REPORT DOCUMENTATION PAGE

Form Approved
OMB NO. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	9/18/96	Technical

4. TITLE AND SUBTITLE
Numerical Solution of Nonlinear Oscillatory Multibody Dynamic Systems

5. FUNDING NUMBERS
DAAL03-92-G-0247

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8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)
U.S. Army Research Office
P.O. Box 12211
Research Triangle Park, NC 27709-2211

10. SPONSORING / MONITORING AGENCY REPORT NUMBER
ARO 29850.3-MA

11. SUPPLEMENTARY NOTES
The views, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy or decision, unless so designated by other documentation.

12a. DISTRIBUTION / AVAILABILITY STATEMENT
Approved for public release; distribution unlimited.

19961024 111

13. ABSTRACT *(Maximum 200 words)*

One of the outstanding problems in the numerical simulation of mechanical systems is the development of efficient methods for dealing with highly oscillatory systems. These types of sytems arise for example in vehicle simulation in modeling the suspension system or tires, in models for contact and impact, in flexible body simulation from vibrations in the structural model, and in molecular dynamics. Simulations involving high frequency vibration can take a huge number of time steps, often as a consequence of oscillations which are not physically important. The components causing the oscillations cannot usually be eliminated from the model because in some situations they are critical to the simulation. The equations of motion of a multibody mechanical system are described by a system of differential-algrebraic equations (DAEs). In this paper, we will explore two types of methods. The first class of methods damps out the oscillation via highly stable implicit methods. Even in this relatively simple approach, unforseen problems may arise for Newton iteration convergence, due to the nonlinearities. The second class of methods involves linearizing the system around the smooth solution. The linearized system can be solved rapidly via a number of different methods.

14. SUBJECT TERMS	15. NUMBER IF PAGES
constrained dynamics, multibody systems, differential-algebraic equations, numerical methods, highly oscillatory systems.	15
	16. PRICE CODE

17. SECURITY CLASSIFICATION OR REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL

DEPARTMENT OF COMPUTER SCIENCE

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TR 95-066

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Technical Report

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August 24, 1995

Abstract

One of the outstanding problems in the numerical simulation of mechanical systems is the development of efficient methods for dealing with highly oscillatory systems. These types of systems arise for example in vehicle simulation in modeling the suspension system or tires, in models for contact and impact, in flexible body simulation from vibrations in the structural model, and in molecular dynamics. Simulations involving high frequency vibration can take a huge number of time steps, often as a consequence of oscillations which are not physically important. The components causing the oscillations cannot usually be eliminated from the model because in some situations they are critical to the simulation. The equations of motion of a multibody mechanical system are described by a system of differential-algebraic equations (DAEs). In this paper, we will explore two types of methods. The first class of methods damps out the oscillation via highly stable implicit methods. Even in this relatively simple approach, unforseen problems may arise for Newton iteration convergence, due to the nonlinearities. The second class of methods involves linearizing the system around the smooth solution. The linearized system can be solved rapidly via a number of different methods.

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†Department of Computer Science and Army High Performance Computing Research Center, University of Minnesota, Minneapolis, MN 55455. The work of this author was partially supported by the Army High Performance Computing Research Center, and by ARO contract number DAAL03-92-G-0247 and ARO contract number DAAH04-94-G-0208, and by the Minnesota Supercomputer Institute.

1 Introduction

Much recent work has focused on the development of numerical methods and underlying theory for the solution of multibody dynamic systems (MBS) consisting of fast and slow subsystems [17, 22]. These types of systems occur frequently as initial value problems in the computer-aided design and modeling of constrained mechanical systems, molecular dynamics, and in many other applications [1, 31]. It is well-known that the characteristics of fast or slow solution are determined not only by the modeling aspects, e.g., the coefficients of stiffness and damping, but also by the initial conditions and events that may excite stiff components in the system during the simulation. As an example, the governing equations of motion of a mechanical system of stiff or highly oscillatory force devices may be written as a system of differential-algebraic equations (DAE) [5]:

$$M(q)\ddot{q} + G^T(q)\lambda - (f^s + f^n) = 0 \quad (1a)$$

$$g(q) = 0 \quad (1b)$$

where $q = [q_1, ..., q_n]^T$ is the generalized coordinate, $\dot{q} = \frac{dq}{dt}$ is the generalized velocity, $\ddot{q} = \frac{d^2q}{dt^2}$ is the acceleration and $\lambda = [\lambda_1, ..., \lambda_m]^T$ is the Lagrange multiplier. The stiff or oscillatory force is $f^s = \sum_i^{n_f} f_i^s$, and f^n includes all the field forces and the external forces which are non-stiff compared to the stiff components, e.g.,

$$\|\frac{\partial f^s}{\partial q}\| \gg \|\frac{\partial f^n}{\partial q}\|. \quad (2)$$

The kinematic constraints are g, and $G = \frac{\partial g}{\partial q}$, and M is the mass-inertia matrix. For the stiff force components in (1a), we assume that

$$f_i^s = -B_i(q)(K_i\eta_i(q) + C_i\frac{d\eta_i}{dt}) \quad (3)$$

where η_i is smooth $\forall i \in \{1, ..., n_f\}$ and $B_i = \frac{\partial \eta_i}{\partial q}^T$, and K_i, C_i are the associated stiffness and damping factors. For some generalized coordinate sets, the function η may be linear or even identities, e.g., for instance $\eta_i = q_{i_l}$ for some $i \in \{1, ..., n_f\}$ and $i_l \in \{1, ..., n\}$. When the components of the coefficient matrices K_i and C_i become large, these force components may cause rapid decay or high frequency oscillation in the solution of (1). The purpose of this article is to study these systems and their numerical solution. In this notation, the stiff force term in (1a) can be written as

$$f^s = -B(q)(K\eta(q) + CB(q)v). \quad (4)$$

To demonstrate the problem of oscillation and the recent developments in this area, we present two examples: a stiff pendulum and a 2D *bushing* problem. The former

is a very simple example of a type of system often seen in modeling molecular dynamic systems, and the latter is a general form of modeling force devices in multibody mechanical systems.

Stiff pendulum

In Cartesian coordinates, a simple stiff pendulum model, with unit mass and gravity, may be expressed as

$$
\begin{aligned}
0 &= \dot{x} - u && (5a)\\
0 &= \dot{y} - v && (5b)\\
0 &= \dot{u} + x\lambda && (5c)\\
0 &= \dot{v} + y\lambda - 1.0 && (5d)\\
\epsilon^2\lambda &= \frac{\sqrt{x^2+y^2} - 1.0}{\sqrt{x^2+y^2}} && (5e)
\end{aligned}
$$

where the stiff spring of natural length 1.0 and stiffness $\frac{1}{\epsilon^2}$ is attached to the center of mass of the pendulum. Preloading the spring by using $\epsilon = \sqrt{10^{-3}}$, the initial condition $(x_0, y_0) = (0.9, 0.1)$ and the zero initial velocity $(u_0, v_0) = (0, 0)$, the results of the states (x, y, u, v) in the 0 to 10 second simulation are shown in Fig. 1. The corresponding eigenvalues of the underlying ODE of (5), i.e., substituting (5e) into (5c, 5d), are illustrated in Fig. 2, where the 3D figures contain all the eigenvalues on the complex plane drawn along the time-axis. The dominant pair of eigenvalues in the example are $\pm\frac{1}{\epsilon}i$, as shown in Fig. 2. As $\epsilon \to 0$, the pair of eigenvalues approaches $\pm\infty$ along the imaginary axis. The other pair of eigenvalues oscillates on the complex plane with the amplitude and frequency approaching $\pm\infty$. Decreasing ϵ to $\sqrt{10^{-5}}$, the eigenvalues of the underlying ODE of (5) are times the magnitude of those in Fig. 2, and the oscillating pair increases its frequency proportional to the size of ϵ.

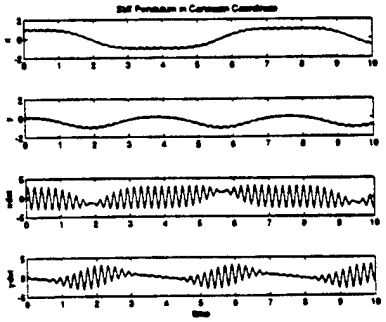

Figure 1: Stiff Pendulum in Cartesian Coordinates

Lubich [17] shows that the numerical solution by a class of Runge-Kutta methods of stiff mechanical systems of a strong potential energy, e.g., stiff spring force such as the stiff pendulum (5), converges to the slowly varying part of the solution, with the

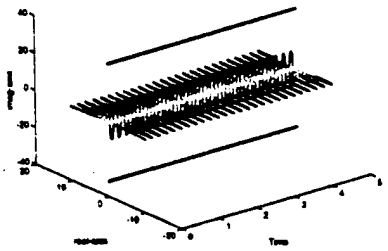

Figure 2: Eigenvalues of Stiff Pendulum in Cartesian Coordinates. epsilon = 10e-1.5

stepsize independent of the parameter ϵ in (5). Reich [22] extends the *principle* of *slow manifold* [9, 15] to DAE of MBS with highly oscillatory force terms. Algebraic constraints corresponding to the slow motion were introduced with a relaxation parameter to preserve the slow solution while adding flexibility to it in the slow manifold approach.

It is not clear that a *slow* solution appears in the above example. In fact, we can only identify the *slow* solution of (5) using a proper nonlinear coordinate transformation. In *polar coordinates* (r, θ), we obtain the equations of motion of (5):

$$0 = \dot{r} - z \tag{6a}$$

$$0 = \dot{\theta} - \omega \tag{6b}$$

$$0 = \dot{z} + r\omega^2 + \frac{1}{\epsilon^2}(r-1) - \sin\theta \tag{6c}$$

$$0 = \dot{\omega} - \frac{1}{r}(2z\omega - \cos\theta) \tag{6d}$$

where (z, ω) is the velocity. In the 0 to 10 second simulation, using the same initial conditions as the previous examples, i.e., $r_0 = \sqrt{x_0^2 + y_0^2}$, $\theta_0 = \arctan\frac{y_0}{x_0}$, and $(z_0, \omega_0) = (0, 0)$, where (x_0, y_0) is the initial position of (5), we obtain the solution in Fig. 3. It is clear that the solutions of (r, z) represent the *fast* motion, and the solutions of (θ, ω) are the *slow* ones. The eigenvalues along the solution trajectory are presented in Fig. 4. Note the dominate eigenvalues are of the same magnitude as those in (5), see Fig. 2. This is because the coordinate transformation, $x = r\cos\theta$, $y = r\sin\theta$, is linear with respect to the fast moving r. The eigenvalues of (6) with $\epsilon = \sqrt{10^{-5}}$ are similar to the comparison in the Cartesian formulation, i.e., we obtain the eigenvalues of 10 times magnification. However, the eigenvalues corresponding to the slow motion have near zero imaginary parts, therefore, the oscillations along the imaginary axis of eigenvalues 3,4 in Fig. 4 remain insignificant.

Although there are ongoing developments to extend the results of Lubich to multistage multistep methods [23], and impressive application of the slow manifold technique in some molecular dynamic models. it is not clear that these results may apply

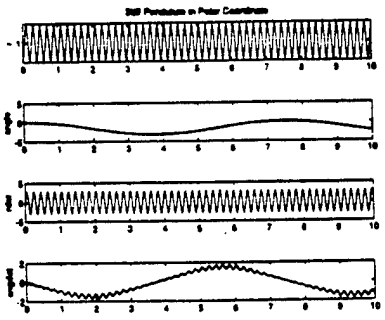

Figure 3: Stiff Pendulum in Polar Coordinates

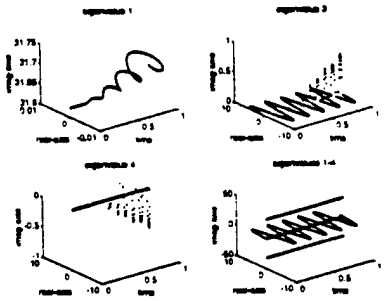

Figure 4: Eigenvalues of Stiff Pendulum in Polar Coordinates, epsilon = 10e-1.5

directly to all the types of oscillatory components in MBS. As indicated in [17], the representation of stiff or oscillatory components in an appropriate coordinate system of MBS is not always possible. i.e., the constraints associated with the stiff or oscillatory potential force can be difficult to obtain in general. Nevertheless, for (1), an approximation of the dynamics of such local coordinates can be obtained for the oscillatory components in the form of (4). We are also concerned with the convergence of Newton's method for the numerical solution with large stepsize of (1), which may be an obstacle in obtaining efficient numerical solution of an oscillatory MBS in either of the above-mentioned approaches.

Bushing force

We have been studying more general MBS of nonlinear oscillatory components such as a bushing force, which is often used in modeling vehicle suspension systems. Different from the linear spring, this element is usually an *anisotropic* force, i.e., it has different spring coefficients along the principle axes of the bushing local coordinate frame. The bushing force between *body-i* and *body-j* may be defined using the relative displacement d_{ij}, its time derivative $\dot{d}_{ij}$, and the relative angle θ_{ij} and its time derivative $\dot{\theta}_{ij}$ of two body-fixed local coordinate frames at the bushing location on two bodies. Using the vectors s'_i and s'_j representing the bushing location in the *body-i*'s and *body-j*'s centroid

local coordinate systems, respectively, we have

$$d_{ij} = \begin{bmatrix} x_i \\ y_i \end{bmatrix} - \begin{bmatrix} x_j \\ y_j \end{bmatrix} + A_i s'_i - A_j s'_j \tag{7}$$

where the orientation transformation matrices A_i and A_j are

$$A_i = A(\theta_i) = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix}$$

$$A_j = A(\theta_j) = \begin{bmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{bmatrix}$$

and $[x_i, y_i, \theta_i]$ and $[x_j, y_j, \theta_j]$ are Cartesian coordinates at body-fixed frames. The bushing force f_b can then be written as

$$f_b = \begin{bmatrix} f_b^x \\ f_b^y \end{bmatrix} = A_i \begin{bmatrix} k^x & 0 \\ 0 & k^y \end{bmatrix} A_i^T d_{ij} + A_i \begin{bmatrix} c^x & 0 \\ 0 & c^y \end{bmatrix} A_i^T \dot{d}_{ij} \tag{8}$$

and the applied torque is

$$\tau_b = k^\theta \theta_{ij} + c^\theta \omega_{ij} \tag{9}$$

where $\omega_{ij} = \frac{d\theta_{ij}}{dt}$, k^x, k^y, and k^θ are the spring coefficients associated with the x, y, and θ coordinates, and c^x, c^y, and c^θ are the corresponding damping coefficients.

A simple example may be obtained from this model using unit mass-inertia and gravity, grounding the first body, and setting the bushing location on the second body to $s' = [-\frac{1}{2}, 0]$. A bushing element with no damping attached at the global position of $[\frac{1}{2}, 0]$ yields

$$0 = \ddot{x} - k^x(\frac{1}{2} - x + \frac{\cos\theta}{2}) \tag{10a}$$

$$0 = \ddot{y} + k^y(y - \frac{\sin\theta}{2}) + 1 \tag{10b}$$

$$0 = \ddot{\theta} + k^\theta\theta - \frac{\sin\theta}{2}k^x(\frac{1}{2} - x + \frac{\cos\theta}{2}) - \frac{\cos\theta}{2}k^y(y - \frac{\sin\theta}{2}). \tag{10c}$$

It is easy to see from (10) that the local eigenstructure of the system may change rapidly, depending on the size of the stiffness coefficients.

Using the initial values of $(x, y, \theta) = (1.1, 0.1, 0.0)$ with $(k^x, k^y, k^\theta) = (10^4, 10^4, 10^3)$, the solution of (10) exhibits high frequency oscillation for all coordinates, as shown in Fig. 5. Solving the eigenvalue problem of (10) at each time step yields three pairs as illustrated in Fig. 6.

Many methods for efficient solution of oscillatory dynamic systems are predicated on a nearly linear form of the equation. For example, the *method* of *averaging* [4] requires the linear part of the oscillation equations of motion to be dominant. and the *mode-acceleration* method for structural dynamics, which eliminates higher modes

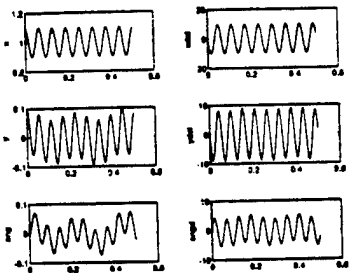

Figure 5: Bushing Problem in Cartesian Coordinates

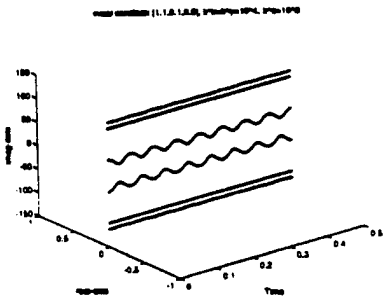

Figure 6: Eigenvalues of Bushing Problem

in the computation of the *mode-displacement* solution [29], is based on the time-invariant eigenvalues of the structural dynamic equations. Our aim is to treat the class of general nonlinear stiff and oscillatory forces represented in the MBS of (1). One approach is based on the study of a class of MBS DAE solvers [27] and the energy dissipative method, which may damp out the oscillation where the amplitude is small. The other approach is to approximate the system by a nearly linear system, whose solution approximates the solution to the original system.

The main technical problem in either approach arises from the fact that for these nonlinear oscillating problems, the eigenstructure of the local Jacobian varies rapidly, on the scale of the high-frequency oscillation, in a neighborhood of the smooth solution. We will used the fact that the eigenstructure varies slowly along the smooth solution to construct efficient numerical methods and accurate approximate solutions.

2 Damping the oscillation

Given the possibility of a rapidly changing local eigenvalue structure, perhaps the simplest strategy is to consider damping the oscillation whenever it is not important via highly stable implicit numerical methods. Since the amount of damping is controlled by the time-step, and automatic stepsize selection increases the time-step whenever the

solution is slowly-varying (i.e. if the amplitude of the oscillation is small in comparison with the local error tolerances), the stepsize is increased when the oscillation is no longer important.

In recent work [27] we have considered the solution of mechanical systems with high frequency vibrations via this type of technique. In experiments with the bushing problem (10) solved directly by low-order BDF methods, we found that the methods experienced severe problems with Newton convergence. To overcome these problems, we proposed a *coordinate-split* (CS) formulation of the equations of motion, and a Newton-type iteration for solving the coordinate-split equations at each time step. The coordinate-split formulation eliminates problems due to obtaining an accurate predictor for the Lagrange multiplier variables because these variables are no longer present in the computation. We found that the coordinate-split formulation worked well for several test problems involving mechanical systems with high frequency oscillations. However, for problems with very high-frequency oscillations, there are still difficulties in Newton convergence with this method.

The Jacobian matrix for solving the nonlinear equations of the coordinate-split formulation at each time step involves several terms which are complicated to compute and which are small at the solution of the nonlinear system. These are terms of second-order which correspond to the derivative of the projection operator onto the constraints. Away from the smooth solution, these terms are highly oscillatory. By neglecting these terms, we found that the resulting Newton-type method converged much faster for oscillating test problems like the bushing problem. We called the resulting method the *modified coordinate-split*, or CM method. In [28], convergence of the CM iteration is analyzed, and the improved convergence for oscillatory multibody systems is explained. Intuitively, by neglecting these terms the CM-iteration approximates the Jacobian along the smooth solution, thus yielding more reliable Newton directions.

The modified coordinate-split (CM) method performed extremely well in numerical experiments described in [27]. The constraints $g(p) = [g_1, ..., g_5]^T$ of a two-body pendulum may be written as

$$g_1 = x_1 \tag{11a}$$
$$g_2 = y_1 \tag{11b}$$
$$g_3 = \theta_1 \tag{11c}$$
$$g_4 = (x_1 - x_2)^2 + (y_1 - y_2)^2 - 1 \tag{11d}$$
$$g_5 = \theta_2 \tag{11e}$$

where x_i and y_i, $i = 1, 2$ are Cartesian coordinates of the center of mass of body i, and θ_i is the orientation coordinate of the body centroid reference coordinate system, and the length of the pendulum is 1. Applying the bushing force (8) with $[k^x, k^y, k^\theta]$ $= [1000, 1000, 1000]$ and $[c^x, c^y, c^\theta] = [10, 10, 10]$ to the pendulum, small oscillations of

the numerical solution appear. Using the initial values q [0.0,0.9.9989e-1.-1.4852e-2.0] and v = [0,0,0,-6.75e-5,-4.5444e-3], numerical results from the BDF code DASSL [21] are contained in Table 1, in which error test failures ($etf - s$) and convergence test failures ($ctf - s$) are listed. We denoted by CS the coordinate-splitting formulation, LG the stabilized index-2 formulation proposed by Gear [10], CM the coordinate-split form using a modified iteration matrix with the second-order derivative terms omitted, and LM the modified LG using the new predictor of the multipliers by the CS method. Using simplified Newton iterations and the corresponding modified local error estimate, CS, CM, LG and LM obtain consistent results.

Method	TOL	*step*	$f-s$	$j-s$	$etf-s$	$ctf-s$
CS	10^{-3}	62	156	48	0	13
CM	10^{-3}	62	156	48	0	13
LM	10^{-3}	62	154	48	0	13
LG	10^{-3}	59	141	46	0	12
CS	10^{-4}	77	193	65	1	16
CM	10^{-4}	77	193	65	1	16
LG	10^{-4}	61	136	27	1	5
LM	10^{-4}	77	193	65	1	16
CS	10^{-5}	87	215	54	0	13
CM	10^{-5}	87	215	54	0	13
LG	10^{-5}	108	259	77	1	21
LM	10^{-5}	87	215	54	0	13
CS	10^{-6}	138	343	97	1	25
CM	10^{-6}	138	343	97	1	25
LG	10^{-6}	131	308	65	0	16
LM	10^{-6}	138	343	97	1	25

Table 1: Simple Pendulum with a Bushing Force. Spring Constant = 10^3

To see the effect of more severe oscillation, we increased the spring constant of the bushing to 10^5. Time steps of these methods selected by DASSL are shown in Figure 7. Clearly, CM took much larger steps than the other methods. Moreover, if the spring constant is increased to 10^6, we found severe convergence problems for LG, CS and LM; the results are contained in Table 2. Further details on the numerical experiments are given in [27].

3 Smooth Linearization

Often in multibody systems the components exhibiting high frequency oscillation result from the potential forces induced by material deformations. In flexible multibody dynamic systems, for example, there are usually nonlinear transformations applying

Method	TOL	time	$step$	$f-s$	$j-s$	$etf-s$	$ctf-s$
CS_{sn}	10^{-4}	0-0.1	2252	4901	3361	1	1120
CM_{sn}	10^{-4}	0-0.1	20	40	7	0	0
LG_{sn}	10^{-4}	0-0.1	5267	10536	7899	0	2633
LM_{sn}	10^{-4}	0-0.1	2251	4900	3360	1	1120

Table 2: Results of Bushing Problem, Spring Constant = 10^6, Damping = 10

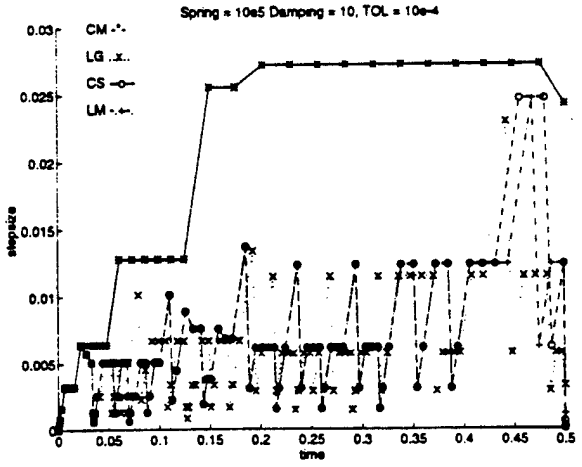

Figure 7: Time Steps Used in Solving the Bushing Problem, Spring Constant = 10^5

to the internal and inertia forces of the flexible structure to obtain equations of motion in the generalized coordinate space [29]. The resulting system of equations contains nonlinear high frequency oscillatory forces.

One approach to solving the high frequency oscillation problem is to carry out modal analysis and then eliminate the higher modes, since lower modes may preserve the slowly varying part of the solution [8, 13]. For example, the extreme high modes of a structure are often rejected in modeling flexible effects of mechanisms, since the details of the oscillating solution are not so important as the long-term solution behavior. A similar approach has been developed in recent work on molecular dynamics simulation [31]. However, due to nonlinear oscillatory forces in the multibody formulation, the modal analysis needs to be carried out at each time step to resolve the rapidly varying local eigenstructure of the system, resulting in very costly computations.

Another approach is to resolve the oscillation efficiently via *look-up tables*. This idea is frequently used in a real-time simulation environment. One such example is the modeling of contact compliance in rigid body simulation, where the localized oscillation may be of interest. Applying linear constitutive laws to modeling the contact compliance, e.g., the elastic half space theory, Boussinesq's influence functions and Hertz' contact model leads to linear spring forces between contact bodies [26, 14, 18]. The spring coefficients may be very large since the contact deformations are small compared to the gross motion of the contacting bodies. The advantage of using table-look-up is efficiency, however it is not clear how the variable stepsize and order numerical integration should interact with the tables to maintain efficiency and accuracy.

In a numerical method such as multistep or Runge-Kutta, which are based on approximating the solution locally, the stepsize must be chosen very small to resolve the high-frequency oscillation in the system. Moreover, due to the nonlinear transformation that places oscillating components in the space of the generalized coordinates, e.g., in the form of (1), the numerical method may become ineffective since the eigenstructure may change rapidly as shown in the previous examples. Our goal of treating large-scale MBS with highly oscillatory components (1) is to develop numerical methods that approximate the high frequency components properly.

Modal analysis in structural dynamics is well-developed and implemented in production software [8, 19]. As shown in [7], combining structural dynamic subsystems with the DAE of MBS, high frequency nonlinear oscillating solutions may occur. Based on the solution of this class of nonlinear oscillating problems, we propose a new approach which is based on linearizing the oscillating components around the *smooth* solution.

Under the assumption that the oscillatory forces are of the form (3), we can choose the local coordinates $\tilde{q} = [\tilde{q}_1, ..., \tilde{q}_{n_f}]^T$ and the velocity $\tilde{v} = \frac{d\tilde{q}}{dt}$ such that

$$\tilde{q} - \eta(q) \;=\; 0 \qquad (12a)$$

$$\tilde{v} - B(q)^T v = 0 \tag{12b}$$

at each force component. Note that the corresponding potential energy becomes

$$V^s = \tilde{q}^T K \tilde{q} \tag{13}$$

where $\|K\|$ is the dominant term of the highly oscillatory forces. The oscillations in the solution of q are coupled with those of the local coordinate $\tilde{q}$. Since the potential (13) is quadratic in $\tilde{q}$, the dynamic equations of $\tilde{q}$ contain a (nearly) linear term that characterize the local oscillations due to the forces.

To derive the equations of motion for $\tilde{q}$, we can eliminate λ in (1) using the *acceleration* equations,

$$G(q)\ddot{q} = -v^T \frac{dG}{dq} v \equiv \gamma \tag{14}$$

obtained from twice differentiating (1b). Let $M(q)$ be nonsingular and $f^s = B(q)K\tilde{q}$, solving for $\ddot{q}$ from (1a) and (14) yields

$$\ddot{q} = M^{-1}(I - G^T(GM^{-1}G^T)^{-1}GM^{-1})(f^n + f^s) + M^{-1}G^T(GM^{-1}G^T)^{-1}\gamma. \tag{15}$$

Differentiating (12b) with respect to time yields

$$\ddot{\tilde{q}} - B(q)^T\ddot{q} - \dot{q}^T \frac{dB^T}{dq}\dot{q} = 0. \tag{16}$$

Substituting (15) into (16) we obtain

$$\ddot{\tilde{q}} + \tilde{K}\tilde{q} = \tilde{f}(\tilde{q}, \dot{\tilde{q}}, t) \tag{17}$$

in which we denote $\tilde{M}^{-1}K = \tilde{K}$ such that

$$\tilde{M}^{-1} = B(q)^T \Lambda(q) B(q) \tag{18}$$

where $\Lambda(q) = -M^{-1}(I - G^T(GM^{-1}G^T)^{-1}GM^{-1})$. According to the assumption (2), the right-hand side of (17)

$$\tilde{f} = \gamma^B - B^T \Lambda B f^n \tag{19}$$

where

$$\gamma^B \equiv G^T(GM^{-1}G^T)^{-1}\gamma + \dot{q}^T \frac{dB^T}{dq}\dot{q},$$

satisfies

$$\|\frac{\partial \tilde{f}}{\partial \tilde{q}}\| \ll \|\tilde{K}\|. \tag{20}$$

Thus, the solution of (17) has an *asymptotic expansion* with a dominant first term for a fixed q [4], i.e., a smooth solution of $\tilde{q}$ can be expressed as

$$\hat{q} = \sum_{j=0}^{k} \epsilon^j \tilde{q}_j \tag{21}$$

where $\tilde{q}_j$ are smooth functions and $\|\epsilon\tilde{K}\| \approx 1$.

As shown in the stiff pendulum and the bushing examples, the nonlinear oscillatory force in (1) can be written, using proper local coordinates, in a nearly linear form. In the stiff pendulum, the proper choice of the local coordinate is the polar system. In the bushing example, this is achieved using the local displacement and relative angle at the bushing reference frame. Using this approach, the general form of flexible MBS can be written as

$$f(y', y, z, t) = 0 \tag{22a}$$

$$z' + H(y)z + h(y, z, t) = 0 \tag{22b}$$

where $y = [q, \dot{q}]$ and $z = [\tilde{q}, \dot{\tilde{q}}]$, and (22a) is often a system of differential-algebraic equations. Note that (22) usually contains a large number of equations, i.e., there are many of the *free vibration modes*.

Our objective is to develop a method that takes large time-steps relative to the high-frequency oscillations. Several methods are currently under investigation. One approach is to solve a linearization of (22) directly by damped numerical methods. The linearization is carried out around a *smooth* solution $\hat{y}$ such that $\hat{y} = [\hat{q}, \dot{\hat{q}}]$ is near the *equilibrium* of the highly oscillatory forces, i.e., $\eta(\hat{q}) \approx 0$ and $B^T(\hat{q})\dot{\hat{q}} \approx 0$. To obtain the numerical solution, we apply the above-mentioned CM method to (22a), and the Lanczos types or Arnoldi algorithm to the resulting linear system. These methods for nonsymmetric linear systems with proper modifications, for example in [12], are used to acquire a stable reduced order model for the linearized form of (22). Another possibility is to apply modal analysis to (22b) along the smooth solutions, yielding a reduced system of (22b) that is described by the Ritz vectors or the Lanczos vectors [25, 16, 30, 24, 19]. The key idea is to obtain a smooth linearization by choosing the proper local coordinates and linearizing along the smooth solution.

References

[1] S. S. Ashour and O. T. Hanna, *Explicit exponential method for the integration of stiff ordinary differential equations*, J. Guidance 14 (1991), 1234-1239.

[2] U. Ascher and L. Petzold, *Projected implicit Runge-Kutta methods for differential-algebraic equations*, SIAM J. Numerical Analysis 28 (1991), 1097-1120.

[3] U. Ascher and L. Petzold, *Stability of computational methods for constrained dynamics systems*, SIAM J. SISC 14 (1993), 95-120.

[4] N. N. Bogoliubov and Y. A. Mitropolski, *Asymptotic Methods in the Theory of Nonlinear Oscillations*, Hindustan Publishing Corp., Delhi, India, 1961.

[5] K. E. Brenan, S. L. Campbell and L. R. Petzold, *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*, Elsevier Science Publishers, 1989.

[6] P. N. Brown, A. C. Hindmarsh and L. R. Petzold, *Using Krylov methods in the solution of large-scale differential-algebraic systems*, to appear, SIAM J. on Scientific Computing.

[7] A. Cardona and M. Geradin, *Time integration of the equations of motion in mechanism analysis*, Comput. and Struct. **33**(1989), pp. 801-810.

[8] R. R. Craig, *Structural Dynamics*, Wiley, 1981.

[9] N. Fenichel, *Geometric singular perturbation theory for ordinary differential equations*, J. Diff. Eq. (31), 53-98, 1979.

[10] C.W. Gear, G.K. Gupta and B.J. Leimkuhler, *Automatic integration of the Euler-Lagrange equations with constraints*, J. Comp. Appl. Math., vol. 12 & 13, 1985, 77-90.

[11] G. H. Golub and V. Pereyra, *The differentiation of pseudo-inverses and nonlinear least squares problems whose variables separate*, SIAM J. Numer. Anal., Vol. 10, No. 2, April 1973, pp. 413-432.

[12] E. Grimme, D. Sorensen, and P. Van Dooren, *Stable partial realizations via an implicitly restarted Lanczos method*, to appear Proc. American Control Conf., (Baltimore, MD), 1994.

[13] T.J.R. Hughes, *The Finite Element Method*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1987.

[14] K. L. Johnson, *Contact Mechanics*, Cambridge University Press, 1985.

[15] N. Kopell, *Invariant manifolds and initialization problem for some atmospheric equations*, Physica D., (14), 203-215, 1985.

[16] P. Leger and E. L. Wilson, *Generation of load dependent Ritz transformation vectors in structural dynamics*, Eng. Comput. 4 (1987), 309-318.

[17] Ch. Lubich, *Integration of stiff mechanical systems by Runge-Kutta methods*, ZAMP, Vol. 44, 1022-1053, 1993.

[18] A. I. Luré, *Three-dimensional Problems of the Theory of Elasticity.* Interscience, New York, English translation by J.R.M. Radok, 1964.

[19] B. Nour-Omid, *Applications of the Lanczos algorithm*, Comp. Phy. Comm., 53, 1989.

[20] L. R. Petzold, *An efficient numerical method for highly oscillatory ordinary differential equations*, SIAM J. Numer. Anal. 18 (1981), 455-479.

[21] L.R. Petzold, *A description of DASSL: a differential/algebraic system solver*, Proc. 10th IMACS World Congress, August 8-13 Montreal 1982.

[22] S. Reich, *A free energy approach to the torsion dynamics of macromolecules*, ZIB preprint SC 95-17, June, 1995.

[23] S. Schneider, *private communication*, 1994.

[24] H. C. Chen and R. L. Taylor, *Using Lanczos vectors and Ritz vectors for computing dynamic responses*. Eng. Comput. 6 (1989), 151-157.

[25] E. L. Wilson, M. Yuan and J. M. Dickens, *Dynamic analysis by direct superposition of Ritz vectors*, Earthquake Engineering and Structural Dynamics 10 (1982), 813-821.

[26] S. C. Wu, S. M. Yang, E. J. Haug, *Dynamics of mechanical systems with Coulomb friction, stiction, impact and constraint addition-deletion II. Planar systems*. Mech. Mach. Theory 21, 407-416, 1986.

[27] J. Yen and L. Petzold, *On the numerical solution of constrained multibody dynamic systems*. University of Minnesota AHPCRC 94-038, 1994.

[28] J. Yen and L. Petzold, *Convergence of the iterative methods for coordinate-splitting formulation in multibody dynamics*, TR 95-052, Tech. Rept., Dept. of Comput. Sci., University of Minnesota, July, 1995.

[29] W. S. Yoo and E. J. Haug, *Dynamics of articulated structures. Part I. Theory*, J. Struct. Mech., 14 (1986), 105-126.

[30] M. Yuan, P. Chen, S. Xiong, Y. Li and E. Wilson, *The WYD method in large eigenvalue problems*, Eng. Comput. 6 (1989), 49-57.

[31] G. Zhang and T. Schlick, *The Langevin/implicit Euler/normal-mode scheme for molecular dynamics at large time steps*, J. Chem. Phys. 101 (1994), 4995-5012.