FACTORIZATIONS AND REPRESENTATIONS OF SECOND ORDER LINEAR RECURRENCES WITH INDICES IN ARITHMETIC PROGRESSIONS

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Abstract. In this paper we consider second order recurrences $\{V_k\}$ and $\{U_n\}$. We give second order linear recurrences for the sequences $\{V_{\pm kn}\}$ and $\{U_{\pm kn}\}$. Using these recurrence relations, we derive relationships between the determinants of certain matrices and these sequences. Further, as generalizations of the earlier results, we give representations and trigonometric factorizations of these sequences by matrix methods and methods relying on Chebyshev polynomials of the first and second kinds. We give the generating functions and some combinatorial representations of these sequences.

1. Introduction

Let A and B be nonnegative integers such that $A^2 + 4B \neq 0$. The generalized Lucas sequence $\{V_n(A,B)\}$ and the generalized Fibonacci sequence $\{U_n(A,B)\}$ are defined by: for $n > 0$

$$\begin{aligned} V_{n+1}(A,B) &= AV_n(A,B) + BV_{n-1}(A,B) \\ U_{n+1}(A,B) &= AU_n(A,B) + U_{n-1}(A,B) \end{aligned}$$

where $V_0(A,B) = 2, V_1(A,B) = A$ and $U_0(A,B) = 0, U_1(A,B) = 1$, respectively. We will frequently use the notations V_n and U_n instead of $V_n(A,B)$ and $U_n(A,B)$. The Binet formulas of the sequences $\{U_n\}$ and $\{V_n\}$ are given by

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n$$

where α and β are the roots of the equation $t^2 - At - B = 0$.

When $A = B = 1$, then $V_n(1,1) = L_n$ (nth Lucas number) and $U_n(1,1) = F_n$ (nth Fibonacci number).

Lind (cf. [8, p. 478]) first gave the following trigonometric factorization of Fibonacci numbers and then, two years later Zeitlin derived a factorization of the Lucas numbers using trigonometric factorizations of the Chebyshev polynomials of the first kind [19]:

$$\begin{aligned} F_n &= \prod_{k=1}^{n} (1 - 2i\cos(k\pi/n), \\ L_n &= \prod_{k=0}^{n-1} (1 - 2i\cos(2k+1)\pi/2n). \end{aligned}$$

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14. ABSTRACT
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In [17] and [4], the authors gave complex factorization of the Fibonacci numbers by considering the roots of Fibonacci polynomials. In [10], the author established the following representations:

$$F_n = i^{n-1}\frac{\sin\left(n\cos^{-1}\left(-\frac{i}{2}\right)\right)}{\sin\left(\cos^{-1}\left(-\frac{i}{2}\right)\right)},\ L_n = 2i^n\cos\left(n\cos^{-1}\left(-\frac{i}{2}\right)\right),\ n\geq 1$$

Also in [3], the authors obtained the same results on the trigonometric factorizations of the Fibonacci and Lucas numbers by matrix methods. The matrix method was first used by them.

Recently, in [7], the authors consider the backward second order linear recurrences and they gave the trigonometric factorizations and representations of these sequences. Note that this case will be special case with $k=1$ of the present paper.

The second order linear recurrences have been studied by many authors. For example, in [1], the author gave the following combinatorial representation:

$$V_n = \sum_{k=0}^{\lfloor n/2\rfloor}\frac{n}{n-k}\binom{n-k}{k}A^{n-2k}B^k \tag{1.1}$$

$$U_{n+1} = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}A^{n-2k}B^k \tag{1.2}$$

There are many relationships between linear recurrence relations and determinants of certain matrices. For example, the generalized Lucas sequence can be obtained by the following determinant (see [15, 16, 5, 6]):

$$\begin{vmatrix} A & -2B & & & \\ 1 & A & -B & & \\ & 1 & A & \ddots & \\ & & \ddots & \ddots & -B \\ & & & 1 & A \end{vmatrix} = V_n.$$

Especially the case $A=B=1$ can be found in [2]. Furthermore one can find similar special relationships in [16, 15, 9, 5, 6].

In this paper, we consider the positively and negatively kn subscripted terms of the sequences $\{V_n\}$ and $\{U_n\}$, and we derive relationships between these and the determinants of certain tridiagonal matrices. Then we give the general trigonometric factorizations and representations of terms of these sequences $\{V_{\mp kn}\}$ and $\{U_{\mp kn}\}$. We also present generating functions and combinatorial representations of these sequences.

2. Forward and Backward Generalized Lucas Sequences $\{V_{kn}\},\{V_{-kn}\}$

In this section, for an arbitrary positive integer k, we consider the terms V_{kn} and give a second order linear recurrence relation for the sequence $\{V_{kn}\}$. We start with the following useful lemma.

Lemma 1. *For $k>0$, and $n>1$, the terms V_{kn} satisfy the following recurrence relation*

$$V_{kn} = V_kV_{k(n-1)} + (-1)^{k+1}B^kV_{k(n-2)}.$$

Proof. From the Binet formula and since $\alpha\beta = -B$, we can write

$$\begin{aligned} & V_k V_{k(n-1)} + (-1)^{k+1} B^k V_{k(n-2)} \\ = \; & V_k V_{k(n-1)} - (-B)^k V_{k(n-2)} \\ = \; & \left(\left(\alpha^k + \beta^k\right) \left(\alpha^{k(n-1)} + \beta^{k(n-1)}\right) - (\alpha\beta)^k \left(\alpha^{k(n-2)} + \beta^{k(n-2)}\right) \right) \\ = \; & \alpha^{kn} + \beta^{kn} \\ = \; & V_{kn}, \end{aligned}$$

which is as desired. □

Now we describe a relationship between the terms of sequence $\{V_{kn}\}$ and the determinant of a certain tridiagonal matrix.

Define the $n \times n$ tridiagonal matrix $T_n = (t_{ij})$ by

$$T_n = \begin{bmatrix} V_k & 2(-B)^{k/2} & & & \\ (-B)^{k/2} & V_k & (-B)^{k/2} & & \\ & (-B)^{k/2} & V_k & \ddots & \\ & \ddots & & \ddots & (-B)^{k/2} \\ & & & (-B)^{k/2} & V_k \end{bmatrix}.$$

Theorem 1. *For $n > 1$*

$$\det T_n = V_{kn},$$

where $\det T_1 = V_k$.

Proof. We will use the principle of mathematical induction to show that $\det T_n = V_{kn}$. If $n = 2$, then, by Lemma 1, we obtain

$$\det T_2 = \begin{vmatrix} V_k & 2(-B)^{k/2} \\ (-B)^{k/2} & V_k \end{vmatrix} = V_{2k}.$$

Suppose that the equation holds for $n - 1$. Then we show that the equation holds for n. Expanding $\det T_n$ by the Laplace expansion of a determinant according to the last row, we obtain

$$\det T_n = V_k \det T_{n-1} - (-B)^k \det T_{n-2}.$$

By our assumption and the result of Lemma 1, we have the required conclusion:

$$\det T_n = V_k V_{k(n-1)} - (-B)^k V_{k(n-2)} = V_{kn}.$$

□

In the remaining of the section we consider the terms of the backward Lucas sequence $\{V_{-kn}\}$, and we give a second order linear recurrence relation for these, similar to the positively subscripted terms. Then we determine a certain matrix whose successive determinants equal the terms V_{-kn}.

Lemma 2. *For $k \geq 1$ and $n > 1$,*

$$V_{-kn} = (-B)^{-k} \left(V_k V_{-k(n-1)} - V_{-k(n-2)}\right).$$

Proof. From the Binet formulas of sequence $\{V_{-n}\}$, we have that $\alpha\beta = -B$ and so $V_{-n} = V_n (-B)^{-n}$. The proof follows from Lemma 1. □

Now we give a relationship between the determinant of a certain tridiagonal matrix and the terms of the backward general Lucas sequence.

Define the $n \times n$ tridiagonal matrix H_n by

$$H_n = \begin{bmatrix} V_{-k} & 2(-B)^{-k/2} & & & \\ (-B)^{-k/2} & V_{-k} & (-B)^{-k/2} & & \\ 0 & (-B)^{-k/2} & V_{-k} & \ddots & \\ & & \ddots & \ddots & (-B)^{-k/2} \\ & & & (-B)^{-k/2} & V_{-k} \end{bmatrix}.$$

As a consequence of Theorem 1, we have the following result.

Corollary 1. *For* $n > 1$,

$$\det H_n = V_{-kn}$$

where $\det H_1 = V_{-k}$.

Proof. From the definitions of the matrices H_n and T_n, using the identity $V_{-n} = (-B)^{-n} V_n$, it is seen that $H_n = (-B)^{-k} T_n$, and so

$$\det H_n = (-B)^{-kn} \det T_n. \tag{2.1}$$

By Theorem 1 and equation 2.1, we obtain

$$\det H_n = (-B)^{-kn} V_{kn} = V_{-kn}.$$

The proof is complete. □

3. Trigonometric factorizations of the General Lucas sequences $\{V_{kn}\}$ and $\{V_{-kn}\}$

In this section we give the trigonometric factorizations and representations of the generalized Lucas sequences $\{V_{kn}\}$ and $\{V_{-kn}\}$ by matrix methods.

Define the $n \times n$ tridiagonal matrix Q as below:

$$Q = \begin{bmatrix} 0 & 2 & & \\ 1 & 0 & \ddots & \\ & \ddots & & 1 \\ & & 1 & 0 \end{bmatrix}.$$

The characteristic equation of the matrix Q satisfies the following equation

$$t_{n+1}(\lambda) = -\lambda t_n(\lambda) - t_{n-1}(\lambda), \ n > 0,$$

where $t_0(\lambda) = -\lambda$ and $t_1(\lambda) = \lambda^2 - 2$.

The Chebyshev polynomials of the first kind are defined by the following equation

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \ n > 1,$$

where $T_0(x) = 1, \ T_1(x) = x$.

The zeros of the Chebyshev polynomials of the first kind are given by (for more details see [11, 12, 14])

$$x_k = \cos \tfrac{(2k-1)\pi}{2n}, \ k = 1, 2, \ldots, n.$$

If we take $\lambda \equiv -2x$, then the sequence $\{t_n(\lambda)\}$ is reduced to the sequence of Chebyshev polynomials of the first kind, $\{2T_n(x)\}$. Therefore the zeros of the characteristic equation of matrix Q are given by

$$\lambda_k = -2\cos\tfrac{(2k-1)\pi}{2n}, \text{ for } k = 1, 2, \ldots, n \tag{3.1}$$

From the definitions of Q and T_n, we can write $T_n = V_k I_n + (-B)^{k/2} Q$ where I_n is the $n \times n$ unit matrix.

Theorem 2. *For $n > 1$,*

$$V_{kn} = \prod_{r=1}^{n} \left[V_k - 2(-B)^{k/2} \cos\left(\tfrac{(2r-1)\pi}{2n}\right) \right].$$

Proof. Let λ_r, $r = 1, 2, \ldots, n$, be the eigenvalues of Q with respect to eigenvectors x_r. Then, for all r

$$T_n x_k = \left[V_k I_n + (-B)^{k/2} Q \right] x_r = V_k I_n x_r + (-B)^{k/2} Q x_r = \left[V_k + (-B)^{k/2} \lambda_r \right] x_r.$$

Thus $\mu_r = V_k + (-B)^{k/2} \lambda_r$, $r = 1, 2, \ldots, n$, are the eigenvalues of T_n. Thus by (3.1)

$$\det T_n = \prod_{r=1}^{n} \left[V_k + (-B)^{k/2} \lambda_r \right] = \prod_{r=1}^{n} \left[V_k - 2(-B)^{k/2} \cos\left(\tfrac{(2r-1)\pi}{2n}\right) \right],$$

and the proof is complete. □

As a corollary, we obtain Lind's result [8, p. 478].

Corollary 2. *When $A = B = k = 1$, then by the above theorem, we obtain*

$$L_n = \prod_{r=1}^{n} \left[1 - 2i \cos\left(\tfrac{(2r-1)\pi}{2n}\right) \right].$$

As a consequence of Theorem 2, we give the following corollary.

Corollary 3. *For $n > 1$,*

$$V_{-kn} = \prod_{r=1}^{n} \left[V_{-k} - 2(-B)^{-k/2} \cos\left(\tfrac{(2r-1)\pi}{2n}\right) \right].$$

Proof. From Theorem 2, we have

$$V_{kn} = \prod_{r=1}^{n} \left[V_k - 2(-B)^{k/2} \cos\left(\tfrac{(2r-1)\pi}{2n}\right) \right].$$

Multiplying the above equation by $(-B)^{kn}$, we have the conclusion since $V_{-n} = (-B)^{-n} V_n$. □

Alternatively, one can consider the equation $H_n = V_{-k} I_n + (-B)^{-k/2} Q$, and the next result follows.

Theorem 3. *For $k \geq 1$ and $n > 1$,*

$$V_{\mp kn} = (-B)^{\mp kn/2} \cos\left(n \cos^{-1}\left(\tfrac{V_{\mp k}}{2(-B)^{\mp k/2}} \right) \right).$$

Proof. First, we consider the case V_{kn}. If the $n \times n$ matrix G_n has the following form

$$G_n(x) = \begin{bmatrix} 2x & 2 & & 0 \\ 1 & 2x & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 2x \end{bmatrix}, \tag{3.2}$$

then it is seen that $\det G_n(x) = 2T_n(x)$ where $\{T_n(x)\}$ is the sequence of the Chebyshev polynomials of the first kind. Thus

$$\det T_n = (-B)^{kn/2} \det G_n\left(\tfrac{V_k}{2(-B)^{k/2}}\right) = (-B)^{kn/2} T_n\left(\tfrac{V_k}{2(-B)^{k/2}}\right)$$

Defining $x = \cos\theta$ allows the Chebyshev polynomials of the second kind to be written as (see [12])

$$T_n(x) = \cos n\theta. \tag{3.3}$$

Then by (3.3) and the value of determinant of matrix T_n, we obtain

$$V_{kn} = (-B)^{kn/2} \cos\left(n \cos^{-1}\left(\tfrac{V_k}{2(-B)^{k/2}}\right)\right).$$

For the case of V_{-kn}, we consider

$$\det H_n = (-B)^{-kn/2} \det G\left(\tfrac{V_{-k}}{2(-B)^{-k/2}}\right) = (-B)^{-kn/2} T_n\left(\tfrac{V_{-k}}{2(-B)^{-k/2}}\right),$$

from which the proof follows. □

Corollary 4. *For $k \geq 1$ and $n > 1$ even,*

$$V_{\mp kn} = \prod_{r=1}^{\lfloor n/2 \rfloor} \left[V_{\mp k}^2 - 4(-B)^{\mp k} \cos^2\left(\tfrac{(2r-1)\pi}{2n}\right)\right]$$

and for $n > 1$ odd

$$V_{\mp kn} = V_{\mp k} \prod_{k=1}^{(n-1)/2} \left[V_{\mp k}^2 - 4(-B)^{\mp k} \cos^2\left(\tfrac{(2r-1)\pi}{2n}\right)\right].$$

Proof. These are immediate consequences of Theorem 2 and Corollary 3, since, for $1 \leq k < n/2$, $\cos(k\pi/n) = -\cos((n-k)\pi/n)$. □

4. The Generalized Fibonacci Sequence $\{U_n(A, B)\}$

In this section, we consider the recurrence $\{U_n\}$ and then obtain two recurrence relations for the sequences $\{U_{kn}\}$ and $\{U_{-kn}\}$. Also we determine certain tridiagonal matrices and then we obtain relationships between the determinants of these matrices and the sequences $\{U_{kn}\}$ and $\{U_{-kn}\}$. Therefore, we obtain trigonometric factorizations and representations of these sequences. We start with the following useful lemma.

Lemma 3. *For $k \geq 1$ and $n > 1$,*

$$U_{kn} = V_k U_{k(n-1)} + (-1)^{k+1} B^k U_{k(n-2)}.$$

Proof. From the Binet formula of the sequences $\{U_n\}$, $\{V_n\}$ and since $\alpha\beta = -B$, we can write

$$\begin{aligned}
& V_k U_{k(n-1)} + (-1)^{k+1} B^k U_{k(n-2)} \\
= \;& V_k U_{k(n-1)} - (-B)^k U_{k(n-2)} \\
= \;& \left(\left(\alpha^k + \beta^k\right) \left(\tfrac{\alpha^{k(n-1)} - \beta^{k(n-1)}}{\alpha - \beta}\right) - (\alpha\beta)^k \left(\tfrac{\alpha^{k(n-2)} - \beta^{k(n-2)}}{\alpha - \beta}\right)\right) \\
= \;& \tfrac{\alpha^{kn} - \beta^{kn}}{\alpha - \beta} \\
= \;& U_{kn}.
\end{aligned}$$

The proof is complete. $\square$

Define the $n \times n$ tridiagonal Toeplitz matrix M_n by

$$M_n = \begin{bmatrix} V_k & (-B)^{k/2} & & \\ (-B)^{k/2} & V_k & \ddots & \\ & \ddots & \ddots & (-B)^{k/2} \\ & & (-B)^{k/2} & V_k \end{bmatrix}.$$

Since $U_{2k} = U_k V_k$, we have immediately the following result.

Theorem 4. *For $n > 1$,*

$$\det M_n = \frac{U_{k(n+1)}}{U_k},$$

where $\det M_1 = U_{2k}/U_k$.

Proof. It is known that the tridiagonal Toeplitz matrices satisfy

$$\det M_n = V_k \det M_{n-1} - (-B)^k \det M_{n-2}.$$

From the principle of mathematical induction and Lemma 3, the result follows. $\square$

Lemma 4. *For $k \geq 1$ and $n > 1$,*

$$U_{-k(n+1)} = (-B)^{-k} \left(V_k U_{-kn} - U_{-k(n-1)}\right) = V_{-k} U_{-kn} - (-B)^{-k} U_{-k(n-1)}$$

Proof. By the Binet formulas of $\{U_{-n}\}$ and $\{V_{-n}\}$, we can write

$$\begin{aligned}
& (-B)^{-k} V_k U_{-kn} - (-B)^{-k} U_{-k(n-1)} \\
= \;& (-B)^{-k} \left(V_k U_{-kn} - U_{-k(n-1)}\right) \\
= \;& (-B)^{-k} \left(\left(\alpha^k + \beta^k\right) \left(\tfrac{\alpha^{-kn} - \beta^{-kn}}{\alpha - \beta}\right) - \left(\tfrac{\alpha^{-k(n-1)} - \beta^{-k(n-1)}}{\alpha - \beta}\right)\right) \\
= \;& (-B)^{-k} \left(\tfrac{\alpha^{-kn+k} - \beta^{-kn+k} - \alpha^k \beta^{-kn} + \beta^k \alpha^{-kn} - \alpha^{-kn+k} + \beta^{-kn+k}}{\alpha - \beta}\right) \\
= \;& \left(\alpha^{-k}\beta^{-k}\right) \left(\tfrac{-\alpha^k \beta^{-kn} + \beta^k \alpha^{-kn}}{\alpha - \beta}\right) \\
= \;& \tfrac{\alpha^{-kn-k} - \beta^{-kn-k}}{\alpha - \beta} \\
= \;& U_{-k(n+1)},
\end{aligned}$$

and the conclusion follows. $\square$

Define the $n \times n$ tridiagonal Toeplitz matrix E_n as shown below:

$$E_n = \begin{bmatrix} V_{-k} & (-B)^{-k/2} & & \\ (-B)^{-k/2} & V_{-k} & \ddots & \\ & \ddots & \ddots & (-B)^{-k/2} \\ & & (-B)^{-k/2} & V_{-k} \end{bmatrix}.$$

Corollary 5. *For* $n > 1$,

$$\det E_n = \frac{U_{-k(n+1)}}{U_{-k}}$$

where $\det E_1 = U_{2k}/U_k$.

Proof. Since $E_n = (-B)^{-k} M_n$, $\det E_n = (-B)^{kn} \det M_n$. By Theorem 4, the proof follows easily. □

5. Trigonometric factorization of the General Fibonacci sequences $\{U_{kn}\}$ and $\{U_{-kn}\}$

In this section, we give the trigonometric factorizations and representations of sequences $\{U_{kn}\}$ and $\{U_{-kn}\}$ by matrix methods and the Chebyshev polynomials of the second kind.

Define the $n \times n$ tridiagonal matrix W as shown:

$$W = \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & \ddots & \\ & \ddots & & 1 \\ & & 1 & 0 \end{bmatrix}.$$

The characteristic equation of the matrix W satisfies the following recurrence

$$f_{n+1}(y) = -y f_n(y) - f_{n-1}(y), \ n > 0,$$

where $f_0(y) = -y$ and $f_1(y) = y^2 - 1$.

The Chebyshev polynomials of the second kind are defined by the recurrence relation for $n > 1$

$$U_n(x) = 2x U_{n-1}(x) - U_{n-2}(x)$$

where $U_0(x) = 1, \ U_1(x) = 2x$.

The zeros of the Chebyshev polynomials of the second kind is given by (see [11, 12, 14]):

$$x_k = \cos \tfrac{k\pi}{n+1}, \quad k = 1, 2, \ldots, n,$$

Taking $\lambda \equiv -2x$, the sequence $\{f_n(y)\}$ is reduced to the sequence $\{U_n(x)\}$. Then the zeros of the characteristic equation of matrix W are given by

$$y_k = -2\cos \tfrac{k\pi}{n}, \quad \text{for } k = 1, 2, \ldots, n. \tag{5.1}$$

By the definitions of W, M_n and E_n, we write $M_n = V_k I_n + (-B)^{k/2} W$ and $E_n = V_{-k} I_n + (-B)^{-k/2} W$ where I_n is the $n \times n$ unit matrix.

Theorem 5. *Then for* $n > 1$,

$$U_{\mp k(n+1)} = U_{\mp k} \prod_{r=1}^{n} \left[V_{\mp k} - 2(-B)^{\mp k/2} \cos\left(\tfrac{r\pi}{n+1}\right) \right].$$

Proof. Let y_r, $r = 1, 2, \ldots, n$, be the eigenvalues of matrix W with respect to the eigenvectors x_r. Then, for all $r = 12, \ldots, n$

$$M_n x_k = \left[V_k I_n + (-B)^{k/2} W\right] x_r = V_k I_n x_r + (-B)^{k/2} W x_r = \left[V_k + (-B)^{k/2} y_r\right] x_r$$

Thus $\omega_r = V_k + (-B)^{k/2} y_r$, $r = 1, 2, \ldots, n$, are the eigenvalues of M_n. Thus by (5.1) and Theorem 4,

$$\det M_n = U_k \prod_{r=1}^{n} \left[V_k + (-B)^{k/2} y_r\right] = U_k \prod_{r=1}^{n} \left[V_k - 2(-B)^{k/2} \cos\left(\tfrac{r\pi}{n}\right)\right].$$

Similarly, one can obtain that $c_r = V_{-k} + (-B)^{-k/2} y_r$, $r = 1, 2, \ldots, n$, are the eigenvalues of the matrix E_n. Thus we obtain

$$\det E_n = U_{-k} \prod_{r=1}^{n} \left[V_{-k} + (-B)^{-k/2} y_r\right] = U_{-k} \prod_{r=1}^{n} \left[V_{-k} - 2(-B)^{-k/2} \cos\left(\tfrac{r\pi}{n+1}\right)\right].$$

Considering the value of $\det E_n$, the proof is complete. □

For example, when $k = 5$, $A = 1$, $B = 1$ in sequence $\{U_n(A, B)\}$, then

$$F_{5(n+1)} = 5 \prod_{r=1}^{n} \left[11 - 2i \cos\left(\tfrac{r\pi}{n+1}\right)\right].$$

Corollary 6. *For an arbitrary positive integer k and $n > 1$ even,*

$$U_{\mp k(n+1)} = U_{\mp k} \prod_{r=1}^{\lfloor n/2 \rfloor} \left[V_{\mp k}^2 - 4(-B)^{\mp k} \cos^2\left(\tfrac{r\pi}{n+1}\right)\right]$$

and for $n > 1$ odd

$$U_{\mp k(n+1)} = U_{\mp 2k} \prod_{k=1}^{\lfloor n/2 \rfloor} \left[V_{\mp k}^2 - 4(-B)^{\mp k} \cos^2\left(\tfrac{r\pi}{n+1}\right)\right].$$

Proof. The proof follows from Theorem 5, since, for $1 \le k < n/2$, $\cos(k\pi/n) = -\cos((n-k)\pi/n)$ and $U_n V_n = U_{2n}$. □

Theorem 6. *For $k \ge 1$ and $n > 1$,*

$$U_{\mp k(n+1)} = U_{\mp k} \frac{(-B)^{\mp kn/2} \sin\left[(n+1)\cos^{-1}\left(\frac{V_{\mp k}}{2(-B)^{\mp k/2}}\right)\right]}{\sin\left(\cos^{-1}\left(\frac{V_{\mp k}}{2(-B)^{\mp k/2}}\right)\right)}.$$

Proof. Let the matrix K_n be defined by

$$K_n(x) = \begin{bmatrix} 2x & 1 & & 0 \\ 1 & 2x & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 2x \end{bmatrix}_{n \times n}. \tag{5.2}$$

It is known that $\det K_n(x) = U_n(x)$, where $\{U_n(x)\}$ is the sequence of the Chebyshev polynomials of the second kind. Thus we obtain

$$\det M_n = U_k(A, B)(-B)^{kn/2} \det K_n\left(\tfrac{V_k(A,B)}{2(-B)^{k/2}}\right) = (-B)^{kn/2} U_n\left(\tfrac{V_k(A,B)}{2(-B)^{k/2}}\right)$$

If $x = \cos\theta$, the Chebyshev polynomials of the second kind can be written as (see [12])

$$U_n(x) = \frac{\sin[(n+1)\theta]}{\sin\theta}. \tag{5.3}$$

Then by (5.3) and the value of the determinant of the matrix M_n, we obtain

$$U_{k(n+1)}(A,B) = \frac{U_k(A,B)(-B)^{kn/2}\sin\left[(n+1)\cos^{-1}\left(\frac{V_k(A,B)}{2(-B)^{k/2}}\right)\right]}{\sin\left(\cos^{-1}\left(\frac{V_k(A,B)}{2(-B)^{k/2}}\right)\right)}.$$

From

$$\det E_n = (-B)^{-kn/2}\det K_n\left(\frac{V_{-k}}{2(-B)^{-k/2}}\right) = (-B)^{-kn/2}\,U_n\left(\frac{V_{-k}}{2(-B)^{-k/2}}\right)$$

and the values of the determinants of matrices E_n and $K_n(x)$, we obtain the conclusion. □

6. Generating Functions

In this section, we give combinatorial representations and generating functions for the terms of sequences $\{V_{kn}\}$ and $\{V_{-kn}\}$, thus generalizing in one direction some results of [13].

Theorem 7. *For an arbitrary positive integer k and $n > 0$,*

$$\begin{aligned} V_{\mp kn} &= \sum_{r=0}^{\lfloor n/2\rfloor} \frac{n}{n-r}\binom{n-r}{r} V_{\mp k}^{n-2r} B^{\mp kr} \\ U_{\mp k(n+1)} &= U_{\mp k}\sum_{r=0}^{\lfloor n/2\rfloor} \binom{n-r}{r} V_{\mp k}^{n-2r} B^{\mp kr} \end{aligned}$$

Proof. We leave the proof of this theorem to the reader. □

Generating functions are useful tools for solving linear homogeneous recurrence relations with constant coefficients (for more details about generating functions of recurrence relations see [18]). Now we give the generating functions for any power of the sequences $\{V_{kn}\}$ and $\{V_{-kn}\}$, generalizing known identities.

Theorem 8. *Let $\{V_n\}$ be the generalized Lucas sequence.*

(a) *If r is odd, then*

$$\sum_{n=0}^{\infty} V_{\pm kn}^r x^n = \sum_{i=0}^{(r-1)/2}\binom{r}{i}\frac{2-(-B)^{ki}V_{\pm k(r-2i)}x}{1-(-B)^{ki}V_{\pm k(r-2i)}x+(-B)^{\pm kr}x^2}.$$

(b) *If r is even, then*

$$\begin{aligned} \sum_{n=0}^{\infty} V_{\pm kn}^r x^n &= \sum_{i=0}^{r/2-1}\binom{r}{i}\frac{2-(-B)^{ki}V_{\pm k(r-2i)}x}{1-(-B)^{ki}V_{\pm k(r-2i)}x+(-B)^{\pm kr}x^2} \\ &\quad + \binom{r}{r/2}\frac{1}{1-(-B)^{\pm kr/2}x}. \end{aligned}$$

Proof. For easy writing, let

$$G_{k,r}(x) = \sum_{n=0}^{\infty} V_{kn}^{r} x^{n}.$$

We deal with the case $r = 1$ separately, as it can be handled easily by a slightly different method from the general case. First we consider the positively subscripted case:

$$\begin{aligned}
&\left(1 - V_k x + (-B)^k x^2\right) G_{k,1}(x) \\
= \; & V_0 + (V_k - V_k V_0) x + \left(V_{2k} - V_k V_k + (-B)^k V_0\right) x^2 \\
&+ \cdots + \left(V_{kn} - V_k V_{k(n-1)} + (-B)^k V_{k(n-2)}\right) x^n + \cdots .
\end{aligned}$$

Since $V_0 = 2$ and by the recurrence relation of $\{V_{kn}\}$ of Lemma 1 the coefficients of x^n for $n > 1$ are all 0. Thus

$$\left(1 - V_k x + (-B)^k x^2\right) G_{k,1}(x) = V_0 + V_k (1 - V_0) x$$

and so

$$\sum_{n=0}^{\infty} V_{kn} x^n = \frac{V_0 + V_k (1 - V_0) x}{1 - V_k x + (-B)^k x^2}.$$

For the negatively subscripted case, the proof for $r = 1$ case follows from Lemma 2 and an argument similar to the one above.

We shall give the proof for arbitrary power r, only for positively subscripted case, since the negatively subscripted one is similar. We write

$$V_{kn}^{r} = \left(\alpha^{kn} + \beta^{kn}\right)^r = \sum_{i=0}^{r} \binom{r}{i} \alpha^{kni} \beta^{kn(r-i)},$$

and so,

$$\begin{aligned}
G_{k,r}(x) &= \sum_{n=0}^{\infty} \sum_{i=0}^{r} \binom{r}{i} \left(\alpha^{ki} \beta^{k(r-i)} x\right)^n \\
&= \sum_{i=0}^{r} \binom{r}{i} \sum_{n=0}^{\infty} \left(\alpha^{ki} \beta^{k(r-i)} x\right)^n \\
&= \sum_{i=0}^{r} \binom{r}{i} \frac{1}{1 - \alpha^{ki} \beta^{k(r-i)} x}.
\end{aligned}$$

We will deal with the case of r odd, only, since the case of r even is similar. Thus, under r odd, using $\binom{r}{i} = \binom{r}{r-i}$ and $\alpha\beta = -B$, we get

$$\begin{aligned}
G_{k,r}(x) &= \sum_{i=0}^{(r-1)/2} \binom{r}{i} \left(\frac{1}{1 - \alpha^{ki} \beta^{k(r-i)} x} + \frac{1}{1 - \alpha^{k(r-i)} \beta^{ki} x} \right) \\
&= \sum_{i=0}^{(r-1)/2} \binom{r}{i} \frac{2 - \alpha^{k(r-i)} \beta^{ki} x - \alpha^{ki} \beta^{k(r-i)} x}{1 - \alpha^{k(r-i)} \beta^{ki} x - \alpha^{ki} \beta^{k(r-i)} x + (\alpha\beta)^{kr} x^2} \\
&= \sum_{i=0}^{(r-1)/2} \binom{r}{i} \frac{2 - (-B)^{ki} V_{k(r-2i)} x}{1 - (-B)^{ki} V_{k(r-2i)} x + (-B)^{kr} x^2},
\end{aligned}$$

since $\alpha^{k(r-i)}\beta^{ki} = (-B)^{ki}\alpha^{k(r-2i)}$, $\alpha^{ki}\beta^{k(r-i)} = (-B)^{ki}\beta^{k(r-2i)}$, and $\alpha^{k(r-2i)} + \beta^{k(r-2i)} = V_{k(r-2i)}$. $\square$

Regarding the generalized Fibonacci sequence, we can show the following theorem that generalizes Theorem 1 of [13], and [8, Formulas 1 and 17 on p. 230]. Since the proof is somewhat similar to the proof of Theorem 8, we will leave it to the interested reader.

Theorem 9. *Let $\{U_n\}$ be the generalized Fibonacci sequence.*

(*a*) *If r is odd, then*

$$\sum_{n=0}^{\infty} U_{\pm kn}^{r} x^n = \delta^{r-1} \sum_{i=0}^{(r-1)/2} (-1)^i \binom{r}{i} \frac{(-B)^{\pm ki} U_{k(r-2i)} x}{1-(-B)^{\pm ki} V_{\pm k(r-2i)} x + (-B)^{\pm kr} x^2}.$$

(*b*) *If r is even, then*

$$\begin{aligned}\sum_{n=0}^{\infty} U_{\pm kn}^{r} x^n &= \delta^{r} \sum_{i=0}^{r/2-1} (-1)^i \binom{r}{i} \frac{2-(-B)^{\pm ki} V_{\pm k(r-2i)} x}{1-(-B)^{\pm ki} V_{\pm k(r-2i)} x + (-B)^{\pm kr} x^2} \\ &\quad + \delta^{r}(-1)^{r/2}\binom{r}{r/2}\frac{1}{1-B^{\pm kr/2}x}.\end{aligned}$$

There is nothing special about the arithmetic progression kn, so one can obtain similar formulas for indices in any other arithmetic progression modulo k. We chose this particular one, namely kn, since it is consistent with the first part of the paper, and the results are easier to state.

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