Consensus, Polarization and Clustering of Opinions in Social Networks

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***Abstract*—We consider a variation of the Deffuant-Weisbuch model introduced by Deffuant et al. in 2000, to provide new analytical insights on the opinion dynamics in a social group. We model the trust that may exist between like-minded agents through a trust function, which is a discontinuous (*hard-interaction*) non-increasing function of the opinion distance. In this model, agents exchange their opinions with their neighbors and move their opinions closer to each other if they are like-minded (that is, the distance between opinions is smaller than a threshold). We first study the dynamics of opinion formation under random interactions with a *fixed* rate of communication between pairs of agents. Our goal is to analyze the convergence properties of the opinion dynamics and explore the underlying characteristics that mark the phase transition from opinion polarization to consensus. Furthermore, we extend the hard-interaction model to a *strategic interaction model* by considering a time-varying rate of interaction. In this model, social agents themselves decide the time and energy that should be expended on interacting each of their neighbors, based on their utility functions. The aim is to understand how and under what conditions clustering patterns emerge in opinion space. Extensive simulations are provided to validate the analytical results of both the hard-interaction model and the strategic interaction model. We also offer evidence that suggests the validity of the proposed model, using the location and monthly survey data collected in the Social Evolution experiment over a period of nine months.**

***Index Terms*—opinion dynamics, continuous opinions, opinion clusters, consensus, polarization, opinion formation, social networks, opinion diffusion, information aggregation, network learning, non-Bayesian models, information aggregation**

I. Introduction

TO OBTAIN a fundamental understanding of opinion formation in social networks, one natural question to ask is that how the initial scattered information is shared and diffused in social networks. Indeed, information aggregation and diffusion generally involve interactions between agents in the network. Given the role of social interactions in the diffusion of information, it is important to obtain a thorough understanding of how the structure of social interactions affects the formation of opinion and in shaping individual behaviors.

Indeed, various mathematical models of social interactions have been formulated and studied historically in Economics [1]–[3] and Social Sciences [4]–[7]. Studies in these disciplines have focused on providing theoretical tools and analyzing experimental results to interpret social trends. More recently, an increasing number of disciplines that relate to information processing, such as computer science, optimization and control as well as signal processing, have taken a close interest in the study of social interactions. Another motivation for this study is from the increasingly dispersed social data in clouds of interacting computing systems. An interesting question is whether learning and parallel decision making in such environments lead to herding (a phenomenon that is referred to as *information cascade* [4], [8]) or fragmentation.

A popular approach in Economics and Social Sciences is to model social interactions via Bayesian learning. Specifically, Bayesian models describe the interactions between rational agents by postulating that agents' actions are driven by the objective of maximizing their own expected utilities, which depend on the state of the world θ. Agents observe the actions of their peers, and update their beliefs using Bayes rule, optimally fusing public information obtained by observing their peers' behaviors, and private information about θ. While it is possible to quantify formation and propagation of opinions with Bayesian learning, the complexity of computing the chain of actions and resulting belief updates under the pure Bayesian learning model complicates the analysis. In contrast, the description which we refer to as *non-Bayesian learning*, that emerged from the field of statistical physics [9], radically reduces the complexity of the interaction model, by removing the strategic action and directly postulating a rule to update agent opinions. In this case, interactions between agents are often random and local while the learning rule is designed to approximate the resulting change in the agents' beliefs, leaving out the agents' decisions. A clear exposition of the difference between the two approaches, which also provides an excellent set of references on the topics is in [10].

The interaction models discussed in this paper fall in this last class of models with continuous opinions (see [9] for a survey) which are embodied by a probability distribution for the state of the world θ that is assumed to be a discrete random variable. Our goal is to present a simple but plausible model for the evolution of opinion in a population of agents, and provide answers to the following fundamental questions concerning the formation of opinions in social networks:

- What kinds of interactions will lead to consensus and which ones will lead to polarization of opinions?
- Under what conditions will a social group split into clusters of opinions?

In the following, we use opinion and belief interchangeably.

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14. ABSTRACT

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An early formulation of continuous opinion dynamics was given by DeGroot [11]. Indeed, in DeGroot's model, individuals start with an initial opinion profile on a binary decision $\theta \in \{0,1\}$, represented by a vector of the probabilities that each agent attributes to one of the two events, say $x_i = P(\theta = 1)$, which is agent i's private information. The update process is captured by a fixed stochastic matrix T, where the $(i,j)^{\text{th}}$ element T_{ij} represents the relative trust that agent i places on agent j's opinion. Beliefs of individuals are updated linearly by taking a weighted average of their neighbors' beliefs (T_{ij} are the weights). The analysis of such a model is relatively straightforward, since the belief updates take the same form as the well-known average consensus algorithm [12]. Therefore, considering a social graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V}$ representing the set of agents, and such that $\mathcal{E}$ includes an edge (i,j) if and only if $T_{ij} > 0$, consensus of opinions is attained globally only if $\mathcal{G}$ is connected, otherwise, each connected component will attain, in general, a different consensus in opinions.

Another related model is the *Hegselmann-Krause* (HK) model [13], [14]. In the HK model as well, the opinion of each agent is represented by a real number $0 \leq x_i \leq 1$ that is updated synchronously, similar to the DeGroot model. However, the HK model introduces a confidence level (or a threshold) τ_0 (set to 1 in [14]) to model the lack of influence among agents whose beliefs are too far apart. Specifically, let $\mathcal{N}_i$ be the set of neighbors (i.e., agents) with which agent i can directly communicate. Let $I_i[k;\tau_0] = \{j \in \mathcal{N}_i \cup i : \ |x_i[k] - x_j[k]| \leq \tau_0\}$ be the set of trusted neighbors, i.e., whose absolute opinion distances are less than τ_0 from agent i's opinion $x_i[k]$ after the k^{th} interaction. Then after exchanging beliefs with their neighbors, individual opinions are updated synchronously according to the following rule:

$$x_i[k+1] = \frac{1}{|I_i[k;\tau_0]|} \sum_{j \in I_i[k;\tau_0]} x_j[k] \ ,$$

where $|S|$ denotes the cardinality of the set S. If agent i has no credible neighbors (i.e., $I_i[k;\tau_0] = i$), then it does not change its opinion. Though similar to the DeGroot model, a significant difference is that the HK update is nonlinear with respect to the current opinion profile. In particular, the set of neighbors with which agent i updates its belief may change with k. An extension to the HK model, as discussed in [13], is to assume asymmetric confidence intervals $[-\tau_l, \tau_r]$ such that the trusted set $I_i[k;\tau_0]$ is replaced by $I_i[k;\tau_l,\tau_r] = \{j \in \mathcal{N}_i \cup i : \ -\tau_l \leq x_j[k] - x_i[k] \leq \tau_r\}$. Another related extension, called the heterogeneous HK (htHK) model [15], is to introduce diversity of confidence bounds $\bar{\tau} = [\tau_1, \cdots, \tau_n]$. While the convergence properties of the HK model can be observed numerically, analytical results are limited. Recently, [16] extended the analysis of the HK model to multidimensional beliefs and provided an upper bound on the convergence time.

The *Deffuant-Weisbuch* (DW) model [17], [18], on which our proposed analysis is based, explores the effects of simple random pair-wise interactions between agents whose opinion distance is smaller than a threshold. In this model as well, each individual's opinion is represented by a real number $x_j \in [0,1]$. Agents i and j are randomly selected for interaction, which is assumed to be symmetric (i.e., if $i \in \mathcal{N}_j$, then $j \in \mathcal{N}_i$). Let $I_i[k;\tau_0] = \{j \in \mathcal{N}_i : \ |x_i[k] - x_j[k]| < \tau_0\}$. If $j \in I_i[k;\tau_0]$ and thus $i \in I_j[k;\tau_0]$, then after the interaction, opinions are updated pair-wise as follows

$$\begin{aligned} x_i[k+1] &= (1-\bar{\mu})x_i[k] + \bar{\mu}x_j[k] \\ x_j[k+1] &= (1-\bar{\mu})x_j[k] + \bar{\mu}x_i[k], \end{aligned} \tag{1}$$

where $\bar{\mu} \in (0, 0.5]$ is called the mixing parameter. Specifically, Deffuant et al. in [17] explored this system over a square grid in which individuals are only connected with their four immediate neighbors. Weisbuch in [18] extended this simple lattice topology to a scale free network topology. Though different from the HK model, the DW model also relies on the idea of bounded confidence. Both models focus on a one dimensional belief so they exhibit similar behavior.

The model proposed in this paper is based on the DW model, but rather than binary decision making, as assumed in [13], [14], [17], [18], our model facilitates the analysis of multi-alternative decision making (decision between multiple alternatives), similar to [16], [19]. Specifically, we treat each agent's opinion as a vector of probabilities, in a probability simplex of arbitrary dimension; each element of the opinion vector represents the probability that a certain alternative is true. We introduce another generalization of the DW model: a state-dependent trust function $\mu(d)$. Although similar in spirit to the parameter μ_0 defined in [13], [14], [18], [20], [21], the trust function $\mu(d)$ in our model varies with the squared opinion distance d between the interacting agents. Clearly, the effect of $\mu(d)$ is time varying since agent opinions evolve over time and its value depends on how distance is defined. This leads to the so-called *hard-interaction* model [22] that includes both generalizations. Moreover, we give an explicit mathematical characterization of the existence of a phase transition from a society of polarized opinions to one with a convergent opinion, and its relation to the society's initial opinion profile. Motivated by understanding how the rate of interaction affects the outcome of the interactions in the non-Bayesian setting, we extend the hard interaction model in [22] by considering a time-varying rate of interaction, which is called the *strategic interaction* model. In this model, agents determine how much effort they want to invest in interacting with others. Specifically, at each time instant, agents choose an interaction pattern to select neighbors (connected agents), with whom interaction produces a positive net profit. Our focus is on understanding whether simple incentive schemes for interaction can lead to clustering behaviors in opinion space. Finally, all of our findings are validated numerically. In particular, the Social Evolution dataset [23] collected from 80 people in a student dormitory, is used to validate the model.

II. Social Interaction Model

Let $\mathcal{V} = \{1, 2, \cdots, n\}$ denote a set of social agents, who interact with each other over a fixed undirected communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{E}$ is the set of edges. We focus on the case where $\mathcal{G}$ is arbitrarily connected in the sense that there exists at least one path connecting any two agents in $\mathcal{G}$. Denote $\mathcal{N}_i$ the set of agents (also called neighbors) connected to agent i in $\mathcal{G}$, i.e., $\mathcal{N}_i = \{j \in \mathcal{V} \backslash i \mid (i,j) \in \mathcal{E}\}$. Agent interactions are modeled as pair-wise random encounters. We define a positive and time-invariant vector $\mathbf{p}$ whose ith element p_i is the probability of agent i initiating an interaction and a stochastic matrix $\mathbf{P}$ whose $(i, j)^{th}$ element P_{ij} is the probability that agent i chooses to interact with agent j. We assume:

Assumption 1: The stochastic matrix $\mathbf{P}$ is time-invariant and it has the same sparsity as the undirected network graph $\mathcal{G}$, *i.e.*, $P_{ij} > 0 \leftrightarrow (i, j) \in \mathcal{E}$.

We postpone a discussion of the effect of time-varying communication rates to Section IV.

A. Opinion Updates

The opinion evolution runs as follows. Each agent in the network starts with an initial opinion profile which is modeled as a q-dimensional vector of probabilities $\mathbf{x}_j[0] = [x_{j1}[0], \cdots, x_{jq}[0]]$ in opinion space

$$\mathcal{X} = \left\{ \mathbf{x} = [x_1, \cdots, x_q]^T \mid \sum_{\ell=1}^{q} x_\ell = 1 \text{ and } x_\ell \in [0, 1] \right\}.$$

The opinion $\mathbf{x}_j[0]$ can be thought of as a set of probabilities that the agent attributes to outcomes of a discrete random variable, or a mixed strategy. At each step, the network selects an agent[1], say agent i, with probability p_i to initiate an interaction; and with probability P_{ij}, agent i will then choose to interact with agent $j \in \mathcal{N}_i$. If neither of the agents finds the opinion of the other agent trust-worthy (according to a trust function), then nothing happens. Otherwise, the agent who is open-minded to the other agent's opinion, will update its opinion in a way that decreases the opinion distance between the two agents. Motivated by the application in [24], we introduce a distance function $s(\mathbf{x}_i, \mathbf{x}_j) : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ to measure the difference between opinions, i.e.,

$$s(\mathbf{x}_i, \mathbf{x}_j) = \|\mathbf{x}_i - \mathbf{x}_j\|_A = [(\mathbf{x}_i - \mathbf{x}_j)^T A (\mathbf{x}_i - \mathbf{x}_j)]^{1/2}$$

where $A \in \mathbb{R}^{q \times q}$ is a positive definite matrix. The set $\mathcal{X}$ is bounded with respect to the *norm* $\|\mathbf{x}_i\|_A := s(\mathbf{x}_i, 0)$, i.e., $\sup_i \|\mathbf{x}_i\|_A < \infty$ for $\forall \mathbf{x}_i \in \mathcal{X}$ and $\forall i \in \mathcal{V}$. Hence, the triangular inequality implies $s(\mathbf{x}_i, \mathbf{x}_j) \le 2\sup_i \|\mathbf{x}_i\|_A := s_{\text{sup}}$, where $\sup_i$ denotes the supremum of the value $\mathbf{x}_i$ for $\forall i \in \mathcal{V}$.

For ease of notation, let $d_{ij}[k] = s^2(\mathbf{x}_i[k], \mathbf{x}_j[k])$ denote the squared opinion distance after k network-wide interactions have occurred and $d_{\text{sup}} = s_{\text{sup}}^2$ denotes the supremum of squared distance. Given that agents i and j interact at the $(k+1)^{\text{th}}$ step, the opinions are updated as follows:

$$\mathbf{x}_i[k+1] = \mathbf{x}_i[k] + \mu\left(d_{ij}[k]\right)\left(\mathbf{x}_j[k] - \mathbf{x}_i[k]\right) \quad (2)$$

$$\mathbf{x}_j[k+1] = \mathbf{x}_j[k] + \mu(d_{ij}[k])(\mathbf{x}_i[k] - \mathbf{x}_j[k]) \quad (3)$$

where $\mu(d)$ defines the trust function of each agent in terms of d. Note that $\mu(d) > 0$ when the agent is open-minded to the opinion of the interacting agent and $\mu(d) = 0$ otherwise.

We say that the network attains *consensus* (*herding*) when all agents have the same opinion vector with respect to the distance measure, *i.e.*, $\forall i, j \in \mathcal{V},\ s(\mathbf{x}_i, \mathbf{x}_j) = \sqrt{d_{ij}} = 0$, but need not believe in only one outcome, i.e., consensus does not require $x_{i\theta}[k] = 1$ for $\forall i$ and some $\theta \in [1, \ldots, q]$. In contrast, *polarization* describes a process in which the society is asymptotically divided into non-interacting sub-groups (herds), each of which is internally in consensus with respect to the distance measure. More precisely, $s(\mathbf{x}_i, \mathbf{x}_j) = 0$ if i, j belong to the same sub-group. Finally, we say that a system devolves into *clusters* of opinions if there exist distinct groups of agents with *similar* opinions within the group, while the opinion distances between different clusters are relatively large. Hence, the opinion distances between agents in the same cluster need not be zero. Note that herding or clustering is in opinion space; it does not make any assumption about $\mathcal{G}$.

The dynamics of individual opinions clearly depend upon $\mu(d)$. We present here one specific construction of $\mu(d)$.

B. Hard-Interaction Model

The *hard-interaction* model considers a trust function that satisfies the following assumption.

Assumption 2: The trust function $\mu(d) \in [0, 0.5]$ is a non-increasing function of the squared opinion distance d.

The function $\mu(d)$ models the trust that may exist between like-minded agents. Given that the amount of trust one places in another reflects the amount of risk one is willing to take in social interactions, it is reasonable that as the difference between the opinions of two interacting agents increases, the trust $\mu(d)$ between the agents decreases. Simply put, agents interact more strongly with agents that have closer opinions. Note also that $\mu(d)$ is similar in spirit to the fixed mixing parameter $\bar{\mu}$ defined in [13], [14], [18], [20], [21], but here it is dependent on the opinions of the two agents. Hence, $\mu(d)$ is stochastic: its value depends on which two agents are communicating and on their opinion distance. The constraint $\mu(d) \le 0.5$ implies that an agent trusts its own prior opinion at least as much as that of the opinion of its neighbor. The technical reason will become apparent soon.

The hardness of the hard-interaction model stems from the existence of a threshold τ, such that if the squared opinion distance between interacting agents is larger than τ, then neither of the two agents updates its opinion.

Assumption 3: There exists a threshold $\tau : d \ge \tau \to \mu(d) = 0$ and $\mu(0)/\mu(\tau^-) \le \beta < \infty$.

This implies that agents will interact with their neighboring agents if and only if d is less than a given $\tau \in (0, d_{\text{sup}})$. Therefore, the system in (2) and (3) might not change at all after an opinion exchange, implying that the agents have no influence over each other when they are not like-minded[2]. For reasons that will soon become apparent (Lemma 6), even though our results are not restricted to the step-function in (1), the condition $\mu(0)/\mu(\tau^-) < \infty$ means that there must be a discontinuity of $\mu(d)$ at $d = \tau$ for the results to hold.

Assumption 4: The trust function $\mu(d)$ is concave and C^2-differentiable for $\forall d \in (0, \tau)$.

The regularity condition imposed in Assumption 4 is needed for analytical reasons, made clear in the next section.

[1]The phrase 'randomly selected by the network' is not meant to imply that the "network" chooses the initiating agent; rather this is a model for agents randomly getting activated. In earlier work, we had used the model of a common rate Poisson clock at each agent, which dictates when the agent initiates an interaction

[2]The system will not change if $\mathbf{x}_i[k] = \mathbf{x}_j[k]$, i.e., agents are already in agreement.

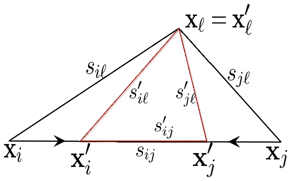

Fig. 1. Opinions of agents i, j and ℓ before and after the interaction of agents i, j, where $\mathbf{x}'$ and $s' = \sqrt{d'}$ denote the opinion and opinion distance after a generic update.

With all of the above in mind, it can be checked that the DW update in (1) is a special case of our proposed model in (2) and (3). Let $f(x)$ be the step function. It follows from (1) that the squared opinion distance after the update is given by

$$\begin{aligned} &|x_i[k+1] - x_j[k+1]|^2 \qquad (4)\\ &= \left[1 - 2\bar{\mu} f\left(\tau_0^2 - |x_i[k] - x_j[k]|^2\right)\right]^2 \cdot |x_i[k] - x_j[k]|^2. \end{aligned}$$

Let $\mathbf{x}_i = [x_i, 1-x_i]^T$ for $\forall i \in \mathcal{V}$. Then with $q = 2$, $A = I_2/2$, $\mu(d) = \bar{\mu}$ for $0 \le d < \tau_0^2$, our model in (2) and (3) reduces to the DW model in (1) in terms of the change in squared opinion distance. Note also that for the DW case, the concavity condition (Assumption 4) is met.

III. Opinion Dynamics

In this section, we study the asymptotic convergence property of the opinion dynamics under the modeling assumptions described in the previous section. Our aim is to provide analytical insights on what may happen as the initial scattered information is diffused in the network, with an emphasis on finding global conditions under which a society will converge to one opinion or it will polarize into isolated groups.

Suppose that the $(k+1)^{\text{th}}$ interaction is between agents i and j. It follows from (2) and (3) that

$$\begin{aligned} d_{ij}[k+1] &= \left[1 - 2\mu(d_{ij}[k])\right]^2 d_{ij}[k] \qquad (5)\\ &= d_{ij}[k] - 4\mu(d_{ij}[k])\left(1 - \mu(d_{ij}[k])\right) d_{ij}[k]. \end{aligned}$$

Thus the condition $|1 - 2\mu(d_{ij})| < 1$ (see Assumption 2) is needed to ensure that distances do not increase after an interaction. We call this decrease in d_{ij} the *private marginal benefit* that is caused by the interaction between agents i and j. Note that the condition $1/2 \le \mu(d_{ij}) \le 1$ also leads to a decrease in d_{ij}, but the "ordering" of the opinions $\mathbf{x}_i$ and $\mathbf{x}_j$ reverses after the interaction, implying that the agent puts more trust in the prior opinion of its neighbor than in its own.

Because of the triangular inequality (see Fig. 1), if opinions of the interacting agents (i, j) are moving closer, then the perimeter of the triangle with vertices i, j and any other point ℓ shrinks after the update[3]. In other words, the sum of opinion distances can only shrink after an interaction; any interaction between the pair $(i, j) \in \mathcal{E}$ also changes the opinion distances between these two nodes and their neighbors. This creates a network effect called *network externality*.

Now consider the dynamic effects that arise from network externality. Since opinions of the non-interacting agents do not change after the interaction, i.e., $x_\ell[k+1] = x_\ell[k]$ for $\forall \ell \ne i, j$, a simple manipulation of (2) and (3) yields

$$\begin{aligned} d_{i\ell}[k+1] &= d_{i\ell}[k] + \mu^2(d_{ij}[k]) d_{ij}[k]\\ &\quad + 2\mu(d_{ij}[k])(\mathbf{x}_i[k] - \mathbf{x}_\ell[k])^T A(\mathbf{x}_j[k] - \mathbf{x}_i[k]) \qquad (6)\\ d_{j\ell}[k+1] &= d_{j\ell}[k] + \mu^2(d_{ij}[k]) d_{ij}[k]\\ &\quad + 2\mu(d_{ij}[k])(\mathbf{x}_j[k] - \mathbf{x}_\ell[k])^T A(\mathbf{x}_i[k] - \mathbf{x}_j[k]). \qquad (7) \end{aligned}$$

Let $D^{ij}[k+1] := \sum_{r=1}^{n} \sum_{m=r+1}^{n} d_{rm}[k+1]$ be the sum of squared distances over all possible pairs of agents in the network, given that agents i and j interacted at $k+1$. Let $D[k] = \sum_{r=1}^{n} \sum_{m=r+1}^{n} d_{rm}[k]$ be the sum of squared distances at time k which is prior to the interaction. Thus $D[k]$ is a measure of the disparity of opinion in the society. The change in D after the interaction equals (see Appendix)

$$D^{ij}[k+1] - D[k] = -2n\mu(d_{ij}[k])\left[1 - \mu(d_{ij}[k])\right] d_{ij}[k]. \qquad (8)$$

Notice that the change in the overall sum of squared distances depends entirely on $d_{ij}[k]$ between the two interacting agents and it is $n/2$ times as large as the change in $d_{ij}[k]$ of the interacting pair (see Eqn. (5)).

A. ODE Approximation of the Distance Dynamic

Agents in the network interact at random: at each time, agent i is selected with nonzero probability p_i to initiate an interaction and with probability P_{ij}, agent i will then choose to interact with agent j. Let $\overline{\mathbf{P}} \in \mathbb{R}^{n \times n}$ be the matrix of probabilities of the pair (i, j) performing any exchange, *i.e.*, $\overline{P}_{ij} = p_i P_{ij} + p_j P_{ji}$. The first term corresponds to the case of agent i initiating an interaction with agent j and the second to that the probability of agent j initiating the interaction with agent i. By Assumption 1, $\overline{\mathbf{P}} = \left[\overline{P}_{ij}\right]$ has the same sparsity as $\mathcal{G}$, *i.e.*, $\overline{P}_{ij} > 0 \leftrightarrow (i, j) \in \mathcal{E}$. Hence, given that $\mathcal{G}$ is connected, information can be propagated directly or indirectly from any agent $\mathcal{G}$ to any other agent with nonzero probability.

Let $f_k(d)$ be the distribution of d between any pair of agents at time k. For n sufficiently large, the conditional expectation of $d[k+1]$ with respect to $f_{k+1}(d)$ can be approximated by the sample mean and we have the following relation: $\mathbb{E}\{d[k+1]\} = \sum_{(i,j)\in\mathcal{E}} \overline{P}_{ij}\, \mathbb{E}\left\{d[k+1] \,|\, | \,(i,j) \text{ interacts}\right\} \approx \sum_{(i,j)\in\mathcal{E}} \overline{P}_{ij} \frac{D^{ij}[k+1]}{n(n-1)/2}$. With this sample mean approximation and the relation in (8), we have

$$\begin{aligned} &\mathbb{E}\left\{d[k+1]\right\} - \mathbb{E}\left\{d[k]\right\}\\ &\approx \sum_{(i,j)\mathcal{E}} \overline{P}_{ij} \frac{D^{ij}[k+1] - D[k]}{n(n-1)/2}\\ &= -\frac{4}{n-1} \sum_{(i,j)\in\mathcal{E}} \overline{P}_{ij} \mu(d_{ij}[k])\left[1 - \mu(d_{ij}[k])\right] d_{ij}[k]. \end{aligned}$$

Our general approach to studying the asymptotic behavior of opinion dynamics here is to map the difference equation onto an ordinary differential equation using an Euler-type approximation, i.e., the derivative of a continuous function $d(t)$ equals $\dot{d}(t) \approx (d[k+1] - d[k])/h$ where $h > 0$ denotes the discretization step size and $d[k]$ is the value of $d(t)$ at

[3]Of course, this does not imply that the opinion distances of ℓ to i and j both shrink. It is quite possible that one (but not both) of the interacting agents moves away from ℓ.

$t = kh$. Using Euler's approximation and setting $h = \frac{4}{n-1}$, the following *ordinary differential equation* (ODE) can be derived

$$\begin{aligned}\dot{\overline{d}}(t) &= -\sum_{(i,j)\in\mathcal{E}} \overline{P}_{ij} \underbrace{\mu(d_{ij}(t))\left[1-\mu(d_{ij}(t))\right]}_{:=\rho(d_{ij}(t))} d_{ij}(t) \qquad (9)\\ &= -\sum_{(i,j)\in\mathcal{E}} \overline{P}_{ij}\rho(d_{ij}(t))d_{ij}(t).\end{aligned}$$

where $\overline{d}(t) = \mathbb{E}\{d(t)\}$ denotes the mean of the average distribution $f_t(d)$. For convenience, we do not explicitly show the time variable t whenever it does not cause confusion in the rest of the paper.

Taking the expectation on both sides of Eqn. (9) with respect to the distribution $f_t(d)$, we get

$$\begin{aligned}\dot{\overline{d}} &= -\sum_{(i,j)\in\mathcal{E}} \mathbb{E}\left\{\overline{P}_{ij}\rho(d_{ij})d_{ij}\right\} \qquad (10)\\ &= -\left(\sum_{(i,j)\in\mathcal{E}} \overline{P}_{ij}\right)\mathbb{E}\left\{\rho(d_{ij})d_{ij}\right\} = -\,\mathbb{E}\left\{\rho(d_{ij})d_{ij}\right\}.\end{aligned}$$

The second equality holds true because it follows from Assumption 1 that given a network $\mathcal{G}$, the pairwise communication rates $\overline{P}_{ij}$ are fixed prior to any interaction. We are now ready to develop the main results regarding opinion dynamics in our model.

B. System Analysis and Results

The next property gives a lower bound on the dynamic of the expected squared distance $\overline{d}$.

Lemma 5: Under Assumptions 2 – 4, the dynamic in (10) is lower bounded by $\dot{\overline{d}} \geq -\beta\rho(\overline{d})\overline{d}$.

Proof: See Appendix. ■

It is to be noted that Assumptions 2– 4 allow us to use the expected squared distance $\overline{d}(t)$ as a surrogate for studying the evolution of total disparity $D(t)$ between agents' opinions. From Lemma 5, it is expected that (10) will not converge if the lower bound does not converge. For convenience, we express the dynamic of the lower bound system by $\dot{b} = -\beta\rho(b)b$. Indeed, the ODE of b locally resembles the form of the logistic equation, as initially investigated in [22]. However, the model in this paper includes network externalities that were not accounted for in [22] and hence analysis is quite different.

Lemma 6: Under Assumptions 2 and 4, the system $\dot{b} = -\beta\rho(b)b$ converges if and only if $\tau > b(0)$.

Proof: See Appendix. ■

Using Lemma 6, a necessary condition for (10) to converge under the hard-interaction model can be established and it generalizes the result in [22] to the model with externalities as defined in this paper.

Lemma 7: Under Assumptions 1 – 4, when the number of agents n is sufficiently large, a necessary condition for the interaction model in (2) and (3) in a connected network to converge almost surely is $\tau > \overline{d}[0]$.

Lemma 7 indicates that if the system converges almost surely, then the threshold must be above $\overline{d}[0]$, the average initial squared distance. If $\tau < \overline{d}[0]$, then agents will remain polarized with a positive probability. One way to interpret the threshold value τ is that it represents how open-minded a society is. If it is open-minded enough (relative to this initial dissonance in opinions), then the system will converge. This also implies the existence of a phase transition τ_{pt} from polarization to consensus for τ_{pt} sufficiently above $\overline{d}[0]$. However, we will show next that the necessary condition developed in Lemma 7 is not a sufficient condition.

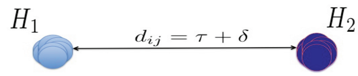

Fig. 2. Group polarization

Let there be two groups of opinions H_1 and H_2 in a society (See Fig. 2). Within each group, the opinions are in consensus, but for $\forall(i,j) : i \in H_1, j \in H_2$, the distance $d_{ij} = \tau + \epsilon$ is above the threshold. Let $\mathcal{E}_{12} = \{(i,j) \mid i \in H_1 \text{ and } j \in H_2\}$. Since the two groups do not communicate (as the trust function $\mu(d_{ij}) = 0\ \forall(i,j) : i \in H_1, j \in H_2$), the agents cannot come to a consensus. However, if the number of intergroup pairs of agents normalized by the total number of pairs is $\frac{|\mathcal{E}_{12}|}{n(n-1)/2} < \frac{\tau}{\tau+\delta}$, then $\overline{d}[0] = (\tau+\sigma)\frac{|\mathcal{E}_{12}|}{n(n-1)/2} < \tau$ and thus $\mu\left(\overline{d}[0]\right) > 0$. This contradicts the assumption that the two group do not communicate. Hence, $\tau > \overline{d}[0]$ is not a sufficient condition for consensus.

A final remark pertains to the effect of Assumption 1 on the network dynamics. Recall that in the derivation of (10), we have shown that the rate of interaction $\overline{P}_{ij}$ does not affect the dynamic of the mean $\overline{d}(t)$ when $\overline{P}_{ij}$ is fixed (as required in Assumption 1). If this is the case, then for n sufficiently large, there is no graph selection that can mitigate the effect of a large initial average distance $\overline{d}[0]$ (relative to the threshold) since the necessary requirement for consensus in Lemma 7 does not change. Then the natural question to ask is that what asymptotic opinion profile can be observed if P_{ij} is time-varying. We will examine this in the following section.

IV. Strategic Opinion Formation

While the interaction model discussed in Section III studies the opinion dynamics under random interactions with a fixed rate of communication, there are many social settings in which agents dynamically determine how much effort and time they want to consume in interacting with their neighbors. A *strategic model of interaction* for an agent is the process of defining a goal through a utility function and allocating resources (i.e., time and energy) for interactions with its neighbors to maximize the utility function. In this model, we contemplate the possibility of agents not wanting to interact at all with their peers and, therefore, we allow the sparsity pattern of $\overline{\mathbf{P}}$ to be different from that of $\mathcal{G}$ and the communication rates $\overline{P}_{ij}$ for $\forall(i,j) \in \mathcal{E}$ to vary with time.

There are two steps in modeling strategic interactions for a network of agents. First, one needs to explicitly model the cost and benefit that each agent receives as a result of a particular action, which define the incentives agents have to interact more or less often with neighboring agents. In this case, it is assumed that the initiator of an interaction knows the opinions of all its neighbors, before it starts the interaction.

Second, the strategic model should be tractable so that it can provide insights or predictions on how individual incentives affect the aggregation of information and the formation of the asymptotic opinion profile.

A. Strategic Interaction Model

As highlighted earlier, the utility function of an agent should reflects its goal for interactions. The mathematical description of goal-attainment thus involves the cost and benefit for participating in a particular interaction. We define the following utility function: for some $\kappa > 1$,

$$U_i([P_{ij}]_{j\in\mathcal{N}_i}) = \sum_{j\in\mathcal{N}_i} P_{ij}\left[r(d_{ij}) - \xi(d_{ij})\right] - P_{ij}^{\kappa} \tag{11}$$

where $r(d_{ij})$ is the benefit or reward agent i receives from interacting with agent j and $\xi(d_{ij})$ represents the cost of interacting with agent j. Both the cost and reward are functions of the squared opinion distance. Note also that *without* the second term P_{ij}^{κ}, maximizing the utility function is equivalent to assigning $P_{ij} = 1$ to agent j that gives the *highest* positive profit, measured by the reward received minus the cost. It is however, important to emphasize that the focus of the strategic interaction is not to model the agents as selfish individuals who calculate their overall profits from each potential interaction, but rather to study their tendency to initiate interactions that are beneficial to them. This tendency is measured by P_{ij} and the term P_{ij}^{κ} in (11) should be interpreted as the force behind such incentives. The mathematical formulation of the utility function thus allows us to translate individual's tendency to interact into a tractable process that can potentially provide answers to why herding behavior occurs.

Recall that the underlying assumption is that agents do seek to arrive at consensus, but subject to the interaction rules. Also recall from Eqn. (8) that the private marginal benefit (5) enjoyed by the interacting pair (i, j) is amplified $n/2$ times (n is the number of nodes) in terms of the network-wide average squared distance. Motivated by this, we set the reward function to be proportional to this private marginal benefit, i.e.,

$$r(d_{ij}) = \alpha\rho(d_{ij})d_{ij},$$

where $\rho(d)$ is defined in (9) and α is called the *reward coefficient*.

Let us now consider the cost of interaction $\xi(d_{ij})$, which should specify the energy for communicating with agent j whose opinion is $\sqrt{d_{ij}}$ away from agent i. Here we present two interesting constructions of the energy function $\xi(d_{ij})$:

(i) $\xi(d_{ij}) = \gamma_1 d_{ij}$ if $d_{ij} < \tau$ and $\xi(d_{ij}) = +\infty$ if $d_{ij} \geq \tau$

(ii) $\xi(d_{ij}) = \gamma_2\mu^2(d_{ij})d_{ij}$ if $d_{ij} < \tau$ and $\xi(d_{ij}) = +\infty$ if $d_{ij} \geq \tau$

where γ_1 and γ_2 are called the *cost coefficients* associated with the two energy functions. The first construction of the energy function assumes that, if the (squared) opinion distance d_{ij} is large, then agent i consumes more energy to communicate to j than when d_{ij} is small. The notion here is that it will take more effort to dialogue with someone farther away in opinion and convince them to move their opinion profile closer. Rather than choosing a linear cost function, we could have chosen a function of the form $\xi(d_{ij}) = \gamma_1 f(d_{ij})d_{ij}$ where $f(d)$ is any positive non-decreasing function of d. As will become evident from the analysis in the next section, the choice $f(d) = 1$ does not lead to any loss of generality. In contrast, the second construction of the energy function captures the overall cost of opinion changes for both agents. Note that given $d_{ij} < \tau$, the amount of energy spent in interacting with agent j is proportional to agent j's change in opinion, i.e., $d(x_j[k], x_j[k+1]) = \mu^2(d_{ij}[k])d_{ij}[k]$, which also equals agent i's change in opinion $d(x_i[k], x_i[k+1])$. In both cases, agents assign infinite energy for interactions with neighbors who are not like-minded. Each of the two cases will lead to different opinion formation processes, as we shall see shortly.

Agents interact with each other according to the following strategic model. An agent is randomly activated at each time instant with equal probability, i.e., $p_i = 1/n$ for $\forall i \in \mathcal{V}$. If agent i is selected, it needs to make a strategic decision on an interaction pattern for selecting neighbors. The decision is in the form of probabilities P_{ij} obtained by maximizing the utility function defined in (11). Then based on the decision, agent i interacts with one of the neighbors in $\mathcal{N}_i$, and this event is followed by the opinion update rule given in (2) and (3). At the next time instant, the same procedure is applied.

B. System Analysis

In this section, we explore how the individual incentives for interactions affect the opinion formation process. Suppose that agent i is chosen at time k. Recall that the first step in the strategic interaction model is to determine P_{ij} for $\forall j \in \mathcal{N}_i$ maximizing the utility function, i.e.,

$$\max_{P_{ij}} \sum_{j\in\mathcal{N}_i} P_{ij}\left[r(d_{ij}[k]) - \xi(d_{ij}[k])\right] - P_{ij}^{\kappa}$$

under the constraint that $\sum_{j\in\mathcal{N}_i\cup i} P_{ij} = 1$ and $P_{ij} \geq 0$. Let $[a]^+ = a$ if $a > 0$, and 0 otherwise. Solving the above optimization with respect to P_{ij} yields, for $\forall j \in \mathcal{N}_i$

$$P_{ij}[k] = \frac{1}{S_i}\left(\left[r(d_{ij}[k]) - \xi(d_{ij}[k])\right]^+\right)^{\frac{1}{\kappa-1}} \tag{12}$$

where the scaling factor S_i is defined as follows: let $c = \sum_{m\in\mathcal{N}_i}\left(\left[r(d_{im}[k]) - \xi(d_{im}[k])\right]^+\right)^{\frac{1}{\kappa-1}}$,

$$S_i := \begin{cases} c & \text{if } c \neq 0 \\ 1 & \text{otherwise.} \end{cases} \tag{13}$$

In the case that $P_{ij}[k] = 0$ for $\forall j \in \mathcal{N}_i$, we have $P_{ii}[k] = 1$ and $P_{ii}[k] = 0$ otherwise. That is, if interacting with any of its neighbors will bring zero or negative net benefit to agent i, then agent i will choose to be a recluse for the moment.

1) **Case Study (i):** Consider the case when the energy function is defined as $\xi(d) = \gamma_1 d$ for $d < \tau$ and $+\infty$ otherwise. Note that under Assumption 2, the term $\rho(d) = \mu(d)(1 - \mu(d))$ is a non-increasing function of d, with maximum value $\rho(0)$. When $d_{ij} < \tau$, replacing $\xi(d_{ij})$ with $\gamma_1 d_{ij}$ in (12), we observe that, if the ratio γ_1/α of the cost coefficient to the reward coefficient is greater than or equal to the product $\rho(0)$, then

$$\frac{\gamma_1}{\alpha} \geq \rho(d_{ij}) \tag{14}$$

for $\forall d_{ij}$. This situation implies that the agents have insufficient incentives to interact with each other. In this case, it follows from equation (12) that $P_{ij} = 0$ for $\forall j \in \mathcal{N}_i$, given that $d_{ij} < \tau$. Thus agents always choose to not interact with the other agents and remain inactive, i.e., $P_{ii}[k] = 1$ for $\forall k$. As a result from (14) and the fact that $P_{ij} = 0$ whenever $d_{ij} \geq \tau$, the system will not evolve over time and the initial opinion profile is the one and only fixed point. For a more interesting scenario, we have the following assumption.

Assumption 8: The reward and cost coefficients satisfy the relation $\frac{\gamma_1}{\alpha} < \rho(0)$.

Under the strategic model, it follows from equation (12) and Assumption 8 that the rate of interaction P_{ij} changes over time, depending on the relative distances between agent i and its neighbors. If the cost of interacting with an agent exceeds the benefit, then the agent who is to initiate an interaction will impose a zero rate of interaction with that agent. On the contrary, more probability weight will be put on the neighbors yielding higher (positive) utilities. Hence, the probability distribution of pairwise interactions $\overline{P}_{ij} = (P_{ij} + P_{ji})/n$ is also dependent on the opinion distances between agents.

It follows from (9) and (12) that the dynamic of $\overline{d}$ equals

$$\dot{\overline{d}} = -\frac{1}{n} \sum_{(i,j)\in\mathcal{E}} \left(\frac{1}{S_i} + \frac{1}{S_j}\right) \eta(d_{ij})^{\frac{1}{\kappa-1}} \rho(d_{ij}) d_{ij}^2, \qquad (15)$$

where $\eta(d) = [\alpha\rho(d) - \gamma_1]^+$. From (15), one can clearly see that the system stops evolving (i.e., $\dot{\overline{d}} = 0$) if d_{ij} for $\forall (i,j) \in \mathcal{E}$ satisfies one of the two conditions: (i) $d_{ij} = 0$; (ii) $\eta(d_{ij}) = 0$. The first condition is satisfied if the interacting agents are in consensus. The second condition implies that agents will not interact if their squared opinion distance d_{ij} lies in the union $\mathcal{D}_1 \cup [\tau, d_{\sup})$ where

$$\mathcal{D}_1 = \left\{ d \in (0,\tau) \mid \rho(d) \leq \frac{\gamma_1}{\alpha} \right\}. \qquad (16)$$

Note that $\mathcal{D}_1$ is an empty set when the ratio $\gamma_1/\alpha < \rho(\tau^-)$. In this case, agents will not update their opinions only if $d_{ij} > \tau$ for $\forall (i,j) \in \mathcal{E}$. Under this situation, the network follows the same opinion update rule as in the hard-interaction model with a threshold τ, but it allows the rate of interaction P_{ij} to vary with agent opinion profiles.

On the other hand, as shown in Fig. 3, when $\gamma_1/\alpha \geq \rho(\tau^-)$, the set is nonempty and the threshold $\tau > \inf(\mathcal{D}_1)$, where inf denotes the infimum. Since $\rho(d)$ is a non-increasing function of d, the range of the set $\mathcal{D}_1$ goes from $\inf(\mathcal{D}_1)$ to τ. Clearly, $d_{ij} \geq \inf(\mathcal{D}_1)$ implies that the associated agents will not update their opinions. The opinion diffusion in this case will not converge to a consensus not only because the agents may not be sufficiently like-minded (i.e., $d_{ij} \geq \tau$), but also because γ_1/α is too big to warrant sufficient incentives between pairs of like-minded agents whose (squared) opinion distances lie in the interval $d_{ij} \in [\inf(\mathcal{D}_1), \tau)$.

2) **Case Study (ii):** We now examine the case when the energy function is defined as $\xi(d) = \gamma_2\mu^2(d)d$ for $d < \tau$ and $+\infty$ otherwise. Under Assumption 2, it can be shown that the expression $1/\mu(d) - 1$ is a non-decreasing function of d, with a minimum value $1/\mu(0) - 1$. Hence, if the ratio γ_2/α is less

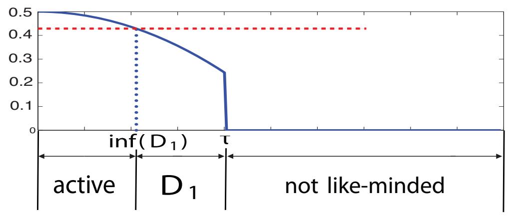

Fig. 3. The solid line represents the trust function $\mu(d)$ and the vertical line maps the value of μ such that $\gamma_1/\alpha = \rho(d)$ to the point $d = \inf(\mathcal{D}_1)$. The "active" region means that agents i and j will interact if d_{ij} falls in this region.

than $1/\mu(0) - 1$, then

$$\frac{\gamma_2}{\alpha} < \frac{1}{\mu(d_{ij})} - 1$$

for $\forall d_{ij}$. When $d_{ij} < \tau$, replacing $\xi(d_{ij})$ with $\gamma_2\mu^2(d_{ij})d_{ij}$ in equation (12), observe that if the preceding relation is true, then $P_{ij} = 0$ for $\forall j \in \{m \in \mathcal{N}_i \mid d_{im} < \tau\}$. This result, together with the fact that $P_{ij} = 0$ whenever $d_{ij} \geq \tau$, implies that the initial opinion profile of the system is the one and only fixed point because none of the agents is willing to interact with its neighbors. Hence, we make the following assumption.

Assumption 9: The reward cost coefficients satisfy the relation $\frac{\gamma_2}{\alpha} \geq \frac{1}{\mu(0)} - 1$.

We now consider the opinion evolution of a system satisfying Assumption 9. Since $\xi(d) = \gamma_2\mu^2(d)d$ for $d < \tau$, the dynamic of the expected squared distance $\dot{\overline{d}}$ has the same expression as (15) except that $\eta(d_{ij}) = \left[\alpha\mu(d_{ij}) - (\alpha+\gamma_2)\mu^2(d_{ij})\right]^+$. As mentioned before, the system reaches a fixed point if d_{ij} for $\forall (i,j) \in \mathcal{E}$ satisfies one of the two conditions: (i) $d_{ij} = 0$; (ii) $\eta(d_{ij}) = 0$. The second condition indicates that agents will not interact if d_{ij} lies in the union $\mathcal{D}_2 \cup [\tau, d_{\sup})$ where

$$\mathcal{D}_2 = \left\{ d \in (0,\tau) \;\middle|\; \frac{1}{\mu(d)} - 1 \geq \frac{\gamma_2}{\alpha} \right\}. \qquad (17)$$

Or equivalently, $d_{ij} \in (0, \sup(\mathcal{D}_2)] \cup [0, d_{\sup})$ since $1/\mu(d) - 1$ is a non-decreasing function of d.

Clearly, if the threshold $\tau \leq \sup(\mathcal{D}_2)$ is small relative to the supremum of the set $\mathcal{D}_2$, then agents in the network will not update their opinions because either they are too closed-minded to the opinions of the others or they do not have sufficient incentives to interact. In contrast, as depicted in Fig. 4, if the threshold $\tau \gg \sup(\mathcal{D}_2)$, agents will interact if d_{ij} lies in the open interval $(\sup(\mathcal{D}_2), \tau)$. Hence, it can be deduced that the system will form one or multiple opinion clusters. Within each cluster, the squared opinion distances are upper bounded by $\sup(\mathcal{D}_2)$. Between the clusters, the squared opinion distances are lower bounded by the threshold τ. Comparing Figures 3 and 4, we observe that the two cost functions lead to distinct active regions (under appropriate conditions on the cost-reward coefficients): one case can lead to consensus; the other case leads to clustering.

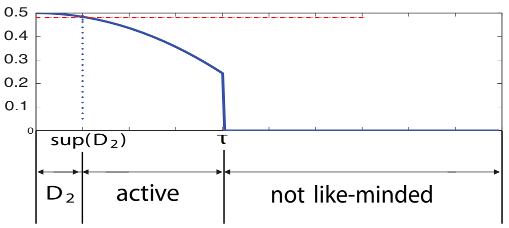

Fig. 4. The solid line represents the trust function $\mu(d)$ and the vertical line maps the value of μ such that $\gamma_2/\alpha = 1/\mu(d) - 1$ to the point $d = \sup(\mathcal{D}_2)$.

V. Numerical Results

In this section, we first numerically validate the analytical results of both the hard-interaction model (Sec. V-A) and the strategic interaction model (Sec. V-B) using synthetic data. Then we offer evidence that the proposed model for opinion evolution could portray real phenomena using the Social Evolution dataset [23] (see Sec. V-C).

A. Hard-Interaction Model

Recall that the choice of the underlying communication network $\mathcal{G}$ is arbitrary and the analytical results hold for any connected network. We start by generating $\mathcal{G}$ using a random geometric graph (RGG), i.e., $\mathcal{G} = \mathcal{G}(n, r)$, which consists of n randomly distributed social agents over an unit disk with a radius of communication r. While $r = 1$ implies a fully connected network, $r = 0$ represents a completely disconnected network. Each initial opinion profile $\mathbf{x}_i[0]$ for $\forall i \in \mathcal{V}$ is uniformly distributed in the opinion space $\mathcal{X}$ with $q = 3$ possible decision states. Without loss of generality, the ℓ_2-norm is used to measure the opinion distance between agents, i.e., $s(\mathbf{x}_i, \mathbf{x}_j) = \|\mathbf{x}_i - \mathbf{x}_j\|_A$ with $A = I_q$.

For the hard-interaction model, we set the trust function to be $\mu(d) = 0.4$ for $d < \tau$ and 0 otherwise. Three connected RGG graphs $\mathcal{G}(n, 0.8)$ are generated with $n = 50, 100, 200$ number of agents and are kept fixed for all trials. We define a subgraph $\mathcal{G}_{\text{eff}}[k] = (\mathcal{V}, \mathcal{E}_{\text{eff}}[k])$ of $\mathcal{G}$ at each time instant, where $\mathcal{E}_{\text{eff}}[k] = \{(i, j) \in \mathcal{E} \mid d_{ij}[k] < \tau\}$. A normalized algebraic connectivity of the subgraph $\mathcal{G}_{\text{eff}}[k]$ is defined as the algebraic connectivity of $\mathcal{G}_{\text{eff}}[k]$ divided by the algebraic connectivity of $\mathcal{G}$. For k sufficiently large, the normalized algebraic connectivity approaches either 0 or 1. A value of 0 indicates that $\mathcal{G}_{\text{eff}}$ is disconnected; A value of 1 indicates that $\mathcal{E}_{\text{eff}}[k] = \mathcal{E}$ and moreover, $d_{ij}[k] \approx 0$ for $\forall (i, j) \in \mathcal{E}$, implying that consensus is reached. The top panel in Fig. 5 shows the (normalized) algebraic connectivity of the subgraph $\mathcal{G}_{\text{eff}}[k]$, where k is sufficiently large, averaged over 300 realizations for various values of τ. Each realization starts with an uniformly distributed initial opinion profile. Uniform communication rate (*i.e.*, $p_i = 1/n$ and $P_{ij} = 1/(n-1)$) is used. Observe that as τ approaches 0.64, society reaches a consensus almost surely and thus a phase transition occurs approximately at $\tau = 0.64$. Moreover, as n increases, the phase transition boundary becomes sharper; the normalized algebraic connectivity decreases at $\tau = \bar{d}[0]$. The bottom panel in Fig. 5 shows the corresponding histograms of the 300 asymptotic

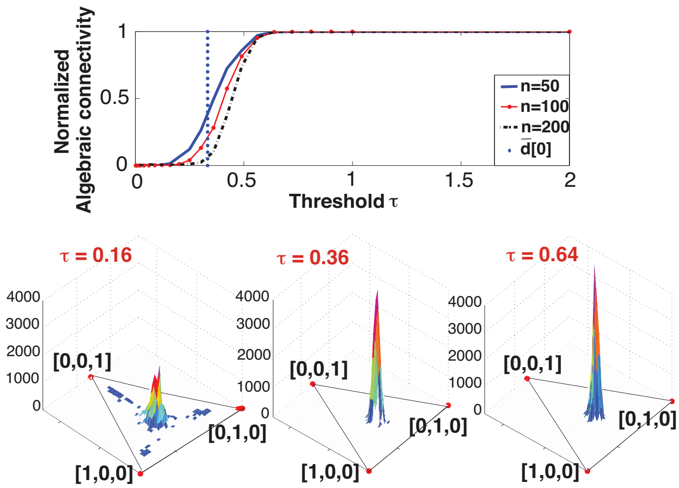

Fig. 5. Phase transition (top) of a society from polarized beliefs to a consistent belief. Histogram (bottom) of 300 asymptotic opinion profiles with $n = 100$ at $\tau = 0.16, 0.36, 0.64$.

opinion profiles at $\tau = 0.16, 0.36, 0.64$, respectively. Notice the evidence of fragmentation or clustering at smaller values of τ, and consensus at larger values.

Fig. 6 compares the phase transitions for three different network configurations: RGG, Erdos-Renyi graph and small-world graph. Specifically, $n = 50$ agents are generated for each of the graphs. The radius of communication r in the RGG is set to be 0.8. The Erdos-Renyi graph [25] is generated uniformly at random from the collection of all graphs which have $n = 50$ nodes and $M = 120$ edges. The small-world graph [26] is generated by setting $p_{\text{sw}} = 1$, $q_{\text{sw}} = 1$ and $\alpha_{\text{sw}} = 2$, where q_{sw} denotes the maximum distance of short-range connections, q_{sw} represents the number of random connections to add per node and α_{sw} is the clustering exponent. All of the three graphs used in the simulation are connected. Uniform rate of communication $\overline{P}_{ij}$ is adopted. The (normalized) algebraic connectivity of $\mathcal{G}_{\text{eff}}[k]$ for each graph is averaged over 400 realizations, each of which starts with an uniformly distributed initial opinion profile. Observe from Fig. 6 that the phase transitions almost overlap with one anther. As discussed towards the end of Section III, specific choices of the fixed graph ($\overline{P}$) do not impact the dynamics

B. Strategic Interaction Model

For the strategic interaction model, we set $\kappa = 2$ and define the trust function to be $\mu(d) = 0.5 - 0.4d^2$ for $\forall d < \tau$ as shown in Fig. 7 for different values of τ, i.e., $\tau = 0.25, 0.64, 0.81$. The underlying communication graph is a RGG graph with $n = 50$ and $r = 0.8$. The initial opinion profiles for the individual agents are generated uniformly over the opinion space $\mathcal{X}$ with $q = 3$. Opinion distances between agents are measured by the ℓ_2-norm.

1) **Case Study (i):** Consider the subgraph $\mathcal{G}_{\text{eff}}[k] = (\mathcal{V}, \mathcal{E}_{\text{eff}}[k])$ of $\mathcal{G}$ at time k, where $\mathcal{E}_{\text{eff}}[k]$ contains all the edges $(i, j) \in \mathcal{E}$ whose corresponding distances $d_{ij}[k]$ are such that $d_{ij}[k] < \tau$ if $\mathcal{D}_1 = \emptyset$ and $d_{ij} < \inf(\mathcal{D}_1)$ otherwise. Fig. 8 shows the (normalized) algebraic connectivity of the graph $\mathcal{G}_{\text{eff}}[k]$ for k sufficiently large. The plot is averaged over

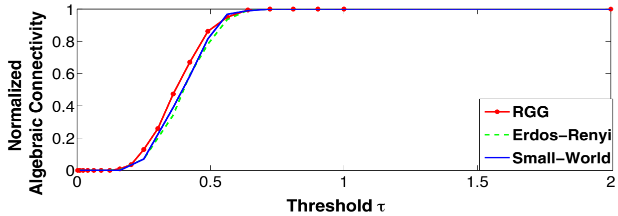

Fig. 6. Phase transitions using three different random graphs, $n = 50$.

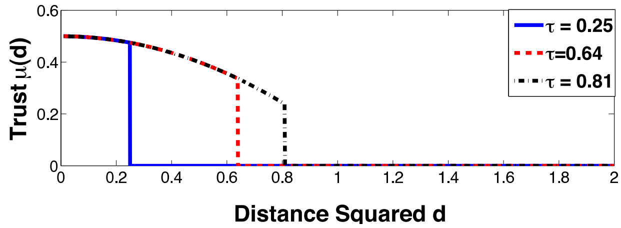

Fig. 7. Trust Function with $\tau = 0.25, 0.64, 0.81$

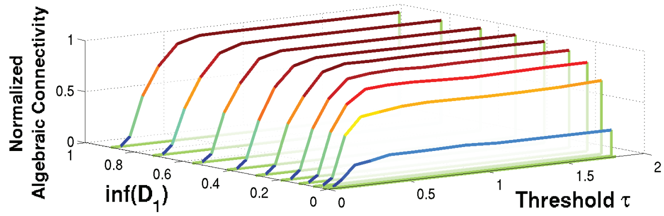

Fig. 8. Phase Transitions for different values of $\inf(\mathcal{D}_1)$

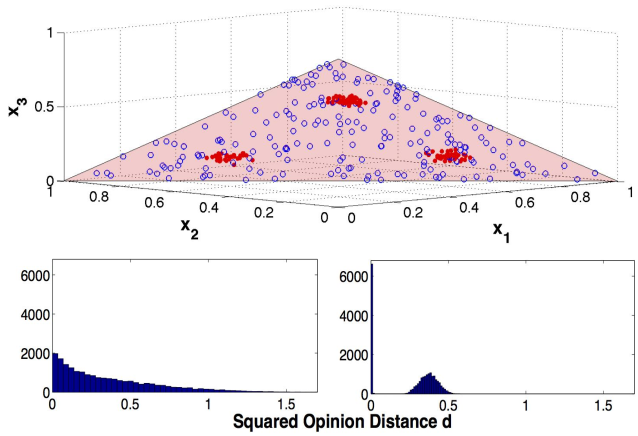

Fig. 9. Final opinion landscape (top), initial (bottom left) and final (bottom right) squared distance distributions, with $\tau = 0.09$ and $\sup(\mathcal{D}_2) = 0.0158$.

400 realizations for different values of τ and $\inf(\mathcal{D}_1)$. Each realization starts with an uniformly distributed initial opinion profile. Observe that when $\inf(\mathcal{D}_1)$ is small, i.e., γ_1/α is large, the agents are less likely to reach a consensus for any value of τ. In contrast, when $\inf(\mathcal{D}_1)$ is large, i.e., $\inf(\mathcal{D}_1) > 0.64$ approximately, the society tends to form a convergent opinion almost surely for large values of τ (approximately above 0.64).

2) **Case Study (ii):** Fig. 9 shows the final outcome of the interactions (top panel) and the squared distance distribution (bottom panel) with $\tau = 0.09$ and $\sup(\mathcal{D}_2) = 0.0158$ (i.e., $\gamma_2/\alpha = 4.0016$). Observe from the top panel that three opinion clusters are formed. Within each cluster, the (squared) opinion distances are upper bounded by $\sup(\mathcal{D}_2) = 0.0158$, as shown in the bottom right panel of Fig. 9. Also, the (squared) distances between clusters are at least 0.18, which is much larger than the threshold $\tau = 0.09$. Fig. 10 shows the final outcome of the interactions (top panel) and the squared distance distribution (bottom panel) when $\tau = 0.64$ and $\sup(\mathcal{D}_2) = 0.0158$. In this case, agents form a single opinion cluster, as shown in the top panel; the squared distances within this cluster are computed to be upper bounded by 0.0156, which is less than $\sup(\mathcal{D}_2)$. Fig. 11 shows the time evolution of the expected squared opinion distance $\overline{d}[k]$ as k increases for the two scenarios. In both cases, $\overline{d}$ decreases until it reaches an equilibrium point.

C. Social Evolution Dataset

To demonstrate the validity of the proposed interaction models, we use data collected in the Social Evolution experiment [23]. Specifically, the experiment monitored more than 80% of the residents in a dormitory, with a population of approximately 30 freshmen, 20 sophomores, 10 juniors, 10 seniors and 10 graduate student tutors. Interactions between individuals were tracked by their proximity, location, SMS and call records. Surveys were conducted monthly on social relationships, health-related habits, on-campus activities, political views and common cold symptoms. The friendship or communication network is connected; the exact network graph, although not significant to this research, is provided in [27] and we assume it stays constant. While many insights can be gleaned from an analysis of the full data, for this research, the data that we choose to exploit are the location data and the survey data on health-related habits.

The location data are collected through a mobile phone application that scans the wireless local area network (WLAN) access points within a certain range. We use the location data to study how social interactions shape individuals preferences in terms of how frequently they visit a place. There are 32724 identified WLAN access points; we choose to use the top 100 most frequently visited access points (approximately 10 locations) to extract individual opinions. The opinion space $\mathcal{X}$ is of dimension $q = 101$, where the first 100 states correspond to the top 100 frequently visited WLAN access points and the last state represents all other locations. Opinions are estimated once every four weeks. Specifically, we assume a Binomial distribution for visiting an access point with n_{ac} being the total number of places a student could visit in four weeks. We count the total number of times a student visited one of the 100 access points. Then the probability of visiting each place is the maximum likelihood estimator (MSE) of the Binomial distribution, given the actual count. The relative number of visits to different location is an indication of the importance of the locations. A question of interest is how this relative importance evolves over time, and whether this stabilizes. Opinion distances are measured by the ℓ_2-norm with $A = I_q$. Although the distance calculations are carried out with $q = 101$, it is important to note that the intrinsic dimensionality of the opinion space $\mathcal{X}$ can be estimated by $n_{\text{loc}} + 1$ where n_{loc} is the number of locations with which

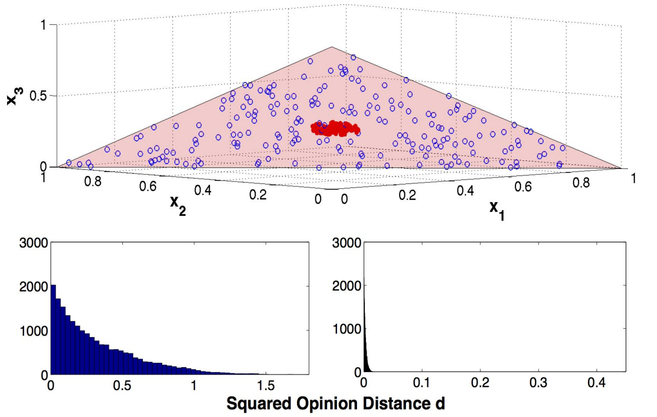

Fig. 10. Final opinion landscape (top), initial (bottom left) and final (bottom right) squared distance distributions, with $\tau = 0.64$ and $\sup(\mathcal{D}_2) = 0.0158$.

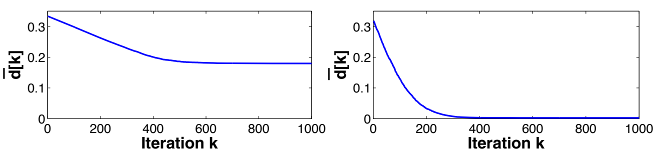

Fig. 11. Dynamics of expected (squared) distance $\overline{d}$ for $\tau = 0.09$ (left) and $\tau = 0.64$ (right).

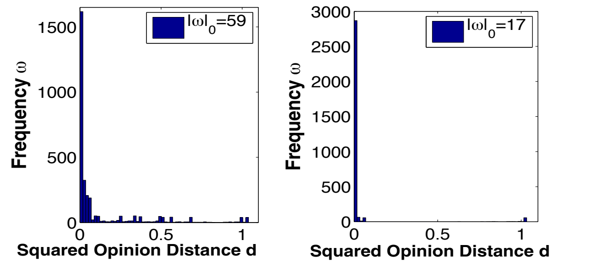

Fig. 12. Histograms of squared opinion distance d in P1 and P6. The ℓ_0 norm $|\omega|_0$ denotes the number of "quantized" opinion distances d that have nonzero probability.

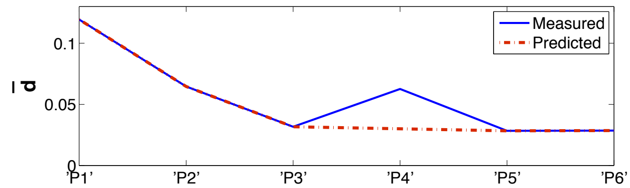

Fig. 13. Evolutions of the expected $\overline{d}$: measured v.s. predicted dynamics from P1 to P6.

the WLAN access points are associated; according to [27], $n_{\text{loc}} \approx 10$ for these top 100 WLAN access points.

Fig. 12 shows the histograms of the squared opinion distances d in the following six time periods: (P1) 10/1/08 - 10/25/08; (P2) 10/26/08 - 11/22/08; (P3) 11/23/08-12/20/08; (P4) 12/21/08-1/17/09; (P5) 1/18/09-2/14/09; (P6) 2/15/09-2/28/09. We observe that in P1, d can take almost all the values ranging from 0 to about 0.7 and a few values around 1, while by P6, the values of d are concentrated around 0 and 1. We compute a quantity $|\omega|_0$, the number of "quantized" distances d that have nonzero probability: the value $|\omega|_0$ drops from 59 in P1 to 17 in P6, indicating that opinions are clustered. Fig. 13 shows the time evolution of the expected (squared) opinion distance $\overline{d}$. We observe a similar downward trend in $\overline{d}$ (see Fig. 11) except in the weeks around Christmas, i.e., P4. The dotted line in Fig. 13 shows a predicted curve for $\overline{d}$ once the data in P4 were discarded.

Consider now the monthly survey data on health-related habits, in particular, the eating habits and the exercising habits. Eating habits are tracked by the number of healthy items a person eats (i.e., number of salads per week and number of fruits and vegetables per day). Exercising habits are recorded by the number of aerobic exercises (each lasting at least 20 minutes) per week and the number of times one participates in team sports per week. Thus individual opinions can be constructed by computing the probabilities of the following four states: (1) eating healthy but not exercising; (2) exercising but not eating healthy; (3) eating healthy and exercising; (4) none of the above. The opinions are again computed on a monthly basis. Assume that both events (eating healthy and exercising) are independent and each follows a Binomial distribution with n_1 and n_2 being the total number of items a person could eat per month and the total number of exercise activities one could do per month. Then the probability of each event is the maximum likelihood estimator (MSE) of the Binomial distribution given the actual number of healthy items a person eaten in a month or the number of exercises a person done in a month. Using the independence assumption, the probabilities of the four states can be computed.

Fig. 14 shows the histograms of the squared opinion distances in selected months[4]. Clearly, they show no significant change over time. Unlike the previous experiment (on the frequently visited places) which exhibits clustering behavior as time evolves, social interactions between students do not seem to affect individual health-related habits. This suggests that social interactions do not influence a person's opinions toward different issues at the same level, in terms of the rate of behavioral changes. There are two possible explanations. One is that, generally speaking, health-related habits usually change over a long time scale and this might be caused by a small threshold τ in the trust function; people are close-minded toward changing habits. Hence, it is difficult for the survey data to capture such minuscule changes in opinion. Another reason is that students are often not actively involved in exchanging relevant health advices during social interactions. If this is the case, then P_{ij} is in fact, equal to zero for $i \neq j$ and thus opinions stay constant.

VI. Conclusions

We proposed a generalization of the Deffuant-Weisbuch model and studied opinion dynamics in a connected network under two related interaction models, i.e., the hard-interaction model and the strategic interaction model. Under the hard-interaction model, we provided a necessary condition that guarantees opinion convergence. We showed that the necessary condition for convergence does not change as long as the

[4]Survey results are not available for Nov. 08, Jan. 09, Feb. 09 and May 09.

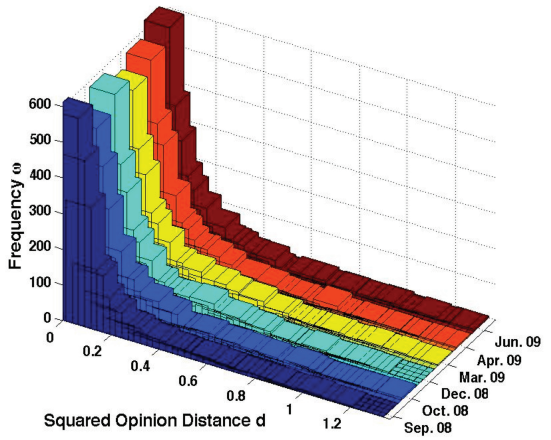

Fig. 14. Histogram of squared opinion distance d for selected months.

communication rates are time-invariant. Under the strategic interaction model, we showed how opinion formation processes are affected by the individual incentives behind interactions. In particular, we explored two specific utility functions, which lead to two different asymptotic opinion patterns. Our analysis of a rich social data indicate evolution towards clustering behaviors, measured by the frequency of visiting a location; however, the data do not indicate clustering with respect to other behavioral traits which may evolve at a longer time scale.

VII. Appendix

Derivation of Eqn. (8): Given that agents i and j interacted at $k+1$, the change in D after the interaction is only affected by the opinion distances between the pairs (i,j), (i,ℓ) and (j,ℓ) for $\forall\ell$. It then follows from (5), (6) and (7) that $D^{ij}[k+1]-D[k]$ equals

$$\begin{aligned}
&d_{ij}[k+1]-d_{ij}[k]\\
&\quad+\sum_{\forall\ell\neq i,j} d_{i\ell}[k+1]-d_{i\ell}[k]+d_{j\ell}[k+1]-d_{j\ell}[k]\\
=&-4\mu(d_{ij}[k])(1-\mu(d_{ij}[k]))d_{ij}[k]\\
&\quad+\sum_{\forall\ell\neq i,j}\left[2\mu^2(d_{ij}[k])d_{ij}[k]-2\mu(d_{ij}[k])d_{ij}[k]\right]\\
=&-4\mu(d_{ij}[k])(1-\mu(d_{ij}[k]))d_{ij}[k]\\
&\quad+(n-2)\left[2\mu^2(d_{ij}[k])d_{ij}[k]-2\mu(d_{ij}[k])d_{ij}[k]\right]\\
=&-2n\mu(d_{ij}[k])\left[1-\mu(d_{ij}[k])\right]d_{ij}[k].
\end{aligned}$$

Lemma 5: Let $h(d)=\rho(d)d=\mu(d)(1-\mu(d))d$. Then under Assumptions 2 and 4, its second derivative equals $\ddot{h}=-2(\dot{\mu})^2d+(1-2\mu)\ddot{\mu}d+2(1-2\mu)\dot{\mu}\leq 0$ for $d\in(0,\tau)$ and thus $h(d)$ is concave for $d\in(0,\tau)$. Let $P_{\text{eff}}=\int_0^\tau f_t(u)\mathrm{d}u\leq 1$ and $\overline{d}_{\text{eff}}=\int_0^\tau uf_t(u)\mathrm{d}u$. Then

$$\mathbb{E}\{h(d)\}=\int_0^{d_{\sup}}h(u)f_t(u)\mathrm{d}u=P_{\text{eff}}\int_0^\tau h(u)\frac{f_t(u)}{P_{\text{eff}}}\mathrm{d}u.$$

Using Jensen's inequality and the relation $\overline{d}_{\text{eff}}\leq\overline{d}$ yields

$$\mathbb{E}\{h(d)\}\leq P_{\text{eff}}h\left(\frac{\overline{d}_{\text{eff}}}{P_{\text{eff}}}\right)\leq\rho\left(\frac{\overline{d}_{\text{eff}}}{P_{\text{eff}}}\right)\overline{d}\leq\rho(\overline{d}_{\text{eff}})\overline{d}.$$

The last inequality holds because $\rho(d)$ is a non-increasing function of d (i.e., its derivative $\dot{\rho}=(1-2\mu)\dot{\mu}\leq 0$). We have $\rho(\overline{d}_{\text{eff}})/\rho(\overline{d})\leq\rho(0)/\rho(\tau^-)\leq\beta$ and subsequently, $\mathbb{E}\{h(d)\}\leq\beta h(\overline{d})$.

Lemma 6: For s sufficiently small, Taylor's expansion gives $\rho\left(b(t+s)\right)\approx\rho\left(b(t)\right)+\left(b(t+s)-b(t)\right)\dot{\rho}\left(b(t)\right)$. Then system $\dot{b}(t+s)=-\beta\rho(b(t+s))b(t+s)$ becomes

$$\dot{b}(t+s)=\begin{cases}-\beta\rho(b(t))b(t+s) & \text{if } \dot{\rho}\left(b(t)\right)=0;\\ -f(s;t) & \text{if } \dot{\rho}\left(b(t)\right)<0.\end{cases}$$

where $f(s;t)=\beta\left[\rho\left(b(t)\right)-b(t)\dot{\rho}\left(b(t)\right)\left(1-\frac{b(t+s)}{b(t)}\right)\right]b(t+s)$ and $\dot{\rho}(b):=d\rho/db$. In the first case, $\rho(b(t))$ is locally constant and hence, the local rate of convergence around $b(t)$ is exponential and is equal to $\rho\left(b(t)\right)$ when $\rho\left(b(t)\right)>0$, *i.e.*, $b(t)<\tau$. For the second case when $\dot{\rho}\left(b(t)\right)<0$, define $\xi(s;t):=b(t+s)/b(t)$. With respect to s, the dynamics of ξ become

$$\dot{\xi}(s;t)=\frac{1}{b(t)}\dot{b}(t+s)=-\beta R(t)\xi(s;t)\left(1-\frac{\xi(s;t)}{K(t)}\right)\ , \tag{18}$$

$$\text{where}\quad R(t)=\rho\left(b(t)\right)-b(t)\dot{\rho}\left(b(t)\right)\ , \tag{19}$$

$$K(t)=1+\frac{\rho\left(b(t)\right)}{-b(t)\dot{\rho}\left(b(t)\right)}>1\ . \tag{20}$$

Note that the dynamics of $\xi(s;t)$ resemble the logistic equation. There are two equilibria at $\xi_1=0$ (stable) and $\xi_2=K(t)$ (unstable). When $R(t)>0$, as shown in Fig. 15, the system will converge if $K(t)>\xi(0;t)=1$. Indeed, if $b(t)<\tau$, then $K(t)=1-\frac{\rho(b(t))}{b(t)\dot{\rho}(b(t))}>1$. and (18) converges. The exponential rate of convergence equals $R(t)$. Moreover, since $b(t)$ is a monotonically decreasing function of t, an equivalent condition for convergence is $b(0)<\tau$. On the contrary, when $b(t)=\tau$, we get $K(t)=1$ because $\rho(\tau)=0$. For $b(t)>\tau$, both $\rho\left(b(t)\right)$ and $\dot{\rho}\left(b(t)\right)$ are zero, which implies $R(t)=0$ in (18) and bifurcation occurs. Thus the system may not converge when $b(t)\geq\tau$ for $\forall t$.

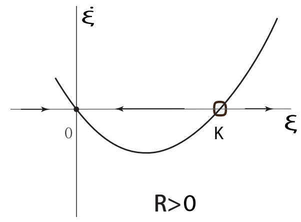

Fig. 15. Vector Field: the point $\xi(s;t)$ moves toward the stable fixed point $\xi_1=0$ if $\xi(0;t)<K$ and moves away, otherwise.

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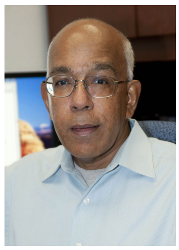

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