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## NEW FINITE-DIFFERENCE TECHNIQUE FOR SOLUTION OF THE HEAT-CONDUCTION EQUATION, ESPECIALLY NEAR SURFACES WITH CONVECTIVE HEAT TRANSFER



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ABSTRACT

Finite-difference methods have come into wide use for solving special problems including transient-heat conduction. Dusinberre[1] has ably presented the possibilities of finite-difference methods. The success of most such methods depends on the existence of a certain degree of uniformity of behavior of the temperature over the finite intervals of both space and time selected for the computation process. In some cases, however, this required uniformity constitutes a handicap since temperatures are changing so rapidly that inconveniently short time intervals have to be chosen. This paper represents an effort to develop a finite-difference method free from the foregoing defect.

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[1] "Numerical Analysis of Heat Flow," by G. M. Dusinberre, McGraw-Hill Book Company, Inc., New York, N. Y., 1949.

# NEW FINITE-DIFFERENCE TECHNIQUE FOR SOLUTION OF THE HEAT-CONDUCTION EQUATION, ESPECIALLY NEAR SURFACES WITH CONVECTIVE HEAT TRANSFER

By H. G. Elrod, Jr.

## TABLE OF SYMBOLS

$A_j$ temperature influence coefficient defined in eq. 26

$B_j$ temperature influence coefficient defined in eq. 26

$C_j$ temperature influence coefficient defined in eq. 26

$C_1(0,t)$ and $C_2(0,t)$ contributions to the temperature at $x = 0$ from the regions $x > 0$ and $x < 0$, respectively.

$D_j$ temperature influence coefficient defined in eq. 26

$E_j$ temperature influence coefficient defined in eq. 26

$f$ "function of"

$f^r$ "rth derivative of"

$F_n$ heat-conduction function defined by eqs. 43 and 44, $ft^{-n}$

$F_n^*$ heat-conduction function defined by eq. 78

$G_n$ heat-conduction function defined by eq. 52

$\underline{h}$ convective heat-transfer coefficient, Btu/hr ft$^2$ F

$\mathcal{h}$ the ratio, $(\underline{h})/k$, as used in ref. 2, ft$^{-1}$

$H_n$ heat-conduction function defined by eq. 53

$I_j$ integral defined by eqs. 58, 59, and 60, °F.

$k$ thermal conductivity, Btu/hr ft °F

$M$ heat-condution modulus, $(\Delta x)^2/\kappa\Delta t$

$N'$ heat-conduction Nusselt number, $(\underline{h}\,\Delta x)/k$

$t$ time, hr

$T$ temperature, °F

$x$ distance in one-dimensional medium, ft

$x'$ dummy space variable, ft

$\Delta$ "finite difference of"

$\epsilon$ base of natural logarithms, 2.718

$\zeta, \xi$ dummy variables

$\kappa$ thermal diffusivity, $ft^2/hr$

$\tau$ dummy time variable, hr

$\phi$ function describing the ambient temperature, °F

Numerical subscripts applied to temperatures, as in $T_1$, $T_0$, $T_{-1}$, refer to temperatures located at $x = \Delta x$, $x = 0$, and $x = -\Delta x$, etc., at the time $t = 0$.

Positive and negative superscripts applied to temperatures, mean that the temperatures are to be evaluated at the plus and minus sides of the points in question.

## Introduction:

In recent years finite-difference methods have been used to solve a vast number of special problems involving transient heat conduction. In flexibility of use and simplicity of concept these methods excel those of classical mathematical analysis, and, indeed, on many occasions they are not to be regarded as substitutes for more precise methods, but as the only possible methods to use. The possibilities of finite-difference methods for many problems akin to those considered in this paper are well presented in the book by Dusinberre[1].

The success of most finite-difference methods depends on the existence of a certain degree of uniformity of behavior of the temperature over the finite intervals of both space and time selected for the computation process. There are, however, a number of occasions when this requirement of uniformity is a handicap, since temperatures are changing so rapidly that inconveniently-short time intervals have to be chosen. This awkward feature usually arises near the boundary of the computation region; for example, it may arise near the surface of a casting during quenching.

The present investigation represents an effort to develop a finite-difference method free from the foregoing defect. Formulas are derived which do not imply a uniformity of behavior with respect to time. Within the interior of a solid, these formulas reduce, in general, to those obtainable by the simpler technique of heat balances. However, near a convective heat-transfer surface they do not reduce to earlier formulas. In this region they possess greater potentiality, in that they will handle with uniform precision cases of variable, and even discontinuous, ambient temperature, with the heat-transfer coefficient ranging from zero (insulation) to infinity (perfect contact.)

In actual practice, the new formulas merely introduce new weighting factors into the standard finite-difference equations. A numerical table of such weighting factors is given for the temperatures on, and adjacent to, a convective heat-transfer surface when the space and time intervals are chosen to conform with the Binder-Schmidt selection of $M = (\Delta x)^2/\kappa\Delta t = 2$.

Derivation of Formulas for the Infinite Medium:

Formulas will here be derived which are appropriate for use in computing the transient conduction of heat in an infinite medium of uniform, constant properties. The differential equation to be applied is as follows:

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \qquad (1)$$

This is a linear differential equation, and the response at some time "t" is linearly related to the temperature distribution input "f(x)" at time zero by the solution given below[2].

$$T(x,t) = \frac{1}{2\sqrt{\pi \kappa t}} \int_{-\infty}^{+\infty} f(x') \, \epsilon^{-\frac{(x-x')^2}{4\kappa t}} \, dx' \qquad (2)$$

At the point x=0 this last equation reduces to:

$$T(0,t) = \frac{1}{2\sqrt{\pi \kappa t}} \int_{-\infty}^{+\infty} f(x') \, \epsilon^{-\frac{x'^2}{4\kappa t}} \, dx' \qquad (3)$$

Now let the contribution to T(0,t) originating within the region $x > 0$ be denoted by $C_1(0,t)$. This contribution can be written in the following manners:

$$C_1(0,t) = \frac{1}{2\sqrt{\pi \kappa t}} \int_{0}^{\infty} f(x') \, \epsilon^{-\frac{x'^2}{4\kappa t}} \, dx' \qquad (4)$$

Or:

$$C_1(0,t) = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} f(2\xi\sqrt{\kappa t}) \, \epsilon^{-\xi^2} \, d\xi \qquad (5)$$

The integral in eq. 5 can be successively integrated by parts to give:

$$C_1(0,t) = \frac{1}{2}\left[\sum_{r=0}^{n} f^{r}(0)\,(2\sqrt{\kappa t})^{r}\, i^{r}erfc(0) + \int_{0}^{\infty} (2\sqrt{\kappa t})^{n+1} f^{n+1}(2\xi\sqrt{\kappa t})\; i^{n}erfc(\xi)\, d\xi\right] \qquad (6)$$

where the $i^n erfc(\xi)$ are the iterated error functions defined by:

$$i^{-1}erfc(\xi) \equiv \frac{2}{\sqrt{\pi}} \epsilon^{-\xi^2} \qquad (7)$$

and:

$$i^{n}erfc(\xi) \equiv \int_{\xi}^{\infty} i^{n-1}erfc(\zeta)\, d\zeta \qquad (8)$$

The presumption in writing eq. 6 is that the first "n" derivatives of f(x') are continuous on the plus side of x'=0. These derivatives may, or may not, be continuous <u>through</u> x'= 0. If they are continuous, then when the contribution

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[2] "Conduction of Heat in Solids," by Carslaw and Jaeger, Oxford University Press, 1947.

$C_2(0,t)$ is added to $C_1(0,t)$, all odd temperature derivatives cancel, leaving the following result for $T(0,t)$.

$$T(0,t)=\sum_{r=0}^{n} f^{2r}(0)(4\kappa t)^r i^{2r}\mathrm{erfc}(0) + \frac{(2\sqrt{\kappa t})^{2n+1}}{2}\int_0^\infty\left\{f^{2n+1}(2\xi\sqrt{\kappa t})-f^{2n+1}(2\xi\sqrt{\kappa t})\right\} i^{2n}\mathrm{erfc}(\xi)\,d\xi \tag{9}$$

When the last integral in eq. 9, which is the error term, is neglected, and when use is made of the fact that:

$$i^{2r}\mathrm{erfc}\,(0) = \frac{1}{4^r(r!)} \tag{10}$$

eq. 9 then reduces to:

$$T(0,t) = \sum_{r=0}^{n} \frac{(\kappa t)^r f^{2r}(0)}{r!} \tag{11}$$

Thus when the temperature distribution in an infinite medium conforms at t=0 with some polynomial in "x", eq. 11 gives the exact answer for all subsequent time. In terms of the modulus "M" this equation becomes:

$$T(0,t) = \sum_{r=0}^{n} M^{-r}\left[(\Delta x)^{2r}\frac{f^{2r}(0)}{r!}\right] \tag{12}$$

where:

$$t = t-0 = \Delta t \tag{13}$$

Equation 13 is still not a convenient expression to use numerically. An improvement can be made by evaluating the derivatives in this expression in terms of discretely-spaced temperatures with the uniform interval, $\Delta x$. Thus, for a second-degree polynomial:

$$\begin{aligned}(\Delta x)^0 f^0(0) &= T_0\\ (\Delta x)^2 f^2(0) &= T_1 - 2T_0 + T_{-1}\end{aligned} \tag{14}$$

For higher-degree polynomials, the second and higher derivatives take on more complicated, but similar, form. Substitution of eqs. 14 into eq. 12 gives the standard finite-difference form, widely used in heat-conduction studies. Thus:

$$T(0,\Delta t) = T_0 + \frac{1}{M}\left(T_1 - 2T_0 + T_{-1}\right) \tag{15}$$

<u>The Special Case of M = π</u>:

Equation 15 can be deduced very much more quickly by direct use of heat balances. In this case the ability of the present, more elaborate analysis to accomodate non-uniform time behavior is not made evident. To bring this ability into evidence, let it be supposed now that at time zero neither the temperature nor its derivatives is continuous through $x=0$. Such a situation can arise physically when two plates of similar material, but dissimilar temperatures, are suddenly brought into good thermal contact. It obviously includes as a special case the more usual situation analyzed above.

In the present, more general case, $C_1(0,t)$ and $C_2(0,t)$ must be evaluated separately. Now let it be assumed that the temperature distribution at $t=0$ for $x>0$ can be well represented by a second-degree polynomial. Furthermore, let the temperatures on the plus side of various stations be identified with "plus" superscripts, and the temperatures on the minus side with "minus" superscripts. Then:

$$\begin{aligned} f^{0}(+0) &= T_0^{+} \\ (\Delta x) f'(+0) &= \tfrac{1}{2}\left(4T_0^{+} - 3T_1 - T_2\right) \\ (\Delta x)^2 f^{2}(+0) &= T_2 - 2T_1 + T_0^{+} \end{aligned} \qquad (16)$$

Since $\text{ierfc}(0) = 1/\sqrt{\pi}$, use of eq. 16 in eq. 6 gives:

$$C_1(0,t) = \frac{1}{2}\left[T_0^{+}\left(1 - \frac{3}{\sqrt{M\pi}} + \frac{1}{M}\right) + T_1\left(\frac{4}{\sqrt{M\pi}} - \frac{2}{M}\right) + T_2\left(\frac{1}{M} - \frac{1}{\sqrt{M\pi}}\right)\right] \qquad (17)$$

This last expression is valid <u>regardless</u> of the temperature distribution for $x<0$; that is, regardless of the magnitude of the time derivatives induced by discontinuities at $x=0$.

Inspection of eq. 17 shows that a rather remarkable simplification occurs if $M = \pi$. In this event, only the temperatures $T_0^{+}$ and $T_1$ need to be known in order to obtain exact results for an initial quadratic temperature profile. If the profile for $x<0$ is also quadratic, though different, addition of the contributions $C_1$ and $C_2$ leads again to eq. 15, provided the temperature $T_0$ in that equation be interpreted as:

$$T_o \equiv \frac{1}{2}\left(T_o^+ + T_o^-\right) \qquad (18)$$

Thus the new method of deducing the finite-difference equations has brought out the unique property of $M = \pi$;

namely, that it can accomodate space discontinuities in the temperature and its derivatives, if these discontinuities occur at a central grid point.

Although the interpretation of $T_0$ according to eq. 18 makes eq. 15 for $M = \pi$ highly accurate when a temperature-jump occurs at the central grid point, there remains the problem of how to weight the temperatures at station 1, say, if a temperature jump occurs there, instead. As before, let the temperature profiles to the right and left of station 1 be quadratic, though different. Then $C_2(0,t)$ can immediately be written as:

$$C_2(0,t) = \frac{1}{2}\left[T_o\left(1-\frac{2}{\pi}\right) + \frac{2}{\pi}T_{-1}\right] \qquad (19)$$

On the other hand, $C_1(0,t)$ requires special treatment.

Let $f^*(x')$ be the smooth, or analytic, continuation into the region $x'>\Delta x$ of the actual temperature profile existing in the region $x'<\Delta x$. Then $C_1(0,t)$ can be written as:

$$C_1(0,t) = \frac{1}{2\sqrt{\pi\kappa t}}\left\{\int_0^{\Delta x} f(x')\,\epsilon^{-\frac{x'^2}{4\kappa t}}\,dx' + \int_{\Delta x}^{\infty} f^*(x')\,\epsilon^{-\frac{x'^2}{4\kappa t}}\,dx' + \right\}$$

$$+ \frac{1}{2\sqrt{\pi\kappa t}}\int_{\Delta x}^{\infty}\left\{f(x') - f^*(x')\right\}\epsilon^{-\frac{x'^2}{4\kappa t}}\,dx' \qquad (20)$$

The first two integrals in this last equation can be summed for $M = \pi$ to give: $\frac{1}{2}\left[T_o - \frac{2}{\pi}T_o + \frac{2}{\pi}T_1^-\right]$

The third integral might be integrated by parts, as on earlier occasions, to yield a series. However, since such a procedure would complicate any formula by introducing temperatures beyond $T_1^+$, it is not followed. Instead, the functional difference $f(x') - f^*(x')$ is treated as stationary compared with the fast-attenuating exponential, and the last integral is approximated by the following expression:

$$(T_1^+ - T_1^-)\frac{1}{2}\,\mathrm{erfc}\left(\frac{\sqrt{\pi}}{2}\right) = 0.1053\,(T_1^+ - T_1^-) \qquad (21)$$

Combination of the foregoing results gives the complete expression for $C_1$; i. e.,

$$C_1(0,t) = \frac{1}{2}\left[T_0\left(1-\frac{2}{\pi}\right) + \frac{2}{\pi}T_1^-\right] + 0.1053\,(T_1^+ - T_1^-) \qquad (22)$$

Since $0.1053\,\pi = 0.331 = 1/3$, the sum of $C_1+C_2$ can be written with high accuracy as:

$$T(0,\Delta t) = T_0 + \frac{T_{-1} - 2T_0 + \frac{2T_1^- + T_1^+}{3}}{\pi} \qquad (23)$$

Comparison of eq. 23 with eq. 15 shows that the standard form of eq. 15 can be retained if, when predicting $T(0,\Delta t)$ with a temperature jump at station 1 ($x = \Delta x$), the "inside" temperature at the jump is weighted twice as heavily as the "outside" temperature. Or, in other words, $T_1$ in eq. 15 is to be interpreted as:

$$T_1 = \frac{2\,T_1^- + T_1^+}{3} \qquad (24a)$$

Put alternatively, the following extended form of eq. 15 yields excellent accuracy when $M = \pi$ and when the temperature profile can be represented by second-degree curves in the intervals:

$$x < -\Delta x\ ;\ -\Delta x < x < 0\ ;\ 0 < x < \Delta x\ ;\ x > \Delta x$$

Thus:

$$T(0,\Delta t) = \frac{1}{2}(T_0^+ + T_0^-) + \left\{\frac{T_{-1}^- + 2T_{-1}^+}{3} + \frac{2T_1^- + T_1^+}{3} - (T_0^+ + T_0^-)\right\}\frac{1}{\pi} \qquad (24b)$$

To illustrate the practical use of the foregoing rules, let us consider the following example. "A semi-infinite slab

$(x>0)$ at a temperature of 100 degrees is suddenly brought into perfect thermal contact with a second semi-infinite slab $(x<0)$ at a temperature of -100 degrees. The temperature distribution for all time for $x>0$ is desired."

The above example is solved in ref. 1 (p. 121) for $M = 2, 3$ and 4. Table I of this paper shows the numerical results obtained when $M = \pi$. For the chosen mathematical model, exact values are shown in parentheses. At no tabulated point does the absolute error of the finite-difference method exceed 3.2%.

TABLE I

| | $-\Delta x$ | 0 | $\Delta x$ | $2\Delta x$ | $3\Delta x$ | $4\Delta x$ |
|---|---|---|---|---|---|---|
| 0 | -100 | | 100 | 100 | 100 | 100 |
| $\Delta t$ | | 0 | 78.7<br>(79.0) | 100<br>(98.8) | 100 | 100 |
| $2\Delta t$ | | 0 | 60.5<br>(62.4) | 93.3<br>(92.3) | 100 | 100 |
| $3\Delta t$ | | 0 | 51.7<br>(53.0) | 85.0<br>(85.2) | 97.9<br>(97.0) | 100 |

It is interesting to note that the temperature at $x=0$ is, for the purposes of computing future temperatures at the same point, taken as zero right from the start. However, for purposes of computing future temperatures at station 1, the initial temperature at $x=0$ is, by eq. 24, to be taken as:

$$\frac{2 \times 100 + (-100)}{3} = 33.3 \text{ deg.}$$

The major source of improvement of the present computational accuracy over the cited examples in ref. 1 lies in the treatment of temperatures at a point of discontinuity.

If automatic computing machinery is to be used for the finite-difference computations, the selection of $M = \pi$ should introduce negligible inconvenience. However, for hand computation the use of M = 3, a value quite close to $\pi$, would appear to be preferable because of the simpler arithmetical manipulations required. The rules given by eqs. 18 and 24 should be retained. In the foregoing example, use of M = 3 with these rules increases the maximum error by only 0.3%.

## Formulas for the Neighborhood of a Convective Heat-Transfer Surface:

The formulas obtained in the previous section are valid for the infinite medium, or for finite regions which can be mimicked by an infinite medium through the use of superposition of symmetric and anti-symmetric temperature distributions. In the general case of heat convection from a surface, however, the heat transfer coefficient is usually neither so small that heat transfer can be neglected, nor so large that perfect thermal contact can be assumed. This general case does not, unfortunately, lend itself readily to the superposition technique, and special formulas are required. Suitable formulas of high accuracy will be given in this section. Their detailed derivation is given in the Appendix, and, because no new principles are involved, it will suffice here to summarize and illustrate the results.

The short-term behavior of all slabs of finite thickness is, with respect to changes at their surfaces, like that of corresponding semi-infinite slabs. Accordingly, the results obtained for the semi-infinite slab whose surface is exposed to convective heat-transfer, can also be used for the finite slab, so long as $\Delta t$ for the time interval of computation is not too large. Consider, therefore, a semi-infinite solid medium having uniform, constant properties. Within the solid the temperatures must obey Fourier's equation (eq. 1). At the surface of the solid, cooling takes place according to Newton's "Law" of Cooling; i. e.,

$$\underline{h}(T_0 - T_a) = k\left(\frac{\partial T}{\partial x}\right)_{x=0} \tag{25}$$

Now let the temperature within the solid at t = 0 be expressible as a second-degree polynomial in "x" for all "x" beyond the surface. Further, let the ambient temperature between t = +0 and t = $\Delta$t - 0 be a linear function of time. Then because of the linearity of the governing differential equation and its boundary conditions, the temperatures $T(0,\Delta t)$ and $T(\Delta x, \Delta t)$ can be linearly expressed in terms of $T_0^+$, $T_1$, $T_2$, $T_a(+0)$ and $T_a(\Delta t - 0)$. The weighting factors for the various temperatures are arrived at in a manner similar to that used for the case of the infinite medium. The results are:

$$T(0,\Delta t) = A_0T_0^+ + B_0T_1 + C_0T_2 + D_0T_a(+0) + E_0T_a(\Delta t - 0) \tag{26}$$

$$T(\Delta x,\Delta t) = A_1T_0^+ + B_1T_1 + C_1T_2 + D_1T_a(+0) + E_1T_a(\Delta t - 0) \tag{27}$$

The coefficients defined by eqs. 26 and 27 are given by the following formulas:

$$A_j = -N\,F_1^* - \frac{3}{2}F_1^* - N\,F_3^* + \frac{1}{M} \qquad j = 1,2 \tag{28}$$

$$B_j = 2F_1^* - \frac{2}{M} + 2N\,F_3^* \qquad '' \tag{29}$$

$$C_j = -\frac{F_1^*}{2} + \frac{1}{M} - N\,F_3^* \qquad '' \tag{30}$$

$$D_j = N\,F_1^* - NM\,F_3^* \qquad '' \tag{31}$$

$$E_j = NM\,F_3^* \qquad '' \tag{32}$$

In these equations "N" is a Nusselt number defined by:

$$N = \underline{h}(\Delta x)/k \tag{33}$$

The various F's are dependent on the choice of "j", and are to be calculated by the following relations:

$$F_0^* = \epsilon^{\frac{-j^2 M}{4}} \left( \epsilon^{\xi^2} \operatorname{erfc} \xi \right) \tag{34}$$

where:

$$\xi = \frac{1}{2} j \sqrt{M} + \frac{N}{\sqrt{M}} \tag{35}$$

$$F_1^* = \frac{1}{N} \left[ \operatorname{erfc}\left( \frac{j\sqrt{M}}{2} \right) - F_0^* \right] \tag{36}$$

$$F_2^* = \frac{1}{N} \left[ \frac{2}{\sqrt{M}} \operatorname{ierfc}\left( \frac{j\sqrt{M}}{2} \right) - F_1^* \right] \tag{37}$$

$$F_3^* = \frac{1}{N} \left[ \frac{4}{M} i^2\operatorname{erfc}\left( \frac{j\sqrt{M}}{2} \right) - F_2^* \right] \tag{38}$$

The coefficients in eq. 26 and 27 are functions of M, N and the position parameter, j. Their calculation involves a fair amount of numerical work. However, for specific and widely-used values of M, tables of these coefficients can be prepared for universal use. One such table, for M = 2, is given in the next section. (Table II) It occupies little space, yet is suitable for linear interpolation over the entire range of possible values of the heat transfer coefficient. When such a table is available, use of the new coefficients is very straightforward. Questions of stability do not arise, and discontinuities of temperature in both space and time at the surface are handled automatically.

For grid points more than distance $\Delta$x from the surface, the standard finite-difference formula appropriate to the chosen M should be used. (See eq. 15). At a sacrifice of accuracy, this formula can also be used to compute the temperature history at $x = \Delta x$.

The Special Case of M = 2:

To illustrate the capabilities of the new finite-difference formulas, a table for M = 2 will now be given, and used to solve typical problems. Table II gives the needed coefficients. Equations 26 and 27 are repeated, making use of the table nearly self-explanatory. The argument to be used in entering the table is $1/(N+1)$, which is the ratio of the surface resistance to the sum of the surface resistance plus the resistance of a slab of thickness, $\Delta x$. A simple check which all such tables must satisfy is that the sum of the coefficients for any given "N" must be unity. That this statement must be true can be seen from the fact that if all temperatures within the solid are the same as the constant ambient temperature, the temperatures at $x = 0$ and $x = \Delta x$ must be the same at the end of time $\Delta t$ as at the beginning. When the ambient temperature is constant, very often it can be used as the datum temperature (i. e., taken as zero), and in this event the coefficients $D_j$ and $E_j$ do not enter the computation.

Consider the following problem to illustrate the use of the foregoing table. "A semi-infinite medium of uniform, constant properties is everywhere at a temperature of 1000 degrees at time zero. At this time convective cooling commences at its exposed surface to an ambient temperature of 0 degrees. The thermal properties of the medium are known, as well as the value of the surface heat-transfer coefficient. Find the temperature history within the slab."

This problem is solved by the new technique by selecting first a size of space interval suitable for sampling the tem-

## TABLE II

### BOUNDARY INFLUENCE COEFFICIENTS FOR M = 2

$$T(0,\Delta t) = A_oT_o^+ + B_oT_1 + C_oT_2 + D_oT_a(+0) + E_oT_a(\Delta t - 0)$$

| $\frac{1}{N+1}$ | $A_o$ | $B_o$ | $C_o$ | $D_o$ | $E_o$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 1 |
| 0.1 | 0.0129 | 0.0480 | 0.0267 | 0.0672 | 0.8452 |
| 0.2 | 0.0334 | 0.1080 | 0.0474 | 0.1087 | 0.7025 |
| 0.3 | 0.0609 | 0.1749 | 0.0628 | 0.1276 | 0.5738 |
| 0.4 | 0.0935 | 0.2438 | 0.0743 | 0.1292 | 0.4592 |
| 0.5 | 0.1290 | 0.3116 | 0.0826 | 0.1189 | 0.3579 |
| 0.6 | 0.1654 | 0.3765 | 0.0888 | 0.1009 | 0.2684 |
| 0.7 | 0.2017 | 0.4375 | 0.0933 | 0.0783 | 0.1892 |
| 0.8 | 0.2370 | 0.4943 | 0.0967 | 0.0531 | 0.1189 |
| 0.9 | 0.2709 | 0.5471 | 0.0991 | 0.0266 | 0.0563 |
| 1.0 | 0.3032 | 0.5957 | 0.1011 | 0 | 0 |

| $\frac{1}{N+1}$ | $A_1$ | $B_1$ | $C_1$ | $D_1$ | $E_1$ |
|---|---|---|---|---|---|
| 0 | 0.1074 | 0.1507 | 0.4246 | 0.1666 | 0.1507 |
| 0.1 | 0.1256 | 0.1802 | 0.4248 | 0.1491 | 0.1203 |
| 0.2 | 0.1443 | 0.2074 | 0.4243 | 0.1286 | 0.0954 |
| 0.3 | 0.1624 | 0.2315 | 0.4234 | 0.1077 | 0.0750 |
| 0.4 | 0.1792 | 0.2527 | 0.4222 | 0.0877 | 0.0582 |
| 0.5 | 0.1944 | 0.2710 | 0.4212 | 0.0692 | 0.0442 |
| 0.6 | 0.2081 | 0.2869 | 0.4201 | 0.0524 | 0.0325 |
| 0.7 | 0.2204 | 0.3008 | 0.4191 | 0.0372 | 0.0225 |
| 0.8 | 0.2314 | 0.3130 | 0.4182 | 0.0235 | 0.0139 |
| 0.9 | 0.2414 | 0.3234 | 0.4176 | 0.0114 | 0.0062 |
| 1.0 | 0.2500 | 0.3333 | 0.4167 | 0 | 0 |

perature distribution in the regions of interest. The time interval must then be chosen so that $M = 2$. Also, from the space-interval selection, the surface Nusselt Number, $N$, can be calculated. This last parameter determines the coefficients which are read from Table II. In the present case, suppose $N = 1/2$. Table III gives computed results for six time intervals Certain exact results are given in parentheses to provide a gauge of the computational accuracy. At no point of comparison does the error of the finite-difference process exceed 0.7%.

TABLE III

| | $T_a$ | 0 | $\Delta x$ | $2\Delta x$ | $3\Delta x$ | $4\Delta x$ | $5\Delta x$ | $6\Delta x$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1000 | 1000 | 1000 | 1000 | 1000 | 1000 | 1000 |
| $\Delta t$ | 0 | 699.2 (699.2) | 932.4 | 1000 | 1000 | 1000 | 1000 | 1000 |
| $2\Delta t$ | 0 | 613.8 (615.7) | 847.1 | 966.2 | 1000 | 1000 | 1000 | 1000 |
| $3\Delta t$ | 0 | 558.9 (562.6) | 789.2 | 923.6 | 983.1 | 1000 | 1000 | 1000 |
| $4\Delta t$ | 0 | 520.4 (523.2) | 742.3 | 886.2 | 961.8 | 991.6 | 1000 | 1000 |
| $5\Delta t$ | 0 | 490.1 (492.6) | 704.4 | 852.0 | 938.9 | 980.9 | 995.8 | 1000 |
| $6\Delta t$ | 0 | 465.3 (467.2) | 672.2 (674.7) | 821.7 (822.9) | 916.4 | 967.4 | 990.5 | 997.9 |

To assess the worth of the present adaptation of the method of finite-differences, one must compare it with alternatives. For example, as shown in ref. 1, p. 129, a heat-balance at the surface, made on the assumption of constant temperature gradients throughout one time interval, gives coefficients equivalent in application to the present $A_0$, $B_0$ and $D_0$. The formula is as follows:

$$T(0,\Delta t) = \frac{2N}{M} T_a + \frac{2}{M} T_1 + \left[1 - \frac{2}{M}(N+1)\right] T_0 \qquad (39)$$

It is to be used in conjunction with eq. 15 for <u>all</u> interior points. Dusinberre[1] shows that stability in the numerical calculations requires that M be greater than 2(N+1) in eq. 39, and greater than two in eq. 15. (Thus M=2 used in the above example is at the limit of stability of eq. 15 and beyond the limit of stability of eq. 39.) When N = 1/2, M = 4 meets the foregoing stability criteria, and this value of the modulus was used with eqs. 15 and 39 to solve the example problem. Retention of the same space interval meant the use of twice as many time intervals to achieve the same real time; that is, twice as many computation points were required. The error in this alternative calculation was almost uniformly twice as great as in the calculation tabulated in Table III.

To illustrate further the use of the table of coefficients, two other problems will be solved for the first few time intervals. As a first illustration, suppose that the heat-transfer coefficient in the problem just solved were essentially infinite. Then the solution of the problem would start as shown in Table IV.

Table IV

| | $T_a$ | +0 | $\Delta x$ | $2\Delta x$ | $3\Delta x$ |
|---|---|---|---|---|---|
| 0 | 0 | 1000 | 1000 | 1000 | 1000 |
| $\Delta t$ | 0 | 0 | 683 | 1000 | 1000 |
| $2\Delta t$ | 0 | 0 | 528 | 842 | 1000 |

Table V

| | $T_a$ | +0 | $\Delta x$ | $2\Delta x$ | $3\Delta x$ |
|---|---|---|---|---|---|
| 0 | no influence | 100 | 200 | 300 | 400 |
| $\Delta t$ | | 180 | 217 | 300 | 400 |
| $2\Delta t$ | | 214 | 242 | 308 | 400 |

As a second example, consider a semi-infinite slab having uniform thermal properties. Let the exposed surface be insulated, and let the temperature vary linearly with distance from the exposed surface. The calculations begin as in Table V.

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## APPENDIX

In this Appendix detailed derivations are given for the functions and formulas useful in calculating heat conduction in a solid near a convective heat-transfer surface.

### PROPERTIES OF THE FUNCTIONS $F_n(x, t, h)$ and $G_n(x)$

#### The Iterated Error Functions:

The Iterated Error Functions are defined by eqs. 7 and 8. They are tabulated in refs. 2. From eq. 7 differentiation gives:

$$\frac{d}{dx}\, i^n \operatorname{erfc}(x) = -\, i^{n-1} \operatorname{erfc}(x) \qquad (40)$$

This last relation can be used to give meaning to iterated functions with indices less than minus one (-1). Thus:

$$i^{-2} \operatorname{erfc}(x) = \frac{4x}{\sqrt{\pi}}\, \epsilon^{-x^2} \qquad (41)$$

The recursion equation satisfied by these functions is[2]:

$$2n\, i^n \operatorname{erfc}(x) = i^{n-2} \operatorname{erfc}(x) - 2x\, i^{n-1} \operatorname{erfc}(x) \qquad (42)$$

Equations 7 and 8 can be used to establish the validity of eq. 42 for all "n".

#### Definition of $F_n(x,t,h)$:[2]

The functions "$F_n$" are defined by the recursion equation:

$$F_{n+1} = h^{-1}\left[(2\sqrt{\kappa t})^n i^n erfc \frac{x}{2\sqrt{\kappa t}} - F_n\right] \tag{43}$$

with:

$$F_0 \equiv \epsilon^{hx+\kappa t h^2} erfc\left\{\frac{x}{2\sqrt{\kappa t}} + h\sqrt{\kappa t}\right\} \tag{44}$$

Since both $F_0$ and all the $(2\sqrt{\kappa t})^n i^n erfc\left(\frac{x}{2\sqrt{\kappa t}}\right)$ satisfy the heat-conduction equation, it follows that:

$$\frac{\partial^2 F_n}{\partial x^2} = \frac{1}{\kappa}\frac{\partial F_n}{\partial t} \tag{45}$$

Now suppose that, for a moment:

$$\frac{\partial F_{n+1}}{\partial x} = -F_n \tag{46}$$

then:

$$-F_n = h^{-1}\left[-(2\sqrt{\kappa t})^{n-1} i^{n-1} erfc\left(\frac{x}{2\sqrt{\kappa t}}\right) - \frac{\partial F_n}{\partial x}\right] \tag{47}$$

But also: $$-F_n = h^{-1}\left[-(2\sqrt{\kappa t})^{n-1} i^{n-1} erfc\left(\frac{x}{2\sqrt{\kappa t}}\right) + F_{n-1}\right] \tag{48}$$

Therefore:

$$\frac{\partial F_n}{\partial x} = -F_{n-1} \tag{49}$$

Thus eq. 49 is true if eq. 46 is true. But likewise, if eq. 49 is true, eq. 46 can be proved to be true. Finally, eq. 46 can be directly verified for n = 0. Hence, by induction, eq. 46 is true in general.

Also, through the use of eqs. 45 and 46, a second useful relation can be found. Thus:

$$\frac{\partial^2 F_{n+1}}{\partial x^2} = -\frac{\partial F_n}{\partial x} = F_{n-1} = \frac{1}{\kappa}\frac{\partial F_{n+1}}{\partial t} \tag{50}$$

Or:

$$F_{n-1} = \frac{1}{\kappa}\frac{\partial F_{n+1}}{\partial t} \quad \text{for all "n"} \tag{51}$$

<u>Recursion Formula for the G Functions</u>:

A second type of function appears in the heat-conduction formulas to be developed. It is defined by:

$$G_n(x) \equiv i^n erfc\,(x) + i^n erfc\,(-x) \tag{52}$$

A complementary set of functions is defined by:

$$H_n(x) \equiv i^n erfc\,(x) - i^n erfc\,(-x) \tag{53}$$

The recursion equation for the Iterated Error Functions easily gives:

$$2n\,G_n - G_{n-2} = -2x\,H_{n-1} \tag{54}$$

and:

$$2n\,H_n - H_{n-2} = -2x\,G_{n-1} \tag{55}$$

When values of "$H_n$" from eq. 54 are substituted into eq. 55, the following recursion equation is obtained for the "$G_n$".

Thus:

$$G_{n+1} = \frac{(2x^2+2n-1)G_{n-1} - \frac{1}{2}G_{n-3}}{2n(n+1)} \tag{56}$$

The first few functions are given below.

$$G_{-2} = 0 \qquad G_0 = 2 \qquad G_1 = 2x + 2\,i\,erfc\,(x)$$

$$G_{-1} = \frac{4}{\sqrt{\pi}}\,\epsilon^{-x^2} \qquad G_2 = \frac{1}{2}(1+2x^2)$$

RESPONSE NEAR SOLID SURFACE TO VARIABLE AMBIENT TEMPERATURE

Analytical Solution:

The following problem is solved in ref. 2, p. 297. "The region $x>0$. Initial temperature $f(x)$. Radiation (Newtonian cooling) at the surface into medium at $\phi(t)$."

$$T = I_1 + I_2 + I_3 \tag{57}$$

where:

$$I_1 = \int_0^\infty \frac{1}{2\sqrt{\pi\kappa t}}\left[\epsilon^{-\frac{(x-x')^2}{4\kappa t}} + \epsilon^{-\frac{(x+x')^2}{4\kappa t}}\right] f(x')\,dx' \tag{58}$$

$$I_2 = -h\int_0^\infty \epsilon^{\kappa t h^2 + h(x+x')}\,erfc\left\{\frac{x+x'}{2\sqrt{\kappa t}} + h\sqrt{\kappa t}\right\} f(x')\,dx' \tag{59}$$

and:

$$I_3 = \int_0^t \left[\frac{\epsilon^{\frac{-x^2}{4\kappa(t-\tau)}}}{\sqrt{\pi\kappa(t-\tau)}} - h\,\epsilon^{\kappa h^2(t-\tau)+hx}\,erfc\left\{\frac{x}{2\sqrt{\kappa(t-\tau)}} + h\sqrt{\kappa(t-\tau)}\right\}\right]\phi(\tau)\,d\tau \tag{60}$$

Each of the above three integrals will now be expressed in terms of the functions $F_n$ and $G_n$.

Evaluation of $I_1$:

The first of these integrals is handled in precisely the same manner as was used to evaluate $T(0,t)$ in eqs. 3 to 9. The result is:

$$I_1 = \frac{1}{2}\sum_{r=0}^{r=n} f^{r}(0)(2\sqrt{\kappa t})^{r}\,G_r\!\left(\frac{x}{2\sqrt{\kappa t}}\right) + \frac{1}{2}(2\sqrt{\kappa t})^{n}\int_0^\infty f^{n+1}(x')\left[\begin{matrix} i^n erfc\,\frac{x'-x}{2\sqrt{\kappa t}} \\ +i^n erfc\,\frac{x'+x}{2\sqrt{\kappa t}}\end{matrix}\right]dx' \tag{61}$$

Evaluation of $I_2$:

The second integral can be rewritten as:

$$I_2 = -h \int_0^\infty F_0(x+x', t, h)\, f(x')\, dx' \tag{62}$$

$$= h \int_0^\infty f(x')\, (\partial F_1/\partial x')\, dx' \tag{63}$$

$$= h \left[ -f(0) F_1(x,t,h) - \int_0^\infty F_1\, f'(x')\, dx' \right] \text{ etc.} \tag{64}$$

Or:

$$I_2 = -h \left[ \sum_{r=0}^{r=n} f^r(0)\, F_{r+1}(x,t,h) + \int_0^\infty f^{n+1}(x')\, F_{n+1}\, dx' \right] \tag{65}$$

Evaluation of $I_3$:

The third integral can also be expressed simply in terms of the "F" functions. Thus:

$$I_3 = \kappa h \int_0^t F_1(x, t-\tau, h)\, \phi(\tau)\, d\tau \tag{66}$$

Or, with the use of eq. 51, one obtains:

$$I_3 = h \int_0^t \frac{\partial F_1}{\partial t} \phi(\tau)\, d\tau = -h \int_0^t \frac{\partial F_1}{\partial \tau} \phi(\tau)\, d\tau \tag{67}$$

$$= h \left[ -F_1 \phi(\tau) \right]_0^t + h \int_0^t F_1\, \phi'(\tau)\, d\tau \tag{68}$$

$$= h F_1(x,t,h)\, \phi(0) - \frac{h}{\kappa} \int_0^t \frac{\partial F_2}{\partial \tau} \phi'(\tau)\, d\tau \tag{69}$$

$$= h \left[ F_1 \phi(0) + \frac{1}{\kappa} F_3 \phi'(0) \right] + \frac{h}{\kappa} \int_0^t F_3\, \phi''(\tau)\, d\tau \tag{70}$$

The general result is:

$$I_3 = h \sum_{r=0}^{n} F_{2r+1}(x,t,h) \frac{\phi^r(0)}{\kappa^r} + \frac{h}{\kappa^n} \int_0^t F_{2n+1}\, \phi^{n+1}(\tau)\, d\tau \tag{71}$$

DETERMINATION OF THE TEMPERATURE COEFFICIENTS

The final result for T(x,t) is, exclusive of error terms:

$$T(x,t) = \frac{1}{2} \sum_{r=0}^{n} f^r(0)\, (2\sqrt{\kappa t})^r\, G_r\!\left(\frac{x}{2\sqrt{\kappa t}}\right) - h \sum_{r=0}^{n} f^r(0)\, F_{r+1}(x,t,h) + h \sum_{r=0}^{n} \frac{\phi^r(0)}{\kappa^r} F_{2r+1}(x,t,h) \tag{72}$$

The derivatives appearing in eq. 72 can be expressed in terms of finite differences[3]. If, at time zero, the space distribution of temperature can be expressed by a second-degree polynomial in "x", and the ambient temperature as a linear function

[3] "Numerical Calculus," by W. E. Milne, Princeton University Press, Princeton, N. J., 1949.

of "t", the following expressions apply to the various derivatives:

$$f^{o}(0) = T_o^{+} \qquad (73)$$

$$f'(0) = (4T_1 - 3T_o^{+} - T_2)(\Delta x)^{-1} \qquad (74)$$

$$f^{2}(0) = (T_o^{+} - 2T_1 + T_2)(\Delta x)^{-2} \qquad (75)$$

$$\phi^{o}(0) = T_a(+0) \qquad (76)$$

$$\phi'(0) = \{T_a(\Delta t - 0) - T_a(+0)\}(\Delta t)^{-1} \qquad (77)$$

These expressions are used in this paper, although the result contained in eq. 72 applies to polynomials of arbitrarily-high degree.

When the finite-difference expressions 73-77 are substituted into eq. 72, the coefficients of the various equally-spaced temperatures can be assembled. For the case where $t = \Delta t$ and $x = j(\Delta x)$ these coefficients are given in eqs. 28-32 of the text. In presenting these coefficients, it is convenient to use the dimensionless sequence of functions defined by:

$$F_n^{*} = F_n/(\Delta x)^n \qquad (78)$$