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Office of Scientific Research and Development

UNCLASSIFIED

MONTHLY REPORT NO. AES-12 (OSRD NO. 5393)

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# AIR AND EARTH SHOCK

Volume 12. June 25 to July 25, 1945

**A Compilation of Informal Reports Submitted in Advance of Formal Reports**





Pertinent Service Project
OD-03







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Volume 12. June 25 to July 25, 1945

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DECLASSIFIED - DOD Directive No. 5200.9, 27 September 1958

Approved on August 1, 1945
for submission to the Committee

UNCLASSIFIED

E. Bright Wilson Jr.

E. Bright Wilson, Jr., Chief
Division 2
Effects of Impact and Explosion

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## Preface

This report is the twelfth monthly report of Division 2, NDRC, on Air and Earth Shock, covering the period from June 25 to July 25, 1945. These monthly reports are compilations of informal reports submitted in advance of formal reports. In no case is it to be presumed that the work is complete or that the results reported are other than tentative.

The work described in the report is pertinent to the project designated by the War Department Liaison Officer as OD-03 and was performed under Contract OEMsr-260 with Princeton University.

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July 25, 1945
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Project OD-03

Princeton University
W. Bleakney, Supervisor

# REACTIONS OF SIMPLE SYSTEMS UNDER BLAST LOADING

by D. Montgomery and A. H. Taub

## Abstract

The differential equation $M\ddot{x} + F(x) = p(t)$ is considered for some simple cases of blast loading. The right-hand side is assumed linear, and $F(x)$ on the one hand is taken as constant and on the other is taken as linear from the origin to the constant and then as remaining constant for larger values of $x$. It is shown that the situations in the two cases differ moderately. An approximation formula is developed by which certain information in the latter case can be obtained from the former.

## 1. Introduction

In discussing the behavior of various targets under blast loading it is often possible to reduce the mathematical problem to that of a one-dimensional system governed by the equation

$$M \frac{d^2x}{dt^2} + F(x) = p(t), \tag{1}$$

where $x$ is the displacement of the system, $F(x)$ is the restoring force, $M$ is the equivalent mass of the system, and $p(t)$ is the force [ = pressure × area] acting on the system where $p(t)$ is dependent on time.

Equations of this form arise in many problems; for example, if the target is elastic and has various modes of vibration, its response is determined by solving a set of equations of the type of Eq. (1) where $F(x)$ is of the form $k_n x$. Again, this equation is found in the treatment given by Christopherson[1/] in R.C. 349 of the action of brick walls. There it is shown that $F(x)$ may be replaced by a constant.

In the application we have in mind (blast wave) the function $p(t)$ is zero for negative time, has a finite initial value $p_0$ at $t = 0$, decreases to

1/ "A modification of the impulse criterion for blast damage," by D. G. Christopherson, R.C. 349, Sept. 1942 (Confidential).

zero again at time $t_o$, and becomes negative thereafter, rising to zero at some later time. The problem with which we are mainly concerned may be stated as follows:

> What relation must exist between $p_o$ and $t_o$ in order that the maximum of the solution of Eq. (1) be a specified quantity, say $x_3$?

If the solution of this problem is known, then for a target such as a brick wall we can determine the relation between $p_o$ and $t_o$ that will just cause the target to fail -- that is, reach a critical displacement with zero velocity. The quantity $p_o$ is called the peak pressure acting on the target, and $t_o$ is called the duration of the pressure wave. The area under the pressure-time curve between $t = 0$ and $t = t_o$, called the positive impulse, may be related to $p_o$ and $t_o$. The result may then be expressed in terms of the peak pressure and positive impulse just necessary to cause failure. If the relation between peak pressure and impulse acting on the target and the same quantities in the blast wave are known, then for any charge weight a distance can be determined that is the limiting distance at which the target is destroyed. In order to perform the last calculation the dependence of peak pressure and impulse in the blast wave on weight of charge and distance must be known. These quantities must be corrected for reflection, diffraction, and motion of the target in order to obtain the peak pressure and impulse acting on the target.

This paper will be concerned with the determination of the relation between peak pressure and impulse <u>acting on the target</u> for a given maximum displacement for special cases of Eq. (1). The specializations made are as follows:

$$\text{A:}\quad p(t) = p_o\left(1 - \frac{t}{t_o}\right).$$

$$\text{Case I:}\quad F(x) = \text{constant} = P,$$

or

$$\text{Case II:}\quad F(x) = \begin{cases} \dfrac{P}{x_1}\,x, & 0 \leq x \leq x_1; \quad \text{(a)} \\ P, & x \geq x_1. \quad \text{(b)} \end{cases}$$

Case I is a limiting case of case II. If the desired deflection is $x_3$ and if $x_1/x_3$ approaches zero, then general existence theorems guarantee that the solutions in case II approach those in case I. However, there is no guarantee that a given value of $x_1/x_3$, say 0.01 for example, will bring the solutions near each other to an accuracy of about the same size. Actually we find that the solutions can differ to a greater degree than 0.01 in this case, although the difference is not excessive. We exhibit numerical calculations bearing on this point, and we also develop a formula that makes it easy to calculate from the limiting case what the situation is for a given value of $x_1/x_3$ provided this value is not too large. We consider only cases where $x_3 > x_1$ since in such cases a target will be destroyed when it reaches a deflection $x_3$ with zero velocity. At the end we also take up a related question whose description we postpone.

## 2. Solution for case II

We shall treat case I as a special case of case II and proceed first to obtain the solutions in the latter case.

In the interval from 0 to $x_1$ the solution is as follows:

$$x = \frac{P_o}{M\omega^2}\left[\frac{\sin \omega t}{\omega t_o} - \frac{t}{t_o} + 1 - \cos \omega t\right], \tag{2}$$

where $\omega^2 = P/Mx_1$, and hence in this interval

$$\dot{x} = \frac{P_o}{M\omega^2 t_o}[\cos \omega t + \omega t_o \sin \omega t - 1]. \tag{3}$$

Let $t_1$ be the time at which the displacement reaches $x_1$. Making use of the fact that $\omega^2 = P/Mx_1$, we see that

$$x_1 = \frac{P_o x_1}{P}\left[\frac{\sin \omega t_1}{\omega t_o} - \frac{t_1}{t_o} + 1 - \cos \omega t_1\right].$$

Dividing by $x_1$ and rearranging, we find as the equation determining $t_1$

$$\omega t_o \cos \omega t_1 + \omega t_1 - \sin \omega t_1 = \omega t_o\left(1 - \frac{P}{P_o}\right). \tag{4}$$

We let $\dot{x}_1$ be the value of $\dot{x}$ at $t_1$ and we denote by $I$ the quantity $\frac{1}{2}p_o t_o$ which is the area under the curve $p(t)$ from 0 to $t_o$. Then from Eq. (3)

$$\frac{M\dot{x}_1}{I} = \frac{2}{\omega^2 t_o^2}(\omega t_o \sin \omega t_1 + \cos \omega t_1 - 1). \tag{5}$$

When $t$ is greater than $t_1$ and $x$ is greater than $x_1$ Eq. (1) becomes

$$M\ddot{x} = p(t) - P = (p_o - P) - \frac{p_o}{t_o}t.$$

Making use of the fact that $x = x_1$ and $\dot{x} = \dot{x}_1$ when $t = t_1$, we find that

$$M\dot{x} = M\dot{x}_1 + \left[p_o\left(1 - \frac{t_1}{t_o}\right) - P\right](t - t_1) - \frac{p_o}{2t_o}(t - t_1)^2 \tag{6}$$

and

$$Mx = Mx_1 + M\dot{x}_1(t - t_1) + \frac{1}{2}\left[p_o\left(1 - \frac{t_1}{t_o}\right) - P\right](t - t_1)^2 - \frac{p_o}{6t_o}(t - t_1)^3. \tag{7}$$

Let $t_3$ be the value of $t$ at which the solution given by Eq. (7) has its maximum value, and let $x_3$ be this maximum. When $t = t_3$ the left-hand side of Eq. (6) is zero and we obtain

$$M\dot{x}_1 = \frac{p_o}{2t_o}(t_3 - t_1)^2 - \left[p_o\left(1 - \frac{t_1}{t_o}\right) - P\right](t_3 - t_1).$$

Let $\tau_3 = (t_3 - t_1)/t_o$. Rearrangement gives

$$\frac{M\dot{x}_1}{I} = \tau_3^2 - 2\left[\left(1 - \frac{t_1}{t_o}\right) - \frac{P}{p_o}\right]\tau_3.$$

Solving for $\tau_3$,

$$\tau_3 = \left(1 - \frac{t_1}{t_o} - \frac{P}{p_o}\right) \pm \left[\frac{M\dot{x}_1}{I} + \left(1 - \frac{t_1}{t_o} - \frac{P}{p_o}\right)^2\right]^{1/2}, \tag{8}$$

where we must choose the sign before the square root so as to make $\tau_3$ positive. In developing the approximation formula we consider the case where $1 - t_1/t_o - P/p_o$ is positive.

To find $x_3$ we substitute this value in Eq. (7):

$$Mx_3 = Mx_1 + t_o\tau_3\left\{M\dot{x}_1 + \frac{1}{2}\left[p_o\left(1 - \frac{t_1}{t_o}\right) - P\right]t_o\tau_3 - \frac{p_o t_o}{6}\tau_3^2\right\}.$$

Replace $M\ddot{x}_1$ by its value from the equation preceding Eq. (8):

$$M\ddot{x}_3 = M\ddot{x}_1 + \frac{p_o t_o^2 \tau_3^2}{6}\left[2\tau_3 - 3\left(1 - \frac{t_1}{t_o} - \frac{P}{p_o}\right)\right]. \qquad (8a)$$

Now $p_o = \frac{p_o}{P} P = \frac{p_o}{P} M\omega^2 x_1$, and hence

$$x_3 = x_1 + \frac{1}{6}\frac{p_o}{P}\omega^2 t_o^2 x_1 \tau_3^2\left[2\tau_3 - 3\left(1 - \frac{t_1}{t_o} - \frac{P}{p_o}\right)\right]. \qquad (9)$$

Instead of plotting $p_o$ against $t_o$ it is more convenient to plot $I/\sqrt{2PM\ddot{x}_3}$ against $p_o/P$, and we shall next derive a formula expressing the first of these quantities in terms of the second in the limiting case. In this case $t_1 = x_1 = 0$. Here Eq. (9) becomes meaningless because in Eq. (9) we have used the expression $P/x_1$ for a slope. However, from Eq. (8a), in this case

$$M\ddot{x}_3 = \frac{t_o^2 \tau_3^2 p_o}{6}\left[2\tau_3 - 3\left(1 - \frac{P}{p_o}\right)\right],$$

and also in this same case by Eq. (8)

$$\tau_3 = 2\left(1 - \frac{P}{p_o}\right).$$

Hence

$$2PM\ddot{x}_3 = \frac{16PI^2}{3p_o}\left(1 - \frac{P}{p_o}\right)^3 \qquad (10)$$

and

$$\frac{I^2}{2PM\ddot{x}_3} = \frac{3}{16}\frac{p_o}{P}\frac{1}{\left(1 - \frac{P}{p_o}\right)^3}. \qquad (11)$$

From Eq. (11) a table of values may be computed relating $p_o/P$ and $I/\sqrt{2PM\ddot{x}_3}$. Table I and the graph of $x_1/x_3 = 0$ in Fig. 1 present these values.

Notice that for some computations in case II it is convenient to use the following relation

$$\frac{I}{\sqrt{2PM\ddot{x}_3}} = \frac{\frac{1}{2}\frac{p_o}{P}\omega^2 x_1 M t_o}{\sqrt{2M^2\omega^2 x_1 x_3}} = \frac{1}{2\sqrt{2}}\frac{p_o}{P}\sqrt{\frac{x_1}{x_3}}\,\omega t_o. \qquad (12)$$

Table I. Values of $I/\sqrt{2PM\ddot{x}_3}$ in the limiting case.

| $\frac{p_o}{P}$ | $\frac{I}{\sqrt{2PM\ddot{x}_3}}$ | $\frac{p_o}{P}$ | $\frac{I}{\sqrt{2PM\ddot{x}_3}}$ |
|---|---|---|---|
| 1.25 | 5.413 | 6 | 1.394 |
| 1.5 | 2.756 | 7 | 1.444 |
| 2 | 1.732 | 8 | 1.496 |
| 3 | 1.378 | 9 | 1.550 |
| 4 | 1.333 | 10 | 1.604 |
| 5 | 1.353 | | |

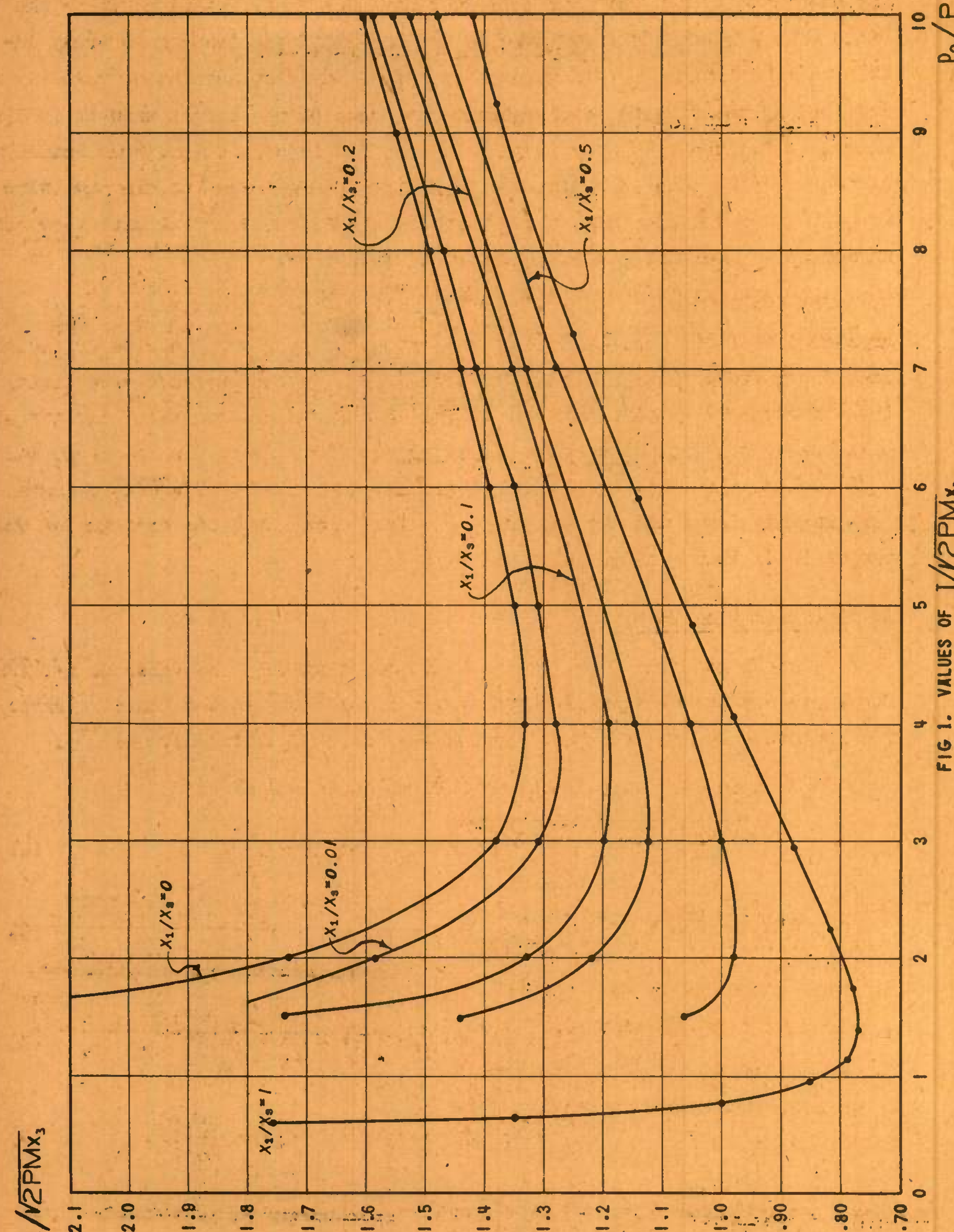


FIG 1. VALUES OF $1/\sqrt{2PMx_3}$

Princeton University Station

Division 2, NDRC

The quantity $I^2/2PMx_3$ has the following physical interpretation; it is the ratio of the kinetic energy given to the target if the loading is truly impulsive (the impulse $\underline{I}$ is communicated before any displacement or velocity is acquired by the target) to the static work done on the target when it is displaced to failure. This ratio would be one for impulsive loading. Actually in the limiting case this ratio is a function of $P/p_o$ and its minimum value is 4/3. Thus the fact that the loading is spread out over a finite time has an appreciable effect on the behavior of the system.

The fact that the value of $I/\sqrt{2PMx_3}$ rises slowly for values of $p_o/P$ greater than four, implies that in this range of $p_o/P$ the "impulse criterion" is approximately true. That is, if the impulse in the pressure wave acting on the structure is greater than and approximately the minimum value the target will break, provided of course $p_o/P$ is greater than four. In the range where $p_o/P$ is less than two, but greater than one, the value of $I/\sqrt{2PMx_3}$ changes very rapidly for small changes in $p_o/P$. This means that the breaking of the target is following a pressure criterion.

## 3. Approximation formulas

We shall now find a method to obtain approximately the value of $I/\sqrt{2PMx_3}$ for a given value of $x_1/x_3$ from the value of $I/\sqrt{2PMx_3}$ in the limiting case. The equations we need for this purpose are Eqs. (4), (5), (8), and (9).

For convenience we collect these formulas in one place

$$\omega t_o \cos \omega t_1 + \omega t_1 - \sin \omega t_1 = \omega t_o\left(1 - \frac{P}{p_o}\right), \tag{4}$$

$$\frac{M\dot{x}_1}{I} = \frac{2}{\omega^2 t_o^2}\left(\omega t_o \sin \omega t_1 + \cos \omega t_1 - 1\right), \tag{5}$$

$$\tau_3 = \left(1 - \frac{t_1}{t_o} - \frac{P}{p_o}\right) + \left[\frac{M\dot{x}_1}{I} + \left(1 - \frac{t_1}{t_o} - \frac{P}{p_o}\right)^2\right]^{1/2}, \tag{8}$$

$$x_3 = x_1 + \frac{x_1}{6}\frac{p_o}{P}\omega^2 t_o^2 \tau_3^2\left[2\tau_3 - 3\left(1 - \frac{t_1}{t_o} - \frac{P}{p_o}\right)\right]. \tag{9}$$

Suppose that $P/p_0$ is fixed. If we also fix $\omega t_0$, then $\omega t_1$ is determined by Eq. (4). Then $M\dot{x}_1/I$ is given by Eq. (5) and, since $t_1/t_0 = \omega t_1/\omega t_0$, $\tau_3$ is given by Eq. (8) and $x_1/x_3$ is determined by Eq. (9). Hence for each value of $P/p_0$ there will be a value of $\omega t_0$ which gives $x_1/x_3$ a fixed value.

Assuming that $x_1/x_3$ and $P/p_0$ are fixed we now estimate what the value of $\omega t_0$ will be when $x_1/x_3$ is small.

For a first rough estimate assume that $t_1 = 0$ and that $\tau_3 = 2(1 - P/p_0)$ as in the limiting case. Then from Eq. (9)

$$\frac{x_3}{x_1} = 1 + \frac{2}{3}\frac{p_0}{P}\,\omega^2 t_0^2\left(1 - \frac{P}{p_0}\right)^3.$$

We may drop the one as unimportant when $x_3/x_1$ is large and get as a first estimate $(\omega t_0)_1$ of $\omega t_0$,

$$(\omega t_0)_1^2 = \frac{3\,\frac{x_3}{x_1}\,\frac{P}{p_0}}{2\left(1 - \frac{P}{p_0}\right)^3}. \tag{13}$$

We now assume that a second approximation $(\omega t_0)_2$ is given by

$$(\omega t_0)_2 = \beta(\omega t_0)_1, \tag{14}$$

where $\beta$ is a quantity to be determined.

From Eq. (4) it is seen that a good approximation to $\omega t_1$ is

$$\omega t_1 = \arccos\left(1 - \frac{P}{p_0}\right), \tag{15}$$

and from Eq. (5) an approximation for $M\dot{x}_1/I$ is

$$\frac{M\dot{x}_1}{I} = \frac{2\sin\omega t_1}{\omega t_0}. \tag{16}$$

We also make the following estimates for $\tau_3$ and $\tau_3^2$:

$$\tau_3 = 2\left(1 - \frac{t_1}{t_0} - \frac{P}{p_0}\right) + \frac{M\dot{x}_1}{I}\,\frac{1}{2\left(1 - \frac{t_1}{t_0} - \frac{P}{p_0}\right)} \tag{17}$$

and

$$\tau_3^2\left[2\tau_3 - 3\left(1 - \frac{t_1}{t_0} - \frac{P}{p_0}\right)\right] = 4\left(1 - \frac{P}{p_0}\right)^3 - 12\left(1 - \frac{P}{p_0}\right)^2\frac{t_1}{t_0} + 6\,\frac{M\dot{x}_1}{I}\left(1 - \frac{P}{p_0}\right). \tag{18}$$

The value $(\omega t_o)_1$ has been chosen so that the desired value of $x_3/x_1$ is given by

$$\frac{x_3}{x_1} = \frac{2}{3}\frac{p_o}{P}(\omega t_o)_1^2\left(1 - \frac{P}{p_o}\right)^3, \tag{19}$$

but substitution of Eq. (18) in Eq. (9) and use of the second approximation $(\omega t_o)_2 = \beta(\omega t_o)_1$ gives

$$\frac{x_3}{x_1} = 1 + \frac{1}{6}\frac{p_o}{P}\beta^2(\omega t_o)_1^2\left[4\left(1 - \frac{P}{p_o}\right)^3 - 12\left(1 - \frac{P}{p_o}\right)^2\frac{t_1}{t_o} + 6\frac{M\dot{x}_1}{I}\left(1 - \frac{P}{p_o}\right)\right].$$

Equating these two values, again dropping the one as unimportant when $x_3/x_1$ is large, and dividing, we find

$$4\left(1 - \frac{P}{p_o}\right)^3 = \beta^2\left[4\left(1 - \frac{P}{p_o}\right)^3 - 12\left(1 - \frac{P}{p_o}\right)^2\frac{t_1}{t_o} + 6\frac{M\dot{x}_1}{I}\left(1 - \frac{P}{p_o}\right)\right].$$

Hence

$$\beta^2 = \frac{1}{1 + \alpha},$$

where

$$\alpha = \frac{1}{1 - \frac{P}{p_o}}\left[\frac{3/2}{1 - \frac{P}{p_o}}\frac{M\dot{x}_1}{I} - 3\frac{t_1}{t_o}\right].$$

Substituting the relation given by Eq. (16) and remembering that $t_1/t_o = \omega t_1/\omega t$,

$$\alpha = \frac{3}{\left(1 - \frac{P}{p_o}\right)^2}\left[\sin\omega t_1 - \left(1 - \frac{P}{p_o}\right)\omega t_1\right]\frac{1}{(\omega t_o)_2},$$

and then using the relations given in Eqs. (13), (14), and (15),

$$\alpha = \frac{1}{\beta}\frac{3}{\left(1 - \frac{P}{p_o}\right)^2}\left[\sin\left\{\arccos\left(1 - \frac{P}{p_o}\right)\right\} - \left(1 - \frac{P}{p_o}\right)\arccos\left(1 - \frac{P}{p_o}\right)\right]\left[\frac{2\left(1 - \frac{P}{p_o}\right)^3\frac{x_1}{x_3}}{3\frac{P}{p_o}}\right]^{1/2}$$

or

$$\alpha = \frac{\gamma\sqrt{\frac{x_1}{x_3}}}{\beta},$$

where

$$\gamma = 3\left[\frac{2\frac{p_o}{P}}{3\left(1-\frac{P}{p_o}\right)}\right]^{1/2}\left[\sin\left\{\arc\cos\left(1-\frac{P}{p_o}\right)\right\} - \left(1-\frac{P}{p_o}\right)\arc\cos\left(1-\frac{P}{p_o}\right)\right].$$

Hence

$$\beta^2 = \frac{1}{1+\alpha} = \frac{1}{1+\frac{\gamma\sqrt{\frac{x_1}{x_3}}}{\beta}},$$

$$\beta = -\frac{\gamma\sqrt{\frac{x_1}{x_3}}}{2} + \sqrt{1+\frac{\gamma^2 x_1/x_3}{4}},$$

$$\beta \cong 1 - \frac{\gamma}{2}\sqrt{\frac{x_1}{x_3}} + \frac{\gamma^2}{8}\frac{x_1}{x_3}. \qquad (20)$$

Table II. Values of $\gamma$ and $\beta$.

| $\frac{p_o}{P}$ | $\gamma$ | $\frac{\gamma}{2}$ | $\frac{\gamma^2}{8}$ | $\beta$ for $\frac{x_1}{x_3} = 0.01$ |
|---|---|---|---|---|
| 1.25 | 4.323 | 2.162 | 2.337 | 0.760 |
| 1.5 | 2.767 | 1.384 | 0.958 | .852 |
| 2 | 1.677 | 0.838 | .351 | .920 |
| 3 | 0.959 | .480 | .115 | .953 |
| 4 | .688 | .344 | .0592 | .966 |
| 5 | .522 | .261 | .0341 | .974 |
| 6 | .425 | .212 | .0225 | .979 |
| 7 | .359 | .180 | .0162 | .982 |
| 8 | .311 | .156 | .0122 | .984 |
| 9 | .274 | .137 | .0094 | .986 |
| 10 | .245 | .122 | .0074 | .988 |

Thus we have achieved our purpose — to find the approximate value of $\beta$. A table of values of $\gamma/2$ and $\gamma^2/8$ as well as the values of $\beta$ when $x_1/x_3 = 0.01$ is given in Table II.

If we replace $\omega t_o$ in Eq. (12) by $(\omega t_o)_1$ we see that we obtain the value of $I/\sqrt{2PMx_3}$ for the limiting case. Hence the factor $\beta$ is also the factor which when multiplied by $I/\sqrt{2PMx_3}$ in the limiting case yields the value (approximately) for any value of $x_1/x_3$.

Table III shows $I/\sqrt{2PMx_3}$ as computed in certain cases and as given by the approximation formula derived above. It can be seen that the approximation formula is quite accurate in these cases.

Table III. Comparison of exact and approximate values of $I/\sqrt{2PMx}$

| $\frac{p_o}{P}$ | $\frac{x_1}{x_3}$ | Exact Value of $\frac{I}{\sqrt{2PMx_3}}$ | Value Given b Approximation Formula |
|---|---|---|---|
| 1.5 | 0.0205 | 2.280 | 2.265 |
| 2.0 | .0424 | 1.457 | 1.458 |
| 2.0 | .0124 | 1.577 | 1.578 |
| 3.0 | .0150 | 1.300 | 1.299 |
| 3.0 | .0069 | 1.324 | 1.324 |
| 4.0 | .0223 | 1.269 | 1.266 |
| 4.0 | .0129 | 1.283 | 1.281 |
| 7.0 | .0091 | 1.420 | 1.420 |
| 10.0 | .0220 | 1.575 | 1.575 |
| 10.0 | .0125 | 1.583 | 1.582 |

Hence, to get a good estimate of $I/\sqrt{2PMx_3}$ for a given small value of $x_1/x_3$, compute $\beta$ from Eq. (20), using in many cases the values of $\gamma/2$ and $\gamma^2/8$ from Table II. Then multiply $\beta$ and the limiting value of $I/\sqrt{2PMx_3}$ from Table I. This approximation formula is quite accurate when $x_1/x_3$ is small, and is fairly accurate for values of $x_1/x_3$ as large as 0.3 or 0.4. Figure 1 gives the values of $I/\sqrt{2PMx_3}$ for various values of $x_1/x_3$ between 0 and 1. The case $x_1/x_3 = 1$ may be handled as follows. In this case F(x) is a straight line and[2/]

$$x_3 = \frac{2p_o}{M\omega^2}\left[1 - \frac{\text{arc tan } \omega t_o}{\omega t_o}\right];$$

using the fact that $\omega^2 = \frac{P}{Mx_3}$, we obtain

$$\frac{P}{p_o} = 2\left[1 - \frac{\text{arc tan } \omega t_o}{\omega t_o}\right].$$

We find also in this case, by Eq. (12),

$$\frac{I}{\sqrt{2PMx_3}} = \frac{1}{2\sqrt{2}}\,\frac{p_o}{P}\,\omega t_o.$$

The graphs for this case and for the cases where $x_1/x_3 = 0.0$, 0.01, 0.1, 0.2, and 0.5 are shown in Fig. 1.

The curves of Fig. 1 all have vertical asymptotes on the left. To find them proceed as follows. Looking at the Eqs. (4), (5), (8), (9), and (12),

2/ The following equation is derived by solving the differential equation for x(t). Determine the time at which the maximum is obtained from the equation $\dot{x}(t_1) = 0$. Substitute this value of $t_1$ in the equation $x(t_1) = x_3$. See R.C. 6, "The design of buildings against air attack (Part 2)," March 1939. Restricted.

let $\omega t_0$ approach infinity. Then $\tau_3$ approaches zero. Let $\frac{p_0}{P}$, $\frac{1}{2} \leq \frac{p_0}{P} \leq 1$, be fixed, and attempt to find the corresponding value of $x_1/x_3$. Since $1 - t_1/t_0 - P/p_0$ is negative, the approximation for $\tau_3$ given by Eq. (17) is to be replaced by the expression obtained by the choice of signs in Eq. (8) which makes $\tau_3$ positive. This gives

$$\tau_3 = \frac{-M\dot{x}}{I} \frac{1}{2(1 - t_1/t_0 - P/p_0)}.$$

Since $t_1/t_0 = \omega t_1/\omega t_0$ is negligible compared to $1 - P/p_0$, we obtain

$$\tau_3 = \frac{\sin \omega t_1}{\omega t_0(P/p_0 - 1)}.$$

Substituting in Eq. (19) and replacing $\sin^2 \omega t_1$ by $2P/p_0 - (P/p_0)^2$ we obtain

$$\frac{x_3}{x_1} = 1 + \frac{1}{6} \frac{p_0}{P} \frac{2P/p_0 - (P/p_0)^2}{(P/p_0 - 1)^2} [\tau_3 - 3(1 - P/p_0)].$$

Making use of the fact that $\tau_3$ is small compared to $3(1 - P/p_0)$ we write

$$\frac{x_3}{x_1} = 1 + \frac{1}{6} \frac{2 - P/p_0}{(P/p_0 - 1)^2} 3(P/p_0 - 1)$$

$$= 1 + \frac{2 - P/p_0}{P/p_0 - 1}$$

and

$$\frac{x_1}{x_3} = \frac{2P/p_0 - 2}{P/p_0}.$$

This may be written

$$\frac{p_0}{P} = 1 - \frac{1}{2} \frac{x_1}{x_3}.$$

Hence this is the location of the vertical asymptotes for the curve associated with $x_1/x_3$. When $x_1/x_3 = 0$ the asymptote is at 1, and as $x_1/x_3$ increases to 1 the position of the asymptote shifts linearly to $\frac{1}{2}$.

As we have seen, the curves all have a vertical asymptote given as above. After this they drop rather soon to a minimum and then rise gradually. The position of the minimum varies from about 1.5 for $x_1/x_3 = 1$ to 4 for $x_1/x_3 = 0$.

4. Comparison of case II to an elastic system

Returning to the differential equation Eq. (1), we discuss next a problem which arises in connection with $F(x)$ as given in case II and as given in still another case called case III.

The function $F(x) = kx$ where $k = P/x_1$ is the value given in (a) of case II Thus the curve of case III is merely a continuation of the straight line with which the curve in case II begins. Suppose that the desired maximum deflection in case II is $x_3$ and that the desired maximum deflection in case III is $x_3'$. The area $A_2$ under the curve II from 0 to $x_3$ is

$$A_2 = kx_1x_3 - \frac{kx_1^2}{2}.$$

The area under the curve III from 0 to $x_3'$ is

$$A_3 = \frac{kx_3'^2}{2}.$$

Under some conditions it is reasonable to suppose that if $A_2 = A_3$, then the $p_0$ and $t_0$ which produce a maximum deflection $x_3$ in case II will produce a maximum deflection $x_3'$ in case III. This conjecture will be examined below, and it will be shown that it is not always true.

For $A_2 = A_3$, the following must hold

$$x_3'^2 = 2x_1x_3 - x_1^2,$$

or when $x_1$ is small,

$$x_3'^2 = 2x_1x_3. \tag{21}$$

Let $x_{III}$ be the maximum deflection in case III. Then

$$x_{III} = \frac{2p_0}{M\omega^2}\left[1 - \frac{\arc\tan \omega t_0}{\omega t_0}\right]$$

$$= 2\,\frac{p_0}{P}\,x_1\left[1 - \frac{\arc\tan \omega t_0}{\omega t_0}\right].$$

For small values of $x_1$ this is approximated fairly well by

$$x_{III} = 2\,\frac{p_0}{P}\,x_1.$$

In case II assume that $x_1/x_3$ is so small that the maximum deflection $x_{II}$ is approximately equal to what it would be in the limiting case,

$$x_{II} = \frac{1}{M}\frac{2}{3}\left(1 - \frac{P}{p_o}\right)^3 p_o t_o^2.$$

According to Eq. (21) we wish to compare the quantities

$$4\left(\frac{p_o}{P}\right)^2 x_1^2 \quad \text{and} \quad \frac{4}{3M}\left(1 - \frac{P}{p_o}\right)^3 p_o t_o^2 x_1.$$

It is clear that in general these two quantities do not approximate each other, and as a further check it is easy to choose special values of the constants which show a substantial discrepancy between the two quantities.

As a numerical example suppose that $M = 1$, $t_o = 1$, $\frac{P}{p_o} = \frac{1}{2}$ and $\frac{x_{II}}{x_1} = 100$. Then

$$x_{II} = \frac{p_o}{12},$$

$$x_{III} = 4x_1,$$

and

$$\frac{x_{II}}{x_1} = 100 = \frac{p_o}{12x_1},$$

$$p_o = 1200x_1.$$

We wish to compare the quantities

$$16x_1^2 \quad \text{and} \quad \frac{p_o x_1}{6},$$

or

$$16x_1 \quad \text{and} \quad \frac{p_o}{6},$$

or

$$16x_1 \quad \text{and} \quad 200x_1.$$

These quantities differ by a factor of more than 12. For this numerical case practically all of the action takes place while the right-hand side of the differential equation is positive.

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ABSTRACT:

The differential equation Mx + F(x) = p(t) is considered for some simple cases of blast loading. The right-hand side is assumed linear, and F(x) on the one hand is taken as constant and on the other is taken as linear from the origin to the constant and then as remaining constant for larger values of x. It is shown that the situation in the two cases differ moderately. An approximation formula is developed by which certain information in the lstter case can be obtained from the former.

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