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ON DETERMINING THE FULL SET OF SOLUTIONS
OF A FINITE GAME

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ON DETERMINING THE FULL SET OF SOLUTIONS OF A FINITE GAME

L. S. Shapley

Summary. A detailed account, on the elementary level, of a procedure for establishing whether a given set of solutions to a finite game is complete. The aim has been to make the determination a mechanical matter, not depending on the geometric intuition of the computer. An illustrative example is worked through.

I

We will be concerned throughout this discussion with a finite two-person zero-sum game, represented as usual by an m x n matrix $A = (a_{ij})$. The mixed strategies x and y are vectors subject to

$$x_i \geq 0, \qquad y_j \geq 0, \qquad \sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = 1 .$$

The auxiliary vectors h(y) and k(x) are defined by

$$h_i(y) = \sum_{j=1}^{n} a_{ij} y_j , \qquad k_j(x) = \sum_{i=1}^{m} a_{ij} x_i .$$

A solution is defined to be any pair x, y satisfying

$$\min_j k_j(x) = \max_i h_i(y) ;$$

this amount is called the value, v, of the game. The individual vectors x and y of a solution are known as optimal mixed strategies (OMST), and any pair of opposing OMST constitute a solution. The sets of OMST for the two players we denote by X and Y respectively.

II

A (pure) strategy i or j is said to be essential if $x_i > 0$ or $y_j > 0$ for some $x \in X$ or $y \in Y$. A strategy i or j is said to be inadmissible if $h_i(y) < v$ or $k_j(x) > v$ for some $y \in Y$ or $x \in X$. The number of essential strategies for player K we denote by e_K; of inadmissible strategies by u_K. The submatrix obtained by considering just the essential strategies is the essential matrix, denoted A_1.

THEOREM 1. Every (pure) strategy is either essential or inadmissible, but not both. Hence,

$$e_1 + u_1 = m, \qquad e_2 + u_2 = n .$$

(For a proof, see AMS-24,* page 54, Theorem 1.)

III

We now introduce some geometrical concepts. A convex set is a set of points that contains the line-segment joining any pair of its points. An extreme point of the set is one that does not lie on the line between any two other points of the set. If points are made to correspond in the usual way with vectors in a vector space, the dimension of a convex set of the type we shall be considering** is one less than the largest number of linearly independent vectors that can be found in the set. A convex polytope is the generalization of a convex polygon (polyhedron) in 2 (3) dimensions; it is characterized by

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*Annals of Mathematics Studies No. 24: "Contributions to the Theory of Games," Princeton, 1950.

**That is, sets that can not be linearly extended to include the origin of the vector space.

having a finite number of extreme points, which we call vertices. When these are known, the entire convex polytope is uniquely determined. The faces of an r-dimensional convex polytope are the polytopes of dimension $r-1$ making up the boundary; they are themselves convex and no two of them lie in the same $r-1$-dimensional hyperplane. A knowledge of all the faces also suffices to determine the polytope uniquely. An r-dimensional simplex is a special kind of polytope which has exactly $r+1$ vertices and $r+1$ faces. (It is the "simplest" kind, in the sense that any other polytope of the same dimension has more vertices and more faces.) Every face of a simplex is itself a simplex; likewise every face of a face, and so on. The sets of mixed strategies in the game under consideration are simplices of dimension $m-1$ and $n-1$.

THEOREM 2. *X and Y are convex polytopes.*

(For a proof, see AMS-24, page 34, Corollary 2.)

THEOREM 3. *If v is not zero, then*

$$\dim X = e_1 - \operatorname{rank} A_1$$

$$\dim Y = e_2 - \operatorname{rank} A_1 \ .$$

(Proved as Theorem 2, AMS-24, page 55. Since there is no loss in generality in supposing $v \neq 0$, we state this theorem (and Theorem 4) only for that case.)

IV

The set of solutions is the cartesian product $X \times Y$ of the sets of OMST. It is also a convex polytope, and we call its vertices the basic solutions.

Basic solutions may be formed by pairing any vertex of X with any vertex of Y.

THEOREM 4. <u>If</u> v <u>is not zero</u>, <u>then a solution</u> x, y <u>is basic if and only if there is a square</u>, <u>non-singular submatrix</u> $(a_{i_\mu j_\nu})$ <u>of</u> (a_{ij}) <u>whose inverse</u> $(b_{j_\nu i_\mu})$ <u>satisfies</u>

$$v = 1 \Big/ \sum_{\mu=1}^{r} \sum_{\nu=1}^{r} b_{j_\nu i_\mu}$$

$$x_{i_\mu} = v \sum_{\nu=1}^{r} b_{j_\nu i_\mu}$$

$$y_{j_\nu} = v \sum_{\mu=1}^{r} b_{j_\nu i_\mu} .$$

(Proved as Theorem 2, AMS 24, page 30.)

The square submatrix of the theorem in general neither contains, nor is contained in, the essential matrix A_1. But it must always contain the rows and columns for which the particular x_i and y_j are non-zero. A basic solution may have several associated square matrices; but different basic solutions cannot arise from the same square submatrix.

If X and Y have s and t vertices, respectively, then the game will have st basic solutions. Unless these numbers are small, it becomes tedious and profitless to attempt to find all the basic solutions by inverting submatrices. Most of them would simply repeat the known vertices of X and Y in new combinations. What is needed is a way of checking a set of vertices to discover whether the convex polytope they determine is or is not the complete set of OMST. Such a test exists; in order to state it we must first describe more closely the faces of X and Y.

V

X lies in the $m-1$-dimensional simplex of all mixed strategies of the first player, as we have seen. But X also lies in the e_1-1-dimensional sub-simplex E_1 determined by the essential (pure) strategies. E_1 is, in general, a lower-dimensional component of the boundary of the full simplex. Now, each face of X must lie either inside E_1, or in some face of E_1. A face of the first kind we shall call _inner_, a face of the second kind, _outer_.

An outer face can be "explained" by the observation that the points beyond it are not mixed strategies; they fall outside of E_1 and outside of the full strategy simplex as well. They are characterized analytically by having $x_i < 0$ for some i. The i in question must be an essential (pure) strategy, since the other x_i are identically zero throughout X and the linear extension of X.

On the other hand, an inner face can occur only because the strategies on the outside of the face are not optimal. That is,

$$k_j(x) < v \quad \text{for some } j .$$

The offending j must come from the other player's set of inadmissible (pure) strategies since the other $k_j(x)$ are identically equal to v over X and its linear extension.

The following definition is now natural:

A face of X is _justified_ if either

(a) $x_i = 0$ on the face, for some essential i, or

(b) $k_j(x) = v$ on the face, for some inadmissible j.

Similarly, a face of Y is justified if either

(c) $y_j = 0$ on the face, for some essential j, or

(d) $h_i(y) = v$ on the face, for some inadmissible i.

To verify these relations "on the face," it is always sufficient to verify them just at the vertices of the face.

We then have the theorem:

THEOREM 5. _Every face of_ X _is justified_. _Hence_ X _has at most_ $e_1 + u_2$ _faces_. _If_ K _is a subset of_ X, _of the same dimension as_ X, _and if every face of_ K _is justified_, _then_ K = X. _Similarly for_ Y.

(The proof is straightforward, and is essentially covered by the preceding discussion. For further illumination, see AMS-24, pages 60-67.)

VI

TEST. The test for completeness may now be stated in three stages. Let us suppose that we have discovered a certain number of vertices of X and Y, by means of Theorem 4 or otherwise, and that they determine the polytopes $X_?$ and $Y_?$.* We wish to know whether or not $X_? = X$ and $Y_? = Y$.

Stage 1: The sets $X_?$ and $Y_?$ must be sufficient to determine the essential submatrix A_1. That is, in the light of Theorem 1, they must reveal whether each pure strategy is essential or inadmissible.

Stage 2: The dimensions of $X_?$ and $Y_?$ must satisfy the equations of Theorem 3. If the value of the game happens to be zero, however, a constant must be added to the matrix before computing the rank of A_1.

Stage 3: All faces of both $X_?$ and $Y_?$ must be justified, in the sense of Theorem 5.

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*Read "X-doubt," "Y-doubt."

Whenever the sets $X_?$ and $Y_?$ fail the test, at any stage, a clue is left as to where to search for the missing basic solutions. We shall not go into this matter in detail: the reading of the clues is more or less obvious once the test itself is well understood.*

The only stage of the test which may give rise to difficulty of application is the third. The possibly difficult question is that of determining the faces of a convex polytope, when just the vertices and the dimension are known. In practice we can replace that question by the somewhat easier question: when do a given set of faces (the justifiable faces) "close out" a convex polytope - i.e., form the complete boundary? We shall show how this question is answered in the low dimensional cases. We assume in the discussion that the candidates $X_?$ and $Y_?$ have passed the first two stages of the test. The third stage involves X and Y independently; without loss of generality we limit our discussion to X alone.

Case I. $\dim X_? = \dim X = 0$.

Since the only 0-dimensional polytope is a point, we automatically have $X_? = X$.

Case II. $\dim X_? = \dim X = 1$.

The only 1-dimensional convex polytope is a line segment, with exactly two vertices. Again we automatically have $X_? = X$.

Case III. $\dim X_? = \dim X = 2$.

Here the third stage of the test at last becomes non-trivial. X is a plane polygon, with as many faces (sides) as vertices. We will have $X_? = X$ if and only if $X_?$ has as many justified faces as it has vertices. (It follows that the

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*Several instances of clue-reading appear in the Example beginning on page 8.

justified faces will then fit together in a closed, polygonal curve.) It may not always be obvious which pairs of vertices of $X_?$ constitute faces, and which merely represent interior chords. But since a chord will never satisfy the conditions for justification, a simple and rigorous procedure is to apply these conditions to all pairs of vertices. (Thus, the pair of vertices x, x' must have $x_i = x_i' = 0$ for an essential i, or $k_j(x) = k_j(x') = v$ for an inadmissible j.)

For dim $X_? = \dim X \geq 3$, no such simple counting procedure is available. The more detailed calculation that is necessary in 3 or more dimensions will be described in a separate paper.

VII

EXAMPLE. The matrix

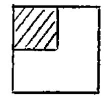
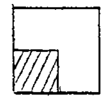
$$A = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \\ \end{array} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \\ \hline 3 & 0 & 6 & 0 \\ 0 & 3 & -3 & 3 \\ 2 & 1 & 3 & 3 \\ 1 & 2 & 0 & 1 \\ \hline k_1 & k_2 & k_3 & k_4 \end{array} \begin{array}{c} \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ \\ \end{array}$$

is given; the full sets X and Y of OMST are to be determined.

1st attempt: Perhaps the most obvious OMST of the game are

$x^{(1)} = (1/2, 1/2, 0, 0)$, $\qquad y^{(1)} = (1/2, 1/2, 0, 0)$,

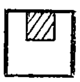
$x^{(2)} = (0, 0, 1/2, 1/2)$,

arising from the square submatrices which we may indicate by  and .

The value of the game is $v = 3/2$. By simple calculation we obtain:

$$k(x^{(1)}) = (v, v, v, v), \qquad h(y^{(1)}) = (v, v, v, v),$$
$$k(x^{(2)}) = (v, v, v, 2).$$

Applying "Stage 1," we see that all of $i = 1, 2, 3, 4$ are essential, that $j = 1, 2$ are essential, and that $j = 4$ is inadmissible. But the character of $j = 3$ is not determined. Therefore our first attempt has not passed even Stage 1 of the test.

<u>2nd attempt</u>: Looking for an OMST involving $j = 3$, we discover that the submatrix  produces a new extreme point of Y:

$$y^{(2)} = (0, 3/4, 1/4, 0)$$

(together with the OMST $x^{(1)}$ already known). This establishes the essentiality of $j = 3$, and gives us the essential matrix:

$$A_1 = \begin{vmatrix} 3 & 0 & 6 \\ 0 & 3 & -3 \\ 2 & 1 & 3 \\ 1 & 2 & 0 \end{vmatrix} \qquad e_1 = 4, \quad e_2 = 3,$$

rank $A_1 = 2$ (since the first column is half the sum of the second and third).

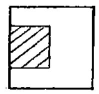
Hence, by Theorem 3,

$$\dim X = e_1 - \operatorname{rank} A_1 = 2,$$
$$\dim Y = e_2 - \operatorname{rank} A_1 = 1.$$

But considering our present candidates $X_?$ and $Y_?$, each having just two vertices, we see that

$$\dim X_? = \dim Y_? = 1 .$$

Therefore the present $X_?$ does not pass Stage 2 of the test.

<u>3rd attempt</u>: Looking for an OMST that will increase the dimension of $X_?$, we soon find that the submatrix  yields the extreme OMST:

$$x^{(3)} = (0, 1/4, 3/4, 0) .$$

This gives $\dim X_? = 2$, as demanded by Stage 2 of the test. Passing to Stage 3, we first observe that $Y_?$ is equal to Y automatically, since Y is a 1-dimensional polytope. We turn therefore to our latest $X_?$, which has three vertices. We summarize our information:

$$\begin{aligned} x^{(1)} &= (1/2,\ 1/2,\ 0,\ 0\ ), & k(x^{(1)}) &= (v,\ v,\ v,\ v\ ); \\ x^{(2)} &= (\ 0,\ 0,\ 1/2,\ 1/2), & k(x^{(2)}) &= (v,\ v,\ v,\ 2\ ); \\ x^{(3)} &= (\ 0,\ 1/4,\ 3/4,\ 0\ ), & k(x^{(3)}) &= (v,\ v,\ v,\ 5/2). \end{aligned}$$

Of the three possible faces we find that

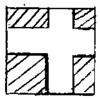
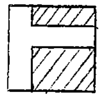
$\left[x^{(1)}x^{(2)}\right]$ ...is not justified,

$\left[x^{(1)}x^{(3)}\right]$ ...is justified (outer) by $x_4 = 0$,

$\left[x^{(2)}x^{(3)}\right]$ ...is justified (outer) by $x_1 = 0$.

Since, for two dimensions, Stage 3 demands that the number of justified faces be equal to the number of vertices, our present $X_?$ fails to pass the test.

<u>4th attempt</u>: Outer faces corresponding to $i = 1$ and $i = 4$ have already been found. This means that x_1 and x_4 must be strictly positive for any

undiscovered vertices of X. After a few trials, we hit upon one of the 3x3 submatrices  or  , either one of which gives us

$$x^{(4)} = (1/10,\ 0,\ 3/10,\ 6/10), \qquad k(x^{(4)}) = (v, v, v, v) .$$

This new vertex produces three more possible faces of X, as follows:

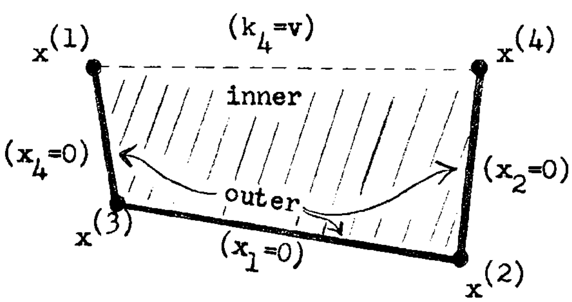
$\left[x^{(1)}x^{(4)}\right]$ ....is justified (inner) by $k_4 = v$,

$\left[x^{(2)}x^{(4)}\right]$ ....is justified (outer) by $x_2 = 0$,

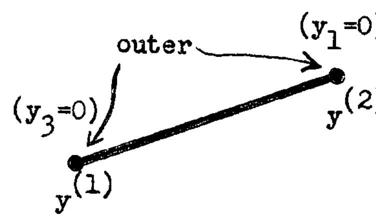
$\left[x^{(3)}x^{(4)}\right]$ ....is not justified.

We now have four vertices and four justified faces. The solution is therefore complete. We may represent X and Y schematically as follows:

X:



Y:



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