compensate for the drift and the electrons will follow the helices again.

Still for the case of $k\, r_i \gg 1$, the slow waves are waves of fluid velocity which are circularly polarized perpendicular to the applied field. The magnetic field is negligible for these wavelengths. With $k_z r_i \gg 1$, the dispersion relation for the fast wave simplifies to become:

$$\omega_k \cong \hat{u}_o \cdot \hat{k} + m_k \; V_A \, r_i k_z k \tag{3.15}$$

The group velocity in the x-direction becomes:

$$v_x = u_{o_x} + m_{zk} V_A \, r_i k_z \; \frac{k_x}{k} \tag{3.16}$$

For slow waves:

$$\omega_k \cong m_k \frac{k_z}{|k_z|} \; \omega_{c_{e_i}} + \hat{k} \cdot \hat{u}_o \tag{3.17}$$

As we increase k_x and k_z proportionately, taking $k_y = 0$ for the moment, we see that for fast waves v_x is nearly proportional to $k_z r_i$ $v_x/V_A = 7$ for $k_z r_i = 10$ for $\hat{k}$ at about 45^o with $\hat{B}_o$.

With $k_z r_i \gg 1$, the eigen vectors greatly simplify. The magnetic vector for the fast wave is nearly circularly polarized perpendicular to the wave-vector. The departure from circularity is of the order of $1/kr_i$. The fluid-velocity part for the fast wave is $1/kr_i$ times smaller than the circular magnetic part and the electric part is much smaller still. For $k_z r_i \gg 1$:

$$\left[\frac{\text{Fluid Kinetic Energy in Wave}}{\text{Magnetic Energy in Wave}}\right]^{1/2} = \frac{V_A}{V_{ph}} = \begin{cases} 1/k_z r_i & \text{fast wave} \\ kr_i & \text{slow wave} \end{cases}$$

$$\left[\frac{\text{Electric Energy in Wave}}{\text{Magnetic Energy in Wave}}\right]^{1/2} = \frac{V_{ph}}{c} = \begin{cases} k_z r_i \; \dfrac{V_A}{c} & \text{fast wave} \\ \dfrac{1}{kr_i} \; \dfrac{V_A}{c} & \text{slow wave} \end{cases}$$

With $k_z r_i \gg 1$, the fluid velocity part of the slow wave is nearly circularly-polarized about $\hat{B}_o$. The departure from circularity and the magnetic components are both about $1/kr_i$ times smaller.

Landau damping of a wave is based upon the transference of electrostatic energy from the wave to charged particles caught in the potential troughs of the wave. That is, a particle is trapped because its thermal velocity nearly equals the phase-velocity of the wave. If the particle moves slightly slower than the phase-velocity it will gain energy from the wave. The net effect is damping only if there are more particles moving slightly slower than the phase-velocity than there are particles moving slightly faster than the wave. That is, there is damping if the velocity distribution* has a negative slope at a particle velocity equal to the wave phase-velocity.

If we consider waves with frequencies much less than the electron gyro-frequency, the electron motion in the wave will be perpendicular to any electric field because of the usual drift $c \frac{\hat{E} \times \hat{B}}{B^2}$.[21] Then we do not expect the electron to be trapped in an electrostatic potential trough of the wave and we do not expect the Landau damping to work.

However, the slow waves have frequencies near the ion cyclotron frequency for $kr_i \gg 1$ and their phase-velocity is of the order of the thermal velocity of the ions. Also, the polarization of the electric field of the slow wave rotates in the same direction as the ions. Hence, ions can be "trapped" and the wave is heavily Landau damped. Similar arguments hold for the fast wave when its frequency is close to the electron gyro-frequency. The electric polarization of the wave rotates with the electrons, and the fast wave is then heavily damped by the electrons.

For any frequency and wavelength wave, the amount of Landau damping is proportional to the number of particles having a thermal velocity equal to the appropriate phase velocity of the wave. This phase velocity is $(-\omega+\omega_{c_i})/k$ for damping of the slow wave by ions and $(\omega+\omega_{c_e})/k$ for damping of the slow wave by electrons. This velocity is $(\omega+\omega_{c_i})/k$ for damping of the fast wave by ions and $(-\omega+\omega_{c_e})/k$ for damping of the fast wave by electrons. Instead of ω_{c_i} or ω_{c_e} in these expressions we can have any integer multiple of these frequencies. Our conclusions concerning the damping is based upon the following facts: one, for slow waves, $\omega \lesssim \omega_{c_i}$ for $kr_i \simeq 1$; two, for fast waves, $\omega \lesssim \omega_{c_e}$ for $kr_e \simeq 1$; and three, the thermal velocities are given by a Maxwell distribution for each species so that there is an exponentially small number of particles having a thermal velocity different from the mean thermal velocity. Therefore, Landau damping is exponentially small for the above phase velocities differing from the mean thermal velocity.

* Landau damping, which puts particle energy into waves for those particles moving faster than the phase velocity, looked at from the particle's point of view, is radiation loss by Čerenkov radiation. This radiation consists of the appropriate waves in the plasma.

SECTION IV

THE WAVE TRANSPORT EQUATION

AND WAVE AMPLIFICATION

If thermodynamic equilibrium existed between fast MHD waves and the particles, then a negligible amount of energy is invested in the waves, only about 10^{-11} times the particle thermal energy. The ratio of particle thermal energy to wave energy at equilibrium is, within a factor of two, the corresponding ratio of the number of degrees of freedom which can be excited, since $(1/2)kT$ of energy goes with each degree of freedom. This ratio is then $N(r_e)$, the number of particles contained in a volume r_e^3, since we only count degrees of freedom for the fast MHD waves which belong to wavelengths greater than r_e. As already discussed, the fast waves are heavily damped by the electrons for wavelengths shorter than r_e; $N(r_e)=nr_e^3=(nh^3)\left(\frac{m_e c^2}{kT}\right)^{3/2} = N(h)\left(\frac{m_e c^2}{kT}\right)^{3/2}$ is about 10^7 times $N(h)$ the number of particles contained in a Debye-length cube, which is about 10^4 particles, and is the ratio of particle energy to energy in plasma oscillations. Therefore, unless there is some amplification mechanism to build up the waves to some non-equilibrium distribution whose energy is comparable to or greater than the particle thermal energy, fast MHD waves will not be important.

We now describe an amplification mechanism by which the plasma non-uniformities give up their energy to fast MHD waves. Because the waves scatter with other waves, this amplification is irreversible and, for some cases, can account for a total wave energy of about ten times the particle energy behind a MHD shock wave moving with about twice the Alfvén speed.

Let us now describe a non-uniform plasma whose properties are not changing with time except for the fact that fast MHD waves are propagating in it. We make approximations concerning the behavior of the plasma which are consistent only with MHD flow, i.e., we only consider cases for which the Maxwell displacement current is small compared to the current density in the plasma, and for which any charge separation occurs over distances of the order of a Debye length which is small compared to the scale-length of the flow.

The objective of this analysis is to describe the effects of the non-uniformities of the magnetic field and of the plasma velocity, which we now include in order to generalize our earlier discussion of the normal waves, on the propagation of a single fast MHD wave-packet. We are not interested in the non-linear effects of waves interacting with other waves—this we treat in the next chapter—but in the compression of the wave-packet by the changing density of the plasma and in the Doppler shift of the wave-frequency due to the plasma velocity and in the changes in group velocity because of the changes of the plasma velocity as the wave-packet moves to different positions of the plasma.

We represent the total magnetic field as $\hat{B}_o(\hat{x}) + \hat{B}(\hat{x}, t)$. $\hat{B}_o$ is the magnetic field which varies with position in the plasma and is present regardless of the wave-packet, whose magnetic field is $\hat{B}(\hat{x}, t)$. The total plasma velocity is $\hat{u}_o(\hat{x}) + \hat{u}(\hat{x}, t)$, the total density is $\rho_o(\hat{x}) + \rho(\hat{x}, t) = \rho_T$; the total current density is $\hat{j}_o(\hat{x}) + \hat{j}(\hat{x}, t)$; and the total electric field is $\hat{E}_o(\hat{x}) + \hat{E}(\hat{x}, t) = \hat{E}_T$. We have introduced the overall plasma properties for the sake of convenience. The subscript "o" denotes the average plasma properties, which change with position in the plasma regardless of the wave-packet. We repeat the definitions of fluid velocity, density and current density

$$\hat{u}_T = \hat{u}_o + \hat{u} = \frac{m_i \hat{V}_i + m_e \hat{V}_e}{m_i + m_e} \tag{4.1}$$

$$\rho_T = \rho_o + \rho = m_i n_i + m_e n_e \tag{4.2}$$

$$\hat{j}_T = \hat{j}_o + \hat{j} = en_i\hat{V}_i - en_e\hat{V}_e \tag{4.3}$$

We substitute these quantities into equations (2.3) through (2.9) having neglected displacement current and charge separation ($n_i = n_e$). Another approximation is made namely, that the scale-lengths of the phenomena of interest, either that of the non-uniformities or of the wavelengths of the waves, are large compared to the electron gyroradii so that $m_e\left(\frac{\partial \hat{V}_e}{\partial t} + \hat{V}_e \cdot \hat{\nabla} \hat{V}_e\right)$ is small compared to either $e\hat{E}$ or $\frac{e\hat{V}_e}{c} \times \hat{B}_o$, so that we may take in Equation (2.8)

$$E_T + \frac{\hat{V}_e}{c} \times \hat{B}_T = 0 \tag{4.4}$$

This states, in effect, that the electrons make tight helices around the magnetic field lines, except for their local drift velocity $c\,\frac{E_T \times B_T}{B_T^2}$ where $\hat{E}_T$ and $\hat{B}_T$ are the total fields.

We now write down the equations just for the average properties of the plasma in the absence of waves and then for the total fields including the waves.

$$\hat{\nabla} \cdot \rho_o \hat{u}_o = 0 \tag{4.5}$$

$$\hat{\nabla} \times \hat{E}_o = 0 \tag{4.6}$$

$$\hat{\nabla} \times \hat{B}_o = \frac{4\pi}{c}\hat{j}_o \tag{4.7}$$

$$\hat{\nabla} \cdot \hat{B}_o = 0 \tag{4.8}$$

$$\rho_T\left(\frac{\partial \hat{u}_T}{\partial t} + \hat{u}_T \cdot \hat{\nabla}\hat{u}_T\right) = \frac{1}{c}\hat{j}_T \times \hat{B}_T \tag{4.9}$$

$$\hat{\nabla} \times \hat{E} = -\frac{1}{c}\frac{\partial \hat{B}}{\partial t} \tag{4.10}$$

(4.11)

$$\hat{\nabla} \times \hat{B} = \frac{4\pi}{c} \hat{j} \tag{4.12}$$

$$\hat{\nabla} \cdot \hat{B} = 0 \tag{4.13}$$

We call a linear term one which is proportional to the amplitude of the wave field and a non-linear term, one which is proportional to a higher power of the wave amplitude. If we collect all of the non-linear terms on the right-hand side of the equations and concentrate on the linear terms, and if we consider fast waves with wavelengths sufficiently less than r_i that they are nearly all magnetic field energy compared to particle kinetic energy ($kr_i \gtrsim 3$) and electric field energy, then we use Equations (4.9) to (4.13) but ignore $\hat{u}$ and ρ. We have

$$\frac{\partial \hat{B}}{\partial t} - \hat{\nabla} \times \left[\hat{u}_o \times \hat{B} - \frac{m_i}{e\rho_o}\left(\hat{j} \times \hat{B}_o + \hat{j}_o \times \hat{B}\right)\right] = \left(\frac{\partial \hat{B}}{\partial t}\right)_{n-\ell} \tag{4.14}$$

where $\left(\frac{\partial \hat{B}}{\partial t}\right)_{n-\ell}$ is the time rate of change of wave magnetic field due to the non-linear terms which we have collected on the right-hand side of the equation. The non-linear terms will be considered in detail later. However, to avoid complication, we do not include in them the effects of the space variations of ρ_o, $\hat{u}_o$ and $\hat{B}_o$ under the assumption that the space Fourier components of ρ_o, $\hat{u}_o$ and $\hat{B}_o$ are small for wavelengths comparable to those of the waves.

Equation (4.14) is the basic equation for examining the propagation of a wave-packet of fast MHD of wavelengths somewhat less than r_i but considerably greater than $r_e = r_i \sqrt{m_e/m_i}$ in a plasma having steady non-uniformities on a scale also considerably larger than r_e.

For the sake of both mathematical simplicity and physical interest, we take the flow $\hat{u}_o$ perpendicular to $\hat{B}_o$, the magnetic field, which is always directed along the z axis but changing magnitude with x. We have a case of one-dimensional steady MHD flow perpendicular to a magnetic field. We take $\hat{u}_o$ always in the x-direction. All properties vary only with x.

In this case,

$$\rho_o u_o = \text{constant} \tag{4.15}$$
$$u_o B_o = \text{constant} \tag{4.16}$$
$$E_{yo} = \text{constant} \tag{4.17}$$

Equation (4.15) states that the mass is conserved since $\rho_o u_o$ is the rate of mass flow. Equation (4.17) follows from $\hat{\nabla} \times \hat{E}_o = 0$, since the magnetic field B_o is not time-varying. Equation (4.16) follows from $\hat{\nabla} \times (\hat{u}_o \times \hat{B}_o) = 0$, which is equivalent to $\hat{E}_o + \frac{\hat{u}_o}{c} \times \hat{B}_o = 0$. We do not assume infinite conductivity, although that would result in the previous expression, but we do assume that the electrons move adiabatically, i.e., they preserve their magnetic moment as they pass through the non-uniformity. This is true if the non-uniformities vary slowly in distance of the order of the electron gyroradii, which is the case we have in hand. This means that the electrons drift with a velocity cE_{yo}/B_o. Since there is no total current-density in the flow direction, since $\hat{\nabla} \times \hat{B}_o$ has no component in that direction, the ions must be moving with the same velocity as the drift velocity of the electrons so that the fluid velocity $u_o = cE_{yo}/B_o$; from this and Equation (4.17) we deduce Equation (4.16). Equation (4.15) and (4.16) imply that B_o/ρ_o = constant.* We will use this assumption whenever necessary. We discuss in Chapter VI in somewhat greater detail the conditions under which B_o/ρ_o is not strictly constant. There are other conservation laws implied by our Equations (4.9) through (4.13) but these are complicated by the presence of the magnetic pressure tensor and energy flux tensor of the waves. We also deal with these equations in Chapter VI when discussing an MHD shock. The above Equations (4.15), (4.16) and (4.17) will suffice for the description of the propagation of the wave packet if we treat ρ_o, u_o and B_o as given functions of x and solve for the wave amplitude in terms of them. Remember $\hat{B}(\hat{x}, t)$ is the magnetic field of a distribution of fast MHD waves. $\hat{B}(\hat{x}, t)$ satisfies a vector partial differential equation, Equation (4.14) with

* B_o/ρ_o must equal the same constant far behind the non-uniform part of the flow as it is far ahead of the non-uniform part of the flow. If the electrical conductivity is not very high B_o/ρ_o may depart from this constant within the non-uniform flow. H. E. Petschek has shown that wave scattering produces a friction between electrons and ions which results in a sufficiently high conductivity for our plasma conditions so that B_o/ρ_o can be taken as nearly constant everywhere for our plasma conditions. See proceedings of the American Physical Society Meeting, Gatlinburg, Tennessee, November 2-4.

coefficients that depend only on x. We expand $\hat{B}(\hat{x}, t)$ in a Fourier series in time t and space y and z so that

$$\hat{B}(\hat{x},t) = \sum_{\omega, k_y, k_z} \hat{a}_{\omega,k_y,k_z}(x)\, e^{-i(k_y y + k_z z - \omega t)} \tag{4.18}$$

where

$$\omega = \frac{2\pi}{T} n_\omega$$

$$k_y = \frac{2\pi}{L_y} n_y \qquad\qquad n_\omega, n_y, n_z = \pm 1, \pm 2, \cdots$$

$$k_z = \frac{2\pi}{L_z} n_z$$

(4.19)

$$\hat{a}_{\omega,k_y,k_z}(x) = \frac{1}{T L_y L_z} \underset{\substack{|t-\bar{t}| < \frac{T}{2} \\ |y-\bar{y}| < \frac{L_y}{2} \\ |z-\bar{z}| < \frac{L_z}{2}}}{\iiint} dt\, dy\, dz\; \hat{B}(\hat{x},t)\, e^{-i(k_y y + k_z z - \omega t)}$$

(4.20)

We will henceforth omit to mention the region of integration as indicated under the triple integration of equation (4.20). $\hat{a}_{\omega,k_y,k_z}(x)$ is the vector amplitude of a single fast MHD wave of constant frequency ω and constant wave-vector components k_y and k_z.

Substituting (4.18) into (4.14) and using equation (4.20), we obtain the following ordinary vector differential equation for $\hat{a}_{\omega,k_y,k_z}(x)$ which we call $\hat{a}$.

$$\begin{aligned} -i\omega\hat{a} - \hat{e}_y \frac{m_i c}{4\pi e}\frac{d}{dx}\left(\frac{1}{\rho_0}\frac{dB_0}{dx}\right) a_x + \frac{du_0}{dx}(\hat{a} - \hat{e}_x a_x) + \frac{m_i c}{4\pi e}\frac{1}{\rho_0}\frac{dB_0}{dx} k_y \hat{a} \\ -i\frac{m_i c}{4\pi e}\frac{B_0}{\rho_0} k_z (i k_y \hat{e}_y + i k_z \hat{e}_z) \times \hat{a} + i\frac{m_i c}{4\pi e} k_z \hat{e}_x \times \frac{d\hat{a}}{dx} \\ + u_0 \frac{d\hat{a}}{dx} = \frac{1}{T L_y L_z}\iiint dt\, dy\, dz \left(\frac{\partial \hat{B}}{\partial t}\right)_{n-\ell} e^{-i(k_y y + k_z z - \omega t)} \end{aligned} \tag{4.21}$$

In order to demonstrate wave amplification, we solve Equation (4. 21), with the right-hand side set equal to zero, by an approximation that resembles the WKB approximation of quantum mechanics. We introduce a form for the solution of (4. 21) that resembles the wave solution for a uniform plasma except that we allow the wave-vector and amplitude to vary with position in the plasma. Wherever we so indicate, we omit terms which depend upon second derivatives of the plasma properties. Just as in the WKB approximation, these terms are unimportant when the wave length of our approximate wave solution is short compared to the distance over which the plasma properties change appreciably. One usually finds in quantum mechanical problems that the WKB method predicts large amplitudes where the approximations no longer hold, that is, where the wavelength becomes very long. However, we are dealing with a case for which the amplitude becomes infinitely large, were it not for the non-linear terms on the right-hand side of (4. 21), when, at the same time, the wavelength becomes very short and so our approximation is quite valid. The points at which the amplitude becomes infinite are points at which the wave-packet group velocity approaches zero. This loosely resembles the quantum mechanical case of a classical turning point at which the particle has zero velocity. The WKB approximation incorrectly predicts an amplitude which is infinite. The exact solution of the Schrödinger equation gives a large but finite amplitude. However, equation (4. 21), with the right-hand side equal to zero, does predict an infinite amplitude. This is not the fault of the approximate wave solution that resembles the WKB approximation. The fact is that equation (4. 21) is singular in the domain of integration. That is, (4. 21) can be reduced to a singular second-order homogeneous linear differential equation, if use is made of $\hat{\nabla}\cdot\hat{B}(\hat{x}, t) = 0$. The equation is singular because the coefficient of the highest derivative of the new equation is zero, exactly where we predict wave amplification. On the contrary, the Schrödinger equation is not singular and therefore does not predict infinite amplitudes. The wave-packet for our case is continually being compressed by the surrounding medium at a place where the packet has come to a standstill; however, it does not turn around but merely grows in amplitude and shrinks in width.

We consider a single wave-packet travelling in one direction with one polarization. The steady state for waves in the plasma also consists of waves travelling in the opposite direction. These arise from reflections from the non-uniformities. We do not discuss these reflections but limit our attention to the time-dependent behavior of a single wave-packet.

Now multiply (4.20) by a^+ and add the complex conjugate equation.

$$
\begin{aligned}
-i(\omega-\omega^+)|a|^2 &- \frac{m_i c}{4\pi e}\frac{d}{dx}\left(\frac{1}{\rho_0}\frac{dB_0}{dx}\right)(a_y^+ a_x + a_y a_x^+) + 2(|a|^2-|a_x|^2)\frac{du_0}{dx} \\
&+ i\frac{m_i c}{4\pi e}\frac{B_0}{\rho_0}k_z\frac{d}{dx}\,\hat{e}_x\cdot\hat{a}\times\hat{a}^+ + u_0\frac{d|a|^2}{dx} \\
&= \frac{1}{TL_yL_z}\iiint dt\,dy\,dz\left[\hat{a}^+\cdot\left(\frac{\partial\hat{B}}{\partial t}\right)_{n-\ell} e^{-i(k_y y+k_z z-\omega t)} + \text{c.c.}\right]
\end{aligned} \tag{4.22}
$$

We are dealing with a single fast MHD wave of constant frequency ω whose wavelength is sufficiently smaller than r_i so that it is mainly circularly polarized magnetic field. Now, if we specify that $\hat{a}$ is circularly-polarized perpendicular to some vector $\hat{k} = k_x\hat{e}_x + k_y\hat{e}_y + k_z\hat{e}_z$, where k_y and k_z are the same Fourier parameters as before, then

$$
\begin{aligned}
|a_x|^2 &= \frac{1}{2}\left(1-\frac{k_x^2}{k^2}\right)|a|^2 \\
a_y^+ a_x + a_y a_x^+ &= -\frac{k_x k_y}{k^2}\,|a|^2 \\
\hat{e}_x\cdot\hat{a}\times\hat{a}^+ &= -i\frac{k_x}{k}\,|a|^2
\end{aligned} \tag{4.23}
$$

so that (4.22) becomes

$$
\begin{aligned}
2(\mathrm{Im}\,\omega)|a|^2 &+ \left[\frac{m_i c}{4\pi e}\frac{d}{dx}\left(\frac{1}{\rho_0}\frac{dB_0}{dx}\right)\frac{k_x k_y}{k^2} + \frac{k_x^2}{k^2}\frac{du_0}{dx}\right]|a|^2 \\
&+ \frac{d}{dx}\left[\left(u_0 + \frac{m_i c}{4\pi e}\frac{B_0}{\rho_0}k_z\frac{k_x}{k}\right)|a|^2\right] \\
&= \frac{1}{TL_yL_z}\iiint dt\,dy\,dz\left[\hat{a}^+\cdot\left(\frac{\partial\hat{B}}{\partial t}\right)_{n-\ell} e^{-i(k_y y+k_z z-\omega t)} + \text{c.c.}\right]
\end{aligned} \tag{4.24}
$$

The introduction of k_x and k is quite natural. It is easy to show from equations (4. 21) and (4. 24) that if $a_{\omega, k_y, k_z}(x)$ is written as $\alpha_{\omega, k_y, k_z}(x)\ \hat{\mathcal{E}}(\hat{k})\ e^{i\int^x k_x(x')dx'}$ where $\alpha_{\omega, k_y, k_z}$ can be taken as real-valued or at most with constant phase, then $k_x(x)$ is the function of x defined by the equations:

$$\mathcal{R}e\ \omega = u_o k_x + \frac{m_i c}{4\pi e}\frac{B_o}{\rho_o} k k_z + \frac{m_i c}{4\pi e}\frac{1}{\rho_o}\frac{dB_o}{dx} k_y = \text{constant}$$

$$k = (k_x^2 + k_y^2 + k_z^2)^{1/2} \tag{4. 25}$$

$\hat{\mathcal{E}}(\hat{k})$ is the unit circular-polarization vector.

We are dealing with a single fast MHD wave of constant frequency whose wavelength is sufficiently smaller than r_i so that we may use the approximate dispersion relation $\omega = u_o k_x + (m_i c/4\pi e)(B_o/\rho_o)(k_z k)$ in a uniform medium. The eigen vector is then nearly circularly-polarized magnetic field and negligible electric-field and fluid velocity.

If we say that $\hat{a}_{\omega, k_y, k_z}(x) = \alpha_{\omega, k_y, k_z}(x)\ \hat{\mathcal{E}}(\hat{k}(x))\ e^{i\int^x k_x(x')dx'}$ where $\hat{k} = (k_x(x), k_y, k_z)$, then $\hat{k}$ corresponds, in the case of a uniform medium, to the wave-vector of a circularly-polarized transverse wave. The polarization vector is $\hat{\mathcal{E}}(\hat{k})$ and rotates as a function of x. We see that k_x (x) is the x-component of the space-varying wave vector $\hat{k}$ about which the wave is always circularly-polarized. As the wave moves to different positions x, the wave-vector component k_x changes but the other components, k_y and k_z do not. Therefore $\hat{k}$ is changing direction and magnitude and the circular-polarization vector $\hat{\mathcal{E}}(\hat{k})$ is changing correspondingly. The amplitude of the wave $\alpha_{\omega, k_y, k_z}(x)$ is also changing with x, and this is described by equation (4. 24).

The dispersion relation, equation (4. 25), has as its first term the Doppler shift of frequency due to the motion of the fluid u_o carrying the wave. u_o is a function of x and so is k_x so that the amount of Doppler shift may vary from point to point in the plasma. The second term of equation (4. 25) is the un-Doppler-shifted frequency of the wave, i. e., the one measured by an observer moving with the local speed of the fluid. The third term is the effect of the gradient of the magnetic field on the un-Doppler-shifted frequency.

Equation (4.24) is the most important equation of this discussion. It contains on the left-hand side the description of wave amplification, which we next discuss, and on the right-hand side the non-linear effects, which we represent as wave scattering and discuss in the next chapter.

Let us consider equations (4.24) and (4.25). The group velocity in the x-direction is

$$v_x = \frac{\partial \omega}{\partial k_x} = u_0 + \frac{m_i c}{4\pi e} \frac{B_0}{\rho_0} k_z \frac{k_x}{k} \qquad (4.26)$$

Then equation (4.24) can be rewritten as

$$\frac{d}{dx}(v_x |a|^2) + \left(\frac{k_x^2}{k^2} \frac{du_0}{dx} + \frac{m_i c}{4\pi e} \frac{d}{dx}\left(\frac{1}{\rho_0}\frac{dB_0}{dx}\right) \frac{k_x k_y}{k^2} \right) |a|^2 = \text{RHS} \qquad (4.27)$$

with Im $\omega = 0$, since damping of the normal waves is not considered.

If we define t_v as the time for the wave-packet to move a distance x then

$$t_v = \int_{x_0}^{x_0+x} \frac{dx'}{v_x(x')}$$

and $\quad dt_v = dx/v_x$

For finite damping, $(v_x |a|^2)$ has a factor $e^{-(\text{Im}\,\omega)\, t_v}$.

We show in detail later that the wave-packet can be amplified exponentially with time, t_v, by the gradient of velocity: $\frac{du_0}{dx}$. If we multiply equation (4.27) by $(v_x |a|^2)^{-1}$ we have

$$\frac{1}{v_x |a|^2} \frac{d}{dt_v} (v_x |a|^2) + \left(\frac{k_x^2}{k^2} \frac{du_0}{dx} + \frac{m_i c}{4\pi e} \frac{d}{dx}\left(\frac{1}{\rho_0}\frac{dB_0}{dx}\right) \frac{k_x k_y}{k^2} \right) = 0 \qquad (4.28)$$

when we set the R.H.S. to zero for the moment. We show later that $(v_x |a|^2)$ becomes infinitely large at places in the flow for which $u_0^2 = \left(\frac{m_i c\, B_0}{4\pi e\, \rho_0} k_z \right)^2$. k_x and k also become infinitely large there. In this case, equation (4.28) gives, approximately:

$$(v_x |a|^2) \propto e^{-\frac{du_0}{dx}(x_0)\, t_v}$$

where $x = x_c$ is the point for which $u_o^2 = \left(\frac{m_i c B_o}{4\pi e \rho_o} \quad k_z \quad\right)^2$. This is the point for which the group velocity of the wave is approaching zero, so that the wave packet is coming to a gradual standstill against the plasma flow in the opposite direction. When $\left({du_o}/{dx}\right)_{x} t_v$ is negative we have a region of increasing density in the direction of propagation of the wave-packet and it is being compressed, which accounts for an exponential rise. If we now differentiate equation (4. 25) with respect to x, using Re ω = constant, Im ω = 0, we have

$$0 = \frac{du_o}{dx} k_x + \frac{m_i c}{4\pi e} \frac{d}{dx}\left(\frac{1}{\rho_o}\frac{dB_o}{dx}\right) k_y + v_x \frac{dk_x}{dx}$$

so that (4. 27) becomes

$$\frac{d}{dx}\left(v_x |a|^2\right) - \frac{k_x}{k^2}\frac{dk_x}{dx}\left(v_x |a|^2\right) = \text{RHS} \tag{4. 29}$$

If we set (R. H. S.) equal to zero for the while to simplify the discussion we see that equation (4. 29) yields:

$$|a|^2 \propto \frac{k}{|v_x|} \tag{4. 30}$$

We now discuss in detail the behavior of a wave-packet with a narrow band of frequency centered about ω. We have already shown that the wave propagates with a wave-vector whose x-component varies with x because the wave packet moves into regions of different properties. The squared amplitude of the wave packet envelope depends upon the factor k/v_x, where k is the magnitude of the wave-vector and v_x is the group velocity of the wave-packet. We show that k/v_x can become infinitely large as the wave-packet moves along in the plasma. This is due to the continual compression of the packet by the surrounding medium at a place where it has almost come to a standstill.

For given ω, k_y and k_z, equation (4. 25) states that

$$u_o(x) = \frac{\omega'' - Vk(x)}{k_x(x)} \qquad V = \frac{m_i c}{4\pi e}\frac{B_o}{\rho_o} k_z \tag{4.31}$$

where $\omega'' = \omega - \frac{k_y}{k_z} V \frac{d \ln B_o}{dx}$. We will take B_o/ρ_o = constant, a consequence of equations (4.15) and (4.16), and neglect $\frac{k_y}{k_z} V \frac{d \ln B_o}{dx}$ compared to ω, all for the sake of convenience for the ensuing arguments. B_o/ρ_o is not strictly constant if there are some particle collisions to make non-zero electrical resistivity and gas viscosity. To elaborate upon this point is beyond the scope of this paper. For sufficiently small regions of x the derivative term in ω'' will be nearly constant.

There are two interesting regions for frequency and wave-vector components. These regions are: $\omega'' > Vk_1$ and $\omega'' \leq Vk_1$ where $k_1^2 = k_y^2 + k_z^2$. From equation (4.25) we see that $k_x(x)$, as a function of $u_o(x)$, becomes infinitely large for $u_o^2 = V^2$. Figure 1 shows $k_x(x)$ versus $u_o(x)$ for the three cases $\omega'' > Vk_1$, $\omega'' = Vk_1$, and $\omega'' < Vk_1$. Let us always consider k_x positive. The arguments are similar for k_x negative and the velocities with opposite sign.

For a wave with $\omega'' > Vk_1$, for positive k_x, the wave has positive phase-and group-velocity in the x-direction. As $u_o(x)$ approaches -V (V is assumed positive for the moment) k_x becomes infinitely large, behaving asymptotically as: $\omega''/(u_o + V)$. The group-velocity v_x approaches zero, being nearly equal to $u_o + V$. The wave packet is moving against the plasma velocity, its group velocity is zero when its motion in the fluid exactly cancels the plasma velocity, u_o. Hence, k_x/v_x behaves like: $\omega''/(u_o + V)^2$. If $u_o(x) \simeq -V + (du_o/dx)_{x_c}(x - x_c)$, where $x = x_c$ is the point at which $u_o = -V$, then $t_v = \int^x \frac{dx'}{v_x} \sim \ln(x - x_c)$ and $k_x/v_x \sim k/v_x \sim e^{-2(\frac{du_o}{dx})_{x_c} t_v}$ If $(du_o/dx)_{x_c}$ is negative, that is, the ρ_o is increasing with x, since $\rho_o u_o$ = constant, then we have exponential growth in time for the amplitude of the wave-packet, as given by equation (4.28).

For a wave with $\omega'' = Vk_1$ either k_x is identically zero regardless of $u_o(x)$ and the group velocity is $u_o(x)$, i.e., the packet is being carried by the plasma and not moving of its own accord, or k_x becomes infinitely large for u_o = -V as in the previous case, and that discussion applies here.

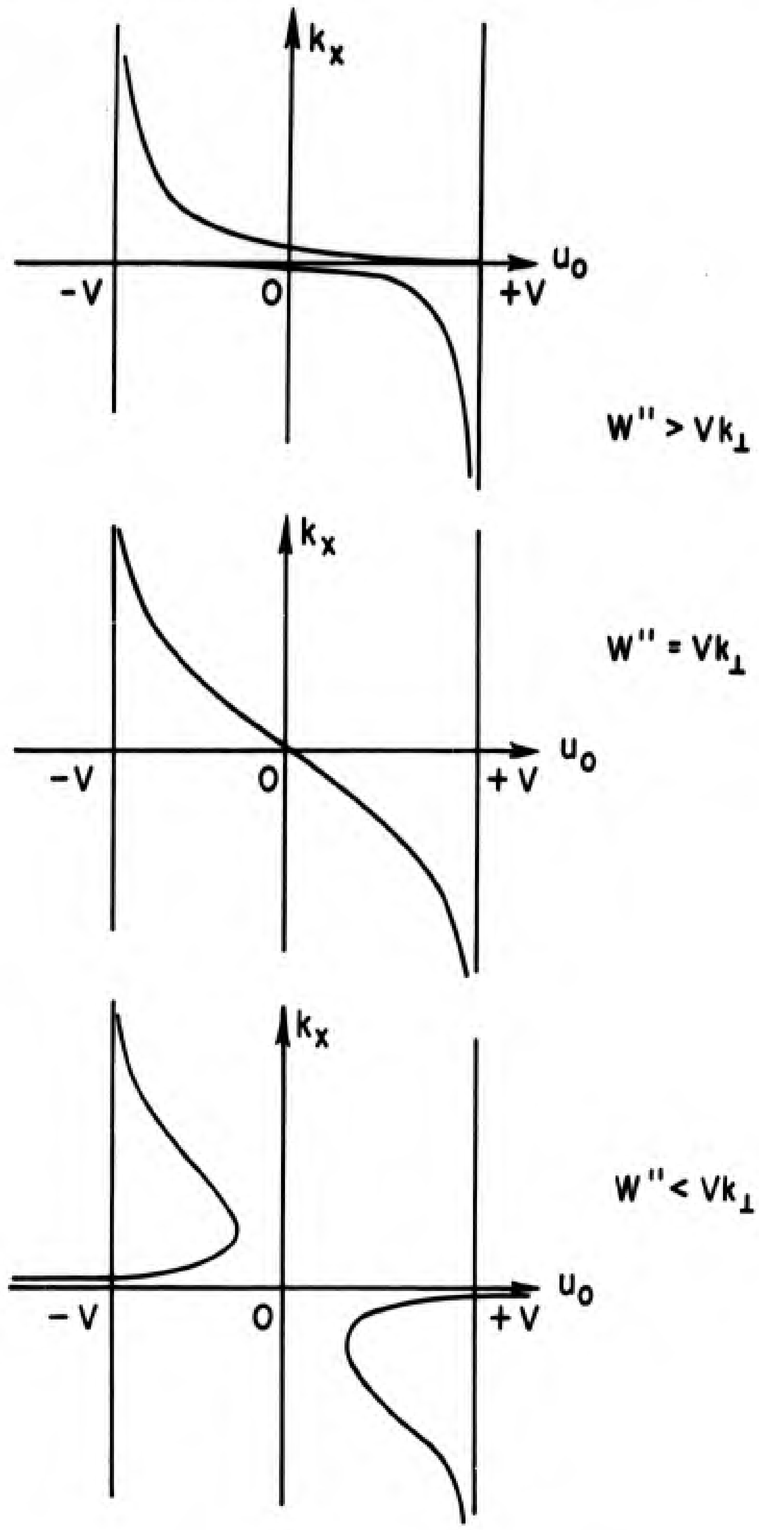


Fig. 1 The behavior of k_x the component of the packet wave-vector along the direction of the plasma velocity u_o.

5118

Also of interest is the fact that the group velocity vanishes for $u_o = 0$. If this happens, $k/v_x \sim e^{-(du_o/dx)_{x_m} t_v}$ where $x = x_m$ is the point for which $u_o = 0$. There will be exponential growth if $(du_o/dx)_{x_m}$ is negative. If $(du_o/dx)_{x_m}$ is zero but $(d^2u_o/dx^2)_{x_m}$ is not zero, then k/v_x increases as the second power of t_v.

For a wave with $\omega'' < Vk_1$ we now have the case of the wave packet with positive phase velocity but its group velocity is either positive or negative, depending upon whether its group velocity $V^{k_x/k}$ in the fluid itself is greater than or less than the fluid velocity. Again k_x becomes infinitely large when $u_o = -V$, and the analysis is the same as before. However, for the point $x = x_m$ for which $k = \frac{V k_1^2}{\omega''}$, the group velocity vanishes. This case is similar to the one already discussed when $u_o = 0$. We find that $k/v_x \sim e^{-(du_o/dx)_{x_m} t_v}$.

We now return to equation (4. 27) in order to derive the left hand side of the wave transport equation. The derivative with respect to x is performed with ω, k_y, and k_z constant. Let us make k_x, k_y, and k_z the independent variables. We use equation (4. 25), which relates k_x to ω, to do this.

$$\left.\frac{\partial}{\partial x}\right)_\omega = \left.\frac{\partial}{\partial x}\right)_{k_x} + \left.\frac{\partial k_x}{\partial x}\right)_\omega \left.\frac{\partial}{\partial k_x}\right)_x$$

$\left.\frac{\partial k_x}{\partial x}\right)_\omega$ is obtained from equation (4. 28) so that equation (4. 27) becomes, after we multiply by v_x,

$$v_x \frac{\partial}{\partial x}(v_x |a|^2) - k_x \frac{du_o}{dx} \frac{\partial}{\partial k_x}(v_x |a|^2) + \frac{k_x^2}{k^2} \frac{du_o}{dx} (v_x |a|^2) = v_x \, \mathrm{RHS} \qquad (4.\,32)$$

we have neglected $\frac{m_i c}{4\pi e} k_y \frac{d}{dx} \frac{1}{\rho_o} \frac{dB_o}{dx}$, a higher derivative term, compared to $k_x \frac{du_o}{dx}$ because we now assume that $\frac{1}{V_A r_i} \left|\frac{du_o}{dx}\right|$ is much greater than $\left|\frac{k_y}{k_x} \frac{d^2}{dx^2} \ln B_o\right|$ and that $\frac{d}{dx} B_o/\rho_o = 0$. That is, we disallow the possibility that $du_o/dx = 0$ at places in the plasma where wave-growth is expected, i.e. where the group velocity $v_x = u_o + V^{k_x/k}$ vanishes.

Equation (4. 32) has a close resemblance to the left-hand side of the time-independent particle Boltzman equation. The dependent variable is $v_x|a|^2$, $a = a_{\omega, k_y, k_z}(x)$. We have substituted k_x for ω as independent variables so that it is convenient to define $E_{k_x', k_y, k_z}(x)$ as $\frac{v_x}{(L_x/T)}|a_{\omega, k_y, k_z}|^2$. Then

$$\sum_{k_x, k_y, k_z} E_{k_x, k_y, k_z}(x) = \sum_{\omega, k_y, k_z} |a_{\omega, k_y, k_z}(x)|^2 = \iiint \frac{dt\,dy\,dz}{T L_y L_z} |\hat{B}(\hat{x}, t)|^2$$

if we use equations (4.19) and introduce $k_x = \frac{2\pi}{L_x} n_x$, $n_x = \pm 1, \pm 2, \ldots$ where L_x is sufficiently small that the group-velocity doesn't change appreciably in a distance comparable to L_x. In that case, $\Delta n_\omega = \Delta n_x \frac{\Delta\omega}{\Delta k_x} / \frac{L_x}{T}$ Remember ω, as a function of k_x and x, is given by equation (4.25). Equation (4.33) enables us to interpret $E_{\hat{k}}(x)$ as the energy-density spectrum function at the position x. That is, a wave-packet consisting of waves over a wave-vector range k_x, k_y, k_z to $k_x + dk_x$, $k_y + dk_y$, and $k_z + dk_z$ will have, averaged over a very short time, an energy per unit volume equal to $E_{\hat{k}}(x)\, dk_x\, dk_y\, dk_z$. As the packet moves to different positions in the plasma this energy-density spectrum function will change. We must remember that k_x is now treated as varying quite independently from x. We see a shifting of energy-density into or out of the range $(\hat{k}, \hat{k} + d\hat{k})$ if $E_{\hat{k}}(x)$ increases or decreases, respectively.

If we define $N_{\hat{k}} \equiv \frac{E_{\hat{k}}(x)}{\omega_{\hat{k}}^0}$, where $\omega_{\hat{k}}^0$ is the frequency of the wave of the given k in a uniform quiescent plasma, $\omega_{\hat{k}}^0 \simeq \frac{m_i c B_0}{\sqrt{\pi} e \rho_0} k_z k$ then equation (4.32) becomes

$$v_x \frac{\partial N_{\hat{k}}}{\partial x} - k_x \frac{du_0}{dx} \frac{\partial N_{\hat{k}}}{\partial k_x} = \frac{1}{\omega_{\hat{k}}^0} \frac{v_x}{L_x/T} \, \mathrm{RHS} \tag{4.33}$$

$N_{\hat{k}}$ corresponds to the time-independent particle distribution function f $(\hat{r}, \hat{v})$. Equation (4.33) corresponds to $\left(\hat{v} \cdot \frac{\partial}{\partial \hat{r}} + \frac{\hat{F}}{m} \cdot \frac{\partial}{\partial \hat{v}}\right) f = \left(\frac{\partial f}{\partial t}\right)_{\text{collisions}}$. The first term of (4.33) is the convective derivative which states that $N_{\hat{k}}$ changes because the wave-packet moves to new positions with the group velocity.

The second term is like the force term of the Boltzman equation, except that $\hat{k}$ is the independent variable instead of $\hat{v}$. The 'force' is $-k_x\, du_0/dx$. If we were dealing with quantum mechanics, $N_{\hat{k}}$ would be proportional to the number of quanta in the k^{th} state. Equation (4. 33) would correspond to the transport equation for the number of quanta in this state. In the absence of the R. H. S., equation (4. 33) becomes $\frac{d}{dt_v} N_{\hat{k}} = 0$ where $\frac{d}{dt_v}$ is the time derivative along the path of the wave-packet in its phase-space: $(\hat{x}, \hat{k})$.* We deal with large numbers of quanta and we are not concerned with quantum effects. Classically $N_{\hat{k}}$ can be interpreted as action, the action of all the waves with wave-vector $\hat{k}$.

We return to equation (4. 32) and interpret the third term by comparing it to the corresponding term of the Boltzman equation, which we write in terms of the energy density $1/2\, mv^2 f$ as:

$$\hat{v} \cdot \frac{\partial}{\partial \hat{r}} \left(\tfrac{1}{2} mv^2 f\right) + \frac{\hat{F}}{m} \cdot \frac{\partial}{\partial \hat{v}} \left(\tfrac{1}{2} mv^2 f\right) - \frac{\hat{F} \cdot \hat{v}}{\frac{1}{2} mv^2} \left(\tfrac{1}{2} mv^2 f\right) = \frac{\partial}{\partial t} \left(\tfrac{1}{2} mv^2 f\right)_{collisions}$$

so that $E_{\hat{k}}(x)$ corresponds to $\frac{1}{2} mv^2 f(\hat{r}, \hat{v})$. We see then that $\frac{du_0}{dx} \frac{k_x^2}{k^2}$ corresponds to $-\hat{F} \cdot \hat{v} / \frac{1}{2} mv^2$ which equals minus the fractional rate of increase of the kinetic energy due to the force $\hat{F}$ on the particle. Hence, $-\frac{k_x^2}{k^2} du_0/dx$ must be the fractional rate of increase of the wave-packet energy due to the force of the surrounding medium. In fact it is easy to show that $\frac{k_x k_x}{k^2} E_{\hat{k}}(x)$ is the (x, x) component of the average magnetic stress tensor for a wave packet consisting of circularly-polarized waves of magnetic field, which is the case for the fast waves we are considering. The magnetic stress tensor $P_{\alpha\beta} = \frac{-1}{4\pi} \left[B_\alpha B_\beta - \tfrac{1}{2} \delta_{\alpha\beta} B^2\right]$ where $B_\alpha = \alpha^{th}$ component of the average magnetic field in the wave-packet. Then $-\frac{k_x^2}{k^2} \frac{du_0}{dx} E_{\hat{k}} = -P_{xx} \frac{du_0}{dx}$; this is the rate of work done on the packet by the surrounding medium in compressing the packet against the magnetic pressure P_{xx} exerted by the packet.

It is shown in Appendix 6 that the R. H. S. of equation (4. 27), multiplied by $V_x(x)/\frac{L_x}{T}$, which is

$$RHS = \frac{V_x}{L_x/T} \hat{a}^{+}_{\omega, k_y, k_z}(x) \cdot \iiint \frac{dt\, dy\, dz}{T L_y L_z} \left(\frac{\partial \hat{B}(\ell, t)}{\partial t}\right)_{n-\ell}$$

* The left hand side of the wave transport equation, in general, must be $\frac{dN}{dt_v}$. When this is zero this corresponds to the adiabatic approximation in quantum mechanics that any changes in the external parameters of the system as, for example, the volume and magnetic field, are sufficiently slow so that the number of quanta in each state stays the same, as the frequency and wave number of each state slowly shift.

when averaged over x, for $|x-\bar{x}| < L_x/2$, is just the average rate of change of $E_{\mathbf{k}}(\bar{x})$, the energy-density spectrum function at the point $\bar{x}$, due to non-linear wave interaction. The rate is the average value in the time interval $|t-\bar{t}| < T/2$ where T is very short compared to the relaxation time of the wave collision process.

We neglect the term in equation (4.27) $\frac{m_i c}{4\pi e} \frac{k_x k_y}{k^2} \frac{d}{dx} \frac{1}{r_c} \frac{dB_0}{dx}$ because it contains a second derivative and because it is multiplied by a factor which we know becomes small in regions where wave amplification is important.

We average equation (4.27) over x, so that it becomes

$$\left[V_x(\bar{x}) \frac{\partial}{\partial x} - k_x \frac{du_0}{dx}(\bar{x}) \frac{\partial}{\partial k_x} + \frac{k_x^2}{k^2} \frac{du_0}{dx}(\bar{x}) \right] E_{\mathbf{k}}(\bar{x}) = \left(\frac{\partial E_{\mathbf{k}}(\bar{x})}{\partial t} \right)_{n-\ell}$$

where $(\partial E_{\mathbf{k}}(\bar{x}) / \partial t)_{n-\ell}$ is the above-mentioned average rate of change of $E_{\mathbf{k}}(x)$ due to non-linear wave interaction. Terms of order $L_x \, d \ln|u_0| / dx$ have been neglected because the plasma properties do not vary much across a distance L_x.

SECTION V

WAVE COLLISIONS

NON-LINEAR INTERACTIONS

In deriving equation (4. 14), the non-linear terms, that is terms involving the second and higher powers of the time dependent fields, were collected on the right-hand side and called $\left(\frac{\partial \hat{B}}{\partial t}\right)_{n-\ell}$. We now investigate in detail, these terms. We also discuss the corresponding non-linear terms of equation (4. 9) which can be called $\left(\frac{\partial \hat{u}}{\partial t}\right)_{n-\ell}$.

To complicate matters these terms also contain the time-independent but space varying quantities, denoted by subscript "o", which are the non-uniform plasma properties. The non-uniformity has a scale-length r_i; we assume that the waves which are involved have wavelengths somewhat shorter than this. As done in the last chapter, we Fourier analyze with periodic boundary conditions the fields in boxes whose sides have lengths L_x, L_y, and L_z, which are small compared to distance over which the non-uniformities vary appreciably. In this way, the Fourier components of the non-uniformities which correspond to the wavelengths comparable to those of the waves are small. That is, ths plasma appears uniform inside the Fourier integration box. We then use the normal waves of a uniform plasma to describe the scattering. This approximation omits effects which are of the order of $\frac{d \ln |u_o|}{dx/L_x}$. These effects are not the usual reflections of the waves from the inhomogenities of the medium for these have been accounted for by the left-hand side of equations (4. 14) and (4. 9). Instead, the effects of the non-uniformities can be represented by non-linear processes which violate the selection rule, to be discussed later, that the sum of the wave-vectors in a wave collision process must be conserved. In some ways this violation is similar to the 'Umklapp' processes in phonon scattering in crystals. However, the violation of the wave-vector rule in our case is due to the lack of any translational invariance of the equations of motion, whereas, the 'Umklapp' processes arise because of the discreteness of the crystal, i.e., translational invariance occurs only for lattice vector translations instead of infinitesimal translations, as in a uniform continuum. The amount by which the sum of the wave vectors is not conserved in our case must be of the order of wave-vectors which are typical of the Fourier decomposition of the non-uniform fields, which we have assumed are small compared to the wave-vectors of the wave field. Also this effect only applies to the x-components of the wave-vectors since the non-uniformity varies only with x. If k_x is a typical wave-vector for the non-uniform fields, then our choice of L_x is to make $L_x k_x \ll 1$. k_x is of the order of $\frac{d}{dx} \ln |u_o|$, so that $L_x \left| \frac{d}{dx} \ln |u_o| \right| \ll 1$, as stated earlier.

Now that we have replaced the non-uniform time-independent fields by their average values inside the Fourier-box, we make the approximations made previously that the wavelengths of the waves lie between r_e and r_i so that

$$\hat{E} + \frac{\hat{V}_e}{c} \times (\hat{B}_o + \hat{B}) = 0$$

and we also assume, as before, that for these wavelengths the Maxwell displacement current is small, so that we do not consider plasma oscillations and the phase-velocities are of the order of the Alfvén speed, V_A. Equations (2.3) to (2.8) become

$$\frac{\partial}{\partial t} \sqrt{\tfrac{\rho_o}{2}}\,\hat{u} - \left(\hat{\nabla} \times \frac{\hat{B}}{\sqrt{8\pi}}\right) \times \hat{e}_z V_A + \hat{u}_o \cdot \hat{\nabla} \sqrt{\tfrac{\rho_o}{2}}\,\hat{u} \, \frac{1}{\sqrt{\frac{\rho_o}{2}}} = \left(\hat{\nabla} \times \frac{\hat{B}}{\sqrt{8\pi}}\right) \times \hat{e}_z \left(\frac{1}{1+\rho/\rho_o} - 1\right) V_A$$
$$+ \left(\hat{\nabla} \times \frac{\hat{B}}{\sqrt{8\pi}}\right) \times \frac{\hat{B}}{\sqrt{8\pi}} \, \frac{1}{1+\rho/\rho_o} \, \frac{1}{\sqrt{\rho_o/2}} - \sqrt{\tfrac{\rho_o}{2}}\,\hat{u} \cdot \hat{\nabla} \sqrt{\tfrac{\rho_o}{2}}\,\hat{u} \, \frac{1}{\sqrt{\frac{\rho_o}{2}}} \tag{5.1}$$

$$\frac{\partial}{\partial t} \frac{\hat{B}}{\sqrt{8\pi}} - \hat{\nabla} \times \left[\sqrt{\tfrac{\rho_o}{2}}\,\hat{u} \times \hat{e}_z V_A + \hat{u}_o \times \frac{\hat{B}}{\sqrt{8\pi}} - \left(\hat{\nabla} \times \frac{\hat{B}}{\sqrt{8\pi}}\right) \times \hat{e}_z V_A r_i \right] =$$
$$\hat{\nabla} \times \left[\sqrt{\tfrac{\rho_o}{2}}\,\hat{u} \times \frac{\hat{B}}{\sqrt{8\pi}} \, \frac{1}{\sqrt{\frac{\rho_o}{2}}} - \left(\hat{\nabla} \times \frac{\hat{B}}{\sqrt{8\pi}}\right) \times \frac{\hat{B}}{\sqrt{8\pi}} \, \frac{r_i}{1+\rho/\rho_o} \, \frac{1}{\sqrt{\rho_o/2}} - \left(\hat{\nabla} \times \frac{\hat{B}}{\sqrt{8\pi}}\right) \times \hat{e}_z \left(\frac{1}{1+\rho/\rho_o} - 1\right) V_A r_i \right] \tag{5.2}$$

$$\frac{\partial}{\partial t} \rho/\rho_o + \hat{\nabla} \cdot \sqrt{\tfrac{\rho_o}{2}}\,\hat{u} \, \frac{1}{\sqrt{\frac{\rho_o}{2}}} + \hat{u}_o \cdot \hat{\nabla}\, \rho/\rho_o = - \hat{\nabla} \cdot \sqrt{\tfrac{\rho_o}{2}}\,\hat{u}\, \rho/\rho_o \, \frac{1}{\sqrt{\rho_o/2}} \tag{5.3}$$

where $V_A = \frac{B_o}{\sqrt{4\pi\rho_c}}$

$$r_i = \sqrt{\frac{m_i c^2}{4\pi n_o e^2}}$$

On the left-hand sides are the linear terms which have given us the normal waves discussed earlier. We use the left-hand side to make clear the properties of the normal waves in the absence of the non-linear terms. Our aim is to obtain the time rate of change of the amplitude of the normal wave with wave-vector $\hat{k}$ due to interactions with waves of wave-vector $\hat{p}$, $\hat{q}$, $\hat{r}$, $\hat{s}$,... and to use this as the right-hand side of the transport equation (4. 34). As already mentioned, the wave interaction is analogous to anharmonic corrections to the motion of a simple harmonic oscillator. Because of these effects, if the oscillator starts its motion in only two of its normal modes it will, in some characteristic time τ, excite motion in several of its other modes. In the case of the interaction of lattice vibrations τ will be the relaxation time which enters into the kinetic theory expression for the thermal conductivity of pure crystals of large size and of not too lot temperature, in which the anharmonic effects would be expected to dominate[17]. For the plasma case, we obtain estimates of relaxation times τ both for fast and fast wave interaction and for fast and slow wave interaction and apply this in the next chapter to an estimate of the thickness of an MHD shock for the plasma conditions we have outlined previously. Other kinetic-theoretic plasma phenomena, such as electrical and thermal conductivity, could be described with these wave relaxation times, but this is beyond the scope of this work at present.

The idea of a relaxation time is introduced because the expression for $\partial A_{\hat{k}}/\partial t$ the rate of change of the amplitude $A_{\hat{k}}$ of the $\hat{k}^{th}$ wave is an integral equation which depends upon the amplitude of the $\hat{k}^{th}$ wave and all other waves which are allowed to interact with the $\hat{k}^{th}$ wave by the selection rules. This rate is evaluated approximately by assuming that a distribution of waves at time $t = 0$, say, is only slightly changed due to the interaction in times comparable to the periods of the waves. Time-dependent perturbation theory, in common usage in quantum theory, is applied in first and second order. The rate of change of amplitude then is an integral over terms that explicitly depend upon the wave amplitudes at $t = 0$. By dividing the rate by the value of the amplitude at $t = 0$ we have what may be called 'the transition probability per unit time' for transitions of the wave of the k^{th} state to any other wave state allowed by the selection rules on the wave-vector and wave-frequency. Though we deal with no quantum effects, since the number of quanta is always large, the expressions are very similar to those of quantum theory. The coefficients of the amplitudes in the integral expression for the rates are the matrix elements of the interaction Hamiltonian between the interacting wave states. The interaction Hamiltonian resembles the anharmonic corrections to the potential of a simple collection of harmonic oscillators, the normal waves, as we have already mentioned[20]. The relaxation time we speak of is just the reciprocal of the transition probability per unit time. This relaxation time depends upon the assumed initial distribution of waves, i.e., it is not shown that there exists a relaxation time which is characteristic of the wave interactions regardless of the shape of the wave distribution in wave vector space[3]. One can compute relaxation times for wave distributions which deviate from spherical uniformity. Generally, there will be a different relaxation time for each spherical harmonic.* Depending upon what property of the wave distribution is important, say, energy flux, the appropriate relaxation time τ_k must be chosen. Then the right-hand side of equation (4.34), the rate due to wave-interaction, is represented by $-\frac{1}{\tau_{\hat{k}}}\left(|A_{\hat{k}}|^2 - \overline{|A|^2}\right)$. $\overline{|A|^2}$ is the average of the squared amplitude over a surface in $\hat{k}$-space. This surface is analogous to the energy shell encountered in scattering theory because it is the locus of all $\hat{k}$ such that the sum of the frequencies of the interacting waves is conserved. This selection rule of frequency arises because the interaction occurs in times during which the waves go through many cycles, and only those interactions are important which do not depend, in time, upon the rapid oscillation of the waves.

* Because of the external magnetic field the linearized collision integral is invariant only under rotations in $\hat{k}$-space about the magnetic field direction. Therefore, strictly speaking, one should consider only the azial harmonics $\{e^{im\phi}\}$ where ϕ is the azimuthal angle. A doubly-infinite denumerable set of orthogonal polynomials in the variables $(k_x^2 + k_y^2)^{1/2}$ and k_z, with respect to the weighting function $(N_k^o)^2$ can be found. (v. p. 60 for definition of N_k^o). There is then a single relaxation time for a given orthogonal polynomial.

The validity of the perturbation calculation depends upon the average wave pressure being small compared to the magnetic pressure in the plasma. We call the ratio of these pressures β . The relaxation time $\tau_{\mathbf{k}}$ is of the order of magnitude of $(\beta\omega_{\mathbf{k}})^{-1}$. For the perturbation to be small, $\tau_{\mathbf{k}}$ must be large compared to $\omega_{\mathbf{k}}{}^{-1}$, so that β must be small, as we stated. We may define a mean free path Λ_k for the waves by taking the group velocity times $\tau_{\mathbf{k}}$. Λk then is of the order of magnitude of $(\beta\,\omega_k)^{-1}$ which must be large compared to the wavelength, which it is if β is small. Estimates of β based upon energy and momentum considerations for the case of MHD compression fronts and not too strong MHD shocks show β is at most three-tenths, which is sufficiently small to justify the perturbation calculation.

The condition that $\tau_{\mathbf{k}}$ be large compared to $\omega_{\mathbf{k}}{}^{-1}$, or $\Lambda_{\mathbf{k}}$ large compared to $\omega_k{}^{-1}$, also justifies the use of the random phase approximation since the waves go through many cycles or move many wavelengths in the time $\tau_{\mathbf{k}}$, which is the mean-free time between successive collisions. The phase correlations between three waves which have collided once, will be obliterated by collisions with other waves by the time these same three waves collide again. Hence, we expect that any three waves which do collide will be uncorrelated in phase. We cannot prove this and we use random phases as an assumption that our observations are coarse-grained with respect to detection of phase correlation. Appendix 3 contains the estimate that the wave interaction energy, taking into account phase correlations at a wave equilibrium temperature T, is of the order of $\frac{1}{r_e{}^3}\,kT/(B_o{}^2/8\pi)$ times the mean wave energy. This is 10^{-15} times the mean wave energy for our plasma conditions.*

Randomly-distributed phases at any given initial time characterizes the statistical ensemble underlying our kinetic theory of waves. The transport equation which we derive contains no phase interaction. We concern ourselves only with the behavior of the squared-amplitude of the wave. Our concern is the departure from or approach to, the stationary state of the squared-amplitude. The stationary state, we show, is that of equal wave amplitude over all wave-vector space. It should be noted that random phases

* This implies that we must deal with a wave distribution which is drastically out of equilibrium with the particles.

do not eliminate correlations of the fields at different points in the plasma, at a given instant. As pointed out in Appendix 4, the random phase distribution is equivalent to a normal distribution of the field components.

WAVE SCATTERING EQUATIONS

We now derive the rate of change of the amplitude for the wave-vector $\hat{k}$. We expand $1/(1 + \rho/\rho_o)$ into the power series $1 - \rho/\rho_o + (\rho/\rho_o)^2 + \ldots$ in equations (5.1) and (5.2). We then substitute the Fourier eigenvector sums, equations (3.21), for $\rho(\hat{x}, t)$, $\hat{u}(\hat{x}, t)$, and $\hat{B}(\hat{x}, t)$ in equations (5.1), (5.2) and (5.3). The left-hand sides of equations (5.1) and (5.2) are as follows:

$$\sum_k \mathcal{Y}_k \, e^{i(\hat{k}\cdot\hat{x} - \omega_k t)} \, \dot{A}_k(t), \qquad k = (\hat{k}, m_k = \pm 1)$$

since $\mathcal{Y}_k e^{i(\hat{k}\cdot\hat{x} - \omega_k t)}$ is an eigen solution of the linear equations, when written in the six-dimensional notation explained in chapter 3. The right-hand sides of equations (5.1) and (5.2) become

$$\sum_{p,q} S^{(1)}_{pq} A_p(t) A_q(t) \, e^{i[(\hat{p}+\hat{q})\cdot\hat{x} - (\omega_p+\omega_q)t]}$$
$$+ \sum_{p,q,r} S^{(2)}_{pqr} A_p(t) A_q(t) A_r(t) \, e^{i[(\hat{p}+\hat{q}+\hat{r})\cdot\hat{x} - (\omega_p+\omega_q+\omega_r)t]}$$
$$+ \ldots \tag{5.4}$$

where $S^{(1)}$, $S^{(2)}$, ..., are six-dimensional vectors which contain all p, q, r,... dependent factors resulting from the "$\hat{\nabla}$x...", "x$\hat{e}_z$". etc. operations on the field quantities. $A_p(t)$, $A_q(t)$, $A_r(t)$,... are the time-dependent wave amplitudes. In the absence of the non-linear terms these would be constant. That the product of more than two amplitudes appears is due to the power expansion of $(1 + \rho/\rho_o)^{-1}$, otherwise the non-linearity is quadratic, corresponding to a cubic term in the Hamiltonian.

We now invert the Fourier sums by multiplying by $e^{-i(\hat{k}\cdot\hat{x} - \omega_k t)} \mathcal{Y}_k^+$ and integrating over $\hat{x}$ so that our equations become

$$\dot{A}_k(t) = \sum_{p,q} S^{(1)}_{kpq} A_p(t) A_q(t) \Delta^{(3)}(\hat{k} - \hat{p} - \hat{q}) \, e^{-i(\omega_p + \omega_q - \omega_k)t}$$

$$+ \sum_{p,q,r} S^{(2)}_{kpqr} A_p(t) A_q(t) A_r(t) \Delta^{(3)}(\hat{k}-\hat{p}-\hat{q}-\hat{r}) e^{-i(\omega_p+\omega_q+\omega_r-\omega_k)t} \tag{5.5}$$
$$+ \cdots$$

where $\Delta^{(3)}(\hat{k}-\hat{p}-\hat{q})$ and $\Delta^{(3)}(\hat{k}-\hat{p}-\hat{q}-\hat{r})$ are generalized Kronecker-deltas which are unity when their arguments are zero and zero otherwise. They express the selection rule on wave-vectors in a collision.

$$S^{(1)}_{kpq} = \mathcal{Y}^+_k \cdot S^{(1)}_{pq} \quad ; \quad S^{(2)}_{kpqr} = \mathcal{Y}^+_k \cdot S^{(2)}_{pqr} \quad ; \cdots$$

Equation (5.5) is then integrated:

$$A_k(t) \simeq A_k(t_0) + \sum_{p,q} S^{(1)}_{kpq} A_p(t_0) A_q(t_0) \Delta^{(3)}(\hat{k}-\hat{p}-\hat{q}) \int_{t_0}^{t} e^{-i(\omega_p+\omega_q-\omega_k)t'} dt'$$
$$+ \sum_{p,q,r} S^{(2)}_{kpqr} A_p(t_0) A_q(t_0) A_r(t_0) \Delta^{(3)}(\hat{k}-\hat{p}-\hat{q}-\hat{r}) \int_{t_0}^{t} e^{-i(\omega_p+\omega_q+\omega_r-\omega_k)t'} dt' \tag{5.6}$$
$$+ \cdots$$

if we have the amplitudes varying slowly with respect to the exponential oscillating factors. The amplitudes at $t = t_o$ are denoted by the argument "t_o". That $t = t_o$ is our reference for initial data is of no consequence. It is only essential that $t - t_o$ is not comparable to times over which the amplitudes vary appreciably.

Then, using equations (5.5) and (5.6) we find that

$$|\dot{A}_k(t)|^2 = A^+_k(t) \dot{A}_k(t) + \text{c.c.} =$$
$$\sum_{p,q,p',q'} S^{(1)+}_{kpq} S^{(1)}_{kp'q'} A^+_p(t_0) A^+_q(t_0) A_{p'}(t_0) A_{q'}(t_0) \Delta^{(3)}(\hat{k}-\hat{p}-\hat{q}) \Delta^{(3)}(\hat{k}-\hat{p}'-\hat{q}') \cdot$$
$$\cdot e^{-i(\omega_{p'}+\omega_{q'}-\omega_k)t} \int_{t_0}^{t} e^{i(\omega_p+\omega_q-\omega_k)t'} dt' + \cdots$$
$$+ \sum_{pqr} S^{(2)}_{kpqr} A_p(t) A_q(t) A_r(t) A^+_k(t_0) \Delta^{(3)}(\hat{p}+\hat{q}+\hat{r}-\hat{k}) e^{-i(\omega_p+\omega_q+\omega_r-\omega_k)t}$$
$$+ \cdots \tag{5.6'}$$

Equation (5.6') must be expanded still further, by means of equation (5.6), so that all terms which involve up to four amplitude factors evaluated at time t_o are displayed.

$$|\dot{A}_k(t)|^2 = \sum_{p,q,p',q'} S^{(1)+}_{kpq} S^{(1)}_{kp'q'} A^+_p(t_0) A^+_q(t_0) A_{p'}(t_0) A_{q'}(t_0) \Delta^{(3)}(\hat{p}+\hat{q}-\hat{k}) \Delta^{(3)}(\hat{p}'+\hat{q}'-\hat{k}) \cdot$$
$$\cdot e^{-i(\omega_{p'}+\omega_{q'}-\omega_k)t} \int_{t_0}^{t} e^{i(\omega_p+\omega_q-\omega_k)t'} dt' \tag{5.7}$$
$$+ \sum_{p,q,p',q'} S^{(1)}_{kpq} S^{(1)}_{pp'q'} A_{p'}(t_0) A_{q'}(t_0) A_q(t_0) A^+_k(t_0) \quad \Delta^{(3)}(\hat{p}+\hat{q}-\hat{k}) \Delta^{(3)}(\hat{p}'+\hat{q}'-\hat{p}) \cdot$$

$$\cdot\, e^{-i(\omega_p+\omega_q-\omega_k)t}\int_{t_0}^{t} e^{-i(\omega_{p'}+\omega_{q'}-\omega_p)t'}dt'$$

$$+\sum_{p,q,p',q'} S^{(1)}_{kpq} S^{(1)}_{qp'q'} A_p(t_0)A_{p'}(t_0)A_{q'}(t_0)A^{+}_{k}(t_0)\ \Delta^{(3)}(\hat p+\hat q-\hat k)\,\Delta^{(3)}(\hat p'+\hat q'-\hat q)\cdot$$

$$\cdot\, e^{-i(\omega_p+\omega_q-\omega_k)t}\int_{t_0}^{t} e^{-i(\omega_{p'}+\omega_{q'}-\omega_q)t'}dt'$$

$$+\sum_{p,q,r} S^{(2)}_{kpqr} A_p(t_0)A_q(t_0)A_r(t_0)A^{+}_{k}(t_0)\ \Delta^{(3)}(\hat p+\hat q+\hat r-\hat k)\, e^{-i(\omega_p+\omega_q+\omega_r-\omega_k)t}$$

$$+\ \text{c.c.}\ +\ \ldots$$

We now apply the random phase approximation to amplitudes at $t=t_0$. Any pair of amplitudes, $A_p(t_0)\,A_{p'}(t_0)$, averaged over the phases, is $|A_p(t_0)|^2\Delta^{(3)}(\hat p+\hat p')\,\delta\, m_p, m_{p'}$. We recall that $A_{-p', m_p} = A^{+}_{p', m_p}$. It is not difficult to show that the term involving $S^{(2)}_{kpqr}$ of equation (5.7) will vanish when the random phases are averaged over, making k equal to one of the other three: p, q, or r and the remaining two becoming negatives of each other. We then have:

$$\begin{aligned} |A_k(t)|^2 \simeq \sum_{p,q}\Big\{ & S^{(1)+}_{kpq}\left(S^{(1)}_{kpq}+S^{(1)}_{kqp}\right)|A_p|^2|A_q|^2 \\ & + S^{(1)}_{kpq}\left(S^{(1)}_{p,k,-q}+S^{(1)}_{p,-q,k}\right)|A_q|^2|A_k|^2 \\ & + S^{(1)}_{kpq}\left(S^{(1)}_{q,k,-p}+S^{(1)}_{q,-p,k}\right)|A_p|^2|A_k|^2\Big\}. \\ & \cdot\, \Delta^{(3)}(\hat p+\hat q-\hat k)\ 2\pi\,\delta(\omega_p+\omega_q-\omega_k) \end{aligned} \tag{5.8}$$

where we have used $2\,R_e\left\{e^{-i(\omega-\omega_0)t}\int_{t_0}^{t} e^{i(\omega-\omega_0)t'}dt'\right\} \underset{|t-t_0|\gg|\omega-\omega_0|^{-1}}{\sim} 2\pi\,\delta(\omega-\omega_0)$

and we have dropped the argument "t_0" in the amplitudes.

It is shown in Appendix 2 that

$$\begin{aligned} \left(S^{(1)}_{kpq}+S^{(1)}_{kqp}\right)\omega_q &= -\left(S^{(1)}_{q,-p,k}+S^{(1)}_{q,k,-p}\right)^{+}\omega_k \\ \left(S^{(1)}_{kpq}+S^{(1)}_{kqp}\right)\omega_p &= -\left(S^{(1)}_{p,k,-q}+S^{(1)}_{p,-q,k}\right)^{+}\omega_k \end{aligned} \tag{5.9}$$

when k, p, and q all belong to the same modes. If q belongs to a different mode from k and p then

$$S^{(1)}_{kpq}\,\omega_q = -\left(S^{(1)}_{q,-p,k} + S^{(1)}_{q,k,-p}\right)^{+}\omega_k$$

$$S^{(1)}_{kpq}\,\omega_p = -\,S^{(1)\,+}_{p,k,-q}\;\omega_k$$

We also have the condition that

$$S^{(1)}_{-k,-p,-q} = S^{(1)\,+}_{kpq}$$

because the fields $\hat{u}(\hat{x},\,t)$, $\hat{B}(\hat{x},\,t)$ are real-valued and the eigenvectors are chosen so that $\mathcal{Y}_{-p} = \mathcal{Y}_{p}^{+}$. We then can write equation (5.8) as follows*

$$\dot{N}_k = \sum_{p,q} \Delta^{(3)}(\hat{p}+\hat{q}-\hat{k})\,2\pi\,\delta(\omega_p+\omega_q-\omega_k)\,M_{kpq}\left[N_pN_q - N_k(N_p+N_q)\right] \qquad (5.10)$$

where $N_k = \dfrac{|A_k|^2}{\omega_k^{\,o}}$ is the action density of the kth state. $\omega_k^{\,o}$ is the frequency of the wave, of wave-vector $\hat{k}$, relative to the fluid, i.e., the Doppler shift is excluded.

* Using the methods of second quantization, one may derive equation (5.10) as follows: using the free-field Hamiltonian and the interaction Hamiltonian as given in Appendix 2 and the usual creation and annihilation operators in the interaction representation $a_k^{+} = \frac{1}{\sqrt{\omega_k}}A_k$; $[a_k, a_{k'}^{+}] = \delta_{kk'}$, we may write the time rate of change of a many-wave state-vector Ψ in momentum space as follows:

$$i\frac{\partial}{\partial t}\Psi = H_1(t)\,\Psi$$

$$\langle t|N_k|t\rangle \equiv \Psi^{+}(t)\,N_k\,\Psi(t) \qquad ; \qquad N_k = a_k^{+}a_k$$

$$i\frac{\partial}{\partial t}\langle t|N_k|t\rangle = \langle t|[N_k, H_1(t)]|t\rangle = \langle t_o|U^{+}[N_k, H_1(t)]|U|t_o\rangle$$

$$U \equiv U(t,t_o) = P\left\{e^{-i\int_{t_o}^{t} H_1(t')dt'}\right\} \sim 1 + \frac{(-i)}{1!}\int_{t_o}^{t} H_1(t')dt' + \frac{(-i)^2}{2!}\iint_{t_o t_o}^{t\,t} dt_1 dt_2\, P\{H_1(t_1)H_1(t_2)\} + \cdots$$

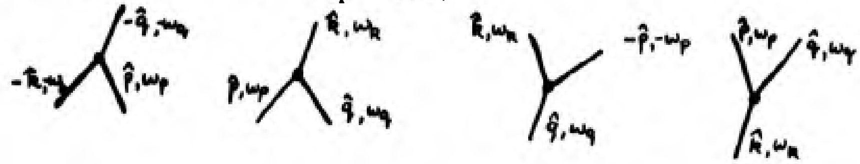
$$\therefore \frac{\partial}{\partial t}\langle t|N_k|t\rangle \sim \langle t_o|[\textstyle\int_{t_o}^{t} H_1(t')dt', [N_k, H_1(t)]]|t_o\rangle + \cdots$$

$$\cong \sum_{kpq} M_{kpq}\,\Delta^{(3)}(\hat{p}+\hat{q}-\hat{k})\,2\pi\,\delta(\omega_p+\omega_q-\omega_k)\times$$
$$\times\left[N_pN_q(N_k+1) - N_k(N_p+1)(N_q+1)\right]$$

$$\because [N_k, a_{k'}] = -a_k\Delta^{(3)}(\hat{k}-\hat{k}') \;;\; [N_k, a_{k'}^{+}] = a_k^{+}\Delta^{(3)}(\hat{k}-\hat{k}')$$

$$[N_k, H_1(t)] = H_{1k}(t) - H_{1k}^{+}(t) \;;\; H_{1k}(t) \equiv \sum_{pq} M_{kpq}\,\Delta^{(3)}(\hat{p}+\hat{q}-\hat{k})\,e^{i(\omega_k-\omega_p-\omega_q)(t)}\,a_k^{+}a_pa_q.$$

There is an elementary derivation by Peierls[17] using the usual perturbation expression for the transition probability per unit time for the four first order processes:



$$M_{kpq} = \begin{cases} \left| S^{(1)}_{kpq} + S^{(1)}_{kqp} \right|^2 \dfrac{\omega_p^o \omega_q^o}{\omega_k^o} & \text{if p, q, k refer to the same modes} \\ \left| S^{(1)}_{kpq} \right|^2 \dfrac{\omega_p^o \omega_q^o}{\omega_k^o} & \text{if q mode differs from p and k modes} \end{cases} \tag{5.11}$$

We see that, due to equations (5.9) and (5.10),

$$\dot{N}_k = -\dot{N}_p = -\dot{N}_q \tag{5.12}$$

if we have only three waves k, p, q interacting. This states that the rates of transfer of action are equal for the three states, the sign of the rate of a state depending upon the sign of the frequency of that state in the $\delta(\omega_p + \omega_q - \omega_k)$ factor.

The rate of transfer of action is identically zero if $N_k \propto \frac{1}{\omega_k^o}$ or $|A_k|^2 =$ constant for all $\hat{k}$, because of $\delta(\omega_p + \omega_q - \omega_k)$. This stationary state corresponds to a uniform energy spectrum. Though this is a stationary state, it is not unique because the total momentum density of the waves is conserved, as shown below. The general stationary distribution is

$$N_k \propto \frac{1}{\omega_k^o - \hat{k}\cdot\hat{w}}$$

the total momentum being proportional to $\hat{w}$, for small $\hat{w}$, analogous to the motion of the whole fluid with a velocity $\hat{w}$. Non-zero $\hat{w}$ implies energy flux exists for the stationary distribution. Hence, the relaxation time for energy conductivity cannot involve only the fast waves.

If we consider some function of $\hat{k}$, $f(\hat{k})$, and sum it with N_k, we have some total property of the wave distribution. If $f(\hat{k}) = \omega_k$, then $\Sigma f(k) N_k = E$, wave energy-density. If $f(\hat{k}) = \hat{k}$, then $\Sigma f(\hat{k}) N_k = \hat{M}$, wave momentum-density, and so on. We later are interested in the time rate of change of such total quantities.

We see from equation (5.10) and the symmetries implied by (5.9) that for any $f(\hat{k})$ for which the sums can be performed,

$$\frac{\partial}{\partial t}\sum_k f(\hat{k}) N_k = \sum_k f(\hat{k})\dot{N}_k$$
$$= \sum_{kpq} \Delta^{(3)}(\hat{p}+\hat{q}-\hat{k})\, 2\pi\delta(\omega_p+\omega_q-\omega_k)\cdot$$
$$\cdot M_{kpq} N_p N_q \left[f(\hat{k}) - f(\hat{p}) - f(\hat{q}) \right] \tag{5.13}$$

We easily conclude from equation (5.13) that energy and momentum densities are conserved.

We must now consider in detail the scattering of specific waves. As we have mentioned several times, the fast waves are more important than the slow waves with regard to their ability to diffuse and to randomize by scattering. There are several reasons. First, the fast waves are not heavily damped by either the ions or electrons for the range of wavelengths of physical interest, i.e., between r_e and r_i, while the slow waves are heavily damped. This means that the fast wave and not the slow wave distribution has a chance to amplify at the expense of plasma non-uniformities at an exponential rate that competes favorably against the damping rate. Second, the velocities of the fast waves equal or exceed the Alfvén speed while those of the slow waves never exceed the Alfvén speed. Therefore, the fast waves, and not the slow waves, can move along with the non-uniformities of an MHD flow in order to make the amplification mechanism last a sufficiently long time for non-isentropic compression of their wave packets. And third, the fast waves scatter with fast waves at a rate which we show later, is faster by at least a factor of $(kr_i)^2$, or about ten to one-hundred, than the rate at which fast waves scatter with slow waves. $\hat{k}$ is a mean wave-vector of the wave-distribution.

As we have written down equation (5.10) we have not specified the modes over which we sum or to which N_p, N_q, and N_k refer.

Let us consider first the case of fast waves only. We obtain from equations (5.1) and (5.2) the general expression for $S_{kpq}^{(1)}$, the scattering matrix element. We will then use approximations that rely on k_z, p_z, or q_z

being large compared to r_i^{-1}, so that only the largest terms of the general matrix element are retained.

In terms of the eigen-vectors of the linear equations which are given in equations (3.19) and (3.20),

$$S^{(1)}_{kpq} = \hat{\eta}_k^{+} \cdot \left[(i\hat{p} \times \hat{\mathcal{E}}_p) \times \hat{e}_z \, \xi_q V_A + (i\hat{p} \times \hat{\mathcal{E}}_p) \times \hat{\mathcal{E}}_q \frac{1}{\sqrt{\frac{\rho_0}{2}}} - \hat{\eta}_p \cdot i\hat{q} \, \hat{\eta}_q \frac{1}{\sqrt{\frac{\rho_0}{2}}} \right]$$
$$+ \hat{\mathcal{E}}_k^{+} \cdot i\hat{k} \times \left[\hat{\eta}_p \times \hat{\mathcal{E}}_q \frac{1}{\sqrt{\frac{\rho_0}{2}}} - (i\hat{p} \times \hat{\mathcal{E}}_p) \times \hat{\mathcal{E}}_q \frac{r_i}{\sqrt{\frac{\rho_0}{2}}} + (i\hat{p} \times \hat{\mathcal{E}}_p) \times \hat{e}_z \xi_q V_A r_i \right] \qquad (5.14)$$

where $\hat{\eta}_k$ and $\hat{\mathcal{E}}_k$ are the polarization vectors for fluid velocity and magnetic field, respectively. $\xi_k = \frac{\hat{k} \cdot \hat{\eta}_k}{\omega_k^0} \frac{1}{\sqrt{\rho_0/2}}$ is the fourier coefficient of ρ/ρ_0.

For fast waves $\hat{\eta}_k$ is of the order $1/kr_i$ smaller than $\hat{\mathcal{E}}_k$. $\hat{\mathcal{E}}_k$ is the circular-polarization vector to order $1/kr_i$. Hence, the largest terms of $S_{kpq}^{(1)}$ are

$$- \hat{\mathcal{E}}_k^{+} \cdot i\hat{k} \times \left[(i\hat{p} \times \hat{\mathcal{E}}_p) \times \hat{\mathcal{E}}_q \frac{r_i}{\sqrt{\rho_0/2}} \right]$$

Since $i\hat{k} \times \hat{\mathcal{E}}_k = \dfrac{\omega_k^0}{V_A r_i k_z} \hat{\mathcal{E}}_k$ plus terms of order $1/kr_i$ smaller, we have, to highest order in kr_i, pr_i, and qr_i

$$S^{(1)}_{kpq} \simeq \frac{\omega_k^0 \omega_p^0}{k_z p_z} \frac{r_i}{(V_A r_i)^2 \sqrt{\rho_0/2}} \quad \hat{\mathcal{E}}_k^{+} \cdot \hat{\mathcal{E}}_p \times \hat{\mathcal{E}}_q$$

Hence, M_{kpq}, which appears as the kernel of the scattering equation (5.10), and is defined by equation (5.11), is approximately, to highest order in kr_i, pr_i and qr_i:

$$M_{kpq} \simeq \frac{\omega_k^0 \omega_p^0 \omega_q^0}{(V_A^2 r_i)^2 \frac{\rho_0}{2} (k_z p_z q_z)^2} (\omega_p^0 q_z - \omega_q^0 p_z)^2 \left| \hat{\mathcal{E}}_k^{+} \cdot \hat{\mathcal{E}}_p \times \hat{\mathcal{E}}_q \right|^2 \qquad (5.15)$$

Expression (5.15) is completely symmetric under permutation of the indices, as it should be for k, p, q all referring to fast waves. We will use this expression later to obtain an estimate of the relaxation time due to the scattering of three fast waves.

Now let us take k and p as fast waves and q as a slow wave. For $kr_i \gg 1$, the magnetic polarization vector is of order $1/kr_i$ smaller than the fluid velocity polarization vector which is nearly circularly-polarized around the z-axis, the direction of the magnetic field. Corrections to the circular-polarization are of order $1/kr_i$. Hence,

$$S^{(1)}_{kpq} \simeq (i\hat{k}r_i \times \hat{\mathcal{E}}_k)^{+} \cdot (i\hat{p} \times \hat{\mathcal{E}}_p) \times \left[\hat{e}_z \xi_q V_A - \hat{\mathcal{E}}_q \frac{1}{\sqrt{\rho_0/2}}\right] + (i\hat{k}r_i \times \hat{\mathcal{E}}_k)^{+} \cdot \left[-i\hat{q}r_i \times \hat{\mathcal{E}}_q + \hat{\eta}_q\right] \times \hat{\mathcal{E}}_p \frac{1}{\sqrt{\rho_0/2}} \tag{5.16}$$

However, this can be simplified to highest order in kr_i, pr_i, and qr_i.

$$S^{(1)}_{kpq} \simeq \frac{\omega_k^0}{V_A r_i k_z \omega_q^0 \sqrt{\rho_0/2}} \left(\frac{\omega_p^0}{p_z} - \frac{\omega_q^0}{q_z}\right) \hat{q}\cdot\hat{\eta}_q \; \hat{\mathcal{E}}_k^{+} \cdot \hat{\mathcal{E}}_p \times \hat{e}_z \tag{5.17}$$

Hence, M_{kpq} for the scattering equation, by equations (5. 11) and (5. 17) is approximately, to highest order in kr_i, pr_i, and qr_i:

$$M_{kpq} \simeq \frac{\omega_k^0 \omega_p^0}{\omega_q^0 \frac{\rho_0}{2} (k_z p_z q_z)^2} (\omega_q^0 p_z - \omega_p^0 q_z)^2 \, |\hat{q}\cdot\hat{\eta}|^2 \, |\hat{\mathcal{E}}_k^{+} \cdot \hat{\mathcal{E}}_p \times \hat{e}_z|^2 \tag{5.18}$$

which is symmetric under exchange of p and q indices. We use expression (5. 18) to estimate the relaxation time for the collisions of two fast waves with a slow wave. At first sight, the expression for M_{kpq} for three fast waves, (5. 15), looks to be the same order of magnitude as expression (5. 18) for two fast waves and a slow wave. This would mean that the relaxation times would also be of the same order of magnitude for the two types of collisions. However, we show that in (5. 18) there is a cancellation of two nearly equal terms leaving a difference which is $1/(kr_i)^2$ smaller than expression (5. 15) for the three fast waves.*

RELAXATION TIMES FOR WAVE COLLISIONS

Let us consider a wave distribution whose action N_k is a small deviation by an amount N_k' from the stationary distribution N_k^o. That is

* The total rate of change of action in the fast waves due to this mechanism is zero. However, the corresponding total rates of change of momentum and energy in the fast waves are not zero. The estimate of the slower relaxation time by a factor $(kr_i)^2$ applies to the relaxation of energy.

$$N_k = N_k^0 + N_k' \tag{5.19}$$

where $N_k^{\,0} = \dfrac{A^2}{\omega_k^{\,0}}$. Then equation (5.10), if we neglect products of the small deviations, becomes

$$\dot{N}_k' = \sum_{p,q} \Delta^{(3)}(\hat{p}+\hat{q}-\hat{k})\, 2\pi\, \delta(\omega_p+\omega_q-\omega_k)\, M_{kpq} \cdot \cdot \left[N_p'(N_q^0 - N_k^0) + N_q'(N_p^0 - N_k^0) - N_k'(N_p^0 + N_q^0) \right] \tag{5.20}$$

where $\omega_k^0 = \omega_p^0 + \omega_q^0$ and $N_p^0 N_q^0 - N_k^0 (N_p^0 + N_q^0) = 0$

Equation (5.20) is of the form: •

$$\dot{N}_k' = -\frac{1}{\tau_k}\left[N_k' - \overline{N'} \right] \tag{5.21}$$

where

$$\frac{1}{\tau_k} = \sum_{p,q} \Delta^{(3)}(\hat{p}+\hat{q}-\hat{k})\, 2\pi\, \delta(\omega_p+\omega_q-\omega_k)\, M_{kpq}\, (N_p^0 + N_q^0) \tag{5.22}$$

and

$$\frac{1}{\tau_k}\overline{N'} = \sum_{p,q} \Delta^{(3)}(\hat{p}+\hat{q}-\hat{k})\, 2\pi\, \delta(\omega_p+\omega_q-\omega_k)\, M_{kpq} \left[N_p'(N_q^0 - N_k^0) + N_q'(N_p^0 - N_k^0) \right] \tag{5.23}$$

τ_k is an effective relaxation time of the action of waves of wave-vector k. Without difficulty, we replace the sums over p, q by integrals.

The integration in (5.23) is performed only over a surface in $\hat{p}$-space because of the $\delta^{(3)}(\hat{p} + \hat{q} - \hat{k})$ which eliminates the summation over $\hat{q}$ and the $\delta(\omega_p + \omega_q - \omega_k)$ which eliminates one of the integrations in $\hat{p}$-space, expresses the eliminated integration variable, say one of the components of $\hat{p}$, in terms of the other two, and introduces a group velocity into the denom-

• If we write $N_k' = (N_k^0)^2\, \Phi(k)$, equation (5.20) becomes $(N_k^0)^2 \frac{\partial \Phi}{\partial t} = I(\Phi) = \sum_{pq} K(k,p,q)(\Phi_k - \Phi_p - \Phi_q)$ where $K(\hat{k},\hat{p},\hat{q}) = 2\pi\, \delta(\omega_p+\omega_q-\omega_k)\, \Delta^{(3)}(\hat{p}+\hat{q}-\hat{k})\, M_{kpq}\, N_k^0\, N_p^0\, N_q^0$ is a symmetric kernel.
$(\Psi,\Phi) \equiv \int \Psi\, I(\Phi)\, d^3k = \sum_{pq} K(\hat{k},\hat{p},\hat{q})\, \delta\Psi\, \delta\Phi$ where $\delta\Phi = \Psi(k) - \Psi(p) - \Psi(q)$
Hence, (Φ, Φ) is a positive semi-definite bilinear form.

$I(\Phi) \equiv 0$ is a homogeneous integral equation of the second kind. Solutions of $I(\Phi) \equiv F$, the inhomogeneous equation, must be orthogonal to solutions of $I(\Phi) = 0$.

If we consider $\lambda (N_k^0)^2 \Phi = I(\Phi)$, the eigenfunctions $\{\Phi_N\}$, belonging to the eigenvalues $\{\lambda_N\}$, form an orthogonal set with respect to the weighting function $(N_k^0)^2$, i. e. $\int \Phi_{N'} (N_k^0)^2 \Phi_N\, d^3k = \delta_{NN'}$
The solutions of $I(\Phi) = 0$, i. e., the eigenfunctions belonging to $\lambda = 0$ are the collision invariants: ω_k^0 and k_x, k_y, k_z.

inator, analogous to the density of final energy states which appears in scattering theory, i.e.,

$$\delta(\omega_p + \omega_q - \omega_k) = \frac{1}{\left|\frac{\partial}{\partial p_\alpha}(\omega_{\hat{p}} + \omega_{\hat{k}-\hat{p}})\right|}\,\delta(p_\alpha - p_\alpha(p_\beta, p_\gamma)) \tag{5.24}$$

where p_α, p_β, and p_γ are the three components of $\hat{p}$ and $p_\alpha = p_\alpha(p_\beta, p_\gamma)$ is the expression for p_α in terms of p_β and p_γ because $\omega_{\hat{p}} + \omega_{\hat{k}-\hat{p}} = \omega_{\hat{k}}$, for fixed $\hat{k}$.

τ_k of equation (5.22) is the relaxation time we estimate here. However for the three fast wave collisions, there is a complication because part of (5.22) diverges, because of the high powers of p and q contained in M_{kpq}. When we integrate over the part of the surface for which p can become infinitely large we find that the integrand has a second order pole. In order to eliminate this difficulty N_k^o is assumed to have the value• A^2/ω_k^o only for $k < k_c$, a cut-off radius, and N_k^o is zero for $k > k_c$. This corresponds to a wave-distribution that has established wave equilibrium only for $k < k_c$. Other calculations are done with a Gaussian distribution multiplying A^2/ω_k^o so that the distribution has appreciable amplitude only for $k < k_c$. The results for the two assumed distributions are nearly equal. The dependence of the relaxation time on the size of the cut-off indicates low rates for wave vectors near or beyond the cut-off and high rates well inside the cut-off.

The case of a sphere in wave-vector space of constant wave energy up to a cut-off, as described above, is interesting because the important wave-vectors lie near the cut-off surface and these have a nearly stationary distribution because equation (5.20) vanishes identically if the surface of integration over $\hat{p}$ lies entirely within a region of constant wave energy, i.e., lies entirely within the cut-off sphere. This is not exactly the case for the waves somewhat below the cut-off surface. Whatever scattering of these waves there is adjusts the low wave vector region well inside the cut-off sphere but

• The cut-off is introduced so that the total action and higher moments of the wave distribution, which we introduce later, will be finite. The stationary distribution is the high temperature (or classical) limit of the Planck distribution. The divergence of the integrals over $\hat{k}$-space is associated with the ultraviolet catastrophe. The Planck distribution is obtained as the stationary distribution for the quantum expression for the rate of change of action, therefore, no cut-off is needed. However, the wave distribution will be cut-off at high wave numbers by damping into particle modes long before quantum effects are operative.

not the large wave-vector region outside the sphere. The rates of change of physical properties depend upon high powers of the wave-vector because the kernel M_{kpq} of equation (5. 20) depends upon high powers of the wave-vector, on the sixth power to be precise. Together with the surface element for $\hat{p}$ integration and the density of states per unit frequency (the group velocity factor of equation (5. 24), this makes a total dependence on the seventh power of k for the factor which multiplies $(N_p^{\,o} + N_q^{\,o})$ in equation (5. 22) for τ_k^{-1}. Hence the rates associated with the wave-vector region inside the cut-off sphere are unimportant compared with those in the region near the cut-off surface. Therefore it makes sense to use N_k' as a perturbation on the nearly stationary distribution of waves near the cut-off surface and to use the relaxation time given by (5. 22) for these wave-vectors as also the characteristic time for the relaxation of any deviation of the action from the true stationary state.

Let us now calculate the relaxation time given by equation (5. 22) for the three fast wave collision process. Because of the magnetic field and the wave-vector $\hat{k}$ we have two preferred directions, making the integration difficult. Therefore, we first consider the case for which the wave-vector $\hat{k}$ lies along the magnetic field and then find the next two terms of an expansion in powers of the angle that $\hat{k}$ makes with the magnetic field, so that we have an estimate of the angle dependence of τ_k^{-1} on the direction of $\hat{k}$.

When we take $\hat{k}$ along the magnetic field, $k_x = k_y = 0$. We find that the integration, with which we replace the summation, in (5. 22) is over one variable p_z since the surface of integration is axi-symmetric about the z-axis, and the integration over φ_p, the azimuthal angle is trivial.

The selection rules for this case state that

$$\begin{aligned} p_z + q_z &= k_z \\ p_x + q_x &= 0 \\ p_y + q_y &= 0 \\ m_p p p_z + m_q q q_z &= m_k k k_z \end{aligned} \qquad (5.25)$$

where $m_k, m_p, m_q = \pm 1$ determines the polarization for each of the fast waves.

Figure 2 shows the surface of integration in p-space for $m_k = +1$ and $k_z > 0$. We see that there are three parts to the surface. One is bounded

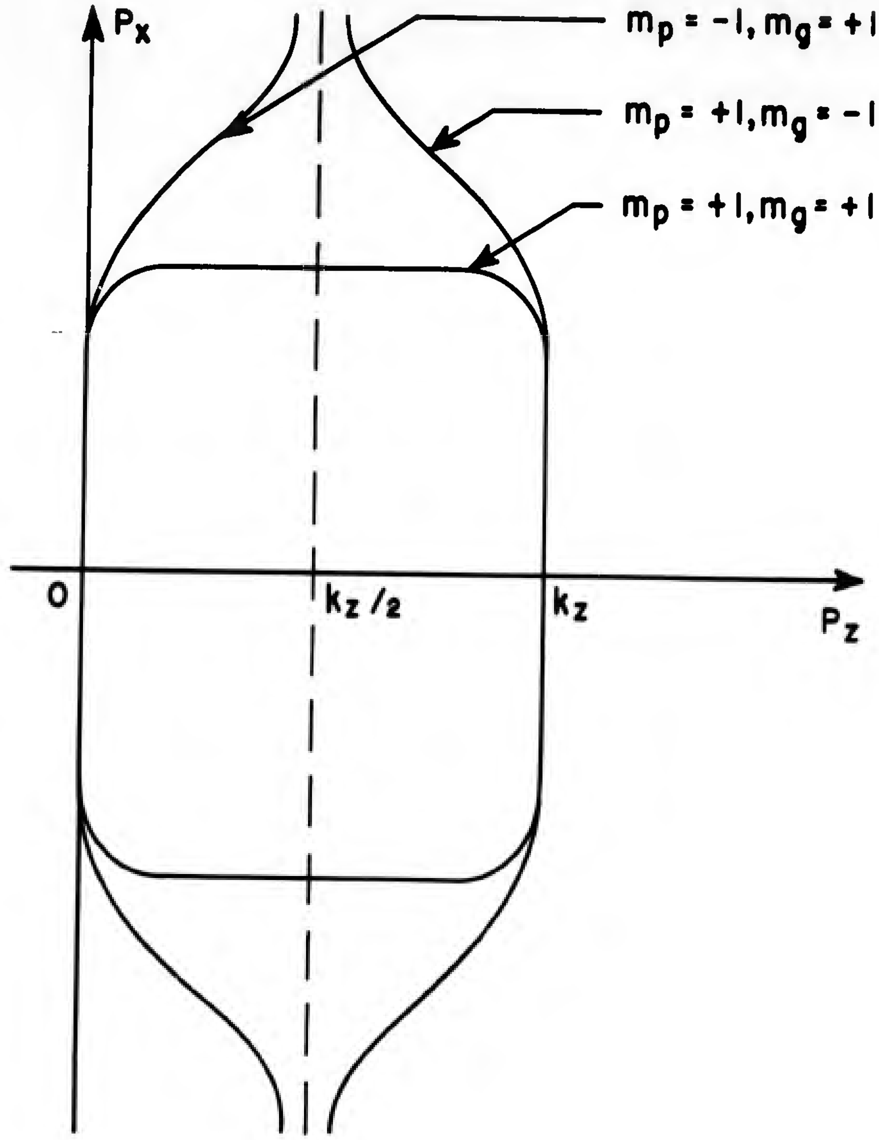


Fig. 2 Integration surface for fast wave interaction for a wave propagating along the magnetic field.

such that p doesn't exceed k_z by more than about 10%. This corresponds to $m_p = m_q = +1$. The second part corresponds to $m_q = +1$ and $m_p = -1$. It is tangent to the first surface at $p = 0$. p get infinitely large for $p_z = q_z = k_z/2$. The third part corresponds to $m_q = -1$ and $m_p = +1$. It is tangent to the first surface at $\hat{p} = \hat{k}$. p on the third part also gets infinitely large for $p_z = q_z = k_z/2$. All parts are axi-symmetric about the p_z-axis. We find that because p gets infinitely large for the second and third parts of the integration surface the integrand for integration over p_z has a pole of second order at $p_z = k_z/2$, which has led to the introduction of the previousle mentioned cut-off. The cut-off radius is chosen so that the cut-off sphere is tangent to the bounded part of the integration surface. This cuts off most of the two unbounded parts. The cut-off radius for this is about $1.1\,k_z$. This leaves integration over the unbounded parts, which we estimate amounts to less than 5% of the contribution from the bounded part.

Equation (5.22) can be written as

$$\frac{1}{\omega_k \tau_k} = \frac{k_z^3}{\frac{\rho_0}{2} V_A^2}\, 4\pi^2 A^2 \int_0^1 dx \frac{\left[x^2+(1-x)^2-\sqrt{x(1-x)}\right]^3}{(2x-1)^2\left[2\{x^2+(1-x)^2\}^2-\sqrt{x(1-x)}\{1+4x(1-x)\}\right]} \tag{5.26}$$

if we include integration over the bounded surface alone. The unbounded surfaces have the same integrand except that the square roots have the opposite signs and the integration is not over the whole interval $0 \le x \le 1$ but over the intervals $0 \le x \le .04$ and $0 \le 1 - x \le .04$, where $x = p_z/k_z$.

From (5.26) we estimate that

$$\omega_k \tau_k \simeq \frac{1}{\pi\beta} \tag{5.27}$$

where β is the ratio of wave pressure to magnetic pressure $Bo^2/8\pi$. We have used the fact that the total wave energy density in the cut-off sphere is

$$4\pi \int_0^{1.1 k_z} dp\, p^2 A^2 = 3\beta \frac{B_0^2}{8\pi} \tag{5.28}$$

The factor of three appears on the right hand side of (5.28) because the wave pressure is one-third of the total wave energy density.

We estimate the angle dependence of (5.22) and find that

$\frac{1}{\omega_k \tau_k} \approx \beta\pi\left\{1 + \frac{3}{2}\theta_k^{\,2}\right\}$ for $\theta_k^{\,2} \ll 1$ where θ_k is the angle $\hat{k}$ makes with the magnetic field. This estimate is somewhat difficult to obtain unless one integrates along $\hat{k}$ instead of along the magnetic field. The surface of integration for $\hat{k}$ with a non-zero angle with the magnetic field is very similar to the surface of integration for $\hat{k}$ along the magnetic field except that it has rotated with $\hat{k}$ through the angle θ_k.

We now obtain an estimate of the relaxation time for two fast waves, k and p, interacting with a slow wave, q.

For $qr_i > 1$, ω_q, the frequency of the slow wave is $m_q \frac{q_z}{|q_z|} \omega_{ci}$, i.e., plus or minus the ion gyro-frequency.

The selection rule on frequencies in this case states that we must integrate over two surfaces of constant frequency for the p wave, because ω_k is fixed by ω_q is a constant independent of $\hat{q}$. The two surfaces correspond to $\omega_p = \omega_k \pm \omega_{ci}$. However, ω_{ci} is much smaller than ω_k by a factor $(kr_i)^{-2}$. Therefore, the two surfaces are very close together. The rate of change of action in the fast wave state k is given by equations (5.10), (5.11), and (5.18). If we look at the different terms on the right hand side of (5.10) we find that the $M_{kpq}N_kN_p$ term changes sign in going from $\omega_q = +\omega_{ci}$ to $-\omega_{ci}$. The other two terms may not because they involve N_q which changes sign, too. The slow waves are heavily damped and N_q is expected to be small compared to either N_p or N_k, and we omit terms with N_q. Of course, without damping, $\omega_q N_q = \omega_p N_p = \omega_k N_k$ is the stationary state for this wave interaction. In that case N_q would be large compared to N_p and N_k. However the relaxation time defined by (5.22) for $k_x = k_y = 0$ can be shown to be given by:

$$\frac{1}{\tau_k} \cong \frac{\omega_k^{9/2}\,\pi^2}{\frac{\rho_0}{2} k_z^2 (V_A r_i)^5} \, \frac{1}{2\omega_{ci}} \sum_{m=\pm 1} m \int_0^1 \frac{dx}{x^6}(1-x^4)(1+x^2)^2 N_p^0 \tag{5.29}$$

where $x = \sqrt{\frac{\omega_p}{V_A r_i k_z^2}}$. If we approximate the difference in (5.29) by a derivative with respect to m, and if we use $N_p^{\,o}\,\omega_p^{\,o} \propto e^{-cp^2}$ we find that

$$\frac{1}{\omega_{ci}\tau_k} \simeq 3\pi\beta \frac{1}{ck^2} \tag{5.30}$$

However, $ck^2 \simeq 1$ for wave-vectors near or inside the main portion of the Gaussian wave distribution. The result (5.30) gives a relaxation time for the two fast wave and slow wave collisions which is larger by a factor ω_k/ω_{ci}, which is $(kr_i)^2$, than the relaxation time for the three fast wave collisions given by equation (5.27). For fast wave colliding with a slow wave we estimate:

$$\omega_k \tau_k \simeq \frac{1}{3\pi} (kr_i)^2 \frac{1}{\beta} \tag{5.31}$$

SECTION VI

MHD SHOCK WAVE

Let us view the structure of a shock wave in our plasma from the coordinate system moving with the shock. We then see time - stationary non-uniformities of plasma properties in the shock region. We have dealt with such non-uniformities in the previous chapters. We take the shock as progressing along the x-axis, perpendicular to the magnetic field along the z-axis. In the shock coordinates the undisturbed fluid ahead of the shock is then seen streaming in the negative x-direction with the shock's super-Alfvén velocity. Within the shock front the stream velocity decreases in magnitude monotomically to its sub-Alfvén velocity, also in the negative x-direction, behind the shock.

Let us view the situation from the standpoint of the particles of the plasma. The particles, both ions and electrons, which simply gyrate around the magnetic lines in the still undisturbed fluid are seen to have a drift velocity, equal to minus the shock velocity, perpendicular to the magnetic field in the moving coordinate system. Because we have transformed to a system moving with the shock velocity, $\hat{u}_s$, the magnetic field, $\hat{B}$ by the Lorentz transformation, is nearly the same, neglecting terms of order $(u_s/c)^2$, in both systems. However, an electric field does appear, which is $\hat{E}_s = \frac{\hat{u}_s}{c} \times \hat{B}$, in the moving system. The charged particles drift perpendicular to $\hat{B}$ and $\hat{E}_s$, in a cycloidal motion, with a velocity $c\,\frac{\hat{E}_s \times \hat{B}}{B^2}$ which is independent of mass and charge. By using the definition of $\hat{E}_s$ we find the drift velocity to be $-\hat{u}_s$, as it should be, since we have made the particles drift by transforming to a coordinate system moving with the opposite velocity. For this reason, the drift is present regardless of whether the particles have many or few collisions although the cycloidal motions occur only between collisions.

There are other drifts of the particles, but these drift velocities depend upon the charge to mass ratio of the particles, including the sign of the charge. These drifts are due to the bending or spreading apart of the field lines. Due to the bending of the lines, particles which move in the helices wound around the field lines on the average feel a centrifugal force due to the curvature of the field line. The spreading of the field lines, i.e., the gradient of B in a direction perpendicular to $\hat{B}$ produces a cycloidal motion because the gyro-radius is smaller in a field on one side of the gyro-motion than on the other side of the gyro-motion in the weaker field a gyro-radius away.[21]

If the shock is very weak and its thickness is large compared to the gyro-radius then the particles will move so that their magnetic moments are nearly constant. The magnetic moment is constant if space-variations of the magnetic field occur over distances large compared to the gyro-radii or if time-variations of the magnetic field occur in times long compared to the period of a gyration of the particles around a field line. If we call the component of particle velocity perpendicular to the field line $V_\perp$ we then have that the magnetic moment $\mu = \frac{1}{2}\frac{mV_\perp^2}{B}$. If we define a temperature $T_\perp$ = average $(\frac{1}{2}mV_\perp^2)$, then $T_\perp$ must be proportional to B. We show below that B is proportional to the density, so that $T_\perp$ is, too.

Because the magnetic field changes slowly, the electric field acting on a particle must be $\hat{E} = -\frac{\hat{V}}{c} \times \hat{B}$ where $\hat{v}$ is the particle velocity averaged over the gyro-motion, i.e., the drift velocity. In other words, the Lorentz transformation which we applied for the shock velocity relative to the particles in the undisturbed fluid ahead of the shock also applies to the particle anywhere in the shock. $\hat{v}$ then is the drift velocity seen in the moving system. It is $-\hat{u}_s$ for particles of the fluid ahead of the shock and $-\hat{u}_s'$ some sub-Alfvén velocity behind the shock. If the non-uniformities are spread over distances large compared to the electron gyro-radius but comparable or smaller than the ion gyro-radius then this applies to the electrons only, as we have done in Chapter III. If the ion gyro-radius has to be taken into account the acceleration of the ions must be included, i.e., $m_i\left(\frac{\partial \hat{V}_i}{\partial t} + \hat{V}_i \cdot \hat{\nabla}\hat{V}_i\right) = e\left(\hat{E} + \frac{\hat{V}_i}{c} \times \hat{B}\right)$. However, if both gyro-radii are small then $\hat{E} + \frac{\hat{V}}{c} \times \hat{B} = 0$. For the geometry outlined above, $-\frac{1}{c}\frac{\partial \hat{B}}{\partial t}\cdot\hat{e}_z = \frac{\partial E_y}{\partial x} = -\frac{\partial}{\partial x}\frac{vB}{c}$ so that $\frac{\partial B}{\partial t} + \frac{\partial}{\partial x}vB = 0$. However, conservation of particles states that $\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}vn = 0$, so that $\frac{\partial}{\partial t}(B/n) + v\frac{\partial}{\partial x}(B/n) = d(B/n)/dt = 0$. This conclusion depends on the straightness of the field lines; we have assumed that $\hat{B}$ always points along the z-direction. Hence B/n is conserved. Since $T_\perp$, the temperature of velocities perpendicular to the magnetic field, is proportional to B because μ is constant, $T_\perp$ is proportional to n. Since n does not change much across a weak shock, $T_\perp$ cannot change much either.

For a shock wave, the magnetic moments of all the particles cannot be strictly constant or else there will be no entropy increase through the shock and the conditions in front and behind the wave are exactly the same and we have no shock at all. If we compare conditions on the high density of the shock wave with those for an isentropic compression producing the same density then the entropy and the discrepancy of pressure and temperature between isentropic and non-isentropic compression will depend on the third power of the shock strength, which we measure by the non-dimensional number: M_A^2-1, where M_A is the shock Mach number, the ratio of shock speed to the Alfvén speed for conditions ahead of the shock. For weak shocks then, the entropy increase will be very small and the magnetic moments of the particles will be nearly constant.

As viewed from shock coordinates, the streaming of the undisturbed fluid into the shock front has kinetic energy which has to be partially dissipated, in order for thermodynamic conditions to be different between front and back of the shock. In the case of ordinary aerodynamic shocks, the dissipation is by particle collisions which randomize the particle motion and produce an increase of thermal energy at the expense of the fluid kinetic energy. It is proposed that for an MHD shock the dissipation occurs in two steps: first, fast MHD waves are excited, grow at the expense of fluid kinetic energy and scatter each other causing a randomization of their distribution and, second, the fast waves are Landau damped or produce slow waves which are more heavily Landau damped. Nevertheless the waves eventually transfer their energy to individual particles.

The relaxation time for fast wave-fast wave scattering is shorter than that for fast wave-slow wave scattering and shorter than that for other processes which eventually involve the individual particles. These slower processes occur behind the shock front as a "lag" phenomenon, as for example, the vibrational lag in aerodynamic shocks in CO_2.

The fast waves with large k are also heavily damped and it is expected that fast waves scattering on fast waves generally spreads the wave distribution to high values of k after many collisions. Yet, for the shock front, the waves are to be randomized in only a few collisions, i.e., the shock front is only a few mean free paths thick. The randomization must be primarily that of angle of $\hat{k}$ and not magnitude of $\hat{k}$. Hence the damping of the fast wave distribution due to spreading to very large k is not important for this shock model, though it would be for a theory of turbulence which requires large scale motion decaying into small scale motion. However, we have estimated that the relaxation times for diffusion in angle and magnitude of $\hat{k}$ are nearly equal.

The primary dissipation mechanism then is the scattering of fast waves with fast waves. The randomness of the wave distribution, when viewed from a sufficiently gross scale corresponds to an increase of entropy of the fluid.

The investment of fluid energy mainly in waves is a highly non-equilibrium state. There are relatively few degrees of freedom associated with the waves as compared with the degrees of freedom of all of the individual particles, as we have already shown in Chapter IV.

The essential ingredients of the theory of wave dissipation are: one, a mechanism for the growth of waves by amounts much greater than that corresponding to isentropic changes, and two, a scattering mechanism that changes the wave distribution in times comparable to that for appreciable wave growth. These mechanisms are described by the wave transport equation. And, it is easily shown from the dispersion relation, equation (3.13) that only the fast waves have super-Alfvén group velocities perpendicular to the magnetic field. This property is essential for the waves to catch and keep up with the compressive front, in order for the waves to be amplified by the compression. In short, a wave must stay in the shock front for times much longer than those spent by a particle.

We now apply the wave transport equation (4. 34) to the structure of an MHD shock. Instead of solving for the wave distribution as a function of x for each $\hat{k}$, we consider continuity equations for total properties of the wave distribution by taking moments of the wave distribution, as we had done with the Boltzmann equation to obtain the equations of motion.

In a later paper we will solve these equations using transport coefficients so that fluxes of properties due to waves are set proportional to the gradients of the overall wave properties, e.g. energy flux at a point in the plasma is proportional to the gradient of the total wave energy at that point by a coefficient which is analogous to the heat conductivity.

Another method of solving for the shock structure is described by Mott-Smith[14]. He assumes, for the case of an aerodynamic shock, that the particle velocity distribution is the sum of two Maxwell distributions with the parameters of the distributions as functions of position in the gas. He then uses the continuity equations which he derives from the particle Boltzmann equation. These continuity equations involve quantities which are conserved in particle collisions. He then derives a moment equation for a quantity which is not conserved in a particle collision. This last equation is the only one which introduces the rates of particle collisions. The equations are then solved simultaneously, yielding the shock profile, from which the shock thickness is easily estimated.

We derive a similar set of moment equations. Those which correspond to quantities which are invariant under all wave collisions, including those with slow waves, can be integrated and result in the algebraic Rankine-Hugoniot relations. We derive, in addition, moment equations that involve the properties of the fast wave distribution alone. Some of these involve quantities which are invariant under a fast wave collision with other fast waves and the equations for these will involve only the rates for collisions of fast waves with slow waves. Other equations involve quantities which are not invariant under fast wave collisions with fast waves, and these equations will involve the rates for all of the wave collision processes.

CONTINUITY EQUATIONS

We derive from the wave transport equations the continuity equations for action, momentum, energy and magnetic stress for the fast MHD wave distribution.

We take equation (4.34) and multiply by some function of $\hat{k}$, $f(\hat{k})$, and then sum over $\hat{k}$. We obtain

$$\sum_k v_x f(\hat{k}) \frac{\partial E_{\hat{k}}}{\partial x} + \frac{du_o}{dx} \sum_k E_{\hat{k}} \left[\frac{\partial}{\partial k_x} k_x f(\hat{k}) + \frac{k_x^2}{k^2} f(\hat{k}) \right] = \frac{\partial}{\partial t} \left(\sum_k f(\hat{k}) E_{\hat{k}} \right)_{n-\ell} \quad (6.1)$$

For $f(\hat{k}) = 1/\omega_k^o$, k_x/ω_k^o, 1, and k_x^2/k^2, we obtain, respectively, the continuity equations for action, N; x-component of momentum, M_x; energy, E; and x -x component of the magnetic stress tensor P_{xx} for the fast wave distribution.* These are:

$$\frac{d}{dx} \left(\sum_k v_x N_k \right) = \left(\frac{\partial N}{\partial t} \right)_{n-\ell} \quad (6.2)$$

$$\frac{d}{dx} \left(\sum_k v_x M_{kx} \right) + M_x \frac{du_o}{dx} = \left(\frac{\partial M_x}{\partial t} \right)_{n-\ell} \quad (6.3)$$

$$\frac{d}{dx} \left(\sum_k v_x E_k \right) + P_{xx} \frac{du_o}{dx} = \left(\frac{\partial E}{\partial t} \right)_{n-\ell} \quad (6.4)$$

$$\frac{d}{dx} \left(\sum_k v_x P_{kxx} \right) + (2P_{xx} - \bar{Q}_x) \frac{du_o}{dx} = \left(\frac{\partial P_{xx}}{\partial t} \right)_{n-\ell} \quad (6.5)$$

where

$$N_k = E_k/\omega_k^o \qquad N = \sum_k N_k \quad (6.6)$$

$$\hat{M}_k = \frac{\hat{k}}{\omega_k^o} E_k \qquad M_x = \sum_k M_{kx} \quad (6.7)$$

$$P_{kxx} = \frac{k_x^2}{k^2} E_k \qquad P_{xx} = \sum_k P_{kxx} \quad (6.8)$$

$$E = \sum_k E_k \quad (6.9)$$

$$\bar{Q}_x = \sum_k \frac{k_x^4}{k^4} E_k \quad (6.10)$$

The terms on the right-hand sides are the rates due to wave scattering. If we sum k only over fast waves then the rates for momentum and energy are due to fast waves colliding with slow waves. The fast waves conserve momentum and energy among themselves, as was shown in Chapter V.

The derivative terms on the left-hand side are the divergences of the fluxes of the quantities in question, v_x being the x-component of the group velocity, which includes the velocity of the fluid.

* Equations (6.2) to (6.5) implicitly use the assumption that the action for a wave vanishes exponentially with wave number, so that surface terms in $\hat{k}$-space vanish in the integration-by-parts in equation (6.1).

We return to the equations of motion, equations (4.9) - (4.13), and derive continuity equations for mass, momentum, and energy for all the MHD phenomena, namely, fluid non-uniformities, fast waves, and slow waves. Appendix 5 contains a derivation of these. Because all phenomena of interest are included these quantities are conserved and the continuity equations are easily integrated to give the algebraic Rankine-Hugoniot relations which must apply everywhere in the shock. These use the assumption that particle pressure is negligible compared to wave energy in the shock. These conservation relations are:

$$\rho_0 u_0 = a_1 \tag{6.11}$$

$$u_0 B_0 = a_2 \tag{6.12}$$

$$P_{xx} + \rho_0 u_0^2 + \frac{B_0^2}{8\pi} = a_3 \tag{6.13}$$

$$(E + P_{xx}) u_0 + q_x + \frac{1}{2}\rho_0 u_0^3 + \frac{B_0^2}{4\pi} u_0 = a_4 \tag{6.14}$$

where a_1, a_2, a_3 and a_4 are constants and

$$q_x = \sum_k (V_x - u_0) E_k \tag{6.15}$$

Though the quantities associated with the waves are somewhat small compared to those of the fluid, denoted by subscript "o", as in Chapter. IV, they must appear in these equations in order to account for the required increase of entropy across a shock.*

We note that there are nine unknowns: u_0, ρ_0, B_0, M_x, E, P_{xx}, q_x, N, and Q_x, where q_x, given by equation (6.15), is the energy flux and

$$Q_x = \sum_k (V_x - u_0) N_k \tag{6.16}$$

which appears in equation (6.2). We have also used the relationship

$$P_{xx} = \sum_k (V_x - u_0) M_{kx} \tag{6.17}$$

to reduce the number of unknowns, and we have omitted (6.5).

* The H-theorem for the waves is as follows: entropy $S = \sum_k \ln |N_k|$. This is the classical expression for entropy obtained from the usual statistical mechanics of Bose and Einstein particles: $S = \sum_j [(C_j + N_j)\ln(C_j + N_j) - N_j \ln N_j - C_j \ln C_j]$, $C_j \gg 1$ (see Mayer & Mayer, Statistical Mechanics, p. 112, John Wiley & Sons, Inc. N.Y. (1940)). Where $|N_k| = N_j / C_j$, which is large compared to one. The rate of change of entropy due to wave collisions is then shown to be:

$$\dot{S} = \frac{1}{3} \sum_{kpq} |S_{kpq}|^2 (E_k E_p E_q)^{-1} \left[E_p E_q - E_k \left(E_p \frac{\omega_q}{\omega_k} + E_q \frac{\omega_p}{\omega_k}\right)\right]^2 \Delta^{(3)}(\hat{p} + \hat{q} - \hat{k}) 2\pi \delta(\omega_p + \omega_q - \omega_k) \geq 0$$

where $E_k = |A_k|^2 = \omega_k N_k \geq 0$.

Hence, the entropy always increases for arbitrary wave distribution except for the stationary distribution.

There are only seven equations, four of which are algebraic. An approximate solution is obtained if a reasonable assumption is made relating two of the unknown wave properties to the other wave properties. This requires the introduction of transport coefficients obtained from the steady-state, non-equilibrium solutions of the wave transport equation. The one quantity which is left undetermined is the shock velocity, which is left as a parameter. An approximate transport coefficient has been obtained in the form of an average relaxation time for the wave collision processes. This is unsatisfactory because different relaxation times are appropriate for different phenomena, and the ratios of these will relate the different moments.

An approximate solution of only equations (6.11) to (6.14) are reported elsewhere[6]. This solution is based upon the assumption that $q_x = -u_0 (P_{xx} - \frac{1}{3}E)$ This occurs for a particular numerical value of the ratio of relaxation times associated with energy flux and momentum flux.

The analysis presented here is not sufficiently advanced to predict the actual value of the ratio of relaxation times based upon the collision mechanisms analyzed in Chapter V.

Nevertheless, the average relaxation time gives a wave mean free path Λ_k defined as the relaxation time τ_k times the wave group velocity V^o_x. Thus

$$\Lambda_k \simeq \frac{1}{\pi} \frac{1}{\beta} \frac{1}{k}$$

which agrees with a crude estimate made elsewhere, which is based upon the random-walk of a wave vector of a constant frequency wave in the magnetic field which is perturbed by the presence of other waves. This estimate gives $\Lambda_k \sim \frac{3}{\beta k}$.

The random walk of the wave-number has been described as follows: At constant frequency the wave-number k changes by an amount Δk if the magnetic field through which the wave propagates changes by an amount ΔB_o. The change ΔB_o is due to other waves in the vicinity of the wave in question. Using the approximate dispersion relation $\omega \simeq \frac{m_i c B_o}{4\pi e \rho_o} k^2 \cos\theta$ for fast waves with $k_z r_i \gg 1$, we obtain $2\frac{\Delta k}{k} = -\frac{\Delta B_o}{B_o}$ neglecting the change of the angle of $\hat{k}$ with the magnetic field. Assuming that the wave interaction is coherent only over a distance $1/k_m$ where k_m is the average wave-number of the fast wave distribution, then the phase of the wave changes by an amount $\frac{\Delta B_o}{2B_o} \cdot k/k_m$ in the distance $1/k_m$. The 'optical path length' Λ_k required to obtain an r.m.s. change of phase of about $\pi/2$ radians by a random walk with a change of phase per step $\frac{\Delta B_o}{2B_o}\frac{k}{k_m}$ is then given by: $\frac{\pi}{2} = \frac{\Delta B_o k}{2B_o k_m}\sqrt{\Lambda_k k_m}$ having set the total phase change equal to the product of the phase change per step times the square-root of the number of steps $\Lambda_k k_m$. Hence

$$\Lambda_k = \pi^2 \frac{k_m}{k^2}\left(\frac{\Delta B_o}{B_o}\right)^{-2} \sim 3\,\frac{1}{\beta k}$$

where $\beta = \frac{1}{3}\frac{(\Delta B_o)^2}{B_o^2}$ is approximately the ratio of wave pressure to magnetic pressure. Λ_k is then defined as the mean free path of the wave. This is interpreted as the average distance to produce convolutions of wavelength size in an initially perfectly plane wave front.

To predict the shock thickness and its dependence on shock velocity the following crude arguments have been presented: The shock thickness is approximately two mean free paths since this is about the length in which the diffusion and dissipation mechanism will be operative. This also happens to be the shock thickness in particle mean free paths for strong aerodynamic shocks. The wave number is then chosen so that the mean group velocity in the shock direction is equal to the super-Alfvén velocity behind the shock. This sets $kr_i \simeq 2M_A$, where M_A is the shock Mach number, the ratio of shock speed to Alfvén speed ahead of the shock. These estimates give the shock thickness as approximately $\frac{4r_i}{\beta M_A}$. This formula agrees with the shock thickness data of shock tube experiments in the range of shock Mach numbers between two and three. β, a fairly strong function of M_A, has been calculated elsewhere based upon $q_x = -u_o(P_{xx} - \frac{E}{3})$ as mentioned previously.

APPENDIX 1

Numerical Values of Characteristic Lengths, Speeds and Frequencies.

With the following plasma conditions:

Temperature, T, equals 10^{5} °K.

Density, n, equals 10^{15} particles/cc.

Magnetic field, B_o, equals 10^4 gauss

We have the following lengths:

Debye length, h, = $\sqrt{kT/4\pi ne^2} \simeq 10^{-4}$ cm.

Electron gyro-radius (based on Alfvén speed), $r_e = \sqrt{m_e c^2/4\pi ne^2} \simeq 10^{-2}$ cm.

Ion gyro-radius (based on Alfvén speed), $r_i = \left(m_i c^2/4\pi ne^2\right)^{1/2} \simeq 1$ cm.

We have the following speeds:

Alfvén speed, V_A, = $B_o/\sqrt{4\pi m_i n} \simeq 10^8$ cm/sec

Electron thermal speed = $\sqrt{2kT/m_e} \simeq 10^8$ cm/sec

Ion thermal speed = $\sqrt{2kT/m_i} \simeq 10^6$ cm/sec

We have the following frequencies:

Electron gyro-frequency, $\omega_{c_e} = eB_o/m_e c \simeq 10^{11}$ sec^{-1}

Electron plasma frequency, $\omega_{p_e} = \sqrt{4\pi ne^2/m_e} \simeq 10^{13}$ sec^{-1}

Ion gyro-frequency, $\omega_{c_i} = eB_o/m_i c \simeq 10^{7}$ sec^{-1}

Ion plasma frequency, $\omega_{p_i} = \sqrt{4\pi ne^2/m_i} \simeq 10^{10}$ sec^{-1}

APPENDIX 2

Hamiltonian Formalism for Non-Linear Wave Interactions

Equation of Motion for scattering of three waves (see Eq. (5.5)):

$$\dot{A}_k e^{-i\omega_k t} = \sum_{p,q} S_{kpq} A_p e^{-i\omega_p t} A_q e^{-i\omega_q t} \tag{A2.1}$$

Letting $\phi_k = A_k e^{-i\omega_k t}$ (A2.2)

We obtain:

$$\dot{\phi}_k + i\omega_k \phi_k = \sum_{p,q} S_{kpq} \phi_p \phi_q \tag{A2.3}$$

Lagrangian: $L = L_o + L_1$ (A2.4)

Free-Field Lagrangian: $L_o = \sum_k \phi_k^+ \left(i\frac{\dot{\phi}_k}{\omega_k} - \phi_k \right)$ (A2.5)

Interaction Lagrangian: $L_1 = \sum_k \frac{i\dot{\phi}_k}{\omega_k} \sum_{p,q} S_{kpq}^+ \phi_p^+ \phi_q^+$ (A2.6)

$= -\sum_k \frac{i\dot{\phi}_k^+}{\omega_k} \sum_{p,q} S_{kpq} \phi_p \phi_q$ (A2.7)

$\phi_k^+ = \phi_{-k}$; $S_{kpq}^+ = S_{-k,\,-p,\,-q}$, must obtain because L is real;
the original particle velocities, magnetic field, etc. are real-valued.

Π_k, the conjugate momentum (to ϕ_k) $= \frac{\partial L}{\partial \dot{\phi}_k} = i\frac{\phi_k^+}{\omega_k}$ (A2.8)

Hamiltonian: $H = H_o + H_1 = \sum_k \Pi_k \dot{\phi}_k - L$ (A2.9)

Free-Field Hamiltonian, $H_o = \frac{1}{2} \sum_k (\phi_k \phi_k^+ + \omega_k^2 \Pi_k \Pi_k^+)$ (A2.10)

Interaction Hamiltonian, $H_1 = -L_1 = \sum_k \Pi_k \sum_{p,q} S_{kpq} \phi_p \phi_q$ (A2.11)

Equations of Motion:

$$\dot{\phi}_k = \frac{\partial H}{\partial \Pi_k} = \omega_k^2 \Pi_k^+ + \sum_{p,q} S_{kpq} \phi_p \phi_q = -i\omega_k \phi_k + \sum_{p,q} S_{kpq} \phi_p \phi_q \tag{A2.12}$$

$$\dot{\Pi}_k = -\frac{\partial H}{\partial \phi_k} = -\phi_k^+ - \sum_{p,q} \Pi_p (S_{pkq} + S_{pqk}) \phi_k \tag{A2.13}$$

Since $\Pi_k = i\dot{\phi}_k^+/\omega_k$ and the two above equations must agree

$$i\dot{\phi}_k^+/\omega_k + \phi_k^+ = \frac{i}{\omega_k} (\dot{\phi}_k + i\omega_k \phi_k)^+ \tag{A2.14}$$

and

$$-\sum_{p,q} \Pi_p (S_{pkq} + S_{pqk}) \phi_q = \frac{i}{\omega_k} \left(\sum_{p,q} S_{kpq} \phi_p \phi_q \right)^+ \tag{A2.15}$$

or $$-\frac{1}{\omega_p} (S_{p,k,-q} + S_{p,-q,k}) = \frac{1}{\omega_k} S_{kpq}^+ \tag{A2.16}$$

and similarly $-\frac{1}{\omega_p} S_{p,k,-q} = \frac{1}{\omega_k} S_{kpq}^+$ if q mode is different (A2.17)

APPENDIX 3

Phase Correlation by the Non-Linear Interactions at Thermal Equilibrium

$$Z(T) = \prod_{n=1}^{N} \int d\phi_n \, d\phi_n^+ \, d\Pi_n \, d\Pi_n^+ \, e^{-H_n/kT}$$

$$H_n = \tfrac{1}{2}\left(\phi_n \phi_n^+ + \omega_n^2 \Pi_n \Pi_n^+\right) + \mu \Pi_n \sum_{\ell,m} \phi_\ell \phi_m S_{n\ell m}$$

T is the temperature, and μ is the coupling constant.

Appendix 2 describes the Hamiltonian, Hn.

The interaction energy, $E_1 = \mu kT \dfrac{\partial \ln Z(T)}{\partial \mu}$

$$= \frac{\mu}{Z} \int \Sigma_{n,\ell,m} \Pi_n \phi_\ell \phi_m S_{n\ell m} e^{-H_n/kT} (d\phi_n d\phi_n^+ \cdots)$$

We compare one term of the sum in Eq. (A3.1) with the mean oscillator energy kT under the assumption that the interaction Hamiltonian is small compared to kT.

Then,

$$\int \mu \Pi_n \phi_\ell \phi_m S_{n\ell m} e^{-\frac{1}{2kT}\left[|\phi_n|^2 + \omega_n^2 |\Pi_n|^2\right]} e^{-\frac{\mu}{kT} \Pi_n^+ \phi_\ell^+ \phi_m^+ S_{n\ell m}} d\phi_\ell d\phi_\ell^+ d\phi_n d\phi_n^+ d\Pi_n d\Pi_n^+ d\phi_m d\phi_m^+ \quad \text{(A3.2)}$$

is approximately:

$$\frac{\mu^2}{kT} \int |\Pi_n|^2 |\phi_\ell|^2 |\phi_m|^2 |S_{n\ell m}|^2 e^{-\frac{1}{2kT}\left[|\phi_n|^2 + \omega_n^2 |\Pi_n|^2\right]} d\frac{|\Pi_n|^2}{2} d\frac{|\phi_n|^2}{2} d\frac{|\phi_\ell|^2}{2} d\frac{|\phi_m|^2}{2} d\psi_n d\theta_n d\theta_\ell d\theta_m \quad \text{(A3.3)}$$

having expanded the second exponential factor of (A3.2) in powers of μ/kT and then integrated over the phase angles of the complex coordinates and conjugate momenta.

$$\Pi_n = |\Pi_n| e^{i\psi_n} \qquad \phi_n = |\phi_n| e^{i\theta_n}$$

Hence, this one term of the interaction energy divided by kT is approximately

$$\left| \frac{\mu S_{n\ell m}}{\omega_n} \right|^2 kT \quad \text{(A3.4)}$$

where $\omega_n \simeq V_{A} r_i k_n^2$

$\mu S_{n\ell m}$ is approximately

$$\frac{r_i}{\sqrt{\frac{1}{2}\rho_0 V}}\, k_n (k_\ell \pm k_m)\, \Delta^{(3)}(\hat{k}_n - \hat{k}_\ell - \hat{k}_m) \tag{A3.5}$$

where V is the normalization volume for the oscillators, i.e., V contains N oscillators. $N/V \simeq r_e^{-3}$, the number of wave degrees of freedom per unit volume.

If we sum over ℓ and m such that the Δ-function of (A3.5) is satisfied, we have that the interaction energy between the oscillator and all other oscillators is approximately

$$\frac{N}{V}\,\frac{kT}{B_0^2/8\pi}\cdot kT$$

We now estimate the phase correlation introduced by the interaction Hamiltonian. We find that

$$\phi_n(t) \simeq \phi_n(t_0)\, e^{-i\omega_n(t-t_0)} + \mu \sum_{\ell,m} \phi_m(t_0)\,\phi_\ell(t_0)\, S_{n\ell m} \int_{t_0}^{t} e^{i(\omega_\ell+\omega_m-\omega_n)(t_0-t')}\, dt'\; e^{-i\omega_n(t-t_0)}$$

for $\omega_n(t-t_0) \gg 1$.

$$|\phi_n(t)|^2 \simeq |\phi_n(t_0)|^2 + (t-t_0)\sum_{\ell,m} |\phi_\ell|^2 |\phi_m|^2\, 2\pi\, \delta(\omega_\ell+\omega_m-\omega_n)\, |\mu S_{n\ell m}|^2$$
$$+ 2\sum_{m',\ell',m,\ell} \mathrm{Re}\; \phi_m(t_0)\,\phi_\ell(t_0)\,\phi_{m'}(t_0)\,\phi_{\ell'}(t_0)\, \mu^2 S_{n\ell m} S_{n\ell' m'}\, \frac{1}{(\omega_\ell+\omega_m-\omega_n+i\epsilon)(\omega_{\ell'}+\omega_{m'}-\omega_n+i\epsilon)} \tag{A3.6}$$
$$+ 2\sum_{m,\ell} \mathrm{Re}\; \phi_m(t_0)\,\phi_\ell(t_0)\,\phi_n^{+}(t_0)\,\mu S_{n\ell m}\, \frac{1}{i(\omega_\ell+\omega_m-\omega_n+i\epsilon)}.$$

We now take the ensemble average of Equation (A3.6) with respect to $e^{-Hn/kT}$. We find that the ensemble averages of the terms on the right-hand side of (A3.6) are respectively:

$$\sim kT \tag{A3.7}$$

$$\sim kT\left(\frac{N}{V}\,\frac{kT}{B_0^2/8\pi}\right)\left(\frac{N}{V}\lambda_n^3\right)^{-1/3} \omega_n(t-t_0) \tag{A3.8}$$

$$\sim kT\left(\frac{N}{V}\,\frac{kT}{B_0^2/8\pi}\right)\left(\frac{N}{V}\lambda_n^3\right)^{-2/3} \tag{A3.9}$$

$$\sim kT\left(\frac{N}{V}\,\frac{kT}{B_0^2/8\pi}\right)\left(\frac{N}{V}\lambda_n^3\right)^{-1/3} \tag{A3.10}$$

where λ_n is the wavelength of the nth oscillator.

(A3.10) is the largest term in powers of $\frac{N}{V}\frac{kT}{B_0^2/8\pi} \equiv \beta$ which is omitted by the random phase approximation. Aside from the factor

$\omega_n (t - t_o)$ in (A3.8), (A3.8) and (A3.10) are the same order of magnitude. Of course, $\omega_n (t - t_o)$ is much greater than one because the interaction occurs over many cycles of the nth oscillator. Hence, the omission of (A3.10) by the random phase approximation is not justified by the smallness of the interaction energy which correlates phases but by the appearance of the $\omega_n (t - t_o)$ factor in (A3.8) which term is retained by the random phase approximation.

(A3.9) is omitted by the random phase approximation. We see that (A3.9) is smaller than (A3.8) by the factor

$$\frac{N}{V}\frac{kT}{B_o^2/8\pi}\left(\frac{N}{V}\lambda_n^3\right)^{-1/3}\left(\omega_n(t-t_o)\right)^{-1} \ll 1$$

where

$$\frac{N}{V}\frac{kT}{B_o^2/8\pi} \sim 10^{-13}$$

$$\left(\frac{N}{V}\lambda_n^3\right)^{-1/3} \sim r_e/\lambda_n \sim .01 - .10$$

$$1 \ll \omega_n(t-t_o) \gtrsim \omega_n\tau_n \sim 1/\beta \sim \left(\frac{N}{V}\frac{kT}{B_o^2/8\pi}\right)^{-1}$$

where τ_n is the relaxation time of the non-linear interaction discussed in Chapter V. 3β is the ratio of oscillator (wave) energy per unit volume to magnetic pressure $B_o^2/8\pi$.

The factor $(\frac{N}{V}\lambda_n^3)^{-1/3}$ arises from the δ-function in frequency which eliminates one of the variables of summation (integration) in (A3.6).

(A3.8) is approximately $t-t_o/\tau_n$ times (A3.7) as it should be in a perturbation expansion valid for times short compared to the relaxation time of the interaction.

Rate of Change of the Phase of the Complex Wave-Amplitude

$$\dot{A}_n = \sum_{m,\ell} A_m A_\ell S_{n\ell m} e^{i(\omega_n - \omega_m - \omega_\ell)t} \qquad \text{(A3.11)}$$

is the rate of change of the complex amplitude:

$$A_n = r_n e^{i\theta_n} \qquad \text{(A3.12)}$$

r_n is the magnitude and Θ_n is the phase. We then use first and second order perturbations on the initial amplitudes, denoted by superscripts "o", to obtain the following rate of change of the phase if we assume the initial phases are random:

$$\dot{\theta}_n \simeq \sum_{m,\ell} |A_m^o|^2 |S_{n\ell m}|^2 \frac{2\sin^2 \Delta\omega t/2}{\Delta\omega}\frac{\omega_\ell}{\omega_n} \qquad \text{(A3.13)}$$

where $\Delta\omega = \omega_n - \omega_m - \omega_\ell$. This should be compared with the expression for τ_n the relaxation time for the magnitude of the complex amplitude, obtained in Chapter V:

$$\frac{1}{\tau_n} \simeq \sum_{m,\ell} |A_m^0|^2 |S_{n\ell m}|^2 \frac{\sin \Delta\omega t}{\Delta\omega} \frac{\omega_\ell}{\omega_n} \qquad \text{(A3.14)}$$

The function $\sin\Delta\omega t / \Delta\omega$ is $\pi\,\delta(\Delta\omega)$ for $\Delta\omega t \gg 1$, so that the summation in (A3.14) is limited to the surface corresponding to $\Delta\omega = 0$. However, (A3.13) is not limited to such a surface because $(\sin^2 \Delta\omega t/2 / \Delta\omega)$ is not a δ-function. Its behavior as a function of $\Delta\omega$ for $\Delta\omega t \gg 1$ enables us to write (A3.13) as a sum over n' of $1/\tau_{n'}$ corresponding to values of $\Delta\omega$ ranging from 0 to ω_n . That is, we find that

$$\dot{\theta}_n \simeq \int_0^{\omega_n} \frac{1}{\tau(\omega)} \frac{d\omega}{\omega_n + \frac{\pi}{t} - \omega} \simeq \frac{1}{\tau_n}\left[\ln \frac{\omega_n t}{\pi} - 1\right] \qquad \text{(A3.15)}$$

if we include only the relaxation times determined by the bounded surface of integration described in Chapter V. The fact that $\Delta\omega$ ranges from 0 to ω_n means that we are dealing with rates for waves whose wave vectors are small compared to the cut-off radius and these rates are much higher so that (A3.15) is a conservative estimate of the rate of phase change. It is probably higher by a factor of 10. This indicates rapid change of the phase, so that in the time τ_n the phase goes through many multiples of 2π , which motivates the choice of uniform a priori probability for the phase. We say that the rapid rate of phase change does not depend strongly on the initial conditions for the phases. The terms omitted by assuming initial random phases are small. Because the rate of phase change is much greater than the rate of magnitude change, we suppose that the phases will have ample opportunity to randomize even if there did exist phase correlations at some initial instant.

APPENDIX 4

Space Correlation Implied by Random Phases.

$Bi_d(\hat{X}_d, t)$ is the i_dth component of the magnetic field of the wave distribution to be found at the point $\hat{X}_d$ at the time t.

$Bi_d(\hat{X}_d, t)$ is Fourier analyzed as in Chapter III, so that

$$Bi_d(\hat{X}_d, t) = \int d^3k\, A_k\, e^{i(\hat{k}\cdot\hat{X}_d - \omega_k t)}\, [\hat{\mathcal{E}}(k)]_{i_d}$$

where $\hat{\mathcal{E}}$ is the unit circular-polarization vector transverse to $\hat{k}$ and A_k is the complex amplitude. The phase of $A_{\hat{k}}$ is a random variable with equal probability over 0 to 2π. This makes the m quantities $Bi_d(\hat{X}_d, t)$, $d = 1, 2, \ldots, m$, random variables, too. The joint probability distribution for these m variables is denoted by $P(B_1, \ldots, B_m)$. This is the probability that $B_{i_1}(\hat{x}_1, t) = B_1$, $B_{i_2}(\hat{x}_2, t) = B_2, \ldots, B_{i_m}(\hat{X}_m, t) = B_m$.

The expectation value of some polynomial in the m-variables, say $B_1^{n_1} B_2^{n_2} \cdots B_m^{n_m}$ is given as follows:

$$\int dB_1 dB_2 \cdots dB_m\, B_1^{n_1} B_2^{n_2} \cdots B_m^{n_m}\, P(B_1, B_2, \ldots, B_m) = \mathrm{Avg.}\left[\int d^3k_1 A_{k_1} e^{i(\hat{k}_1\cdot\hat{x}_1 - \omega_{k_1} t)} \mathcal{E}_1\right]^{n_1} \cdots \left[\int d^3k_m A_{k_m} e^{i(\hat{k}_m\cdot\hat{x}_m - \omega_{k_m} t)} \mathcal{E}_m\right]^{n_m}$$

where "Avg." means average over the phases of $A_{k_1}, A_{k_2}, \cdots, A_{k_m}$.

It is not difficult to show that $P(B_1, \ldots, B_m)$ is the normal distribution, i.e.

$$P(B_1, B_2, \cdots, B_m) = \frac{e^{-\frac{1}{2}\mathbb{B}^T \mathbb{M}^{-1}\mathbb{B}}}{(2\pi)^{m/2}\,[\det \mathbb{M}]^{1/2}}$$

$$\mathbb{B} = (B_1, B_2, \cdots, B_m)$$

$$\mathbb{M} = \begin{bmatrix} \mu_{11} & \cdots & \mu_{1m} \\ \vdots & & \\ \mu_{m1} & \cdots & \mu_{mm} \end{bmatrix}$$

$$\mu_{\alpha\beta} = \iint dB_\alpha dB_\beta\, B_\alpha B_\beta\, P(B_\alpha, B_\beta)$$

$$= E\,\delta_{\alpha\beta} + R_{\alpha\beta}(1 - \delta_{\alpha\beta})$$

In particular, $P(B) = \frac{1}{\sqrt{2\pi E}} e^{-B^2/2E}$

$$P(B_1, B_2) = \frac{1}{2\pi} \frac{1}{\sqrt{E^2 - R_{12}^2}} e^{-\frac{E B_1^2 - 2 R_{12} B_1 B_2 + E B_2^2}{2(E^2 - R_{12}^2)}}$$

where $E = \int d^3k \, |A_k|^2$, the energy density, and

$R_{12} = \int d^3k \, |A_k|^2 e^{i\hat{k}\cdot(\hat{x}_1 - \hat{x}_2)} = \int B_{i_1}(\hat{x}_1, t) B_{i_2}(\hat{x}_2, t) \, d^3\left(\frac{\hat{x}_1 + \hat{x}_2}{2}\right)$ the correlation function.

APPENDIX 5

Conservation Laws for a Non-Uniform Plasma

$$\hat{E}_T = E_o \hat{e}_y + \hat{E}(\hat{x}, t)$$
$$\hat{B}_T = B_o(x)\, \hat{e}_z + \hat{B}(\hat{x}, t)$$
$$\hat{u}_T = u_o(x)\, \hat{e}_x + \hat{u}(\hat{x}, t)$$
$$\rho_T = \rho_o(x) + \rho(\hat{x}, t)$$

$$\hat{E}_T + \frac{\hat{u}_T}{c} \times \hat{B}_T + \frac{1}{4\pi n e} \hat{B}_T \times (\hat{\nabla} \times \hat{B}_T) = 0 \qquad \text{(A5.1)}$$

$$\rho_T \left(\frac{\partial \hat{u}_T}{\partial t} + \hat{u}_T \cdot \hat{\nabla} \hat{u}_T \right) = n \frac{\hat{j}_T \times \hat{B}_T}{c} \qquad \text{(A5.2)}$$

$$\hat{\nabla} \times \hat{E}_T = -\frac{1}{c} \frac{\partial \hat{B}_T}{\partial t} \qquad \text{(A5.3)}$$

$$\hat{\nabla} \times \hat{B}_T = \frac{4\pi}{c} \hat{j}_T + \frac{1}{c} \frac{\partial \hat{E}_T}{\partial t} \qquad \text{(A5.4)}$$

$$\hat{\nabla} \cdot \hat{B}_T = 0 \qquad \text{(A5.5)}$$

$$n_i = n_e \qquad \text{(A5.6)}$$

$$\frac{\partial \rho_T}{\partial t} + \hat{\nabla} \cdot \rho_T \hat{u}_T = 0 \qquad \text{(A5.7)}$$

We denote by " $\overline{}$ " an average over t, y, z, and x : $|t - \bar{t}| < T/2$, $|\hat{x} - \hat{\bar{x}}| < L_{\hat{x}}/2$

$$\overline{\rho_T u_{T_x}} = \left\{ \rho_o u_o + \overline{\rho u_x} \right\}_{(\bar{x}, \bar{t})} = a_1 \qquad \text{(A5.8)}$$

$$\overline{\rho\hat{u}\cdot\hat{\nabla}\hat{u}} - \overline{\tfrac{1}{4\pi}(\hat{\nabla}\times\hat{B}_T)\times\hat{B}_T} = 0 \tag{A5.9}$$

$$-\overline{(\hat{\nabla}\times\hat{B}_T)\times\hat{B}_T} = \overline{\hat{\nabla}\frac{B_T^2}{2}} - \overline{\hat{B}_T\cdot\hat{\nabla}\hat{B}_T} = \frac{d}{dx}\left(\frac{\overline{B_o^2}}{2} + \frac{\overline{B^2}}{2} - \overline{B_x^2}\right) \tag{A5.10}$$

$$E = \frac{\overline{B^2}}{8\pi} = \sum_k |A_k|^2 \tag{A5.11}$$

$$P_{xx} = -\frac{2\overline{B_x^2} - \overline{B^2}}{8\pi} = \sum_k \frac{k_x^2}{k^2}|A_k|^2 \tag{A5.12}$$

$$\overline{\rho\hat{u}\cdot\hat{\nabla}\hat{u}} = \overline{\rho_o u_o \frac{du_o}{dx}} \tag{A5.13}$$

$$\therefore \left\{\rho_o u_o^2 + \frac{B_o^2}{8\pi} + P_{xx}\right\}_{(x,t)} = a_2 \tag{A5.14}$$

$$\hat{j}_T\cdot\hat{E}_T = -\hat{j}_T\cdot\frac{\hat{u}_T}{c}\times\hat{B}_T = \hat{u}_T\cdot\frac{\hat{j}_T}{c}\times\hat{B}_T = \rho\hat{u}\cdot\left(\frac{\partial\hat{u}}{\partial t} + \hat{u}\cdot\hat{\nabla}\hat{u}\right) \tag{A5.15}$$

$$\frac{\partial}{\partial t}\frac{1}{8\pi}(E_T^2 + B_T^2) + \frac{c}{4\pi}\hat{\nabla}\cdot\hat{E}_T\times\hat{B}_T + \hat{j}_T\cdot\hat{E}_T = 0 \tag{A5.16}$$

$$\frac{c}{4\pi}\hat{\nabla}\cdot\overline{\hat{E}_T\times\hat{B}_T} + \overline{\hat{j}_T\cdot\hat{E}_T} = \frac{d}{dx}\left(\frac{c}{4\pi}\overline{\hat{E}_T\times\hat{B}_T\cdot\hat{e}_x} + \overline{\rho\hat{u}_T\cdot\hat{e}_x\frac{u_T^2}{2}}\right) \tag{A5.17}$$

$$\therefore \left\{\frac{c}{4\pi}\overline{\hat{E}_T\times\hat{B}_T\cdot\hat{e}_x} + \frac{\rho u_T^3}{2}\right\}_{(x,t)} = a_3 \tag{A5.18}$$

$$\frac{c}{4\pi}\overline{\hat{E}_T\times\hat{B}_T\cdot\hat{e}_x} = -\overline{(\hat{u}\times\hat{B})\times\hat{B}\cdot\frac{\hat{e}_x}{4\pi}} - \overline{\frac{(\hat{B}\times\hat{j})\times\hat{B}}{n}\cdot\hat{e}_x\frac{c}{4\pi e c}}$$

$$= \frac{\overline{u_o}}{4\pi}\left(\overline{B^2} - \overline{B_x^2}\right) + \overline{\frac{u_o B_o^2}{4\pi}} + \frac{c B_o}{4\pi n_o}\overline{\frac{B_z}{4\pi}\frac{\partial B_y}{\partial z}} \tag{A5.19}$$

$$q_x = \frac{c B_o}{4\pi n_o e}\overline{\frac{B_z}{4\pi}\frac{\partial B_y}{\partial z}} = \sum_k |A_k|^2 (v_x - u_o) \tag{A5.20}$$

$$\frac{1}{8\pi}\overline{B_z\frac{\partial B_y}{\partial z}} = \frac{1}{2}\sum_k\left[-ik_z(\hat{\mathcal{E}}_k\cdot\hat{e}_z)(\hat{\mathcal{E}}_k\cdot\hat{e}_y)^+ + \text{c.c.}\right]|A_k|^2$$

$$= \sum_k m_k \frac{k_z k_x}{2k}|A_k|^2 \tag{A5.21}$$

$$\therefore \left\{ u_0(E+P_{xx}) + q_x + \frac{u_0 B_0^2}{4\pi} + \frac{1}{2}\rho_0 u_0^3 \right\}_{(\hat{x},\bar{t})} = a_3 \tag{A5.22}$$

APPENDIX 6

Relating the Right-Hand Side (R. H. S.) of (4. 27) to Wave Scattering

The R. H. S. of (4. 27) is:

$$\hat{a}^{+}_{\omega,k_y,k_z}(x) \cdot \int \frac{dt\,dy\,dz}{T L_y L_z}\, e^{i\omega t} e^{-i(k_y y + k_z z)} \frac{\delta \hat{B}(\hat{x},t)}{\delta t} + \text{c.c.} \tag{A6. 1}$$

where $\delta/\delta t$ denotes the rate of change due to non-linear wave interaction and $\hat{a}_{\omega,k_y,k_z}(x) = \int \frac{dt\,dy\,dz}{T L_y L_z} \hat{B}(\hat{x},t) \exp i\{(k_y y + k_z z) + \omega t\}$ is the Fourier coefficient of the wave magnetic field $\hat{B}(\hat{x},t)$. We also express $\hat{B}(\hat{x},t)$ as a Fourier sum of normal waves in a small box with periodic boundary conditions. The two Fourier expansions agree for $|t-\bar{t}| < T/2$, $|\hat{x}-\bar{\hat{x}}| < L_{\hat{x}}/2$.

$$\hat{B}(\hat{x},t) = \sum_{\omega,k_y,k_z} \hat{a}_{\omega,k_y,k_z}(x)\, e^{i(k_y y + k_z z)} e^{-i\omega t} = \sum_{\hat{k}} A_{\hat{k}}(t)\, e^{i(\hat{k}\cdot\hat{x} - \omega_k t)} \hat{\varepsilon}_{\hat{k}} \tag{A6. 2}$$

$$\hat{a}_{\omega,k_y,k_z}(x) = \sum_{k_x',k_y',k_z'} \int \frac{dt}{T} A_{\hat{k}'}(t)\, e^{-i(\omega_{\hat{k}'} - \omega)t}\, e^{i(\hat{k}'\cdot\hat{x} - k_y y - k_z z)}\, \Delta(k_y - k_y')\,\Delta(k_z - k_z')$$

$$\frac{\delta}{\delta t} |\hat{a}_{\omega,k_y,k_z}|^2 = \sum_{k_x',k_x''} e^{i(k_x' - k_x'')x} \left(\int \frac{dt}{T} A_{k'}(t)\, e^{i(\omega_{k'} - \omega)t} \hat{\varepsilon}_{\hat{k}'}\right)^{+} \cdot \left(\int \frac{dt}{T} A_{k''}(t)\, e^{i(\omega_{k''} - \omega)t} \hat{\varepsilon}_{\hat{k}''}\right) + \text{c.c.} \tag{A6. 3}$$

$$\int \frac{dx}{L_x} \frac{\delta}{\delta t} |\hat{a}_{\omega,k_y,k_z}|^2 = \sum_{k_x'} \left(\int A_{k'}(t)\, e^{-i(\omega_{k'} - \omega)t} \hat{\varepsilon}_{\hat{k}'} \frac{dt}{T}\right)^{+} \cdot \left(\int \dot{A}_{k'}(t)\, e^{-i(\omega_{k'} - \omega)t} \hat{\varepsilon}_{\hat{k}'} \frac{dt}{T}\right) + \text{c.c.} \tag{A6. 4}$$

$$\simeq \frac{d}{dt} |A_k(\bar{t})|^2 \frac{L_x/T}{V_x(\bar{x})}$$

$$V_x(\bar{x}) = \frac{\partial \omega_{\hat{k}}}{\partial k_x}(\bar{x})$$

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