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In this paper we investigate the problem of selecting the treatment that has the strongest relationship between the response variable and an explanatory variable in a linear measurement error model. A selection procedure based on moment estimates has been developed and the large sample performance of the derived selection rule has also been analyzed. At the end of this paper, a simulation study is carried out to illustrate the large sample performance of the selection procedure.

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CENTER FOR STATISTICAL DECISION SCIENCES AND DEPARTMENT OF STATISTICS

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Abstract

Measurement errors are the differences between the actual desired values and the observed values. In the real world, it is usually very difficult to obtain exactly the "true" values. Instead, one may only get the observed values that are related to the true values through the measurement errors.

In this paper we investigate the problem of selecting the treatment that has the strongest relationship between the response variable and an explanatory variable in a linear measurement error model. A selection procedure based on moment estimates has been developed and the large sample performance of the derived selection rule has also been analyzed. At the end of this paper, a simulation study is carried out to illustrate the large sample performance of the selection procedure.

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1 Introduction

Measurement errors are the differences between the actual desired values and the observed values. In the real world, it is usually very difficult to obtain exactly the "true" values. Instead, one may only get the observed values that are related to the true values through the measurement errors. Extra care must be taken to deal with the mesurement errors in the analysis because the data become more noisy and error-prone when the measurement errors are taken into consideration.

Measurement error models are those in which one or more of the explanatory variables cannot be observed directly and are measured with error. Fuller (1987) gave a comprehensive introduction to measurement error models. Carroll, Ruppert, and Stefanski (1995) discussed nonlinear measurement error models and the corresponding approaches.

In this paper, we are concerned with a problem of selecting a treatment that has the strongest relationship between an explanatory variable and the response variable in a linear measurement error model. For the general approaches to statistical selection problems, references can be made to Bechhofer, Santner, and Goldsman (1996) and Gupta and Panchapakesan (1996).

The following is the measurement error model that we are interested in. Suppose there are k treatments $\Pi_i, i = 1, \ldots, k$ and n observations from each treatment. For each treatment $\Pi_i, i = 1, \ldots, k$ and each observation $j = 1, \ldots, n$, we have the following model:

$$Y_{ij} = \beta_{0i} + \beta_{1i} X_{ij} + \epsilon_{ij}, \qquad W_{ij} = X_{ij} + U_{ij}. \tag{1}$$

For each $i = 1, \ldots, k$, the intercept β_{0i} and the slope β_{1i} are both unknown, and $\{(X_{ij}, U_{ij}, \epsilon_{ij}), 1 \leq j \leq n\}$ are assumed independent with mean $(0,0,0)$ and covariance $\text{diag}(\sigma_{xxi}, \sigma_{uui}, \sigma_{\epsilon\epsilon i})$, where $\text{diag}(\sigma_{xxi}, \sigma_{uui}, \sigma_{\epsilon\epsilon i})$ refers to a 3×3 matrix whose diagonal elements are σ_{xxi}, σ_{uui}, and $\sigma_{\epsilon\epsilon i}$ while the rest of the elements are all 0. We assume that for each i, σ_{uui} is known.

We are interested in the relationship between the explanatory variable X and the response variable Y. However, X_{ij} cannot be observed directly, instead we observe W_{ij}, which is X_{ij} mixed with a linear error term U_{ij}. An interesting question here is: how to select the treatment that has the strongest relationship between the explanatory variable X and the response variable Y?

In this selection problem, the slope β_{1i} is important. It is the rate of the change in the mean value of Y with respect to X and therefore a measurement of the strength of the relationship between X and Y. Gupta and Lin (1997) studied a selection problem in which the selection criterion was to select the one that has the largest slope under this modeling setting.

However, sometimes the relationship between X and Y can take opposite directions and the k slopes can have different signs. In other words, some slopes may be negative while some are positive. If we stick to the criterion of selecting the largest slope in this situation, then we are essentially excluding the negative slopes and only considering those positive ones. From this point of view, it is necessary to broaden the scope of our consideration and generalize the selection problem studied in Gupta and Lin (1997). In this paper, we studied the problem and derived a selection procedure of which the selection criterion is to select the treatment that has the largest absolute value of the slope.

A treatment Π_i is said to be the best if the absolute value of the slope $|\beta_{1i}|$ is the largest, i.e., $|\beta_{1i}| = \max_{1 \le j \le k} |\beta_{1j}|$. Otherwise the treatment is said to be non-best. The selection goal is to select the best treatment.

Let $\Omega = \{\underline{\beta}_1 = (\beta_{11}, \beta_{12}, \ldots, \beta_{1k}) | \beta_{1i} \in R, i = 1, \ldots, k\}$ be the parameter space and $\underline{a} = (a_1, \ldots, a_k)$ be an action, where $a_i = 0$ or 1, $i = 1, \ldots, k$. When action $\underline{a}$ is taken, $a_i = 1$ means treatment Π_i is selected as the best and $a_i = 0$ means Π_i is excluded as the non-best. For $i = 1, \ldots, k$, let $W_i = (W_{i1}, \ldots, W_{in})$, $Y_i = (Y_{i1}, \ldots, Y_{in})$, $\underline{W} = (W_1, \ldots, W_k)$, and $\underline{Y} = (Y_1, \ldots, Y_k)$. Let χ be the sample space generated by $(\underline{W}, \underline{Y})$. Since the true order of $|\beta_{11}|, \ldots, |\beta_{1k}|$ is unknown, we denote $|\beta_{1[1]}| \le |\beta_{1[2]}| \le \ldots < |\beta_{1[k]}|$. For simplicity, we assume that $|\beta_{1[k]}| - |\beta_{1[k-1]}| = 2\delta > 0$, where δ is unknown.

A selection rule $d(\underline{w}, \underline{y}) = (d_1(\underline{w}, \underline{y}), \ldots, d_k(\underline{w}, \underline{y}))$ is a mapping defined on χ, where $d_i(\underline{w}, \underline{y})$ is the probability that given $\underline{W} = \underline{w}$ and $\underline{Y} = \underline{y}$, Π_i is selected as the best. Also, $\sum_{i=1}^{k} d_i(\underline{w}, \underline{y}) = 1$, for all $(\underline{w}, \underline{y}) \in \chi$. In other words, only one of the k treatments will be selected as the best.

We consider the following loss function:

$$L(\underline{\beta}_1, \underline{a}) = \begin{cases} 1, & \text{if the best treatment is not selected,} \\ 0, & \text{if the best treatment is selected.} \end{cases} \tag{2}$$

2 Formulation of the Selection Procedure

Before we develop a selection procedure for this problem, let us first look at the estimation of these slopes. Fuller (1987) has shown that the ordinary least square regression analysis will not work in this case because the ordinary regression slope estimate is always biased toward 0. We will use the moment estimators instead.

The population moments of (W_{ij}, Y_{ij}) satisfy

$$(\mu_{wi}, \mu_{yi}) = (EW_{ij}, EY_{ij}) = (0, \beta_{0i}), \tag{3}$$

and

$$\begin{aligned}
& (\sigma_{wwi}, \sigma_{wyi}, \sigma_{yyi}) \\
= & (VarW_{ij}, Cov(W_{ij}, Y_{ij}), VarY_{ij}) \\
= & (\sigma_{xxi} + \sigma_{uui}, \beta_{1i}\sigma_{xxi}, {\beta_{1i}}^2\sigma_{xxi} + \sigma_{\epsilon\epsilon i}).
\end{aligned} \tag{4}$$

The sample means $(\bar{W}_i, \bar{Y}_i)$ and the sample covariances $(S_{wwi}, S_{wyi}, S_{yyi})$, where, for example,

$$S_{wyi} = \frac{1}{n-1}\sum_{j=1}^{n}(W_{ij} - \bar{W}_i)(Y_{ij} - \bar{Y}_i), \tag{5}$$

will be the basis of our selection procedure.

We estimate the parameters by replacing the unknown population moments with their sample moments. Note that σ_{xxi} should be positive. Otherwise X_{ij} can take only one value for all $j = 1, \ldots, n$ and there is no point to study the quantitative relationship between X_{ij} and Y_{ij}. Therefore, estimator $\hat{\sigma_{xxi}}$ should be positive as well. Let $\hat{\sigma_{xxi}} = S_{wwi} - \sigma_{uui}$ when $S_{wwi} - \sigma_{uui}$ is positive, otherwise let $\hat{\sigma_{xxi}} = S_{yyi}^{-1}S_{wyi}^2$. Also define

$$\hat{\beta_{1i}} = \begin{cases} (S_{wwi} - \sigma_{uui})^{-1}S_{wyi}, & \text{if } \mathrm{S_{wwi}} - \sigma_{\mathrm{uui}} > 0, \\ S_{wyi}^{-1}S_{yyi}, & \text{otherwise.} \end{cases} \tag{6}$$

We construct selection procedure $d_n(\underline{w}, \underline{y}) = (d_{1n}(\underline{w}, \underline{y}), d_{2n}(\underline{w}, \underline{y}), \ldots, d_{kn}(\underline{w}, \underline{y}))$ as follows:

$$d_n(\underline{w}, \underline{y}) = \begin{cases} 1, & \text{if } |\hat{\beta_{1\mathrm{i}}}| = \max_{1 \le \mathrm{j} \le \mathrm{k}} |\hat{\beta_{1\mathrm{j}}}|, \\ 0, & \text{otherwise,} \end{cases} \tag{7}$$

when $\underline{W} = \underline{w}$ and $\underline{Y} = \underline{y}$ are observed. In other words, the treatment associated with the largest estimated absolute value of the slope $\max_{1 \le i \le k} |\hat{\beta_{1i}}|$ will be selected as the best.

3 Performance of the Selection Procedures

We now study the performance of the selection procedure developed in (7). A measure of the performance of this decision rule is the probability of making a wrong decision when using this rule. Since in this case the loss function is the 0–1 loss, the probability of making a wrong decision is the expected risk of the proposed procedure. We would like the probability of making a wrong decision to be as small as possible.

Denote P_n to be the probability measure generated by the random observations $(\underset{\sim}{W}, \underset{\sim}{Y})$, and for each $(\underset{\sim}{w}, \underset{\sim}{y}) \in \chi$, let

$$i^* = \{i|\ |\beta_{1i}| = \max_{1\le j\le k} |\beta_{1j}| = |\beta_{1[k]}|,\ i = 1, \dots, k\}, \tag{8}$$

and

$$i_n^* = \{i|\ |\hat{\beta_{1i}}| = \max_{1\le j\le k} |\hat{\beta_{1j}}|,\ i = 1, \dots, k\}. \tag{9}$$

Then, the expected risk of the proposed selection procedure is

$$\begin{aligned}
& E^{(\underset{\sim}{w}, \underset{\sim}{y})} L(\underset{\sim}{\beta}, d_n(\underset{\sim}{w}, \underset{\sim}{y})) \qquad (10)\\
= & \sum_{i=1}^{k} \sum_{j=1, j\neq i}^{k} P_n\{i^* = i, i_n^* = j\} \\
\le & \sum_{i=1}^{k} \sum_{j=1, j\neq i}^{k} P_n\{i^* = i, i_n^* = j, S_{wwi} - \sigma_{uui} > \frac{\sigma_{xxi}}{2}, S_{wwj} - \sigma_{uuj} > \frac{\sigma_{xxj}}{2}\} \\
& + \sum_{i=1}^{k} \sum_{j=1, j\neq i}^{k} P_n\{i^* = i, i_n^* = j, S_{wwi} - \sigma_{uui} \le \frac{\sigma_{xxi}}{2}\} \\
& + \sum_{i=1}^{k} \sum_{j=1, j\neq i}^{k} P_n\{i^* = i, i_n^* = j, S_{wwj} - \sigma_{uuj} \le \frac{\sigma_{xxj}}{2}\} \\
\le & \sum_{i=1}^{k} \sum_{j=1, j\neq i}^{k} P_n\{|\beta_{1i}| - |\hat{\beta_{1i}}| > \delta, S_{wwi} - \sigma_{uui} > \frac{\sigma_{xxi}}{2}, S_{wwj} - \sigma_{uuj} > \frac{\sigma_{xxj}}{2}\} \\
& + \sum_{i=1}^{k} \sum_{j=1, j\neq i}^{k} P_n\{|\hat{\beta_{1j}}| - |\beta_{1j}| > \delta, S_{wwi} - \sigma_{uui} > \frac{\sigma_{xxi}}{2}, S_{wwj} - \sigma_{uuj} > \frac{\sigma_{xxj}}{2}\} \\
& + \sum_{i=1}^{k} \sum_{j=1, j\neq i}^{k} P_n\{S_{wwi} - \sigma_{uui} \le \frac{\sigma_{xxi}}{2}\} \\
& + \sum_{i=1}^{k} \sum_{j=1, j\neq i}^{k} P_n\{S_{wwj} - \sigma_{uuj} \le \frac{\sigma_{xxj}}{2}\} \\
\le & \sum_{i=1}^{k} \sum_{j=1, j\neq i}^{k} P_n\{|\beta_{1i}| - |\hat{\beta_{1i}}| > \delta, S_{wwi} - \sigma_{uui} > \frac{\sigma_{xxi}}{2}\} \\
& + \sum_{i=1}^{k} \sum_{j=1, j\neq i}^{k} P_n\{|\hat{\beta_{1j}}| - |\beta_{1j}| > \delta, S_{wwj} - \sigma_{uuj} > \frac{\sigma_{xxj}}{2}\} \\
& + 2k \sum_{i=1}^{k} P_n\{S_{wwi} - \sigma_{uui} \le \frac{\sigma_{xxi}}{2}\} \\
\le & \ 2k \sum_{i=1}^{k} P_n\{|\beta_{1i} - \hat{\beta_{1i}}| > \delta, S_{wwi} - \sigma_{uui} > \frac{\sigma_{xxi}}{2}\}
\end{aligned}$$

$$+2k\sum_{i=1}^{k} P_n\{S_{wwi} - \sigma_{uui} \leq \frac{\sigma_{xxi}}{2}\}.$$

From above we observe that it suffices to analyze the performance of the followings two sequences:

$$P_n\{S_{wwi} - \sigma_{uui} \leq \frac{\sigma_{xxi}}{2}\}, \quad P_n\{|\hat{\beta_{1i}} - \beta_{1i}| \geq \delta, S_{wwi} - \sigma_{uui} > \frac{\sigma_{xxi}}{2}\}. \tag{11}$$

Definition 1 A sequence of selection procedures $\{d_n(\underline{w}, \underline{y})\}_{n=1}^{\infty}$ is said to be asymptotically optimal of order e_n if $E^{(\underline{W},\underline{Y})}L(\underline{\beta}, d_n(\underline{w}, \underline{y})) = O(e_n)$, where e_n is a sequence of positive numbers such that $\lim_{n\to\infty} e_n = 0$.

The large sample performance of the derived selection rule $d_n(\underline{w}, \underline{y})$ will be analyzed in two situations.

3.1 When The α-th Moment Exists ($\alpha > 2$)

In this subsection, we suppose that the α-th ($\alpha > 2$) moments of $(X_{ij}, U_{ij}, \epsilon_{ij})$ exist, that is,

$$E|X_{ij}|^{\alpha} < \infty, \qquad E|U_{ij}|^{\alpha} < \infty, \qquad E|\epsilon_{ij}|^{\alpha} < \infty. \tag{12}$$

We will show that the expected risk of the proposed selection procedure converges to 0 at the rate of $o(n^{-(\alpha/2-1)})$.

We introduce some useful lemmas. The first lemma is well known, a similar result can be found in Baum and Katz (1965).

Lemma 1 Let $X_1, \ldots, X_n$ be independent random variables with mean 0. Suppose for a fixed number $\alpha > 1$, $E|X_i|^{\alpha} < \infty$, for $i = 1, \ldots, n$, then for any $\epsilon > 0$,

$$P\{|\sum X_i/n| \geq \epsilon\} = o(n^{-(\alpha-1)}). \tag{13}$$

As a consequence of Lemma 1, we have

Lemma 2 Let $X_1, \ldots, X_n$ be independent random variables with mean $EX_i = \mu$ and variance $\text{Var}X_i = \sigma^2$, for $i = 1, \ldots, n$. Let $\bar{X} = \frac{1}{n}\sum X_i$ and $S_n^2 = \frac{1}{n-1}\sum(X_i - \bar{X})^2$. Suppose for $i = 1, \ldots, n$ and a fixed number $\alpha > 2$, $E|X_i|^{\alpha} < \infty$, then for any $\epsilon > 0$,

$$P\{|S_n^2 - \sigma^2| \geq \epsilon\} = o(n^{-(\alpha/2-1)}). \tag{14}$$

Proof.

$$P\{|S_n^2 - \sigma^2| \geq \epsilon\} \quad = \quad P\{|\frac{1}{n-1}\sum_{i=1}^{n} X_i^2 - \frac{n}{n-1}\bar{X}^2 - \sigma^2| \geq \epsilon\} \tag{15}$$

$$
\begin{aligned}
&\leq P\{|\frac{1}{n-1}\sum X_i^2 - \frac{n}{n-1}(\mu^2+\sigma^2)| \geq \frac{\epsilon}{2}\} \\
&\quad +P\{|\frac{n}{n-1}\bar{X}^2 - \frac{n\mu^2+\sigma^2}{n-1}| \geq \frac{\epsilon}{2}\} \\
&= P\{|\frac{1}{n}\sum(X_i^2 - (\mu^2+\sigma^2))| \geq \frac{n-1}{n}\frac{\epsilon}{2}\} \\
&\quad +P\{|\bar{X}^2 - \mu^2| \geq \frac{\epsilon}{2} - \frac{1}{n}\sigma^2\} \\
&= P\{|\frac{1}{n}\sum(X_i^2 - (\mu^2+\sigma^2))| \geq \frac{\epsilon}{4}\} \\
&\quad +P\{|\bar{X}^2 - \mu^2| \geq \frac{\epsilon}{4}\} \\
&:= I_1 + I_2,
\end{aligned}
$$

for n large enough, that is, when $n \geq \max(2, [\frac{4\sigma^2}{\epsilon}]+1)$, we have $\frac{n-1}{n}\frac{\epsilon}{2} \geq \frac{\epsilon}{4}$ and $\frac{\epsilon}{2} - \frac{1}{n}\sigma^2 \geq \frac{\epsilon}{4}$. From Lemma 1, we have

$$
\begin{aligned}
I_1 &= P\{|\frac{1}{n}\sum(X_i^2 - (\mu^2+\sigma^2))| \geq \frac{\epsilon}{4}\} \\
&= o(n^{-\alpha/2-1})
\end{aligned}
\tag{16}
$$

and

$$
\begin{aligned}
I_2 &= P\{|\bar{X}^2 - \mu^2| \geq \frac{\epsilon}{4}\} \\
&= P\{|(\bar{X}+\mu)(\bar{X}-\mu)| \geq \frac{\epsilon}{4} \text{ and } (\bar{X}+\mu) > (2\mu+1)\} \\
&\quad +P\{|(\bar{X}+\mu)(\bar{X}-\mu)| \geq \frac{\epsilon}{4} \text{ and } (\bar{X}+\mu) \leq (2\mu+1)\} \\
&\leq P\{(\bar{X}-\mu) > 1\} + P\{4(2\mu+1)|(\bar{X}-\mu)| \geq \epsilon\} \\
&= o(n^{-(\alpha-1)}).
\end{aligned}
\tag{17}
$$

From Lemma 2, we can see that

$$
\begin{aligned}
P\{S_{wwi} - \sigma_{uui} \leq \frac{\sigma_{xxi}}{2}\} &= P\{S_{wwi} - \sigma_{wwi} \leq -\frac{\sigma_{xxi}}{2}\} \\
&= o(n^{-(\alpha/2-1)}).
\end{aligned}
\tag{18}
$$

Moreover,

$$
\begin{aligned}
&P_n\{|\hat{\beta_{1i}} - \beta_{1i}| \geq \delta, S_{wwi} - \sigma_{uui} > \frac{\sigma_{xxi}}{2}\} \\
&= P_n\{|\frac{S_{wyi}}{S_{wwi} - \sigma_{uui}} - \beta_{1i}| \geq \delta, S_{wwi} - \sigma_{uui} > \frac{\sigma_{xxi}}{2}\}
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&\leq P_n\{|S_{wyi} - \beta_{1i}(S_{wwi} - \sigma_{uui})| \geq \delta\frac{\sigma_{xxi}}{2}\} \\
&= P_n\{|\frac{1}{n-1}\sum_{j=1}^{n}(W_{ij} - \bar{W}_i)(Y_{ij} - \bar{Y}_i) - \beta_{1i}(\frac{1}{n-1}(\sum_{j=1}^{n}(W_{ij} - \bar{W}_i)^2 - \sigma_{uui})| \\
&\qquad \geq \delta\frac{\sigma_{xxi}}{2}\} \\
&= P_n\{|\frac{1}{n-1}\sum_{j=1}^{n}W_{ij}Y_{ij} - \frac{n}{n-1}\bar{W}_i\bar{Y}_i - \frac{1}{n-1}\beta_{1i}\sum_{j=1}^{n}W_{ij}^2 + \frac{n}{n-1}\beta_{1i}\bar{W}_i^2 \\
&\qquad +\beta_{1i}\sigma_{uui}| \geq \delta\frac{\sigma_{xxi}}{2}\} \\
&\leq P_n\{|\frac{1}{n-1}\sum_{j=1}^{n}W_{ij}Y_{ij} - \frac{n}{n-1}\beta_{1i}\sigma_{xxi}| \geq \delta\frac{\sigma_{xxi}}{6}\} \\
&\quad +P_n\{|\frac{n}{n-1}\bar{W}_i\bar{Y}_i - \frac{n}{n-1}\beta_{1i}\bar{W}_i^2 - \frac{1}{n-1}\beta_{1i}\sigma_{uui}| \geq \delta\frac{\sigma_{xxi}}{6}\} \\
&\quad +P_n\{|\frac{1}{n-1}\beta_{1i}\sum_{j=1}^{n}W_{ij}^2 - \frac{n}{n-1}\beta_{1i}(\sigma_{xxi} + \sigma_{uui})| \geq \delta\frac{\sigma_{xxi}}{6}\} \\
&:= J_1 + J_2 + J_3.
\end{aligned}
$$

For any $i = 1, \ldots, k$, $\{W_{ij}Y_{ij}, j = 1, \ldots, n\}$ are independent random variables with mean $E(W_{ij}Y_{ij}) = \beta_{1i}\sigma_{xxi}$. By Holder's inequality,

$$
E|W_{ij}Y_{ij}|^{\alpha/2} \leq \sqrt{E|W_{ij}|^{\alpha}E|Y_{ij}|^{\alpha}} < \infty, \tag{20}
$$

therefore, we have

$$
\begin{aligned}
J_1 &= P_n\{|\frac{1}{n}\sum_{j=1}^{n}(W_{ij}Y_{ij} - \beta_{1i}\sigma_{xxi})| \geq \frac{n-1}{n}\delta\frac{\sigma_{xxi}}{6}\} \qquad (21) \\
&= o(n^{(-\alpha/2-1)}).
\end{aligned}
$$

Since

$$
\begin{aligned}
&\bar{W}_i\bar{Y}_i - \beta_{1i}\bar{W}_i^2 \qquad (22) \\
&= \bar{W}_i(\bar{Y}_i - \beta_{1i}\bar{W}_i) \\
&= \bar{W}_i(\beta_{0i} + \bar{\epsilon}_i + \beta_{1i}\bar{U}_i) \\
&= \beta_{0i}\bar{W}_i + \bar{\epsilon}_i\bar{W}_i + \beta_{1i}\bar{X}_i\bar{U}_i + \beta_{1i}\bar{U}_i^2,
\end{aligned}
$$

we observe that

$$
\begin{aligned}
J_2 &= P_n\{|\beta_{0i}\bar{W}_i + \bar{\epsilon}_i\bar{W}_i + \beta_{1i}\bar{X}_i\bar{U}_i + \beta_{1i}\bar{U}_i^2 - \frac{1}{n}\beta_{1i}\sigma_{uui}| \geq \frac{n-1}{n}\delta\frac{\sigma_{xxi}}{6}\} \\
&\leq P_n\{|\beta_{0i}\bar{W}_i| \geq \frac{n-1}{n}\delta\frac{\sigma_{xxi}}{24}\} \qquad (23)
\end{aligned}
$$

$$
\begin{aligned}
&+P_n\{|\bar{\epsilon}_i\bar{W}_i| \geq \frac{n-1}{n}\delta\frac{\sigma_{xxi}}{24}\} \\
&+P_n\{|\beta_{1i}\bar{X}_i\bar{U}_i| \geq \frac{n-1}{n}\delta\frac{\sigma_{xxi}}{24}\} \\
&+P_n\{|\beta_{1i}\bar{U}_i^2 - \frac{1}{n}\beta_{1i}\sigma_{uui}| \geq \frac{n-1}{n}\delta\frac{\sigma_{xxi}}{24}\} \\
\leq\ & P_n\{|\beta_{0i}\bar{W}_i| \geq \frac{n-1}{n}\delta\frac{\sigma_{xxi}}{24}\} \\
&+P_n\{|\bar{\epsilon}_i| \geq \sqrt{\frac{n-1}{n}\delta\frac{\sigma_{xxi}}{24}}\} + P_n\{|\bar{W}_i| \geq \sqrt{\frac{n-1}{n}\delta\frac{\sigma_{xxi}}{24}}\} \\
&+P_n\{|\sqrt{\beta_{1i}}\bar{X}_i| \geq \sqrt{\frac{n-1}{n}\delta\frac{\sigma_{xxi}}{24}}\} + P_n\{|\sqrt{\beta_{1i}}\bar{U}_i| \geq \sqrt{\frac{n-1}{n}\delta\frac{\sigma_{xxi}}{24}}\} \\
&+P_n\{|\beta_{1i}(\bar{U}_i^2 - \frac{1}{n}\sigma_{uui})| \geq \frac{n-1}{n}\delta\frac{\sigma_{xxi}}{24}\}.
\end{aligned}
$$

Then by Lemma 1, we have

$$
\begin{aligned}
P_n\{|\beta_{0i}\bar{W}_i| \geq \frac{n-1}{n}\delta\frac{\sigma_{xxi}}{24}\} &= \begin{cases} o(n^{-(\alpha-1)}), & \text{if } \beta_{0\mathrm{i}} \neq 0, \\ 0, & \text{if } \beta_{0\mathrm{i}} = 0, \end{cases} \\
&= o(n^{-(\alpha-1)}),
\end{aligned} \tag{24}
$$

$$
P_n\{|\bar{\epsilon}_i| \geq \sqrt{\frac{n-1}{n}\delta\frac{\sigma_{xxi}}{24}}\} = o(n^{-(\alpha-1)}), \tag{25}
$$

$$
P_n\{|\bar{W}_i| \geq \sqrt{\frac{n-1}{n}\delta\frac{\sigma_{xxi}}{24}}\} = o(n^{-(\alpha-1)}), \tag{26}
$$

$$
P_n\{|\sqrt{\beta_{1i}}\bar{X}_i| \geq \sqrt{\frac{n-1}{n}\delta\frac{\sigma_{xxi}}{24}}\} = o(n^{-(\alpha-1)}), \tag{27}
$$

$$
P_n\{|\sqrt{\beta_{1i}}\bar{U}_i| \geq \sqrt{\frac{n-1}{n}\delta\frac{\sigma_{xxi}}{24}}\} = o(n^{-(\alpha-1)}), \tag{28}
$$

$$
P_n\{|\beta_{1i}(\bar{U}_i^2 - \frac{1}{n}\sigma_{uui})| \geq \frac{n-1}{n}\delta\frac{\sigma_{xxi}}{24}\} \tag{29}
$$

$$
\begin{aligned}
&\le P_n\{|\beta_{1i}\bar{U}_i^2| \ge \frac{n-1}{n}\delta\frac{\sigma_{xxi}}{24} - \frac{1}{n}\beta_{1i}\sigma_{uui}\} \\
&= P_n\{|\sqrt{\beta_{1i}}\bar{U}_i| \ge \sqrt{\frac{n-1}{n}\delta\frac{\sigma_{xxi}}{24} - \frac{1}{n}\beta_{1i}\sigma_{uui}}\} \\
&= o(n^{-(\alpha-1)}),
\end{aligned}
$$

Therefore, $J_2 = o(n^{-(\alpha-1)})$. Similarly,

$$
\begin{aligned}
J_3 &= P_n\{|\frac{1}{n}\beta_{1i}\sum_{j=1}^{n} W_{ij}^2 - \beta_{1i}(\sigma_{xxi}+\sigma_{uui})| \ge \frac{n-1}{n}\delta\frac{\sigma_{xxi}}{6}\} \\
&= o(n^{-(\alpha/2-1)}).
\end{aligned} \tag{30}
$$

Hence, by combining the above arguments, we have the following theorem.

Theorem 1 The selection procedure $d_n(\underline{w}, \underline{y})$, defined in (7), is asymptotically optimal with a convergence rate of order $o(n^{-(\alpha/2-1)})$ under condition (12). That is,

$$
E^{(\underline{w},\underline{y})}L(\underline{\beta}, d_n(\underline{w}, \underline{y})) = o(n^{-(\alpha/2-1)}). \tag{31}
$$

3.2 When The Moment Generating Function Exists

In this subsection, we suppose the moment generating functions of $\{X_{ij}^2, U_{ij}^2, \epsilon_{ij}^2\}$ exist in a neighborhood of the origin, that is, for $-T \le t \le T$,

$$
Ee^{tX_{ij}^2} < \infty, \qquad Ee^{tU_{ij}^2} < \infty, \qquad Ee^{t\epsilon_{ij}^2} < \infty. \tag{32}
$$

where T is a positive constant.

We first introduce the following lemma, which can be found in Petrov (1995).

Lemma 3 Let $\{X_1, \ldots, X_n\}$ be independent random variables with mean $EX_i = 0, i = 1, \ldots, n$. Suppose there exist positive constants $g_1, \ldots, g_n$ and T such that

$$
Ee^{tX_i} \le e^{g_i t^2/2} \qquad (i = 1, \ldots, n) \tag{33}
$$

for $-T \le t \le T$. Let $G_n = \sum_{i=1}^{n} g_i$, then

$$
P(|\sum_{i=1}^{n} X_i| \ge x) \le \begin{cases} e^{-(x^2/2G_n)}, & \text{if } 0 \le \mathrm{x} \le \mathrm{G_n T}, \\ e^{-(Tx/2)}, & \text{if } \mathrm{x} > \mathrm{G_n T}. \end{cases} \tag{34}
$$

The following lemma clarifies the probabilistic meaning of the conditions of Lemma 3.

Lemma 4 Let X be a random variable with mean $EX = 0$. The following two assertions are equivalent:

(I) There exist positive constants g and H such that

$$Ee^{tX} \leq e^{gt^2/2} \qquad \text{for } -\mathrm{H} \leq \mathrm{t} \leq \mathrm{H}, \tag{35}$$

(II) There exists a positive constant T such that

$$Ee^{tX} < \infty \qquad \text{for } -\mathrm{T} \leq \mathrm{t} \leq \mathrm{T}. \tag{36}$$

Proof. It is clear that (I) implies (II). We now prove that (II) also implies (I). If (II) holds, then the random variable X has the moments of all orders, and the following relation holds:

$$\log Ee^{tX} = \frac{1}{2}\sigma^2 t^2 + o(t^2) \tag{37}$$

as $t \to 0$, where $\sigma^2 = EX^2$. For any constant $g > \sigma^2$, the inequalities $\log Ee^{tX} \leq gt^2/2$ and $Ee^{tX} \leq e^{gt^2/2}$ hold for all sufficiently small t, that is, (I) is true. This completes the proof of Lemma 4. As we can see in the proof, we can always set $g = 2\sigma^2$.

We further assume that the 4-th moments of $\{X_{ij}, U_{ij}, \epsilon_{ij}\}$ are uniformly bounded, that is, there exists a positive constant C such that

$$EX_{ij}^4 < C, \qquad EU_{ij}^4 < C, \qquad E\epsilon_{ij}^4 < C. \tag{38}$$

We can see from (38) that EW_{ij}^4, EY_{ij}^4 and $E(W_{ij}Y_{ij})^2$ are all bounded.

We analyze $P_n\{S_{wwi} - \sigma_{uui} \leq \frac{\sigma_{xxi}}{2}\}$ first.

$$\begin{aligned} & P_n\{S_{wwi} - \sigma_{uui} \leq \frac{\sigma_{xxi}}{2}\} \\ \leq \;& P_n\{|S_{wwi} - \sigma_{wwi}| \geq -\frac{\sigma_{xxi}}{2}\} \\ \leq \;& P_n\{|\frac{1}{n}\sum_{j=1}^{n} W_{ij}^2 - \sigma_{wwi}| \geq \frac{n-1}{n}\frac{\epsilon}{2}\} \\ & + P_n\{|\bar{W}_i^2| \geq \frac{\epsilon}{2} - \frac{1}{n}\sigma^2\} \\ = \;& P\{|\frac{1}{n}\sum_{j=1}^{n}(W_{ij}^2 - \sigma_{wwi})| \geq \frac{\epsilon}{4}\} \\ & + P\{|\bar{W}_i| \geq \sqrt{\frac{\epsilon}{4}}\} \\ := \;& K_1 + K_2, \end{aligned} \tag{39}$$

for n large enough, that is, when $n \geq \max(2, [\frac{4\sigma^2}{\epsilon}] + 1)$, we have $\frac{n-1}{n}\frac{\epsilon}{2} \geq \frac{\epsilon}{4}$ and $\frac{\epsilon}{2} - \frac{1}{n}\sigma^2 \geq \frac{\epsilon}{4}$. Since for $j = 1, \ldots, n$, $E(W_{ij} - \sigma_{wwi}) = 0$ and for $-T/2 \leq t \leq T/2$,

$$Ee^{tW_{ij}^2} \leq Ee^{|t(X_{ij}+U_{ij})^2|} \leq E(e^{2|t|X_{ij}^2}e^{2|t|U_{ij}^2}) = E(e^{2|t|X_{ij}^2})E(e^{2|t|U_{ij}^2}) < \infty. \tag{40}$$

By Lemma 3 and Lemma 4, we have

$$K_1 = P\{|\frac{1}{n}\sum_{j=1}^{n}(W_{ij}^2 - \sigma_{wwi})| \geq \frac{\epsilon}{4}\} \tag{41}$$
$$\leq \begin{cases} e^{-(n^2\epsilon^2/32G_n)}, & \text{if } \epsilon \leq 2\mathrm{TG_n/n}, \\ e^{-(T\epsilon/8)n}, & \text{if } \epsilon > 2\mathrm{TG_n/n}, \end{cases}$$

where G_n is twice the sum of the n variances of $(W_{ij}^2 - \sigma_{wwi})$, $j = 1, \ldots, n$. Since $(EW_{ij}^4, j = 1, \ldots, n)$ are bounded, $G_n = O(n)$. Therefore,

$$K_1 = P\{|\frac{1}{n}\sum_{j=1}^{n}(W_{ij}^2 - \sigma_{wwi})| \geq \frac{\epsilon}{4}\} \tag{42}$$
$$\leq \begin{cases} e^{-(n^2\epsilon^2/32G_n)}, & \text{if } \epsilon \leq 2\mathrm{TG_n/n}, \\ e^{-(T\epsilon/8)n}, & \text{if } \epsilon > 2\mathrm{TG_n/n}, \end{cases}$$
$$= O(e^{-c_{K_1}^* n}),$$

where $c_{K_1}^*$ is a positive constant. Similarly, for $-T \leq t \leq T$,

$$Ee^{tW_{ij}} \leq Ee^{|tW_{ij}|} \leq Ee^{|t|(W_{ij}^2+1)} < \infty. \tag{43}$$

$$K_2 = P\{|\bar{W}_i| \geq \sqrt{\frac{\epsilon}{4}}\} \tag{44}$$
$$= O(e^{-c_{K_2}^* n}),$$

where $c_{K_2}^*$ is also a positive constant.

Next we consider $P_n\{|\hat{\beta_{1i}} - \beta_{1i}| \geq \delta, S_{wwi} - \sigma_{uui} > \frac{\sigma_{xxi}}{2}\}$. We have

$$P_n\{|\hat{\beta_{1i}} - \beta_{1i}| \geq \delta, S_{wwi} - \sigma_{uui} > \frac{\sigma_{xxi}}{2}\} \tag{45}$$
$$\leq P_n\{|\frac{1}{n-1}\sum_{j=1}^{n} W_{ij}Y_{ij} - \frac{n}{n-1}\beta_{1i}\sigma_{xxi}| \geq \delta\frac{\sigma_{xxi}}{6}\}$$
$$+P_n\{|\frac{n}{n-1}\bar{W}_i\bar{Y}_i - \frac{n}{n-1}\beta_{1i}\bar{W}_i^2 - \frac{1}{n-1}\beta_{1i}\sigma_{uui}| \geq \delta\frac{\sigma_{xxi}}{6}\}$$
$$+P_n\{|\frac{1}{n-1}\beta_{1i}\sum_{j=1}^{n} W_{ij}^2 - \frac{n}{n-1}\beta_{1i}(\sigma_{xxi} + \sigma_{uui})| \geq \delta\frac{\sigma_{xxi}}{6}\}$$
$$:= L_1 + L_2 + L_3.$$

For any $i = 1, \ldots, k$, $\{W_{ij}Y_{ij}, j = 1, \ldots, n\}$ are independent, by Cauchy-Schwarz's inequality, we have, for $-T/2 \leq t \leq T/2$,

$$Ee^{tW_{ij}Y_{ij}} \leq Ee^{|tW_{ij}Y_{ij}|} \leq Ee^{|t|\frac{(W_{ij}^2+Y_{ij}^2)}{2}} \leq \sqrt{Ee^{|t|W_{ij}^2}Ee^{|t|Y_{ij}^2}} < \infty. \tag{46}$$

Besides, for each i and j, the variance of $W_{ij}Y_{ij}$ is bounded, therefore,

$$\begin{aligned} L_1 &= P_n\{|\frac{1}{n}\sum_{j=1}^{n}(W_{ij}Y_{ij} - \beta_{1i}\sigma_{xxi})| \geq \frac{n-1}{n}\delta\frac{\sigma_{xxi}}{6}\} \\ &= O(e^{-c^*_{L_1}n}), \end{aligned} \tag{47}$$

where $c^*_{L_1}$ is a positive constant. Next we analyze L_2 and L_3. Similarly,

$$\begin{aligned} L_2 &\leq P_n\{|\beta_{0i}\bar{W}_i| \geq \frac{n-1}{n}\delta\frac{\sigma_{xxi}}{24}\} \\ &\quad +P_n\{|\bar{\epsilon}_i| \geq \sqrt{\frac{n-1}{n}\delta\frac{\sigma_{xxi}}{24}}\} + P_n\{|\bar{W}_i| \geq \sqrt{\frac{n-1}{n}\delta\frac{\sigma_{xxi}}{24}}\} \\ &\quad +P_n\{|\sqrt{\beta_{1i}}\bar{X}_i| \geq \sqrt{\frac{n-1}{n}\delta\frac{\sigma_{xxi}}{24}}\} + P_n\{|\sqrt{\beta_{1i}}\bar{U}_i| \geq \sqrt{\frac{n-1}{n}\delta\frac{\sigma_{xxi}}{24}}\} \\ &\quad +P_n\{|\beta_{1i}(\bar{U}_i^2 - \frac{1}{n}\sigma_{uui})| \geq \frac{n-1}{n}\delta\frac{\sigma_{xxi}}{24}\} \\ &= O(e^{-c^*_{L_2}n}), \end{aligned} \tag{48}$$

and

$$\begin{aligned} L_3 &= P_n\{|\frac{1}{n}\beta_{1i}\sum_{j=1}^{n}W_{ij}^2 - \beta_{1i}(\sigma_{xxi} + \sigma_{uui})| \geq \frac{n-1}{n}\delta\frac{\sigma_{xxi}}{6}\} \\ &= O(e^{-c^*_{L_3}n}), \end{aligned} \tag{49}$$

where $c^*_{L_2}$ and $c^*_{L_3}$ are positive constants. Hence, by the above argument, if we set $c^* = \min(c^*_{K_1}, c^*_{K_2}, c^*_{L_1}, c^*_{L_2}, c^*_{L_3})$, then $c^* > 0$. We have the following theorem.

Theorem 2 The selection procedure $d_n(\underline{w}, \underline{y})$, as defined in (7), is asymptotically optimal with convergence rate of order $O(e^{-c^*n})$ under conditions (32) and (38). That is,

$$E^{(\underline{W},\underline{Y})}L(\underline{\beta}, d_n(\underline{w}, \underline{y})) = O(e^{-c^*n}), \tag{50}$$

where $c^* > 0$ is defined as above. We consider two special situations next.

Two special situations.

1. $\{(X_{ij}, U_{ij}, \epsilon_{ij}), 1 \leq j \leq n\}$ are normally distributed. In this case, $\{(X_{ij}, U_{ij}, \epsilon_{ij})\}$ are i.i.d. $N_3((0,0,0), \text{diag}(\sigma_{xxi}, \sigma_{uui}, \sigma_{\epsilon\epsilon i}))$. Since $(X_{ij}^2/\sigma_{xxi}, U_{ij}^2/\sigma_{uui}, \epsilon_{ij}^2/\sigma_{\epsilon\epsilon i})$ follow χ^2 distributions, their moment generating functions exist. By Theorem 2, the selection procedure $d_n(\underline{w}, \underline{y})$ in this case is asymptotically optimal with the rate of convergence of order $O(e^{-c^*n})$.

2. $\{(X_{ij}, U_{ij}, \epsilon_{ij}), 1 \leq j \leq n\}$ are bounded. Then conditions (32) and (38) always hold and therefore, the selection procedure $d_n(\underline{w}, \underline{y})$ is asymptotically optimal with the convergence rate of order $O(e^{-c^*n})$.

4 Simulations

We carried out a simulation study to investigate the performance of the selection procedure d_n. The expected risk $E^{(\underline{\mathrm{W}},\underline{\mathrm{Y}})}L(\underline{\beta}, d_n(\underline{\mathrm{w}}, \underline{\mathrm{y}}))$ is used as a measure of the performance of the selection rule. In this study, we considered normal distributions and there are $i = 3$ treatments. The simulation scheme is described as follows:

1. For each $j = 1, \ldots, n$ and $i = 1, 2$ and 3, we generated independent random observations $(X_{ij}, U_{ij}, \epsilon_{ij})$ from multivariate normal $N_3((0, 0, 0)^T, \mathrm{diag}(\sigma_{xxi}, \sigma_{uui}, \sigma_{\epsilon\epsilon i}))$.

2. Let $W_{ij} = X_{ij} + U_{ij}$ and $Y_{ij} = \beta_{0i} + \beta_{1i}X_{ij} + \epsilon_{ij}$.

3. Based on (W_{ij}, Y_{ij}), we obtained the estimator of β_{1i}, then made the selection using d_n and computed $D(\underline{\mathrm{W}}, \underline{\mathrm{Y}})$ which is as follows:

$$D(\underline{\mathrm{W}}, \underline{\mathrm{Y}}) = \begin{cases} 1, & \text{if we make a wrong selection,} \\ 0, & \text{if we make a correct selection.} \end{cases} \tag{51}$$

4. Step 1, 2 and 3 were repeated 10000 times. With each set of observations, $D(\underline{\mathrm{W}}, \underline{\mathrm{Y}})$ would be either 0 or 1, as we might make a right or wrong decision. When we take the sample $(\underline{\mathrm{w}}, \underline{\mathrm{y}})$ repeatedly, by the law of large numbers, the average of $D(\underline{\mathrm{W}}, \underline{\mathrm{Y}})$ would be getting very close to the expected risk $E^{(\underline{\mathrm{W}},\underline{\mathrm{Y}})}L(\underline{\beta}, d_n(\underline{\mathrm{w}}, \underline{\mathrm{y}}))$ and can be used as an estimator of the expected risk when the number of iterations is large enough.

We specified the number of iterations to be 10000 to make sure that the deviation between the estimated value and the true value is less than 0.01 with 95% confidence. The following is a brief introduction to the power calculation in this study. We are interested in the unknown probability of making a wrong decision. So we take the sample repeatedly and each time the result can be either right or wrong. Therefore, we have a binomial setting here: we use the sample proportion (denoted by $\hat{p}$) to estimate the population proportion (denoted by p). When the number of iterations (denoted by N) is large enough,

$$\frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{N}}} \sim N(0, 1).$$

With 95% confidence, $|\hat{p} - p| \leq 2\sqrt{\frac{p(1-p)}{N}}$. When $N = 10000$, since $p(1 - p) \leq 0.25$,

$$|\hat{p} - p| \leq 2\sqrt{\frac{p(1-p)}{N}} \leq 2 \times \sqrt{\frac{0.25}{10000}} = 0.01$$

This is the reason why the number of iterations was set to be 10000. The results from the simulation study are listed in Table 1 for the case where

$$\sigma_{xx1} = \sigma_{xx2} = \sigma_{xx3} = 1,$$

$$\sigma_{uu1} = \sigma_{uu2} = \sigma_{uu3} = 1,$$
$$\sigma_{\epsilon\epsilon 1} = \sigma_{\epsilon\epsilon 2} = \sigma_{\epsilon\epsilon 3} = 1,$$
$$\beta_{01} = \beta_{02} = \beta_{03} = 0,$$
$$\beta_{11} = 0.4, \beta_{12} = 0.5, \beta_{13} = -0.6.$$

and

$$\begin{aligned} n &= 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 200, 300, 400, 500, \\ & \quad 600, 700, 800, 900, 1000, 1100, 1200, 1500, 2000. \end{aligned} \tag{52}$$

The curve of the estimated probability of making a wrong decision with respect to n is attached in Figure 1 at the end of this paper. It bears out our conclusions that the rate of convergence of the probability of making a wrong decision should be $O(e^{-c^* n})$ in this case.

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Table 1

n	D_n
10	0.5443
20	0.4851
30	0.4425
40	0.3926
50	0.3667
60	0.3578
70	0.3334
80	0.3219
90	0.2862
100	0.2743
200	0.1627
300	0.1103
400	0.0684
500	0.0488
600	0.0315
700	0.0271
800	0.0246
900	0.0163
1000	0.0139
1100	0.0089
1200	0.0064
1500	0.0035
2000	0.0004

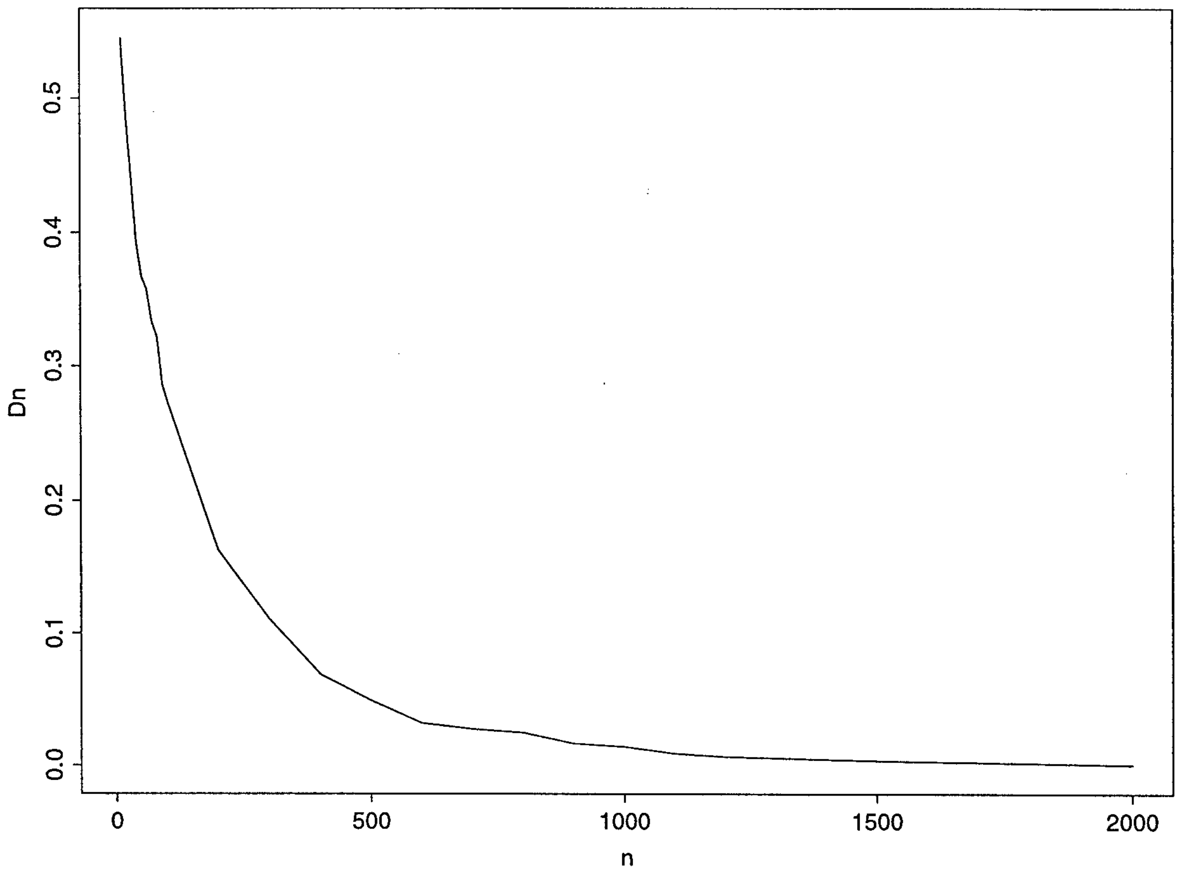

Figure 1: Figure 1

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