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A BASIC APPROACH TO THE USE OF CANONICAL VARIABLES AND VON ZEIPEL'S METHOD IN PERTURBATION THEORY

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A BASIC APPROACH TO THE USE OF CANONICAL VARIABLES AND VON ZEIPEL'S METHOD IN PERTURBATION THEORY

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PREFACE

Systematic methods of obtaining approximate solutions to complicated non-linear problems are the subject of perturbation theory. This theory probably reaches its highest development in celestial mechanics where detailed complexities must be taken into account in order to achieve the desired accuracies. Within recent years there has been a considerable revival of interest in the application of classical mechanics theory to practical problems involving satellites and space flight, using the Hamilton-Jacobi theory, canonical variables, and von Zeipel's perturbation method to provide some very elegant solutions to artificial satellite problems.

However, many engineers without considerable training or experience in the subject have difficulty in following this work. Probably most of the difficulty arises because of their unfamiliarity with both the mathematical and astronomical jargon involved (e.g., "determining functions," "secular," "periodic," "canonic," etc.) and because of the complexity introduced when several degrees of freedom are involved.

The von Zeipel perturbation technique at the time of this writing is a specialized tool available principally to astronomers and those trained in celestial mechanics. It is a powerful method which should have wide application to problems involved in other fields. Accordingly, this Memorandum presents the subject from an engineering rather than mathematical point of view, using relatively simple examples to illustrate the method.

SUMMARY

This Memorandum is concerned with the von Zeipel perturbation method for finding an approximate solution to differential equations such as occur in non-linear mechanics. The method is illustrated by briefly introducing some fundamental principles of Hamilton-Jacobi mechanics relating to the formation of the Hamiltonian, canonic equations, canonic variables, transformations, etc.

The von Zeipel method consists of making successive mathematical transformations of variables of a canonical system of differential equations. The transformations are performed in a methodical way according to established general rules so that the final solution is obtained in a certain desired form. Two examples are worked out in detail, one is for the non-linear spring equation where energy is conserved and the other is for the Van der Pol equation to illustrate how an energy-dissipating system is handled. Also there is a discussion of some basic principles and details regarding the variation of parameters method and finally, a comparison is made of the von Zeipel method with that of Kryloff-Bogoliuboff (which is widely applied in the fields of engineering and physics).

There are several important advantages to the use of a canonical system of equations and von Zeipel's method. Perhaps the most important is that solutions may be carried out to a high degree of accuracy and may be obtained in a very methodical way, involving no great mathematical difficulty. These solutions will have relevance to practical problems of space flight guidance.

CONTENTS

LIST OF SYMBOLS

ω or k	angular frequency of a simple harmonic system
q_j	generalized coordinate $j = 1, 2, 3, \cdots$
H	the Hamiltonian--equal to the sum of the kinetic and potential energies of the system (sometimes designated F by astronomers)
L	canonical variable representing momentum conjugate to the angle variable ℓ
ℓ	canonical variable representing a coordinate angle conjugate to the momentum variable L
m	mass of a particle
p_j	generalized momentum $j = 1, 2, 3, \cdots$
ε	a small positive number ($\varepsilon < 1$)

Dots above symbols represent derivatives with respect to time

I. INTRODUCTION

For more than a century astronomers and mathematicians have used the Hamilton-Jacobi theory and related perturbation techniques in explaining and predicting the motion of celestial bodies. Many of the powerful auxiliary perturbation methods and techniques developed by men such as Delaunay, Hill, Brown, Shook, and von Zeipel were evolved in solving highly specialized problems in celestial mechanics related to lunar theory, planetary theory, and solar system stability. These problems were very complicated, involving three (and sometimes four) degrees of freedom and entailing an enormous amount of detailed derivation and calculation. Accordingly, it is not surprising that some of these mathematical tools have been familiar over the years only to persons astronomically oriented and indoctrinated in the theory of classical mechanics. Recently, wider interest has developed as engineers and applied mathematicians have encountered practical problems in space flight guidance.

In the past, it was the astronomer who was interested in predicting the motion of some celestial body for an extensive period of time, say 50 or 100 years. Today, space flight or satellite systems are being planned to operate for periods of several years. The procedure of using a crude mathematical model and making many iterations on an electronic computer usually breaks down in problems of this type because of truncation and round-off errors. There has, therefore, been a need for the use of more sophisticated mathematics. Within the past four years notable contributions along these lines have been made by Brouwer,[1] Garfinkel,[2] Hori,[3] Kozai,[4] and many others using the Hamilton-Jacobi theory and a perturbation technique usually ascribed to von Zeipel,[5] although others[6,7] contributed methods along very similar lines. Brouwer, Hori, and Garfinkel developed a complete first-order* solution for the perturbed motion of an artificial satellite without air drag. Complete second-order solutions of this problem have been developed by others.[4] These publications appeared in 1959; since then a number of papers have appeared describing the adaptation of the technique to various perturbation problems in celestial mechanics (see references). Unfortunately, many of these problems are complicated in detail, and by the very nature of their solution require a tedious amount of analytic derivation and development, whereas the method itself is simple and straightforward. A thorough knowledge of classical mechanics and the calculus of variations is necessary to understand the underlying theory involved, but only an elementary acquaintance is necessary in order to apply the technique successfully to many practical engineering problems.

The intent here is to outline the method from an engineering, rather than a mathematical, point of view, using relatively simple examples involving only one degree of freedom (i.e., the harmonic oscillator and the Van der Pol equations). The mathematical theory and derivation may be found in texts on classical or celestial mechanics.[6,7,9,10]

Because of the similarity of the von Zeipel perturbation method to that of Kryloff and Bogoliuboff method[12,13] which is well known in the fields of physics and engineering, one section is devoted to a comparison of the two.

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*Referring to the first power of the coefficient of the second harmonic of the earth's potential function, variously designated J_2, k_2, B_2, etc.

II. CANONICAL VARIABLES AND TRANSFORMATIONS

The von Zeipel method consists of making successive mathematical transformations of variables of a canonical system of differential equations. The transformations are performed in a methodical way according to established general rules, so that the final solution is obtained in a certain desired form. For example, in celestial mechanics it is expedient to describe the variation of an orbital parameter, say $q_{(t)}$, in the form:

$$q_{(t)} = q_o + \dot{q}_o(t-t_o) + \text{long-period trigonometric terms} + \text{short-period trigonometric terms}$$

where q_o, $\dot{q}_o$, and t_o are constants of the motion. The linear time varying term is referred to as being "secular." The long-period terms may involve periodicities very great in magnitude relative to the unperturbed two-body period, whereas the short-period terms are usually commensurable with this period. To an astronomer, it is important to be able to separate mathematically the long-period from the secular terms, for if the "long period" were say, 100 or 1000 years, the periodic effect would be difficult to separate from the secular effect by observational means.

A system of differential equations describing a mechanical system is said to be canonical if:

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \qquad \dot{p}_j = \frac{-\partial H}{\partial q_j} \qquad j = 1 \cdots n \tag{1}$$

where H is the Hamiltonian equal to the sum of the kinetic and potential energies of the system; p_j is the generalized momentum and q_j the generalized coordinate. The system has n degrees of freedom. We will assume that H may be written so that it does not contain the time explicitly. That is,

$$H = H(p_j, q_j) \qquad j = 1 \ldots n$$

Equations (1) are a set of 2n first-order differential equations. Their solution gives p_j and q_j as functions of the time and 2n arbitrary constants. As an example, consider the simple linear spring where a particle of mass m is constrained to move along a path, its displacement from a fixed point being q. The spring constant is k^2.

The kinetic energy is: $\frac{1}{2} m\dot{q}_x^2$

The potential energy is: $\frac{1}{2} k^2 q_x^2$

Designating the momentum as $p_x = m\dot{q}_x$, the Hamiltonian is:

$$H_o(p_x, q_x) = \frac{p_x^2}{2m} + \frac{1}{2} k^2 q_x^2 \tag{2}$$

Note that p_x and q_x are both functions of time but t is not explicitly contained in H_o. The 2n canonical equations are

$$\begin{aligned} \dot{q}_x &= \frac{\partial H_o}{\partial p_x} = \frac{p_x}{m} \\ \dot{p}_x &= - \frac{\partial H_o}{\partial q_x} = - k^2 q_x \end{aligned} \tag{3}$$

These two first-order equations may be readily solved for $q_x(t)$ and $p_x(t)$ in terms of 2n = 2 arbitrary constants. The first appears to be rather trivial, merely restating the definition of momentum; the second equation can be combined with the first to form one single second-order equation which is that of the spring or simple harmonic oscillator.

$$m\ddot{q}_x + k^2 q_x = 0 \tag{4}$$

We shall be concerned here with obtaining solutions to the first order Eqs. (3) rather than Eq. (4). For the perturbed case (i.e., non-linear spring) to be considered in the next section, the Hamiltonian of Eq. (3) is not in a form amenable for perturbation techniques. A fundamental theorem in the transformation theory of classical mechanics is that a transformation of variables may be made such that the functional form of the Hamiltonian is changed, although its value is not, providing t is not explicitly present in H. If an appropriate transformation method is used, the differential equations describing the system will remain canonical although of a different functional form. For example, let us assume that the Hamiltonian H_o is to be transformed to one having a new functional form H_o' with new variables p_x' and q_x'. Then

$$H_o(p_x, q_x) = H_o'(p_x', q_x') \tag{5}$$

$$\dot{q}_x' = \frac{\partial H_o'}{\partial p_x'} \quad \dot{p}_x' = - \frac{\partial H_o'}{\partial q_x'} \tag{6}$$

In stationary state Hamilton-Jacobi (H-J) perturbation theory, the first step is to transform the Hamiltonian of the unperturbed problem (i.e., H_o) so that it contains the "action" variables only (i.e., momenta, in our case). In practice, the mathematics of this step can usually be circumvented by intuitive reasoning or, better yet, by starting with someone else's results that will guarantee the desired functional form of H as well as the H-J equations remaining canonical. This procedure will be illustrated later in Section V, dealing with applications to celestial mechanics.

Returning to the example of the linear spring, the Hamiltonian may be transformed so as to contain only a momentum variable by intuitive reasoning. That is

$$H_o(p_x, q_x) \rightarrow H_o'(p_x', -) \quad \text{where } H_o = H_o' \tag{7}$$

First consider the solution of Eq. (3) with the conditions m = 1, $\dot{q}_{xo} = 0$

$$\begin{aligned} q_x(t) &= q_{xo} \cos k(t-t_o) \\ p_x(t) &= \dot{q}_x(t) = -\, k q_{xo} \sin k(t-t_o) \end{aligned} \tag{8}$$

At time t_o, the kinetic energy is zero and the total energy of the system is equal to the initial potential energy or

$$H_o = \frac{k^2 \, q_{xo}{}^2}{2} = H_o' = \text{total system energy} \tag{9}$$

For convenience, we shall designate p_x' and its conjugate q_x' as L and ℓ respectively. The canonical conditions of Eq. (6) and the desired transformation of Eq. (7) may be determined intuitively by letting:

$$L = \frac{k \, q_{xo}{}^2}{2} \tag{10}$$

$$\ell = k(t-t_o) \tag{11}$$

so that

$$H_o(p_x,q_x) = H_o'(L, -) = k\ L \tag{12}$$

Then, the canonical equations are

$$\dot{L} = -\frac{\partial H_o'}{\partial \ell} = 0 \qquad \text{or } L = \text{const.} \tag{13}$$

$$\dot{\ell} = \frac{\partial H_o'}{\partial L} = k \tag{14}$$

The solution to the linear, i.e., unperturbed, problem may be considered as auxiliary equations connecting the transformation. That is

$$\begin{aligned} q_x &= \sqrt{\frac{2L}{k}} \cos \ell \\ \dot{q}_x &= -\sqrt{2\ k\ L}\ \sin \ell \end{aligned} \tag{15}$$

In the unperturbed case, L and the angular variable ℓ are "elements," or "parameters," specifying and defining the simple harmonic motion of Eq. (15). It is logical to assume that if this simple motion is suddenly perturbed by a small disturbing force, Eqs. (15) could still be used to describe or approximate the resulting motion; however, the elements would require a slight adjustment to represent the motion accurately for a given short interval of time. This subject is discussed further in the next section.

III. THE VARIATION OF PARAMETERS TECHNIQUE

In order to clarify the material which is to be presented in the next section, it may be worthwhile to review briefly the variation of parameters technique. In the preceding section the solutions of Eqs. (3) were of the form

$$q_x(t) = A \cos (kt + B)$$

$$p_x(t) = \dot{q}_x(t) = - A\, k \sin (kt + B) \tag{16}$$

where A and B are constants of integration and k is the angular frequency or in terms of other variables which have been used

$$A = q_{xo} = \sqrt{\frac{2L}{k}}$$

$$B = - kt_o$$

so that Eq. (16) corresponds to Eqs. (8) or (15). It was suggested that, if the simple harmonic motion described by these equations was perturbed by a small disturbing force, these solutions could still be used to describe or approximate the resulting motion. However, the elements or constants A and B would require slight adjustments to represent the motion accurately for a given short interval of time. Moreover, as time progresses, one would expect that adjustments to the elements would be continuously or periodically necessary to achieve this purpose. Therefore, these "constants" of the above equations may be thought of in the perturbed case as slowly time-varying. In perturbation theory this concept is usually referred to as the "variation of parameters" method. Generally, one may expect that the adjustments to the elements will consist of additive terms: a linear time-varying term plus periodic terms of several different frequencies as in a Fourier series. To illustrate this technique, consider an equation of the form

$$\ddot{q}_x + F(q_x, p_x, t) = 0 \tag{17}$$

where the solution is nearly sinusoidal or quasi-harmonic so that

$$F(q_x, p_x, t) = k^2 q_x + \varepsilon X(q_x, p_x, t) \tag{18}$$

In the above, ε is a parameter characterizing the smallness of the deviation of F from $k^2 q_x$. Next, if we examine the case where t is not explicitly present in X, Eq. (17) becomes

$$\ddot{q}_x + k^2 q_x + \varepsilon X(q_x, p_x) = 0 \tag{19}$$

where, for $\varepsilon = 0$, Eqs. (16) are the solution. For ε small and positive, A and B may be considered new unknown functions of time and determined in such a manner that Eqs. (16) are forced to be solutions of Eq. (19). To do this we assume

$$q_x = q_x(t, A, B)$$

where

$$A = A(t) \text{ and } B = B(t)$$

so that differentiating the first of Eqs. (16) with respect to time

$$\frac{dq_x}{dt} = \frac{dA}{dt} \cos (kt + B) - Ak \sin (kt + B)$$

$$- A\frac{dB}{dt} \sin (kt + B)$$

Now examining the above equation and considering the requirement for $p_x(t)$ of Eqs. (16), it is necessary that

$$p_x(t) = \frac{dq_x}{dt} = - Ak \sin (kt + B)$$

and

$$\frac{dA}{dt} \cos (kt + B) - A \frac{dB}{dt} \sin (kt + B) = 0 \qquad (20)$$

Similarly, differentiating p_x of Eqs. (16) with respect to time results in

$$\dot{p}_x = \ddot{q}_x = - k \frac{dA}{dt} \sin (kt + B) - Ak^2 \cos (kt + B)$$

$$- Ak \frac{dB}{dt} \cos (kt + B)$$

Examining the above equation and considering the requirements for $\ddot{q}_x$ of Eq. (19) and for q_x (see Eqs. 16), it is necessary that

$$k \frac{dA}{dt} \sin (kt + B) + Ak \frac{dB}{dt} \cos (kt + B) = \varepsilon X(q_x, p_x) \qquad (21)$$

Solving Eqs. (20) and (21) for $\frac{dA}{dt}$ and $\frac{dB}{dt}$

$$\frac{dA}{dt} = \frac{\varepsilon X}{k} \sin (kt + B)$$

$$\frac{dB}{dt} = \frac{\varepsilon X}{Ak} \cos (kt + B) \qquad (22)$$

where

$$X = X(q_x, p_x)$$

Note that $\frac{dA}{dt}$ and $\frac{dB}{dt}$ being proportional to ε will be slowly time-varying during the period $2\pi/k$.

Summarizing, if Eqs. (22) can be integrated so that A and B are known or approximated as functions of time, then Eqs. (16) would represent a solution to Eq. (19). In other words, Eqs. (16) can serve to represent the perturbed $\varepsilon \neq 0$ solution if the parameters A and B are considered to be time-varying according to Eqs. (22). Now returning to the canonical variables L and ℓ, it is clear that since

$$A = q_{xo} = \sqrt{\frac{2L}{k}}$$

then

$$\frac{dL}{dt} = \sqrt{2kL}\ \frac{dA}{dt}$$

and since

$$B = -kt_o$$

then

$$\ell = k(t-t_o) = kt + B$$

$$\frac{d\ell}{dt} = k + \frac{dB}{dt}$$

Note that Eqs. (22) could be substituted in the above expressions for $\frac{dL}{dt}$ and $\frac{d\ell}{dt}$. Therefore to obtain a solution to the canonic equations

$$\frac{dL}{dt} = -\frac{\partial H}{\partial \ell}$$

$$\frac{d\ell}{dt} = \frac{\partial H}{\partial L}$$

where H is the Hamiltonian of the non-linear or perturbed problem, is equivalent to solving Eqs. (22). In the next section we shall consider von Zeipel's method for approximating the solution of the above canonic equations. The results, of course, with Eq. (16) (equivalent to Eqs. (8) or (15)) constitute a complete solution to the problem.

IV. THE VON ZEIPEL PERTURBATION TECHNIQUE

NON-LINEAR SPRING EXAMPLE

So far the equation of motion of a mass attracted to an equilibrium position by a force proportional to the distance has been used as an example (i.e., the linear spring). Suppose now that a perturbation term proportional to the cube of the displacement is added to approximate a non-linear spring or the motion of a pendulum. The equation of the system is now

$$m\ddot{q}_x + k^2 q_x + \varepsilon q_x^3 = 0 \tag{23}$$

where ε is a small parameter. It is necessary to add the term $\frac{\varepsilon q_x^4}{4}$ to the Hamiltonian of Eq. (2) so that the canonical equation for $\dot{p}_x$ becomes equivalent to the above. That is

$$H = \frac{p_x^2}{2} + \frac{1}{2} k^2 q_x^2 + \frac{\varepsilon q_x^4}{4} \tag{24}$$

so that

$$\dot{p}_x = -\frac{\partial H}{\partial q_x} = \ddot{q}_x = -k^2 q_x - \varepsilon q_x^3 \tag{25}$$

Taking the results of section II in expressing the new Hamiltonian H

$$H(L, \ell) = H_o (L) + \left[H(L, \ell) - H_o (L)\right]$$

where

$$H-H_o = \frac{\varepsilon q_x^4}{4}$$

The portion of the Hamiltonian representing the perturbation $(H-H_o)$ contains both action and angle variables. The unperturbed portion contains only the action or momentum variable. Using the definitions of L and ℓ and the unperturbed solution for q_x of Eqs. (15)

$$H(L, \ell) = kL + \frac{\varepsilon L^2}{k^2} \cos^4 \ell \tag{26}$$

The first step of von Zeipel's method consists of transforming a Hamiltonian H (which may be a function of several action and angle variables) into a new Hamiltonian H^* so that one or more of the angular variables present in H are eliminated from H^*. The solution may progress in stages by transforming H^* into H^{**}, etc., until no angle variables are present in the final result. For example, in the above equation $H(L, \ell)$ would be transformed to $H^*(L', -)$, (the dash indicating the absence of ℓ'). L' is now a new action variable. The resulting H-J canonical equations are

$$\dot{\ell}' = \frac{\partial H^*}{\partial L'} = \text{const.} \qquad \dot{L}' = -\frac{\partial H^*}{\partial \ell'} = 0$$

The time variation in ℓ' is linear (secular) whereas L' is a constant of the motion. It may appear that part of the solution has been lost, but it will become apparent later in carrying through the transformation from H to H^* that the periodic part of the solution is obtained in the process. The final results of the problem will be in the form

$$\ell = \dot{\ell}'(t-t_0) + \text{trigonometric terms periodic in } \ell$$

$$L = L' + \text{trigonometric terms periodic in } \ell$$

$$q_x = \sqrt{\frac{2L}{k}} \cos \ell \text{ (using } \ell, L \text{ of above)}$$

The transformation technique is described as follows. Assume that $H = H_0 + (H-H_0)$ may be expanded in an infinite series, each term having a coefficient involving some power of the small parameter ε.

That is

$$H = H_o + H_1 + H_2 + \cdots$$

where the subscript refers to the power of e involved in the coefficient and $H_2(e^2)$, for example, is designated as "of order 2".

Without defining its mathematical significance, (6, 9-11) we introduce a "determining function" having a new variable L' and the old variable ℓ

$$S = S(L', \ell) = S_o + S_1 + S_2 + \cdots$$

The subscripts have the same meaning as before. The following relationships are imposed on S:

$$L = \frac{\partial S}{\partial \ell} \qquad \ell' = \frac{\partial S}{\partial L'} \tag{27}$$

It is desirable that $\ell = \ell'$ when $e = 0$ (the unperturbed case) Therefore S_o is arbitrarily defined as

$$S_o = L' \ell \tag{28}$$

The canonical transformation is to be

$$H(L, \ell) = H^*(L', -)$$

or

$$H_o(L) + \left[H(L, \ell) - H_o(L)\right] = H_o^*(L') + H_1^*(L') + \cdots \tag{29}$$

Now, using the relationships from Eqs. (27) and (28)

$$H(L, \ell) = H(L' + \frac{\partial S_1}{\partial \ell} + \frac{\partial S_2}{\partial \ell} + \cdots, \ell) \tag{30}$$

$$= H_o^* + H_1^* + H_2^* + \cdots$$

and expanding the Hamiltonian of Eq. (30) in a Taylor series about L' and collecting terms of corresponding order in both right and left members

$$\underline{\text{Zero order}} \qquad H_o(L') = H_o^*(L') \tag{31}$$

$$\underline{\text{First order}} \qquad \frac{\partial H_o}{\partial L'} \frac{\partial S_1}{\partial \ell} + H_1(L', \ell) = H_1^*(L') \tag{32}$$

$$\underline{\text{Second order}} \qquad \frac{\partial H_o}{\partial L'} \frac{\partial S_2}{\partial \ell} + \frac{1}{2} \frac{\partial^2 H_o}{\partial L'^2} \left(\frac{\partial S_1}{\partial \ell}\right)^2 \tag{33}$$

$$+ \frac{\partial H_1}{\partial L'} \frac{\partial S_1}{\partial \ell} = H_2^*(L')$$

The object of this expansion is to solve the resulting partial differential equations (ordinary differential equations in this special case) for S_1, S_2, etc., depending on the order of accuracy desired. When this is done, the main difficulty of the problem is overcome and the solution follows at once. It is very important to observe in the work that follows that, S_1, S_2, etc., will be obtained by solving <u>linear</u> equations. That is, S_1 will be determined from Eq. (32) and then substituted into Eq. (33) which is linear in S_2. Similarly the third order equations can be solved from an equation linear in S_3 using the previously determined values of S_1 and S_2.

Consider the first-order term

$$H_1(L', \ell) = \frac{\varepsilon L'^2}{k^2} \cos^4 \ell \tag{34}$$

It may be expressed as the sum of two terms, one constant (i.e., "secular" or non-periodic in ℓ) and the other periodic in ℓ. These terms are easily obtained by expressing

$$\cos^4 \ell = \frac{3}{8} + \frac{1}{2}\cos 2\ell + \frac{1}{8}\cos 4\ell$$

so

$$H_1(L', \ell) = H_{1s} + H_{1p}$$

where

$$H_{1s} = \frac{3\varepsilon L'^2}{8k^2} \qquad H_{1p} = \frac{\varepsilon L'^2}{2k^2}\left[\cos 2\ell + \frac{1}{4}\cos 4\ell\right] \tag{35}$$

or alternatively

$$H_{1s} = \frac{\varepsilon L'^2}{\pi k^2}\int_0^{\pi} \cos^4 \ell \, d\ell = \frac{3\varepsilon L'^2}{8k^2}$$

$$H_{1p} = H_1 - H_{1s}$$

The first-order Eq. (32) may be expressed as

$$\frac{\partial H_o}{\partial L'}\frac{\partial S_1}{\partial \ell} + H_{1s} + H_{1p} = H_1^*$$

and separated into two equations:

$$H_{1s} = H_1^*(L') = \frac{3\varepsilon L'^2}{8k^2} \tag{36}$$

$$H_{1p} = -\frac{\partial H_o}{\partial L'}\frac{\partial S_1}{\partial \ell} \tag{37}$$

In the above, $\frac{\partial H_o}{\partial L'}$ stands for $\frac{\partial H_o}{\partial L}$ with L' replacing L after the derivative is taken or in our example, simply:

$$\frac{\partial H_o}{\partial L'} = k$$

Hence from Eqs. (35) and (37)

$$\frac{\partial S_1}{\partial \ell} = -\frac{\varepsilon L'^2}{2k^3}\left(\cos 2\ell + \frac{1}{4}\cos 4\ell\right) \tag{38}$$

and

$$S_1 = -\frac{\varepsilon L'^2}{4k^3}\left(\sin 2\ell + \frac{1}{8}\sin 4\ell\right) + \text{const.} \tag{39}$$

The constant of integration is not significant to the problem since we are only concerned with the derivatives of S_1. Now the part of the solution periodic in ℓ is obtained to first order (i.e., $O(\varepsilon)$ from Eqs. (27)).

$$L = \frac{\partial S}{\partial \ell} = \frac{\partial(S_o + S_1)}{\partial \ell} = L' - \frac{\varepsilon L'^2}{2k^3}\left(\cos 2\ell + \frac{1}{4}\cos 4\ell\right) \tag{40}$$

$$\ell' = \frac{\partial S}{\partial L'} = \frac{\partial(S_o + S_1)}{\partial L'} = \ell - \frac{\varepsilon L'}{2k^3}\left(\sin 2\ell + \frac{1}{8}\sin 4\ell\right) \tag{41}$$

or

$$\ell = \ell' + \frac{\varepsilon L'}{2k^3}\left(\sin 2\ell + \frac{1}{8}\sin 4\ell\right)$$

If first-order accuracy is sufficient in calculating the periodic terms, there is no interest in determining S_2. At this point, the Hamiltonian H^* is from Eqs. (31) and (36)

$$H^*(L') = H_o^* + H_1^* + \ldots$$

$$= kL' + \frac{3\,\varepsilon L'^2}{8k^2} + \ldots$$

H_2^* may be obtained in a similar manner from Eq. (33). Now, S_1, S_2, etc., will all be periodic in ℓ and since

$$\frac{\partial^2 H_o}{\partial L'^2} = 0$$

$$H_2^*(L') = \text{the part independent of } \ell \text{ of } \frac{\partial H_1}{\partial L'} \frac{\partial S_1}{\partial \ell}$$

Now $\frac{\partial S_1}{\partial \ell}$ is given by Eq. (38), and from Eq. (26)

$$\frac{\partial H_1}{\partial L'} = \frac{2\varepsilon L'}{k^2} \cos^4 \ell$$

so it is necessary to determine the non-periodic part of

$$\frac{2\varepsilon^2 L'^3}{k^5} \cos^4 \ell \left(\frac{3}{8} - \cos^4 \ell\right)$$

Considering the infinite series for $\cos^4 \ell$ (see prior to Eq. 35)

$$\frac{3}{8} \cos^4 \ell \Big|_{sec} = \frac{9}{64}$$

$$- \cos^8 \ell \Big|_{sec} = -\left(\frac{9}{64} + \frac{1}{8} + \frac{1}{128}\right) = -\frac{35}{128}$$

Hence

$$H_2^* = -\frac{17}{64} \frac{\varepsilon^2 L'^3}{k^5}$$

and H^* is given to second order, $O(\varepsilon^2)$, by

$$H^* = kL' + \frac{3\varepsilon L'^2}{8k^2} - \frac{17}{64} \frac{\varepsilon^2 L'^3}{k^5} \tag{42}$$

The secular part of the solution is given by the canonical H-J equations

$$\frac{d\ell'}{dt} = \frac{\partial H^*}{\partial L'} = k + \frac{3}{4}\,\frac{\varepsilon L'}{k^2} - \frac{51}{64}\,\frac{\varepsilon^2 L'^2}{k^5} \tag{43}$$

$$\frac{dL'}{dt} = -\frac{\partial H^*}{\partial \ell} = 0$$

Summarizing the complete perturbed solution we have

$$q_x(t) = \sqrt{\frac{2L}{k}}\cos \ell \tag{44}$$

where

$$\ell = \dot{\ell}'(t-t_o) + \frac{\varepsilon L'}{2k^3}\left(\sin 2\ell' + \frac{1}{8}\sin 4\ell'\right) \tag{45}$$

$$\ell' = \dot{\ell}'(t-t_o) = \left[k + \frac{3}{4}\frac{\varepsilon L'}{k^2} - \frac{51}{64}\frac{\varepsilon^2 L'^2}{k^5}\right](t-t_o) \tag{46}$$

and

$$L = L' - \frac{\varepsilon L'^2}{2k^3}\left(\cos 2\ell' + \frac{1}{4}\cos 4\ell'\right)$$

Note that ℓ has been replaced by ℓ' in Eqs. (40) and (41) since the difference between ℓ and ℓ' is $O(\varepsilon)$. The coefficients of the right-hand terms are also $O(\varepsilon)$, making the discrepancy small.

Referring back to the original variables and substituting in

$$L = \frac{k\, q_{xo}^2}{2} \qquad L' = \frac{k q'^2_{xo}}{2} \tag{47}$$

$$q_{xo}^2 = q'^2_{xo}\left[1 - \varepsilon\,\frac{q'^2_{xo}}{4k^2}\left(\cos 2\ell + \frac{1}{4}\cos 4\ell'\right)\right] \tag{48}$$

The physical interpretation of these results is as follows: q'_{xo} may be interpreted as the actual initial displacement of a mass attached to a nonlinear spring at time t_o. The displacement after an interval $(t-t_o)$ may be determined from the simple linear spring equation by considering that the elements (i.e., parameters) defining the linear motion are time-varying due to the perturbation. The variation of the linear spring parameter q_{xo} defining the amplitude is given by Eq. (48). The variations in frequency of the motion are inherent in Eq. (45).

In the above example, the perturbation term involves only the displacement q_x, and the energy of the system is conserved. The solution of Eq. (48) shows the amplitude of the oscillatory motion to have no secular terms. That is, it does not increase or decrease without bound but is represented by a constant plus trigonometric terms having frequencies which are multiples of the fundamental frequency of the system. As will be shown later, this is true whenever the perturbation term is independent of p_x.

Note that the secular part of the solution (Eq. 43) is accurate to $O(\varepsilon^2)$ whereas the periodic part (Eq. 41) is accurate to $O(\varepsilon)$. In theory, the solution may be extended indefinitely by collecting higher-order terms in the Taylor series expansion of H and solving the resulting linear partial equations for S_2, S_3, etc. For example, continuing on with Eqs. (31) through (33), the third-order terms are:

$$\frac{\partial H_o}{\partial L'}\frac{\partial S_3}{\partial \ell} + \frac{1}{2}\frac{\partial^2 H_1}{\partial L'^2}\left(\frac{\partial S_1}{\partial \ell}\right)^2 + \frac{\partial H_1}{\partial L'}\frac{\partial S_2}{\partial \ell} = H_3^*(L') \qquad (49)$$

(Note: The first term involving S_3 is periodic in ℓ).

S_1 has been determined from Eq. (32) and S_2 may be determined by finding the part of Eq. (33) periodic in ℓ. That is, the periodic part of

$$\frac{\partial H_o}{\partial L'}\frac{\partial S_2}{\partial \ell} + \frac{1}{2}\frac{\partial^2 H_o}{\partial L'^2}\left(\frac{\partial S_1}{\partial \ell}\right)^2 + \frac{\partial H_1}{\partial L'}\frac{\partial S_1}{\partial \ell} = 0 \qquad (50)$$

gives a partial differential equation which may be solved for S_2.

(Note: $\frac{\partial^2 H_o}{\partial L'^2} = 0$.) H_3^* may be determined by separating Eq. (49)

$$H_3^* = \text{secular part of} \left[\frac{1}{2}\frac{\partial^2 H_1}{\partial L'^2}\left(\frac{\partial S_1}{\partial \ell}\right)^2 + \frac{\partial H_1}{\partial L'}\frac{\partial S_2}{\partial \ell}\right] \qquad (51)$$

The details of this calculation are carried out in Appendix A.

NON-CONSERVATIVE SYSTEMS

So far the discussion has been restricted to conservative systems where there is neither creation nor disipation of energy. In the non-linear spring example, the perturbation term was a function of a position coordinate only and did not involve the time explicitly. Next, let us consider a mechanical system to be perturbed by external forces such that the equations describing the system are:

$$\frac{dq_x}{dt} = \frac{\partial H}{\partial p_x} + X(q_x, p_x, t)$$

$$(52)$$

$$\frac{dp_x}{dt} = -\frac{\partial H}{\partial q_x} - Y(q_x, p_x, t)$$

where

$$H = H(q_x, p_x)$$

As in the preceding example, let us suppose that we will wish to transform Eqs. (52) so that they are in terms of some new variables L, ℓ and so that

$$H(q_x, p_x) = H'(L, \ell)$$

and

$$\frac{d\ell}{dt} = \frac{\partial H'}{\partial L} + X'$$

$$\frac{dL}{dt} = -\frac{\partial H'}{\partial \ell} - Y' \qquad (53)$$

First it will be necessary to express X′ and Y′ in terms of the new variables L, ℓ. This transformation of variables can be done in the following manner. First multiply Eqs. (52) by $\frac{\partial p_x}{\partial L}$, $-\frac{\partial q_x}{\partial L}$ and add. The sum is

$$\frac{dq_x}{dt}\frac{\partial p_x}{\partial L} - \frac{dp_x}{dt}\frac{\partial q_x}{\partial L} = \frac{\partial H'}{\partial L} + X\frac{\partial p_x}{\partial L} + Y\frac{\partial q_x}{\partial L}$$

If on the left side we substitute

$$\frac{dq_x}{dt} = \frac{\partial q_x}{\partial \ell}\frac{d\ell}{dt} + \frac{\partial q_x}{\partial L}\frac{dL}{dt}$$

$$\frac{dp_x}{dt} = \frac{\partial p_x}{\partial \ell}\frac{d\ell}{dt} + \frac{\partial p_x}{\partial L}\frac{dL}{dt}$$

the result is

$$\left[\frac{\partial q_x}{\partial \ell}\frac{\partial p_x}{\partial L}-\frac{\partial p_x}{\partial \ell}\frac{\partial q_x}{\partial L}\right]\frac{d\ell}{dt}=\frac{\partial H'}{\partial L}+X\frac{\partial p_x}{\partial L}+Y\frac{\partial q_x}{\partial L}$$

The bracket on the left side is a Lagrangian bracket $[\ell, L]$ and equals unity for a canonical transformation where the Hamiltonian is unchanged.(7) (This may be illustrated by carrying out the above indicated operations on q_x, p_x as given by Eqs. (15) of the previous section). Hence

$$\frac{d\ell}{dt}=\frac{\partial H'}{\partial L}+X\frac{\partial p_x}{\partial L}+Y\frac{\partial q_x}{\partial L}$$

Comparing the above with the first of Eqs. (53) it is clear that

$$X'(L, \ell)=X(p_x, q_x)\frac{\partial p_x}{\partial L}+Y(p_x, q_x)\frac{\partial q_x}{\partial L} \tag{54}$$

If Eqs. (52) are multiplied by $\frac{\partial p_x}{\partial \ell}$, $-\frac{\partial q_x}{\partial \ell}$ and similar operations carried out the result is

$$Y'(L, \ell)=X\frac{\partial p_x}{\partial \ell}+Y\frac{\partial q_x}{\partial \ell} \tag{55}$$

At this point we can consider an example of a non-conservative system such as a relaxation oscillator. Let this system be described by van der Pol's equation which is of the form

$$\ddot{q}_x+\omega^2 q_x+\varepsilon(q_x^2-1)\,p_x=0$$

Suppose the solution to be periodic of the form

$$q_x = q_{xo} \cos \omega(t - t_o)$$
$$p_x = \dot{q}_x = - q_{xo}\, \omega \sin \omega(t - t_o) \tag{56}$$

and as before let

$$L = \frac{\omega}{2} q_{xo}^2$$
$$\ell = \omega(t - t_o) = \omega t + B \tag{57}$$

so that Eqs. (56) are transformed to

$$q_x = \sqrt{\frac{2L}{\omega}} \cos \ell$$
$$p_x = - \sqrt{2L\omega} \sin \ell \tag{58}$$

and the canonical equations are now

$$\dot{q}_x = \frac{\partial H_o}{\partial p_x}$$
$$\dot{p}_x = - \frac{\partial H_o}{\partial q_x} - Y \tag{59}$$

where

$$H_o = \frac{p_x^2}{2} + \frac{1}{2} \omega^2 q_x^2 = H_o'(L) = \omega L$$

$$X = 0$$

$$Y = \varepsilon(q_x^2 - 1)p_x$$

Upon differentiating Eqs. (58) it is clear that

$$\frac{\partial q_x}{\partial \ell} = \frac{p_x}{\omega}$$

$$\frac{\partial q_x}{\partial L} = \frac{1}{\omega q_x}$$

Substituting these results in Eqs. (54) and (55) results in

$$X' = \frac{\varepsilon(q_x^2 - 1)p_x}{\omega q_x} = \varepsilon\left[\tan \ell - q_{xo}^2 \sin \ell \cos \ell\right]$$

$$Y' = \frac{\varepsilon(q_x^2 - 1)p_x^2}{\omega} = \varepsilon\omega \left[\left(q_{xo}^4 + q_{xo}^2\right) \cos^2\ell - q_{xo}^4 \cos^4\ell - q_{xo}^2\right]$$

Referring to Eqs. (53), and expressing the above equations as a trigonometric series, we find that

$$\frac{d\ell}{dt} = \omega + \varepsilon \left[\tan \ell - \frac{q_{xo}^2}{2} \sin 2\ell\right]$$

$$\frac{dL}{dt} = -\frac{\varepsilon\omega}{2}\left[\left(\frac{q_{xo}^4}{4} - q_{xo}^2\right) + q_{xo}^2 \cos 2\ell - \frac{q_{xo}^4}{4} \cos 4\ell \right] \qquad (60)$$

Note that the first equation which involves the angular rate (and hence the periodicity) contains only short periodic terms whereas the second equation involving the amplitude contains a secular term plus short periodic terms. Taking the values averaged over one period

$$\dot{\ell}_{Av} = \omega$$

$$\dot{L}_{Av} = -\frac{\varepsilon\omega}{2}\left[\frac{q_{xo}^4}{4} - q_{xo}^2\right]$$

Using these average values in order to obtain a first approximation to the solution, we may obtain

$$\dot{q}_{xo} = \frac{\varepsilon}{2} q_{xo}\left(1 - \frac{q_{xo}^2}{4}\right)$$

(since $\dot{L} = \omega q_{xo}\, \dot{q}_{xo}$)

Performing the integration(14)

$$q'_{xo} = \frac{q_{xo} e^{\frac{1}{2}\varepsilon(t-t_o)}}{\sqrt{1 + \frac{1}{4} q_{xo}^2 \left(e^{\varepsilon(t-t_o)} - 1\right)}} \tag{61}$$

The first approximation is a harmonic oscillation

$$q_x = q'_{xo} \cos \omega(t-t_o)$$

where the amplitude varies according to Eq. (61) and the frequency is constant.

The preceding example was relatively simple because of the simple form of the Hamiltonian H_o (a function of L only). Instead of Eq. 53, consider the case where the canonical equations are

$$\frac{d\ell}{dt} = \frac{\partial H}{\partial L} + X'$$

$$\frac{dL}{dt} = -\frac{\partial H}{\partial \ell} + Y'$$

where $H = H_o(L) + H_1(\ell,L) + \cdots$

the subscripts having the same significance as in the example of the previous section. One or more successive canonical transformations may be performed by the von Zeipel method so as to eliminate the angle variables from the new Hamiltonian.

For example

$$H(L,\ell) \longrightarrow H^*(L' -)$$

$$\dot{\ell}' = \frac{\partial H^*}{\partial L'} + X''$$

$$\dot{L}' = -\frac{\partial H^*}{\partial \ell'} - Y''$$

Or, if two transformations are involved

$$\dot{\ell}'' = \frac{\partial H^{**}}{\partial L'} + X'''$$

$$\dot{L}'' = -\frac{\partial H^{**}}{\partial \ell''} - Y'''$$

V. APPLICATIONS TO CELESTIAL MECHANICS

The non-linear spring and Van der Pol equations were chosen because of the simplicity involved in having only one degree of freedom whereas perturbation problems in celestial mechanics usually involve either three or four degrees of freedom. As in the preceding examples, the ultimate objective is to express the effects of the perturbation as time variations in the two-body orbital parameters. As a first step, the Hamiltonian is written in terms of a convenient set of canonical variables. A usual procedure for doing this is to start with the well-known and basic Delaunay variables: L,G,H representing momenta and l,g,h representing the conjugate angle variables. That is

MOMENTA	COORDINATES
$L = \sqrt{\mu a}$	$l = M$ (mean anomaly)
$G = L\sqrt{1-e^2}$	$g = \omega$ (long. of perigee)
$H = G \cos i$	$h = \Omega$ (node)

where a, e, i, Ω, ω, and T are the familiar non-canonical Keplerian elements.(7) These canonical elements may then be regarded as a key which allows the Hamiltonian to be written at once and leads to the canonical H-J equations. A very important point is that the Delaunay variables may be transformed in a prescribed manner by simple algebraic manipulation to one of many other different sets which may be more adaptable to the particular problem (e.g., Poincaré variables and variations thereof).(7)

In celestial mechanics, the letter F rather than H is usually used to denote the Hamiltonian* in order to avoid confusion with Delaunay's H.

Then letting

$$F_o = F_o(L) = \frac{\mu^2}{2L^2}$$

will give the correct set of 2n = 6 canonical H-J differential equations leading to the unperturbed two-body solution. Since

$$\dot{q}_x = -\frac{\partial F}{\partial p_x} \qquad \dot{p}_x = \frac{\partial F}{\partial q_x}$$

when $F = F_o$, the time derivatives of all but one of the canonical elements are zero, i.e.,

$$\dot{l} = -\frac{\partial F_o}{\partial L} = \frac{\mu^2}{L^3} = \frac{\sqrt{\mu}}{a^{3/2}} = n$$

The familiar two-body elliptical equations are used to obtain position and velocity as a function of the orbital elements and time.

In writing the Hamiltonian F in the general (i.e., perturbed) case, if the time is present explicitly in F, an additional pair of conjugate variables K, k may be introduced for convenience as a device to eliminate t.(6) The solution proceeds by successively transforming the Hamiltonian F so as to eliminate the angle variables l, g, h, or k. Each transformation may involve the removal of one (or more) of these variables. Such a transformation might go as follows

*Actually, for convenience a minus sign is associated with F. That is, F = - H (the Hamiltonian).

$$F(L,G,H,\ \ell,g,h) \rightarrow F^{*}(L',G',H',-,g,-)$$

$$\text{using } S = S_o + S_1 + \cdots$$

$$F^{*}(L',G',H',-,g,-) \rightarrow F^{**}(L'',G'',H'',-,-,-)$$

$$\text{using } S^{*} = S_o^{*} + S_1^{*} + \cdots$$

The short periodic part of the solution involving ℓ and h is obtained by solving the partial differential equations for S. Similarly, the long periodic terms involving g come from S^{*}. The secular (linear time-varying) changes in the orbital parameters are obtained from the resulting H-J equations.

$$\frac{d\ell''}{dt} = -\frac{\partial F^{**}}{\partial L''} \qquad \frac{dL''}{dt} = \frac{\partial F^{**}}{\partial \ell} = 0$$

$$\frac{dg''}{dt} = -\frac{\partial F^{**}}{\partial G''} \qquad \frac{dG''}{dt} = \frac{\partial F^{**}}{\partial g} = 0$$

$$\frac{dh''}{dt} = -\frac{\partial F^{**}}{\partial H''} \qquad \frac{dH''}{dt} = \frac{\partial F^{**}}{\partial h} = 0$$

VI. THE KRYLOFF AND BOGOLIUBOFF METHOD

It was shown in Section III that the effects of a small perturbing force on a simple harmonic motion could be described or approximated by considering the elements or constants of the linear solution to be time variables. This was expressed by Eqs. (22) which are repeated below using the more common notation of ω for angular frequency instead of k.

$$\frac{dA}{dt} = \frac{\varepsilon X}{\omega} \sin (\omega t + B)$$

$$\frac{dB}{dt} = \frac{\varepsilon X}{A\omega} \cos (\omega t + B)$$

where

$$X = X(p_x, q_x)$$

The Kryloff-Bogoliuboff (K-B) first-order approximation consists of considering $\frac{dA}{dt}$ and $\frac{dB}{dt}$ constant during the interval from zero to $2\pi/\omega$. Accordingly, the objective is to find the average value of the functions on the right-hand sides of the above equations during this time interval. It may easily be shown that if X is a function of q_x only, the average value of $\frac{dA}{dt}$ will be zero and hence A will be a constant.

Next, consider the canonical equations as discussed in Section IV

$$\frac{d\ell}{dt} = \frac{\partial H}{\partial L}$$

$$\frac{d\ell'}{dt} = \frac{\partial H^*}{\partial L'}$$

where

$$H = H(L, \ell) = H_o + H_1$$

$$H^* = H^*(L', -) = H_o^* + H_1^* + H_2^* + \cdots.$$

$$H_1^* = H_{1s}$$

$$L = \frac{\omega q_{xo}^2}{2}$$

$$\ell = \omega(t-t_o) = \omega t + B$$

$$\frac{\partial H}{\partial L} = \frac{\partial H}{\partial q_{xo}} \frac{\partial q_{xo}}{\partial L} = \frac{1}{\omega q_{xo}} \frac{\partial H}{\partial q_{xo}}$$

Referring to Eq. (19), it is clear that if the perturbation term X is a function of q_x only

$$H_1 = \int_0^{q_x} \varepsilon\, X(q_x)\, dq_x$$

and the secular or constant part of H_1 is given by

$$H_{1s} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{q_x} \varepsilon\, X(q_x)\, dq_x\, d\ell \qquad (62)$$

Using the $\varepsilon = 0$ solution

$$q_x = q_{xo} \cos \ell$$

Equation (62) becomes

$$H_{1s} = \frac{-1}{2\pi} \int_0^{2\pi} \int_{a=\pi/2}^{a=\ell} \varepsilon q_{xo} X(q_{xo} \cos \ell) \sin \ell \, d\ell \, da \tag{63}$$

where a is a dummy variable of integration. Therefore, $\frac{d\ell'}{dt}$ is given to first order in ε by

$$\frac{d\ell'}{dt} = \frac{1}{\varpi q_{xo}} \frac{\partial}{\partial q_{xo}} \left[H_o + H_{1s}\right] \tag{64}$$

It is interesting to compare the above procedure with that of the K-B first-order approximation. First it should be noted that by definition

$$\frac{d\ell}{dt} = \varpi + \frac{dB}{dt}$$

Using a prime, as in the preceding, to indicate the perturbed value, the K-B first-order approximation is determined by averaging $\frac{dB}{dt}$ over the period $\frac{2\pi}{\varpi}$ so that

$$\frac{d\ell'}{dt} = \varpi + \frac{\varepsilon}{\varpi q_{xo}} \frac{1}{2\pi} \int_0^{2\pi} X(q_{xo} \cos \ell) \cos \ell \, d\ell \tag{65}$$

It is clear that for the special case where $X(q_x)$ is of the form

$$X(q_x) = \varepsilon q_x^{\alpha} = \varepsilon(q_{xo} \cos \ell)^{\alpha}$$

where α is some constant

Then H_{1s} of Eq. (64) becomes

$$H_{1s} = \frac{q_{xo}^{\alpha+1}}{\alpha+1} \left(+ \frac{\varepsilon}{2\pi}\right) \int_0^{2\pi} \cos^{\alpha+1} \ell \, d\ell \qquad (66)$$

and Eqs. (64) and (65) give the same result

$$\frac{d\ell'}{dt} = \omega + \frac{\varepsilon q_{xo}^{\alpha-1}}{\omega} \frac{1}{2\pi} \int_0^{2\pi} \cos^{\alpha+1} \ell \, d\ell$$

The preceding may be extended for those cases where $X(q_x)$ may be represented as a power series or polynomial in q_x.

NON-CONSERVATIVE SYSTEMS USING THE K-B METHOD

In Section IV, the Van der Pol equation is considered, using a canonical system of equations. The same solution may be obtained using the K-B first approximation by assuming, as before, that the solution is of the form

$$q_x = q_{xo} \cos \omega(t-t_o)$$

$$\dot{q}_x = - q_{xo} \omega \sin \omega(t-t_o)$$

Then

$$X(q_x, p_x) = - (1 - q_x^2) \dot{q}_x = \left(1 - q_{xo}^2 \cos^2 \ell\right) q_{xo} \omega \sin \ell$$

$$= q_{xo} \omega \left[\left(1 - \frac{q_{xo}^2}{4}\right) \sin \ell - \frac{q_{xo}^2}{4} \sin 3\ell\right]$$

Applying the general relationships of Eqs. (22) and finding the mean value over the interval $\ell = \omega(t-t_o) = 2\pi$

$$\frac{d\ell}{dt} = \omega + \frac{\mathcal{E}}{2\pi}\int_0^{2\pi}\left[\left(1 - \frac{q_{xo}^2}{4}\right)\sin\ell\cos\ell - \frac{q_{xo}^2}{4}\sin 3\ell\cos\ell\right]d\ell$$

$$= \omega$$

$$\frac{dL}{dt} = \omega q_{xo}\,\dot{q}_{xo} = \frac{\mathcal{E}\omega q_{xo}^2}{2\pi}\int_0^{2\pi}\left[\left(1 - \frac{q_{xo}^2}{4}\right)\sin^2\ell - \frac{q_{xo}^2}{4}\sin 3\ell\sin\ell\right]d\ell$$

$$= -\frac{\mathcal{E}\omega}{2}\left[\frac{q_{xo}^4}{4} - q_{xo}^2\right]$$

These results agree with the average or secular terms as derived from Eqs. (60).

VII. CONCLUSIONS

The preceding material primarily concerns the use of canonical variables and the Hamilton-Jacobi equations in finding approximate solutions to non-linear mechanics problems. Since von Zeipel's perturbation technique is similar in many respects to the Kryloff-Bogoliuboff method, some discussion and comparison of the two is given, particularly since the latter is popular for solving problems in the fields of engineering and physics whereas the former is popular in astronomy. Both methods allow the solution of a complicated non-linear problem to be separated into a preconceived desirable form (e.g., separation of periodic terms into convenient groups).

As pointed out in Ref. (7), the use of a canonical system of equations permits a transformation to be applied to any number of terms in the disturbing function simultaneously, and the Hamiltonian is obtained directly in the course of finding the determining function. Moreover, the new canonical variables resulting from the transformations are related to the original non-canonical set by simple formulas arising from the unperturbed solution. The solution may be carried out theoretically to any desired degree of accuracy in terms of powers of a small parameter ϵ. From a user's standpoint it is convenient because the solution is obtained in a very methodical way involving no great mathematical difficulty.

Another interesting point is that the method of solution does not contribute erroneous secular terms to the solution. As pointed out by Poincaré[11] and others, the classical perturbation technique (due to Poisson) gives rise to linear time-varying terms which do not have physical meaning (e.g., a linear time-increasing amplitude in the non-linear spring problem).

Appendix A

HIGHER-ORDER SOLUTION OF NON-LINEAR SPRING PROBLEM

As outlined in Section IV, the solution may be carried out to higher orders of accuracy by evaluating successively the partial differential equations resulting from the Taylor Series expansion of the Hamiltonian. S_1 has been determined from Eq. (32) as

$$S_1 = -\frac{\varepsilon L'^2}{4k^3} (\sin 2l + 1/8 \sin 4l)$$

S_2 may be determined by finding the part of Eq. (33) periodic in l. That is, the periodic part of

$$\frac{\partial H_o}{\partial L'} \frac{\partial S_2}{\partial l} + 1/2 \frac{\partial^2 H_o}{\partial L'^2} \left(\frac{\partial S_1}{\partial l}\right)^2 + \frac{\partial H_1}{\partial L'} \frac{\partial S_1}{\partial l} = 0$$

First we accumulate the following relationships

$$\frac{\partial H_o}{\partial L'} = k$$

$$\frac{\partial^2 H_o}{\partial L'^2} = 0$$

$$\frac{\partial H_1}{\partial L'} = \frac{2\varepsilon L'}{k^2} \cos^4 l \qquad \text{(from Eq. 34)}$$

$$\frac{\partial S_1}{\partial l} = -\frac{\varepsilon L'^2}{k^3} (\cos^4 l - 3/8) \qquad \text{(from Eq. 38)}$$

Hence, from the above

$$\frac{\partial S_2}{\partial \ell} = + \text{ periodic part of } \frac{2\varepsilon^2 L'^3}{k^6} \left[\cos^8 \ell - \frac{3}{8} \cos^4 \ell\right]$$

The trigonometric identity for $\cos^4 \ell$ is given prior to Eq. (35). The identity for $\cos^8 \ell$ is

$$\cos^8 \ell = \frac{1}{128} \left[\cos 8\ell + 8 \cos 6\ell + 28 \cos 4\ell + 56 \cos 2\ell + 35\right]$$

from which we may derive the following:

$$\frac{\partial S_2}{\partial \ell} = \frac{\varepsilon^2 L'^3}{2k^6} \left[\cos 2\ell + \frac{11}{16} \cos 4\ell + \frac{1}{4} \cos 6\ell + \frac{1}{32} \cos 8\ell\right]$$

$$S_2 = - \frac{\varepsilon^2 L'^3}{4k^6} \left[\sin 2\ell + \frac{11}{32} \sin 4\ell + \frac{1}{12} \sin 6\ell + \frac{1}{128} \sin 8\ell\right]$$

$$\frac{\partial S_2}{\partial L'} = - \frac{3\varepsilon^2 L'^2}{4k^6} \left[\sin 2\ell + \frac{11}{32} \sin 4\ell + \frac{1}{12} \sin 6\ell + \frac{1}{128} \sin 8\ell\right]$$

Note that $\frac{\partial S_1}{\partial \ell}$, $\frac{\partial S_2}{\partial \ell}$, etc., being constrained to be purely periodic in ℓ, cannot contribute to H_1^* or H_2^* respectively. To determine H_3^*, we refer to Eq. (49) which is repeated below:

$$\frac{\partial H_o}{\partial L'} \frac{\partial S_3}{\partial \ell} + \frac{1}{2} \frac{\partial^2 H_1}{\partial L'^2} \left(\frac{\partial S_1}{\partial \ell}\right)^2 + \frac{\partial H_1}{\partial L'} \frac{\partial S_2}{\partial \ell} = H_3^*(L')$$

and

$$H_3^* = \text{part independent of } l \text{ of } \frac{1}{2}\frac{\partial^2 H_1}{\partial L'^2}\left(\frac{\partial S_1}{\partial l}\right)^2 + \frac{\partial H_1}{\partial L'}\frac{\partial S_2}{\partial l}$$

$$= \text{part independent of } l \text{ of } \frac{\varepsilon}{k^2}\cos^4 l\left(\frac{\partial S_1}{\partial l}\right)^2 + \frac{2\varepsilon L'}{k^2}\cos^4 l\,\frac{\partial S_2}{\partial l}$$

Carrying out the indicated multiplication and retaining only the constant or non-periodic terms, we obtain

$$H_3^* = \frac{375}{1024}\frac{\varepsilon^3 L'^4}{k^8}$$

and

$$\frac{\partial H_3^*}{\partial L'} = \frac{375}{256}\frac{\varepsilon^3 L'^3}{k^8} = \frac{375}{2048}\frac{\varepsilon^3 q_{xo}^{\ 6}}{k^5}$$

The secular solution of Eqs. (43) may now be modified to include terms of $O(\varepsilon^3)$

$$\dot{l}' = \frac{\partial H^*}{\partial L'} = k + \frac{3}{4}\frac{\varepsilon L'}{k^2} - \frac{51}{64}\frac{\varepsilon^2 L'^2}{k^5} + \frac{375}{256}\frac{\varepsilon^3 L'^3}{k^8}$$

The periodic portion of the solution (Eqs. 40 and 41) may be similarly modified to include terms of $O(\varepsilon^2)$. The mean angular rate is expressed in terms of the initial displacement by using the substitution

$$L' = \frac{k q_{xo}'^{\,2}}{2}$$

to obtain

$$\dot{l}' = k\left[1 + \frac{3}{8}\varepsilon\left(\frac{q_{xo}}{k}\right)^2 - \frac{51}{256}\varepsilon^2\left(\frac{q_{xo}}{k}\right)^4 + \frac{375}{2048}\varepsilon^3\left(\frac{q_{xo}}{k}\right)^6 + \dots\right]$$

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