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A model for the hydrodynamic wake of a surface ship is developed. Neglecting only ambient stratification and ambient surface films, we begin with general equations and boundary conditions for the flow field and reduce the problem to a solvable one through a series of approximations. Among the assumptions are:

- the fluid dynamics are independent of the gas particle (bubble) dynamics;
- the ship undergoes only small motions about a uniform, mean motion;
- the free surface elevations are small; and
- the ship-induced turbulence is significantly stronger than the ambient turbulence.

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The Hydrodynamic Wake of a Surface Ship: Theoretical Foundations

19. ABSTRACT (Continued)

With these approximations, the overall problem becomes separable into one governing the ship-induced flow in otherwise still water and one governing the interaction of the background with the ship-induced flow.

The former problem is examined by application of the method of matched asymptotic expansions. We find that the ship produces an outer flow field that satisfies the traditional Kelvin-Neumann problem for calculating a ship's Kelvin and radiated wave systems. We find also that the ship induces an inner flow field, its turbulent wake. This flow field is governed by the steady, three-dimensional, parabolic Navier-Stokes equations with plane of symmetry boundary conditions at the free surface. The overall, or composite, ship-induced flow field is derived from these two asymptotic flow fields.

Besides the plane of symmetry boundary conditions at the free surface for the turbulent wake problem, the more significant results of this report include:

- Surface tension effects are unimportant to determine the uniform, first order, ship-induced flow field in otherwise still water; and
- The inner flow field produces a first order modification to the traditionally calculated ship wave elevations. This result could explain the experimentally observed differences between the transverse wave systems of model and full-scale ships.

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The Hydrodynamic Wake of a Surface Ship: Theoretical Foundations

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June 28, 1984

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CONTENTS

THE HYDRODYNAMIC WAKE OF A SURFACE SHIP: THEORETICAL FOUNDATIONS

INTRODUCTION

A model for describing the hydrodynamic wake of a surface ship is developed. The model accounts, in some manner, for all significant wake flow processes. Simultaneously, the model is formulated to be computationally realizable. This is necessary because small scale flow behavior on a large spatial domain is characteristic of the wake.

During the course of the derivation, we observe that several approximations must be made to obtain the model. The approximations result from a lack of firm theoretical foundations in certain areas. Besides turbulent flow dynamics, these areas include descriptions of multiphase flows, turbulence/free-surface interactions, and ambient background characteristics.

Other assumptions and constraints are required to achieve computational realizability. These include the decomposition of the flow field into potential and viscous dominated parts, limits on allowable ship motions, and certain linearizations and simplifications of boundary conditions.

The model does not include several flow phenomena considered of secondary or intermittent importance. These phenomena are associated with ambient stratification and ambient surface films.

EQUATIONS OF MOTION

Visually, the wake of a surface ship contains, among other things, a foamy mass of white water caused by sprays, turbulent entrainment of air, and bubbles. To the modeler, the wake is a multiphase flow field consisting of a gas-water mixture. This leads to immediate modeling difficulties because a firm theoretical foundation for multiphase flow dynamics does not exist [1,2]. As noted in Ref. 1, this results partially from the particle/continuum dichotomy of multiphase flows and partially from the inability to ascertain fluid constants in multiphase flow systems.

Therefore, to make any progress in describing the wake of a surface ship, we must develop an approximate treatment of the multiphase flow field. To this end, we adopt an "inertialess particle" hypothesis [3]. For our problem, the validity of this hypothesis requires, first, that the gas particles (bubbles) entrained in the water column be dispersed enough to be essentially noninteracting, and second, that the inertia of the gas particles be much less than the inertia of the entraining water. In ship wakes, the first requirement is observed empirically to be met away from boundaries. The satisfaction of the second requirement then follows from the satisfaction of the first because of the relative density ratio between gas and water.

In addition to the above requirements, the inertialess particle hypothesis presumes, from the continuum viewpoint, that any infinitesimal volume of water is still large enough to allow a meaningful specification of gas particle distribution in that volume. We denote this specification by the gas particle (or bubble) volume distribution function f_{0V_B} which gives the number of bubbles per unit of water volume, per unit of bubble volume increment dV_B. Here, V_B represents the volume of a gas particle.

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From the definition of f_{0V_B}, certain useful quantities related to the bubble population can be obtained directly. In particular, the number of bubbles per unit of water volume having a gas volume between V_{B1} and V_{B2} is

$$N_{0V_{B1},0V_{B2}} = \int_{V_{B1}}^{V_{B2}} f_{0V_B} dV_B, \tag{1}$$

and the total gas volume per unit of water volume, or void fraction, is

$$\gamma_{0B} = \int_0^\infty f_{0V_B} V_B dV_B. \tag{2}$$

Within the inertialess particle hypothesis, the velocity u_{0Bi} of the gas particle is assumed to be the velocity u_{0i} of the entraining water, plus the rise velocity u_{R3} of the gas particle in otherwise undisturbed water. Here, we have adopted indicial notation and defined the "3" direction as upwards. We find

$$u_{0Bi} = u_{0i} + u_{R3}\delta_{i3} \tag{3a}$$

where δ_{ij} is the Kronecker delta. The rise velocity itself is calculated independently of the water velocity by methods such as those discussed in Ref. 4. The position vector R_{0Bi} of the gas particle is obtained from the integral of

$$\frac{d}{dt} R_{0Bi} = u_{0Bi}. \tag{3b}$$

The equations of motion for the water in the inertialess particle limit are developed straightforwardly by noting that inertialess implies that no mass or momentum fluxes are carried by the gas particles. Hence, the primary influence of the gas particles is to change the density of the water from its true value ρ to an apparent value $\rho(1-\gamma_{0B})$. Substituting this apparent value for the density in the flux (or conservation) forms of the continuity and Navier-Stokes equations [5], we obtain the equations of motion for the water as

$$\frac{\partial}{\partial t}[(1-\gamma_{0B})\rho] + \frac{\partial}{\partial x_{0k}}[(1-\gamma_{0B})\rho u_{0k}] = 0 \tag{4a}$$

and

$$\frac{\partial}{\partial t}[(1-\gamma_{0B})\rho u_{0i}] + \frac{\partial}{\partial x_{0k}}[(1-\gamma_{0B})\rho u_{0i}u_{0k}]$$
$$= -\frac{\partial \Pi_0}{\partial x_{0i}} - (1-\gamma_{0B})\rho g\delta_{i3} + \frac{\partial}{\partial x_{0k}}\left\{(1-\gamma_{0B})\,\mu\left[\frac{\partial u_{0i}}{\partial x_{0k}} + \frac{\partial u_{0k}}{\partial x_{0i}} - \frac{2}{3}\,\delta_{ik}\,\frac{\partial u_{0j}}{\partial x_{0j}}\right]\right\}. \tag{4b}$$

In Eq. (4b), Π_0 denotes the total pressure, g the acceleration due to gravity, and μ the viscosity of the water. The appearance of the factor $(1-\gamma_{0B})$ modifying μ follows from the inertialess particle hypothesis [1, Eq. (6.16)].

BOUNDARY CONDITIONS

Consider the two coordinate systems shown in Fig. 1. The $O_0x_{01}x_{02}x_{03}$ system is fixed in the fluid and is the coordinate system in which Eqs. (1) to (4) have been derived. The undisturbed free surface coincides with the $x_{03} = 0$ plane. The $O_Sx_{S1}x_{S2}x_{S3}$ system is fixed in the ship. With respect to this latter system, R_{SHi} denotes a vector from O_S to a point on the surface of the ship. For those points that have motions in addition to the rigid body ship motions (for example, propellors and control surfaces), R_{SHi} is time dependent.

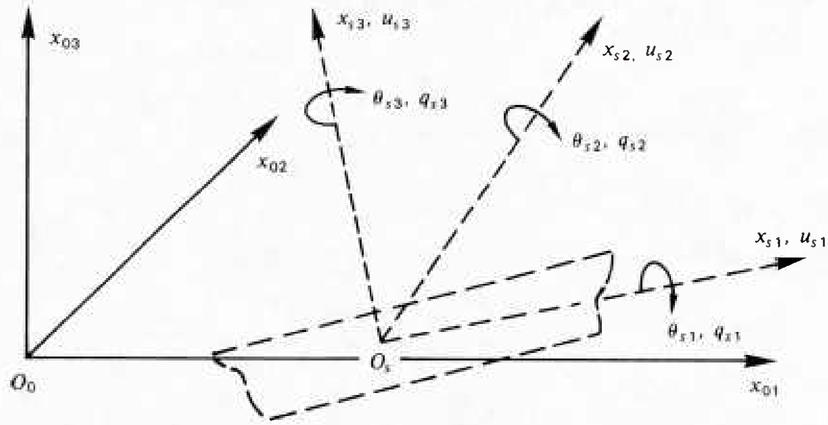

Fig. 1 — Coordinate systems and notation

We also define the vector from O_0 to O_S by R_{0Si}. Then, with respect to the $O_0x_{01}x_{02}x_{03}$ coordinate system, the vector R_{0Hi} from O_0 to the same point on the surface of the ship is obtained [6],

$$R_{0Hi} = R_{0Si} + T_{ij}R_{SHj}. \tag{5}$$

The transformation matrix T is determined from the noncommutative set of rotations—yaw θ_{S3} about x_{S3}, pitch θ_{S2} about x_{S2}, and roll θ_{S1} about x_{S1}—with positive senses, as indicated in Fig. 1. From Ref. 6, we find

$$T = \begin{bmatrix} \cos\theta_{S3}\cos\theta_{S2} & \cos\theta_{S3}\sin\theta_{S2}\sin\theta_{S1} - \sin\theta_{S3}\cos\theta_{S1} & \cos\theta_{S3}\sin\theta_{S2}\cos\theta_{S1} + \sin\theta_{S3}\sin\theta_{S1} \\ \sin\theta_{S3}\cos\theta_{S2} & \sin\theta_{S3}\sin\theta_{S2}\sin\theta_{S1} + \cos\theta_{S3}\cos\theta_{S1} & \sin\theta_{S3}\sin\theta_{S2}\cos\theta_{S1} - \cos\theta_{S3}\sin\theta_{S1} \\ -\sin\theta_{S2} & \cos\theta_{S2}\sin\theta_{S1} & \cos\theta_{S2}\cos\theta_{S1} \end{bmatrix}. \tag{6}$$

The velocity u_{SHi} of this point on the surface of the ship is given, with respect to the ship fixed coordinate system [6], by

$$u_{SHi} = u_{Si} + e_{ijk}q_{Sj}R_{SHk} + dR_{SHi}/dt, \tag{7}$$

where u_{Si} and q_{Si} are, respectively, the linear velocities along, and angular velocities about, the $O_Sx_{S1}x_{S2}x_{S3}$ coordinate system, and e_{ijk} is the permutation symbol defined by

$$e_{ijk} = \begin{cases} 0, & \text{if any two of } i,\ j,\ k \text{ are the same} \\ 1, & \text{if } ijk \text{ is an even permutation of } 1, 2, 3 \\ -1, & \text{if } ijk \text{ is an odd permutation of } 1, 2, 3. \end{cases} \tag{8}$$

In standard terminology, q_{S1} is the roll rate, q_{S2} is the pitch rate, and q_{S3} is the yaw rate. The relationship between the angular rates and rates of change of the heading angles is determined [6] from

$$q_{Si} = R_{ij}\, d\theta_{Sj}/dt, \tag{9}$$

where the rotational matrix R is

$$R = \begin{bmatrix} 1 & 0 & -\sin\theta_{S2} \\ 0 & \cos\theta_{S1} & \cos\theta_{S2}\sin\theta_{S1} \\ 0 & -\sin\theta_{S1} & \cos\theta_{S2}\cos\theta_{S1} \end{bmatrix}. \tag{10}$$

The transform of the velocity u_{SHi} from the ship fixed to the fluid-fixed coordinate system gives the velocity u_{0Hi} in the latter system as

$$u_{0Hi} = T_{ij} u_{SHj}. \tag{11}$$

Applying the no-slip condition over the surface of the ship then yields the boundary condition on the fluid

$$u_{0i} = u_{0Hi} \qquad \text{for } \{x_{01}, x_{02}, x_{03}\} = \{R_{0H1}, R_{0H2}, R_{0H3}\}. \tag{12}$$

In addition, assuming infinitely deep and unrestricted waters, we have

$$u_{0i} \to u_{0\infty i} \qquad \text{as } \sqrt{x_{0i} x_{0i}} \to \infty, \tag{13}$$

where $u_{0\infty i}$ is the velocity of the fluid at infinity referenced to the $0_0 x_{01} x_{02} x_{03}$ coordinate system.

Let us now consider conditions at the free surface of the water. We designate by $\eta_0(x_{01}, x_{02}, t)$ the elevation of this surface above the undisturbed reference plane $x_{03} = 0$. Two conditions prevail at the free surface: the first relating to continuity of stress across the air-water interface and the second relating to continuity of the interface itself. Neglecting the presence of any ambient surface films, the former condition can be written [5] as

$$\begin{aligned}&(\Pi_0 - P_A - 2\alpha H_0) n_{0i} \\ &- (1 - \gamma_{0B})\mu \left(\frac{\partial u_{0i}}{\partial x_{0k}} + \frac{\partial u_{0k}}{\partial x_{0i}} - \frac{2}{3}\,\delta_{ik}\,\frac{\partial u_{0j}}{\partial x_{0j}} \right) n_{0k} = -\sigma_{0ik}^{(w)} n_{0k} \qquad \text{at } x_{03} = \eta_0.\end{aligned} \tag{14}$$

Here, P_A is the ambient atmospheric pressure and $\sigma_{0ik}^{(w)}$ are the components of the wind stress tensor at the surface. The coefficient of surface tension is denoted by α. Also, n_{0i} and H_0 are, respectively, the outward normal to the free surface and the local mean curvature of the free surface. The latter quantity is taken as positive if the center of curvature is within the water. From Ref. 7, we have

$$n_{0i} = \frac{-\dfrac{\partial \eta_0}{\partial x_{01}}\,\delta_{i1} - \dfrac{\partial \eta_0}{\partial x_{02}}\,\delta_{i2} + \delta_{i3}}{\left[1 + \left(\dfrac{\partial \eta_0}{\partial x_{01}}\right)^2 + \left(\dfrac{\partial \eta_0}{\partial x_{02}}\right)^2\right]^{1/2}} \tag{15a}$$

and

$$2H_0 = \frac{-\dfrac{\partial^2 \eta_0}{\partial x_{01}^2}\left[1 + \left(\dfrac{\partial \eta_0}{\partial x_{02}}\right)^2\right] + 2\,\dfrac{\partial \eta_0}{\partial x_{01}}\,\dfrac{\partial \eta_0}{\partial x_{02}}\,\dfrac{\partial^2 \eta_0}{\partial x_{01}\partial x_{02}} - \dfrac{\partial^2 \eta_0}{\partial x_{02}^2}\left[1 + \left(\dfrac{\partial \eta_0}{\partial x_{01}}\right)^2\right]}{\left[1 + \left(\dfrac{\partial \eta_0}{\partial x_{01}}\right)^2 + \left(\dfrac{\partial \eta_0}{\partial x_{02}}\right)^2\right]^{3/2}}. \tag{15b}$$

The second condition, relating to continuity of the air-water interface, is obtained as

$$\frac{\partial \eta_0}{\partial t} + u_{01}\,\frac{\partial \eta_0}{\partial x_{01}} + u_{02}\,\frac{\partial \eta_0}{\partial x_{02}} - u_{03} = 0 \qquad \text{at } x_{03} = \eta_0. \tag{16}$$

TRANSFORMATION TO MOVING COORDINATES

Suppose that, in some mean sense, the origin O_S of the ship fixed coordinate system is moving in the negative x_{01} direction with velocity U. It is then desirable to refer the governing equations of fluid motion to a coordinate system more closely aligned with the position of the ship than the $O_0 x_{01} x_{02} x_{03}$ system. We define this system as the $O x_1 x_2 x_3$ system with

$$x_i = x_{0i} + U\delta_{i1} t \tag{17}$$

and we have taken the origins of the two systems to coincide at $t = 0$. The relationships between the derivatives in the fixed and moving systems are found as

$$\frac{\partial}{\partial x_{0i}} = \frac{\partial}{\partial x_i}, \qquad \left(\frac{\partial}{\partial t}\right)_{\text{fixed}} = \frac{\partial}{\partial t} + U \frac{\partial}{\partial x_1}. \tag{18}$$

We also have that any quantity Q_0 in the fixed system is given in the moving system as

$$Q_0(x_{0i},t) = Q_0(x_i - U\delta_{i1}t,t) \equiv Q(x_i,t) \tag{19a}$$

and, conversely,

$$Q(x_i,t) = Q(x_{0i} + U\delta_{i1}t,t) \equiv Q_0(x_{0i},t). \tag{19b}$$

If, in addition to the above transforms, we write the total pressure Π as the sum of its hydrostatic and dynamic components or

$$\Pi = P_A - \rho g x_3 + P, \tag{20}$$

the equations of fluid motion in the moving coordinate system are obtained from Eq. (4) as

$$\frac{\partial}{\partial t}(1-\gamma_B) + U\frac{\partial}{\partial x_1}(1-\gamma_B) + \frac{\partial}{\partial x_k}[(1-\gamma_B)u_k] = 0 \tag{21a}$$

and

$$\frac{\partial}{\partial t}\{(1-\gamma_B)u_i\} + U\frac{\partial}{\partial x_1}\{(1-\gamma_B)u_i\} + \frac{\partial}{\partial x_k}\{(1-\gamma_B)u_iu_k\}$$
$$= -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + g\gamma_B\delta_{i3} + \frac{1}{\rho}\frac{\partial}{\partial x_k}\left\{(1-\gamma_B)\mu\left[\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial u_j}{\partial x_j}\right]\right\}. \tag{21b}$$

From Eq. (2), the void fraction γ_B is

$$\gamma_B = \int_0^\infty f_{V_B} V_B dV_B. \tag{22}$$

The gas particle velocity u_{Bi} and location $R_{Bi} = R_{0Bi} + U\delta_{i1}t$ in the moving coordinate system are determined from Eqs. (3a) and (3b) as

$$u_{Bi} = u_i + u_{R3}\delta_{i3} \tag{23a}$$

and

$$\frac{d}{dt}R_{Bi} = U\delta_{i1} + u_{Bi}. \tag{23b}$$

To resolve the boundary condition over the surface of the ship, it is convenient to first define

$$\theta_{Si} = \pi\delta_{i3} + \phi_{Si} \tag{24}$$

so that the ship-fixed x_{S1} axis is basically aligned with the negative x_1 axis. The transformation and rotational matrices, Eqs. (6) and (10), become

$$T = \begin{bmatrix} -\cos\phi_{S3}\cos\phi_{S2} & -\cos\phi_{S3}\sin\phi_{S2}\sin\phi_{S1} + \sin\phi_{S3}\cos\phi_{S1} & -\cos\phi_{S3}\sin\phi_{S2}\cos\phi_{S1} - \sin\phi_{S3}\sin\phi_{S1} \\ -\sin\phi_{S3}\cos\phi_{S2} & -\sin\phi_{S3}\sin\phi_{S2}\sin\phi_{S1} - \cos\phi_{S3}\cos\phi_{S1} & -\sin\phi_{S3}\sin\phi_{S2}\cos\phi_{S1} + \cos\phi_{S3}\sin\phi_{S1} \\ -\sin\phi_{S2} & \cos\phi_{S2}\sin\phi_{S1} & \cos\phi_{S2}\cos\phi_{S1} \end{bmatrix} \tag{25a}$$

and

$$R = \begin{bmatrix} 1 & 0 & -\sin\phi_{S2} \\ 0 & \cos\phi_{S1} & \cos\phi_{S2}\sin\phi_{S1} \\ 0 & -\sin\phi_{S1} & \cos\phi_{S2}\cos\phi_{S1} \end{bmatrix}. \tag{25b}$$

Further, we write

$$u_{Si} = -UT_{ij}^{-1}\delta_{j1} + \mathrm{v}_{Si} = -UT_{i1}^{-1} + \mathrm{v}_{Si} \tag{26}$$

to separate the basic motion of the ship in the negative x_{01} direction from the remaining linear velocity components v_{Si} along the ship-fixed axes. The velocity of a point on the surface of the ship then is found from Eq. (7) as

$$u_{SHi} = -UT_{i1}^{-1} + \mathrm{v}_{Si} + e_{ijk}q_{Sj}R_{SHk} + dR_{SHi}/dt. \tag{27}$$

From Eqs. (11) and (12), the boundary condition over the surface of the ship with respect to the $Ox_1x_2x_3$ coordinate system is obtained as

$$u_i = T_{ij}u_{SHj} \qquad \text{for } \{x_1,x_2,x_3\} = \{R_{H1},R_{H2},R_{H3}\} \tag{28a}$$

where, from Eq. (5),

$$R_{Hi} = R_{Si} + T_{ij}R_{SHj} \tag{28b}$$

and where R_{Si} is the vector from O to O_S. In addition, we have from Eqs. (13) and (19a)

$$u_i \rightarrow u_{0\infty i}(x_i - U\delta_{i1}t,t) \qquad \text{as } \sqrt{x_ix_i} \rightarrow \infty. \tag{29}$$

At the free surface, the condition on continuity of stresses is found from Eqs. (14) and (20) as

$$(P - \rho g\eta - 2\alpha H)n_i - (1-\gamma_B)\mu\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial u_j}{\partial x_j}\right)n_k = -\sigma_{ik}^{(w)}n_k \qquad \text{at } x_3 = \eta \tag{30a}$$

where, from Eqs. (15a) and (15b),

$$n_i = \frac{-\dfrac{\partial\eta}{\partial x_1}\delta_{i1} - \dfrac{\partial\eta}{\partial x_2}\delta_{i2} + \delta_{i3}}{\left[1 + \left(\dfrac{\partial\eta}{\partial x_1}\right)^2 + \left(\dfrac{\partial\eta}{\partial x_2}\right)^2\right]^{1/2}} \tag{30b}$$

and

$$2H = \frac{-\dfrac{\partial^2\eta}{\partial x_1^2}\left[1 + \left(\dfrac{\partial\eta}{\partial x_2}\right)^2\right] + 2\dfrac{\partial\eta}{\partial x_1}\dfrac{\partial\eta}{\partial x_2}\dfrac{\partial^2\eta}{\partial x_1\partial x_2} - \dfrac{\partial^2\eta}{\partial x_2^2}\left[1 + \left(\dfrac{\partial\eta}{\partial x_1}\right)^2\right]}{\left[1 + \left(\dfrac{\partial\eta}{\partial x_1}\right)^2 + \left(\dfrac{\partial\eta}{\partial x_2}\right)^2\right]^{3/2}}. \tag{30c}$$

The condition on continuity of the free surface is obtained from Eqs. (16) and (18) as

$$\frac{\partial \eta}{\partial t} + (U + u_1)\frac{\partial \eta}{\partial x_1} + u_2\frac{\partial \eta}{\partial x_2} - u_3 = 0 \qquad \text{at } x_3 = \eta. \tag{31}$$

Within the limits of the inertialess particle hypothesis and the neglect of ambient stratification and ambient surface films, Eqs. (21) through (31) represent an exact description of the hydrodynamic flow field about a surface ship. The set of equations is unsolvable.

DEVELOPMENT OF THE MODEL

Part 1: Decoupling the Fluid and Gas Particle Flow Fields

The first step in constructing a solvable model for the hydrodynamic wake of a surface ship is to decouple the water and entrained gas particle flow fields. This entails approximating the void fraction by $\gamma_B = 0$ in the equations of motion and boundary conditions. Though this decoupling step is not absolutely essential, it does greatly simplify the numerical work involved in the flow field calculations. If higher order approximations are necessary, an iterative scheme such as outlined in Ref. 2 can be used. However, since the equations of motion in the inertialess particle limit are valid only for $\gamma_B << 1$, the error introduced by setting $\gamma_B = 0$ is generally small. This is especially true at distances removed from strong sources of gas particles.

With $\gamma_B = 0$, Eqs. (21a) and (21b) reduce to the standard continuity and Navier-Stokes equations for an incompressible fluid, or

$$\frac{\partial u_k}{\partial x_k} = 0 \tag{32a}$$

and

$$\frac{\partial u_i}{\partial t} + U\frac{\partial u_i}{\partial x_1} + \frac{\partial u_i u_k}{\partial x_k} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \frac{\mu}{\rho}\frac{\partial^2 u_i}{\partial x_k \partial x_k}. \tag{32b}$$

Part 2: Converting the Ship and Free Surface Boundary Conditions to Known Surfaces

A fundamental difficulty preventing the solution of Eqs. (21) through (31) is that the ship and free surface boundary conditions are applied at surfaces whose locations are not known "a priori." Hence, the second step in constructing a solvable model for the hydrodynamic wake of a surface ship is to convert these boundary conditions to surfaces whose locations are known "a priori."

a. Ship Boundary Conditions

Referring to Eqs. (28b) and (25a), we see that both R_{Si}, the vector from O to O_S, and T_{ij}, the transformation matrix, contain terms dependent on the instantaneous location of the ship hull. If we stipulate that the time-averaged (mean) motions of the ship about its equilibrium motion in the negative x_{01} direction are zero and that these motions are themselves suitably small, then the boundary condition on the ship hull can be applied at the mean location of the hull. For those points on the ship hull that have motions in addition to the rigid body ship motions, the boundary condition is to be applied at some appropriate mean location of the points (for example, the equivalent actuator disc of a propeller [8] or the zero deflection location of a control surface).

Taking the time averages of Eqs. (28b) and (25a) and imposing the above stipulations, we find the mean location of the ship hull as

$$\overline{R}_{Hi} = \overline{R}_{Si} + \overline{T}_{ij} R_{SHj}, \tag{33a}$$

where overbars denote time averages (means), and where

$$\overline{T}_{ij} = -\delta_{ij} + 2\delta_{i3}\delta_{j3}. \tag{33b}$$

To the same order of approximation, the boundary condition at the mean location of the ship hull is, from Eq. (28a),

$$u_i = \overline{T}_{ij} u_{SHj} \qquad \text{for } \{x_1,x_2,x_3\} = \{\overline{R}_{H1},\overline{R}_{H2},\overline{R}_{H3}\} \tag{34a}$$

where, from Eq. (27),

$$u_{SHi} = U\delta_{i1} + \mathrm{v}_{Si} + e_{ijk} q_{Sj} R_{SHk} + \mathrm{v}^*_{SHi}. \tag{34b}$$

Here, $dR_{SHi}/dt = \mathrm{v}^*_{SHi}$ indicates that effective, as opposed to actual, velocities may be required at those points having motions in addition to the rigid body ship motions. (See again Ref. 8 for a discussion of propeller flow problems.) Combining Eqs. (34a) and (34b), we obtain

$$u_i = -U\delta_{i1} + \overline{T}_{ij}(\mathrm{v}_{Sj} + e_{jkl} q_{Sk} R_{SHl} + \mathrm{v}^*_{SHj}) \qquad \text{for } \{x_1,x_2,x_3\} = \{\overline{R}_{H1},\overline{R}_{H2},\overline{R}_{H3}\}. \tag{35}$$

This expression is equivalent to the standard linearized ship hull boundary condition. Higher order boundary conditions, based on Taylor series expansions about this mean condition, have been developed [9]. However, the flow fields calculated using these higher order boundary conditions do not differ significantly from those calculated using the mean condition while the numerical work involved does. Hence, we take Eq. (35) as the appropriate ship hull boundary condition in our model for the hydrodynamic wake of a surface ship.

b. Free Surface Boundary Conditions

The exact free surface boundary conditions, taken at the unknown surface $x_3 = \eta$, are given by Eqs. (30) and (31). If we stipulate that η is suitably small, these boundary condtions can be approximated by a Taylor series expansion about $x_3 = 0$. To model the hydrodynamic wake of a surface ship, we retain only the lowest order terms of these expansions—again noting that higher order boundary conditions lead to significant increases in numerical complexity while not significantly affecting the calculated results.

To obtain the appropriate boundary conditions to be applied at $x_3 = 0$, we let $\eta = \epsilon\eta'$ where ϵ is a small dimensionless parameter. Expanding the kinematic free surface boundary condition given by Eq. (31) about $x_3 = 0$ and substituting for η, we have

$$\epsilon \frac{\partial \eta'}{\partial t} + \left[U + u_1 + \epsilon\eta' \frac{\partial u_1}{\partial x_3} + 0(\epsilon^2)\right] \epsilon \frac{\partial \eta'}{\partial x_1} + \left[u_2 + \epsilon\eta' \frac{\partial u_2}{\partial x_3} + 0(\epsilon^2)\right] \epsilon \frac{\partial \eta'}{\partial x_2}$$
$$-\left[u_3 + \epsilon\eta' \frac{\partial u_3}{\partial x_3} + 0(\epsilon^2)\right] = 0 \qquad \text{at } x_3 = 0.$$

We see that for a consistent expansion $u_3 = \epsilon u_3'$ and, with this substitution, find the kinematic free surface boundary condition to lowest order in ϵ as

$$\frac{\partial \eta}{\partial t} + (U + u_1) \frac{\partial \eta}{\partial x_1} + u_2 \frac{\partial \eta}{\partial x_2} - u_3 = 0 \qquad \text{at } x_3 = 0. \tag{36}$$

Consider now Eq. (30a). With $\gamma_B = 0$, this boundary condition becomes

$$(P - \rho g \eta - 2\alpha H) n_i - \sigma_{ik} n_k = -\sigma_{ik}^{(w)} n_k \qquad \text{at } x_3 = \eta \tag{37a}$$

where we have put

$$\sigma_{ik} = \mu \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right). \tag{37b}$$

To estimate σ_{ik} near the free surface, we note that from continuity, Eq. (32a), both $\partial u_1/\partial x_1$ and $\partial u_2/\partial x_2$ are of the order $\partial u_3/\partial x_3 = \epsilon \partial u_3'/\partial x_3$. Hence, $\sigma_{11} = \epsilon \sigma_{11}'$, $\sigma_{22} = \epsilon \sigma_{22}'$, and $\sigma_{33} = \epsilon \sigma_{33}'$. Similarly, since there is no reason for one horizontal direction to be preferred over the other, $\sigma_{12} = \epsilon \sigma_{12}'$. Also, since we have stipulated small surface displacements, both $\partial u_1/\partial x_3$ and $\partial u_2/\partial x_3$ must be small; or, $\sigma_{13} = \epsilon \sigma_{13}'$ and $\sigma_{23} = \epsilon \sigma_{23}'$. Thus, we find $\sigma_{ik} = \epsilon \sigma_{ik}'$. Further, from Eq. (30c), we have

$$2H = -\epsilon \left(\frac{\partial^2 \eta'}{\partial x_1^2} + \frac{\partial^2 \eta'}{\partial x_2^2} \right) + 0(\epsilon^3).$$

Expanding Eq. (37a) about $x_3 = 0$ and substituting the above results, we find

$$\left[\epsilon P' + \epsilon^2 \eta' \frac{\partial P'}{\partial x_3} + 0(\epsilon^3) - \rho g \epsilon \eta' + \alpha \epsilon \left(\frac{\partial^2 \eta'}{\partial x_1^2} + \frac{\partial^2 \eta'}{\partial x_2^2} \right) + 0(\epsilon^3) \right] n_i$$

$$- \left(\epsilon \sigma_{ik}' + \epsilon^2 \eta' \frac{\partial \sigma_{ik}'}{\partial x_3} + 0(\epsilon^3) \right) n_k = -\sigma_{ik}^{(w)} n_k \qquad \text{at } x_3 = 0$$

where, for a consistent expansion, we have set $P = \epsilon P'$. Taking the $i = 1$, 2, and 3 components of this expansion and observing from Eq. (30b) that n_1, n_2, and n_3 are of orders ϵ, ϵ, and 1, respectively, we obtain, to lowest order in ϵ, the conditions on continuity of stress across the free surface as

$$\mu \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) = \sigma_{13}^{(w)} \quad \text{at } x_3 = 0, \tag{38a}$$

$$\mu \left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \right) = \sigma_{23}^{(w)} \quad \text{at } x_3 = 0, \text{ and} \tag{38b}$$

$$P - \rho g \eta + \alpha \left(\frac{\partial^2 \eta}{\partial x_1^2} + \frac{\partial^2 \eta}{\partial x_2^2} \right) - 2\mu \frac{\partial u_3}{\partial x_3} = -\sigma_{33}^{(w)} \quad \text{at } x_3 = 0. \tag{38c}$$

Equations (36) and (38) are the desired free surface boundary conditions for our model of the hydrodynamic wake of a surface ship.

Part 3: Decomposing the Flow Fields

To proceed with the development of our model, we next wish to decompose the flow field into ship-induced and interaction-induced components. This decomposition allows us to both visualize and calculate the overall flow field as two basically different, but interacting, components.

Let us write

$$u_i = \mathrm{v}_i + \mathrm{v}_i' + w_i \tag{39}$$

where we define v_i and v_i', respectively, as the mean and randomly fluctuating flow velocities induced by the ship motions in otherwise still water and w_i as the flow velocities induced by the background and background/ship flow interactions. Several assumptions are implicit in the decomposition of the flow field represented by Eq. (39).

First, we have assumed, through the introduction of v_i', that turbulence modeling will be incorporated into our description of the hydrodynamic wake of a surface ship.

Second, we have assumed that the turbulent velocity fluctuations v_i' produced by the ship motions are significantly larger in magnitude than fluctuations occurring on the same time scale in the background. Hence, the term w_i' has been neglected in the velocity decomposition given by Eq. (39). This assumption is physically reasonable since oceanic sources of turbulence are normally weak compared to the sources resulting from the presence of a ship.

Third, we have assumed that the time scale of the turbulent fluctuations v_i' is small compared to the time scales of both the mean ship-induced flow v_i and the mean interaction flow w_i. This assumption permits the unambiguous decomposition of the velocity field given by Eq. (39). The fundamental limit on this assumption is the time scale of the shortest gravity-capillary waves present in the background. For example, for 3.5-cm waves, we find that the turbulent fluctuations must occur with a frequency $>>20$ Hz which is near the limit of strict validity of the assumption. For much shorter waves, the assumption clearly fails though certain mitigating factors possibly are present because of the very small depth of penetration of such waves.

In a manner similar to the decomposition of the velocity field, we write the dynamic pressure P and surface elevation η as

$$P = P_v + P_v' + P_w \tag{40a}$$

and

$$\eta = \eta_v + \eta_v' + \eta_w, \tag{40b}$$

where the subscripts v and w refer, respectively, to quantities induced by the ship motions in otherwise still water and to quantities induced by the background and background/ship flow interactions. The three assumptions applicable to the decomposition of the velocity field are applicable also to the decompositions of P and η.

Proceeding to substitute Eqs. (39) and (40) into Eqs. (32) governing the fluid motions and Eqs. (29), (35), (36), and (38) governing the fluid behavior at the boundaries of the flow, and time averaging the resultant expressions, the turbulent equations of motion and the turbulent boundary conditions are obtained. We separate these equations of motion and boundary conditions into sets of terms identified, respectively, with the flow field induced by ship motions in otherwise still water and the flow field induced by the background and background/ship flow interactions. Setting these sets of terms individually to zero, since they are individually separable and equal to zero at infinity, produces the governing equations of motion and boundary conditions for the two components of the decomposed flow field.

a. Ship-Induced Flow Field

We find the continuity and turbulent Navier-Stokes equations for the ship-induced flow field to be given by

$$\frac{\partial v_k}{\partial x_k} = 0 \tag{41a}$$

and

$$\frac{\partial v_i}{\partial t} + U\frac{\partial v_i}{\partial x_1} + \frac{\partial v_i v_k}{\partial x_k} = -\frac{1}{\rho}\frac{\partial P_v}{\partial x_i} + \frac{\partial}{\partial x_k}\left[\frac{\mu}{\rho}\frac{\partial v_i}{\partial x_k} - \overline{v_i' v_k'}\right]. \tag{41b}$$

The boundary conditions applicable to this flow field are obtained as

$$v_i \rightarrow 0 \qquad \text{as } \sqrt{x_i x_i} \rightarrow \infty, \tag{41c}$$

$$\mathrm{v}_i = -U\delta_{i1} + \overline{T}_{ij}(\mathrm{v}_{Sj} + e_{jkl}q_{Sk}R_{SHl} + \mathrm{v}^*_{SHj}) \qquad \text{for } \{x_1,x_2,x_3\} = \{\overline{R}_{H1},\overline{R}_{H2},\overline{R}_{H3}\}, \tag{41d}$$

$$\mu\left[\frac{\partial \mathrm{v}_1}{\partial x_3} + \frac{\partial \mathrm{v}_3}{\partial x_1}\right] + \overline{\sigma'_{13\mathrm{v}}} = 0 \qquad \text{at } x_3 = 0, \tag{41e}$$

$$\mu\left[\frac{\partial \mathrm{v}_2}{\partial x_3} + \frac{\partial \mathrm{v}_3}{\partial x_2}\right] + \overline{\sigma'_{23\mathrm{v}}} = 0 \qquad \text{at } x_3 = 0, \tag{41f}$$

$$P_\mathrm{v} - \rho g \eta_\mathrm{v} + \alpha\left[\frac{\partial^2 \eta_\mathrm{v}}{\partial x_1^2} + \frac{\partial^2 \eta_\mathrm{v}}{\partial x_2^2}\right] + \overline{\sigma'_{\tau\mathrm{v}}} - 2\mu\frac{\partial \mathrm{v}_3}{\partial x_3} - \overline{\sigma'_{33\mathrm{v}}} = 0 \qquad \text{at } x_3 = 0, \tag{41g}$$

and

$$\frac{\partial \eta_\mathrm{v}}{\partial t} + U\frac{\partial \eta_\mathrm{v}}{\partial x_1} + \frac{\partial}{\partial x_i}(\mathrm{v}_i\eta_\mathrm{v} + \overline{\mathrm{v}'_i\eta'_\mathrm{v}}) - \mathrm{v}_3 = 0 \qquad \text{at } x_3 = 0. \tag{41h}$$

In developing Eqs. (41e) through (41g), we have allowed for the possibility that the average values of both the fluctuating stresses $\overline{\sigma'_{ij\mathrm{v}}}$ and fluctuating surface tension $\overline{\sigma'_{\tau\mathrm{v}}}$ are not zero. Also, Eq. (41h) has been developed by writing Eq. (36) in the equivalent form

$$\frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial x_1} + \frac{\partial u_i\eta}{\partial x_i} - u_3 = 0$$

which follows from Eq. (32a) and the fact that η is independent of x_3.

b. Interaction-Induced Flow Field

For the interaction-induced flow field, the governing equations of motion and boundary conditions are found as

$$\frac{\partial w_k}{\partial x_k} = 0, \tag{42a}$$

$$\frac{\partial w_i}{\partial t} + U\frac{\partial w_i}{\partial x_1} + \frac{\partial}{\partial x_k}(w_iw_k + w_i\mathrm{v}_k + \mathrm{v}_iw_k) = -\frac{1}{\rho}\frac{\partial P_w}{\partial x_i} + \frac{\mu}{\rho}\frac{\partial^2 w_i}{\partial x_k \partial x_k}, \tag{42b}$$

$$w_i \to u_{0\infty i}(x_i - U\delta_{i1}t,t) \qquad \text{as } \sqrt{x_ix_i} \to \infty, \tag{42c}$$

$$w_i = 0 \qquad \text{for } \{x_1,x_2,x_3\} = \{\overline{R}_{H1},\overline{R}_{H2},\overline{R}_{H3}\}, \tag{42d}$$

$$\mu\left[\frac{\partial w_1}{\partial x_3} + \frac{\partial w_3}{\partial x_1}\right] = \sigma_{13}^{(w)} \qquad \text{at } x_3 = 0, \tag{42e}$$

$$\mu\left[\frac{\partial w_2}{\partial x_3} + \frac{\partial w_3}{\partial x_2}\right] = \sigma_{23}^{(w)} \qquad \text{at } x_3 = 0, \tag{42f}$$

$$P_w - \rho g \eta_w + \alpha\left[\frac{\partial^2 \eta_w}{\partial x_1^2} + \frac{\partial^2 \eta_w}{\partial x_2^2}\right] - 2\mu\frac{\partial w_3}{\partial x_3} = -\sigma_{33}^{(w)} \qquad \text{at } x_3 = 0, \tag{42g}$$

and

$$\frac{\partial \eta_w}{\partial t} + U\frac{\partial \eta_w}{\partial x_1} + \frac{\partial}{\partial x_i}(w_i\eta_w + w_i\eta_\mathrm{v} + \mathrm{v}_i\eta_w) - w_3 = 0 \qquad \text{at } x_3 = 0. \tag{42h}$$

Part 4: A Matched Asymptotic Expansion for the Ship-Induced Flow Field

Equations (41) describing the ship induced flow field in otherwise still water are, with currently available techniques, unsolvable on large domains regardless of the turbulence model used. The principal difficulties preventing a solution are the nonlinear free surface boundary condition given by Eq.

(41h) and the second order viscous derivative terms in Eq. (41b). These derivative terms, which are influential only in the vicinity of the ship hull and the ship turbulent wake, give Eqs. (41) the characteristics of what Van Dyke [10] terms a singular perturbation problem. For such problems, further progress can be made by applying the method of matched asymptotic expansions [10] to the solution procedure. Before applying this method, we must nondimensionalize Eqs. (41) to identify the relevant governing parameters, and before this, introduce the rudiments of a turbulence model.

For our present purposes, it is sufficient to use isotropic eddy viscosity and diffusivity models for the turbulent stresses and diffusivities appearing in Eqs. (41). More complex models can be readily incorporated without changing the basic results of this section. Following Rodi [11], we write

$$-\overline{\mathrm{v}_i'\mathrm{v}_k'} = \frac{\overline{\sigma_{ik\mathrm{v}}'}}{\rho} = \nu_t\left(\frac{\partial \mathrm{v}_i}{\partial x_k} + \frac{\partial \mathrm{v}_k}{\partial x_i}\right) - \frac{2}{3}\,\mathrm{k}\delta_{ik}, \tag{43a}$$

where ν_t is the turbulent viscosity, and k is the turbulent kinetic energy per unit mass,

$$\mathrm{k} = \frac{1}{2}\,\overline{\mathrm{v}_i'\mathrm{v}_i'}. \tag{43b}$$

By analogy with Eq. (43a), we also take

$$\overline{\sigma_{\tau\mathrm{v}}'} = \alpha_t\left(\frac{\partial^2\eta_\mathrm{v}}{\partial x_1^2} + \frac{\partial^2\eta_\mathrm{v}}{\partial x_2^2}\right) \tag{44a}$$

where α_t denotes the coefficient of turbulent surface tension. Further, in analogy with eddy diffusivity models for turbulent heat or mass transport, we put

$$-\overline{\mathrm{v}_i'\eta_\mathrm{v}'} = \sigma_\eta\nu_t\,\frac{\partial\eta_\mathrm{v}}{\partial x_i}, \tag{44b}$$

where σ_η is a dimensionless number. We also set

$$\frac{\mu}{\rho} = \nu = \sigma_\nu\nu_t \tag{45a}$$

and

$$\alpha = \sigma_\alpha\alpha_t. \tag{45b}$$

Here, ν is the kinematic viscosity of the water; σ_ν is the ratio between this and the turbulent viscosity; and σ_α is the ratio between the coefficient of surface tension and the coefficient of turbulent surface tension.

Since our primary interest is in the hydrodynamic wake produced by the ship, we choose nondimensionalizing parameters associated with this wake. As a length scale, we use the dominant Kelvin wavelength [12] $2\pi U^2/g$ and, as the time scale, the dominant Kelvin frequency $2\pi U/g$. This choice of scaling parameters renders the wake Froude number equal to unity. The dimensional quantities appearing in the ship wake problem then are given, in terms of their dimensionless counterparts, as

$$t = \frac{2\pi U}{g}\,\tau, \qquad x_i = \frac{2\pi U^2}{g}\,\xi_i, \qquad \mathrm{v}_i = U\tilde{\mathrm{v}}_i,$$

$$P_\mathrm{v} = \rho U^2\tilde{P}_\mathrm{v}, \qquad \mathrm{v}_{Si} = U\tilde{\mathrm{v}}_{Si}, \qquad \mathrm{v}^*_{SHi} = U\tilde{\mathrm{v}}^*_{SHi},$$

$$q_{Si} = \frac{g}{2\pi U}\,\tilde{q}_{Si}, \qquad R_{SHi} = \frac{2\pi U^2}{g}\,\tilde{R}_{SHi}, \qquad \overline{R}_{Hi} = \frac{2\pi U^2}{g}\,\tilde{\overline{R}}_{Hi},$$

$$\overline{R}_{Si} = \frac{2\pi U^2}{g}\,\tilde{\overline{R}}_{Si}, \qquad \text{and } \eta_\mathrm{v} = \frac{2\pi U^2}{g}\,\tilde{\eta}_\mathrm{v}. \tag{46a}$$

For the turbulent kinetic energy, we take

$$\mathrm{k} = \frac{\nu_t g}{2\pi U}\,\tilde{\mathrm{k}} \tag{46b}$$

which insures that both dimensionless components of Eq. (43a) are of the same order of magnitude, as required for dimensionless analysis.

Substituting Eqs. (43) through (46) into Eqs. (41), we obtain the dimensionless equations governing the ship-induced flow field problem as

$$\frac{\partial \tilde{\mathrm{v}}_i}{\partial \xi_i} = 0, \tag{47a}$$

$$\frac{\partial \tilde{\mathrm{v}}_i}{\partial \tau} + \frac{\partial \tilde{\mathrm{v}}_i}{\partial \xi_1} + \frac{\partial \tilde{\mathrm{v}}_i \tilde{\mathrm{v}}_k}{\partial \xi_k} = -\frac{\partial \tilde{P}_{\mathrm{v}}}{\partial \xi_i} + \frac{\partial}{\partial \xi_k}\left[\frac{1}{R_{Kt}}\left(\sigma_\nu \frac{\partial \tilde{\mathrm{v}}_i}{\partial \xi_k} + \frac{\partial \tilde{\mathrm{v}}_i}{\partial \xi_k} + \frac{\partial \tilde{\mathrm{v}}_k}{\partial \xi_i} - \frac{2}{3}\,\tilde{\mathrm{k}}\delta_{ik}\right)\right], \tag{47b}$$

$$\tilde{\mathrm{v}}_i \to 0 \qquad \text{as} \sqrt{\xi_i \xi_i} \to \infty, \tag{47c}$$

$$\tilde{\mathrm{v}}_i = -\delta_{i1} + \overline{T}_{ij}(\tilde{\mathrm{v}}_{Sj} + e_{jkl}\tilde{q}_{Sk}\tilde{R}_{SHl} + \tilde{\mathrm{v}}^*_{SHj}) \qquad \text{for } \{\xi_1, \xi_2, \xi_3\} = \{\overline{\overline{R}}_{H1}, \overline{\overline{R}}_{H2}, \overline{\overline{R}}_{H3}\}, \tag{47d}$$

$$\left(\frac{\partial \tilde{\mathrm{v}}_1}{\partial \xi_3} + \frac{\partial \tilde{\mathrm{v}}_3}{\partial \xi_1}\right) = 0 \qquad \text{at } \xi_3 = 0, \tag{47e}$$

$$\left(\frac{\partial \tilde{\mathrm{v}}_2}{\partial \xi_3} + \frac{\partial \tilde{\mathrm{v}}_3}{\partial \xi_2}\right) = 0 \qquad \text{at } \xi_3 = 0, \tag{47f}$$

$$\tilde{P}_{\mathrm{v}} - 2\pi\tilde{\eta}_{\mathrm{v}} + (\sigma_\alpha + 1)\,\frac{\alpha_{Kt}}{R_{Kt}}\left(\frac{\partial^2 \tilde{\eta}_{\mathrm{v}}}{\partial \xi_1^2} + \frac{\partial^2 \tilde{\eta}_{\mathrm{v}}}{\partial \xi_2^2}\right) - \frac{2(\sigma_\nu + 1)}{R_{Kt}}\,\frac{\partial \tilde{\mathrm{v}}_3}{\partial \xi_3} + \frac{2}{3}\,\frac{\tilde{k}}{R_{Kt}} = 0 \qquad \text{at } \xi_3 = 0, \tag{47g}$$

and

$$\frac{\partial \tilde{\eta}_{\mathrm{v}}}{\partial \tau} + \frac{\partial \tilde{\eta}_{\mathrm{v}}}{\partial \xi_1} + \frac{\partial}{\partial \xi_i}\left(\tilde{\mathrm{v}}_i \tilde{\eta}_{\mathrm{v}} - \frac{\sigma_\eta}{R_{Kt}}\,\frac{\partial \tilde{\eta}_{\mathrm{v}}}{\partial \xi_i}\right) - \tilde{\mathrm{v}}_3 = 0 \qquad \text{at } \xi_3 = 0. \tag{47h}$$

In these expressions, R_{Kt} and α_{Kt} are, respectively, the turbulent-flow Reynolds number and turbulent-flow inverse capillary number, both based on the Kelvin wake. We have

$$R_{Kt} = \frac{2\pi U^3}{g\nu_t} = \sigma_\nu\,\frac{2\pi U^3}{g\nu} = \sigma_\nu R_K \tag{48a}$$

$$\alpha_{Kt} = \frac{\alpha_t}{\rho U \nu_t} = \frac{\sigma_\nu}{\sigma_\alpha}\,\frac{\alpha}{\rho U \nu} = \frac{\sigma_\nu}{\sigma_\alpha}\,\alpha_K \tag{48b}$$

where R_K and α_K are, respectively, the laminar-flow Reynolds number and laminar-flow inverse capillary number. Typical values of R_K and α_K for water are

$$R_K = 6.4 \times 10^5 U^3, \quad \alpha_K = 73/U$$

with U in m/s (1 m/s ~ 2 knots).

As explained in Ref. 10, the method of matched asymptotic expansions is a solution technique for problems such as given by Eqs. (47) where the higher order derivatives have only limited influence on the flow field. The technique consists of developing two series solutions, one governed by an outer flow parameter that takes into account the unimportance of the higher derivatives for this flow; the other governed by an inner parameter that recognizes the limited influence of these higher derivatives. The final or composite solution is obtained by an appropriate combination of the two series that is valid over the entire flow field.

The terms outer and inner have arisen historically because of the application of the method to boundary layer and wake flows in infinite domains. For these problems, it is easy to visualize two distinct flow regions. For the ship wake problem, as we shall see, this analogy falters, because both outer and inner solutions occupy the flow region aft of the ship. However, the formalism of the method does not depend on defining specific flow regions, and we retain the terms outer and inner only for historical consistency.

a. Ship-Induced Outer Flow Field

Following the formalism laid out in Ref. 10, we seek an outer asymptotic solution to Eqs. (47) for large $R_K (=R_{Kt}/\sigma_v)$ since the inverse of this parameter multiplies the highest order derivatives in the equations. We write

$$\tilde{v}_i(\tau,\xi_i;R_K) \sim \left[\lim_{R_K\to\infty} \delta(R_K)\right] \tilde{v}_i^{\dagger}(\tau,\xi_i), \tag{49a}$$

$$\tilde{P}_v(\tau,\xi_i;R_K) \sim \left[\lim_{R_K\to\infty} \delta(R_K)\right] \tilde{P}_v^{\dagger}(\tau,\xi_i), \tag{49b}$$

$$\tilde{k}(\tau,\xi_i;R_K) \sim \left[\lim_{R_K\to\infty} \delta(R_K)\right] \tilde{k}^{\dagger}(\tau,\xi_i), \tag{49c}$$

and

$$\tilde{\eta}_v(\tau,\xi_1,\xi_2;R_K) \sim \left[\lim_{R_K\to\infty} \delta(R_K)\right] \tilde{\eta}_v^{\dagger}(\tau,\xi_1,\xi_2) \tag{49d}$$

where a superscript † denotes the flow variables associated with the outer flow field, and where $\delta(R_K)$ is an expansion parameter to be determined. Substituting these expansions into Eqs. (47) and taking the limits as $R_K \to \infty$, we find

$$\frac{\partial \tilde{v}_i^{\dagger}}{\partial \xi_i} = 0, \tag{50a}$$

$$\frac{\partial \tilde{v}_i^{\dagger}}{\partial \tau} + \frac{\partial \tilde{v}_i^{\dagger}}{\partial \xi_1} + \left[\lim_{R_K\to\infty} \delta(R_K)\right] \frac{\partial \tilde{v}_i^{\dagger}\tilde{v}_k^{\dagger}}{\partial \xi_k} = -\frac{\partial \tilde{P}_v^{\dagger}}{\partial \xi_i}, \tag{50b}$$

$$\tilde{v}_i^{\dagger} \to 0 \qquad \text{as } \sqrt{\xi_i\xi_i} \to \infty, \tag{50c}$$

$$\tilde{v}_i^{\dagger} = \left[\lim_{R_K\to\infty} \frac{1}{\delta(R_K)}\right] [-\delta_{i1} + \bar{T}_{ij}(\tilde{v}_{Sj} + e_{jkl}\tilde{q}_{Sk}\tilde{R}_{SHl} + \tilde{v}^{*}_{SHj})]$$

$$\text{for } \{\xi_1,\xi_2,\xi_3\} = \{\bar{\bar{R}}_{H1},\bar{\bar{R}}_{H2},\bar{\bar{R}}_{H3}\}, \tag{50d}$$

$$\left[\frac{\partial \tilde{v}_1^{\dagger}}{\partial \xi_3} + \frac{\partial \tilde{v}_3^{\dagger}}{\partial \xi_1}\right] = 0 \qquad \text{at } \xi_3 = 0, \tag{50e}$$

$$\left[\frac{\partial \tilde{v}_2^{\dagger}}{\partial \xi_3} + \frac{\partial \tilde{v}_3^{\dagger}}{\partial \xi_2}\right] = 0 \qquad \text{at } \xi_3 = 0, \tag{50f}$$

$$\tilde{P}_v^{\dagger} - 2\pi\tilde{\eta}_v^{\dagger} = 0 \qquad \text{at } \xi_3 = 0, \tag{50g}$$

and

$$\frac{\partial \tilde{\eta}_v^\dagger}{\partial \tau} + \frac{\partial \tilde{\eta}_v^\dagger}{\partial \xi_1} + \left[\lim_{R_K \to \infty} \delta(R_K)\right] \frac{\partial \tilde{v}_i^\dagger \tilde{\eta}_v^\dagger}{\partial \xi_i} - \tilde{v}_3^\dagger = 0 \qquad \text{at } \xi_3 = 0. \tag{50h}$$

We see from Eq. (50b) that the highest order derivatives have been lost in the outer asymptotic expansion as has $\tilde{k}^\dagger$. The outer problem is thus reduced to an inviscid flow problem (we set $\tilde{k}^\dagger = 0$ since it is indeterminate). Having lost the highest order derivatives, the boundary conditions on the ship hull given by Eq. (50d) cannot be identically satisfied nor can the entirety of free surface stress conditions given by Eqs. (50e) through (50g). We instead specify zero fluid penetration through the hull or, equivalently, zero normal velocity and continuity of the normal stress component across the free surface. Hence, Eqs. (50e) and (50f) are ignored in the outer flow approximation. Additionally, we drop v^*_{SHj} from the outer problem since these effective velocities are associated with vorticity production, which is inconsistent with the inviscid nature of the outer problem.

Defining the normal to the ship hull with respect to the ship fixed coordinate system by n_{SHi}, the normal n_i in the $Ox_1x_2x_3$ coordinate system becomes, consistent with the suitably small restrictions already imposed on the ship motions,

$$n_i = \overline{T}_{ij} n_{SHj}. \tag{51}$$

The appropriate outer flow boundary condition on the ship hull then is obtained as

$$\tilde{v}_i^\dagger n_i = \left[\lim_{R_K \to \infty} \frac{1}{\delta(R_K)}\right] [-n_1 + n_i \overline{T}_{ij} (\tilde{v}_{Sj} + e_{jkl} \tilde{q}_{Sk} \tilde{R}_{SHl})]$$
$$\text{for } \{\xi_1, \xi_2, \xi_3\} = \{\overline{\overline{R}}_{H1}, \overline{\overline{R}}_{H2}, \overline{\overline{R}}_{H3}\}. \tag{52}$$

From this equation, we note that the only nondegenerate outer flow problem results from assigning a finite, nonzero value to $\lim_{R_K \to \infty} \delta(R_K)$. Without loss of generality, we can set this limit to unity.

The outer flow problem, since it is inviscid and irrotational, can be simplified considerably by introducing a dimensionless outer flow velocity potential $\tilde{\phi}^\dagger$ related to its dimensional counterpart $\phi^\dagger$ through

$$\phi^\dagger = \frac{2\pi U^3}{g} \tilde{\phi}^\dagger. \tag{53a}$$

Then

$$\tilde{v}_i^\dagger = \frac{\partial \tilde{\phi}^\dagger}{\partial \xi_i}, \tag{53b}$$

and Eq. (50a) yields Laplace's equation for the potential as

$$\frac{\partial^2 \tilde{\phi}^\dagger}{\partial \xi_i \partial \xi_i} = 0, \tag{54a}$$

while Eq. (50b) can be integrated to give Bernoulli's equation

$$\tilde{P}_v^\dagger + \frac{\partial \tilde{\phi}^\dagger}{\partial \tau} + \tilde{v}_1^\dagger + \frac{1}{2} \tilde{v}_i^\dagger \tilde{v}_i^\dagger = 0. \tag{54b}$$

The appropriate boundary conditions, which all can be cast in terms of $\tilde{\phi}^\dagger$, are given by Eqs. (50c), (50g), (50h), and (52).

The above problem is, essentially, still unsolvable because of the nonlinearities appearing in Eqs. (50h) and (54b). However, as pointed out by Newman [12], neglecting these nonlinearities is *necessary* for consistency with the approximations already made in reducing the free surface boundary condition to the plane $\xi_3 = 0$. This is easy to verify. Recall that for this reduction, we took $\tilde{\eta}_v^\dagger \sim \epsilon\tilde{\eta}_v^{\dagger\prime}$ where ϵ was a small dimensionless parameter. This gave $\tilde{v}_3^\dagger \sim \epsilon\tilde{v}_3^{\dagger\prime}$ and $\tilde{P}_v^\dagger \sim \epsilon\tilde{P}_v^{\dagger\prime}$ in the vicinity of the free surface. The free surface boundary condition was then expanded about $\xi_3 = 0$, and all terms of higher order than ϵ were neglected. Referring to Eq. (54b), we see also that, for the outer flow problem, $\tilde{v}_1^\dagger \sim \epsilon\tilde{v}_1^{\dagger\prime}$ in the vicinity of the free surface. Hence, $\tilde{\phi}^\dagger$ and, consequently, $\tilde{v}_2^\dagger$ are, respectively, of order $\epsilon\phi^{\dagger\prime}$ and $\epsilon\tilde{v}_2^{\dagger\prime}$ in the vicinity of the free surface. Thus, the nonlinear terms in Eqs. (50h) and (54b) are of order ϵ^2 for the outer flow problem and *must be* dropped to ensure consistency with our previous approximations.

Returning to dimensional quantities via Eqs. (46a) and (53a), we can summarize the ship-induced outer flow field problem as

$$\frac{\partial^2\phi^\dagger}{\partial x_i \partial x_i} = 0, \tag{55a}$$

$$v_i^\dagger = \frac{\partial\phi^\dagger}{\partial x_i}, \tag{55b}$$

$$v_i^\dagger \to 0 \qquad \text{as } \sqrt{x_i x_i} \to \infty, \tag{55c}$$

$$v_i^\dagger n_i = -Un_1 + n_i\bar{T}_{ij}(v_{Sj} + e_{jkl}q_{Sk}R_{SHl}) \qquad \text{for } \{x_1,x_2,x_3\} = \{\bar{R}_{H1},\bar{R}_{H2},\bar{R}_{H3}\}, \tag{55d}$$

and

$$\frac{\partial^2\phi^\dagger}{\partial t^2} + 2U\frac{\partial^2\phi^\dagger}{\partial t\partial x_1} + U^2\frac{\partial^2\phi^\dagger}{\partial x_1^2} + g\frac{\partial\phi^\dagger}{\partial x_3} = 0 \qquad \text{at } x_3 = 0, \tag{55e}$$

together with the prognostic relations

$$P_v^\dagger = -\rho\left(\frac{\partial\phi^\dagger}{\partial t} + U\frac{\partial\phi^\dagger}{\partial x_1}\right) \tag{55f}$$

and

$$\eta_v^\dagger = -\frac{1}{g}\left(\frac{\partial\phi^\dagger}{\partial t} + U\frac{\partial\phi^\dagger}{\partial x_1}\right) \qquad \text{at } x_3 = 0. \tag{55g}$$

Equations (55) are recognized as being the traditional Kelvin-Neumann problem [12] for determining a ship's Kelvin and radiated wave systems. This problem, while nontrivial, is solvable.

One cautionary note is in order. For the outer problem to be fully consistent over the entire flow domain, it is necessary that the n_1 component of the ship normal be everywhere small (of order ϵ). For, if at some points on the hull it is of order unity, we find from Eq. (52) that, at those points, $\tilde{v}_1^\dagger \sim 0(1)$ as opposed to order ϵ. Hence, the consistency of the outer flow approximation breaks down at such singular points. Physically, such points give rise to the nonlinear bow wave problem [12] which must, in itself, be treated by other expansions and then incorporated within the traditional Kelvin-Neumann problem.

b. Ship-Induced Inner Flow Field

Having lost the viscous derivatives in the outer asymptotic expansion, we have been forced to neglect ship boundary layer and propeller effects and the ship turbulent wake generated by these effects. These viscous phenomena, which are confined to thin regions adjacent to the ship hull and circumferential to a line (the resultant thrust axis) running aft of the ship, must be recouped in the inner asymptotic solution. The technique leading to this solution uses the thin nature of the viscous regions, and the concomitant knowledge that flow quantities vary much more rapidly across these regions than along other regions, to rescale and simplify the equations and boundary conditions governing the general ship-induced flow field.

Since our interest is in the hydrodynamic wake of the ship, we begin by considering the inner asymptotic solution for the region aft of the ship. Before starting, however, one more cautionary note is in order. The subsequent developments restrict *the resultant thrust axis to be parallel to the direction of motion of the ship*, that is, *canted neither significantly towards nor away from the free surface.*

Following again the formalism laid out in Ref. 10, we introduce an inner expansion parameter $\Delta(R_K)$ that represents, in dimensionless terms, the circumferential extent of the viscous wake region about the resultant thrust axis. This parameter requires the property

$$\Delta(R_K) \rightarrow 0 \quad \text{as } R_K \rightarrow \infty. \tag{56}$$

Since the region is thin, a nonuniform scaling of coordinates is necessary to ensure that derivatives across and along the region are a uniform length scale. We accomplish this by modifying the dimensionless global coordinates ξ_i as

$$\xi_1 = s_1,\ \xi_2 = \Delta(R_K)s_2,\ \xi_3 = \Delta(R_K)s_3 \tag{57}$$

where the s_i give the required uniform inner coordinates. We have, here, tacitly assumed that the wake axis parallels ξ_1, from which assumption the above-noted restriction on the direction of the resultant thrust axis has been obtained.

Based on the scaled coordinates, we seek an inner asymptotic solution to Eqs. (47). By definition, the dimensionless velocity $\tilde{v}_1$ along the wake axis must exist if a long, thin, viscous region running aft of the ship is to exist. Hence, $\tilde{v}_1$ must be relatively independent of the circumferential extent of the wake, and we write

$$\tilde{v}_1(\tau,s_i;R_K) \sim \tilde{v}_1^*(\tau,s_i) \tag{58a}$$

where a superscript $*$ denotes the flow variables associated with the inner flow field. The velocities $\tilde{v}_2$ and $\tilde{v}_3$ transverse to the wake axis are taken, more generally, in terms of the inner expansion parameter as

$$\tilde{v}_2(\tau,s_i;R_K) \sim \left[\lim_{R_K\rightarrow\infty} \Delta^{\lambda}(R_K)\right]\tilde{v}_2^*(\tau,s_i), \tag{58b}$$

and

$$\tilde{v}_3(\tau,s_i;R_K) \sim \left[\lim_{R_K\rightarrow\infty} \Delta^{\lambda}(R_K)\right]\tilde{v}_3^*(\tau,s_i), \tag{58c}$$

where the nonnegative exponent λ is to be determined.

Substituting Eqs. (57) and (58) into Eq. (47a), we obtain the continuity equation for the inner flow field as

$$\frac{\partial \tilde{v}_1^*}{\partial s_1} + \left[\lim_{R_K\rightarrow\infty} \Delta^{\lambda-1}(R_K)\right]\left(\frac{\partial \tilde{v}_2^*}{\partial s_2} + \frac{\partial \tilde{v}_3^*}{\partial s_3}\right) = 0. \tag{59}$$

Thus, for the fundamental property of conservation of mass to be preserved in the inner flow field, we must take $\lambda = 1$. Employing this result and Eqs. (57) and (58) in the dimensionless version of Eq. (43a), we then find

$$\tilde{k}(\tau,s_i;R_K) \sim \tilde{k}^*(\tau,s_i) \tag{60}$$

if both components of the inner form of Eq. (43a) are to be of the same order of magnitude, as required for dimensionless analysis.

Using Eq. (48a), Eqs. (47b) become

$$\frac{\partial \tilde{v}_1^*}{\partial \tau} + \frac{\partial \tilde{v}_1^*}{\partial s_1} + \frac{\partial \tilde{v}_1^* \tilde{v}_i^*}{\partial s_i} = \lim_{R_K \to \infty} \left\{ -\frac{\partial \tilde{P}_v}{\partial s_1} + \frac{\partial}{\partial s_1} \left\{ \frac{1}{\sigma_\nu R_K} \left[(\sigma_\nu + 2) \frac{\partial \tilde{v}_1^*}{\partial s_1} - \frac{2}{3} \tilde{k}^* \right] \right\} \right.$$

$$+ \frac{\partial}{\partial s_2} \left\{ \frac{1}{\sigma_\nu R_K \Delta^2(R_K)} \left[(\sigma_\nu + 1) \frac{\partial \tilde{v}_1^*}{\partial s_2} + \Delta^2(R_K) \frac{\partial \tilde{v}_2^*}{\partial s_1} \right] \right\}$$

$$\left. + \frac{\partial}{\partial s_3} \left\{ \frac{1}{\sigma_\nu R_K \Delta^2(R_K)} \left[(\sigma_\nu + 1) \frac{\partial \tilde{v}_1^*}{\partial s_3} + \Delta^2(R_K) \frac{\partial \tilde{v}_3^*}{\partial s_1} \right] \right\} \right\} \tag{61a}$$

$$\frac{\partial \tilde{v}_2^*}{\partial \tau} + \frac{\partial \tilde{v}_2^*}{\partial s_1} + \frac{\partial \tilde{v}_2^* \tilde{v}_i^*}{\partial s_i} = \lim_{R_K \to \infty} \left\{ -\frac{1}{\Delta^2(R_K)} \frac{\partial \tilde{P}_v}{\partial s_2} + \frac{\partial}{\partial s_1} \left\{ \frac{1}{\sigma_\nu R_K \Delta^2(R_K)} \left[\Delta^2(R_K)(\sigma_\nu + 1) \frac{\partial \tilde{v}_2^*}{\partial s_1} + \frac{\partial \tilde{v}_1^*}{\partial s_2} \right] \right\} \right.$$

$$+ \frac{\partial}{\partial s_2} \left\{ \frac{1}{\sigma_\nu R_K \Delta^2(R_K)} \left[(\sigma_\nu + 2) \frac{\partial \tilde{v}_2^*}{\partial s_2} - \frac{2}{3} \tilde{k}^* \right] \right\}$$

$$\left. + \frac{\partial}{\partial s_3} \left\{ \frac{1}{\sigma_\nu R_K \Delta^2(R_K)} \left[(\sigma_\nu + 1) \frac{\partial \tilde{v}_2^*}{\partial s_3} + \frac{\partial \tilde{v}_3^*}{\partial s_2} \right] \right\} \right\} \tag{61b}$$

$$\frac{\partial \tilde{v}_3^*}{\partial \tau} + \frac{\partial \tilde{v}_3^*}{\partial s_1} + \frac{\partial \tilde{v}_3^* \tilde{v}_i^*}{\partial s_i} = \lim_{R_K \to \infty} \left\{ -\frac{1}{\Delta^2(R_K)} \frac{\partial \tilde{P}_v}{\partial s_3} + \frac{\partial}{\partial s_1} \left\{ \frac{1}{\sigma_\nu R_K \Delta^2(R_K)} \left[\Delta^2(R_K)(\sigma_\nu + 1) \frac{\partial \tilde{v}_3^*}{\partial s_1} + \frac{\partial \tilde{v}_1^*}{\partial s_3} \right] \right\} \right.$$

$$+ \frac{\partial}{\partial s_2} \left\{ \frac{1}{\sigma_\nu R_K \Delta^2(R_K)} \left[(\sigma_\nu + 1) \frac{\partial \tilde{v}_3^*}{\partial s_2} + \frac{\partial \tilde{v}_2^*}{\partial s_3} \right] \right\}$$

$$\left. + \frac{\partial}{\partial s_3} \left\{ \frac{1}{\sigma_\nu R_K \Delta^2(R_K)} \left[(\sigma_\nu + 2) \frac{\partial \tilde{v}_3^*}{\partial s_3} - \frac{2}{3} \tilde{k}^* \right] \right\} \right\}. \tag{61c}$$

Examining these expressions in the limit $R_K \to \infty$, we see that, for the inner asymptotic expansion to be both well behaved and to retain viscous flow effects, $\lim_{R_K \to \infty} R_K \Delta^2(R_K)$ must be a finite, nonzero number. Hence, without loss of generality, we set

$$\lim_{R_K \to \infty} R_K \Delta^2(R_K) = 1 \tag{62a}$$

which yields

$$\Delta(R_K) = 1/R_K^{1/2}. \tag{62b}$$

With this limiting value for $R_K \Delta^2(R_K)$, we see, in addition, that, to eliminate singular terms from the inner asymptotic equations,

$$\tilde{P}_v(\tau, s_i; R_K) \sim \left[\lim_{R_K \to \infty} \Delta^2(R_K)\right] \tilde{P}_v^*(\tau, s_i). \tag{63}$$

Using the identities

$$\frac{\partial}{\partial s_i}\left[\frac{1}{\sigma_\nu}\frac{\partial \tilde{v}_i^*}{\partial s_2}\right] \equiv \frac{\partial \tilde{v}_i^*}{\partial s_2}\frac{\partial}{\partial s_i}\left[\frac{1}{\sigma_\nu}\right] \tag{64a}$$

and

$$\frac{\partial}{\partial s_i}\left[\frac{1}{\sigma_\nu}\frac{\partial \tilde{v}_i^*}{\partial s_3}\right] = \frac{\partial \tilde{v}_i^*}{\partial s_3}\frac{\partial}{\partial s_i}\left[\frac{1}{\sigma_\nu}\right] \tag{64b}$$

which follow from Eq. (59) and assuming (soon to be demonstrated) that the wake region is steady in the frame of reference moving with the ship, the inner flow equations for describing this region reduce to

$$\frac{\partial \tilde{v}_i^*}{\partial s_i} = 0, \tag{65a}$$

$$\frac{\partial \tilde{v}_1^*}{\partial s_1} + \frac{\partial \tilde{v}_1^* \tilde{v}_i^*}{\partial s_i} = \frac{\partial}{\partial s_2}\left[\frac{\sigma_\nu + 1}{\sigma_\nu}\frac{\partial \tilde{v}_1^*}{\partial s_2}\right] + \frac{\partial}{\partial s_3}\left[\frac{\sigma_\nu + 1}{\sigma_\nu}\frac{\partial \tilde{v}_1^*}{\partial s_3}\right] \tag{65b}$$

$$\begin{aligned}\frac{\partial \tilde{v}_2^*}{\partial s_1} + \frac{\partial \tilde{v}_2^* \tilde{v}_i^*}{\partial s_i} = &-\frac{\partial \tilde{P}_v^*}{\partial s_2} + \frac{\partial \tilde{v}_i^*}{\partial s_2}\frac{\partial}{\partial s_i}\left[\frac{1}{\sigma_\nu}\right] \\ &+ \frac{\partial}{\partial s_2}\left[\frac{\sigma_\nu + 1}{\sigma_\nu}\frac{\partial \tilde{v}_2^*}{\partial s_2} - \frac{2}{3}\frac{\tilde{k}^*}{\sigma_\nu}\right] + \frac{\partial}{\partial s_3}\left[\frac{\sigma_\nu + 1}{\sigma_\nu}\frac{\partial \tilde{v}_2^*}{\partial s_3}\right]\end{aligned} \tag{65c}$$

$$\begin{aligned}\frac{\partial \tilde{v}_3^*}{\partial s_1} + \frac{\partial \tilde{v}_3^* \tilde{v}_i^*}{\partial s_i} = &-\frac{\partial \tilde{P}_v^*}{\partial s_3} + \frac{\partial \tilde{v}_i^*}{\partial s_3}\frac{\partial}{\partial s_i}\left[\frac{1}{\sigma_\nu}\right] \\ &+ \frac{\partial}{\partial s_2}\left[\frac{\sigma_\nu + 1}{\sigma_\nu}\frac{\partial \tilde{v}_3^*}{\partial s_2}\right] + \frac{\partial}{\partial s_3}\left[\frac{\sigma_\nu + 1}{\sigma_\nu}\frac{\partial \tilde{v}_3^*}{\partial s_3} - \frac{2}{3}\frac{\tilde{k}^*}{\sigma_\nu}\right].\end{aligned} \tag{65d}$$

We note that all second order derivatives involving s_1, along with axial gradients of the pressure and turbulent kinetic energy, have disappeared from the inner flow problem. Hence, Eqs. (65) are hyperbolic in the s_1 direction and elliptic in the (s_2, s_3) cross plane (i.e., parabolic overall). Mathematically, this means that the values of the inner flow variables on some plane $s_1 + ds_1$ are determined entirely from their values on the plane s_1 together with the boundary conditions on the plane $s_1 + ds_1$. Physically, this implies that if the values of the inner flow variables are steady on some plane s_1 aft of the ship, they remain steady as we progress further downstream from the ship. Since we have already restricted the unsteady motions of the ship about its mean motion to be small, the primary sources of the wake are the boundary layer of the ship in uniform motion and the commensurate propeller effects. Both of these sources are nominally steady; thus, our assumption that the wake region is steady in the frame of reference moving with the ship has been validated.

Let us turn our attention now to the boundary conditions for the inner flow problem. We assume that somehow (computationally, experimentally, or parametrically) the values of the inner flow variables on a plane aft of the ship have been specified. Hence, as follows from the parabolic construction of the inner flow problem, the boundary conditions on the ship, given by Eqs. (47d) are irrelevant regarding the subsequent downstream values of these variables. Substituting Eqs. (57), (58), and (63) into the remaining boundary conditions [Eqs. (47c) and (47e) through (47h)], using Eqs. (48) and (62), and invoking the steady nature of the wake region, we find

$$\tilde{v}_i^* \to 0 \qquad \text{as} \sqrt{s_2^2 + s_3^2} \to \infty \tag{66a}$$

$$\frac{\partial \tilde{v}_1^*}{\partial s_3} = 0 \qquad \text{at } s_3 = 0 \tag{66b}$$

$$\frac{\partial \tilde{v}_2^*}{\partial s_3} + \frac{\partial \tilde{v}_3^*}{\partial s_2} = 0 \qquad \text{at } s_3 = 0 \tag{66c}$$

$$\begin{aligned}\tilde{P}_v^* + \frac{2}{3}\frac{\tilde{k}^*}{\sigma_\nu} - 2\left(\frac{\sigma_\nu + 1}{\sigma_\nu}\right)\frac{\partial \tilde{v}_3^*}{\partial s_3} + \lim_{R_K\to\infty}\left\{R_K^{1-\kappa}\left[-2\pi\tilde{\eta}_v^* + \left(\frac{\sigma_\alpha + 1}{\sigma_\alpha}\right)\alpha_K\frac{\partial^2\tilde{\eta}_v^*}{\partial s_2^2}\right]\right.\\ \left. + \left(\frac{\sigma_\alpha + 1}{\sigma_\alpha}\right)\frac{\alpha_K}{R_K^\kappa}\frac{\partial^2\tilde{\eta}_v^*}{\partial s_1^2}\right\} = 0 \quad \text{at } s_3 = 0\end{aligned} \tag{66d}$$

$$\begin{aligned}\frac{\partial\tilde{\eta}_v^*}{\partial s_1} + \tilde{v}_1^*\frac{\partial\tilde{\eta}_v^*}{\partial s_1} + \tilde{v}_2^*\frac{\partial\tilde{\eta}_v^*}{\partial s_2} - \frac{\partial}{\partial s_2}\left(\frac{\sigma_\eta}{\sigma_\nu}\frac{\partial\tilde{\eta}_v^*}{\partial s_2}\right)\\ - \lim_{R_K\to\infty}\left\{\frac{1}{R_K}\frac{\partial}{\partial s_1}\left(\frac{\sigma_\eta}{\sigma_\nu}\frac{\partial\tilde{\eta}_v^*}{\partial s_1}\right) + R_K^{\kappa-\frac{1}{2}}\tilde{v}_3^*\right\} = 0 \qquad \text{at } s_3 = 0.\end{aligned} \tag{66e}$$

In arriving at the latter two boundary conditions, we have taken

$$\tilde{\eta}_v(\tau,s_1,s_2;R_K) \sim \left[\lim_{R_K\to\infty}\Delta^{2\kappa}(R_K)\right]\tilde{\eta}_v^*(s_1,s_2) = \left[\lim_{R_K\to\infty}\frac{1}{R_K^\kappa}\right]\tilde{\eta}_v^*(s_1,s_2) \tag{67}$$

where the nonnegative exponent κ is to be determined.

Consider Eq. (66d). For κ less than, equal to, and greater than unity, respectively, we have in the limit $R_K \to \infty$

$$\kappa < 1, \quad -2\pi\tilde{\eta}_v^* + \left(\frac{\sigma_\alpha + 1}{\sigma_\alpha}\right)\alpha_K\frac{\partial^2\tilde{\eta}_v^*}{\partial s_2^2} = 0 \quad \text{at } s_3 = 0 \tag{68a}$$

$$\kappa = 1, \quad \tilde{P}_v^* + \frac{2}{3}\frac{\tilde{k}^*}{\sigma_\nu} - 2\left(\frac{\sigma_\nu + 1}{\sigma_\nu}\right)\frac{\partial\tilde{v}_3^*}{\partial s_3} - 2\pi\tilde{\eta}_v^* + \left(\frac{\sigma_\alpha + 1}{\sigma_\alpha}\right)\alpha_K\frac{\partial^2\tilde{\eta}_v^*}{\partial s_2^2} = 0 \quad \text{at } s_3 = 0 \tag{68b}$$

$$\kappa > 1, \quad \tilde{P}_v^* + \frac{2}{3}\frac{\tilde{k}^*}{\sigma_\nu} - 2\left(\frac{\sigma_\nu + 1}{\sigma_\nu}\right)\frac{\partial\tilde{v}_3^*}{\partial s_3} = 0 \quad \text{at } s_3 = 0. \tag{68c}$$

The condition obtained for $\kappa < 1$ is independent of the inner flow field and, except for the solution $\tilde{\eta}_v^* \equiv 0$, gives a continuously increasing elevation for $s_2 > 0$ since $[(\sigma_\alpha + 1)/\sigma_\alpha]\alpha_K$ is positive. Hence, we determine either $\kappa \geqslant 1$ or $\kappa < 1$ with $\tilde{\eta}_v^* \equiv 0$. With either constraint, Eq. (66e) yields, in the limit $R_K \to \infty$,

$$\tilde{v}_3^* = 0 \qquad \text{at } s_3 = 0. \tag{69}$$

Together with this result, Eq. (68c) overspecifies the boundary conditions on the inner flow problem. Thus, we require for a well-stated problem either $\kappa = 1$ or $\kappa < 1$ with $\tilde{\eta}_v^* \equiv 0$. Since the former constraint is less restrictive, we conclude $\kappa = 1$.

Summarizing, the boundary conditions for the inner flow problem are ascertained to be

$$\tilde{v}_i^* \rightarrow 0 \qquad \text{as} \sqrt{s_2^2 + s_3^2} \rightarrow \infty \tag{70a}$$

and

$$\tilde{v}_3^* = \frac{\partial \tilde{v}_1^*}{\partial s_3} = \frac{\partial \tilde{v}_2^*}{\partial s_3} = 0 \qquad \text{at } s_3 = 0 \tag{70b}$$

along with the prognostic relation for $\tilde{\eta}_v^*$ given by Eq. (68b). We note from Eq. (70b) that the free surface acts as a plane of symmetry for the inner flow field. Hence, we arrive at the important result that *the solution to the ship-induced inner flow problem is identical to the lower half space* ($s_3 \leqslant 0$) *solution that would be obtained by solving the inner flow problem for the ship and its image (about* $s_3 = 0$) *in an infinite fluid.*

Returning to uniform dimensionless quantities via Eqs. (57), (58), (60), (63), and (67) and, thence, to dimensional quantities via Eqs. (46), we obtain, with the aid of Eqs. (45) and (48), the dimensional form of the ship-induced inner flow field problem as

$$\frac{\partial v_i^*}{\partial x_i} = 0 \tag{71a}$$

$$U \frac{\partial v_1^*}{\partial x_1} + \frac{\partial v_1^* v_i^*}{\partial x_i} = \frac{\partial}{\partial x_2}\left[(\nu + \nu_t)\frac{\partial v_1^*}{\partial x_2}\right] + \frac{\partial}{\partial x_3}\left[(\nu + \nu_t)\frac{\partial v_1^*}{\partial x_3}\right] \tag{71b}$$

$$\begin{aligned} U \frac{\partial v_2^*}{\partial x_1} + \frac{\partial v_2^* v_i^*}{\partial x_i} &= -\frac{1}{\rho}\frac{\partial P_v^*}{\partial x_2} + \frac{\partial v_i^*}{\partial x_2}\frac{\partial \nu_t}{\partial x_i} \\ &+ \frac{\partial}{\partial x_2}\left[(\nu + \nu_t)\frac{\partial v_2^*}{\partial x_2} - \frac{2}{3}\,k^*\right] + \frac{\partial}{\partial x_3}\left[(\nu + \nu_t)\frac{\partial v_2^*}{\partial x_3}\right] \end{aligned} \tag{71c}$$

$$\begin{aligned} U \frac{\partial v_3^*}{\partial x_1} + \frac{\partial v_3^* v_i^*}{\partial x_i} &= -\frac{1}{\rho}\frac{\partial P_v^*}{\partial x_3} + \frac{\partial v_i^*}{\partial x_3}\frac{\partial \nu_t}{\partial x_i} \\ &+ \frac{\partial}{\partial x_2}\left[(\nu + \nu_t)\frac{\partial v_3^*}{\partial x_2}\right] + \frac{\partial}{\partial x_3}\left[(\nu + \nu_t)\frac{\partial v_3^*}{\partial x_3} - \frac{2}{3}\,k^*\right] \end{aligned} \tag{71d}$$

$$v_i^* \rightarrow 0 \qquad \text{as} \sqrt{x_2^2 + x_3^2} \rightarrow \infty \tag{71e}$$

$$v_3^* = \frac{\partial v_1^*}{\partial x_3} = \frac{\partial v_2^*}{\partial x_3} = 0 \qquad \text{at } x_3 = 0 \tag{71f}$$

together with the prognostic relation for η_v^*

$$(\alpha + \alpha_t)\frac{\partial^2 \eta_v^*}{\partial x_2^2} - \rho g \eta_v^* = 2\rho(\nu + \nu_t)\frac{\partial v_3^*}{\partial x_3} - P_v^* - \frac{2}{3}\rho k^* \qquad \text{at } x_3 = 0. \tag{71g}$$

c. Ship-Induced Composite Flow Field

The ship-induced composite flow field is derived most straightforwardly by the method of additive composition detailed in Ref. 10. Basically, to obtain the uniform, first order, composite solution, we sum the first order outer and first order inner solutions and subtract the part they have in common so that it is not counted twice. Formally, this procedure translates into summing the first order outer and inner solutions and subtracting either the outer expansion of the inner solution or the inner expansion of the outer solution. Because we have developed a matched asymptotic solution, these latter two expressions are, by construction, equal and, in our case, identically zero.

To see this, let us rewrite the inner problem in terms of the independent outer variables ξ_i. Substituting Eqs. (57) into Eqs. (65) and (70) and taking the limit as $R_K \to \infty$, we find the outer representation of the inner flow problem as

$$\frac{\partial \tilde{v}_i^{*\dagger}}{\partial \xi_1} = 0, \tag{72a}$$

$$\tilde{v}_i^{*\dagger} \to 0 \qquad \text{as } \sqrt{\xi_2^2 + \xi_3^2} \to \infty, \tag{72b}$$

and

$$\tilde{v}_3^{*\dagger} = \frac{\partial \tilde{v}_1^{*\dagger}}{\partial \xi_3} = \frac{\partial \tilde{v}_2^{*\dagger}}{\partial \xi_3} = 0 \qquad \text{at } \xi_3 = 0 \tag{72c}$$

where the superscript $*\dagger$ denotes the outer value of the inner variable. Hence, the outer expansion of the inner solution—that is, the solution of Eqs. (72)—is

$$\tilde{v}_i^{*\dagger}(\xi_i) = 0. \tag{73}$$

For completeness, let us also rewrite the outer problem in terms of the independent inner variables s_i. Substituting Eq. (57) into the nondimensional versions of Eqs. (55) and taking the limit as $R_K \to \infty$, we find the inner representation of the outer flow problem as

$$\frac{\partial^2 \tilde{\phi}^{\dagger *}}{\partial s_2^2} + \frac{\partial^2 \tilde{\phi}^{\dagger *}}{\partial s_3^2} = 0, \tag{74a}$$

$$\frac{\partial \tilde{\phi}^{\dagger *}}{\partial s_i} \to 0 \qquad \text{as } \sqrt{s_2^2 + s_3^2} \to \infty, \tag{74b}$$

and

$$\frac{\partial \tilde{\phi}^{\dagger *}}{\partial s_3} = 0 \qquad \text{at } s_3 = 0 \tag{74c}$$

where the superscript $\dagger *$ denotes the inner value of the outer variable. Here, the boundary condition on the ship hull has been neglected since the matched asymptotic solution encompasses fully only the region aft of the ship. Thus, the inner expansion of the outer solution satisfies a two-dimensional Laplace's equation with zero gradient boundary conditions. From Ref. 13 and Eq. (74b), we obtain

$$\tilde{\phi}^{\dagger *}(\tau, s_i) = \text{constant}, \tag{75a}$$

or

$$\tilde{v}_i^{\dagger *}(\tau, s_i) = 0. \tag{75b}$$

Consequently, the method of additive composition, together with Eqs. (58), yield the ship-induced composite flow velocities aft of the ship as

$$\tilde{v}_1(\tau, \xi_i) = v_1^{\dagger}(\tau, \xi_i) + \tilde{v}_1^{*}(\xi_i), \tag{76a}$$

$$\tilde{v}_2(\tau, \xi_i) = \tilde{v}_2^{\dagger}(\tau, \xi_i) + R_K^{-1/2}\tilde{v}_2^{*}(\xi_i), \tag{76b}$$

and

$$\tilde{v}_3(\tau, \xi_i) = \tilde{v}_3^{\dagger}(\tau, \xi_i) + R_K^{-1/2}\tilde{v}_3^{*}(\xi_i), \tag{76c}$$

or, in dimensional terms, as

$$v_i(t, x_i) = v_i^{\dagger}(t, x_i) + v_i^{*}(x_i). \tag{77}$$

Let us now turn our attention to the ship-induced composite pressure and surface elevation fields. From Eqs. (63) and (67), both the inner pressure and elevation are of second order in the inner expansion parameter $\Delta(R_K)$. Hence, they do not contribute to the uniform, first order, composite pressure and elevation fields. However, the inner flow velocities do have a first order corrective effect on the outer elevation field and, as a result, on the outer pressure field.

To demonstrate this, we return to the dimensionless statement of the kinematic, free surface boundary condition for the ship-induced flow field given by Eq. (47h). Substituting Eqs. (76) into the kinematic condition, dropping the products of $\tilde{v}_i^\dagger$ and $\tilde{\eta}_v$ since they are of order ϵ^2 in the surface elevation, and using Eq. (70b), we find, in the limit $R_K \to \infty$,

$$\frac{\partial \tilde{\eta}_v}{\partial \tau} + \frac{\partial \tilde{\eta}_v}{\partial \xi_1} + \tilde{v}_1^* \frac{\partial \tilde{\eta}_v}{\partial \xi_1} - \tilde{v}_3^\dagger = 0 \qquad \text{at } \xi_3 = 0. \tag{78}$$

This expression represents the uniform, first order, kinematic free surface boundary condition. Setting

$$\tilde{\eta}_v = \tilde{\eta}_v^\dagger + \tilde{\zeta}_v \tag{79}$$

where $\tilde{\zeta}_v$ gives the correction to the outer elevation field, we obtain, with the aid of Eq. (50h), the governing relation for $\tilde{\zeta}_v$ as

$$\frac{\partial \tilde{\zeta}_v}{\partial \tau} + (1 + \tilde{v}_1^*) \frac{\partial \tilde{\zeta}_v}{\partial \xi_1} = -\tilde{v}_1^* \frac{\partial \tilde{\eta}_v^\dagger}{\partial \xi_1} \qquad \text{at } \xi_3 = 0. \tag{80}$$

This correction to the outer dynamic head leads to a proportionate correction in the outer pressure field. We have

$$\frac{\tilde{P}_v}{\tilde{P}_v^\dagger} = \frac{\tilde{\eta}_v}{\tilde{\eta}_v^\dagger}$$

or, from Eq. (79),

$$\tilde{P}_v = \tilde{P}_v^\dagger + \frac{\tilde{\zeta}_v}{\tilde{\eta}_v^\dagger} \tilde{P}_v^\dagger \tag{81}$$

which, since $\tilde{P}_v^\dagger / \tilde{\eta}_v^\dagger$ is well behaved, is itself well behaved.

Returning to dimensional variables via Eqs. (46), we find

$$\eta_v = \eta_v^\dagger + \zeta_v \tag{82a}$$

and

$$P_v = P_v^\dagger + \frac{\zeta_v}{\eta_v^\dagger} P_v^\dagger \tag{82b}$$

where ζ_v satisfies

$$\frac{\partial \zeta_v}{\partial t} + (U + v_1^*) \frac{\partial \zeta_v}{\partial x_1} = -v_1^* \frac{\partial \eta_v^\dagger}{\partial x_1} \qquad \text{at } x_3 = 0. \tag{82c}$$

SUMMARY

A model for the hydrodynamic wake of a surface ship has been developed. Neglecting only ambient stratification and ambient surface films, we have begun with the general equations and boundary conditions for the flow field and, through a series of rational approximations, reduced the problem to a solvable one.

The approximations are:

- The fluid dynamics are independent of the gas particle (bubble) dynamics.
- The ship undergoes only small motions about a uniform mean motion.
- The free surface elevations are small.
- The ship-induced turbulence is significantly stronger than the ambient turbulence.

With these approximations, the overall problem becomes separable into one governing the ship-induced flow in otherwise still water and one governing the interaction of the background with the ship-induced flow.

The former problem was examined further by applying the method of matched asymptotic expansions. We found that the ship induces an outer flow field that satisfies the traditional Kelvin-Neumann problem for calculating a ship's Kelvin and radiated wave systems. This outer flow field is determined by Eqs. (55). We found also that the ship induces an inner flow field—its turbulent wake. This flow field is governed by the steady, three-dimensional, parabolic Navier-Stokes equations with plane of symmetry boundary conditions at the free surface. The inner problem is given by Eqs. (71). The overall, or composite, ship-induced flow field is derived from these two asymptotic flow fields via Eqs. (77) and (82).

Besides the plane of symmetry boundary conditions at the free surface for the turbulent wake problem, the more significant results of this paper include:

- Surface tension effects are unimportant in determining the uniform, first order, ship-induced flow field in otherwise still water; and
- The inner flow field produces a first order modification to the traditionally calculated ship wave elevations. This result could explain the experimentally observed differences between the transverse wave systems of model and full-scale ships.

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