AD 210631

Division of Engineering
BROWN UNIVERSITY
PROVIDENCE, R. I.

PREDICTION OF ELASTIC CONSTANTS OF MULTI-PHASE MATERIALS

BY

B. PAUL

FC

Department of the Navy
Office of Naval Research
Contract Nonr 562 (19) NR 039-028
Technical Report No. 3 January 1959

Nonr 562(19)/3

Nonr-562(19)/3

526

PREDICTION OF ELASTIC CONSTANTS OF

MULTI-PHASE MATERIALS

by

B. Paul

Technical Report No. 3
Division of Engineering
Brown University
Providence, Rhode Island

January 1959

Sponsored by
Department of the Navy
Office of Naval Research
Under Contract Nonr-562(19)
Task Order NR 039-028

Nonr-562(19)/3

PREDICTION OF ELASTIC CONSTANTS OF MULTI-PHASE MATERIALS*

B. Paul**
(Brown University)

ABSTRACT

The energy theorems of elasticity theory are used to find upper and lower bounds on the elastic moduli in tension and shear for two-phase materials. A "strength of materials" type of approximate solution is also given. Comparison with experimental data for a particular alloy system shows good correlation for the approximate solution, with the scatter band bounded by the predicted limits. It is shown that the method may also be used for more general multi-phase systems, and to predict temperature dependence of the elastic constants of the composite material.

* The results presented in this paper were obtained in the course of research sponsored by the Office of Naval Research under Contract Nonr-562(19) with Brown University.

** Assistant Professor, Division of Engineering

NOTATION

E	Young's modulus
G	Modulus of elasticity in shear
ν	Poisson's ratio
$(\sigma_x,\ldots,\tau_{xy}\ \ldots)$	Stress components
$(\varepsilon_x,\ldots,\gamma_{xy}\ \ldots)$	Strain components
σ,ε	Uniaxial macroscopic stress and strain
V	Volume
U	Strain energy
f	Fraction by volume of matrix (material 1)
A	Area
x	Axial coordinate
$1,2,\ldots,i$	Subscripts denoting particular phase

1. Introduction:

It is known that the elastic modulus of a metal may be considerably increased by dispersing throughout the volume finely divided particles of an alloying material which has a higher elastic modulus than the base metal. The rational design of such alloys requires some knowledge of the relationship between the elastic constants of the constituent materials and those of the composite material. Some experimental data does exist which gives Young's modulus for particular combinations of materials, such as in Ref. [1] and [2][1].

It is the purpose of this paper to establish some general relationships which will facilitate the prediction of Young's modulus, Poisson's ratio, and shear modulus for a composite material which is assumed to be uniform and isotropic in the large. It will also be assumed that the constituents are distinct and capable of separation by purely mechanical means (e.g., not solid solutions).

If E_1, E_2 and E denote respectively the elastic modulus in tension for the matrix material, the dispersed material and the composite material, it would be desirable to find a functional relationship between these material constants and f, the fraction of matrix material in the alloy. Any such relationship must satisfy the condition that $E = E_1$ when $f = 1$, and $E = E_2$ when $f = 0$.

Perhaps the simplest relationship satisfying these conditions is that which results when it is assumed that both constituent materials contribute to the composite stiffness in proportion to their own stiffness and fractional volume. That is:

$$E = E_1 f + E_2(1-f) \tag{1}$$

It will be shown that Equation (1) does in fact provide an upper bound on the elastic modulus E in those cases where both constituent materials have

1 Numbers in brackets refer to the bibliography at the end of the paper.

the same value of Poisson's ratio.

Since not only the stiffness E, but also the compliance, (1/E), must agree with that of the constituent materials at the limits $f = 0$ and $f = 1$, a second simple relationship of the desired type may be obtained by a linear interpolation between the extreme values, that is

$$1/E = (1/E_1)f + (1/E_2)(1-f) \qquad (2)$$

Equation (2) has been proposed by MacDonald and Ransley [3] on different grounds, and provides, as will be shown, a lower bound to the equivalent elastic modulus E.

In Section 2 upper and lower bounds on E will be developed on the basis of two well-known elastic energy theorems. In Section 3 an approximate solution is derived which in general is neither an upper or lower bound on E but which may be expected to give realistic results. In Section 4 the theoretical results will be compared with experimental data. In the following section formulas will be derived for the equivalent shear modulus G of the composite material, and consideration will be given to finding approximate values for ν, the Poisson's ratio of the composite material. Conclusions are stated in Section 6.

2. Lower and upper bounds on Young's modulus

Both the matrix (material 1) and the dispersed particles (material 2) are assumed to be linearly elastic and isotropic and to obey Hooke's law in the form [4]

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}(\epsilon_x + \epsilon_y + \epsilon_z) + \frac{E}{(1+\nu)}\epsilon_x; \text{ etc.} \qquad (3\text{-a})$$

$$\tau_{xy} = G\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}; \text{ etc.} \qquad (3\text{-b})$$

where similar equations may be obtained by cyclic interchange of (x,y,z); ν denotes Poisson's ratio, and the notation for stress and strain is that of

Timoshenko [4]. It will also be assumed that continuity of displacement is always maintained at the interface of the two materials.

The elastic modulus for a composite material may be determined experimentally by means of a simple tension (or compression) test. It will be assumed that the test consists of an application of uniaxial stress which may be considered uniformly distributed over a volume which includes a great many inclusions. Such a stress distribution will be referred to as a macroscopically uniform distribution of stress. In the immediate neighborhood of an inclusion, the local non-homogeniety of the material prohibits the possibility of a truly uniform stress distribution. However, a suitably averaged value of normal stress over a sufficiently large area must equal the value of the macroscopic uniform stress which will be designated as σ. Similarly, the strain distribution must be non-uniform in the small but essentially uniform in the large. The normal strain component (parallel to the axis of applied force) averaged over a sufficiently large area will equal the macroscopic uniform strain denoted by ϵ. It is the ratio of the macroscopic quantities σ and ϵ which is measured in an actual test, and it is their ratio which defines the equivalent elastic modulus of the composite material, i.e.,

$$E = \sigma/\epsilon \tag{4}$$

The strain energy U absorbed by the specimen is given by

$$U = \frac{1}{2}\int_V (\sigma_x\epsilon_x + \sigma_y\epsilon_y + \sigma_z\epsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})dV \tag{5}$$

where V is the volume of the test specimen. Since σ is the only non-vanishing component of macroscopic stress, Equation (5) may, with the help of Equation (4), be recast into either of the following forms:

$$U = (1/2)(1/E)\sigma^2 V \tag{6-a}$$

$$U = (1/2)(E)\epsilon^2 V \tag{6-b}$$

A lower bound on E may be obtained by using the theorem of least work [4, p. 166], which for our purposes can be formulated as follows:

Theorem 1. Let the tractions be completely specified over the surface of a body, and let σ_x^o, σ_y^o, ..., etc., be a state of stress which satisfies the stress equations of equilibrium and the specified boundary conditions. Define U^o as the strain energy computed from the state σ_x^o, ..., etc., by means of Equations (3) and (5). Then the actual strain energy U in the body due to the specified loads cannot exceed U^o, i.e.,

$$U \leq U^o \tag{7-a}$$

An upper bound on E may be obtained by using the theorem of minimum potential energy [5, p. 171] which for our purposes is most conveniently formulated as follows:

Theorem 2. Let the displacement components be completely specified over the surface of a body (except where the corresponding component of traction vanishes), and let ϵ_x^*, ..., etc., be any compatible state of strain which satisfies the specified displacement boundary conditions. Define U^* as the strain energy computed from the state ϵ_x^*,..., etc., by means of Equations (3) and (5). Then the actual strain energy U in the deformed body cannot exceed U^*, i.e.,

$$U \leq U^* \tag{7-b}$$

Lower Bound

In order to find a lower bound on E, the tensile specimen is assumed to be loaded by the normal stress σ over its two end faces and to have zero stress on its lateral surface. A stress field suitable for the application of Theorem 1 to this problem is given by

$$\sigma_x^o = \sigma; \quad \sigma_y^o = \sigma_z^o = \tau_{xy}^o = \tau_{yz}^o = \tau_{zx} = 0 \tag{8}$$

The strain energy for this system of stress is

$$U^o = \frac{1}{2} \int_V \frac{\sigma_x^o \sigma_x^o}{E} \, dV = \frac{\sigma^2}{2} \int_V \frac{dV}{E} \tag{9}$$

where the integration is performed over the volume V. Introducing the fractional volume quantity f into Equation (9), results in

$$U^{o} = \frac{\sigma^2}{2} \left[(f/E_1) + (1-f)/E_2 \right] V \tag{10}$$

Upon substitution of Equations (6-a) and (10) into Inequality (7-a), there results:

$$\frac{1}{(f/E_1) + (1-f)/E_2} \leq E \tag{11}$$

Inequality (11) shows that MacDonald and Ransley [3] have actually given a lower bound on E.

Upper Bound

In order to find an upper bound on E, it is noted that the tensile specimen elongates by amount ϵL, where L is the length of the specimen. A suitable strain field consistent with this displacement boundary condition is given by

$$\epsilon_x^* = \epsilon, \quad \epsilon_y^* = \epsilon_z^* = -m\epsilon \, ; \quad \gamma_{xy}^* = \gamma_{yz}^* = \gamma_{zx}^* = 0 \tag{12}$$

where m is an unspecified constant. Substitution of Equation (12) into Hooke's law, Equation (3) defines the following set of stresses:

$$\begin{aligned} \sigma_x^* &= \epsilon E(1-\nu-2\nu m)/(1-\nu-2\nu^2) \\ \sigma_y^* &= \sigma_z^* = \epsilon E(\nu-m)/(1-\nu-2\nu^2) \\ \tau_{xy}^* &= \tau_{yz}^* = \tau_{zx}^* = 0 \end{aligned} \tag{13}$$

Equations (5), (12) and (13) may be used to formulate the strain energy as follows:

$$U^* = \frac{\epsilon^2}{2} \int_V \left(\frac{1-\nu-4\nu m+2m^2}{1-\nu-2\nu^2}\right) E\,dV$$

$$= \frac{\epsilon^2}{2} V \left[\frac{(1-\nu_1-4\nu_1 m+2m^2) f E_1}{(1-\nu_1-2\nu_1^2)} + \frac{(1-\nu_2-4\nu_2 m+2m^2)}{1-\nu_2-2\nu_2^2} (1-f) E_2 \right] \tag{14}$$

Utilizing Equations (14) and (6-b) in Inequality (7-b) results in the following upper bound for E.

$$E \leq \frac{1-\nu_1+2m(m-2\nu_1)}{1-\nu_1-2\nu_1^2} E_1 f + \frac{1-\nu_2+2m(m-2\nu_2)}{1-\nu_2-2\nu_2^2} E_2(1-f) \tag{15}$$

Although Inequality (15) is valid for any choice of m, the best results will be obtained when U^* is minimized. Using the well-known fact that $\nu \leq 1/2$, it is easy to see that $\partial^2 U^*/\partial m > 0$. Therefore, U^* has a relative minimum where $\frac{\partial U}{\partial m} = 0$, and since U^* is quadratic in m with positive values for very large positive or negative values of m, the relative minimum is also an absolute minimum and occurs at

$$m = \frac{\nu_1(1+\nu_2)(1-2\nu_2)fE_1 + \nu_2(1+\nu_1)(1-2\nu_1)(1-f)E_2}{(1+\nu_2)(1-2\nu_2)fE_1 + (1+\nu_1)(1-2\nu_1)(1-f)E_2} \tag{16}$$

It should be noted that in the limiting cases where f approaches 1 or 0, m approaches ν_1 or ν_2, respectively, as it should. In the special case where $\nu_1 = \nu_2 = \nu$, it follows from Equation (16) that $m = \nu$. For this special case Inequality (15) reduces to

$$E \leq fE_1 + (1-f)E_2 \tag{17}$$

3. Approximate solution for E

Because of the uniformity in the large of the composite material, it is plausible to assume that the macroscopic stress and strain are reproduced in some average sense in a typical unit volume which consists of a single particle of material 2 imbedded in a cube of the matrix material. This typical cube will be assumed to be loaded over two opposite faces by the force $F = (\sigma)(1)$ as shown in Fig. 1.

In the spirit of strength of materials, it will be assumed that cross-sections originally normal to the axis of applied force remain plane and normal to the axis, and each fiber parallel to the axis undergoes simple tension in the direction of applied force.

A cross-section of the cube, at a distance x from an end face, intersects an area A_1 of matrix material and an area A_2 of dispersed material, as shown in Fig. 1. Since the strain is uniform over such a cross-section, the normal stress on area A_1 will be $E_1\varepsilon$ and that on A_2 will be $E_2\varepsilon$ where ε is the normal strain at the cross-section. The total force on the cross-section must equal the total applied force F, therefore

$$F = E_1\varepsilon A_1 + E_2\varepsilon A_2 = \varepsilon[E_1 + (E_2-E_1)A_2] \tag{18}$$

The total elongation of the cube is given by δ, where

$$\delta = \int_0^1 \varepsilon(x)dx = F\int_0^1 dx/[E_1 + (E_2-E_1)A_2] \tag{19}$$

Defining E as the ratio F/δ for the unit cube, it follows that

$$\frac{1}{E} = \int_0^1 \frac{dx}{E_1 + (E_2-E_1)A_2(x)} \tag{20}$$

For any particular distribution of the imbedded material $A_2(x)$ is a well defined function of x, therefore, Equation (20) gives an approximate value for E for any assumed distribution of the inclusion.

In particular, if the inclusion is of a cubic shape it may easily be verified that Equation (20) predicts

$$\frac{E}{E_1} = \frac{E_1 + (E_2-E_1)\ g^{2/3}}{E_1 + (E_2-E_1)\ g^{2/3}(1-g^{1/3})} \tag{21}$$

where $g = 1-f$ is the fractional volume of material 2.

It is of interest to note that if the inclusion is in the form of a prism of any cross-section, which extends the entire length of the unit volume, as shown for example in Figure 2-a and 2-b, Equation (20) predicts a linear relationship between E and f. The value of E is precisely the upper bound given by Inequality (18) for the special case when inclusion and matrix both have the same Poisson's ratio. This is to be expected since a prismatic distribution of the inclusion would make A_2 = constant, and therefore Equation (18) would predict a uniform strain distribution throughout the volume. Further, the absence of transverse strain implies $\nu_1 = \nu_2$. Therefore, the assumed conditions coincide with those hypothesized in deriving the upper bound on E.

Similarly, it may be verified that if the inclusion takes the form of a slab of uniform thickness in the direction of x (see Fig. 3), the distribution of stress will be uniform throughout the specimen, and the value of E predicted by Equation (20) will coincide exactly with the lower bound given by Inequality (12).

4. Comparison with experiment

Fig. 4 shows experimental data reported by Nishimatsu and Gurland [1], and Kieffer and Schwartzkopf [2].

In order to predict the behavior of the alloy analytically, it is necessary to know the elastic constants of the constituents. The following values are taken from [1]: $E_1 = 30 \cdot 10^6$ psi, $E_2 = 102 \cdot 10^6$ psi, $\nu_1 = 0.3$, $\nu_2 = 0.22$. The use of these values for E_1 and E_2 in Inequalities (11) and (15) leads to the upper and lower bounds shown in Fig. 4. It should be noted that the upper bound differs very little from the straight line which would be predicted by Inequality (17) if ν_1 were equal to ν_2. The deviation of the more rigorous upper bound from the straight line is in fact so small that it is imperceptible on the scale of Fig. 4. This indicates that the upper bound given by Inequality (15) is quite insensitive to the influence of Poisson's ratio.

The middle curve shown in Fig. 4 gives the approximate value for E predicted by the "strength of materials" type formula, Equation (21). This latter equation also ignores the effect of Poisson's ratio and seems to correlate the experimental data quite well, in view of the scatter. It should be noted that the scatter band is bounded by the upper and lower limits predicted by the energy theorems.

5. Equivalent shear modulus and Poisson's ratio

Analogous reasoning to that employed for determining E could be used to determine the equivalent modulus of elasticity in shear G from a knowledge of the constituent shear moduli G_1 and G_2. The only significant difference is that the test specimen of the composite material should be thought of as being subjected to a macroscopically uniform simple shear stress τ which produces the macroscopically uniform shear strain γ. It then follows that analogously to Inequalities (11) and (15) one finds

$$\frac{1}{f/G_1 + (1-f)/G_2} \leq G \leq fG_1 + (1-f)G_2$$

where both inequalities are correct irrespective of the values of ν_1 and ν_2. Similarly, one may formulate a "strength of materials" type of approximation to G by merely replacing E, E_1 and E_2 by G, G_1 and G_2, respectively, in Equations (20) and (21).

Having determined approximate values for G and E, one may then approximate ν by the well-known relationship

$$\nu = (E/2G) - 1 \qquad (22)$$

6. Conclusions

The energy theorems of elasticity theory have been utilized to formulate upper and lower bounds on the elastic moduli in tension and shear for finely dispersed two-phase particle strengthened alloys. These bounds are independent of the shape of the dispersed particles.

Experimental values of Young's modulus, obtained by several investigators, show a certain degree of scatter, which is, however, bounded by the upper and lower bounds obtained in this paper.

An approximate value of elastic modulus, which is neither an upper nor lower bound, has been derived, and at least for the particular alloy system studied this approximation fits the experimental data very well. For this alloy system the effect of ν_1 and ν_2 on E seems negligible.

It may perhaps be of interest to note that the dependence of the elastic moduli of composite materials on temperature may be determined by the results of this paper if the temperature dependence of the elastic constants is known for the constituent materials.

The methods of this paper may be extended to multi-phase alloys with more than two components. For example, if E_i and f_i refer to the elastic modulus and fractional volume of the ith component, Inequalities analogous to (11) and (17) may be written:

$$\frac{1}{\sum_i (f_i/E_i)} \leq E \leq \sum_i E_i f_i \tag{23}$$

Similarly, Equation (20) may be rewritten as

$$1/E = \int_0^1 dx/\sum_i E_i A_i \tag{24}$$

where $A_i(x)$ is the area occupied by the ith component in the cross-section of a unit cube located at section x.

Acknowledgment

The author would like to thank Professors J. Gurland and D. C. Drucker for helpful criticism.

REFERENCES

1. C. Nishimatsu and J. Gurland, "Experimental Survey of the Deformation of a Hard-ductile Two-phase Alloy System," WC-Co, Technical Report No. 2, Division of Engineering, Brown University, September 1958.

2. R. Kieffer and P. Schwartzkopf, "Hartstoffe and Hartmetalle," Springer, Vienna, 1953.

3. N. F. MacDonald and C. E. Ransley, "Preparation of High Modulus Alloys by Powder Metallurgy," Symposium on Powder Metallurgy, 1954, Special Rep. No. 58 of the Iron and Steel Institute, London, 1956.

4. S. Timoshenko and J. N. Goodier, "Theory of Elasticity," McGraw-Hill Book Co., New York, 1951.

5. A. E. H. Love, "Mathematical Theory of Elasticity," Cambridge, 4th Edition, 1927.

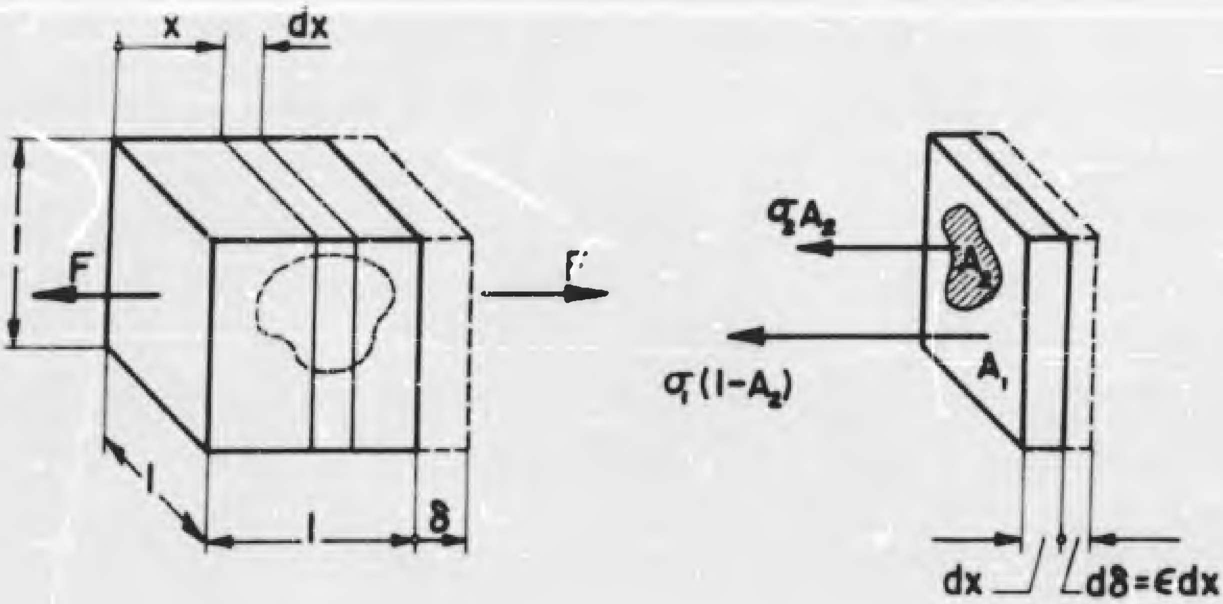
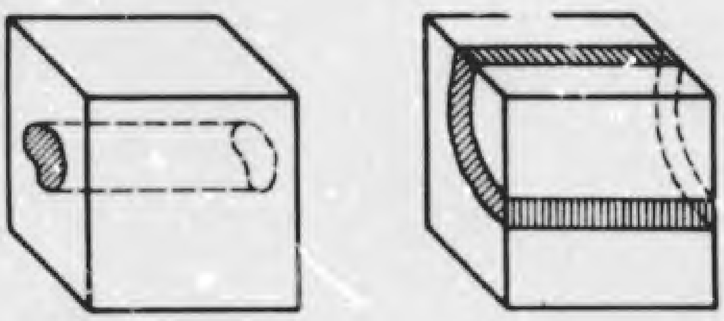


FIG.1 UNIT CUBE WITH INCLUSION
OF ARBITRARY SHAPE.



(a) (b)

FIG.2 UNIT CUBE WITH
PRISMATIC INCLUSION.

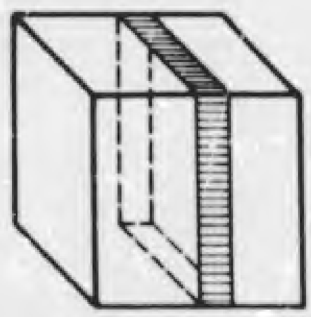


FIG.3 UNIT CUBE WITH
SLAB-LIKE INCLUSION.

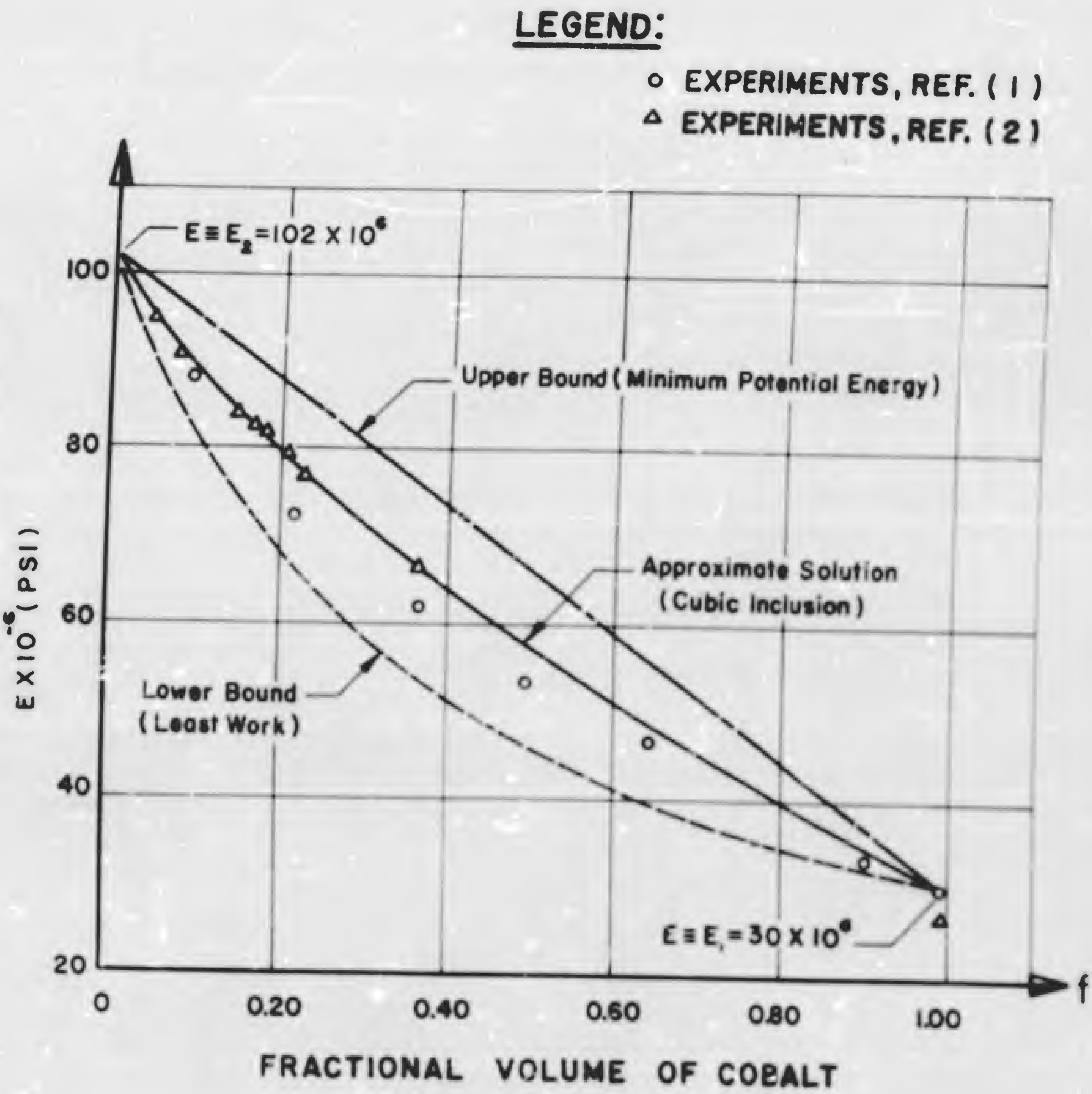


FIG.4 COMPARISON OF THEORY WITH EXPERIMENT.