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THE COMPLEX REPRESENTATION OF SIGNALS

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SUMMARY

The development of the complex representation of signals has simplified the analysis of systems for processing bandpass signals. This report is a collection of some useful properties of the complex representation of signals. It is intended to be a reference work for systems engineers who already have a thorough knowledge of systems theory with real signals. The proofs of the various properties, as well as considerable background theory, are included to emphasize the conditions under which the properties are valid. None of the work is claimed to be original.

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APPROVED K.D. Greenhalgh TITLE Manager, ASD LOCATION CSP 3-49, Syracuse, N.Y.

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THE COMPLEX REPRESENTATION OF SIGNALS

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SECTION I

INTRODUCTION

The purpose of this report is to derive and assemble for reference some useful properties of the complex representation of signals. It is intended for use by systems engineers who already have a thorough knowledge of systems theory with real signals. None of the work is original and the results have been known for many years. The proofs of the various properties are included for the specific purpose of emphasizing the conditions under which these properties are valid. The development begins with a discussion of the Hilbert transform, proceeds to the analytic signal and complex envelope, and ends with sections on the relations between operations on real signals and their complex envelopes, and the complex sampling theorem.

SECTION II

THE HILBERT TRANSFORM

Central to the theory of the complex representation of signals is the notion of the Hilbert transform. Taking the Hilbert transform of a signal amounts to shifting all its frequency components 90° in phase; however, this is not the usual definition of the Hilbert transform, so we shall demonstrate this property.

To avoid mathematical difficulties we shall work throughout this report only with signals that have zero mean. Note that this includes all finite energy signals.

Notation

1. x(t) is a real zero mean signal
2. Integrals written without limits are assumed to be from $-\infty$ to ∞

Definition 1

The Hilbert transform of x(t) is the Cauchy principal value of the integral

$$\hat{x}(t) = \frac{1}{\pi}\int \frac{x(\tau)}{t-\tau}\, d\tau \tag{1}$$

Note that this is a convolution integral, so $\hat{x}(t)$ can be interpreted as the result of passing x(t) through a linear time invariant system with impulse response

$$h(t) = \frac{1}{\pi t} \tag{2}$$

Proposition 1

h(t) has the Fourier transform

$$H(f) = \int \frac{1}{\pi t} e^{-j2\pi ft}\, dt = \begin{cases} -j, & f > 0 \\ j, & f < 0 \end{cases} \tag{3}$$

Note that this amounts to a 90° phase shift at all nonzero frequencies.

Proof

An attempt to evaluate the transform directly leads to difficulties at the infinite limits. An alternate procedure is to define a function

$$h_a(t) = \frac{4\pi t}{a^2 + (2\pi t)^2}$$

which has the Fourier transform

$$H_a(f) = \begin{cases} -je^{-af}, & f > 0 \\ je^{af}, & f < 0 \end{cases}$$

This can be verified by showing that $h_a(t)$ is the inverse transform of $H_a(f)$. Now we note that since

$$h(t) = \lim_{a \to 0} h_a(t)$$

it is reasonable to define

$$H(f) = \lim_{a \to 0} H_a(f) = \begin{cases} -j, & f > 0 \\ j, & f < 0 \end{cases} \tag{4}$$

<u>End of Proof</u>

To be rigorous, it would be necessary to show that the Fourier transform defined by Equation (4) has all the properties of the usual transform, but we shall be content with this justification.

<u>Corollaries</u>

The Fourier transform of the inverse

$$\frac{1}{H(f)} = \begin{cases} j, & f > 0 \\ -j, & f < 0 \end{cases} = -H(f) \tag{5}$$

therefore

$$x(t) = -\frac{1}{\pi}\int \frac{\hat{x}(\tau)}{t - \tau}\, d\tau = -\hat{\hat{x}}(t) \tag{6}$$

i.e., the Hilbert transform of the Hilbert transform of $x(t)$ is $-x(t)$. It is important to remember that this only holds when $x(t)$ is zero mean.

Because of the existence of an inverse Hilbert transform -- Equation (5) -- every zero mean signal has a unique Hilbert transform.

$x(t)$ and $\hat{x}(t)$ are a unique pair (7)

<u>End of Corollaries</u>

We shall now investigate the relationship between the Fourier transform of $x(t)$ and the Fourier transform of $\hat{x}(t)$.

Notation

1. $x(t)$ has the Fourier transform $X(f)$
2. $\hat{x}(t)$ has the Fourier transform $\hat{X}(f)$

Note that $\hat{X}(f)$ is not defined to be the Hilbert transform of $X(f)$.

Corollary

It follows from Equations (1) and (3) that

$$\hat{X}(f) = H(f)\, X(f) = \begin{cases} -j\, X(f), & f > 0 \\ j\, X(f), & f < 0 \end{cases} \tag{8}$$

i.e., the spectrum of the Hilbert transform of $x(t)$ is found by multiplying the positive spectrum of $x(t)$ by $-j$, and the negative spectrum by j.

End of Corollary

We shall now show a useful integral property of Hilbert transforms.

Notation

1. $y(t)$ is a real zero mean signal
2. superscript * denotes complex conjugate

Recall from Fourier transform theory that

$$\int x(t)\, y(t)\, dt = \int X(f)\, Y^*(f)\, df \tag{9}$$

We can use this relation to show that

Proposition 2

$$\int \hat{x}(t)\, \hat{y}(t)\, dt = \int x(t)\, y(t)\, dt \tag{10}$$

Proof

$$\int \hat{x}(t)\, \hat{y}(t)\, dt$$

$$= \int \hat{X}(f)\, \hat{Y}^*(f)\, df \qquad \text{by Equation (9)}$$

$$= \int_{-\infty}^{0} [j\, X(f)]\, [-j\, Y^*(f)]\, df + \int_{0}^{\infty} [-j\, X(f)]\, [j\, Y^*(f)]\, df$$

$$= \int X(f)\, Y^*(f)\, df$$

$$= \int x(t)\, y(t)\, dt \qquad \text{by Equation (9)}$$

End of Proof

Corollaries

From Equations (10) and (6) it follows that

$$\int x(t)\,\hat{y}(t)\,dt = -\int \hat{x}(t)\,y(t)\,dt \tag{11}$$

which implies

$$\int x(t)\,\hat{x}(t)\,dt = 0 \tag{12}$$

End of Corollaries

The following relations will also be useful later in our work.

Notation

$$\bar{y}(t) = y(-t)$$

i.e., the overbar is an operator which turns y(t) into its time reverse.

Proposition 3

$$\hat{\bar{y}}(t) = -\bar{\hat{y}}(t) \tag{13}$$

i.e., the Hilbert transform of the time reverse of y(t) is the negative of the time reverse of the Hilbert transform y(t).

Proof

$$\begin{aligned}
\hat{\bar{y}}(t) &= \frac{1}{\pi}\int \frac{\bar{y}(\tau)}{t-\tau}\,d\tau \\
&= \frac{1}{\pi}\int \frac{y(-\tau)}{t-\tau}\,d\tau \\
&= -\frac{1}{\pi}\int \frac{y(\ \tau)}{-t+\tau}\,d\tau \\
&= -\frac{1}{\pi}\int \frac{y(\sigma)\,d\sigma}{-t-\sigma} \quad \text{where } \sigma = -\tau \\
&= -\bar{\hat{y}}(t)
\end{aligned}$$

End of Proof

Corollaries

From Equations (10) and (13) it follows that

$$\int \hat{x}(t)\,\bar{\hat{y}}(t)\,dt = -\int x(t)\,\bar{y}(t)\,dt \tag{14}$$

which implies

$$\int x(t)\, \overline{\hat{y}}(t)\, dt = \int \hat{x}(t)\, \overline{\hat{\hat{y}}}(t)\, dt = \int \hat{x}(t)\, \overline{y}(t)\, dt \qquad (15)$$

End of Corollaries

The differences between Equations (11) and (14), and between Equations (12) and (15) should be noted.

Finally, we shall have need for relations similar to Equations (10) through (15), but involving ensemble expectations rather than integrations.

Notation

1. x(t) and y(t) are sample functions of a real, zero mean stationary random process
2. E [ ] denotes ensemble expectation

Proposition 4a

$$E\,[\hat{x}(t)\, \hat{y}(s)] = E\,[x(t)\, y(s)] \qquad (16)$$

Proof

$$E\,[\hat{x}(t)\, \hat{y}(s)]$$

$$= E\left[\frac{1}{\pi^2}\int d\tau\, \frac{x(\tau)}{t-\tau}\int d\sigma\, \frac{y(\sigma)}{s-\sigma}\right]$$

$$= E\left[\frac{1}{\pi^2}\int d\xi\, \frac{x(t-\xi)}{\xi}\int d\eta\, \frac{y(s-\eta)}{\eta}\right] \quad \text{where } \xi = t-\tau,\ \eta = s-\sigma$$

$$= \frac{1}{\pi^2}\iint \frac{d\xi}{\xi}\, \frac{d\eta}{\eta}\, E\,[x(t-\xi)\, y(s-\eta)]$$

$$= \frac{1}{\pi^2}\iint \frac{d\xi}{\xi}\, \frac{d\eta}{\eta}\, E\,[x(t)\, y(s+\xi-\eta)]$$

$$= E\left[x(t)\, \frac{1}{\pi}\int \frac{d\xi}{\xi}\, \hat{y}(s+\xi)\right]$$

$$= E\,[x(t)\, y(s)]$$

End of Proof

Corollaries

It follows from Equations (16) and (6) that

$$E\,[\hat{x}(t)\, y(s)] = -E\,[x(t)\, \hat{y}(s)] \qquad (17)$$

which implies

$$E\,[\hat{x}(t)\;x(t)] = 0$$

Also, making use of Equations (16) and (13), it follows that

$$E\,[\hat{x}(t)\;\hat{\bar{y}}(t)] = -E\,[x(t)\;\bar{y}(t)] \tag{18}$$

which implies

$$E\,[x(t)\;\hat{\bar{y}}(t)] = E\,[\hat{x}(t)\;\hat{\hat{\bar{y}}}(t)] = E\,[\hat{x}(t)\;\bar{y}(t)] \tag{19}$$

<u>End of Corollaries</u>

<u>Proposition 4b</u>

$$E\,[x(t)\;\hat{y}(s)] = \hat{E}\,[x(t)\;y(s)] \tag{20}$$

<u>Proof</u>

$$E\,[x(t)\;\hat{y}(s)]$$

$$= E\left[x(t)\;\frac{1}{\pi}\int d\eta\;\frac{y(s+\eta)}{\eta}\right] \quad \text{where } \eta = -s + \sigma$$

$$= \frac{1}{\pi}\int\frac{d\eta}{\eta}\;E\,[x(t)\;y(s+\eta)]$$

$$= \hat{E}\,[x(t)\;y(s)] \quad \text{by stationarity}$$

<u>End of Proof</u>

SECTION III

THE ANALYTIC SIGNAL

Notation

"Re" denotes the real part

In steady state circuit analysis it is common to write

$$A \cos (2\pi ft + \Theta) = \text{Re} \{A e^{j(2\pi ft + \Theta)}\}$$

and to analyze the circuit using only the exponential form, taking real parts when the real signals are wanted. The analytic signal is a natural extension of this technique to a broader class of signals.

Notation

x(t) is a real zero mean signal

Definition 2

The analytic signal of x(t) is

$$\tilde{x}(t) = x(t) + j\,\hat{x}(t) \tag{21}$$

then

$$x(t) = \text{Re}\,\tilde{x}(t)$$

Proposition 5

$x(t)$ and $\tilde{x}(t)$ are uniquely related (22)

Proof

The proposition follows directly from Equations (21) and (7).

End of Proof

Notation

1. $x(t)$ has the Fourier transform $X(f)$
2. $\tilde{x}(t)$ has the Fourier transform $\tilde{X}(f)$

Proposition 6

The spectrum of $\tilde{x}(t)$ is twice the positive frequency spectrum of x(t).

$$\tilde{X}(f) = X(f) + j\,\hat{X}(f) = 2 \begin{cases} X(f), & f > 0 \\ 0, & f < 0 \end{cases} \tag{23}$$

Proof

The proposition follows directly from Equations (21) and (8).

End of Proof

Proposition 7

If the spectrum $X'(f)$ of a complex signal $x(t)$ is zero at nonpositive frequencies, then there exists a real zero mean signal $x(t)$ such that

$$x'(t) = \tilde{x}(t) \tag{24}$$

Prooi

Let $X'(f) = X_r'(f) + j\, X_i'(f)$

where $X_r'(f)$ and $X_i'(f)$ are real

$$\text{Define } X(f) = \begin{cases} \frac{1}{2}\,[X_r(f) + j\, X_i(f)],\ f > 0 \\ \frac{1}{2}\,[X_r(f) - j\, X_i(f)],\ f < 0 \end{cases}$$

Then $x(t)$, the inverse Fourier transform of $X(f)$, is real. Furthermore, the analytic signal $\tilde{x}(t)$ has Fourier transform $\tilde{X}(f) = X'(f)$.

End of Proof

Corollary

A necessary and sufficient condition that a complex signal $x'(t)$ be the analytic signal of a real zero mean signal is that its spectrum $X'(f)$ be zero at nonpositive frequencies.

$$X'(f) = 0,\ f \leq 0 \tag{25}$$

Proof

The corollary follows directly from Equations (23) and (24).

Notation

$y(t)$ is a real signal

Proposition 8

If $z(t) = x(t) + y(t)$

$$\text{then } \tilde{z}(t) = \tilde{x}(t) + \tilde{y}(t) \tag{26}$$

i.e., taking the analytic signal is a linear operation.

Proof

$$\tilde{z}(t) = z(t) + j\,\hat{z}(t)$$
$$= x(t) + j\,\hat{x}(t) + y(t) + j\,\hat{y}(t)$$
$$= \tilde{x}(t) + \tilde{y}(t)$$

End of Proof

SECTION IV

THE COMPLEX ENVELOPE

At the beginning of Section III we noted that the analytic signal was an extension of the use of complex representation of signals in steady state circuit analysis. One of the advantages of this technique is that is is not necessary to carry the exponential term $e^{j2\pi ft}$ through the analysis; it is sufficient to work with the complex amplitude $A\, e^{j\Theta}$. The complex envelope is an extension of this technique which allows the complex amplitude to be time varying.

<u>Notation</u>

x(t) is a real zero mean signal

<u>Definition 3</u>

The complex envelope of x(t) at the frequency f_c is

$$u(t) = \tilde{x}(t)\, e^{-j2\pi f_c t} \tag{27}$$

<u>Note</u>: The complex envelope is often called the "preenvelope" in the literature.

<u>Notation</u>

1. $|u(t)|$ is the magnitude of u(t)
2. $\measuredangle\, u(t)$ is the angle of u(t)

<u>Proposition 9</u>

For any given frequency f_c, any real zero mean signal x(t) can be written uniquely as an amplitude and phase modulation carrier in the form

$$x(t) = |u(t)| \cos\left[2\pi f_c t + \measuredangle\, u(t)\right] \tag{28}$$

<u>Note</u>: There is no bandwidth constraint on this representation.

<u>Proof</u>

$$x(t) = \text{Re}\ \tilde{x}(t) \qquad \text{by Equation (21)}$$

$$= \text{Re}\ u(t)\, e^{j2\pi f_c t} \qquad \text{by Equation (27)}$$

$$= \text{Re}\ \left\{ |u(t)|\, e^{j\measuredangle u(t)}\, e^{j2\pi f_c t} \right\}$$

$$= |u(t)| \cos\left[2\pi f_c t + \measuredangle\, u(t)\right]$$

<u>End of Proof</u>

Notation

1. $u_r(t)$ is the real part of $u(t)$
2. $u_i(t)$ is the imaginary part of $u(t)$

Proposition 10

For any given frequency f_c, any real signal can be written uniquely as the difference of two amplitude modulated carriers in phase quadrature in the form

$$x(t) = u_r(t) \cos 2\pi f_c t - u_i(t) \sin 2\pi f_c t \qquad (29)$$

Note: There is no bandwidth constraint on this representation.

Proof

$$x(t) = \mathrm{Re}\, \tilde{x}(t) \qquad \text{by Equation (21)}$$

$$= \mathrm{Re}\, u(t)\, e^{j2\pi f_c t} \qquad \text{by Equation (27)}$$

$$= \mathrm{Re}\, [u_r(t) + j\, u_i(t)]\, e^{j2\pi f_c t}$$

$$= u_r(t) \cos 2\pi f_c t - u_i(t) \sin 2\pi f_c t$$

End of Proof

Note: The uniqueness of the representations -- Equations (28) and (29) -- results from the precise definition of $\tilde{x}(t)$ by Equation (21). Other representations of the same form as Equations (28) and (29) are possible by defining $x(t)$ to be the real part of some other complex signal.

Notation

1. $\delta(\,)$ is the unit impulse function
2. * between functions denotes convolution

Proposition 11

The Fourier transform of the complex envelope $u(t)$ is

$$U(f) = \tilde{X}(f + f_c) \qquad (30)$$

Proof

$$u(t) = \tilde{x}(t)\, e^{-j2\pi f_c t} \qquad \text{by Equation (27)}$$

Hence $U(f) = \tilde{X}(f) * \delta(f + f_c)$

$$= \tilde{X}(f + f_c)$$

End of Proof

The above proposition shows that the spectrum of the complex envelope u(t) is simply twice the positive frequency spectrum of the real signal shifted downward by frequency f_c, or in the usual terminology, shifted to baseband.

A. "HIGH PASS" AND "NARROW-BAND" MODULATED CARRIERS

It is one thing to define the analytic signal and the complex envelope, and quite another thing to compute them for a given signal, x(t). The potential difficulty lies in carrying out the mathematical operation required to compute the Hilbert transform. For "high pass" modulated carriers we shall demonstrate that the analytic signal and the complex envelope are simple to find; however, we must first define what is meant by a "high pass" modulated carrier.

Definition 4

Let $x(t) = a(t) \cos [2\pi f_c t + \Theta(t)]$ (31)

where a(t) and $\Theta(t)$ are real.

Let $u_1(t) = a(t) e^{j\Theta(t)}$ (32)

Then x(t) is defined to be a high pass modulated carrier if

$U_1(f - f_c) = 0, \; f \leq 0$ (33)

Note: $u_1(t)$ is defined by Equation (32), and is not necessarily the complex envelope of x(t). The next proposition asserts that $u_1(t)$ is the complex envelope of x(t) if, and only if, Equation (33) holds.

Proposition 12

Let x(t) be given by Equation (31). A necessary and sufficient condition that

$\tilde{x}(t) = a(t) e^{j\Theta(t)} e^{j2\pi f_c t}$ (34)

is that x(t) be a high pass modulated carrier.

Proof

Let $u_1(t)$ be given by Equation (32).

Define the complex signal

$$x'(t) = u_1(t)\, e^{j2\pi f_c t}$$

To prove the sufficiency of the assertion, we must show that

$$x'(t) = \tilde{x}(t)$$

Since $X'(f) = U_1(f - f_c)$, and since $x_1(t)$ is a high pass modulated carrier

$$X'(f) = 0, \ f \leq 0$$

But, by Equation (24), this is a sufficient condition for $x'(t)$ to be analytic. Now, since

$$x(t) = \text{Re}\, x'(t) = \text{Re}\, \tilde{x}(t)$$

and since, by Equation (22), $x(t)$ is uniquely related to its analytic signal

$$x'(t) = \tilde{x}(t)$$

To prove the necessity of the assertion, we must show that when $x'(t) = \tilde{x}(t)$, $x(t)$ is a high pass modulated carrier.

Since $x'(t)$ is analytic, it follows from Equation (23) that

$$X'(f) = U_1(f - f_c) = 0, \ f \leq 0$$

Therefore, $x(t)$ is a high pass modulated carrier.

<u>End of Proof</u>

<u>Note</u>: Even if a real zero mean signal given by Equation (31) is not a high pass modulated carrier, it still has a complex envelope defined by Equation (27), but the complex envelope is not given by Equation (32).

The preceding proposition shows how to form the analytic signal and complex envelope of a real signal when the real signal is a high pass modulated carrier whose amplitude and phase modulation are known. The analytic signal is given by Equation (34) and the complex envelope by Equation (32).

In the literature, reference is often made to representing "narrow-band" modulated carriers in this manner. This restriction is clearly sufficient, but not necessary. The "narrow-band" restriction is often imposed to make the magnitude of the complex envelope coincide with the "physical" envelope as seen on an oscilloscope, or because it is desired to represent doppler effects as shifts in the carrier frequency.

B. ZERO-MEAN STATIONARY GAUSSIAN NOISE

Notation

1. n(t) is a real signal which is a sample function of a zero-mean stationary gaussian random process
2. R(t - s) is the autocorrelation function of this process

Proposition 13

$\hat{n}(t)$ is a sample function of a zero-mean stationary random process.

Proof

$\hat{n}(t)$ is the result of a linear time invariant operation on n(t).

End of Proof

Corollaries

By Equation (20) $E[n(t)\,\hat{n}(s)] = \hat{E}[n(t)\,n(s)]$ (35)

By Equation (16) $E[\hat{n}(t)\,\hat{n}(s)] = E[n(t)\,n(s)]$ (36)

Notation

1. k(t) is the complex envelope of n(t) at frequency f_c

. i.e., $k(t) = \tilde{n}(t)\, e^{-j2\pi f_c t}$ (37)

2. $k_r(t)$ and $k_i(t)$ are, respectively, the real and imaginary parts of k(t)

We now investigate the statistics of k(t).

Proposition 14

1. $k_r(t)$ and $k_i(t)$ are zero mean, stationary, and gaussian with autocorrelation function $R(t-s)\cos 2\pi f_c(t-s) + \hat{R}(t-s)\sin 2\pi f_c(t-s)$ and cross-correlation function $R(t-s)\sin 2\pi f_c(t-s) + \hat{R}(t-s)\cos 2\pi f_c(t-s)$.
2. $|k(t)|^2$ is chi-squared two degrees of freedom distributed with mean 2R(0) and variance $4R^2(0)$; i.e., |k(t)| is Rayleigh distributed.
3. $\measuredangle\, k(t)$ is uniformly distributed between 0 and 2π.
4. $|k(t)|^2$ and $\measuredangle\, k(t)$ are independent.

Proof 1

$$k(t) = [n(t) + j\hat{n}(t)]\, e^{-j2\pi f_c t}$$

Hence $k_r(t) = n(t) \cos 2\pi f_c t + \hat{n}(t) \sin 2\pi f_c t$

$k_i(t) = -n(t) \sin 2\pi f_c t + \hat{n}(t) \cos 2\pi f_c t$

Thus, $k_r(t)$ and $k_i(t)$ are sums of zero-mean gaussian random variables, and hence are themselves zero-mean gaussian random variables. We must show that they are from stationary processes. To that end

$$\begin{aligned} E[k_r(t)\, k_r(s)] &= E[n(t)\, n(s)] \cos 2\pi f_c t \cos 2\pi f_c s \\ &+ E[n(t)\, \hat{n}(s)] \cos 2\pi f_c t \sin 2\pi f_c s \\ &+ E[\hat{n}(t)\, n(s)] \sin 2\pi f_c t \cos 2\pi f_c s \\ &+ E[\hat{n}(t)\, \hat{n}(s)] \sin 2\pi f_c t \sin 2\pi f_c s \\ &= R(t-s)[\cos 2\pi f_c t \cos 2\pi f_c s + \sin 2\pi f_c t \sin 2\pi f_c s] \\ &= R(t-s) \cos 2\pi f_c (t-s) + \hat{R}(t-s) \sin 2\pi f_c (t-s) \end{aligned} \tag{38}$$

i.e., stationary. Similarly,

$$E[k_i(t)\, k_i(s)] = E[k_r(t)\, k_r(s)] \tag{39}$$

$$E[k_r(t)\, k_i(s)] = R(t-s) \sin 2\pi f_c (t-s) + \hat{R}(t-s) \cos 2\pi f_c (t-s) \tag{40}$$

End of Proof 1

Proofs 2 and 3

The random variables $|k(t)|^2$, $\measuredangle\, k(t)$ and $k_r(t)$, $k_i(t)$ are related by

$$|k(t)|^2 = k_r^2(t) + k_i^2(t)$$

$$\measuredangle\, k(t) = \arctan \frac{k_i(t)}{k_r(t)}$$

Since the joint distribution of $k_r(t)$ and $k_i(t)$ is known, the joint distribution of $|k(t)|^2$ and $\measuredangle\, k(t)$ can be determined. Without proof

$$p[|k_r|^2, \measuredangle\, k_t] = \begin{cases} \dfrac{1}{4\pi R^2(0)}\, e^{-\frac{|k_t|^2}{2R^2(0)}}, & \begin{array}{l} |k_t|^2 \geq 0 \\ 0 \leq \measuredangle\, k_t \leq 2\pi \end{array} \\ 0 & \text{otherwise} \end{cases}$$

From which the marginal distributions are

$$p[|k_t|^2] = \begin{cases} \dfrac{1}{2R^2(0)} \cdot e^{-\dfrac{|k_t|^2}{2R^2(0)}}, & |k_t|^2 \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{41}$$

$$p[\measuredangle k_t] = \begin{cases} \dfrac{1}{2\pi}, & 0 \leq \measuredangle k_t \leq 2\pi \\ 0 & \text{otherwise} \end{cases} \tag{42}$$

End of Proofs 2 and 3

Proof 4

By inspection

$$p[|k_t|^2, \measuredangle k_t] = p[|k_t|^2]\, p[\measuredangle k_t] \tag{43}$$

End of Proof 4

SECTION V

OPERATIONS ON REAL SIGNALS AND THEIR COMPLEX ENVELOPES

When we add, multiply, correlate, or convolve real signals, we are often interested only in the amplitude and phase modulation of the result, and do not wish to retain any information concerning the carrier. Under these circumstances it is advantageous to work only with the complex envelopes of the signals. When performing the above operations on complex envelopes, we must be confident that we understand what happens to the real signals, and vice versa, or much of the utility is lost. In this section we shall investigate the relation between the real signals and their complex envelopes as they undergo these operations. The theory we have developed in Sections II and III will prove very useful in developing these relationships.

The presentation will be simplified if we agree on the following notation at the onset.

Notation

1. In all that follows, x(t) and y(t) are real zero-mean signals
2. [Env x] (t) is the complex envelope of x(t) at the frequency f_c

$$[\text{Env } x](t) = \tilde{x}(t)\, e^{-j2\pi f_c t} \tag{44}$$

3. "Env" is the operator which transforms x(t) into its complex envelope [Env x] (t); thus,

$$\text{Env } x(t) = [\text{Env } x](t) \tag{45}$$

Note: It is important to distinguish between the operator Env which "takes the complex envelope" and the complex time function [Env x] (t) which is the result of the operation.

A. ADDITION

The complex envelope of the sum of two real signals is the sum of their complex envelopes.

$$\text{Env}\,[x(t) + y(t)] = [\text{Env } x](t) + [\text{Env } y](t) \tag{46}$$

Proof

$$\text{Env}\,[x(t) + y(t)] = [x(t) + y(t)]^{\sim}\, e^{-j2\pi f_c t}$$

$$= [\tilde{x}(t) + \tilde{y}(t)]\, e^{-j2\pi f_c t} \qquad \text{by Equation (26)}$$

$$= \tilde{x}(t)\, e^{-j2\pi f_c t} + \tilde{y}(t)\, e^{-j2\pi f_c t}$$

$$= [\text{Env } x]\ (t) + [\text{Env } y]\ (t)$$

<u>End of Proof</u>

B. TIME CORRELATION

The complex envelope of the time correlation function of two real signals is one-half the time correlation of the complex envelope of one signal and the conjugate complex envelope of the other, as follows.

$$\text{Env} \int x(t)\, y(t-\tau)\, dt = \frac{1}{2} \int [\text{Env } x]\ (t)\ [\text{Env } y]^*(t-\tau)\, dt \qquad (47)$$

<u>Proof</u>

We first note the identity

$$\int \tilde{x}(t)\, \tilde{y}^*(t-\tau)\, dt = \int [x(t) + j\,\hat{x}(t)]\ [y(t-\tau) - j\,\hat{y}(t-\tau)]\ dt$$

$$= \int x(t)\, y(t-\tau)\, dt + \int \hat{x}(t)\, \hat{y}(t-\tau)\, dt$$

$$+ j \int \hat{x}(t)\, y(t-\tau)\, dt - j \int x(t)\, \hat{y}(t-\tau)\, dt$$

$$= 2 \int x(t)\, y(t-\tau)\, dt - 2j \int x(t)\, \hat{y}(t-\tau)\, dt \quad \text{by Equations (10) and (11)}$$

Now we use this to prove the assertion

$$\text{Env} \int x(t)\, y(t-\tau)\, dt = \widetilde{\left[\int x(t)\, y(t-\tau)\, dt \right]} e^{-j2\pi f_c \tau}$$

$$= \left[\left\{ \int x(t)\, y(t-\tau)\, dt \right\} + j\, \widehat{\left\{ \int x(t)\, y(t-\tau)\, dt \right\}} \right] e^{-j2\pi f_c \tau}$$

Using the previously defined notation $\bar{y}(t) = y(-t)$

$$= \left[\int x(t)\, y(t-\tau)\, dt + j \int x(t)\, \hat{\bar{y}}(\tau - t)\, dt \right] e^{-j2\pi f_c \tau}$$

$$= \left[\int x(t)\, y(t-\tau)\, dt - j \int x(t)\, \hat{y}(t-\tau)\, dt \right] e^{-j2\pi f_c \tau} \quad \text{by Equation (13)}$$

$$= \left[\frac{1}{2} \int \tilde{x}(t)\, \tilde{y}^*(t-\tau)\, dt \right] e^{-j2\pi f_c \tau} \quad \text{by the above identity}$$

$$= \frac{1}{2} \int \tilde{x}(t)\, e^{-j2\pi f_c t}\, \tilde{y}^*(t-\tau)\, e^{j2\pi f(t-\tau)}\, dt$$

$$= \frac{1}{2} \int [\text{Env } x]\ (t)\ [\text{Env } y]^*(t-\tau)\, dt$$

<u>End of Proof</u>

C. ENSEMBLE CORRELATION

The complex envelope of the ensemble correlation function of two real signals is one-half the ensemble correlation of the complex envelope of one signal and the conjugate complex envelope of the other, as follows.

$$\text{Env } E\left[x(t)\ y(t-\tau)\right] = \frac{1}{2}\ E\left[[\text{Env } x]\ (t)\ [\text{Env } y]^*(t-\tau)\right] \tag{48}$$

The proof is formally identical to the proof for time correlation given in Section V-B. The proof uses Equations (16), (17), and (18) in place of (10), (11), and (13), respectively.

D. CONVOLUTION

The complex envelope of the convolution of two signals is one-half the convolution of the complex envelopes of the signals.

$$\text{Env}\int x(\tau)\ y(t-\tau)\ d\tau = \frac{1}{2}\int [\text{Env } x]\ (\tau)\ [\text{Env } y]\ (t-\tau)\ d\tau \tag{49}$$

<u>Proof</u>

We first note the identity

$$
\begin{aligned}
[\text{Env } \bar{y}]^*(t) &= \tilde{\bar{y}}^*(t)\ e^{j2\pi f_c t} \\
&= [\bar{y}(t) - j\ \hat{\bar{y}}(t)]\ e^{j2\pi f_c t} \\
&= [\bar{y}(t) + j\ \overline{\hat{y}}(t)]\ e^{-j2\pi f_c(-t)} \qquad \text{by Equation (13)} \\
&= [\text{Env } y]\ (-t)
\end{aligned}
$$

Now the assertion can be proved.

$$
\begin{aligned}
\text{Env}\int x(\tau)\ y(t-\tau)\ d\tau &= \text{Env}\int x(\tau)\ \bar{y}(\tau - t)\ d\tau \\
&= \frac{1}{2}\int [\text{Env } x]\ (\tau)\ [\text{Env } \bar{y}]^*(\tau - t)\ d\tau \qquad \text{by Equation (48)} \\
&= \frac{1}{2}\int [\text{Env } x]\ (\tau)\ [\text{Env } y]\ (t-\tau)\ d\tau \qquad \text{by the identity above}
\end{aligned}
$$

<u>End of Proof</u>

E. LINEAR TIME VARYING SYSTEMS

<u>Notation</u>

$h(\tau, t)$ is the response of a linear system at time t to an impulse at time $t - \tau$

Definition

$$\hat{h}(\tau, t) = \frac{1}{\pi}\int \frac{h(\sigma, t)}{\tau - \sigma}\, d\sigma \tag{50}$$

Then it follows from Equation (49) that

$$\text{Env}\int h(\tau, t)\, x(t - \tau)\, d\tau = \frac{1}{2}\int [\text{Env } h]\ (\tau, t)\, [\text{Env } x]\ (t - \tau)\, d\tau \tag{51}$$

SECTION VI

THE COMPLEX SAMPLING THEOREM

In its usual form, the sampling theorem states that a waveform with no positive frequency components outside a band $0 < f < W$ Hz is completely determined by its time samples taken $\frac{1}{2W}$ seconds apart. This requires 2TW samples in a time interval of T seconds. Intuitively, it would seem that a waveform with no positive frequency components outside a pass band in the range $f_o - \frac{W}{2} < f < f_o + \frac{W}{2}$ Hz should also be completely determined by 2TW samples in every T second interval. The complex sampling theorem shows that this is the case.

Notation

1. $\text{sinc } t = \frac{\sin \pi t}{\pi t}$
2. $\text{rect } t = \begin{cases} 1, & |f| < \frac{1}{2} \\ 0, & |f| > \frac{1}{2} \end{cases}$
3. $\delta(t)$ is the unit impulse function
4. All summations are from $-\infty$ to ∞

Note: The following are Fourier transform pairs.

1. $W \text{ sinc } Wt \longleftrightarrow \text{rect } \frac{f}{W}$ (52)

2. $\frac{1}{W} \sum_k \delta\left[t - \frac{k}{W}\right] \longleftrightarrow \sum_k \delta(f - kW)$ (53)

The Complex Sampling Theorem

Let x(t) be a real waveform with no positive frequency components outside a band in the range $f_o - \frac{W}{2} < f < f_o + \frac{W}{2}$ Hz. Let x(t) have the complex envelope u(t) at a carrier frequency f_c. Then

$$x(t) = \text{Re} \sum_k u\left[\frac{k}{W}\right] \text{sinc}\left[t - \frac{k}{W}\right] e^{j2\pi(f_o + f_c)\left[t - \frac{k}{W}\right]} \quad (54)$$

i.e., x(t) is completely determined by samples of its complex envelope taken $\frac{1}{W}$ seconds apart. Since each sample has a real and imaginary part, there are actually 2TW independent samples in a T second interval.

Proof

Let x(t) have analytic signal $\tilde{x}(t)$ with Fourier transform $\tilde{X}(f)$ Then, $\tilde{X}(f)$ has no frequency components outside the band $f_o - \frac{W}{2} < f < f_o + \frac{W}{2}$ Hz; therefore,

$$\tilde{X}(f) = \left\{ X(f) * \sum_k \delta(t - kW) \right\} \text{ rect } \frac{1}{W}(f - f_o)$$

and taking Fourier transforms of both sides and using Equations (52) and (53)

$$\tilde{x}(t) = \left\{ \tilde{x}(t) \frac{1}{W} \sum_k \delta\left[t - \frac{k}{W}\right] \right\} * W \text{ sinc } Wt\, e^{j2\pi f_o t}$$

$$= \sum_k \tilde{x}\left[\frac{k}{W}\right] \text{ sinc } W\left[t - \frac{k}{W}\right] e^{j2\pi f_o \left[t - \frac{k}{W}\right]}$$

$$= \sum_k \tilde{x}\left[\frac{k}{W}\right] e^{-j2\pi f_c \left[t - \frac{k}{W}\right]} \text{ sinc } W\left[t - \frac{k}{W}\right] e^{j2\pi (f_o + f_c)\left[t - \frac{k}{W}\right]}$$

$$= \sum_k u\left[\frac{k}{W}\right] \text{ sinc } W\left[t - \frac{k}{W}\right] e^{j2\pi (f_o + f_c)\left[t - \frac{k}{W}\right]}$$

End of Proof

SECTION VII

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