

# Adaptive Radar Detection of Extended Gaussian Targets

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**Abstract** We have addressed the derivation and the analysis of an adaptive decision scheme to detect possible extended targets modeled as Gaussian vectors known to belong to a given subspace; noise returns from the cells under test are modeled as independent and identically-distributed Gaussian vectors with one and the same covariance matrix; a set of secondary data, free of signal components is also available; secondary data are Gaussian-distributed and share the same covariance matrix of noise in the cells under test but for a possible different power level.

The proposed detector relies on a two-step design procedure: first we derive the GLRT assuming that the noise covariance matrix is known up to a scale factor; then, we come up with a fully adaptive detector by replacing the structure of the covariance matrix of the noise with the sample covariance matrix based upon the secondary data. The first step requires the maximum likelihood (ML) estimate of the covariance matrix of the useful signal (under the signal-plus-noise hypothesis) which, in turn, has a known structure. That ML estimate has been firstly proposed by Bresler in [3]; a different derivation is also proposed herein.

The performance assessment is conducted resorting to the method proposed in [4–5] to model extended targets: therein an exponential model for fully-polarized returns has been used assuming that each scattering center can be characterized by its (relative) range, amplitude, and polarization ellipse.

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# ADAPTIVE RADAR DETECTION OF EXTENDED GAUSSIAN TARGETS

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## ABSTRACT

**This paper addresses adaptive detection of extended targets whose dominant scatterers are modeled as a linear combination of known modes with jointly Gaussian coefficients. At the design stage we consider the following instances of a priori knowledge about target and noise: the noise is white with unknown power level while the covariance matrix of the coefficients is either known up to its eigenvalues or totally unknown (for the first case we also assume orthogonal modes); or both the noise and the coefficients covariance matrices are unknown. We derive GLRT-based detectors for the first two cases and propose a fully-adaptive detector for the third one by resorting to secondary data possessing the same covariance matrix of the data under test up to a multiplicative factor. Finally, we assess the performance of the last two detectors by resorting to Monte Carlo simulation.**

## 1. INTRODUCTION

A High-Resolution Radar (HRR) can resolve a target into a number of scattering centers depending on the range extent of the target, the range resolution capabilities of the radar, and its operating frequency. Measurements indicate that radar properties of several targets can be modeled in terms of a set of scattering centers each parameterized by its range, amplitude and, possibly, polarization ellipse [1].

Properly designed HRRs allow significant enhancement of the detection performance as shown in [2, and references therein] where adaptive radar detection of extended deterministic targets, embedded in possibly non-Gaussian disturbance, has been addressed. Therein, possible returns from target’s scattering centers are modeled as signals known up to multiplicative factors, possibly different from one scattering center to another, namely supposed to belong to a one-dimensional subspace of the observables. From this

viewpoint, a more general framework is considered in [3] where adaptive detection of point-like deterministic targets, assumed to belong to a known subspace of the observables, has been addressed. That model has also been generalized to deal with point-like stochastic targets in [4]. Herein, following the lead of [2, 4], we address adaptive detection of extended targets modeled as Gaussian signals known to belong to a known subspace of the observables.

To be more definite, assume that an array of  $L$  antennas senses  $K$  range cells and that each antenna collects  $T$  samples from each of those cells; thus, denoting by  $\mathbf{x}_k$  the  $N$ -dimensional vector, with  $N = LT$ , containing returns from the  $k$ -th cell,  $k = 1, \dots, K$ , the problem at hand is that of detecting the possible presence of the target’s scattering centers  $\mathbf{s}_k \in \mathbb{C}^N$ ,  $k = 1, \dots, K$ , modeled as a linear combination of  $r$  linearly independent modes, namely as  $\mathbf{s}_k = \mathbf{H}\boldsymbol{\theta}_k$ , where the columns of the matrix  $\mathbf{H} \in \mathbb{C}^{N \times r}$  are the vectors of the basis while the entries of the vector  $\boldsymbol{\theta}_k \in \mathbb{C}^r$  are the coordinates of the possible target’s scatterer in the  $k$ -th cell. Moreover, we assume that the  $\boldsymbol{\theta}_k$ s are (complex) Gaussian vectors with zero-mean and positive semi-definite covariance matrix  $\mathbf{R}_{\boldsymbol{\theta}\boldsymbol{\theta}} = E[\boldsymbol{\theta}_k \boldsymbol{\theta}_k^\dagger]$ , where  $E$  denotes statistical expectation and  $^\dagger$  Hermitian transpose, i.e.,

$$\boldsymbol{\theta}_k \sim \mathcal{CN}_r(\mathbf{0}, \mathbf{R}_{\boldsymbol{\theta}\boldsymbol{\theta}}),$$

and that the  $\boldsymbol{\theta}_k$ ’s are independent. The noise vectors  $\mathbf{n}_k \in \mathbb{C}^N$ ,  $k = 1, \dots, K$ , are modeled as additive and independent complex Gaussian vectors with zero-mean and covariance matrix given by  $\gamma \mathbf{R}_{nn} \in \mathbb{C}^{N \times N}$ ,  $\gamma > 0$ , i.e.  $\mathbf{n}_k \sim \mathcal{CN}_N(\mathbf{0}, \gamma \mathbf{R}_{nn})$ .

In this paper we design detectors based upon the Generalized Likelihood Ratio Test (GLRT) assuming the following instances for target and noise:

- **Case 1:**  $\mathbf{R}_{nn} = \mathbf{I}_N$ , where  $\mathbf{I}_N$  denotes the  $N \times N$  identity matrix,  $\gamma$  is unknown, and the covariance matrix  $\mathbf{R}_{\boldsymbol{\theta}\boldsymbol{\theta}}$  of the  $\boldsymbol{\theta}_k$ s is known up to its eigenvalues,  $e_1, \dots, e_r$  say. We also assume that the columns of

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$\mathbf{H}$  are each other orthogonal.

- **Case 2:**  $\mathbf{R}_{nn} = \mathbf{I}_N$  while  $\gamma$  and the covariance matrix  $\mathbf{R}_{\theta\theta}$  are unknown.
- **Case 3:** both  $\gamma\mathbf{R}_{nn}$  and  $\mathbf{R}_{\theta\theta}$  are unknown, but  $M > N$  zero-mean Gaussian vectors with covariance matrix  $\mathbf{R}_{nn}$  are available. These vectors will be referred to in the following as secondary data.

We also assess the performance of detectors designed to tackle cases 2 and 3 by resorting to Monte Carlo simulation. The derivation of the GLRT for case 2 (unknown  $\mathbf{R}_{\theta\theta}$ ) is conducted taking advantage of the results reported in [5].

More precisely, the next section addresses the design of the GLRT-based detectors while Section 3 contains a preliminary performance assessment conducted by resorting to Monte Carlo simulation.

## 2. DETECTOR DESIGNS

We cannot implement the likelihood ratio since total or partial ignorance of  $\mathbf{R}_{\theta\theta}$ ,  $\gamma$ , and, possibly,  $\mathbf{R}_{nn}$  is assumed. Leaving aside case 3, we can resort, instead, to the GLRT which is tantamount to replacing, under each hypothesis, the unknown parameters with the maximum likelihood estimates.

To this end, note that the detection problems to be solved can be re-cast in terms of the following binary hypothesis test:

$$\begin{cases} H_0: & \mathbf{x}_k \sim \mathcal{CN}_N(\mathbf{0}, \gamma\mathbf{I}_N), & k = 1, \dots, K, \\ H_1: & \mathbf{x}_k \sim \mathcal{CN}_N(\mathbf{0}, \mathbf{R}_{ss} + \gamma\mathbf{I}_N), & k = 1, \dots, K, \end{cases}$$

where

$$\mathbf{R}_{ss} = E[\mathbf{s}_k \mathbf{s}_k^\dagger] = \mathbf{H} \mathbf{R}_{\theta\theta} \mathbf{H}^\dagger.$$

### 2.1. Case 1: $\gamma$ and the $e_i$ s unknown

This subsection is aimed at designing a GLRT to deal with the case that  $\gamma$  and the eigenvalues of  $\mathbf{R}_{\theta\theta}$  are unknown (while the eigenvectors of  $\mathbf{R}_{\theta\theta}$  are known).

For further developments it is necessary to determine the probability density function (pdf) of the data matrix, i.e.,

$$\mathbf{X} \doteq [\mathbf{x}_1 \cdots \mathbf{x}_K],$$

under each hypothesis. The pdf of  $\mathbf{X}$  under  $H_0$  is given by

$$f(\mathbf{X} | H_0, \gamma) = \frac{1}{\pi^{NK}} \left(\frac{1}{\gamma}\right)^{NK} e^{-\frac{1}{\gamma} \sum_{k=1}^K \mathbf{x}_k^\dagger \mathbf{x}_k}. \quad (1)$$

The pdf of the data under the  $H_1$  hypothesis, instead, can be determined following the lead of [4]. To this end, denote by  $\mathbf{U} \mathbf{E} \mathbf{U}^\dagger$  an eigendecomposition of the positive semidefinite matrix  $\mathbf{R}_{\theta\theta}$ ; hence,  $\mathbf{U} \in \mathbb{C}^{r \times r}$  is a unitary

matrix,  $\mathbf{E} \in \mathbb{C}^{r \times r}$  is a diagonal matrix with non-negative diagonal entries,  $e_1, \dots, e_r$ , and  $\dagger$  denotes conjugate transpose.

It follows that

$$\mathbf{R}_{ss} = \tilde{\mathbf{H}} \mathbf{E} \tilde{\mathbf{H}}^\dagger,$$

where

$$\tilde{\mathbf{H}} = \mathbf{H} \mathbf{U} \doteq [\tilde{\mathbf{h}}_1 \cdots \tilde{\mathbf{h}}_r] \in \mathbb{C}^{N \times r}$$

is a slice of a unitary matrix (remember that  $\mathbf{H}^\dagger \mathbf{H} = \mathbf{I}_r$ ). Moreover, if we denote by  $\tilde{\mathbf{H}}^\perp \in \mathbb{C}^{N \times (N-r)}$  a matrix whose columns are orthonormal vectors spanning the orthogonal complement of the space described by the columns of  $\tilde{\mathbf{H}}$ , the following identity holds true

$$\mathbf{R}_{ss} + \gamma\mathbf{I}_N = \begin{bmatrix} \tilde{\mathbf{H}} & \tilde{\mathbf{H}}^\perp \end{bmatrix} \begin{bmatrix} \mathbf{E} + \gamma\mathbf{I}_r & \\ & \gamma\mathbf{I}_{N-r} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{H}}^\dagger \\ \tilde{\mathbf{H}}^{\perp\dagger} \end{bmatrix}.$$

It follows that the pdf of  $\mathbf{X}$ , under  $H_1$ , is given by

$$f(\mathbf{X} | H_1, \gamma, \mathbf{E}) = \frac{1}{\pi^{NK}} \frac{1}{\gamma^{(N-r)K}} \prod_{i=1}^r \frac{1}{(\gamma + e_i)^K} \times e^{-\frac{1}{\gamma} \sum_{k=1}^K \mathbf{x}_k^\dagger \mathbf{P}_{\tilde{\mathbf{H}}}^\perp \mathbf{x}_k - \sum_{k=1}^K \sum_{i=1}^r \frac{\mathbf{x}_k^\dagger \mathbf{P}_{\tilde{\mathbf{H}}} \mathbf{x}_k}{\gamma + e_i}}, \quad (2)$$

where  $\mathbf{P}_{\tilde{\mathbf{h}}_i}$  and  $\mathbf{P}_{\tilde{\mathbf{H}}}^\perp$  denote the projectors onto the subspace spanned by  $\tilde{\mathbf{h}}_i$  and the orthogonal complement of the subspace spanned by the columns of  $\tilde{\mathbf{H}}$ , respectively.

For further processing, it is also convenient to denote by  $g_i(\cdot)$ ,  $i = 1, \dots, r$ , the function

$$g_i(\mathbf{X} | \gamma, e_i) \doteq \frac{1}{(\gamma + e_i)^K} e^{-\sum_{k=1}^K \frac{\mathbf{x}_k^\dagger \mathbf{P}_{\tilde{\mathbf{h}}_i} \mathbf{x}_k}{\gamma + e_i}},$$

and re-cast the pdf of the data matrix, under  $H_1$ , as

$$f(\mathbf{X} | H_1, \gamma, \mathbf{E}) = \frac{1}{\pi^{NK}} \frac{1}{\gamma^{(N-r)K}} e^{-\frac{1}{\gamma} \sum_{k=1}^K \mathbf{x}_k^\dagger \mathbf{P}_{\tilde{\mathbf{H}}}^\perp \mathbf{x}_k} \times \prod_{i=1}^r g_i(\mathbf{X} | \gamma, e_i).$$

The derivation of the GLRT requires substituting the unknown parameters by the corresponding maximum likelihood estimates under each hypothesis. Maximizing the pdf of the data under  $H_0$  with respect to  $\gamma$  is straightforward; it yields

$$\max_{\gamma > 0} f(\mathbf{X} | H_0, \gamma) = \frac{1}{\pi^{NK}} e^{-NK} \left( \frac{NK}{\sum_{k=1}^K \mathbf{x}_k^\dagger \mathbf{x}_k} \right)^{NK}.$$

Now focus on the more difficult task to maximize the pdf of the data matrix with respect to the  $e_i$ s,  $i = 1, \dots, r$ ,

and  $\gamma$ , under the  $H_1$  hypothesis. We attack this problem by maximizing the pdf with respect to the  $e_i$ s, given  $\gamma$ . To this end, it is easy to check that

$$\max_{e_i \geq 0} g_i(\mathbf{X} | \gamma, e_i) = \begin{cases} e^{-K} \frac{1}{\gamma^{1/K}}, & \text{if } \gamma \leq \gamma_i, \\ \frac{1}{\gamma^K} e^{-\frac{K\gamma_i}{\gamma}}, & \text{if } \gamma \geq \gamma_i, \end{cases}$$

where

$$\gamma_i \doteq \frac{1}{K} \sum_{k=1}^K \mathbf{x}_k^\dagger \mathbf{P}_{\mathbf{h}_i} \mathbf{x}_k, \quad i = 1, \dots, r.$$

Moreover, note that

$$\max_{e_i \geq 0} g_i(\mathbf{X} | \gamma, e_i), \quad i = 1, \dots, r,$$

is constant for  $\gamma \leq \gamma_i$ , while its restriction to  $[\gamma_i, +\infty[$  is a monotonically decreasing function.

Now, denote by  $\gamma_{(i)}$  the  $i$ -th order statistic obtained by ranking  $\gamma_1, \dots, \gamma_r$  in ascending order, i.e.,

$$\gamma_{(1)} \leq \dots \leq \gamma_{(r)}.$$

Moreover, let

$$\tilde{\gamma} \doteq \frac{1}{(N-r)K} \sum_{k=1}^K \mathbf{x}_k^\dagger \mathbf{P}_{\mathbf{H}} \mathbf{x}_k.$$

Then, it is easy to check that the function

$$\max_{e_1 \geq 0, \dots, e_r \geq 0} f(\mathbf{X} | H_1, \gamma, \mathbf{E})$$

attains its maximum (with respect to  $\gamma$ ) at  $\tilde{\gamma}$  if  $\tilde{\gamma} < \gamma_{(1)}$ , while the maximizer is a point of the interval  $[\gamma_{(1)}, \tilde{\gamma}]$  if  $\tilde{\gamma} \geq \gamma_{(1)}$ . More specifically, it is possible to prove the following theorem.

**Theorem 1.** The function

$$\max_{e_1 \geq 0, \dots, e_r \geq 0} f(\mathbf{X} | H_1, \gamma, \mathbf{E}), \quad \gamma > 0,$$

attains its maximum with respect to  $\gamma$  at

$$\begin{aligned} \hat{\gamma}_{(1)}, & \quad \text{if} & \quad \hat{\gamma}_{(1)} \leq \gamma_{(1)}, \\ & \quad \vdots \\ \hat{\gamma}_{(j)}, & \quad \text{if} & \quad \hat{\gamma}_{(i)} > \gamma_{(i)}, i = 1, \dots, j-1, \hat{\gamma}_{(j)} \leq \gamma_{(j)}, \\ & \quad \vdots \\ \hat{\gamma}_{(r+1)}, & \quad \text{if} & \quad \hat{\gamma}_{(i)} > \gamma_{(i)}, i = 1, \dots, r, \end{aligned}$$

where

$$\hat{\gamma}_{(j)} \doteq \frac{(N-r)\tilde{\gamma} + \sum_{i=1}^{j-1} \gamma_{(i)}}{N-r+(j-1)}, \quad j = 1, \dots, r+1. \quad (3)$$

**Proof.** Let

$$\begin{aligned} h_j(\mathbf{X} | \gamma) & \doteq \frac{1}{\pi^{NK}} \frac{1}{\gamma^{(N-r)K}} e^{-\frac{K(N-r)\tilde{\gamma}}{\gamma}} \\ & \times \left[ \prod_{i \in \mathbb{N}: j \leq i \leq r} e^{-K} \frac{1}{(\gamma_{(i)})^K} \right] \\ & \times \left[ \prod_{i \in \mathbb{N}: 1 \leq i < j} \frac{1}{\gamma^K} e^{-\frac{K\gamma_{(i)}}{\gamma}} \right], \end{aligned}$$

$j = 1, \dots, r+1$ , with

$$\prod_{i \in \emptyset} \frac{1}{\gamma^K} e^{-\frac{K\gamma_{(i)}}{\gamma}} = 1$$

and

$$\prod_{i \in \emptyset} e^{-K} \frac{1}{(\gamma_{(i)})^K} = 1.$$

It is easy to check that

$$\begin{aligned} \max_{e_1 \geq 0, \dots, e_r \geq 0} f(\mathbf{X} | H_1, \gamma, \mathbf{E}) & \\ = & \begin{cases} h_1(\mathbf{X} | \gamma), & \text{if } 0 < \gamma \leq \gamma_{(1)}, \\ h_2(\mathbf{X} | \gamma), & \text{if } \gamma_{(1)} \leq \gamma \leq \gamma_{(2)}, \\ \vdots & \\ h_r(\mathbf{X} | \gamma), & \text{if } \gamma_{(r-1)} \leq \gamma \leq \gamma_{(r)}, \\ h_{r+1}(\mathbf{X} | \gamma), & \text{if } \gamma \geq \gamma_{(r)}. \end{cases} \quad (4) \end{aligned}$$

Moreover, the  $h_j(\mathbf{X} | \gamma)$  possess the following properties:

- $h_j(\mathbf{X} | \gamma) \geq h_{j+1}(\mathbf{X} | \gamma)$ ,  $\gamma > 0$ , with equality if  $\gamma = \gamma_{(j)}$ ,  $j = 1, \dots, r$ ;
- $h_j(\mathbf{X} | \gamma)$ ,  $\gamma > 0$ , attains its absolute maximum at  $\hat{\gamma}_{(j)}$ ,  $j = 1, \dots, r+1$ .

It follows that

- if  $\hat{\gamma}_{(1)} \leq \gamma_{(1)}$  then

$$\max_{e_1 \geq 0, \dots, e_r \geq 0} f(\mathbf{X} | H_1, \gamma, \mathbf{E}), \quad \gamma > 0,$$

attains its maximum at  $\hat{\gamma}_{(1)} = \tilde{\gamma}$ .

- If  $\hat{\gamma}_{(1)} > \gamma_{(1)}$  and  $\hat{\gamma}_{(2)} \leq \gamma_{(2)}$ , it is also true that

$$\gamma_{(1)} < \hat{\gamma}_{(2)} \leq \gamma_{(2)};$$

moreover, since

$$\begin{aligned} \max_{\gamma \leq \gamma_{(1)}} h_1(\mathbf{X} | \gamma) & = h_1(\mathbf{X} | \gamma_{(1)}) = h_2(\mathbf{X} | \gamma_{(1)}) \\ & \leq \max_{\gamma_{(1)} \leq \gamma \leq \gamma_{(2)}} h_2(\mathbf{X} | \gamma) \\ & = h_2(\mathbf{X} | \hat{\gamma}_{(2)}), \end{aligned}$$

we conclude that

$$\max_{e_1 \geq 0, \dots, e_r \geq 0} f(\mathbf{X} | H_1, \gamma, e_1, \dots, e_r), \quad \gamma > 0,$$

attains its maximum at  $\hat{\gamma}_{(2)}$ . Moreover, for  $j = 3, \dots, r$ , if  $\hat{\gamma}_{(i)} > \gamma_{(i)}$ ,  $i = 1, \dots, j-1$ , and  $\hat{\gamma}_{(j)} \leq \gamma_{(j)}$ , it is also true that

$$\gamma_{(j-1)} < \hat{\gamma}_{(j)} \leq \gamma_{(j)};$$

moreover, since

$$\begin{aligned} \max_{\gamma \leq \gamma_{(1)}} h_1(\mathbf{X} | \gamma) &= h_1(\mathbf{X} | \gamma_{(1)}) = h_2(\mathbf{X} | \gamma_{(1)}) \\ &\leq \max_{\gamma_{(1)} \leq \gamma \leq \gamma_{(2)}} h_2(\mathbf{X} | \gamma) \\ &= h_2(\mathbf{X} | \gamma_{(2)}) \\ &\vdots \\ &\leq \\ &\vdots \\ &= h_j(\mathbf{X} | \gamma_{(j-1)}) \\ &\leq \max_{\gamma_{(j-1)} \leq \gamma \leq \gamma_{(j)}} h_j(\mathbf{X} | \gamma) \\ &= h_j(\mathbf{X} | \hat{\gamma}_{(j)}), \end{aligned}$$

we conclude that

$$\max_{e_1 \geq 0, \dots, e_r \geq 0} f(\mathbf{X} | H_1, \gamma, \mathbf{E}), \quad \gamma > 0,$$

attains its maximum at  $\hat{\gamma}_{(j)}$ .

- Similarly, if  $\hat{\gamma}_{(i)} > \gamma_{(i)}$ ,  $i = 1, \dots, r$ , we conclude that

$$\max_{e_1 \geq 0, \dots, e_r \geq 0} f(\mathbf{X} | H_1, \gamma, \mathbf{E}), \quad \gamma > 0,$$

attains its maximum at  $\hat{\gamma}_{(r+1)}$ .

The theorem is thus proved. ■

It follows that

$$\max_{\gamma > 0} \max_{e_1 \geq 0, \dots, e_r \geq 0} f(\mathbf{X} | H_1, \gamma, \mathbf{E}),$$

is obtained by evaluating the function (4) at its maximizer; we get

$$\begin{aligned} &\max_{\gamma > 0} \max_{e_1 \geq 0, \dots, e_r \geq 0} f(\mathbf{X} | H_1, \gamma, \mathbf{E}) \\ &= \begin{cases} f_1(\mathbf{X} | H_1), & \hat{\gamma}_{(1)} \leq \gamma_{(1)}, \\ f_2(\mathbf{X} | H_1), & \hat{\gamma}_{(1)} > \gamma_{(1)}, \hat{\gamma}_{(2)} \leq \gamma_{(2)}, \\ \vdots \\ f_j(\mathbf{X} | H_1), & \hat{\gamma}_{(i)} > \gamma_{(i)}, i = 1, \dots, j-1, \hat{\gamma}_{(j)} \leq \gamma_{(j)}, \\ \vdots \\ f_r(\mathbf{X} | H_1), & \hat{\gamma}_{(i)} > \gamma_{(i)}, i = 1, \dots, r-1, \hat{\gamma}_{(r)} \leq \gamma_{(r)}, \\ f_{r+1}(\mathbf{X} | H_1), & \hat{\gamma}_{(i)} > \gamma_{(i)}, i = 1, \dots, r, \end{cases} \end{aligned}$$

where

$$\begin{aligned} f_j(\mathbf{X} | H_1) &\doteq \frac{1}{\pi^{NK}} e^{-NK} \prod_{i=j}^r \left( \frac{1}{\gamma_{(i)}} \right)^K \\ &\times \left( \frac{N-r+(j-1)}{(N-r)\tilde{\gamma} + \sum_{i=1}^{j-1} \gamma_{(i)}} \right)^{(N-r+(j-1))K}, \end{aligned}$$

and  $\hat{\gamma}_{(j)}$  is given by eqn. (3).

### 2.1.1. GLRT-based detector

It is then easy to see that the decision statistic of the GLRT is given by

$$\begin{aligned} \Lambda[\mathbf{X}] &= \begin{cases} \Lambda_1[\mathbf{X}], & \hat{\gamma}_{(1)} \leq \gamma_{(1)}, \\ \Lambda_2[\mathbf{X}], & \hat{\gamma}_{(1)} > \gamma_{(1)}, \hat{\gamma}_{(2)} \leq \gamma_{(2)}, \\ \vdots \\ \Lambda_j[\mathbf{X}], & \hat{\gamma}_{(i)} > \gamma_{(i)}, i = 1, \dots, j-1, \hat{\gamma}_{(j)} \leq \gamma_{(j)}, \\ \vdots \\ \Lambda_r[\mathbf{X}], & \hat{\gamma}_{(i)} > \gamma_{(i)}, i = 1, \dots, r-1, \hat{\gamma}_{(r)} \leq \gamma_{(r)}, \\ 1, & \hat{\gamma}_{(i)} > \gamma_{(i)}, i = 1, \dots, r, \end{cases} \end{aligned}$$

where

$$\Lambda_j[\mathbf{X}] \doteq \frac{\left( \frac{1}{NK} \sum_{k=1}^K \mathbf{x}_k^\dagger \mathbf{x}_k \right)^{NK}}{\left( \frac{(N-r)\tilde{\gamma} + \sum_{i=1}^{j-1} \gamma_{(i)}}{N-r+(j-1)} \right)^{(N-r+(j-1))K} \prod_{i=j}^r (\gamma_{(i)})^K}$$

## 2.2. Case 2: $\gamma$ and $\mathbf{R}_{\theta\theta}$ unknown

Again the pdf of the data matrix  $\mathbf{X}$  under  $H_0$  is given by eqn. (1). It follows that implementation of the GLRT for the case at hand requires determining the pdf of  $\mathbf{X}$  under  $H_1$  and maximizing it with respect to  $\gamma$  and  $\mathbf{R}_{\theta\theta}$ . The pdf of  $\mathbf{X}$  under  $H_1$  is given by

$$f(\mathbf{X} | H_1, \gamma, \mathbf{R}_{\theta\theta}) = \frac{\pi^{-NK}}{|\mathbf{R}_{ss} + \gamma \mathbf{I}_N|^K} e^{-K \text{tr}[\hat{\mathbf{R}}(\mathbf{R}_{ss} + \gamma \mathbf{I}_N)^{-1}]}$$

where  $\text{tr}[\cdot]$  and  $|\cdot|$  denote the trace and the determinant of a square matrix, respectively, while

$$\hat{\mathbf{R}} = \frac{1}{K} \mathbf{X} \mathbf{X}^\dagger.$$

The maximization of the above pdf with respect to  $\gamma > 0$  and  $\mathbf{R}_{\theta\theta} \geq 0$  has been addressed in [5] where it has been shown that

$$\max_{\gamma > 0} \max_{\mathbf{R}_{\theta\theta} \geq 0} f(\mathbf{X} | H_1, \gamma, \mathbf{R}_{\theta\theta}) = \max_{\gamma > 0} \max_{\mathbf{Q} \geq 0} g(\mathbf{X} | H_1, \gamma, \mathbf{Q})$$

where  $g(\mathbf{X}|H_1, \gamma, \mathbf{Q})$  is given by

$$g(\mathbf{X}|H_1, \gamma, \mathbf{Q}) = \frac{e^{-\frac{K}{\gamma} \text{tr}[\mathbf{P}_H^\perp \widehat{\mathbf{R}} + (\mathbf{Q} + \mathbf{I}_r)^{-1} \mathbf{P}]} }{[(\pi\gamma)^N |\mathbf{Q} + \mathbf{I}_r|]^K},$$

with

$$\mathbf{Q} \doteq \frac{1}{\gamma} \mathbf{L}^\dagger \mathbf{R}_{\theta\theta} \mathbf{L},$$

$$\mathbf{P} \doteq \mathbf{L}^{-1} \mathbf{H}^\dagger \widehat{\mathbf{R}} \mathbf{H} \mathbf{L}^{-\dagger},$$

and  $\mathbf{L}$  denoting a nonsingular  $r \times r$  square root factor of  $\mathbf{H}^\dagger \mathbf{H}$ , namely such that  $\mathbf{L} \mathbf{L}^\dagger = \mathbf{H}^\dagger \mathbf{H}$ . Moreover, maximization of  $g(\mathbf{X}|H_1, \gamma, \mathbf{Q})$  has been performed in two steps: first it is shown that

$$\begin{aligned} & \max_{\mathbf{Q} \geq 0} g(\mathbf{X}|H_1, \gamma, \mathbf{Q}) \\ &= \begin{cases} g_r(\mathbf{X} | \gamma), & \gamma \leq \phi_r, \\ g_{r-1}(\mathbf{X} | \gamma), & \phi_r < \gamma \leq \phi_{r-1}, \\ \vdots \\ g_1(\mathbf{X} | \gamma), & \phi_2 < \gamma \leq \phi_1, \\ g_0(\mathbf{X} | \gamma), & \gamma > \phi_1, \end{cases} \end{aligned}$$

where

$$g_j(\mathbf{X} | \gamma) \doteq \frac{e^{-jK}}{(\pi\gamma)^{NK}} \prod_{i=1}^j \left( \frac{\gamma}{\phi_i} \right)^K e^{-\frac{K}{\gamma} \{ \text{tr}[\mathbf{P}_H^\perp \widehat{\mathbf{R}}] + \sum_{i=j+1}^r \phi_i \}},$$

and  $\phi_1, \dots, \phi_r$  are the eigenvalues of the matrix  $\mathbf{P}$  ordered in non-ascending order, i.e.,

$$\phi_r \leq \dots \leq \phi_1.$$

Then, it is shown that the following theorem holds true.

**Theorem 2.** The function

$$\max_{\mathbf{Q} \geq 0} g(\mathbf{X} | H_1, \gamma, \mathbf{Q}), \quad \gamma > 0,$$

attains its maximum with respect to  $\gamma$  at

$$\begin{cases} \widehat{\gamma}_r, & \widehat{\gamma}_r \leq \phi_r, \\ \widehat{\gamma}_{r-1}, & \widehat{\gamma}_r > \phi_r, \widehat{\gamma}_{r-1} \leq \phi_{r-1}, \\ \vdots \\ \widehat{\gamma}_1, & \widehat{\gamma}_i > \phi_i, i = 2, \dots, r, \widehat{\gamma}_1 \leq \phi_1, \\ \widehat{\gamma}_0, & \widehat{\gamma}_i > \phi_i, i = 1, \dots, r, \end{cases}$$

where

$$\widehat{\gamma}_j \doteq \frac{\sum_{i=j+1}^r \phi_i + \text{tr}[\mathbf{P}_H^\perp \widehat{\mathbf{R}}]}{N - j}, \quad j = 0, \dots, r.$$

**Proof.** See [5]. ■

It follows that the statistic of the GLRT is given by

$$\Lambda[\mathbf{X}] = \begin{cases} \Lambda_r[\mathbf{X}], & \widehat{\gamma}_r \leq \phi_r, \\ \Lambda_{r-1}[\mathbf{X}], & \widehat{\gamma}_r > \phi_r, \widehat{\gamma}_{r-1} \leq \phi_{r-1}, \\ \vdots \\ \Lambda_1[\mathbf{X}], & \widehat{\gamma}_i > \phi_i, i = 2, \dots, r, \widehat{\gamma}_1 \leq \phi_1, \\ 1, & \widehat{\gamma}_i > \phi_i, i = 1, \dots, r, \end{cases} \quad (5)$$

where

$$\Lambda_j[\mathbf{X}] \doteq \frac{\left( \frac{1}{NK} \sum_{k=1}^K \mathbf{x}_k^\dagger \mathbf{x}_k \right)^{NK}}{\widehat{\gamma}_j^{(N-j)K} \prod_{i=1}^j \phi_i^K}.$$

This detector will be referred to as a partially-adaptive one.

### 2.3. Case 3: $\gamma$ , $\mathbf{R}_{nn}$ and $\mathbf{R}_{\theta\theta}$ unknown

As a final step observe that the above decision statistic can be modified to address detection in presence of correlated noise. To this end, assume that  $M > N$  additional Gaussian vectors  $\mathbf{r}_k \sim \mathcal{CN}_N(\mathbf{0}, \mathbf{R}_{nn})$ ,  $k = K+1, \dots, K+M$ , free of signals components, but possessing the same statistical characterization of the noise in the cells under test, but for a possible different power level, are available. These vectors are usually referred to as secondary data. Now we have to discriminate between the  $H_0$  hypothesis that can be re-cast as

$$\begin{cases} \mathbf{r}_k \sim \mathcal{CN}_N(\mathbf{0}, \gamma \mathbf{R}_{nn}), & k = 1, \dots, K, \\ \mathbf{r}_k \sim \mathcal{CN}_N(\mathbf{0}, \mathbf{R}_{nn}), & k = K+1, \dots, K+M, \end{cases}$$

and the  $H_1$  hypothesis which is given by

$$\begin{cases} \mathbf{r}_k \sim \mathcal{CN}_N(\mathbf{0}, \mathbf{R}_{ss} + \gamma \mathbf{R}_{nn}), & k = 1, \dots, K, \\ \mathbf{r}_k \sim \mathcal{CN}_N(\mathbf{0}, \mathbf{R}_{nn}), & k = K+1, \dots, K+M, \end{cases}$$

where  $\mathbf{R}_{nn}$  denotes the unknown covariance matrix of the secondary data and, up to a multiplicative factor, that of the data under test.

Then, we can come up with a fully-adaptive detector by resorting to detector (5) fed by the whitened data  $\mathbf{x}_k = \widehat{\mathbf{R}}_{nn}^{-1/2} \mathbf{r}_k$ , where  $\widehat{\mathbf{R}}_{nn}$  is the sample covariance matrix based upon secondary vectors, i.e.,

$$\widehat{\mathbf{R}}_{nn} \doteq \frac{1}{M} \sum_{k=K+1}^{K+M} \mathbf{r}_k \mathbf{r}_k^\dagger.$$

This detector will be referred to as a fully-adaptive detector.

### 2.4. Performance Assessment

In the following we assess the performance of the partially-adaptive and the fully-adaptive detectors by resorting to standard Monte Carlo counting techniques. More precisely, in order to evaluate the threshold necessary to ensure a pre-assigned value of Probability of False Alarm ( $P_{fa}$ ) and the

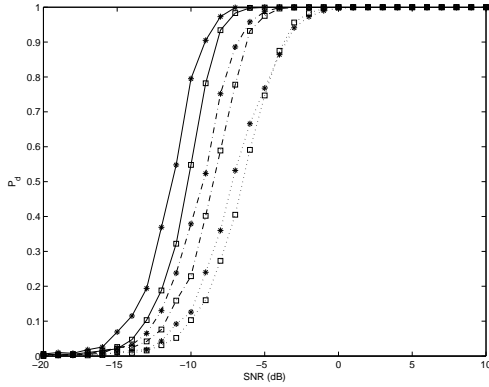


Figure 1:  $P_d$  vs SNR (dB) of the partially-adaptive detector for  $N = 16$ ,  $r = 3$ ,  $P_{fa} = 10^{-3}$ ,  $\gamma = 1$ ,  $K$  and  $\rho_\theta$  as parameters: curves with a square at each data point refer to  $\rho_\theta = 0.5$  while those with a star correspond to  $\rho_\theta = 0.9$ ; solid, dashdot, and dotted refer to  $K = 24$ ,  $K = 12$ , and  $K = 6$ , respectively.

Probability of Detection ( $P_d$ ) we resort to  $100/P_{fa}$  and 1000 independent trials, respectively. It is important to stress that both detectors have the Constant False Alarm Rate (CFAR) property with respect to  $\gamma$ .

Figures 1 and 2 plot  $P_d$  vs the signal-to-noise ratio (SNR) defined as

$$\text{SNR} = \frac{\text{tr}(\mathbf{R}_{ss})}{\text{tr}(\gamma\mathbf{R}_{nn})}$$

with  $\mathbf{R}_{nn} = \mathbf{I}_N$  for curves of Figure 1. More precisely, the curves of Figure 1 refer to the partially-adaptive detector with  $N = 16$ ,  $r = 3$ ,  $\gamma = 1$ , an exponentially-correlated target with one-lag correlation coefficient  $\rho_\theta$ , i.e.,

$$\mathbf{R}_{\theta\theta}(i, j) = \sigma_\theta^2 \rho_\theta^{|i-j|}, \quad i, j \in \{1, \dots, r\},$$

$K$  and  $\rho_\theta$  as parameters. As to  $\mathbf{H}$  its columns are the first  $r$  vectors of the canonical basis in  $\mathbb{R}^N$ . The figure highlights that the performance improves as  $\rho_\theta$  increases: as a matter of fact, since the noise is white the two hypotheses become more and more distinguishable as the signal correlation increases. In addition, the performance of the detector improves as  $K$  increases. In Figure 2, instead, we plot the performance of the fully-adaptive detector implemented by resorting to  $M = 32$  secondary data and assuming  $\gamma = 1$  and an exponentially-correlated noise with one-lag correlation coefficient  $\rho_n$ , i.e.,

$$\mathbf{R}_{nn}(i, j) = \rho_n^{|i-j|}, \quad i, j \in \{1, \dots, N\}.$$

For comparison purposes curves of the partially-adaptive detector, fed by whitened data, are plotted too. The figure highlights the loss of the fully-adaptive detector with respect to the partially-adaptive one where an unknown noise

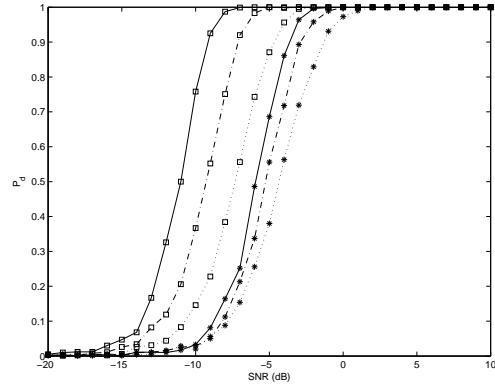


Figure 2:  $P_d$  vs SNR (dB) of the partially-adaptive and the fully-adaptive detector,  $N = 16$ ,  $r = 3$ ,  $\rho_\theta = 0.5$ ,  $\rho_n = 0.6$ ,  $\gamma = 1$ ,  $P_{fa} = 10^{-3}$ ,  $K$  as parameter: curves with a square at each data point refer to the partially-adaptive detector while those with a star correspond to the fully-adaptive one with  $M = 32$ ; solid, dashdot, and dotted refer to  $K = 24$ ,  $K = 12$ , and  $K = 6$ , respectively.

covariance must be estimated from secondary data. A thorough performance assessment is the object of outgoing research activity.

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# Adaptive Radar Detection of Extended Gaussian Targets

by

Giuseppe Ricci and Louis L. Scharf

## Problem Formulation

This paper addresses adaptive detection of extended Gaussian vectors known to belong to a known subspace of the observables.

To this end, we assume that:

- an array of  $L$  antennas senses  $K$  range cells and that
- each antenna collects  $T$  samples from each of those cells.

Denote by  $\mathbf{x}_k$  the  $LT$ -dimensional vector containing returns from the  $k$ -th cell,  $k = 1, \dots, K$ . **The problem herein addressed is** the detection of the possible presence of the target's scattering centers  $\mathbf{s}_k \in \mathbb{C}^N$ ,  $k = 1, \dots, K$ , modeled as a linear combination of  $r$  linearly independent modes, namely as

$$\mathbf{s}_k = \mathbf{H}\boldsymbol{\theta}_k,$$

within the  $K$  range cells under test.

We assume that:

- the matrix  $\mathbf{H} \in \mathbb{C}^{N \times r}$  is known.
- The coordinates vectors  $\boldsymbol{\theta}_k \sim \mathcal{CN}_r(\mathbf{0}, \mathbf{R}_{\theta\theta})$  are each other independent and  $\mathbf{R}_{\theta\theta}$  is a positive semidefinite covariance matrix.
- The noise vectors  $\mathbf{n}_k \sim \mathcal{CN}_N(\mathbf{0}, \gamma\mathbf{R}_{nn})$  are each other independent.

## Problem Formulation (cont.)

We design detectors based upon the Generalized Likelihood Ratio Test (GLRT) assuming the following instances for target and noise:

- **Case 1: we assume that**

- $\gamma \mathbf{R}_{nn} = \gamma \mathbf{I}_N$ ;
- the scale factor  $\gamma$  is unknown;
- $\mathbf{H}$  is a slice of a unitary matrix and is known;
- the covariance matrix  $\mathbf{R}_{\theta\theta}$  of the  $\boldsymbol{\theta}_k$ s is known up to its eigenvalues,  $e_1, \dots, e_r$  say.

- **Case 2: we assume that**

- $\gamma \mathbf{R}_{nn} = \gamma \mathbf{I}_N$ ;
- the scale factor  $\gamma$  is unknown;
- the covariance matrix  $\mathbf{R}_{\theta\theta}$  is unknown.

- **Case 3: we assume that**

both  $\gamma \mathbf{R}_{nn}$  and  $\mathbf{R}_{\theta\theta}$  are unknown, but  $M > N$  zero-mean Gaussian vectors with covariance matrix  $\mathbf{R}_{nn}$  are available. These vectors will be referred to in the following as **secondary data**.

## Detector Design: $\gamma$ and the $e_i$ s unknown

The detection problems to be solved is

$$\begin{cases} H_0 : \mathbf{x}_k \sim \mathcal{CN}_N(\mathbf{0}, \gamma \mathbf{I}_N), & k = 1, \dots, K, \\ H_1 : \mathbf{x}_k \sim \mathcal{CN}_N(\mathbf{0}, \mathbf{H} \mathbf{R}_{\theta\theta} \mathbf{H}^\dagger + \gamma \mathbf{I}_N), & k = 1, \dots, K, \end{cases}$$

where  $\gamma$  and the eigenvalues of  $\mathbf{R}_{\theta\theta}$  are unknown.

It is necessary to introduce the following quantities:

- $\mathbf{U} \mathbf{E} \mathbf{U}^\dagger$  is an eigendecomposition of  $\mathbf{R}_{\theta\theta}$ ;
- $\tilde{\mathbf{H}} = \mathbf{H} \mathbf{U} \doteq [\tilde{\mathbf{h}}_1 \cdots \tilde{\mathbf{h}}_r]$ ;
- $\gamma_{(i)}$  is the  $i$ -th order statistic obtained by ranking the

$$\gamma_i \doteq \frac{1}{K} \sum_{k=1}^K \mathbf{x}_k^\dagger \mathbf{P}_{\tilde{\mathbf{h}}_i} \mathbf{x}_k, \quad i = 1, \dots, r,$$

in ascending order where  $\mathbf{P}_{\tilde{\mathbf{h}}_i}$  is the projector onto the subspace described by  $\tilde{\mathbf{h}}_i$ ;

- $\tilde{\gamma}$  and  $\hat{\gamma}_{(j)}$  are given by

$$\tilde{\gamma} \doteq \frac{1}{(N-r)K} \sum_{k=1}^K \mathbf{x}_k^\dagger \mathbf{P}_{\tilde{\mathbf{H}}}^\perp \mathbf{x}_k$$

and

$$\hat{\gamma}_{(j)} \doteq \frac{(N-r)\tilde{\gamma} + \sum_{i=1}^{j-1} \gamma_{(i)}}{N-r+(j-1)}, \quad j = 1, \dots, r+1,$$

respectively. As to  $\mathbf{P}_{\tilde{\mathbf{H}}}^\perp$  it denotes the projector onto the orthogonal complement of the subspace spanned by the columns of  $\tilde{\mathbf{H}}$ .

## Detector Design: $\gamma$ and the $e_i$ s unknown (cont.)

It is possible to prove that the GLRT is given by

$$\Lambda[\mathbf{x}_1, \dots, \mathbf{x}_k] \underset{H_0}{\overset{H_1}{>}} \lambda$$

where

$$\Lambda[\mathbf{x}_1, \dots, \mathbf{x}_k] \doteq \begin{cases} \Lambda_1[\mathbf{x}_1, \dots, \mathbf{x}_k], & \hat{\gamma}_{(1)} \leq \gamma_{(1)}, \\ \Lambda_2[\mathbf{x}_1, \dots, \mathbf{x}_k], & \hat{\gamma}_{(1)} > \gamma_{(1)}, \hat{\gamma}_{(2)} \leq \gamma_{(2)}, \\ \vdots \\ \Lambda_j[\mathbf{x}_1, \dots, \mathbf{x}_k], & \hat{\gamma}_{(i)} > \gamma_{(i)}, i = 1, \dots, j-1, \hat{\gamma}_{(j)} \leq \gamma_{(j)}, \\ \vdots \\ \Lambda_r[\mathbf{x}_1, \dots, \mathbf{x}_k], & \hat{\gamma}_{(i)} > \gamma_{(i)}, i = 1, \dots, r-1, \hat{\gamma}_{(r)} \leq \gamma_{(r)}, \\ 1, & \hat{\gamma}_{(i)} > \gamma_{(i)}, i = 1, \dots, r, \end{cases}$$

with

$$\Lambda_j[\mathbf{x}_1, \dots, \mathbf{x}_k] \doteq \frac{\left( \frac{1}{NK} \sum_{k=1}^K \mathbf{x}_k^\dagger \mathbf{x}_k \right)^{NK}}{\left( \frac{(N-r)\tilde{\gamma} + \sum_{i=1}^{j-1} \gamma_{(i)}}{N-(r-j+1)} \right)^{(N-(r-j+1))K} \prod_{i=j}^r (\gamma_{(i)})^K}.$$

## Detector Design: $\gamma$ and $\mathbf{R}_{\theta\theta}$ unknown

The detection problems to be solved is

$$\begin{cases} H_0 : \mathbf{x}_k \sim \mathcal{CN}_N(\mathbf{0}, \gamma \mathbf{I}_N), & k = 1, \dots, K, \\ H_1 : \mathbf{x}_k \sim \mathcal{CN}_N(\mathbf{0}, \mathbf{H}\mathbf{R}_{\theta\theta}\mathbf{H}^\dagger + \gamma \mathbf{I}_N), & k = 1, \dots, K, \end{cases}$$

with  $\gamma$  and  $\mathbf{R}_{\theta\theta}$  unknown.

It is necessary to introduce the following quantities:

- $\widehat{\mathbf{R}}$  is given by

$$\widehat{\mathbf{R}} \doteq \frac{1}{K} \sum_{k=1}^K \mathbf{x}_k \mathbf{x}_k^\dagger;$$

- $\mathbf{L}$  is a nonsingular  $r \times r$  square root factor of  $\mathbf{H}^\dagger \mathbf{H}$ , namely such that  $\mathbf{L}\mathbf{L}^\dagger = \mathbf{H}^\dagger \mathbf{H}$ ;
- the  $r \times r$  matrix  $\mathbf{P}$  is defined as

$$\mathbf{P} \doteq \mathbf{L}^{-1} \mathbf{H}^\dagger \widehat{\mathbf{R}} \mathbf{H} \mathbf{L}^{-\dagger};$$

- $\phi_1, \dots, \phi_r$  denote the eigenvalues of  $\mathbf{P}$  ordered in non-ascending order.

## Detector Design: $\gamma$ and $\mathbf{R}_{\theta\theta}$ unknown (cont.)

It is possible to prove that the GLRT is given by

$$\Lambda[\mathbf{x}_1 \cdots \mathbf{x}_k] \underset{H_0}{\overset{H_1}{>}} \lambda \quad (1)$$

where

$$\Lambda[\mathbf{x}_1 \cdots \mathbf{x}_k] = \begin{cases} \Lambda_r[\mathbf{x}_1 \cdots \mathbf{x}_k], & \hat{\gamma}_r \leq \phi_r, \\ \Lambda_{r-1}[\mathbf{x}_1 \cdots \mathbf{x}_k], & \hat{\gamma}_r > \phi_r, \hat{\gamma}_{r-1} \leq \phi_{r-1}, \\ \vdots \\ \Lambda_1[\mathbf{x}_1 \cdots \mathbf{x}_k], & \hat{\gamma}_i > \phi_i, i = 2, \dots, r, \hat{\gamma}_1 \leq \phi_1, \\ 1, & \hat{\gamma}_i > \phi_i, i = 1, \dots, r, \end{cases}$$

where

$$\Lambda_j[\mathbf{x}_1 \cdots \mathbf{x}_k] \doteq \frac{\left( \frac{1}{NK} \sum_{k=1}^K \mathbf{x}_k^\dagger \mathbf{x}_k \right)^{NK}}{\hat{\gamma}_j^{(N-j)K} \prod_{i=1}^j \phi_i^K}$$

and

$$\hat{\gamma}_j \doteq \frac{\sum_{i=j+1}^r \phi_i + \text{tr}[\mathbf{P}_{\mathbf{H}}^\perp \widehat{\mathbf{R}}]}{N - j}, \quad j = 1, \dots, r.$$

This detector will be referred to as a **partially-adaptive one**.

## Detector Design: $\gamma\mathbf{R}_{nn}$ and $\mathbf{R}_{\theta\theta}$ unknown

Assume that  $M > N$  additional Gaussian vectors  $\mathbf{r}_k \sim \mathcal{CN}_N(\mathbf{0}, \mathbf{R}_{nn})$ ,  $k = K + 1, \dots, K + M$ , are available.

The detection problem to be solved is

$$\left\{ \begin{array}{l} H_0 : \left\{ \begin{array}{ll} \mathbf{r}_k \sim \mathcal{CN}_N(\mathbf{0}, \gamma\mathbf{R}_{nn}), & k = 1, \dots, K, \\ \mathbf{r}_k \sim \mathcal{CN}_N(\mathbf{0}, \mathbf{R}_{nn}), & k = K + 1, \dots, K + M, \end{array} \right. \\ \\ H_1 : \left\{ \begin{array}{ll} \mathbf{r}_k \sim \mathcal{CN}_N(\mathbf{0}, \mathbf{R}_{ss} + \gamma\mathbf{R}_{nn}), & k = 1, \dots, K, \\ \mathbf{r}_k \sim \mathcal{CN}_N(\mathbf{0}, \mathbf{R}_{nn}), & k = K + 1, \dots, K + M, \end{array} \right. \end{array} \right.$$

with  $\gamma\mathbf{R}_{nn}$  and  $\mathbf{R}_{\theta\theta}$  unknown.

We can come up with a fully-adaptive detector by resorting to the partially-adaptive detector (1) fed by the whitened data  $\mathbf{x}_k = \widehat{\mathbf{R}}_{nn}^{-1/2}\mathbf{r}_k$ , where

$$\widehat{\mathbf{R}}_{nn} \doteq \frac{1}{M} \sum_{k=K+1}^{K+M} \mathbf{r}_k \mathbf{r}_k^\dagger.$$

This detector will be referred to as a **fully-adaptive detector**.



## Performance Assessment

Performance assessment is conducted via Monte Carlo Simulation: we resort to  $10^5$  and  $10^3$  independent trials in order to set the threshold necessary to ensure  $P_{fa} = 10^{-3}$  and  $P_d$ , respectively.

We assume the following simulation parameters

- the columns of  $\mathbf{H}$  are the first  $r$  vectors of the canonical basis in  $\mathbb{R}^N$ ;
- the target is exponentially-correlated with one-lag correlation coefficient  $\rho_\theta$ , i.e.,

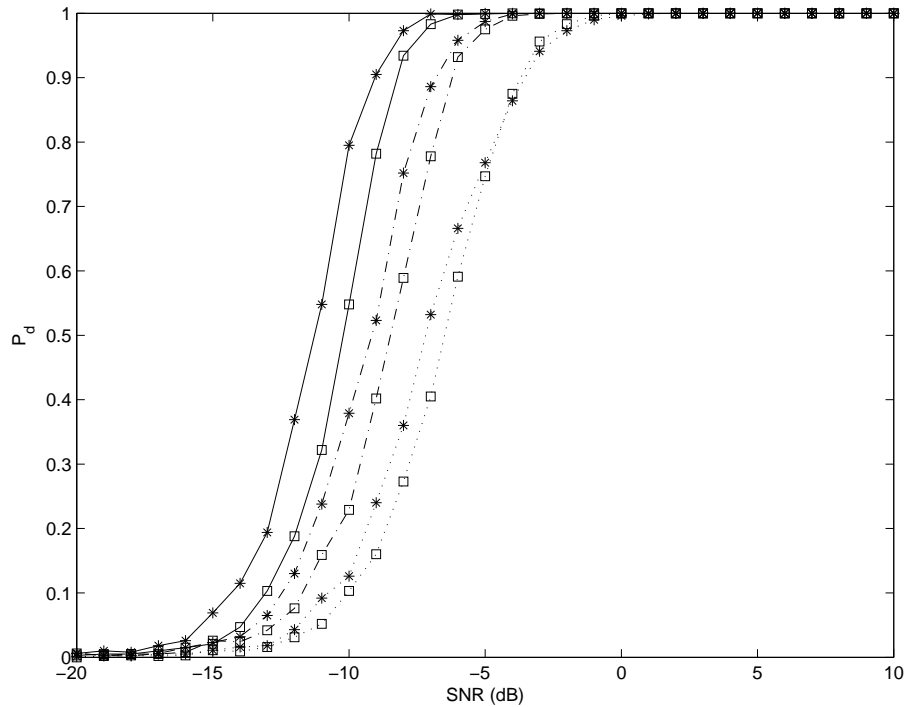
$$\mathbf{R}_{\theta\theta}(i, j) = \sigma_\theta^2 \rho_\theta^{|i-j|}, \quad i, j \in \{1, \dots, r\};$$

- we choose  $N = 16$ ,  $r = 3$ ,  $\gamma = 1$ ;
- curves of Figure 1 assume  $\mathbf{R}_{nn} = \mathbf{I}_N$  while those of Figure 2 refer to exponentially-correlated noise with one-lag correlation coefficient  $\rho_n$ , i.e.,

$$\mathbf{R}_{nn}(i, j) = \rho_n^{|i-j|}, \quad i, j \in \{1, \dots, N\};$$

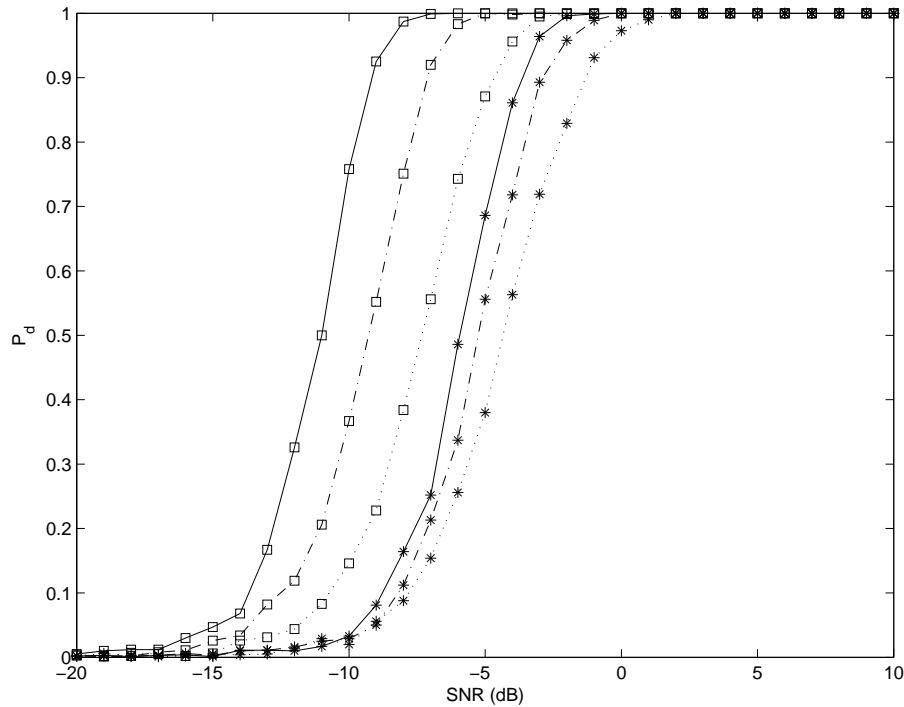
- the fully-adaptive detector of Figure 2 assumes  $M = 32$  secondary data.

## Performance Assessment (cont.)



$P_d$  vs SNR (dB) of the partially-adaptive detector for  $N = 16$ ,  $r = 3$ ,  $P_{fa} = 10^{-3}$ ,  $\gamma = 1$ ,  $K$  and  $\rho_\theta$  as parameters: curves with a square at each data point refer to  $\rho_\theta = 0.5$  while those with a star correspond to  $\rho_\theta = 0.9$ ; solid, dashdot, and dotted refer to  $K = 24$ ,  $K = 12$ , and  $K = 6$ , respectively.

## Performance Assessment (cont.)



$P_d$  vs SNR (dB) of the partially-adaptive and the fully-adaptive detector,  $N = 16$ ,  $r = 3$ ,  $\rho_\theta = 0.5$ ,  $\rho_n = 0.6$ ,  $\gamma = 1$ ,  $P_{fa} = 10^{-3}$ ,  $K$  as parameter: curves with a square at each data point refer to the partially-adaptive detector while those with a star correspond to the fully-adaptive one with  $M = 32$ ; solid, dashdot, and dotted refer to  $K = 24$ ,  $K = 12$ , and  $K = 6$ , respectively.

## Conclusions

- Note that both the partially- and the fully-adaptive detectors have the Constant False Alarm Rate (CFAR) property with respect to  $\gamma$ .
- A thorough performance assessment is the object of outgoing research activity.