MULTIVARIATE LIFE DISTRIBUTIONS INDUCED BY GAMMA PROCESS ENVIRONMENTS

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GWU/IRRA/Serial TR-89/2
1 February 1989

Research Sponsored by

Contact N00014-85-K-0202
Project NR 042-372
Office of Naval Research

and

Grant DAAL03-87-K-0056
U.S. Army Research Office

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Abstract

We develop a class of multivariate life distributions based on the notion that an unknown common environment induces dependencies. The environment is assumed to be dynamic and is described by a gamma process. The multivariate exponential distribution of Marshall and Olkin, motivated via an environment comprising of nonfatal Poisson shock processes wherein the probability of failure associated with each shock is the same, turns out to be a special case of our development. The gamma process environment reduces to a consideration of a Poisson shock model with the shocks having different probabilities of failure.

Key words and phrases. Poisson shock process, system reliability, bivariate exponential, bivariate Weibull, processes with independent increments.

1. Introduction

Consider a two component system which operates in an environment which may or may not be identical to the test bench environment. The component lifelengths, when assessed under the test bench environment, are assumed to have failure rates $\lambda_i(u)$, $i=1, 2$, for $u \geq 0$. The operating environment is supposed to be made up of several stresses whose intensities and presence change over time. It is assumed that the nett effect of the operating environment is to modulate $\lambda_i(u)$ to $\lambda_i(u)\ \eta(u)$, $i=1, 2$. The $\lambda_i(u)$'s may be known or unknown; however, $\eta(u)$ is assumed to be unknown for all $u \geq 0$. If at any time u, the operating environment is judged harsher (gentler) than the test bench environment, then $\eta(u) > (<) 1$; $\eta(u) \equiv 1$ would correspond to the case in which the operating environment and the test bench environments are identical. Given $\lambda_1(u)$, $\lambda_2(u)$ and $\eta(u)$, for all $u \geq 0$, we shall judge the component lifelengths independent. When $\eta(u)$, $u \geq 0$, is unknown, we shall describe our uncertainty about it by a suitable stochastic process $\{\eta(u);\ u \geq 0\}$. Under the above circumstances, there is an induced dependence between the lifetimes of the two components, and the aim of this paper is to describe such lifelengths for a particular choice of $\{\eta(u);\ u \geq 0\}$, namely, that $\{\int_0^t \eta(u)\ du;\ t \geq 0\}$ is a gamma process. In Singpurwalla and Youngren (1989b) we consider the case wherein $\{\eta(u);\ u \geq 0\}$ is described by a "shot-noise process" [cf. Cox and Isham (1980), p. 34]. The case $\lambda_i(u) = \lambda_i$, $i=1, 2$, with $\eta(u) = \eta$, and uncertainty about η described by a gamma distribution, has been considered by Lindley and Singpurwalla (1986), Currit and Singpurwalla (1989) and the references therein. A motivation for describing $\{\int_0^t \eta(u)du;\ t \geq 0\}$ by a gamma process is in Singpurwalla and Youngren (1989a), and in Kalbfleisch and Prentice (1980), p. 203. By way of a brief review, and with the aim of introducing some notation, we have

Definition 1.1. Let $\alpha(t)$ be a nondecreasing left-continuous real valued function on $[0, \infty)$, with $\alpha(0) = 0$, and let $\beta \in (0, \infty)$. A stochastic process $\{x(t);\ t \geq 0\}$ is said to be a *gamma process* with parameters $\alpha(t)$ and β, denoted "$X(t) \in \mathcal{G}(\alpha(t), \beta)$" if

i) $X(0) = 0$

ii) $X(t)$ has independent increments, and

iii) $(X(t) - X(s))$ has a gamma distribution with shape parameter $(\alpha(t) - \alpha(s))$, and scale parameter $1/\beta$, for $0 \leq s \leq t$.

Definition 1.2. [Dykstra and Laud (1981)]. Let $\beta(t)$, $t \geq 0$, be a positive right-continuous real valued function and let $X(t) \in \mathcal{G}(\alpha(t), 1)$. Then the process $\{z(t);\ t \geq 0\}$, with $z(t) \overset{\text{def}}{=} \int_0^t \beta(s)dx(s)$ is called an *extended gamma process* with parameters $\alpha(t)$ and $\beta(t)$, and is denoted "$Z(t) \in \mathcal{G}_E(\alpha(t), \beta(t))$."

A useful property of the extended gamma process is that if $Z(t) \in \mathcal{G}_E(\alpha(t), \beta(t))$, then $G^*_{Z(t)}(w)$, the Laplace Stieltjes transform of the distribution of $Z(t)$, is given as

$$G^*_{Z(t)}(w) = \exp\left[-\int_0^t \ell n\ (1 + w\ \beta(u))\ d\ \alpha(u)\right]; \tag{1.1}$$

see Dykstra and Laud (1981) or Çinlar (1980) [who introduces another generalization of the gamma process and describes an imaginative modeling of the deformation laws of materials].

Let $Y(t) \overset{\text{def}}{=} \int_0^t \eta(u)du$, and suppose that $Y(t) \in \mathcal{G}(\alpha(t), 1/b)$, where $\alpha(t)$ is continuously differentiable with $\frac{d}{dt}\alpha(t) = a(t)$. Then, it follows that $\Lambda_i(t) \overset{\text{def}}{=} \int_0^t \lambda_i(u)\ d\ Y(u)$, the cumulative hazard rate of component i, $i=1, 2$, is such that $\Lambda_i(t) \in \mathcal{G}_E\left[\alpha(t), \frac{\lambda_i(u)}{b}\right]$, and this will be the underlying hypothesis upon which our results will be based.

2. The Bivariate Survival Function and Its Marginals

Let T_1 and T_2 be the lifelengths of the two components, and suppose that $0 \le \tau_1 \le \tau_2 < \infty$. Then, under the hypothesis of the previous section and from the independent increments property of the gamma process, it can be seen that the bivariate survival function of the two components

$$(2.1)\quad P(T_1 > \tau_1, T_2 > \tau_2 | \lambda_1(u), \lambda_2(u), a(u), b) \overset{\text{def}}{=} \overline{F}(\tau_1, \tau_2) =$$

$$\exp\left[-\int_0^{\tau_1} \ell n\left(1 + \frac{\lambda_1(u)+\lambda_2(u)}{b}\right) a(u)du\right] \bullet \exp\left[-\int_{\tau_1}^{\tau_2} \ell n\left(1 + \frac{\lambda_2(u)}{b}\right) a(u)du\right];$$

also, the marginal survival functions $P(T_i > \tau | \lambda_i(u), a(u), b) \overset{\text{def}}{=} \overline{F}_i(\tau) = \exp\left[-\int_0^{\tau} \ell n(1 + \frac{\lambda_i(u)}{b}) a(u)du\right]$, $i=1, 2$.

2.1 The Bivariate Exponential and Weibull as Special Cases

In what follows, we show that one of the most widely discussed multivariate distribution in reliability theory - which has been motivated via considerations which, at least at the surface, appear to be different from ours - is a special case of (2.1). To see this, suppose that $\lambda_i(u) = \lambda_i$, $i=1, 2$, and all $u \ge 0$; then $\Lambda_i(\tau) = \lambda_i Y(\tau) \in \mathcal{G}\left(\alpha(\tau), \frac{\lambda_i}{b}\right)$, from which it can be verified that

$$(2.2)\qquad \overline{F}(\tau_1, \tau_2) = \left(\frac{b}{b+(\lambda_1+\lambda_2)}\right)^{\alpha(\tau_1)} \left(\frac{b}{b+\lambda_2}\right)^{\alpha(\tau_2)-\alpha(\tau_1)}$$

and that

$$(2.3)\qquad \overline{F}_i(\tau) = \left(\frac{b}{b+\lambda_i}\right)^{\alpha(\tau)}.$$

Note that the failure rate of T_i,

$$r_i(\tau) = a(\tau)\ \ell n\left(\frac{b+\lambda_i}{b}\right),\ i=1,\ 2, \tag{2.4}$$

is a constant times the derivative of $\alpha(t)$, the shape parameter of the gamma process. Thus, we may choose different functional forms for $\alpha(t)$, each suggested by a physical scenario of the environment, and obtain different marginal distributions for the component lifelengths. One such choice - perhaps a natural one - is that $\alpha(\tau) = \alpha\tau$, for some constant $\alpha > 0$. For this choice of $\alpha(\tau)$, the marginal distributions are exponential, and the bivariate survival function (2.2) is

$$\left(\frac{b+\lambda_2}{b+\lambda_1+\lambda_2}\right)^{\alpha\tau_1}\left(\frac{b+\lambda_1}{b+\lambda_1+\lambda_2}\right)^{\alpha\tau_2}\left(\frac{b(b+\lambda_1+\lambda_2)}{(b+\lambda_2)(b+\lambda_1)}\right)^{\alpha\tau_2} \tag{2.5}$$

If we set $\lambda_i^* = \alpha\ell n\left(\frac{b+\lambda_1+\lambda_2}{b+\lambda_i}\right)$, $i=1,\ 2$, and $\lambda_{12}^* = \alpha\ell n\left(\frac{(b+\lambda_1)\ (b+\lambda_2)}{b(b+\lambda_1+\lambda_2)}\right)$, then (2.5) becomes,

$$\overline{F}(\tau_1,\tau_2) = \exp\left[-(\lambda_1^*\tau_1 + \lambda_2^*\tau_2 + \lambda_{12}^*\max(\tau_1,\tau_2))\right],\ \text{for } \tau_1,\tau_2 \geq 0; \tag{2.6}$$

this is the bivariate exponential distribution (BVE) of Marshall and Olkin (1967). The marginal distributions corresponding to (2.6), in terms of the parameters α and β, are

$$\overline{F}_i(\tau) = \exp\left[-\alpha\ell n\left(\frac{b+\lambda_i}{b}\right)\tau\right],\ i=1,2,\ \text{and } \tau \geq 0. \tag{2.7}$$

If we let $\alpha(\tau) = \alpha\tau^{\beta}$, for constants α, $\beta > 0$, and proceed as above, then the resulting bivariate survival function is the bivariate Weibull of Marshall and Olkin (1967).

Contrast the above results to those obtained by Lindley and Singpurwalla (1986), who, in effect, describe their uncertainty about $\Lambda_i(t)$, the cumulative rate of component i, by a gamma distribution, and obtain the logistic and the Pareto as their bivariate and univariate survival functions, respectively. The bivariate logistic obtained by the above authors is absolutely continuous, whereas the bivariate exponential, obtained by describing $\Lambda_i(t)$ by a gamma process, has a singular component.

In the next section, we explain as to why a consideration of the gamma leads us to the BVE.

3. The Gamma Process as a Poisson Shock Model with Varying Shock Intensities

The BVE has been motivated [cf. Barlow and Proschan (1975), p. 136] via a consideration of three independent Poisson shock processes, with the shocks within a process being nonfatal and having equal intensities. The latter assumption implies that the probability of failure due to each shock is the same for all the shocks within a process. In what follows, we point out that the gamma process environment boils down to a consideration of a nonfatal Poisson shock process with the shocks having different intensities; that is, each shock induces its own probability of failure.

To ascertain the above, it is useful to recall that a process with independent increments having no Gaussian components and no fixed points of discontinuity - such as a gamma process - can be represented as a sum of a countable number of jumps of random height at a countable number of random points [cf. Ferguson and Klass (1972)]. Furthermore, if for any fixed $\epsilon > 0$, $N(t, \epsilon)$ denotes the number of jumps in $(0, t)$ of magnitude $\geq \epsilon$, then $\{N(t, \epsilon); t \geq 0\}$ is a Poisson process with intensity $(M(\epsilon)$. For the gamma process, $M(\epsilon) = p \int_{\epsilon}^{\infty} u^{-1}e^{-u}\, du$, where p is a parameter of the process; see for example Basawa and Prakasa Rao (1980), p. 106. Let $U(\epsilon)$ denote the magnitude of a jump, given that it is $\geq \epsilon > 0$. Then, for the gamma process, the density of $U(\epsilon)$ is a truncated (at ϵ) gamma density with shape 0.

To invoke the above results for the purpose at hand, we consider jumps of size Δx, and remark that jumps of size X, where $X \geq k\Delta x$, for $k=1, 2, \ldots$, occur in accordance with a Poisson process with intensity $M(k\Delta x)$, and that $U(k\Delta x)$ is a truncated gamma density with shape 0. Thus jumps of size X, where $X \in [k\Delta x, (k+1)\Delta x)$, occur in accordance with a Poisson process with intensity $\Delta M\ (k\Delta x)$, where $\Delta M\ (k\Delta x) = M((k+1)\Delta x) - M(k\Delta x)$, $k=1, 2, \ldots,$. A jump in the cumulative failure rate can be viewed as the result of a discrete shock, the effect of which is to cause the failure of any one, or both components of the system. We would expect that the probability of failure of a component is a function of the size of the shock, and if Δx is sufficiently small, the probability of failure would be approximately the same for all $X \in [k\Delta x, (k+1)\Delta x)$.

Let $P_{01}(k\Delta x)$ $[p_{10}(k\Delta x)]$ be the probability that a shock of size $X \in [k\Delta x, (k+1)\Delta x)$ will cause Component 1 [2] to fail and Component 2[1] to survive. Let $P_{00}(k\Delta x)$ $[p_{11}(k\Delta x)]$ be the probability that a shock will cause both components to fail [survive]. Thus, for shocks of size $X \in [k\Delta x, (k+1)\Delta x)$, $k=1, 2, \ldots$, we have a standard nonfatal shock model with only a common shock applied to both components. Thus, for any $\tau_1, \tau_2 \geq 0$, the probability that the components will survive shocks of magnitude $X \in [k\Delta x, (k+1)\Delta x)$ is,

$$\text{(3.1)} \quad P(T_1 > \tau_1, T_2 > \tau_2 | X \in [k\Delta x, (k+1)\Delta x)) =$$

$$\exp\{-\Delta M(k\Delta x)\, p_{01}(k\Delta x)\tau_1 - \Delta M(k\Delta x)\, P_{10}(k\Delta x)\tau_2 - \Delta M(k\Delta x)\, p_{00}(k\Delta x)$$

$$\max(\tau_1, \tau_2)\}.$$

The probability that the system will survive shocks of *all* magnitudes X is the product of (3.1) over all k. Taking the limit as $\Delta x \downarrow 0$, we have

$$\text{(3.2)} \quad P(T_1 > \tau_1, T_2 > \tau_2) = \exp\Big\{-\int_0^\infty p_{01}(x)dM(x)\,\tau_1 - \int_0^\infty p_{10}(x)dM(x)\tau_2$$

$$- \int_0^\infty p_{00}(x)dM(x)\,\max(\tau_1, \tau_2)\Big\},$$

and this is of the form (2.6).

A comparison of (3.2) with (2.6) enables us to interpret λ_1^*, λ_2^* and λ_{12}^* in terms of the survival probabilities of jumps of magnitude X and their arrival intensities, in a representation of the gamma process.

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