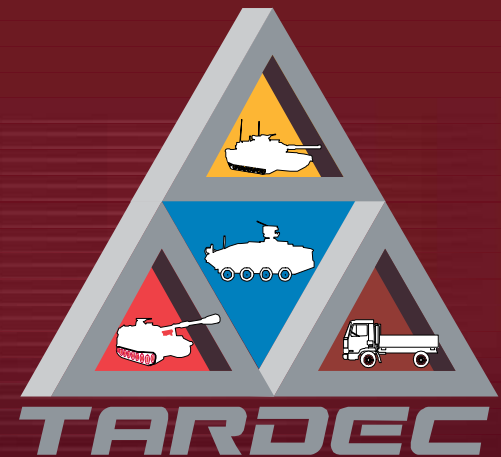




RDECOM

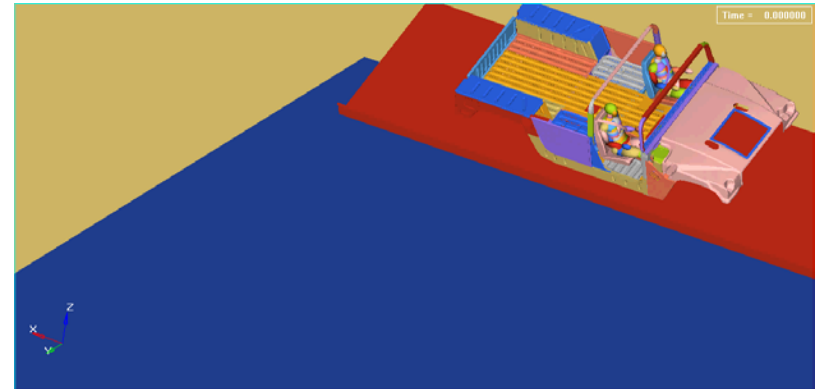


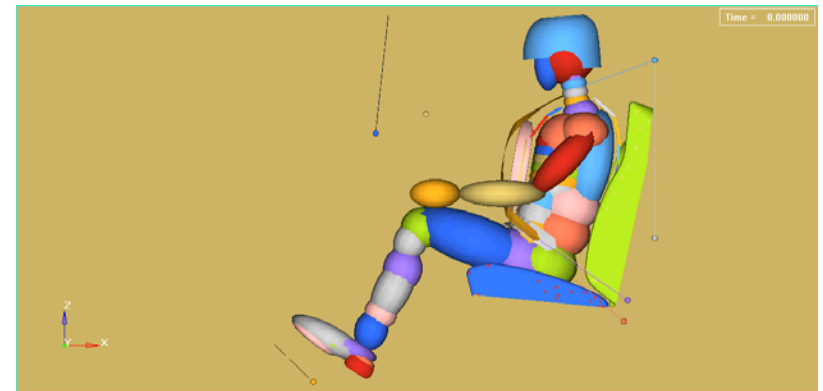
TECHNOLOGY DRIVEN. WARFIGHTER FOCUSED.

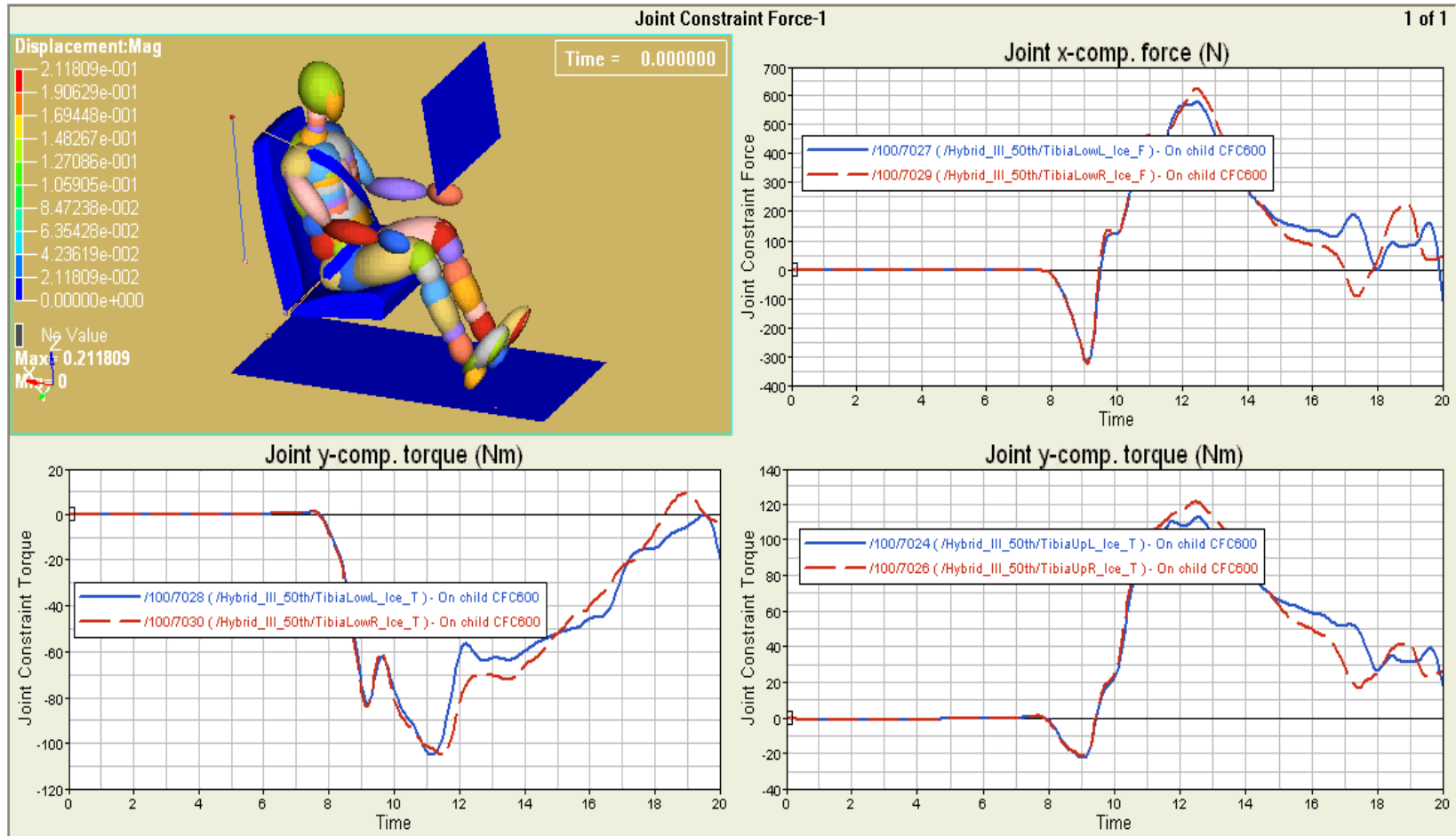
Math-Based Simulation Tools and Methods

Sudhakar Arepally, CRSR Team, US Army RDECOM-TARDEC

10 October 2007









Head

Neck

Chest

Abdomen

Lumbar

Femur

Knee

Tibia

Ankle

Foot

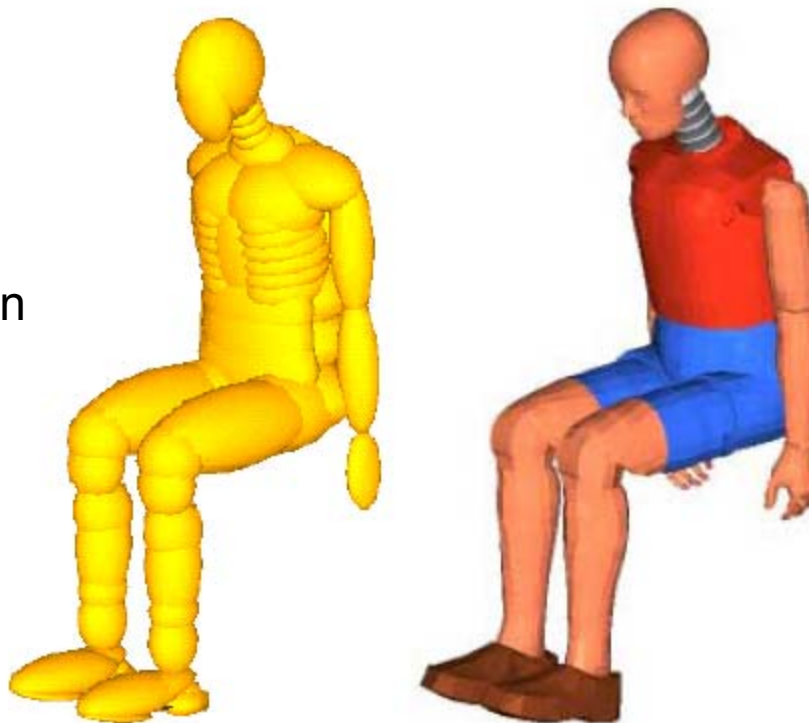
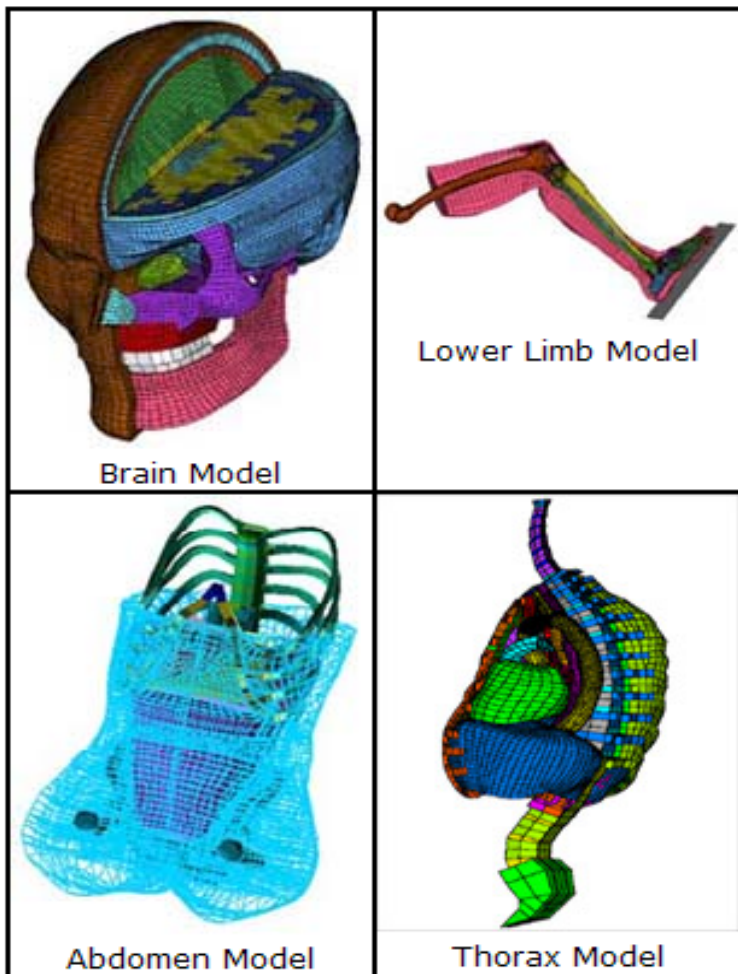
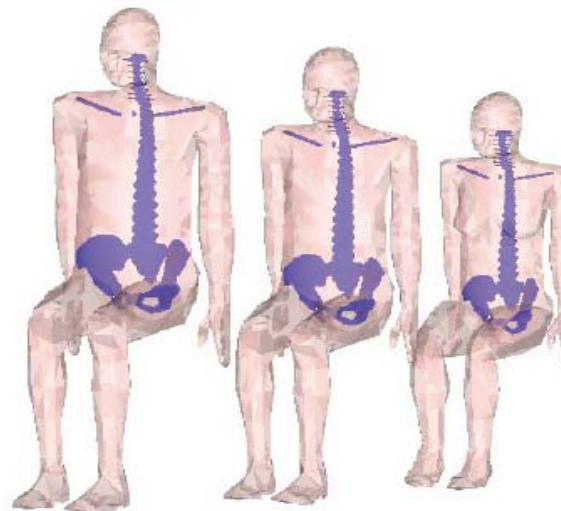


Figure 2.1 Hybrid III 50th percentile dummy model; ellipsoid model (left) and facet model (right).

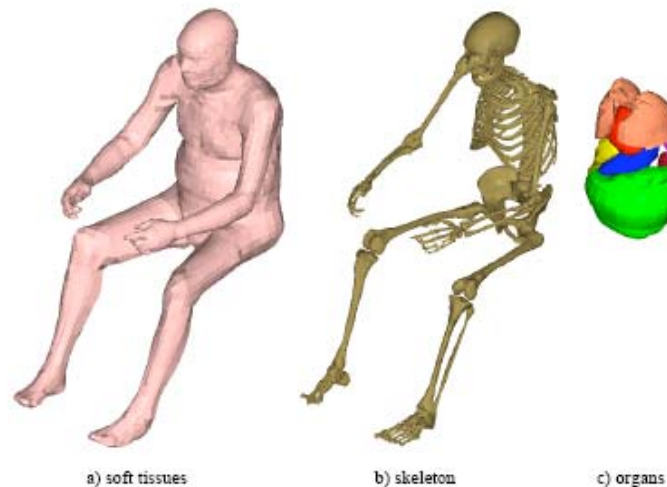


Wayne State - ANSIR

MADYMO- FACET



MADYMO FE- HUMOS2



KINEMATICS OF A RIGID BODY

Position vector of P, a point on body i in reference space is given by,

$$X_i = r_i + x_i$$

or in matrix notation,

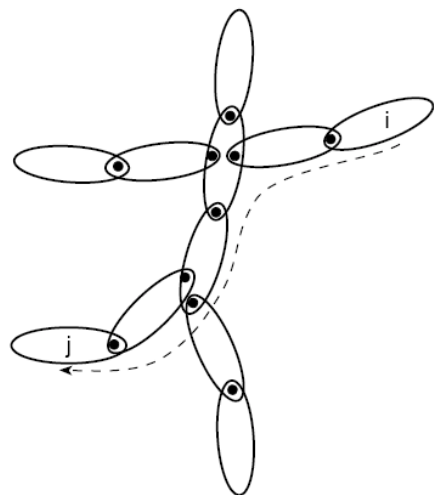
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_i = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}_i + \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}_i \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_i$$

First time derivative equals,

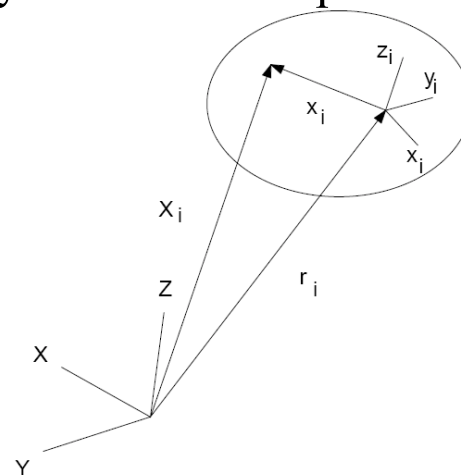
$$\dot{X}_i = \dot{r}_i + \omega_i \times x_i, \text{ where } \omega_i \text{ is the angular velocity vector of body } i$$

And 2nd time derivative equals,

$$\ddot{X}_i = \ddot{r}_i + \dot{\omega}_i \times x_i + \omega_i \times (\omega_i \times x_i)$$



A multibody system is a set of bodies interconnected by a kinematic joints



KINEMATICS OF BODIES CONNECTED BY JOINT

$$\underline{A}_j = \underline{A}_i \underline{C}_{ij} \underline{D}_{ij} \underline{C}_{ji}^T \Rightarrow 1$$

Where,

$\underline{C}_{ij}$ = time dependent direction cosine matrix, joint coordinate system on body i

relative to body local coordinate system on body i

$\underline{D}_{ij}$ = direction cosine matrix, orientation of joint coordinate system on body j with

relative to joint coordinate system on body i

$\underline{A}_i$ = direction cosine matrix, orientation of local coordinate system of body i

$\underline{A}_j$ = direction cosine matrix, orientation of local coordinate system of body j

$$\underline{r}_j = \underline{r}_i + \underline{c}_{ij} + \underline{d}_{ij} - \underline{c}_{ji} \Rightarrow 2$$

$\underline{r}_j$ = position vector of the origin of the local coordinate system on body j

$\underline{r}_i$ = position vector of the origin of the local coordinate system on body i

$\underline{c}_{ij}$ = position vector of the origin of the joint coordinate system on body i relative to the origin

of the local coordinate system of the corresponding body

$\underline{c}_{ji}$ = position vector of the origin of the joint coordinate system on the body j relative to the origin

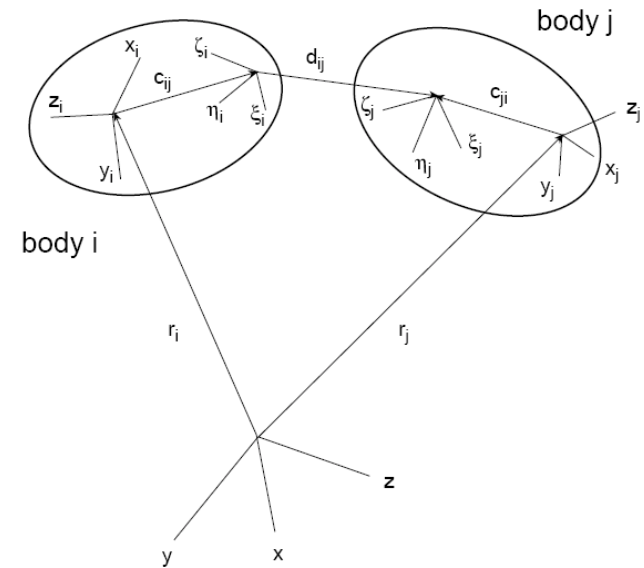
of the local coordinate system of the corresponding body

$\underline{d}_{ij}$ = vector from the origin of the joint coordinate system on body i to the joint coordinate system on body j

Taking first time derivatives, yield the following expressions for angular and linear velocities -

$$\underline{\omega}_j = \underline{\omega}_i + \underline{\omega}_{ij} \Rightarrow 3$$

$$\dot{\underline{r}}_j = \dot{\underline{r}}_i + \underline{\omega}_i \times \underline{c}_{ij} + \dot{\underline{d}}_{ij} - \underline{\omega}_j \times \underline{c}_{ji} \Rightarrow 4$$



Two bodies interconnected by an arbitrary kinematic joint; motion of body j is described relative to parent body i

Equations of motion (Newton - Euler) of rigid body i referred to its center of gravity are :

$$m_i \ddot{r}_i = F_i$$

$$J_i \cdot \dot{\omega}_i + \omega_i \times J_i \cdot \omega_i = T_i$$

Where,

m_i = mass

J_i = inertia tensor w/r to center of gravity

ω_i = angular velocity vector

F_i = resultant force vector relative to the CG

T_i = resultant torque vector relative to the CG

F_i and T_i both include constraint forces and torques due to joints

Using principle of virtual work and variation of position vector, δr_i and a variation of orientation, $\delta \pi_i$, the resulting equations summed for all bodies in the system :

$$\sum \delta r_i \cdot \{m_i \ddot{r}_i - F_i\} + \delta \pi_i \cdot \{J_i \cdot \dot{\omega}_i + \omega_i \times J_i \cdot \omega_i - T_i\} = 0$$

Expression for 2nd derivative of the joint degrees of freedom are obtained :

$$\ddot{\underline{q}}_{ij} = \underline{M}_{ij} \dot{\underline{Y}}_i + \underline{Q}_{ij}$$

Where,

$\dot{\underline{Y}}_i$ is a 6 x 1 column matrix that contains components of linear and angular acceleration of the coordinate system of the parent body i

n_{ij} x 6 matrix $\underline{M}_{ij}$ and the n_{ij} x 1 column matrix of $\underline{Q}_{ij}$ depend on the inertia of the bodies and the instantaneous geometry of the system. $\underline{Q}_{ij}$ depends additionally on the instantaneous geometry of the system and the applied loads.

Numerical integration methods :

The equations of motion form a system of coupled non - linear second order differential equations :

$$\ddot{\underline{q}} = \underline{h}(\underline{q}, \dot{\underline{q}}, t), \text{ with initial values of } \underline{q}_0 \text{ and } \dot{\underline{q}}_0$$

$\underline{q}$ = column matrix with generalized coordinates; joint position degrees of freedom

$\dot{\underline{q}}$ = joint velocity degrees of freedom

$\ddot{\underline{q}}$ = joint acceleration degrees of freedom

Equations of motion are solved numerically using the available three methods :

- a modified Euler method with a fixed or variable time step
- a Runge - Kutta method with a fixed or variable time step
- a Runge - Kutta Merson method with a variable time step

Modified Euler method :

One step method with fixed time step t_s ; second order differential equations are integrated explicit Euler method which gives velocity variables at time point $t_{n+1} = t_n + t_s$,

$$\dot{\underline{q}}_{n+1} = \dot{\underline{q}}_n + t_s \ddot{\underline{q}}_n$$

The velocity variables are integrated using the implicit Euler method which gives solution for the position variables at time point t_{n+1} ,

$$\underline{q}_{n+1} = \underline{q}_n + t_s \dot{\underline{q}}_{n+1}$$

Displacement Matrix

$$\{\delta\}_e = \{u_i, v_i, u_j, v_j, u_m, v_m\}$$

or, displacement function at any point within the element is

$$\{f\}_e = [N]\{\delta\}_e$$

Stress, Strain and Elasticity Matrices

$$\{\sigma\}_e = [D]\{\epsilon\}_e$$

Formulation of Finite Element Methodology using the principle (minimization) of total potential energy of the system -

$$\sum_1^n \int_V (\{\Delta\epsilon\}_e^T \{\sigma\}_e - \{\Delta f\}_e^T \{F\}_e) dV - \sum_1^n \int_s \{\Delta f\}_e^T \{T\}_e ds = 0$$

$$\text{or, } \sum_1^n \{\Delta\delta\}_e^T ([k]_e \{\delta\}_e - \{Q\}_e),$$

the stiffness matrix, $[k]_e$, and nodal force matrix, $\{Q\}_e$,
(due to body force, initial strain, and surface traction) are

$$[k]_e = \int_V [B]^T [D] [B] dV \text{ and}$$

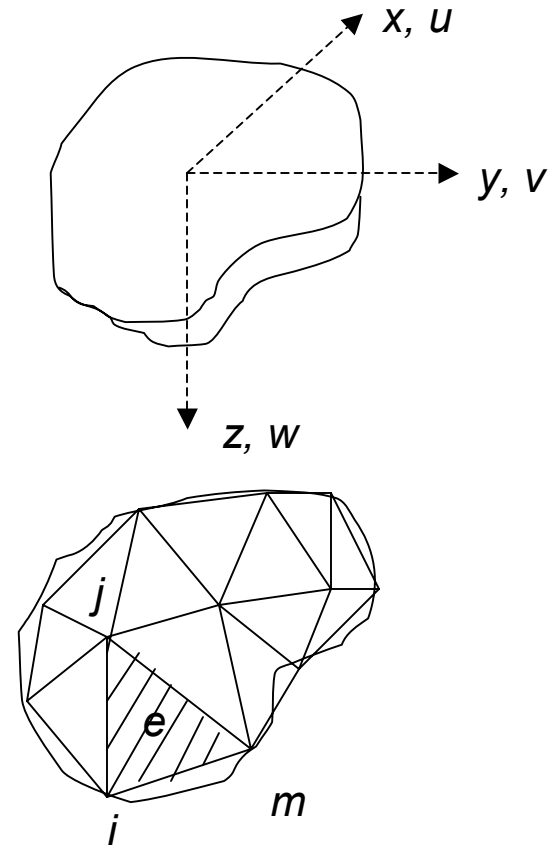
$$\{Q\}_e = \int_V [N]^T \{F\} dV + \int_V [B]^T [D] \{\epsilon_0\} dV + \int_s [N]^T \{T\} ds$$

And the system of equations for equilibrium for nodal forces for the entire structure,

$$[K]\{\delta\} = \{Q\}$$

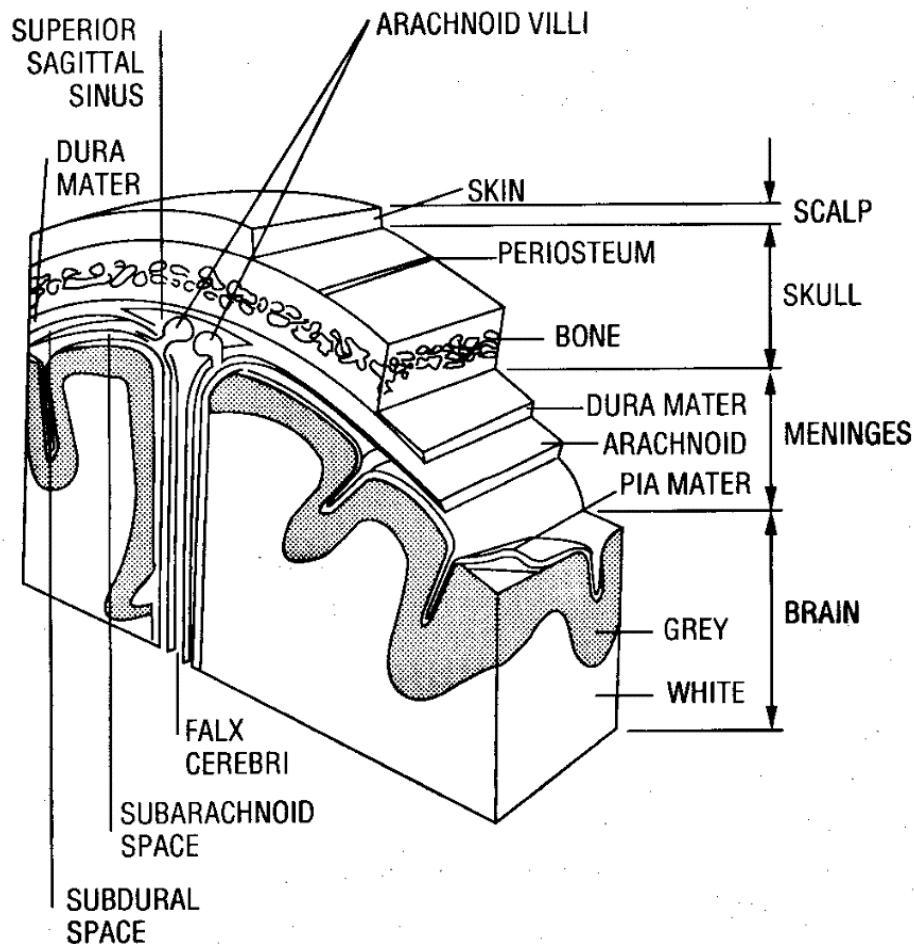
where,

$$[K] = \sum_1^n [k]_e, \text{ and } \{Q\} = \sum_1^n \{Q\}_e$$



Typical finite element 'e'

Scalp, Skull, Meninges and Brain



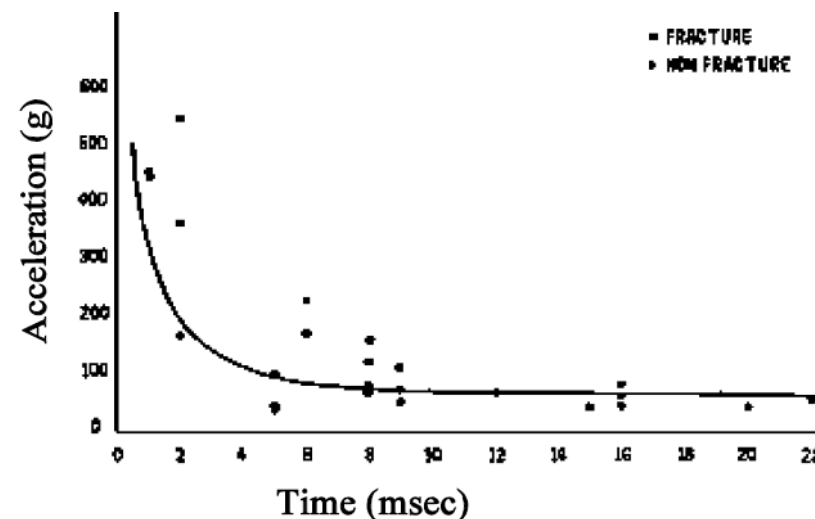
Head Injury Criteria :

$$HIC = \max \left[\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} a(t) dt \right]^{2.5} (t_2 - t_1)$$

Where, t_1 and t_2 are two arbitrary times during acceleration pulse

HIC is limited to 36 ms

Wayne State Tolerance Curve



Lognormal distribution (pdf) :

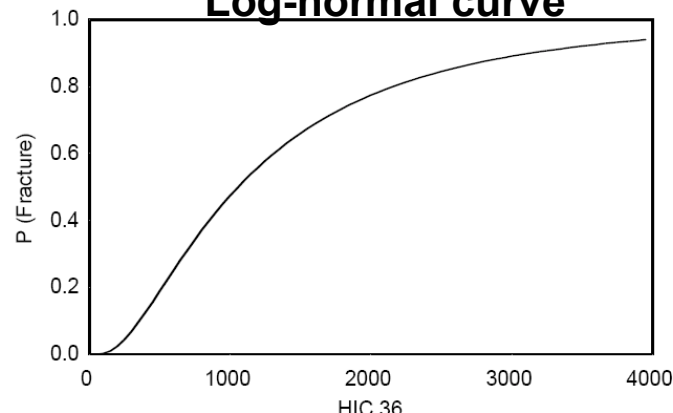
$$f(T') = \frac{1}{\sigma_{T'} \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{T' - \mu'}{\sigma_{T'}} \right)^2}$$

The probability of skull fracture (AIS $\geq$ 2) is given by the formula

$$p(\text{fracture}) = N \left(\frac{\ln(HIC) - \mu}{\sigma} \right),$$

where $N()$ is the cumulative normal distribution, $\mu = 6.96352$ and $\sigma = 0.84664$.

Injury Risk Curve for HIC
Log-normal curve



Non - linear Constrained Optimization Problem :

Minimize : $F(x)$ Objective function

Subject to :

$g_j(X) \leq 0$ $j = 1, m$ Inequality constraint

$h_k(X) = 0$ $k = 1, l$ Equality constraint

$X_i^l \leq X_i \leq X_i^u$ $i = 1, n$ Side constraints

where, $X = \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{Bmatrix}$

Necessary and sufficient conditions for global optimum
for unconstrained problems -

$\nabla F(X) = 0$ (the gradient must vanish)

where,

$$\nabla F(X) = \begin{Bmatrix} \frac{\partial}{\partial X_1} F(X) \\ \frac{\partial}{\partial X_2} F(X) \\ \vdots \\ \frac{\partial}{\partial X_n} F(X) \end{Bmatrix}$$

and the Hessian matrix must be positive definite

$$H = \begin{bmatrix} \frac{\partial^2 F(X)}{\partial X_1^2} & \frac{\partial^2 F(X)}{\partial X_1 \partial X_2} & \cdots & \frac{\partial^2 F(X)}{\partial X_1 \partial X_n} \\ \frac{\partial^2 F(X)}{\partial X_2 \partial X_1} & \frac{\partial^2 F(X)}{\partial X_2^2} & \cdots & \frac{\partial^2 F(X)}{\partial X_2 \partial X_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 F(X)}{\partial X_n \partial X_1} & \frac{\partial^2 F(X)}{\partial X_n \partial X_2} & \cdots & \frac{\partial^2 F(X)}{\partial X_n^2} \end{bmatrix}$$

And for constrained problems, if vector X^* defines the optimum design, Kuhn - Tucker conditions are necessary and sufficient :

1. X^* is feasible

$$2. \lambda_j g_j(X^*) = 0 \quad j = 1, m \quad \lambda_j \geq 0$$

$$3. \nabla F(X^*) + \sum_{j=1}^m \lambda_j \nabla g_j(X^*) + \sum_{k=1}^l \lambda_{m+k} \nabla h_k(X^*) = 0$$

$$\lambda_j \geq 0$$

- Crashworthiness and occupant protection in motor vehicles comprise the following significant mathematics applications –
 - Matrix operations
 - Ordinary and partial differential system of equations
 - Lagrangian operations
 - Fourier transforms
 - Taylor series
 - Finite difference methods
 - Implicit and explicit finite element methods
 - Statistical methods – probabilistic and regression analysis