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THEORETICAL AND EXPERIMENTAL DETERMINATION OF DAMPING CONSTANTS OF ONE-TO THREE-DIMENSIONAL VIBRATING SYSTEMS

THEORETICAL AND EXPERIMENTAL DETERMINATION OF DAMPING CONSTANTS OF ONE- TO THREE-DIMENSIONAL VIBRATING SYSTEMS

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ABSTRACT

Formulas are deduced for vibrating systems of one, two, and three dimensions. Undamped and damped free vibrations and harmonic forced vibrations are treated. Methods are proposed for calculating the damping constants from test observations.

I. INTRODUCTION

In developing formulas for vibration and possible flutter of structures such as rudders,[1,2] it may be necessary to include damping forces. Since these forces are not easy to calculate, methods of determining them from test observations may be needed.[3*] The basic theory for two- and three-dimensional cases will be considered and feasible methods of observation will be sought. First, however, formulas for the one-dimensional system will be written to assist in treating the main problem. For convenience of reference, a summary of the results is given in Table 1; see pages 30 and 31.

II. ONE-DIMENSIONAL VIBRATIONS

Assume as the equation of motion

$$m\ddot{x} + c\dot{x} + kx = P(t) \qquad [1]$$

in which m, c, and k are positive constants, $\dot{x} = dx/dt$, and P(t) denotes an applied force varying with the time t.

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1 References are listed on page 33.

* In Reference 1 (see pages 78 and 83), certain damping terms were omitted from the flutter equations because methods for determining these terms from experiments were unknown to the authors at that time. These flutter equations including the damping terms originally omitted are of the same form as the equations given here for the three-dimensional case.

1. DAMPED FREE VIBRATIONS

If $P = 0$, the general solution of Equation [1] can be written (as is easily verified) as follows in terms of independently arbitrary amplitudes a and b:

If $c^2 < 4\,mk$ (less than critical damping): $x = e^{-\mu t}(a \cos \omega t + b \sin \omega t)$

where $\mu = \frac{c}{2m}$ and $\omega^2 = \frac{k}{m} - \frac{1}{4}\, m^2 c^2$

If $c^2 = 4\,mk$ (critical damping): $x = (a + bt)e^{-\mu t}$, $\mu = \frac{c}{2m}$

If $c^2 > 4\,mk$ (greater than critical damping): $x = ae^{-\mu_1 t} + be^{-\mu_2 t}$

where μ_1 and μ_2 denote the following two values:

$$\mu_{1,2} = \frac{1}{2m}\left(c \pm \sqrt{c^2 - 4\,mk}\right)$$

2. HARMONIC FORCED VIBRATIONS

With $P = p \cos \omega_o t$ in terms of arbitrary constants p and ω_o:

$$x = a \cos \omega_o t + b \sin \omega_o t$$

$$\left[(k - m\omega_o^2)^2 + c^2\omega_o^2\right] a = (k - m\omega_o^2)\, p$$

$$\left[(k - m\omega_o^2)^2 + c^2\omega_o^2\right] b = c\omega_o p$$

$$\left[(k - m\omega_o^2)^2 + c^2\omega_o^2\right](a^2 + b^2) = p^2, \qquad \frac{a}{b} = \frac{k - m\omega_o^2}{c\omega_o}$$

Thus $a = 0$ and the vibration is in time quadrature relative to P when $\omega_o = \sqrt{k/m}$, which is the value of ω for undamped free vibration. The maximum amplitude or maximum of $\sqrt{a^2 + b^2}$ for given p, however, occurs when

$$(d/d\omega_o)\left[(k - m\omega_o^2)^2 + c^2\omega_o^2\right] = 0$$

or when

$$\omega_o^2 = \frac{k}{m} - \frac{c^2}{2m^2}$$

This differs from k/m by twice as much as does ω^2 in a damped free oscillation.

These formulas exhibit several features for which analogs may reasonably be expected in more complicated cases, namely:

(1) Two independent modes of damped free vibration occur. Their amplitudes can be chosen to make x and $\dot{x}$ agree with any assumed initial values.

(2) These free vibrations are oscillatory provided the damping constant c is not too large; in this case, c produces only a second-order change in the oscillatory frequency.

(3) In a harmonic forced vibration, c introduces a component of x in time quadrature relative to the applied force P (proportional to $\sin \omega_o t$ instead of to $\cos \omega_o t$).

(4) x is entirely in quadrature relative to P when the forcing frequency factor ω_o equals the value of ω for <u>undamped</u> free vibration.

(5) The maximum amplitude of x for forcing at given p, when damping is present, occurs at an ω_o differing from the undamped free ω by more than does the oscillatory ω in damped free vibration.

3. EXPERIMENTAL DETERMINATION OF c

If $\mu \neq 0$, its value can easily be determined from a curve showing either x or $\ddot{x}$ as a function of t during damped free motion. Then $c = 2m\mu$.

If ω is also determined from the curve, the ratio k/m can be calculated as $k/m = \omega^2 + \mu^2$. To determine k and m separately, one of them must be known from some other source.

Or, during a damped forced vibration the ratio b/a may be observed as the ratio of the components of x respectively in lagging quadrature to P and in phase with p, or the equal ratio for $\ddot{x}$. (Note that here $\ddot{x} = -\omega_o^2 x$). Then

$$c = \frac{1}{\omega_o} (k - m\omega_o^2) \frac{b}{a}$$

In this case, the values of both k and m must be known.

III. TWO-DIMENSIONAL VIBRATIONS

Assume that the kinetic energy T and potential energy V of a two-dimensional system can be written as [4,5*]

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{y}^2 + m_{12} \dot{x}\dot{y}, \qquad V = \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 y^2 + k_{12} xy$$

in which x and y are generalized coordinates and m_1, m_2, m_{12} are inertial and k_1, k_2, k_{12} elastic constants, of which only m_{12} and k_{12} may be negative. Substitution of first q = x and then q = y in Lagrange's equation or

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}} + \frac{\partial V}{\partial q} = Q_q$$

gives as equations of motion

$$m_1 \ddot{x}_1 + k_1 x + m_{12} \ddot{y} + k_{12} y = \overline{P}(t)$$

$$m_{12} \ddot{x}_1 + k_{12} x + m_2 \ddot{y} + k_2 y = \overline{Q}(t)$$

in which $\overline{P}(t)$ and $\overline{Q}(t)$ represent the total generalized forces acting on the

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* Also see Appendix A of Reference 1.

system (not including internal elastic forces). Part of $\overline{P}$ and $\overline{Q}$ may be due to linear damping forces. Expressing the latter in terms of damping constants c_1, c_2, c_{12}, c_{21}, the equations of motion may be written:

$$\begin{aligned} m_1\ddot{x} + k_1 x + m_{12}\ddot{y} + k_{12}y + c_1\dot{x} + c_{12}\dot{y} &= P(t) \\ m_{12}\ddot{x} + k_{12}x + m_2\ddot{y} + k_2 y + c_{21}\dot{x} + c_2\dot{y} &= Q(t) \end{aligned} \qquad [2]$$

in which P and Q represent possible external forces acting on the system (aside from damping forces).

Certain restrictions on the possible values of the constants are worth noting. Let x and y be so chosen that T and V are never negative. Damping effects can never increase the sum T + V. Multiply the first of Equations [2] by $\dot{x}$ and the second by $\dot{y}$ and add the two equations. The sum of the resulting m and k terms is easily seen to equal (d/dt) (T + V); hence, if P = Q = 0

$$\frac{d}{dt}\,(T + V) = -\,c_1\dot{x}^2 - c_2\dot{y}^2 - (c_{12} + c_{21})\,\dot{x}\dot{y}$$

To keep (d/dt) (T + V) from ever being positive, it is necessary that $c_1 \geq 0$, $c_2 \geq 0$, since either $\dot{x}$ or $\dot{y}$ may vanish. Similarly, to keep $T \geq 0$ and $V \geq 0$, it is necessary that m_1, m_2, k_1, and k_2 all be ≥ 0.

Further restrictions may be inferred from the following theorem. Let α, β, γ, e, g be real numbers. Then

$$\alpha e^2 + \beta g^2 + \gamma eg \geq 0 \ \underline{\text{or}}\ \alpha e^2 + \beta g^2 \geq -\gamma eg \qquad [3]$$

for all values of e and g if and only if

$$\alpha \geq 0, \ \beta \geq 0, \ \gamma^2 \leq 4\alpha\beta$$

To prove this, note first that α and β cannot be negative because of cases in which only $e = 0$ or $g = 0$. Relation [3] then clearly holds if e and g are such that $\gamma eg \geq 0$.

Suppose, however, that $\gamma eg < 0$. Then Equation [3] in its second form is equivalent to the following:

$$(\alpha e^2 + \beta g^2)^2 \geq (\gamma eg)^2 \qquad [3a]$$

provided that positive square roots are taken in passing back from Equation [3a] to Equation [3]. But

$$(\alpha e^2 + \beta g^2) = (\alpha e^2 - \beta g^2)^2 + 4\alpha\beta(eg)^2$$

Hence, if $\alpha > 0$ and $\beta > 0$ and if e and g are chosen so that $\alpha e^2 = \beta g^2$, then $(\alpha e^2 + \beta g^2)^2 = 4\alpha\beta(eg)^2$. Thus Equation [3a] can hold generally only if $4\alpha\beta \geq \gamma^2$. If either α or β vanishes, Equation [3] requires that $\gamma = 0$. Conversely, if the condition that $4\alpha\beta \geq \gamma^2$ is met but $\alpha e^2 \neq \beta g^2$, then $(\alpha e^2 + \beta g^2)^2 > 4\alpha\beta(eg)^2 > \gamma^2(eg)^2$ and Equation [3a] holds, also Equation [3].

Substitute here $\alpha = m_1/2$, $\beta = m_2/2$, $\gamma = m_{12}$, $e = \dot{x}$ and $g = \dot{y}$; next, $\alpha = k_1/2$, $\beta = k_2/2$, $\gamma = k_{12}$, $e = x$ and $g = y$; and finally $\alpha = c_1$, $\beta = c_2$, $\gamma = c_{12} + c_{21}$, $e = \dot{x}$ and $g = \dot{y}$. Compare the resulting expressions with expressions previously written for T, V, and (d/dt) (T + V). It will then be clear that, to prevent T and V from ever becoming negative or (d/dt) (T + V) positive, it is necessary and sufficient that

$$m_{12}^2 \leq m_1 m_2, \; k_{12}^2 \leq k_1 k_2, \; (c_{12} + c_{21})^2 \leq 4c_1 c_2 \qquad [4]$$

These restrictions will be assumed to hold.

It follows then also that

$$2m_{12}k_{12} \leq m_1 k_2 + m_2 k_1, \; c_{12}c_{21} \leq c_1 c_2 \qquad [5a,b]$$

For $(m_1k_2 + m_2k_1)^2 = (m_1k_2 - m_2k_1)^2 + 4m_1m_2k_1k_2 \geq 4m_1m_2k_1k_2 \geq 4m_{12}{}^2k_{12}{}^2$ by relations [4]. (Note that a square cannot be negative.) Similarly, in any case $4c_{12}c_{21} \leq 4c_{12}c_{21} + (c_{12} - c_{21})^2 = (c_{12} + c_{21})^2$; hence, by Equation [4], $4c_{12}c_{21} \leq 4c_1c_2$.

Two other relations that can be inferred in a similar way from relations [4] are:

$$(c_{12} + c_{21})m_{12} \leq c_1m_2 + c_2m_1, \quad (c_{12} + c_{21})k_{12} \leq c_1k_2 + c_2k_1 \qquad [5c,d]$$

1. UNDAMPED FREE VIBRATIONS

Undamped free oscillations merits consideration as background for study of the damped case. Let $c_1 = c_2 = c_{12} = c_{21} = 0$, $P = Q = 0$. Then Equations [2] become

$$m_1\ddot{x} + k_1x + m_{12}\ddot{y} + k_{12}y = 0 \quad m_{12}\ddot{x} + k_{12}x + m_2\ddot{y} + k_2y = 0 \qquad [6]$$

Two special cases may first be noted. According to Equations [6], x can vibrate while $y = 0$ only if $k_1 - m_1\omega^2$ and $k_{12} - m_{12}\omega^2$ are both zero. The first condition fixes ω at $\sqrt{k_1/m_1}$; the second requires that either $m_{12} = k_{12} = 0$ or $m_1k_{12} = m_{12}k_1$. Similarly, y can vibrate with $x = 0$, and $\omega = \sqrt{k_2/m_2}$ only if either $m_{12} = k_{12} = 0$ or $m_2k_{12} = m_{12}k_2$.

If x and y vibrate together in proportion to $\cos \omega t$, the following equations must be satisfied:

$$(k_1 - m_1\omega^2)\, x + (k_{12} - m_{12}\omega^2)\, y = 0$$
$$(k_{12} - m_{12}\omega^2)\, x + (k_2 - m_2\omega^2)\, y = 0$$

Elimination of x and y gives for the determination of ω the following equation:

$$(k_1 - m_1\omega^2)(k_2 - m_2\omega^2) - (k_{12} - m_{12}\omega^2)^2 = 0 \qquad [7a]$$

or

$$(m_1m_2 - m_{12}^2)\omega^4 - (m_1k_2 + m_2k_1 - 2m_{12}k_{12})\omega^2 + (k_1k_2 - k_{12}^2) = 0 \quad [7b]$$

If $k_{12}^2 = k_1k_2$, one root of Equations [7b] is: $\omega^2 = 0$. Alternatively, if $m_1m_2 = m_{12}^2$, only one mode of vibration is possible.

Assume now that $k_{12}^2 < k_1k_2$ and $m_{12}^2 < m_1m_2$. To locate ω^2, consider L, the left-hand member of Equation [7a] or Equation [7b], as a function of ω^2. At $\omega^2 = 0$, $L > 0$; but when ω^2 has increased to ω^2_{min} representing the lesser of the two values k_1/m_1 and k_2/m_2, then it is clear from Equation [7a] that $L < 0$. Hence $L = 0$ at some positive value of ω^2 less than ω^2_{min}. Also at the greater of the values k_1/m_1 and k_2/m_2, $L < 0$, but as $\omega^2 \to \infty$ it is clear from Equations [7b] that $L > 0$. Hence a second root of Equation [7a, b] occurs at a value of ω^2 greater than both k_1/m_1 and k_2/m_2.

Thus two different modes of vibration of the system are possible with both x and y vibrating. In each mode

$$\frac{y}{x} = -\frac{k_{12} - m_{12}\omega^2}{k_2 - m_2\omega^2} = -\frac{k_1 - m_1\omega^2}{k_{12} - m_{12}\omega^2}$$

2. DAMPED FREE VIBRATIONS

Let $P = Q = 0$ so that Equations [2] read

$$m_1\ddot{x} + k_1x + m_{12}\ddot{y} + k_{12}y + c_1\dot{x} + c_{12}\dot{y} = 0 \quad [8a]$$

$$m_{12}\ddot{x} + k_{12}x + m_2\ddot{y} + k_2y + c_{21}\dot{x} + c_2\dot{y} = 0 \quad [8b]$$

In special cases especially if $m_{12} = k_{12} = 0$ and $c_{12} = 0$ so that Equation [8a] reduces to Equation [1] with $P = 0$, x can vary while $y = 0$; or, similarly, if $c_{21} = 0$, y alone may vary. Such cases will not be discussed further here.

For the general case, solutions may be sought in which[6]

$$x = ae^{\lambda t}, \quad y = be^{\lambda t}$$

where a, b, and λ are non-zero constants, real or complex. Substituting in Equations [8a,b] and canceling out $e^{\lambda t}$:

$$(m_1\lambda^2 + k_1 + c_1\lambda)\, a + (m_{12}\lambda^2 + k_{12} + c_{12}\lambda)\, b = 0 \qquad [9a]$$

$$(m_{12}\lambda^2 + k_{12} + c_{21}\lambda)\, a + (m_2\lambda^2 + k_2 + c_2\lambda)\, b = 0 \qquad [9b]$$

The result of eliminating a and b from these equations may be written:

$$\epsilon_4 \lambda^4 + \epsilon_3 \lambda^3 + \epsilon_2 \lambda^2 + \epsilon_1 \lambda + \epsilon_o = 0 \qquad [10]$$

where

$$\epsilon_o = k_1k_2 - k_{12}^2, \quad \epsilon_1 = c_1k_2 + c_2k_1 - (c_{12} + c_{21})k_{12}$$

$$\epsilon_2 = m_1k_2 + m_2k_1 - 2m_{12}k_{12} + c_1c_2 - c_{12}c_{21}$$

$$\epsilon_3 = c_1m_2 + c_2m_1 - (c_{12} + c_{21})m_{12}, \quad \epsilon_4 = m_1m_2 - m_{12}^2$$

The coefficients $\epsilon_o \dots \epsilon_4$ are all ≥ 0, according to Equations [4] and [5a,b,c,d]. Hence no root λ of Equation [10] can be a positive real number. Probably if the damping is strong enough, negative real roots may occur, possibly even four in number, but this difficult question is of little practical interest here.

For the general case, write $\lambda = -\mu + i\omega$ where $i = \sqrt{-1}$ and μ and ω are real numbers. The following two equations result from substituting in Equation [10], then equating the real and imaginary parts separately to zero;

and dividing the imaginary equation by $i\omega$ on the assumption that $\omega \neq 0$:

$$\epsilon_4\omega^4 - (\epsilon_2 - 3\,\epsilon_3\mu + 6\,\epsilon_4\mu^2)\omega^2 + \epsilon_o - \epsilon_1\mu + \epsilon_2\mu^2 - \epsilon_3\mu^3 + \epsilon_4\mu^4 = 0 \qquad [11a]$$

$$\epsilon_1 - \epsilon_3\omega^2 - (2\epsilon_2 - 4\epsilon_4\omega^2)\mu + 3\epsilon_3\mu^2 - 4\epsilon_4\mu^3 = 0 \qquad [11b]$$

These equations determine μ and ω^2. The conjugate quantity $-\mu - i\omega$ is then also a root of Equation [10] . Since there are only four roots in all, there can be only two pairs of values, μ_1 and ω_1, and μ_2 and ω_2. These pairs define two modes of damped oscillation. Since damping cannot increase the total energy, it must turn out that both μ_1 and μ_2 are positive.

To obtain real expressions either the real parts of all quantities (i.e., solutions) may be chosen or the imaginary parts divided by i; the two pairs of real solutions thus obtained are in relative time quadrature. The value of the ratio b/a for each mode may be obtained from Equations [9a,b] . Since usually b/a will turn out complex, there will generally be a difference of phase between x and y as functions of the time.

Thus four real expressions are obtained representing four independent damped oscillations. For these oscillations, x and y can be written thus:

$$x = e^{-\mu_1 t}\,(A_1 \cos\omega_1 t + A_1' \sin\omega_1 t),\quad y = r_1\, e^{-\mu_1 t}\left[A_1 \cos(\omega_1 t + \epsilon_1) + A_1' \sin(\omega_1 t + \epsilon_1)\right]$$

or

$$y = e^{-\mu_2 t}\,(A_2 \cos\omega_1 t + A_2' \sin\omega_2 t),\quad y = r_2\, e^{-\mu_2 t}\left[A_2 \cos(\omega_2 t + \epsilon_2) + A_2' \sin(\omega_2 t + \epsilon_2)\right]$$

Here A_1, A_1', A_2, A_2' are independent arbitrary constants which can be adjusted to fit any assumed initial values of x, $\dot{x}$, y, $\dot{y}$. It should be noted that

$$\ddot{x} = -(\omega_1^2 - \mu_1^2)\, x \text{ and } \ddot{y} = -(\omega_1^2 - \mu_1^2) y$$

in any one mode whereas in the other

$$\ddot{x} = -(\omega_2^2 - \mu_2^2)\, x \text{ and } \ddot{y} = -(\omega_2^2 - \mu_2^2) y$$

Only small damping effects appear to be important in practice. Hence no general discussion of Equations [11a, b] will be undertaken here.

If the c's are sufficiently small, μ will also be small, and the coefficients ϵ_1 and ϵ_3 are likewise small. Consequently all terms in Equation [11a] containing μ are small at least to the second order, and the last three terms in Equation [11b] are small to the third order. For an approximate solution, these terms may all be dropped. Then Equation [11a] becomes: $\epsilon_4 \omega^4 - \epsilon_2 \omega^2 + \epsilon_o = 0$. This agrees with Equations [7a, b] for the case of no damping so that to the degree of approximation under discussion, the oscillation frequencies are the same as if there were no damping. From Equation [11b] the approximate value of μ is

$$\mu = \frac{1}{2} \frac{\epsilon_1 - \epsilon_3 \omega^2}{\epsilon_2 - 2\, \epsilon_4 \omega^2} \qquad [12]$$

More accurate solutions can be obtained from Equations [11a, b] by a process of successive approximation.

3. HARMONIC FORCED VIBRATIONS

If the applied forces are harmonic functions of the time t, they cause harmonic vibrations of x and y. At the start there may also exist superposed damped free oscillations whose amplitudes can be adjusted so as to produce on the whole any initial values of x, y, $\dot{x}$, $\dot{y}$. These damped free oscillations will be assumed to have died out.

Since in the one-dimensional case, the presence of damping introduces a phase difference, assume:

$$P = p\cos\omega_o t + p'\sin\omega_o t, \qquad Q = q\cos\omega_o t + q'\sin\omega_o t$$
$$x = a_1\cos\omega_o t + a_1'\sin\omega_o t, \qquad y = a_2\cos\omega_o t + a_2'\sin\omega_o t \qquad [13]$$

In Equations [2] the $\cos\omega_o t$ and $\sin\omega_o t$ terms must balance separately. After canceling the time factors, the result is the following four equations:

$$(k_1 - m_1\omega_o^2)a_1 + c_1\omega_o a_1' + (k_{12} - m_{12}\omega_o^2)a_2 + c_{12}\omega_o a_2' = p$$
$$-c_1\omega_o a_1 + (k_1 - m_1\omega_c^2)a_1' - c_{12}\omega_o a_2 + (k_{12} - m_{12}\omega_o^2)a_2' = p' \qquad [14]$$
$$(k_{12} - m_{12}\omega_o^2)a_1 + c_{21}\omega_o a_1' + (k_2 - m_2\omega_o^2)a_2 + c_2\omega_o a_2' = q$$
$$-c_{21}\omega_o a_1 + (k_{12} - m_{12}\omega_o^2)a_1' - c_2\omega_o a_2 + (k_2 - m_2\omega_o^2)a_2' = q'$$

Here p, p', q, q', a_1, a_1', a_2, a_2' are eight real numbers. In general, any four of them can be assigned arbitrarily; the equations then fix the values of the other four. Furthermore, since $\cos\omega_o t$ and $\sin\omega_o t$ differ only in phase, the zero for t can be so adjusted that any chosen one of the eight quantities $a_1 \ldots q'$ vanishes, without altering the physical form of the vibration. Thus all cases can be covered while keeping one coefficient zero.

In particular, Equations [14] may be solved for the amplitudes a_1, a_1', a_2, a_2' caused by given applied forces represented by p, p', q, q'. The determinant Δ of the coefficients of a_1, a_1', a_2, a_2' is easily found to have the value

$$\Delta = \left[(k_1 - m_1\omega_o^2)(k_2 - m_2\omega_o^2) - (k_{12} - m_{12}\omega_o^2)^2 \right]^2 - \omega_o^4 (c_1 c_2 - c_{12} c_{21})^2$$

If there is no damping, comparison with Equation [7a] shows that $\Delta = 0$ when ω_o equals the value of ω for either of the frequencies of undamped free vibration of the system.

If c_1, c_2, c_{12}, c_{21} are merely all small, Δ will vanish at two slightly modified frequencies that differ also slightly from the frequencies of damped free oscillation. As ω_o approaches either of these frequencies at which $\Delta = 0$ while p, p', q, q' remain fixed, the amplitude of the forced vibration becomes large (the phenomenon called resonance).

4. EXPERIMENTAL DETERMINATION OF c_1, c_2, c_{12}, c_{21}

One method is to make "bumping" observations by starting a motion and recording it as it decays. By proper adjustment of the initial values of x, $\dot{x}$, y, $\dot{y}$, the system can be made to vibrate in either of its two modes of damped free vibration with the other mode absent. Observations may be made of either x and y or $\ddot{x}$ and $\ddot{y}$ as functions of the time since $\ddot{x} = -(\omega_1^2 - \mu_1^2)\,x$ and $\ddot{y} = -(\omega_1^2 - \mu_1^2)\,y$ in one mode and $\ddot{x} = -(\omega_2^2 - \mu_2^2)\,x$, $\ddot{y} = -(\omega_2^2 - \mu_2^2)\,y$ in the other. From these observations, values can be calculated for each mode of the frequency ω, the damping constant μ, and the amplitude ratio r and phase ϵ of y relative to x, giving the eight known quantities

$$\omega_1 \;\omega_2 \;\mu_1 \;\mu_2 \;r_1 \;r_2 \;\epsilon_1 \;\epsilon_2$$

Insertion of ω_1 and μ_1 and then of ω_2 and μ_2 for ω and μ in Equations [11a, b] then provides four equations which can be solved numerically for c_1, c_2, c_{12}, and c_{21} provided the six constants m_1, m_2, m_{12}, k_1, k_2, and k_{12} are known. It might be more accurate, however, to use equations containing the constants ϵ_1 and ϵ_2 which differ from zero only because of damping. If bumping observations are to be used, further study of the methods of calculation should be made. The damping may be weak enough to justify the use of simplifying approximations.

It may be worth noting that observation of all eight quantities ω_1 to ϵ_2 should make possible the calculation of nine of the ten quantities m_1, m_2, m_{12}, k_1, k_2, k_{12}, c_1, c_2, c_{12}, and c_{21}. For a restriction exists on the possible variation of these quantities. Let Equations [8a, b] be multiplied by an arbitrary constant <u>s</u>. The new equations may then be regarded <u>either</u> as equations in a different form for the original system or as equations for a different system having constants <u>s</u> times as great but the same damping modes as the original system. In order to know which system of this similitude class the observed constants $\omega_1 \, . \, . \, . \, . \, . \, . \, . \, . \, \epsilon_2$ refer to, it is necessary to know at least <u>one</u> of the ten quantities $m_1 \, . \, . \, . \, . \, . \, . \, . \, c_{21}$. Then the remaining nine can all be calculated from the eight observed constants $\omega_1 \, . \, . \, . \, . \, . \, . \, . \, . \, \epsilon_2$.

(If Equations [8a, b] are multiplied by <u>different</u> numbers, they are still valid for the original system but cannot be regarded as equations in the same form as Equations [8a, b] for a different system because the new $m_{12} \neq m_{21}$

and $k_{12} \neq k_{21}$.) Even if the initial values of x, $\dot{x}$, y, $\dot{y}$ cannot be properly adjusted, since one mode will usually die out before the other, both sets of values, μ_1 and ω_1 and μ_2 and ω_2, can be inferred from the same curve of x or y as a function of time. If both modes persist, it is still possible to observe each mode in turn by means of a filter.[7-10] Or a vibrator may be used and adjusted in frequency so as to be in resonance with one mode; then, after the vibrator is removed, a damped free oscillation will occur in this mode only.

If $c_{12} = c_{21} = 0$, c_1 and c_2 can be calculated from μ_1 and μ_2. Otherwise the observed values of μ_1 and μ_2 furnish only two relations among the four quantities c_1, c_2, c_{12}, c_{21}.

An alternative method is to study forced harmonic vibrations produced by applied forces P and Q whose relative amplitudes and phases can be controlled. (Applied forces are pure P when they do no work during variation of y alone, or pure Q when no work is done during variation of x alone.) Two alternative procedures will be described which require no measurements of P or Q. The constants m_1, m_2, m_{12}, k_1, k_2, k_{12}, however, must be known. Either x and y or $\ddot{x}$ and $\ddot{y}$ may be observed since in forced oscillations $\ddot{x} = -\omega_o^2 x$, $\ddot{y} = -\omega_o^2 y$ and ω_o^2 will be seen to cancel out in all final formulas.

First Procedure: Isolation of c_1, c_2, c_{12}, c_{21} in turn. Make observations as follows:

(1) Cause x to vibrate with y = 0. Assume $p' = 0$, so that a_1 denotes

the amplitude of the component of x that is in phase with P and a_1' the amplitude of the quadrature component of x. To do this, apply $P = p \cos \omega_o t$ and adjust the amplitude and phase of Q so that $a_2 = a_2' = 0$. Then Equations [14] reduce to

$$(k_1 - m_1\omega_o^2)\, a_1 + c_1\omega_o a_1' = p \qquad -c_1\omega_o a_1 + (k_1 - m_1\omega_o^2)\, a_1' = 0$$

$$(k_{12} - m_{12}\omega_o^2)\, a_1 + c_{21}\omega_o a_1' = q \qquad -c_{21}\omega_o a_1 + (k_{12} - m_{12}\omega_o^2) a_1' = q'$$

The magnitude of ω_o should be made quite different from $\sqrt{k_1/m_1}$. Only the ratio a_1'/a_1 needs to be observed.

Probably the adjustment of Q can be effected most conveniently by varying its amplitude $\sqrt{(q)^2 + (q')^2}$ until a_2 (or the component of y in phase with P) is zero, then varying the phase of Q (thus varying q') until the quadrature amplitude a_2' of y equals zero, and repeating these adjustments in turn until both a_2 and a_2' remain negligibly small.

Then

$$c_1 = \frac{a_1'}{\omega_o a_1} (k_1 - m_1\omega_o^2)$$

(2) Similarly, to keep $x = 0$, apply $Q = q \cos \omega_o t$, hence $q' = 0$, and with ω_o not near $\sqrt{k_2/m_2}$, adjust p and p' so that $a_1 = a_1' = 0$, and read a_2'/a_2. Then

$$c_2 = \frac{a_2'}{\omega_o a_2} (k_2 - m_2\omega_o^2)$$

(3) Cause x and y to vibrate in phase with P; that is, writing $P = p \cos \omega_o t$ with $p' = 0$, adjust q and q' so that $a_1' = a_2' = 0$. Read a_1/a_2. Then from the second one of Equations [14]

$$c_{12} = -\frac{a_1}{a_2} c_1$$

In this case the simplest way to effect the required adjustment of Q might be to vary its amplitude so as to reduce the larger of a_1' and a_2' until $a_1' = a_2'$, then adjust the phase of Q so as to minimize a_2', and repeat these adjustments until a_1' and a_2' have been made sufficiently small.

(4) Cause x and y to vibrate in phase with Q, assuming $q' = 0$. Adjust p and p' so that $a_1' = a_2' = 0$ nearly enough. Read a_2/a_1. Then from the fourth of Equations [14]

$$c_{21} = -\frac{a_2}{a_1} c_2$$

This procedure should yield the most accurate values of the four c's, but the experimental adjustments required may be considered too tedious.

Second Procedure: Single-phase forcing. Apply P and Q in any known ratio but in the same phase. Write $P = p \cos \omega_o t$, $Q = q \cos \omega_o t$, so that $p' = q' = 0$. Read a_1, a_2 as amplitudes of inphase and a_1', a_2' as amplitudes of quadrature components of x and y. Repeat with a different ratio Q/P, distinguishing the amplitudes thus obtained by a bar.

Substitute each set of a's in turn into the second and fourth of Equations [14], in which $p' = q' = 0$. The resulting equations can be written:

$$\omega_o a_1 c_1 + \omega_o a_2 c_{12} = (k_1 - m_1\omega_o^2)\, a_1' + (k_{12} - m_{12}\omega_o^2)\, a_2'$$

$$\omega_o \bar{a}_1 c_1 + \omega_o \bar{a}_2 c_{12} = (k_1 - m_1\omega_o^2)\, \bar{a}_1' + (k_{12} - m_{12}\omega_o^2)\, \bar{a}_2'$$

$$\omega_o a_1 c_{21} + \omega_o a_2 c_2 = (k_{12} - m_{12}\omega_o^2)\, a_1' + (k_2 - m_2\omega_o^2)\, a_2'$$

$$\omega_o \bar{a}_1 c_{21} + \omega_o \bar{a}_2 c_2 = (k_{12} - m_{12}\omega_o^2)\, \bar{a}_1' + (k_2 - m_2\omega_o^2)\, \bar{a}_2'$$

These two pairs of equations are easily solved for c_1, c_{12}, and c_2, c_{21}.

IV. THREE-DIMENSIONAL VIBRATIONS

Let x, y, z denote the three displacement variables, for example v, γ, α motion of a rudder (see Reference 1). Then linear equations of motion can be written as follows:

$$m_1\ddot{x} + k_1 x + m_{12}\ddot{y} + k_{12} y + m_{13}\ddot{z} + k_{13} z + c_1\dot{x} + c_{12}\dot{y} + c_{13}\dot{z} = P(t) \quad [15a]$$

$$m_{12}\ddot{x} + k_{12} x + m_2\ddot{y} + k_2 y + m_{23}\ddot{z} + k_{23} z + c_{21}\dot{x} + c_2\dot{y} + c_{23}\dot{z} = Q(t) \quad [15b]$$

$$m_{13}\ddot{x} + k_{13} x + m_{23}\ddot{y} + k_{23} y + m_3\ddot{z} + k_3 z + c_{31}\dot{x} + c_{32}\dot{y} + c_3\dot{z} = R(t) \quad [15c]$$

Here P, Q, and R are <u>generalized external forces</u> so defined that the rate at which they do work on the system is always $P\dot{x} + Q\dot{y} + R\dot{z}$. The m's are of the nature of inertial constants and the k's of elastic constants.

Then there may be, as in Equations [15a,b,c], nine linear damping constants c_1, c_2, c_3, c_{12}, c_{13}, c_{21}, c_{23}, c_{31}, c_{32}. The six cross constants c_{12}, etc., will be limited in relative size, as in the two-dimensional case, since the damping necessarily tends to decrease the total energy T + V; they are likely to be relatively small and may be negligible, but this cannot be assumed to be true in general because the magnitudes of all nine constants will vary with the choice of the variables to be called x, y, z.

The situation will be analogous in general to that for two dimensions. If P = Q = R = 0 and all c's are zero, there will be solutions of Equations [15a, b, c] representing three modes of undamped free vibration. If any c's do not vanish, these modes become three modes of damped free oscillation; or, if the c's are sufficiently large, one or more modes may be replaced by two modes of exponential decrease without oscillation, such as were represented by formulas in the one-dimensional case.

In the oscillatory case, on the other hand, there will be three damping constants μ_1, μ_2, μ_3. In any one mode of damped oscillation, the three variables x, y, and z may be assumed to be proportional to $e^{-\mu_1 t} \cos (\omega_1 t + \epsilon)$, in another mode to $e^{-\mu_2 t} \cos (\omega_2 t + \epsilon)$, and in the third to $e^{-\mu_3 t} \cos (\omega_3 t + \epsilon)$, the phase angle ϵ being different in general for x, y and z and different in the three modes.

The frequency factors ω_1, ω_2, ω_3 will not be quite the same as in the undamped vibrations, but the difference will be only of the second order if the damping is relatively small.

A more detailed discussion of these various cases follows:

1. UNDAMPED FREE VIBRATIONS

If $P = Q = R = 0$ and all the c's are zero, a solution of Equations [15a,b,c] is

$x = a_1 \cos \omega t$, $y = a_2 \cos \omega t$, and $z = a_3 \cos \omega t$, a_1, a_2, and a_3 being real numbers; from Equations [15a,b,c] :

$$(k_1 - m_1\omega^2)\, a_1 + (k_{12} - m_{12}\omega^2)\, a_2 + (k_{13} - m_{13}\omega^2)\, a_3 = 0$$

$$(k_{12} - m_{12}\omega^2)\, a_1 + (k_2 - m_2\omega^2)\, a_2 + (k_{23} - m_{23}\omega^2)\, a_3 = 0$$

$$(k_{13} - m_{13}\omega^2)\, a_1 + (k_{23} - m_{23}\omega^2)\, a_2 + (k_3 - m_3\omega^2)\, a_3 = 0$$

Equating the determinant of a_1, a_2, a_3 in these equations to zero gives the equation:

$$\begin{aligned}
&(k_1 - m_1\omega^2)(k_2 - m_2\omega^2)(k_3 - m_{13}\omega^2) \\
&\quad + 2(k_{12} - m_{12}\omega^2)(k_{13} - m_{13}\omega^2)(k_{23} - m_{23}\omega^2) \\
&- (k_1 - m_1\omega^2)(k_{23} - m_{23}\omega^2)^2 - (k_2 - m_2\omega^2)(k_{13} - m_{13}\omega^2)^2 \\
&\quad - (k_3 - m_3\omega^2)(k_{12} - m_{12}\omega^2)^2 = 0
\end{aligned} \qquad [16]$$

This is a cubic equation in ω^2 whose three roots furnish the frequencies for three modes of undamped free vibration. Any two of the original equations can be solved for the ratios of a_1, a_2, and a_3 to each other in any one of the three modes (see, for example, Appendix C of Reference 1).

2. DAMPED FREE VIBRATIONS

Assume $P = Q = R = 0$ and write

$$x = a_1 e^{\lambda t}, \quad y = a_2 e^{\lambda t}, \quad z = a_3 e^{\lambda t}$$

where a_1, a_2, a_3 and λ may all be complex numbers. Substitution in Equations [15a,b,c] then gives:

$$(k_1 + m_1 \lambda^2 + c_1\lambda)\, a_1 + (k_{12} + m_{12}\lambda^2 + c_{12}\lambda)\, a_2$$
$$+ (k_{13} + m_{13}\lambda^2 + c_{13}\lambda)\, a_3 = 0$$

$$(k_{12} + m_{12}\lambda^2 + c_{21}\lambda)\, a_1 + (k_2 + m_2\lambda^2 + c_2\lambda)\, a_2$$
$$+ (k_{23} + m_{23}\lambda^2 + c_{23}\lambda)\, a_3 = 0$$

$$(k_{13} + m_{13}\lambda^2 + c_{31}\lambda)\, a_1 + (k_{23} + m_{23}\lambda^2 + c_{32}\lambda)\, a_2$$
$$(k_3 + m_3\lambda^2 + c_3\lambda)\, a_3 = 0$$

The determinant of a_1, a_2, a_3 in these three equations set equal to zero gives :

$$\begin{aligned}
&(k_1 + m_1\lambda^2 + c_1\lambda)(k_2 + m_2\lambda^2 + c_2\lambda)(k_3 + m_3\lambda^2 + c_3\lambda) \\
&+ (k_{12} + m_{12}\lambda^2 + c_{12}\lambda)(k_{23} + m_{23}\lambda^2 + c_{23}\lambda)(k_{13} + m_{13}\lambda^2 + c_{31}\lambda) \\
&+ (k_{12} + m_{12}\lambda^2 + c_{21}\lambda)(k_{23} + m_{23}\lambda^2 + c_{32}\lambda)(k_{13} + m_{13}\lambda^2 + c_{13}\lambda) \\
&- (k_1 + m_1\lambda^2 + c_1\lambda)(k_{23} + m_{23}\lambda^2 + c_{23}\lambda)(k_{23} + m_{23}\lambda^2 + c_{32}\lambda) \\
&- (k_2 + m_2\lambda^2 + c_2\lambda)(k_{13} + m_{13}\lambda^2 + c_{13}\lambda)(k_{13} + m_{13}\lambda^2 + c_{31}\lambda) \\
&- (k_3 + m_3\lambda^2 + c_3\lambda)(k_{12} + m_{12}\lambda^2 + c_{12}\lambda)(k_{12} + m_{12}\lambda^2 + c_{21}\lambda) = 0
\end{aligned}$$

[17]

This is an equation of the sixth degree in λ. It may have real roots if the c's are large enough, perhaps as many as six real roots. On the other hand, analogy with the two-dimensional case suggests that if the c's are not too large, there will be six complex roots in three pairs: $-\mu_1 \pm i\omega_1$, $-\mu_2 \pm i\omega_2$, $-\mu_3 \pm i\omega_3$.

Two equations for the determination of ω_1, ω_2, ω_3 and μ_1, μ_2, μ_3, analogous to Equations [11a,b] in two dimensions, can be obtained by substituting $\lambda = -\mu + i\omega$ and separating real and imaginary parts. In the three-dimensional case, however, these equations are voluminous and the chance of their ever being put to practical use seems to be very small, hence they will not be written out here in full.

For practical use when the c's and hence also the μ's are small, abbreviated approximate equations can be obtained by omitting all terms of second or higher order, that is, all terms containing a power of μ higher than the first or both μ and one of the c's or the product of two c's. This rule of approximation justifies replacing λ^2 in Equation [17] by $-\omega^2 - 2i\omega\mu$ and also λ by $i\omega$. Furthermore, all products of c terms may be omitted. The first of the six products in Equation [17], for example, is to be replaced by

$$(k_1 - m_1\omega^2 - 2i\omega m_1 \mu + i\omega c_1)(k_2 - m_2\omega^2 - 2i\omega m_2 \mu + i\omega c_2)$$
$$(k_3 - m_3\omega^2 - 2i\omega m_3 \mu + i\omega c_3)$$

and then expanded as

$$(k_1 - m_1\omega^2)(k_2 - m_2\omega^2)(k_3 - m_3\omega^2)$$

$$+ i\omega\,(-2m_1\,\mu + c_1)(k_2 - m_2\omega^2)(k_3 - m_3\omega^2)$$

$$+ i\omega\,(-2m_2\,\mu + c_2)(k_1 - m_1\omega^2)(k_3 - m_3\omega^2)$$

$$+ i\omega\,(-2m_3\,\mu + c_3)(k_1 - m_1\omega^2)(k_2 - m_2\omega^2)$$

It is easily seen that the real part of Equation [17] as thus reduced is the same as Equation [16] for undamped vibration. Hence the frequencies of oscillation in the three damped modes are approximated here by the frequencies of undamped vibration and may be calculated from Equation [16].

To shorten the notation, write now

$$G_1 = k_1 - m_1\omega^2,\quad G_2 = k_2 - m_2\omega^2,\quad G_3 = k_3 - m_3\omega^2$$

$$G_{12} = k_{12} - m_{12}\omega^2,\ G_{13} = k_{13} - m_{13}\omega^2,\ G_{23} = k_{23} - m_{23}\omega^2$$

Then it will be found that the imaginary part of Equation [17] divided by $i\omega$ can be written in its approximated form thus.

$$-2\,\mu\,\Big[m_1(G_2G_3 - G_{23}^2) + m_2(G_1G_3 - G_{13}^2) + m_3(G_1G_2 - G_{12}^2)$$

$$+ 2m_{12}(G_{13}G_{23} - G_3G_{12}) + 2m_{13}\,(G_{12}G_{13} - G_2G_{13})$$

$$+ 2m_{23}(G_{12}G_{13} - G_1G_{23})\Big] + c_1\,(G_2G_3 - G_{23}^2) + c_2(G_1G_3 - G_{13}^2)$$

$$+ c_3(G_1G_2 - G_{12}^2) + (c_{12} + c_{21})(G_{13}G_{23} - G_3G_{12}) + (c_{13} + c_{31})$$

$$(G_{12}G_{23} - G_2G_{13}) + (c_{23} + c_{32})(G_{12}G_{13} - G_1G_{23}) = 0 \qquad [18]$$

After inserting in the G's the proper value of ω^2 for any one of the damped modes, this equation is easily solved for an approximate value of the damping constant μ for that mode.

3. HARMONIC FORCED VIBRATIONS

Assume

$$P = p\cos\omega_o t + p'\sin\omega_o t, \qquad Q = q\cos\omega_o t + q'\sin\omega_o t$$

$$R = r\cos\omega_o t + r'\sin\omega_o t$$

where p, p', q, q', r, r' are any six real amplitudes and ω_o is any positive real number. For the resulting steady vibration write

$$x = a_1\cos\omega_o t + a_1'\sin\omega_o t, \qquad y = a_2\cos\omega_o t + a_2'\sin\omega_o t$$

$$z = a_3\cos\omega_o t + a_3'\sin\omega_o t$$

$a_1 \ldots . a_3'$ being six real numbers.

In any particular motion, by a proper choice of the origin for t, any chosen one of the six variables P, Q, R, x, y, z can be made to vibrate in proportion to $\cos\omega_o t$, or to $\sin\omega_o t$. Thus any one of the twelve amplitudes $p, p' \ldots . a_3, a_3'$ can be assumed to be zero without altering the motion that is represented.

Substitution in Equations [15a,b,c] and separation of sine and cosine terms gives six equations. To shorten the notation, write:

$$F_1 = k_1 - m_1\omega_o^2 \qquad F_2 = k_2 - m_2\omega_o^2 \qquad F_3 = k_3 - m_3\omega_o^2$$

$$F_{12} = k_{12} - m_{12}\omega_o^2 \qquad F_{13} = k_{13} - m_{13}\omega_o^2 \qquad F_{23} = k_{23} - m_{23}\omega_o^2$$

Then the six equations read:

$$F_1a_1 + c_1\omega_o a_1' + F_{12}a_2 + c_{12}\omega_o a_2' + F_{13}a_3 + c_{13}\omega_o a_3' = p \qquad [19a]$$

$$-c_1\omega_o a_1 + F_1 a_1' - c_{12}\omega_o a_2 + F_{12}a_2' - c_{13}\omega_o a_3 + F_{13}a_3' = p' \qquad [19b]$$

$$F_{12}a_1 + c_{21}\omega_o a_1' + F_2a_2 + c_2\omega_o a_2' + F_{23}a_3 + c_{23}\omega_o a_3' = q \qquad [19c]$$

$$-c_{21}\omega_o a_1 + F_{12}a_1' - c_2\omega_o a_2 + F_2a_2' - c_{23}\omega_o a_3 + F_{23}a_3' = q' \qquad [19d]$$

$$F_{13}a_1 + c_{31}\omega_o a_1' + F_{23}a_2 + c_{32}\omega_o a_3' + F_3a_3 + c_3\omega_o a_3' = r \qquad [19e]$$

$$-c_{31}\omega_o a_1 + F_{13}a_1' - c_{32}\omega_o a_2 + F_{23}a_2' - c_3\omega_o a_3 + F_3a_3' = r' \qquad [19f]$$

In general any six of the twelve amplitudes $a_1 \ldots . r'$ can be assigned arbitrarily and the equations then fix the values of the other six.

4. EXPERIMENTAL DETERMINATION OF

c_1, c_2, c_3, c_{12}, c_{13}, c_{21}, c_{23}, c_{31}, c_{32}

The methods described for a two-dimensional system can be extended to three dimensions. Determination of the nine damping constants from general bumping observations, however, will not be discussed here because it appears to involve very complicated observations and calculations.

A feasible alternative might be to lock each of the three coordinates in turn so as to hold it at zero. The given three-dimensional system could thus be studied as a combination of three two-dimensional systems and the methods already described for such systems would be available.

Of three-dimensional motions, only forced harmonic motions will be considered here and only the simplest use of these. In such motions, $\ddot{x}$, $\ddot{y}$, and $\ddot{z}$ are equal respectively to $-\omega_o^2 x$, $-\omega_o^2 y$, and $-\omega_o^2 z$ so that either x, y, or z may be measured.

Apply P, Q, R in any convenient ratio but all in the same phase. Assume $p' = q' = r' = 0$. Read the resulting three inphase amplitudes a_1, a_2, a_3 and the three quadrature amplitudes a_1', a_2', a_3', the latter being relatively small. Repeat with different ratios of P, Q, R, distinguishing the a's thus obtained by a bar, and then with a third set of ratios, marking the a's with a double bar. A possible choice would be to use only P the first time, only Q the second time, and only R the third time.

Substitution of the first set of observed a's in Equations [19b, d, f], then the second set of a's, and finally the third set gives three groups of three equations each for the determination of the nine c's. Since in all cases $p' = q' = r' = 0$, the equations may be slightly rearranged to read as follows:

$$a_1 c_1 + a_2 c_{12} + a_3 c_{13} = \frac{1}{\omega_o} (F_1 a_1' + F_{12} a_2' + F_{13} a_3') \qquad [20]$$

$$\bar{a}_1 c_1 + \bar{a}_2 c_{12} + \bar{a}_3 c_{13} = \frac{1}{\omega_o} (F_1 \bar{a}_1' + F_{12} \bar{a}_2' + F_{13} \bar{a}_3')$$

$$\bar{\bar{a}}_1 c_1 + \bar{\bar{a}}_2 c_{12} + \bar{\bar{a}}_3 c_{13} = \frac{1}{\omega_o} (F_1 \bar{\bar{a}}_1' + F_{12} \bar{\bar{a}}_2' + F_{13} \bar{\bar{a}}_3')$$

$$a_1 c_{21} + a_2 c_2 + a_3 c_{23} = \frac{1}{\omega_o} (F_{12} a_1' + F_2 a_2' + F_{23} a_3')$$

$$\bar{a}_1 c_{21} + \bar{a}_2 c_2 + \bar{a}_3 c_{23} = \frac{1}{\omega_o} (F_{12} \bar{a}_1' + F_2 \bar{a}_2' + F_{23} \bar{a}_3')$$

$$\bar{\bar{a}}_1 c_{21} + \bar{\bar{a}}_2 c_2 + \bar{\bar{a}}_3 c_{23} = \frac{1}{\omega_o} (F_{12} \bar{\bar{a}}_1' + F_2 \bar{\bar{a}}_2' + F_{23} \bar{\bar{a}}_3')$$

$$a_1 c_{31} + a_2 c_{32} + a_3 c_3 = \frac{1}{\omega_o} (F_{13} a_1' + F_{23} a_2' + F_3 a_3')$$

$$\bar{a}_1 c_{31} + \bar{a}_2 c_{32} + \bar{a}_3 c_3 = \frac{1}{\omega_o} (F_{13}\bar{a}'_1 + F_{23}\bar{a}'_2 + F_3\bar{a}'_3)$$

$$\bar{\bar{a}}_1 c_{31} + \bar{\bar{a}}_2 c_{32} + \bar{\bar{a}}_3 c_3 = \frac{1}{\omega_o} (F_{13}\bar{\bar{a}}'_1 + F_{23}\bar{\bar{a}}'_2 + F_3\bar{\bar{a}}'_3)$$

Assuming that the six constants F_1, F_2, F_3, F_{12}, F_{13}, F_{23} have been calculated from the constants of the system and the chosen value of ω_o, the first three of Equations [20] can be solved for c_1, c_{12}, c_{13}, the middle three for c_2, c_{21}, c_{23}, and the last three for c_3, c_{31}, c_{32}.

The computation can be shortened by observing differently. Using chosen p and q, adjust r (p', q', r' being all zero) so that $a_3 = 0$. Read a_1, a_2, a'_1, a'_2, a'_3. Repeat with a different pair of values of p and q, making $\bar{a}_3 = 0$. Read $\bar{a}_1$, $\bar{a}_2$, $\bar{a}'_1$, $\bar{a}'_2$, $\bar{a}'_3$. Then the first two of Equations [20] are easily solved for c_1 and c_{12}, the fourth and fifth for c_2 and c_{21}, and the seventh and eighth for c_{31} and c_{32}.

Repeat using two pairs of values of p and r and adjusting q each time so that $a_2 = \bar{a}_2 = 0$. Read a_1, a_3, a'_1, a'_2, a'_3 and $\bar{a}_1$, $\bar{a}_3$, $\bar{a}'_1$, $\bar{a}'_2$, $\bar{a}'_3$. Then the same three pairs out of Equations [20] yield c_1 and c_{13}, c_{21} and c_{23}, and c_3 and c_{31}.

All of the c's have thus been obtained, with duplicate values of c_1, c_{21} and c_{31}. Other combinations of p, q, r may be used in a similar way.

It will be noted that neither procedure requires actual measurements of P, Q, or R.

V. VARIABLE DAMPING

In practice there seems to be a tendency for high-frequency vibrations to die out more rapidly than low-frequency vibrations. Such differences may result in many ways from the characteristics of the systems. It is worth noting, however, that

(1) A simple increase of scale is likely to lower the damping rate.

(2) The damping rate of a high-frequency mode of vibration can be less than that of a low-frequency mode of the same system.

1. CHANGE OF SCALE

As a simple example, consider a mass on a spring subject to linear damping, its equation of motion being

$$m\ddot{x} + c\dot{x} + kx = 0$$

In a damped vibration

$$x = Ae^{-\mu t} \sin \omega t$$

where $\mu = c/(2m)$ and $\omega^2 = (k/m) - \mu^2$

Now let all linear dimensions be changed in any ratio λ without change of material. Then* $m \propto \lambda^3$, $k \propto \lambda^2$. What happens to c? At given $\dot{x}$, water resistance will tend to vary in proportion to the surface wetted; hence $c \propto \lambda^2$. For simplicity, suppose μ^2 may be dropped in comparison with k/m. Then, approximately,

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* In a change of scale including change of both cross section and length of spring, $k \propto \frac{\lambda^2}{\lambda} = \lambda$. For a mass on a spring, when all dimensions change, $m \propto \lambda^3$, $k \propto \lambda$, $\omega \propto \frac{1}{\lambda}$. If only the length of the spring does not change, $m \propto \lambda^3$ $k \propto \lambda^2$, $\omega \propto \frac{1}{\sqrt{\lambda}}$

$$\omega \propto 1/\sqrt[3]{\lambda}\ ;\ \mu \propto 1/\lambda\ ;\ \therefore \mu \propto \omega^2$$

Or, if c does not change when $\lambda \neq 1$, then approximately when μ is small

$$\omega \propto 1/\sqrt[3]{\lambda};\ \mu \propto 1/\lambda^3;\ \therefore \mu \propto \omega^6$$

In both cases μ and ω both increase if $\lambda < 1$ and decrease if $\lambda > 1$, thus varying "in the same direction."

2. CONTRARY MODES FOR A GIVEN SYSTEM

Since higher frequency tends to mean higher velocities at a given amplitude, it might reasonably be guessed that the damping will be greater in modes of higher frequency. This is not necessarily the case, however, because the components of displacement are in different ratios in different modes and some components may be damped more heavily than others.

As a simple example, suppose

$$m_1\ddot{x} + k_1 x + k_{12} y + c_1 \dot{x} = 0 \qquad m_2\ddot{y} + k_2 y + k_{12} x + c_2 \dot{y} = 0$$

where $k_2/m_2 \gg k_1/m_1$ but $c_2/m_2 \ll c_1/m_1$.

If $k_{12} = 0$ and $c_1 = c_2 = 0$, then in one mode, x vibrates with $y = 0$; in the other, y vibrates at much higher frequency with $x = 0$. If $k_{12} = 0$ but c_1 and c_2 are merely small, then the two frequencies are little altered by c_1 and c_2, and the damping will be much less for the second or y vibration than for the first or x vibration.

Thus higher frequency is accompanied here by lower damping. This conclusion will not be altered if k_{12} is merely kept small but not zero, so that y vibrates a little in the first mode and x vibrates a little in the second mode.

A

Table 1

Summary of Results

One Dimensional Damped Vibrations

$$m\ddot{x} + c\dot{x} + kx = P(t)$$

CASE	SOLUTIONS	REMARKS	EXPERIMENTAL DETERMINATION OF DAMPING
(1) DAMPED FREE VIBRATIONS $P = 0$	When $c^2 < 4mk$ (less than critical damping), $x = e^{-\mu t}(a \cos wt + b \sin wt)$ where $\mu = \frac{c}{2m}$, $w^2 = \frac{k}{m} - \frac{1}{4} m^2 c^2$ $c^2 = 4mk$ (critical damping), $x = (a + bt)e^{-\mu t}$ where $\mu = \frac{c}{2m}$, $c^2 > 4mk$ (greater than critical damping), $x = ae^{-\mu_1 t} + be^{-\mu_2 t}$ where $\mu_{1,2} = \frac{1}{2m}(c \pm \sqrt{c^2 - 4mk})$ a, b arbitrary	(1) Two independent modes of damped free vibration can occur. Their amplitudes can be chosen to make x and $\dot{x}$ agree with any assumed initial values. (2) The free vibrations are oscillatory provided the damping constant c is not too large. Small c has negligible effect on the oscillatory frequency.	(1) Obtain μ from curve of x vs t (when $\mu \neq 0$) Then $c = 2m\mu$.
(2) HARMONIC FORCED VIBRATIONS $P(t) = p \cos w_o t$	$x = a \cos w_o t + b \sin w_o t$ $p^2 = \left[(k - mw_o^2)^2 + c^2 w_o^2\right]$ $\frac{a}{b} = \frac{k - mw_o^2}{cw_o}$ p, w_o arbitrary	(1) c introduces a component of x in time quadrature relative to P. When $w_o = \sqrt{\frac{k}{m}}$, a = 0 and the vibration is in time quadrature *only* relative to P. (2) $\lvert x \rvert_{max}$ for given p occurs when $w_o^2 = \frac{k}{m} - \frac{c^2}{2m^2}$	(1) Observe ratio of quadrature and in phase components of x relative to the phase of P. Then $c = \frac{1}{w_o}(k - mw_o^2)\frac{b}{a}$.

TWO DIMENSIONAL DAMPED VIBRATIONS

$$m_1\ddot{x} + k_1 x + m_{12}\ddot{y} + k_{12} y + c_1\dot{x} + c_{12}\dot{y} = P(t)$$
$$m_{12}\ddot{x} + k_{12} x + m_2\ddot{y} + k_2 y + c_{21}\dot{x} + c_2\dot{y} = Q(t)$$

CASE	SOLUTIONS	REMARKS	EXPERIMENTAL DETERMINATION OF DAMPING
(1) UNDAMPED FREE VIBRATIONS $P = Q = 0$ $c_1 = c_2 = c_{12} = c_{21} = 0$	When x and y vibrate together in proportion to cos wt (i.e. $x, y \propto \cos wt$) $\frac{y}{x} = -\frac{k_{12} - m_{12}w^2}{k_2 - m_2 w^2} = -\frac{k_1 - m_1 w^2}{k_{12} - m_{12}w^2}$ Where the w's are determined from $(k_1 - m_1 w^2)(k_2 - m_2 w^2) - (k_{12} - m_{12}w^2)^2 \equiv (m_1 m_2 - m_{12}^2)w^4$ $-(m_1 k_2 + m_2 k_1 - 2m_{12}k_{12})w^2 + (k_1 k_2 - k_{12})^2 = 0$	(1) When $w = \sqrt{\frac{k_1}{m_1}}$ and either $m_{12} = k_{12} = 0$ or $m_1 k_{12} = m_{12} k_1$ then x vibrates and y = 0. When $w = \sqrt{\frac{k_2}{m_2}}$ and either $m_{12} = k_{12} = 0$ or $m_2 k_{12} = m_{12} k_2$ then y vibrates and x = 0. (2) When x and y vibrate together then in general two independent modes of vibration occur with y and x in fixed ratio depending on the mode. (a) If $k_{12}^2 = k_1 k_2$, then one root of the	

$P = Q = 0$ $c_1 = c_2 = c_{12} = c_{21} = 0$	$\frac{x}{...} = \frac{...}{k_2 - m_2 w^2} = \frac{...}{k_{12} - m_{12} w^2}$ Where the w's are determined from $(k_1 - m_1 w^2)(k_2 - m_2 w^2) - (k_{12} - m_{12} w^2)^2 \equiv (m_1 m_2 - m_{12}^2) w^4$ $-(m_1 k_2 + m_2 k_1 - 2 m_{12} k_{12}) w^2 + (k_1 k_2 - k_{12})^2 = 0$	... then x vibrates and $y = 0$. When $w = \sqrt{\frac{k_2}{m_2}}$ and either $m_{12} = k_{12} = 0$ or $m_2 k_{12} = m_{12} k_2$ then y vibrates and $x = 0$. (2) When x and y vibrate together then in general two independent modes of vibration occur with y and x in fixed ratio depending on the mode. (a) If $k_{12}^2 = k_1 k_2$, then one root of the frequency equation is $w^2 = 0$. (b) If $m_1 m_2 = m_{12}^2$ then only one mode of vibration is possible. (c) If $k_{12}^2 < k_1 k_2$ and $m_{12}^2 < m_1 m_2$, then one root for w^2 occurs at a value less than the smaller of the two quantities $\frac{k_1}{m_1}$ and $\frac{k_2}{m_2}$ and a second root for w^2 occurs at a value greater than these quantities.	
(2) DAMPED FREE VIBRATIONS $P = Q = 0$	$x = ae^{\lambda t}$, $y = be^{\lambda t}$, where λ is determined from $\epsilon_4 \lambda^4 + \epsilon_3 \lambda^3 + \epsilon_2 \lambda^2 + \epsilon_1 \lambda + \epsilon_0 = 0$ a, b, λ non-zero constants real or complex (λ negative real or complex) $\epsilon_i = f$ (m's, k's, c's) positive or zero (see equation 10) For λ complex ($= -\mu \pm iw$) μ and w are positive and determined by successive approximation from $\epsilon_4 w^4 - (\epsilon_2 - 3\epsilon_3 \mu + 6\epsilon_4 \mu^2) w^2 + \epsilon_0 - \epsilon_1 \mu + \epsilon_2 \mu^2 - \epsilon_3 \mu^3 + \epsilon_4 \mu^4 = 0$ $\epsilon_1 - \epsilon_3 w^2 - (2\epsilon_2 - 4\epsilon_4 w^2)\mu + 3\epsilon_3 \mu^2 - 4\epsilon_4 \mu^3 = 0$ For small damping $\mu = \frac{1}{2} \frac{\epsilon_1 - \epsilon_3 w^2}{\epsilon_2 - 2\epsilon_4 w^2}$ and w is determined from $\epsilon_4 w^4 - \epsilon_2 w^2 + \epsilon_0 = 0$ } More accurate solutions are determined by successive approximation from the two equations immediately above. In general $\frac{y}{x} = -\frac{m_1 \lambda^2 + k_1 + c_1 \lambda}{m_{12} \lambda^2 + k_{12} + c_{12} \lambda} = -\frac{m_{12} \lambda^2 + k_{12} + c_{21} \lambda}{m_2 \lambda^2 + k_2 + c_2 \lambda}$	(1) When $m_{12} = k_{12} = c_{12} = 0$ then x vibrates and $y = 0$. When $m_{12} = k_{12} = c_{21} = 0$ then y vibrates and $x = 0$. (2) Since $\frac{y}{x} = \frac{b}{a}$ is complex there is generally a phase difference between x and y as functions of time. (3) The general solution is the sum of four real expressions representing four independent damped oscillations whose amplitudes can be chosen to agree with any assumed initial (or other time) values of $x, \dot{x}, y, \dot{y}$. (4) For small damping μ is small and the oscillation frequencies are the same for the undamped case.	... transient excitation or ... obtain μ_1, w_1 and μ_2, w_2 from decay curve of x ... y vs t if one mode of vibration dies out before the other. If two taped ... modes persist, filter the taped record to separate the modes. (2) Adjust frequency of vibration to be in resonance for either ... Remove vibration and obtain μ_i, w_i (i = 1 or 2) for that mode from curve of x and/or y vs t. (3) Constrain system into configuration corresponding to one mode then remove constraint and obtain μ_i, w_i (i = 1 or 2) for that mode from curve of x and/or y vs t. NOTE 1: If $c_{12} = c_{21} = 0$, c_1 and c_2 can be calculated from μ_1 and μ_2. Otherwise the observed values of μ_1 and μ_2 furnish only two relations among the four quantities c_1, c_2, c_{12}, c_{21}. NOTE 2: More general procedures for calculating from observations values for each mode of the frequency ω, the damping constant μ, the amplitude ratio r, and phase ϕ of y relative to x, are given in the text.
(3) HARMONIC FORCED VIBRATIONS $P = p \cos w_o t + p' \sin w_o t$ $Q = q \cos w_o t + q' \sin w_o t$	$x = a_1 \cos w_o t + a_1' \sin w_o t$, $y = a_2 \cos w_o t + a_2' \sin w_o t$ p, p', q, q', a_1, a_1', a_2, a_2' eight real numbers, four arbitrary, one can be zero. Determination of amplitudes a_1, a_1', a_2, a_2' caused by applied forces p, p', q, q' is made by solving $(k_1 - m_1 w_o^2) a_1 + c_1 w_o a_1' + (k_{12} - m_{12} w_o^2) a_2 + c_{12} w_o a_2' = p$ $-c_1 w_o a_1 + (k_1 - m_1 w_o^2) a_1' - c_{12} w_o a_2 + (k_{12} - m_{12} w_o^2) a_2' = p'$	(1) If there is no damping, then when w_o equals the value of w for either of the frequencies of undamped vibration $\Delta = 0$ and the amplitude of the forced vibration becomes large (i.e. resonance occurs)	(1) Apply $P = p \cos w_o t$ only (i.e. $p' = 0$) and adjust amplitude and phase of Q (as discussed in text) so that $a_2 = a_2' = 0$ (i.e. $y = 0$). With $w_o >$ or $< \sqrt{\frac{k_1}{m_1}}$ observe $\frac{a_1'}{a}$. Then $c_1 = \frac{a_1'}{w_o a_1}(k_1 - m_1 w_o^2)$ (2) Similarly, apply $Q = q \cos w_o t$ only (i.e. $q' = 0$) and make $x = 0$ by adjusting p and p' so that $a_1 = a_1' = 0$.

B

<table>
<tr><td></td><td></td><td>second root for w^2 occurs at a value greater than these quantities.</td><td></td></tr>
<tr>
<td>(2) DAMPED FREE VIBRATIONS
$P = Q = 0$</td>
<td>$x = a e^{\lambda t}$, $y = b e^{\lambda t}$,
where λ is determined from
$\varepsilon_4\lambda^4 + \varepsilon_3\lambda^3 + \varepsilon_2\lambda^2 + \varepsilon_1\lambda + \varepsilon_o = 0$
a, b, λ non-zero constants real or complex (λnegative real or complex)
$\varepsilon_i = f$ (m's, k's, c's) positive or zero (see equation 10)
For λcomplex ($= -\mu \pm iw$) μ and w are positive and determined by successive approximation from
$\varepsilon_4 w^4 - (\varepsilon_2 - 3\varepsilon_3\mu + 6\varepsilon_4\mu^2)w^2 + \varepsilon_o - \varepsilon_1\mu + \varepsilon_2\mu^2 - \varepsilon_3\mu^3 + \varepsilon_4\mu^4 = 0$
$\varepsilon_1 - \varepsilon_3 w^2 - (2\varepsilon_2 - 4\varepsilon_4 w^2)\mu + 3\varepsilon_3\mu^2 - 4\varepsilon_4\mu^3 = 0$
For small damping $\mu = \frac{1}{2}\frac{\varepsilon_1 - \varepsilon_3 w^2}{\varepsilon_2 - 2\varepsilon_4 w^2}$
and w is determined from
$\varepsilon_4 w^4 - \varepsilon_2 w^2 + \varepsilon_o = 0$
More accurate solutions are determined by successive approximation from the two equations immediately above.
In general
$\frac{y}{x} = -\frac{m_1\lambda^2 + k_1 + c_1\lambda}{m_{12}\lambda^2 + k_{12} + c_{12}\lambda} = -\frac{m_{12}\lambda^2 + k_{12} + c_{21}\lambda}{m_2\lambda^2 + k_2 + c_2\lambda}$</td>
<td>(1) When $m_{12} = k_{12} = c_{12} = 0$ then x vibrates and $y = 0$. When $m_{12} = k_{12} = c_{21} = 0$ then y vibrates and $x = 0$.
(2) Since $\frac{\ddot{y}}{x} = \frac{b}{a}$ is complex there is generally a phase difference between x and y as functions of time.
(3) The general solution is the sum of four real expressions representing four independent damped oscillations whose amplitudes can be chosen to agree with any assumed initial (or other time) values of x, $\dot{x}$, y, $\dot{y}$.
(4) For small damping μ is small and the oscillation frequencies are the same for the undamped case.</td>
<td>(1) Using transient excitation on initial deflection obtain μ_1, w_1 and μ_2, w_2 from decay curve of x and/or y vs t if one mode of vibration dies out before the other; if two tape recorded modes persist, filter the taped record to separate the modes.
(2) Adjust frequency of vibration to be in resonance for either mode. Remove vibration and obtain μ_i, w_i (i = 1 or 2) for that mode from curve of x and/or y vs t.
(3) Constrain system into configuration corresponding to one mode then remove constraint and obtain μ_i, w_i (i = 1 or 2) for that mode from curve of x and/or y vs t.
NOTE 1: If $c_{12} = c_{21} = 0$, c_1 and c_2 can be calculated from μ_1 and μ_2. Otherwise the observed values of μ_1 and μ_2 furnish only two relations among the four quantities c_1, c_2, c_{12}, c_{21}.
NOTE 2: More general procedures for calculating from observations values for each mode of the frequency w, the damping constant μ, the amplitude ratio r, and phase ϕ of y relative to x, are given in the text.</td>
</tr>
<tr>
<td>(3) HARMONIC FORCED VIBRATIONS
$P = p\cos w_o t + p'\sin w_o t$
$Q = q\cos w_o t + q'\sin w_o t$</td>
<td>$x = a_1\cos w_o t + a_1'\sin w_o t$, $y = a_2\cos w_o t + a_2'\sin w_o t$
p, p', q, q', a_1, a_1', a_2, a_2' eight real numbers, four arbitrary, one can be zero. Determination of amplitudes a_1, a_1', a_2, a_2' caused by applied forces p, p', q, q' is made by solving
$(k_1 - m_1 w_o^2)a_1 + c_1 w_o a_1' + (k_{12} - m_{12}w_o^2)a_2 + c_{12}w_o a_2' = p$
$-c_1 w_o a_1 + (k_1 - m_1 w_o^2)a_1' - c_{12}w_o a_2 + (k_{12} - m_{12}w_o^2)a_2' = p'$
$(k_{12} - m_{12}w_o^2)a_1 + c_{21}w_o a_1' + (k_2 - m_2 w_o^2)a_2 + c_2 w_o a_2' = q$
$-c_{21}w_o a_1 + (k_{12} - m_{12}w_o^2)a_1' - c_2 w_o a_2 + (k_2 - m_2 w_o^2)a_2' = q'$
For these equations the determinant of the coefficients a_1, a_1', a_2, a_2' is
$\Delta = \left[(k_1 - m_1 w_o^2)(k_2 - m_2 w_o^2) - (k_{12} - m_{12}w_o^2)\right]^2 - w_o^2(c_1 c_2 - c_{12}c_{21})^2$</td>
<td>(1) If there is no damping, then when w_o equals the value of w for either of the frequencies of undamped vibration $\Delta = 0$ and the amplitude of the forced vibration becomes large (i.e. resonance occurs)
(2) If c_1, c_2, c_{12}, c_{21} are all small $\Delta = 0$ at two slightly modified frequencies that differ also slightly from the frequencies of damped free oscillation. As w_o approaches either of these frequencies at which $\Delta = 0$ while p, p', q, q' remain fixed, the amplitude of the forced vibration becomes large (i.e. resonance occurs)</td>
<td>(1) Apply $P = p\cos w_o t$ only (i.e. $p' = 0$) and adjust amplitude and phase of Q (as discussed in text) so that $a_2 = a_2' = 0$ (i.e. $y = 0$).
With $w_o >$ or $< \sqrt{\frac{k_1}{m_1}}$ observe $\frac{a_1'}{a_1}$. Then $c_1 = \frac{a_1'}{w_o a_1}(k_1 - m_1 w_o^2)$
(2) Similarly, apply $Q = q\cos w_o t$ only (i.e. $q' = 0$) and make $x = 0$ by adjusting p and p' so that $a_1 = a_1' = 0$.
With $w_o >$ or $< \sqrt{\frac{k_2}{m_2}}$ observe $\frac{a_2'}{a_2}$. Then $c_2 = \frac{a_2'}{w_o a_2}(k_2 - m_2 w_o^2)$
(3) Apply $P = p\cos w_o t$ only (i.e. $p' = 0$). Adjust Q (as discussed in text) so that $a_1' = a_2' = 0$. Read a_1/a_2. Then $c_{12} = -\frac{a_1}{a_2}c_1$
(4) Similarly, apply $Q = q\cos w_o t$ only (i.e. $q' = 0$). Adjust P so that $a_1' = a_2' = 0$. Read $\frac{a_2}{a_1}$. Then $c_{21} = -\frac{a_2}{a_1}c_2$
(5) Apply $P = p\cos w_o t$ only (i.e. $p' = q' = 0$). Read a_1, a_2, a_1', a_2'. Repeat with different ratio $\frac{Q}{P} = \frac{\overline{Q}}{\overline{P}}$ and read corresponding (barred) values $\overline{a}_1$, $\overline{a}_2$, $\overline{a}_1'$, $\overline{a}_2'$. Substitute each set of a's in turn in second and fourth of the set of four equations given in "solution" in which $p' = q' = 0$. Solve resulting two pairs of equations for c_1, c_{12} and c_2, c_{21}</td>
</tr>
</table>

C

A

Table 1 – Summary of Results (Continued)

Three Dimensional Damped Vibrations

$$m_1\ddot{x} + k_1x + m_{12}\ddot{y} + k_{12}y + m_{13}\ddot{z} + k_{13}z + c_1\dot{x} + c_{12}\dot{y} + c_{13}\dot{z} = P(t)$$
$$m_{12}\ddot{x} + k_{12}x + m_2\ddot{y} + k_2y + m_{23}\ddot{z} + k_{23}z + c_{21}\dot{x} + c_2\dot{y} + c_{23}\dot{z} = Q(t)$$
$$m_{13}\ddot{x} + k_{13}x + m_{23}\ddot{y} + k_{23}y + m_3\ddot{z} + k_3z + c_{31}\dot{x} + c_{32}\dot{y} + c_3\dot{z} = R(t)$$

CASE	SOLUTIONS	REMARKS	EXPERIMENTAL DETERMINATION OF DAMPING
(1) UNDAMPED FREE VIBRATIONS $P = Q = R = 0$ $c_1 = c_2 = c_{12} = c_{21} = c_{13} = c_{31} = c_{23} = c_{32} = 0$	$x = a_1 \cos wt \quad y = a_2 \cos wt \quad z = a_3 \cos wt \quad a_1, a_2, a_3$ real Ratios $\frac{x}{y} = \frac{a_1}{a_2}, \frac{y}{z} = \frac{a_2}{a_3}, \frac{z}{x} = \frac{a_3}{a_1}$ or their inverses are found by solving in any one of the three modes $(k_1 - m_1w^2)a_1 + (k_{12} - m_{12}w^2)a_2 + (k_{13} - m_{13}w^2)a_3 = 0$ $(k_{12} - m_{12}w^2)a_1 + (k_2 - m_2w^2)a_2 + (k_{23} - m_{23}w^2)a_3 = 0$ $(k_{13} - m_{13}w^2)a_1 + (k_{23} - m_{23}w^2)a_2 + (k_3 - m_3w^2)a_3 = 0$ where the w's for these modes are determined from the cubic equation in w^2 $\Delta = (k_1 - m_1w^2)(k_2 - m_2w^2)(k_3 - m_3w^2) + (k_{12} - m_{12}w^2)$ $(k_{13} - m_{13}w^2)(k_{23} - m_{23}w^2) - (k_1 - m_1w^2)(k_{23} - m_{23}w^2)^2$ $- (k_2 - m_2w^2)(k_{13} - m_{13}w^2)^2 - (k_3 - m_3w^2)(k_{12} - m_{12}w^2)^2 = 0$	(1) When x, y and z vibrate together then in general three independent modes of vibration occur with x:y:z in fixed ratio to each other depending upon the mode	
(2) DAMPED FREE VIBRATION $P = Q = R = 0$	$x = a_1e^{\lambda t} \quad y = a_2e^{\lambda t} \quad z = a_3e^{\lambda t} \quad a_1, a_2, a_3, \lambda$ generally complex Ratios x:y:z are found by solving in any one of the three damped modes $(k_1 + m_1\lambda^2 + c_1\lambda)a_1 + (k_{12} + m_{12}\lambda^2 + c_{12}\lambda)a_2 + (k_{13} + m_{13}\lambda^2 + c_{13}\lambda)a_3 = 0$ $(k_{12} + m_{12}\lambda^2 + c_{21}\lambda)a_1 + (k_2 + m_2\lambda^2 + c_2\lambda)a_2 + (k_{23} + m_{23}\lambda^2 + c_{23}\lambda)a_3 = 0$ $(k_{13} + m_{13}\lambda^2 + c_{31}\lambda)a_1 + (k_{23} + m_{23}\lambda^2 + c_{32}\lambda)a_2 + (k_3 + m_3\lambda^2 + c_3\lambda)a_3 = 0$ Where the λ's for these modes are determined from the sixth degree	(1) The sixth degree equation in λ may have some or all roots if the c's are large enough. If the c's are not too large there will be six complex roots $-\mu_k \pm iw_k$, k = 1, 2, 3. (2) For small c's and therefore small μ's the frequencies of oscillation in the three damped modes are approximated by the frequencies of undamped vi-	(1) Methods of measurement similar to those for the two-dimensional case; elaborate procedure necessary to isolate the nine c's in turn.

$$(k_{12} + m_{12}\lambda^2 + c_{21}\lambda)a_1 + (k_2 + m_2\lambda^2 + c_2\lambda)a_2 +$$
$$(k_{23} + m_{23}\lambda^2 + c_{23}\lambda)a_3 = 0$$
$$(k_{13} + m_{13}\lambda^2 + c_{31}\lambda)a_1 + (k_{23} + m_{23}\lambda^2 + c_{32}\lambda)a_2 +$$
$$(k_3 + m_3\lambda^2 + c_3\lambda)a_3 = 0$$

Where the λ's for these modes are determined from the sixth degree equation in λ

$$(k_1 + m_1\lambda^2 + c_1\lambda)(k_2 + m_2\lambda^2 + c_2\lambda)(k_3 + m_3\lambda^2 + c_3\lambda)$$
$$+ (k_{12} + m_{12}\lambda^2 + c_{12}\lambda)(k_{23} + m_{23}\lambda^2 + c_{23}\lambda)(k_{13} + m_{13}\lambda^2 + c_{31}\lambda)$$
$$+ (k_{12} + m_{12}\lambda^2 + c_{21}\lambda)(k_{23} + m_{23}\lambda^2 + c_{32}\lambda)(k_{13} + m_{13}\lambda^2 + c_{13}\lambda)$$
$$- (k_1 + m_1\lambda^2 + c_1\lambda)(k_{23} + m_{23}\lambda^2 + c_{23}\lambda)(k_{23} + m_{23}\lambda^2 + c_{32}\lambda)$$
$$- (k_2 + m_2\lambda^2 + c_2\lambda)(k_{13} + m_{13}\lambda^2 + c_{13}\lambda)(k_{13} + m_{13}\lambda^2 + c_{31}\lambda)$$
$$- (k_3 + m_3\lambda^2 + c_3\lambda)(k_{12} + m_{12}\lambda^2 + c_{12}\lambda)(k_{12} + m_{12}\lambda^2 + c_{21}\lambda) = 0$$

For each mode or value of w^2 found from the cubic equation in w^2 for undamped free vibrations (an approximation for small damping) a corresponding μ is obtained from

$$-2\mu\Big[m_1(G_2G_3 - G_{23}^2) + m_2(G_1G_3 - G_{13}^2) + m_3(G_1G_2 - G_{12}^2)$$
$$+ 2m_{12}(G_{13}G_{23} - G_3G_{12}) + 2m_{13}(G_{12}G_{13} - G_2G_{13})$$
$$+ 2m_{23}(G_{12}G_{13} - G_1G_{23})\Big]$$
$$+ c_1(G_2G_3 - G_{23}^2) + c_2(G_1G_3 - G_{13}^2) + c_3(G_1G_2 - G_{12}^2)$$
$$+ (c_{12} + c_{21})(G_{13}G_{23} - G_3G_{12}) + (c_{13} + c_{31})(G_{12}G_{23} - G_2G_{13})$$
$$+ (c_{23} + c_{32})(G_{12}G_{13} - G_1G_{23}) = 0$$

Where

$$G_1 = k_1 - m_1w^2, \quad G_2 = k_2 - m_2w^2, \quad G_3 = k_3 - m_3w^2$$
$$G_{12} = k_{12} - m_{12}w^2, \quad G_{13} = k_{13} - m_{13}w^2, \quad G_{23} = k_{23} - m_{23}w^2$$

$-\mu_k \pm iw_k$, k = 1, 2, 3.

(2) For small c's and therefore small μ's the frequencies of oscillation in the three damped modes are approximated by the frequencies of undamped vibrations obtained from the cubic equation in w^2.

(3) HARMONIC FORCED VIBRATIONS

$P = p\cos w_ot + p'\sin w_ot$

$Q = q\cos w_ot + q'\sin w_ot$

$R = r\cos w_ot + r'\sin w_ot$

$$x = a_1\cos w_ot + a_1'\sin w_ot, \qquad y = a_2\cos w_ot + a_2'\sin w_ot$$
$$z = a_3\cos w_ot + a_3'\sin w_ot$$

p, p', q, q', r, r', a_1, a_1', a_2, a_2', a_3, a_3' twelve real numbers, six arbitrary, one can be zero. In general, any six of these numbers can be assigned arbitrarily and the remaining six are found by solving

$$F_1a_1 + c_1w_oa_1' + F_{12}a_2 + c_{12}w_oa_2' + F_{13}a_3 + c_{13}w_oa_3' = p$$
$$-c_1w_oa_1 + F_1a_1' - c_{12}w_oa_2 + F_{12}a_2' - c_{13}w_oa_3 + F_{13}a_3' = p'$$

$$F_{12}a_1 + c_{21}w_oa_1' + F_2a_2 + c_2w_oa_2' + F_{23}a_3 + c_{23}w_oa_3' = q$$
$$-c_{12}w_oa_1 + F_{12}a_1' - c_2w_oa_2 + F_2a_2' - c_{23}w_oa_3 + F_{23}a_3' = q'$$
$$F_{13}a_1 + c_{31}w_oa_1' + F_{23}a_2 + c_{32}w_oa_2' + F_3a_3 + c_3w_oa_3' = r$$
$$-c_{31}w_oa_1 + F_{31}a_1' - c_{32}w_oa_2 + F_{23}a_3' - c_3w_oa_3' + F_3a_3' = r'$$

where

$$F_1 = k_1 - m_1w_o^2 \qquad F_2 = k_2 - m_2w_o^2 \qquad F_3 = k_3 - m_3w_o^2$$
$$F_{12} = k_{12} - m_{12}w_o^2 \qquad F_{13} = k_{13} - m_{13}w_o^2 \qquad F_{23} = k_{23} - m_{23}w_o^2$$

(1) Apply P, Q, R in any ratio but in same phase. Assume p' = q' = r' = 0. Read a_1, a_2, a_3, a_1', a_2', a_3'. Repeat with different ratios of P, Q, R, distinguishing a's thus obtained by a bar, and then with a third set of ratios, making the a's with a double bar, (see suggestions for choosing P, Q, R in text). Substitute each set of a's in turn in second, fourth and sixth equations of the set of six equations given in "solution" in which p' = q' = r' = 0. Solve resulting three groups of three equations for the nine

c's (a rearrangement of the nine equations convenient for computation, is given in text (Equation (20)). With this rearrangement each of the three groups of three equations is solved separately, the first group yielding c_1, c_{12}, c_{13}, the second group yielding c_2, c_{21}, c_{23} and the last group c_3, c_{31}, c_{32}.)

(2) Computation is further shortened by <u>observing differently.</u> Choose p and q and adjust r (p' = q' = r' = 0) so that $a_3 = 0$. Read a_1, a_2, a_1', a_2', a_3'. Repeat with different pair of values of p and q making $\bar{a}_3 = 0$. Read $\bar{a}_1$, $\bar{a}_2$, $\bar{a}_1'$, $\bar{a}_2'$, $\bar{a}_3'$. Solve first two of rearranged equations (Equation (20) in text) for c_1 and c_{12},

	For each mode or value of w^2 found from the cubic equation in w^2 for undamped free vibrations (an approximation for small damping) a corresponding μ is obtained from $-2\mu\left[m_1(G_2G_3 - G_{23}^2) + m_2(G_1G_3 - G_{13}^2) + m_3(G_1G_2 - G_{12}^2) + 2m_{12}(G_{13}G_{23} - G_3G_{12}) + 2m_{13}(G_{12}G_{13} - G_2G_{13}) + 2m_{23}(G_{12}G_{13} - G_1G_{23})\right]$ $+ c_1(G_2G_3 - G_{23}^2) + c_2(G_1G_3 - G_{13}^2) + c_3(G_1G_2 - G_{12}^2)$ $+ (c_{12} + c_{21})(G_{13}G_{23} - G_3G_{12}) + (c_{13} + c_{31})(G_{12}G_{23} - G_2G_{13})$ $+ (c_{23} + c_{32})(G_{12}G_{13} - G_1G_{23}) = 0$ Where $G_1 = k_1 - m_1w^2$, $G_2 = k_2 - m_2w^2$, $G_3 = k_3 - m_3w^2$ $G_{12} = k_{12} - m_{12}w^2$, $G_{13} = k_{13} - m_{13}w^2$, $G_{23} = k_{23} - m_{23}w^2$		
(3) HARMONIC FORCED VIBRATIO $P = p\cos w_ot + p'\sin w_ot$ $Q = q\cos w_ot + q'\sin w_ot$ $R = r\cos w_ot + r'\sin w_ot$	$x = a_1\cos w_ot + a_1'\sin w_ot$, $y = a_2\cos w_ot + a_2'\sin w_ot$ $z = a_3\cos w_ot + a_3'\sin w_ot$ p, p', q, q', r, r', a_1, a_1', a_2, a_2', a_3, a_3' twelve real numbers, six arbitrary, one can be zero. In general, any six of these numbers can be assigned arbitrarily and the remaining six are found by solving $F_1a_1 + c_1w_oa_1' + F_{12}a_2 + c_{12}w_oa_2' + F_{13}a_3 + c_{13}w_oa_3' = p$ $-c_1w_oa_1 + F_1a_1' - c_{12}w_oa_2 + F_{12}a_2' - c_{13}w_oa_3 + F_{13}a_3' = p'$ $F_{12}a_1 + c_{21}w_oa_1' + F_2a_2 + c_2w_oa_2' + F_{23}a_3 + c_{23}w_oa_3' = q$ $-c_{12}w_oa_1 + F_{12}a_1' - c_2w_oa_2 + F_2a_2' - c_{23}w_oa_3 + F_{23}a_3' = q'$ $F_{13}a_1 + c_{31}w_oa_1' + F_{23}a_2 + c_{32}w_oa_2' + F_3a_3 + c_3w_oa_3' = r$ $-c_{31}w_oa_1 + F_{31}a_1' - c_{32}w_oa_2 + F_{23}a_3' - c_3w_oa_3' + F_3a_3' = r'$ where $F_1 = k_1 - m_1w_o^2$, $F_2 = k_2 - m_2w_o^2$, $F_3 = k_3 - m_3w_o^2$ $F_{12} = k_{12} - m_{12}w_o^2$, $F_{13} = k_{13} - m_{13}w_o^2$, $F_{23} = k_{23} - m_{23}w_o^2$		(1) Apply P, Q, R in any ratio but in same phase. Assume $p' = q' = r' = 0$. Read a_1, a_2, a_3, a_1', a_2', a_3'. Repeat with different ratios of P, Q, R, distinguishing a's thus obtained by a bar, and then with a third set of ratios, marking the a's with a double bar, (see suggestions for choosing P, Q, R in text). Substitute each set of a's in turn in second, fourth and sixth equations of the set of six equations given in "solution" in which $p' = q' = r' = 0$. Solve resulting three groups of three equations for the nine c's (a rearrangement of the nine equations convenient for computation, is given in text (Equation (20)). With this rearrangement each of the three groups of three equations is solved separately, the first group yielding c_1, c_{12}, c_{13}, the second group yielding c_2, c_{21}, c_{23} and the last group c_3, c_{31}, c_{32}.) (2) Computation is further shortened by observing differently. Choose p and q and adjust r ($p' = q' = r' = 0$) so that $a_3 = 0$. Read a_1, a_2, a_1', a_2', a_3'. Repeat with different pair of values of p and q making $\bar{a}_3 = 0$. Read $\bar{a}_1$, $\bar{a}_2$, $\bar{a}_1'$, $\bar{a}_2'$, $\bar{a}_3'$. Solve first two of rearranged equations (Equation (20) in text) for c_1 and c_{12}, the fourth and fifth for c_2 and c_{21} and the seventh and eighth for c_{31} and c_{32}. Repeat choosing two pairs of values of p and r and adjusting q each time so that $a_2 = \bar{a}_2 = 0$. Then the same three pairs out of the rearranged equations yield c_{21} and c_{23}, and c_3 and c_{31}. All c's thus obtained are duplicate values of c_1, c_{21}, c_{31}. NOTE: Neither (1) nor (2) require actual measurements of P, Q or R.

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