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SOME RESULTS IN NON-LINEAR PROGRAMMING

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Summary: A minimization problem is solved and used to treat a class of maximization problems. Possible applications of the results are discussed.

SOME RESULTS IN NON-LINEAR PROGRAMMING

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Introduction.

Suppose that the components of a vector $x = (x_1, \ldots, x_n)$ are subjected to certain linear inequalities, which restrict x to a region Γ. The linear programming problem is that of determining the maximum or minimum of a linear function of x. If the function to be maximized (or minimized) is non-linear, we have a problem in non-linear programming. In the present memorandum we consider several non-linear programming problems where the non-linear functions treated fall in the class: linear function plus sum of fractions with linear denominators. In section 1, a minimization problem is solved. In section 2, a related maximization problem is introduced. In section 3, a special case (previously solved by a different method by Rufus Isaacs*) is fully solved. An iteration problem is treated and illustrated in sections 4 and 6. Possible applications are considered in section 5.

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*Isaacs, R., "A Wagering Problem and a Maximizing Technique," 1952.

§1. A Minimization Problem.

We wish to minimize the vector function

$$f(x,s,p) = \sum_{i=1}^{n} \frac{s_i p_i}{s_i + x_i} \tag{1}$$

where $x \geq 0$, $s > 0$, $p > 0$, $p_0 = p_1 + \cdots + p_n = 1$, and

$$x_0 = x_1 + \cdots + x_n \tag{2}$$

is given. The vector x therefore varies over a compact set Γ.

The substitution $x_i = s_i y_i$ reduces this to the problem: minimize

$$g(y,p) = \sum_{i=1}^{n} \frac{p_i}{1+y_i} \tag{3}$$

subject to the side conditions $y \geq 0$ and

$$\sum s_i y_i = x_0, \tag{4}$$

which define the domain Γ' of y.

The existence of a minimum (as well as of a maximum) is clear since x varies over a compact set and since f is continuous. The extreme values will be assumed either at some vertex or in the interior of some boundary face.

Let E be a non-void subset of $I = \{1,\ldots,n\}$. Then the vectors x in Γ for which

(5) $$x_i > 0, \quad i \in E; \; x_i = 0, \quad i \notin E$$

constitute a subset Γ_E which is the interior of Γ if $E = I$, which is a vertex if E has one element, and which is otherwise the interior of some face of Γ. Clearly

(6) $$\Gamma = \bigcup_{E \subseteq I} \Gamma_E .$$

Since the Γ_E partition Γ any minimizing vector $\bar{x}$ will lie in exactly one subregion Γ_E.

Suppose that $\bar{x} \in \Gamma_E$ and that Γ_E is not a vertex, say $E = \{i_1, \ldots, i_r\}$. Then $\bar{x}_{i_r} = x_0 - \bar{x}_1 - \ldots - x_{i_{r-1}}$ and

(7) $$\left.\frac{\partial f}{\partial x_j}\right]_{\bar{x}} = \frac{-s_j p_j}{(s_j+\bar{x}_j)^2} + \frac{s_{i_r} p_{i_r}}{(s_{i_r}+\bar{x}_{i_r})^2} = 0, \; (j = i_1, \ldots, i_{r-1})$$

and

(8) $$\frac{-p_j}{s_j} + \frac{s_{i_r} p_{i_r}}{(s_{i_r}+x_{i_r})^2} \geqq 0 \qquad j \notin E$$

are necessary conditions for a minimum at $\bar{x}$.

From (7) we get

$$\frac{s_i p_i}{(s_i+\bar{x}_i)^2} = \frac{s_j p_j}{(s_j+\bar{x}_j)^2} \qquad (i, j \in E)$$

or

(9) $$\frac{s_i p_i}{(s_i+\bar{x}_i)^2} = \frac{1}{\lambda_E{}^2} \qquad (i \in E)$$

hence

(10) $$\begin{cases} \bar{x}_i = \lambda_E \sqrt{s_i p_i} - s_i & (i \in E) \\ \bar{x}_i = 0 & (i \notin E), \end{cases}$$

and also

(11) $$\lambda_E = (x_0 + \sum_{i \in E} s_i) \Big/ \sum_{i \in E} \sqrt{s_i p_i} .$$

Furthermore, we obtain

(12) $$f(\bar{x},s,p) = 1 - \pi_E + \tau_E^2 \Big/ (x_0 + \sigma_E) = f(E)$$

where

(13) $$\tau_E = \sum_{i \in E} \sqrt{s_i p_i}, \quad \sigma_E = \sum_{i \in E} s_i, \quad \pi_E = \sum_{i = E} p_i .$$

Our problem is now reduced to choosing E so as to minimize f(E).

Let $p_i/s_i = \rho_i \qquad (i = 1,\ldots,n)$

and arrange the components so that

(14) $$\rho_1 \leqq \rho_2 \leqq \cdots \leqq \rho_n.$$

From (8) and (9) we see that

(15) $$\rho_j > 1/\lambda_E^2 \qquad j \in E$$
$$\rho_j \leqq 1/\lambda_E^2 \qquad j \notin E.$$

Then from (14) we conclude that E must be one of the sets

(16) $$E_t = \{t, \ldots, n\} \qquad (t=1, \ldots, n)$$

To simplify notation, whenever E_t is used as a subscript we now replace it by t. In particular, the conditions (15) can now be written as

(17) $$\rho_t > 1/\lambda_t^2 \geqq \rho_{t-1}$$

or, equivalently,

(18) $$\sqrt{1/\rho_t} < \lambda_t \leqq \sqrt{1/\rho_{t-1}} \,.$$

In summary, (18) is a necessary condition that the minimizing vector $\bar{x}$ shall lie in Γ_t. We shall show that for each x_0 there is exactly one t for which (18) is satisfied, and hence that there is a unique minimum.

Set

$$(19) \qquad y_j = \sqrt{1/\rho_j}\,\tau_j - \sigma_j \quad (j=1,\ldots,n).$$

Then

$$y_j - y_{j+1} = (\sqrt{1/\rho_j} - \sqrt{1/\rho_{j+1}})\,\tau_{j+1} + \sqrt{1/\rho_j}\,\sqrt{s_j p_j} - s_j > 0,$$

since the first term is positive and the last two cancel. Thus we have

$$(20) \qquad y_1 \geqq y_2 \geqq \ldots \geqq y_n (=0).$$

From (1) we have

$$(21) \qquad \lambda_j = \lambda_j(x_0) = (x_0 + \sigma_j)/\tau_j,$$

and an easy computation shows that

$$(22) \qquad \begin{cases} \lambda_j(y_j) = \lambda_{j+1}(y_j) = \sqrt{1/\rho_j} & (j=1,\ldots,n-1) \\ \lambda_n(y_n) = \sqrt{1/\rho_n} \end{cases}$$

Now $\lambda_j(x_0)$ **is a linear function of** x_0 **with positive** slope; therefore it follows from (22) that $\lambda_j(x_0) > \sqrt{1/\rho_j}$ if and only if $x_0 > y_j$, and $\lambda_j(x_0) \leqq \sqrt{1/\rho_{j-1}}$ if and only if $x_0 \leqq y_{j-1}$. Now from (20) it follows that for each x_0 there is exactly one $t = t(x_0)$ for which (18) holds, namely that t for which

(23) $$y_t < x_0 \leqq y_{t-1}.$$

We summarize these results in the following theorem:

Theorem 1. The function f(x,s,p) of (1) is minimized by the vector x given by

(24) $$\begin{cases} \bar{x}_j = \lambda_t \sqrt{s_j p_j} - s_j & (j=t,\ldots,n) \\ \bar{x}_j = 0 \end{cases}$$

where $t = t(x_0)$ is defined by (23), and the minimum value is

(25) $$f(\bar{x},s,p) = g(x_0) = 1 - \pi_t + \tau_t^2 \Big/ (x_0 + \sigma_t).$$

Theorem 2. $g(x_0)$ is a differentiable function of x_0 in the interval $0 < x_0 < \infty$.

Proof: Both continuity and differentiability are trivial except at the points $y_1,\ldots,y_n$. Consider $x_0 = y_t (t < n)$. We have

(26) $$\lim_{x_0 \to y_t^+} g(x_0) = 1 - \pi_t + \tau_t^2 / (y_t + \sigma_t) = 1 - \pi_t + \tau_t \sqrt{\rho_t}$$

and

(27) $$\lim_{x_0 \to y_t^-} g(x_0) = 1 - \pi_{t+1} + \tau_{t+1}^2/(y_t + \sigma_{t+1}) = 1 - \pi_{t+1} + \tau_{t+1} \sqrt{\rho_t}.$$

The difference is

$$\pi_t - \pi_{t+1} - \sqrt{\rho_t}\,(\tau_t - \tau_{t+1}) = p_t - \sqrt{p_t/s}_t \cdot \sqrt{p_t s_t} = 0;$$

hence $g(x_0)$ is continuous at y_t.

Next

(28) $$\lim_{x_0 \to y_t^+} g'(x_0) = -\tau_t^2 \Big/ (y_t + \sigma_t)^2 = -\rho_t,$$

and for $t < n$

(29) $$\lim_{x_0 \to y_t^-} g'(x_0) = \tau_{t+1}^2 \Big/ (y_t + \sigma_{t+1})^2 = -\rho_t;$$

hence the derivative exists at y_t for $t < n$ and the right hand derivative exists at $0 (= y_n)$.

§ 2. A Maximization Problem.

With notation as above, let $H(t)$ be a function of the positive reals into the positive reals, and then consider the problem of finding the vector x which maximizes the function

(30) $$F(x) = F(x,s,p) = H(x_0+s_0) \sum_{j=1}^{n} \frac{p_j x_j}{x_i + s_j} - x_0.$$

Since $\sum_{j=1}^{n} p_j = 1$, this can be written as

(31) $$F(x) = H(x_0+s_0)\,\left[1 - f(x,s,p)\right] - x_0.$$

For the case of fixed x_0 the maximizing vector $\bar{x}$ will be the one which minimizes $f(x,s,p)$; the solution to this problem is given in Theorem 1.

Next for the case of arbitrary x_0 we argue as follows: if $\bar{x}$ is the maximizing vector then the components $\bar{x}_j$ $(j = 1,\ldots,n)$ can be determined from x_0 according to (24). To determine x_0 let

$$(32) \qquad G(x_0) = H(x_0+s_0)\left[1 - g(x_0)\right] - x_0$$

where $g(x_0)$ (see (25)) is the minimum value of $f(x)$ for all x with $\sum x_i = x_0$. Then the maximum of $F(x)$ as a function of the vector x will be the same as the maximum of $G(x_0)$ considered as a function of x_0, and $\bar{x}_0$ will be the value of x_0 which maximizes $G(x_0)$.

If $H(t)$ is differentiable for all $t > 0$, then by Theorem 2, $G(x_0)$ will be differentiable for $x_0 \geq 0$. Hence the maximum for $G(x_0)$ either does not exist at all, or exists for $x_0 = 0$, or exists at a point for which $G'(x_0) = 0$. For $x_0 > y_1$ we have $g(x_0) = \tau_1^2 / (x_0 + \sigma_1)$ and hence $\lim_{x_0 \to \infty} g(x_0) = 0$. Let $H = \lim_{x_0 \to \infty} H(x_0+s_0)/x_0$. If $H > 1$, $G(x_0)$ will have no maximum. On the other hand, if $H < 1$, for all x_0 sufficiently large $G(x_0)$ is negative and hence there is a maximum. If $H = 1$ there may or may not be a maximum.

We have

(33) $$G'(x_0) = H'(x_0+s_0)(1-g(x_0)) - g'(x_0)H(x_0+s_0)-1,$$

hence

(34) $$\lim_{x_0 \to 0} G'(x_0) = H'(s_0)\cdot 0 - (-\rho_n)H(s_0) - 1$$

from which we conclude that the maximum does not occur at $x_0 = 0$ if

(35) $$p_n H(s_0) < s_n.$$

Now $G(0) = 0$, hence $H < 1$ and (35) are sufficient conditions for existence of a positive maximum.

§3. <u>The Special Case</u> $H(t) = Qt,\ 0 < Q \leqq 1$.

In general it may not be easy to solve the equation $G'(x_0) = 0$. However, this can be done if $H(t) = Qt$ where $0 < Q \leqq 1$. [This special case (except for $Q = 1$) has been solved by Rufus Isaacs in a recent (unnumbered) memorandum, "A Wagering Problem and a Maximizing Technique."]

For $Q = 1$, and $x_0 \geqq y_1$, we have

(36) $$\begin{aligned} G(x_0) &= (x_0+s_0)(1-\tau_1^2/(x_0+s_0)) - x_0 \\ &= s_0 - \tau_1^2 = (s_1+\ldots+s_n) - (\sqrt{p_1 s_1}+\ldots+\sqrt{p_n s_n})^2 , \end{aligned}$$

which, by Schwarz inequality, is positive unless $\rho_1 = \ldots = \rho_n$ in which case $G(x_0) \equiv 0$. Since $H = Q$, this guarantees the existence of a maximum for each Q in the range $0 < Q \leqq 1$. Condition (35) now reduces to

$$(37) \qquad Q > s_n/p_n s_0.$$

Hence, if (36) holds, we are guaranteed a positive maximum for $F(x)$. For $Q = 1$, (36) always holds unless $\rho_1 = \ldots = \rho_n$.

We next study the equation $G'(x_0) = 0$. From (33) we get

$$(38) \qquad G'(x_0) = Q(1-g(x_0)) - g'(x_0)Q(x_0+s_0)-1.$$

If $y_t < x_0 \leqq y_{t-1}$ apply (25) and this reduces to

$$(39) \quad G'(x_0) = Q(\pi_t - \tau_t^2 / (x_0 + \sigma_t)) + Q(x_0+s_0)\frac{\tau_t^2}{(x_0+\sigma_t)^2} - 1$$

and this is zero if and only if

$$(40) \qquad \left(\frac{x_0+\sigma_t}{\tau_t}\right)^2 = \frac{s_0-\sigma_t}{1-\pi_t+q} = \frac{\sum_{j=1}^{t-1} s_j}{q+\sum_{j=1}^{t-1} p_j} \equiv \frac{1}{z_t}$$

where $q = (1-Q)/Q$. Equations (40), (21), and (22) then give

$$(41) \qquad \rho_t > z_t \geqq \rho_{t-1} \qquad (t = 2,\ldots,n).$$

Conversely, if (41) holds, then (40) determines an x_0 for which $G'(x_0) = 0$. Moreover, if (41) holds for t, then $\rho_t > z_{t+1}$, and, inductively,

(42) $$z_h < \rho_{h-1} \qquad (h = t+1,\ldots,n+1).$$

Hence if (41) holds for any t, it can hold for no larger t, i.e., t <u>is unique if it exists</u>. Moreover, (42) for $h = n + 1$ is equivalent to (37), so that this t must give the maximum value to $G(x_0)$. Conversely, if $Q \leqq s_n/p_n s_0$ then (40) has no solution so that $\bar{x} = 0$ is the maximizing vector. Note that $G'(0) = Qs_0p_n/s_n - 1$ so that the maximum value of $F(x,s,p)$ is positive or zero according as $G'(0)$ is positive or non-positive.

We observe that if $x_0 \geqq y_1$ then $G'(x_0) = Q - 1$; hence, if $Q < 1$ the maximum cannot occur for $x_0 \geqq y_1$. For $Q = 1$ all $x_0 \geqq y_1$ give the maximum value $s_0 - \tau_1^2$ (if (36)). This completes the proof of the following theorem.

<u>Theorem 3. Let</u> s, p, x <u>be vectors with</u> $s > 0$, $p > 0$, $x \geqq 0$, <u>and</u> $p_0 = \sum p_i = 1$, <u>and with components so ordered that</u> (14) <u>holds. Let</u>

(42) $$F(x,s,p) = (x_0+s_0)\cdot Q\cdot \sum_{j=1}^{n} \frac{p_j x_j}{s_j+x_j} - x_0, \text{ where } 0 < Q \leqq 1.$$

<u>Then for fixed</u> x_0,s <u>and</u> p <u>the vector</u> x <u>which maximizes</u> $F(x,s,p)$ <u>is given by</u> (24) <u>where</u> t <u>and</u> λ_t <u>are given by</u> (23) <u>and</u> (21).

For fixed s and p the function $F(x,s,p)$ is maximized by a vector $\bar{x}$ for which $\bar{x}_0$ is defined by (40) and (41). If $Q < 1$ there is only one maximizing vector. The maximum value of the function is positive if and only if $Q > s_n/p_n s_0$.

This theorem (except for the case $Q = 1$) was first proved by Isaacs who used somewhat different methods.

We note without giving the proof that for fixed s,p the maximizing x_0 is a continuous function of Q and that for $0 < Q < 1$, $0 \leqq x_0 < y_1$.

§4. Iterated Maximization.

Let $F(x,s,p)$ be given by (30) and suppose that $x^{(o)}$ is the maximizing vector for $F(x,s,p)$, let $s^{(1)} = x^{(o)} + s^{(o)}$, $(s^{(o)}=s)$ and consider the maximizing vector $x^{(1)}$ for $F(x,s^{(1)},p)$. We have

(43) $$\rho_j^{(1)} = \rho_j^{(o)} \quad (j=1,\dots,t-1); \quad \rho_j^{(1)} = \frac{1}{\lambda_t}\sqrt{\rho_j} \quad (j=t,\dots,n)$$

or

(44) $$\rho_j^{(1)}\lambda_t^2 = \sqrt{\rho_j^{(o)}\lambda_t^2} \; .$$

By hypothesis $\rho_t^{(o)} > \frac{1}{\lambda_t^2}$; hence $\rho_t^{(1)} = \frac{1}{\lambda_t}\sqrt{\rho_j^o} > \frac{1}{\lambda_t^2}$.

Since $s_j^{(1)} = s_j^{(o)}$ for $j < t$ we have $\lambda_j^{(1)} = \lambda_j^{(o)}$ for $2 \leq j \leq t$ and hence (17) cannot hold in this new case for

$j < t$. However, by hypothesis $\rho_t^{(o)} > \frac{1}{\lambda_t^2}$; hence, $\rho_t^{(1)} = \sqrt{\rho_j^o} / \lambda_t < \frac{1}{\lambda_t^2}$. Hence, $t^{(1)} = t$, and so the maximizing vector $x^{(1)}$ has the same zero components as $x^{(o)}$ and is given by

$$(45) \quad x_j^{(1)} = 0 \quad (j=1,\ldots,t-1), \; x_j^{(1)} = \lambda_t \sqrt{p_j s_j^{(1)}} - s_j^{(1)} \; (j=t,\ldots,n).$$

Formula (44) shows that this iteration can be continued indefinitely. Let

$$(46) \qquad \rho^* = \lim_{h\to\infty} \rho^{(k)}, \; s^* = \lim_{h\to\infty} s^{(k)}, \; x^* = \lim_{h\to\infty} x^{(k)},$$

$$z = s^* - s^{(o)}.$$

By (43) and (44) we have

$$(47) \quad \rho_j^* = \rho_j^{(o)}, \; s_j^* = s_j^{(o)}, \; z_j = 0 \quad (j=1,\ldots,t-1);$$

$$\rho_j^* = \frac{1}{\lambda_t^2}, \; s_j^* = p_j \lambda_t^2, \; z_j = p_j \lambda_t^2 - s_j^{(o)} \quad (j=t,\ldots,n).$$

Moreover, $F(x,s^*,p)$ has $x = 0$ as its maximizing vector and for any $x \leqq z$ (i.e., $x_j \leqq z_j$ $(j=1,\ldots,n)$) we have

$$(48) \qquad F(x, s^* - x, p) = 0 .$$

We shall give an interpretation of these results in the next section.

§5. Applications.

Suppose we have a situation in which n commodities are to be supplied by a group of manufacturers $\{1,\ldots,n\}$. Suppose that the total profit to all manufacturers is a function only of the total cost of production t; i.e., that we can write

$$\text{Total profit} = H(t) - t \tag{49}$$

where H(t) is the total sales price.

This profit may, of course, be negative. Suppose further that the proportion p_j of the total sales price which accrues to manufacture of the j^{th} commodity is independent of the amount of the j^{th} product which is produced, and also that this amount $H(t)\,p_j$ is divided among producers of the j^{th} commodity in proportion to their allocations to production of this commodity. Thus, if one manufacturer allocates the vector $x = (x_1,\ldots,x_n)$ to the various commodities and if the remaining manufacturers altogether allocate the vector $s = (s_1,\ldots,s_n)$ then the total profit (or loss) for the first manufacturer is

$$F(x,s,p) = H(x_0+s_0)\sum_{j=1}^{n}\frac{p_j x_j}{x_j+s_j} - x_0, \tag{50}$$

(here $x_0 = \sum x_j$ and $s_0 = \sum s_j$).

We would in general expect that for large t, $H(t) < t$ and even that $\lim_{t \to \infty} H(t)/t < 1$. Thus if a manufacturer knows p and s the procedures of section 2 will tell him what allocation of resources (choice of vector x) will maximize his profit. There are three cases which may arise: (1) he must allocate a fixed total amount x_0; (2) he may allocate any amount $x_0 \leqq x_0^*$; (3) he may allocate any amount whatsoever. Case (1) is handled in general by Theorem 1. Case (3) is handled by Theorem 3 for the special selection $H(t) = t \cdot Q$, $0 < Q \leqq 1$. For this selection of H(t) we can also handle Case (2). If the maximizing x_0 does not exceed x_0^* we proceed as in Case (3). If the maximizing x_0 is greater than x_0^* we allocate x_0^* as in Case (1); this procedure is justified by the fact that the function $G'(x_0)$ has at most one zero and hence if $\bar{x}_0 > 0$ maximizes $G(x_0)$ then $G(x_0)$ is monotone increasing in the range $0 \leq x_0 \leq \bar{x}_0$.

The special selection $H(t) = Qt$ treated by Isaacs may be called "the pari-mutual case," since (42) corresponds to the expected profit from a wager $x = (x_1, \ldots, x_n)$ placed just before the window closes assuming that p_j is the bettor's estimate of the probability that the j^{th} horse will win. Here $1 - Q$ is the proportion of each bet that is retained by the track.

§6. <u>An Example</u>.

Let $p = (.2, .3, .2, .3)$, $s = (5, 4, 2, 3)$, $\omega = .9$, then $\rho = (.04, .075, .1, .1)$, $q = \frac{1}{9}$ and the maximizing t is 2. $\lambda_2 = 4.01$ and the maximizing x is $(0, .38, .53, .79)$. $F(x,s,p) = .22$.

Let $s' = s + x$, and repeat the maximizing procedure. Of course, we still have $t = 2$ and $\lambda_2' = \lambda_2 = 4.01$. We get $x' = (0, .22, .31, .47)$ and $F(x', s', p) = .02$. However, the payoff to the first person is now reduced to $F(x, s+x', p) = .08$. Hence the entry of the second person cut the total profit and it would clearly be of advantage to the first person to bribe the second one not to invest (bet).

This is an interesting illustration of the point that maximization by each individual does not necessarily lead to maximization of total profit.

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