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Management Science Research Report Number #600

Balanced 0,  $\pm$  Matrices  
Part I: Decomposition

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January 22, 1994

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This work was support in part by National Science Foundation grants Nos.  
DDM-9201340 and DDM-9001705 and the Office of Naval Research grant  
N00014-89-J-1063.

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### Abstract

A  $0, \pm 1$  matrix is *balanced* if, in every square submatrix with two nonzero entries per row and column, the sum of the entries is a multiple of four. This paper extends the decomposition of balanced  $0, 1$  matrices obtained by Conforti, Cornuéjols and Rao to the class of balanced  $0, \pm 1$  matrices. As a consequence, we obtain a polynomial time algorithm for recognizing balanced  $0, \pm 1$  matrices.

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# 1 Introduction

A 0,1 matrix is *balanced* if it does not contain a square submatrix of odd order with two ones per row and column. This notion was introduced by Berge [1] and extended to  $0, \pm 1$  matrices by Truemper [19].

A  $0, \pm 1$  matrix is *balanced* if, in every square submatrix with two nonzero entries per row and column, the sum of the entries is a multiple of four. This paper extends the decomposition of balanced 0,1 matrices obtained by Conforti, Cornuéjols and Rao [7] to the class of balanced  $0, \pm 1$  matrices. As a consequence, we obtain a polynomial time algorithm for recognizing balanced  $0, \pm 1$  matrices. This algorithm extends the algorithm in [7] for recognizing balanced 0,1 matrices. It is discussed in a sequel paper.

The class of balanced  $0, \pm 1$  matrices properly includes totally unimodular  $0, \pm 1$  matrices. (A matrix is *totally unimodular* if every square submatrix has determinant equal to  $0, \pm 1$ .) The fact that every totally unimodular matrix is balanced is implied, for example, by Camion's theorem [3] which states that a  $0, \pm 1$  matrix is totally unimodular if and only if, in every square submatrix with an *even number* of nonzero entries per row and column, the sum of the entries is a multiple of four. Therefore our work can also be viewed as an extension of Seymour's decomposition and recognition of totally unimodular matrices [18].

In Section 3 we show that, to understand the structure of balanced  $0, \pm 1$  matrices, it is sufficient to understand the structure of the zero-nonzero pattern i.e. the 0,1 matrices that can be signed to be balanced. Such 0,1 matrices are said to be *balanceable*. Clearly balanced 0,1 matrices are bal-

anceable but the converse is not true:  $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$  is balanceable but not

balanced. Section 4 describes the cutsets used in our decomposition theorem and Section 5 states the theorem and outlines its proof. In Section 6, we relate our result to Seymour's [18] decomposition theorem for totally unimodular matrices. The proofs are given in Sections 8 - 13. The necessary definitions and notation are introduced in Section 7.

Interestingly, a number of polyhedral results known for balanced 0,1 matrices and totally unimodular matrices can be generalized to balanced  $0, \pm 1$  matrices. It follows that several problems in propositional logic can be solved in polynomial time by linear programming when the underlying clauses are

“balanced”. These results are reviewed in Section 2.

## 2 Bicoloring, Polyhedra and Propositional Logic

Berge [1] introduced the following notion. A  $0,1$  matrix is *bicolorable* if its columns can be partitioned into blue and red columns in such a way that every row with two or more 1's contains a 1 in a blue column and a 1 in a red column. This notion provides the following characterization of balanced  $0,1$  matrices.

**Theorem 2.1** (Berge [1]) *A  $0,1$  matrix  $A$  is balanced if and only if every submatrix of  $A$  is bicolorable.*

Ghouila-Houri [16] introduced the notion of *equitable bicoloring* for a  $0, \pm 1$  matrix  $A$  as follows. The columns of  $A$  are partitioned into blue columns and red columns in such a way that, for every row of  $A$ , the sum of the entries in the blue columns differs from the sum of the entries in the red columns by at most one.

**Theorem 2.2** (Ghouila-Houri [16]) *A  $0, \pm 1$  matrix  $A$  is totally unimodular if and only if every submatrix of  $A$  has an equitable bicoloring.*

A  $0, \pm 1$  matrix  $A$  is *bicolorable* if its columns can be partitioned into blue columns and red columns in such a way that every row with two or more nonzero entries either contains two entries of opposite sign in columns of the same color, or contains two entries of the same sign in columns of different colors. For a  $0,1$  matrix, this definition coincides with Berge's notion of bicoloring. Clearly, if a  $0, \pm 1$  matrix has an equitable bicoloring as defined by Ghouila-Houri, then it is bicolorable.

**Theorem 2.3** (Conforti, Cornuéjols [6]) *A  $0, \pm 1$  matrix  $A$  is balanced if and only if every submatrix of  $A$  is bicolorable.*

Balanced  $0,1$  matrices are important in integer programming due to the fact that several polytopes, such as the set covering, packing and partitioning polytopes, only have integral extreme points when the constraint matrix is

balanced. Such integrality results were first observed by Berge [2] and then expanded upon by Fulkerson, Hoffman and Oppenheim [14]. In the case of balanced  $0, \pm 1$  matrices, similar integrality results were proved by Conforti and Cornuéjols [6] for the generalized set covering, packing and partitioning polytopes.

Given a  $0, \pm 1$  matrix  $A$ , let  $n(A)$  denote the column vector whose  $i^{\text{th}}$  component is the number of  $-1$ 's in the  $i^{\text{th}}$  row of matrix  $A$ .

**Theorem 2.4** (Conforti, Cornuéjols [6]) *Let  $M$  be a  $0, \pm 1$  matrix. Then the following statements are equivalent:*

- (i)  *$M$  is balanced.*
- (ii) *For each submatrix  $A$  of  $M$ , the generalized set covering polytope  $\{x : Ax \geq 1 - n(A), 0 \leq x \leq 1\}$  is integral.*
- (iii) *For each submatrix  $A$  of  $M$ , the generalized set packing polytope  $\{x : Ax \leq 1 - n(A), 0 \leq x \leq 1\}$  is integral.*
- (iv) *For each submatrix  $A$  of  $M$ , the generalized set partitioning polytope  $\{x : Ax = 1 - n(A), 0 \leq x \leq 1\}$  is integral.*

Several problems in propositional logic can be written as generalized set covering problems. For example, the satisfiability problem in conjunctive normal form (SAT) is to find whether the formula

$$\bigwedge_{i \in S} \left( \bigvee_{j \in P_i} x_j \vee \bigvee_{j \in N_i} \neg x_j \right)$$

is true. This is the case if and only if the system of inequalities

$$\sum_{j \in P_i} x_j - \sum_{j \in N_i} (1 - x_j) \geq 1 \text{ for all } i \in S$$

has a  $0, 1$  solution vector  $x$ . This is a generalized set covering problem

$$\begin{aligned} Ax &\geq 1 - n(A) \\ x &\in \{0, 1\}^n. \end{aligned}$$

Given a set of clauses  $\bigvee_{j \in P_i} x_j \vee \bigvee_{j \in N_i} \neg x_j$  with weights  $w_i$ , MAXSAT consists of finding a truth assignment which satisfies a maximum weight set of clauses. MAXSAT can be formulated as the integer program

$$\begin{aligned}
& \text{Min } \sum_{i=1}^m w_i s_i \\
& Ax + s \geq 1 - n(A) \\
& x \in \{0, 1\}^n, s \in \{0, 1\}^m.
\end{aligned}$$

Similarly, the inference problem in propositional logic can be formulated as

$$\min \{cx : Ax \geq 1 - n(A), x \in \{0, 1\}^n\}.$$

The above three problems are NP-hard in general but SAT and logical inference can be solved efficiently for Horn clauses, clauses with at most two literals and several related classes [4],[20]. MAXSAT remains NP-hard for Horn clauses with at most two literals [15]. A consequence of Theorem 2.4 is the following.

**Corollary 2.5** *SAT, MAXSAT and logical inference can be solved in polynomial time by linear programming when the corresponding  $0, \pm 1$  matrix  $A$  is balanced.*

In fact SAT and logical inference can be solved by repeated application of unit resolution when the underlying  $0, \pm 1$  matrix  $A$  is balanced [5].

### 3 Balanceable $0, 1$ Matrices

In this section, we consider the following question: given a  $0, 1$  matrix, is it possible to turn some of the 1's into  $-1$ 's in order to obtain a balanced  $0, \pm 1$  matrix? A  $0, 1$  matrix for which such a signing exists is called a *balanceable* matrix. It turns out that in order to understand the structure of balanced  $0, \pm 1$  matrices, it is sufficient to concentrate on the zero-nonzero pattern, i.e. it is sufficient to understand the structure of the  $0, 1$  matrices that are balanceable. In fact, if a  $0, 1$  matrix is balanceable, there is a simple algorithm (which we state later) to perform the signing into a balanced  $0, \pm 1$  matrix. So, in effect, the problem of recognizing whether a  $0, 1$  matrix is balanceable is equivalent to the problem of recognizing whether a given  $0, \pm 1$  matrix is balanced.

Given a  $0, 1$  matrix  $A$ , the *bipartite graph representation* of  $A$  is the bipartite graph  $G$  having a node in  $V^r$  for every row of  $A$ , a node in  $V^c$  for every column of  $A$  and an edge  $ij$  joining nodes  $i \in V^r$  and  $j \in V^c$  if and

only if the entry  $a_{ij}$  of  $A$  equals 1. The sets  $V^r$  and  $V^c$  are the *sides* of the bipartition. We say that  $G$  is balanced if  $A$  is.

A *signed graph*  $G$  is a graph together with an assignment of weights  $+1, -1$  to the edges of  $G$ . To a  $0, \pm 1$  matrix corresponds its signed bipartite graph representation. A signed bipartite graph  $G$  is *balanced* if it is the signed bipartite graph representation of a balanced  $0, \pm 1$  matrix. Thus a signed bipartite graph  $G$  is balanced if and only if, in every hole  $H$  of  $G$ , the sum of the weights of the edges in  $H$  is a multiple of four. (A *hole* in a graph is a chordless cycle).

A bipartite graph  $G$  is *balanceable* if there exists a signing of its edges so that the resulting signed graph is balanced.

**Remark 3.1** *Since cuts and cycles of a graph  $G$  have even intersection, it follows that, if a signed bipartite graph  $G$  is balanced, then the signed bipartite graph  $G'$ , obtained by switching signs on the edges of a cut, is also balanced.*

For every edge  $uv$  of a spanning tree, there is a cut containing  $uv$  and no other edge of the tree (such cuts are known as *fundamental cuts*), and every cut is a symmetric difference of fundamental cuts. Thus, if  $G$  is a balanceable bipartite graph, its signing into a balanced bipartite graph is unique up to the (arbitrary) signing of a spanning tree of  $G$ . This was already observed by Camion [3] in the context of  $0, 1$  matrices that can be signed to be totally unimodular. So Remark 3.1 implies that a bipartite graph  $G$  is balanceable if and only if the following signing algorithm produces a balanced signed bipartite graph:

### Signing Algorithm

*Choose a spanning tree of  $G$ , sign its edges arbitrarily and recursively choose an edge  $uv$  which closes a hole  $H$  of  $G$  with the previously chosen edges, and sign  $uv$  so that the sum of the weights of the edges in  $H$  is a multiple of four.*

Note that, in the signing algorithm, the edge  $uv$  can be chosen to close the smallest length hole with the previously chosen edges. Such a hole  $H$  is also a hole in  $G$ , else a chord of  $H$  in  $G$  contradicts the choice of  $uv$ .

It follows from this signing algorithm, and the uniqueness of the signing (up to the signing of a spanning tree), that the problem of recognizing

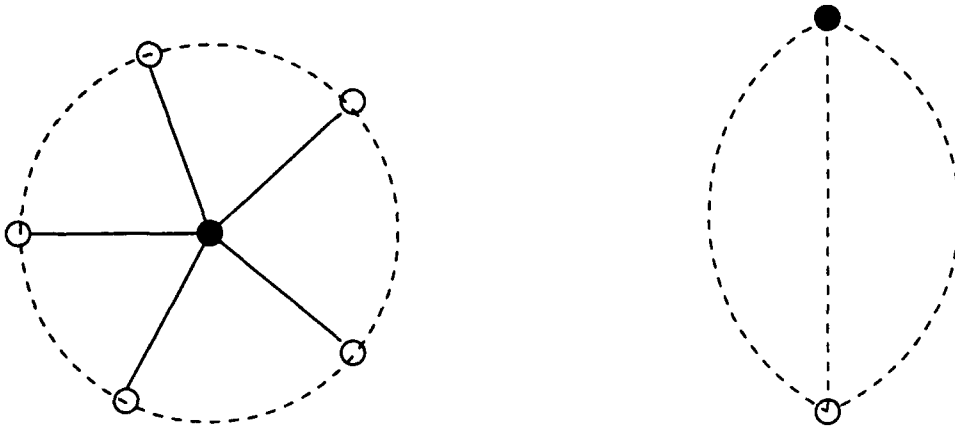


Figure 1: Odd wheel and 3-path configuration

whether a bipartite graph is balanceable is equivalent to the problem of recognizing whether a signed bipartite graph is balanced.

Let  $G$  be a bipartite graph. Let  $u, v$  be two nonadjacent nodes in opposite sides of the bipartition. A *3-path configuration connecting  $u$  and  $v$* , denoted by  $3PC(u, v)$ , is defined by three chordless paths  $P_1, P_2, P_3$  connecting  $u$  and  $v$ , having no common intermediate nodes and such that the subgraph induced by the nodes of these three paths contains no other edges than those of the paths (see Figure 1). Since paths  $P_1, P_2, P_3$  of a 3-path configuration are of length one or three modulo four, the sum of the weights of the edges in each path is also one or three modulo four. It follows that two of the three paths induce a hole of weight two modulo four. So a bipartite graph which contains a 3-path configuration as an induced subgraph is not balanceable.

A *wheel*, denoted by  $(H, x)$ , is defined by a hole  $H$  and a node  $x \notin V(H)$  having at least three neighbors in  $H$ , say  $x_1, x_2, \dots, x_n$ . If  $n$  is even, the wheel is an *even wheel*, otherwise it is an *odd wheel* (see Figure 1). An edge  $xx_i$  is a *spoke*. A subpath of  $H$  connecting  $x_i$  and  $x_j$  is called a *sector* if it contains no intermediate node  $x_l$ ,  $1 \leq l \leq n$ . Consider a wheel which is signed to be balanced. By Remark 3.1, all spokes of the wheel can be assumed to be signed positive. This implies that the sum of the weights of the edges in each sector is two modulo four. Hence the wheel must be an even wheel.

So, balanceable bipartite graphs contain neither odd wheels nor 3-path



configurations. This fact is extensively used in our proofs in this paper. The following important theorem of Truemper [19] states that the converse is also true.

**Theorem 3.2** (Truemper [19]) *A bipartite graph is balanceable if and only if it does not contain an odd wheel or a 3-path configuration.*

## 4 Cutsets

In this section we introduce the operations needed for our decomposition result. A set  $S$  of nodes (edges) of a connected graph  $G$  is a *node (edge) cutset* if the subgraph  $G \setminus S$ , obtained from  $G$  by removing the nodes (edges) in  $S$ , is disconnected.

### Extended Star Cutsets

A *biclique* is a complete bipartite graph containing at least one node from each side of the bipartition and it is denoted by  $K_{BD}$  where  $B$  and  $D$  are the sets of nodes in the two sides of the bipartition.

For a node  $x$ , let  $N(x)$  denote the set of all neighbors of  $x$ . In a bipartite graph  $G$ , an *extended star*  $(x; T; A; N)$  is defined by disjoint subsets  $T$ ,  $A$ ,  $N$  of  $V(G)$  and a node  $x \in T$  such that

- (i)  $A \cup N \subseteq N(x)$ ,
- (ii) the node set  $T \cup A$  induces a biclique (with node set  $T$  on one side of the bipartition and node set  $A$  on the other),
- (iii) if  $|T| \geq 2$ , then  $|A| \geq 2$ .

This concept was introduced in [7]. An *extended star cutset* is one where  $T \cup A \cup N$  is a node cutset.

### Joins

Let  $K_{BD}$  be a biclique with the property that its edge set  $E(K_{BD})$  is a cutset of the connected bipartite graph  $G$  and no connected component of  $G \setminus E(K_{BD})$  contains both a node of  $B$  and a node of  $D$ . Let  $G_B$  be the union of the components of  $G \setminus E(K_{BD})$  containing a node of  $B$ . Similarly, let  $G_D$  be the union of the components of  $G \setminus E(K_{BD})$  containing a node of

$D$ . The set  $E(K_{BD})$  is a *1-join* if the graphs  $G_B$  and  $G_D$  each contains at least two nodes. This concept was introduced by Cunningham and Edmonds [12].

Let  $K_{BD}$  and  $K_{EF}$  be two bicliques of a connected bipartite graph  $G$ , where  $B, D, E, F$  are disjoint node sets and neither  $E(K_{BD})$  nor  $E(K_{EF})$  is a 1-join in  $G$ . Further assume that no connected component of  $G \setminus E(K_{BD}) \cup E(K_{EF})$  has a node in  $B$  and one in  $D$ , or a node in  $E$  and one in  $F$ . Then, we can assume that every component of  $G \setminus E(K_{BD}) \cup E(K_{EF})$  contains either a node of  $B$  and a node of  $E$  or a node of  $D$  and a node of  $F$ . Let  $G_{BE}$  be the union of the components of  $G \setminus E(K_{BD}) \cup E(K_{EF})$  containing a node of  $B$  and a node of  $E$ . Similarly, let  $G_{DF}$  be the union of the components in  $G \setminus E(K_{BD}) \cup E(K_{EF})$  containing a node of  $D$  and a node of  $F$ . The set  $E(K_{BD}) \cup E(K_{EF})$  is a *2-join* if neither of the graphs  $G_{BE}$  and  $G_{DF}$  is a chordless path with all its intermediate nodes in  $V(G) \setminus B \cup D \cup E \cup F$ . This concept was introduced by Cornuéjols and Cunningham [11].

In a connected bipartite graph  $G$ , let  $A_i, i = 1, \dots, 6$  be disjoint nonempty node sets such that, for each  $i$ , every node in  $A_i$  is adjacent to every node in  $A_{i-1} \cup A_{i+1}$  (indices are taken modulo 6), and these are the only edges in the subgraph  $A$  induced by the node set  $\cup_{i=1}^6 A_i$ . Assume that  $E(A)$  is an edge cutset but that no subset of its edges forms a 1-join or a 2-join. Furthermore assume that no connected component of  $G \setminus E(A)$  contains a node in  $A_1 \cup A_3 \cup A_5$  and a node in  $A_2 \cup A_4 \cup A_6$ . Let  $G_{1,3,5}$  be the union of the components of  $G \setminus E(A)$  containing a node in  $A_1 \cup A_3 \cup A_5$  and  $G_{2,4,6}$  be the union of components containing a node in  $A_2 \cup A_4 \cup A_6$ . The set  $E(A)$  constitutes a *6-join* if the graphs  $G_{1,3,5}$  and  $G_{2,4,6}$  each contains at least four nodes (see Figure 2). This concept is new.

## 5 The Main Theorem

A bipartite graph is *restricted balanceable* if its edges can be signed so that the sum of the weights in each cycle is a multiple of four. Restricted balanceable bipartite graphs can be recognized in polynomial time [9], [22].  $R_{10}$  is the balanceable bipartite graph defined by the cycle  $x_1, \dots, x_{10}, x_1$  of length 10 with chords  $x_i x_{i+5}$ ,  $1 \leq i \leq 5$  (see Figure 3).

We can now state the decomposition theorem for balanceable bipartite

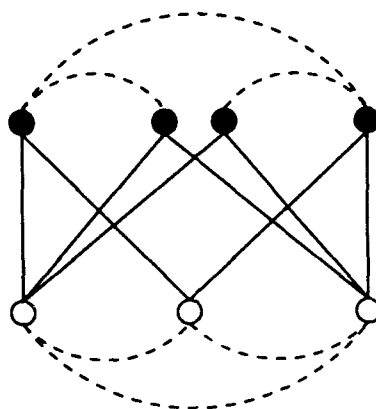


Figure 2: A 6-join

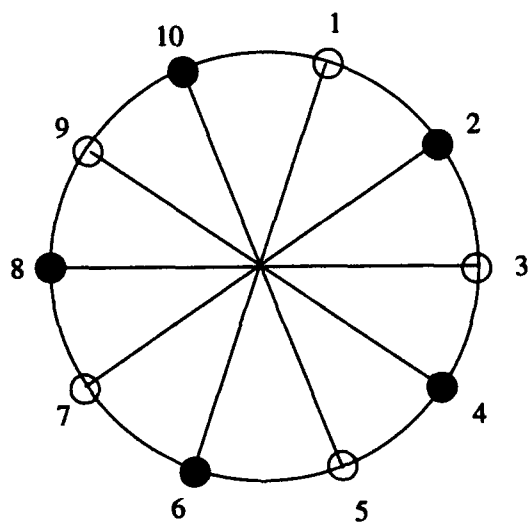


Figure 3:  $R_{10}$

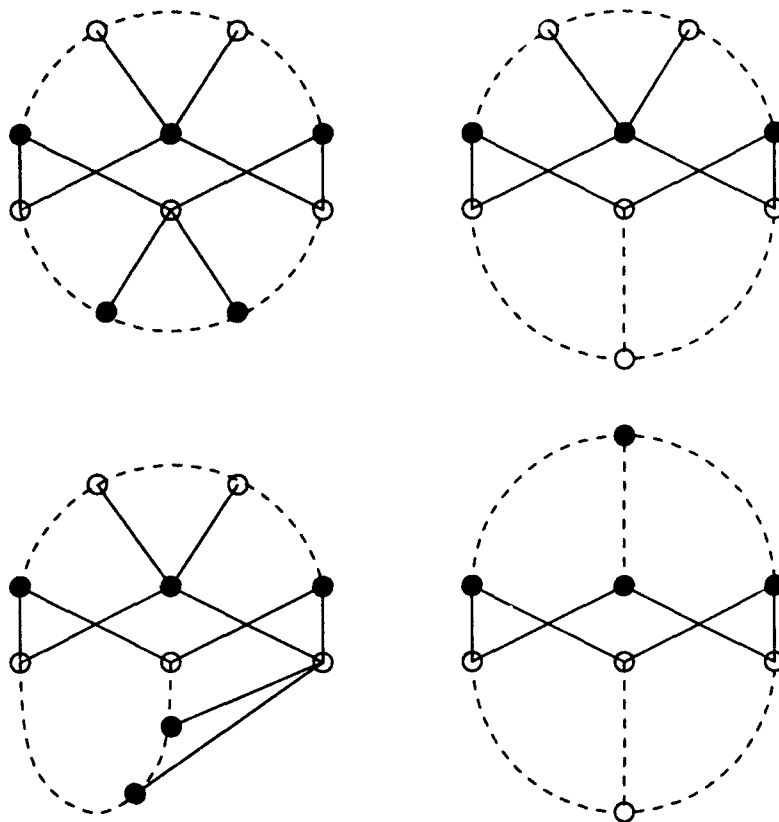


Figure 4: Four kinds of Connected 6-holes

graphs:

**Theorem 5.1** *A balanceable bipartite graph that is not restricted balanceable is either  $R_{10}$  or contains a 2-join, a 6-join or an extended star cutset.*

The key idea in the proof of Theorem 5.1 is that if a balanceable bipartite graph  $G$  is not restricted balanceable, then one of the three following cases occurs: (i) the graph  $G$  contains  $R_{10}$  or (ii) it contains a certain induced subgraph which forces a 6-join or an extended star cutset of  $G$ , or (iii) an earlier result of Conforti, Cornuéjols and Rao [7] applies.

### Connected 6-Holes

A *triad* consists of three internally node-disjoint paths  $t, \dots, u$ ;  $t, \dots, v$  and  $t, \dots, w$ , where  $t, u, v, w$  are distinct nodes and  $u, v, w$  belong to the same side of the bipartition. Furthermore, the graph induced by the nodes of the triad contains no other edges than those of the three paths. Nodes  $u, v$  and  $w$  are called the *attachments* and  $t$  is called the *meet* of the triad.

A *fan* consists of a chordless path  $x, \dots, y$  together with a node  $z$  adjacent to at least one node of the path, where  $x, y$  and  $z$  are distinct nodes all belonging to the same side of the bipartition. Nodes  $x, y$  and  $z$  are called the *attachments* of the fan and  $z$  is the *center*. A *spoke* is an edge connecting  $z$  to a node of the fan.

A *connected 6-hole*  $\Sigma$  is a bipartite graph induced by two disjoint node sets  $T(\Sigma)$  and  $B(\Sigma)$  such that each induces either a triad or a fan, the attachments of  $B(\Sigma)$  and  $T(\Sigma)$  induce a 6-hole and there are no other adjacencies between the nodes of  $T(\Sigma)$  and  $B(\Sigma)$  (see Figure 4).  $T(\Sigma)$  and  $B(\Sigma)$  are the *sides* of  $\Sigma$ ,  $T(\Sigma)$  is the *top* and  $B(\Sigma)$  the *bottom*.

**Theorem 5.2** *A balanceable bipartite graph containing  $R_{10}$  as a proper induced subgraph has a biclique articulation.*

**Theorem 5.3** *A balanceable bipartite graph that contains a connected 6-hole as an induced subgraph has an extended star cutset or a 6-join.*

**Theorem 5.4** [7] *A balanceable bipartite graph not containing  $R_{10}$  or a connected 6-hole as induced subgraphs either is restricted balanceable or contains a 2-join or an extended star cutset.*

Now Theorem 5.1 follows from Theorems 5.2, 5.3 and 5.4.

## 6 Connection with Seymour's Decomposition of Totally Unimodular Matrices

Seymour [18] discovered a decomposition theorem for 0,1 matrices that can be signed to be totally unimodular. The decompositions involved in his

theorem are 1-separations, 2-separations and 3-separations. A matrix  $B$  has a  $k$ -separation if its rows and columns can be partitioned so that

$$B = \begin{pmatrix} A^1 & D^2 \\ D^1 & A^2 \end{pmatrix}$$

where  $r(D^1) + r(D^2) = k - 1$  and the number of rows plus number of columns of  $A^i$  is at least  $k$ , for  $i = 1, 2$ . (here  $r(C)$  denotes the GF(2)-rank of 0, 1 matrix  $C$ ).

For a 1-separation  $r(D^1) + r(D^2) = 0$ . Thus both  $D^1$  and  $D^2$  are identically zero. The bipartite graph corresponding to the matrix  $B$  is disconnected.

For the 2-separation  $r(D^1) + r(D^2) = 1$ , thus w.l.o.g.  $D^2$  has rank zero and is identically zero. Since  $r(D^1) = 1$ , after permutation of rows and columns,  $D^1 = \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}$ , where  $E$  is a matrix all of whose entries are 1. The 2-separation in the bipartite graph representation of  $B$  corresponds to a 1-join.

For the 3-separation  $r(D^1) + r(D^2) = 2$ . If both  $D^1$  and  $D^2$  have rank 1 then, after permutation of rows and columns,

$$D^1 = \begin{pmatrix} 0 & E^1 \\ 0 & 0 \end{pmatrix}, \quad D^2 = \begin{pmatrix} 0 & 0 \\ E^2 & 0 \end{pmatrix}$$

where  $E^1$  and  $E^2$  are matrices whose entries are all 1. This 3-separation in the bipartite graph representation of  $B$  corresponds to a 2-join.

When  $r(D^1) = 2$  or  $r(D^2) = 2$ , it can be shown that the resulting 3-separation corresponds to a 2-join, a 6-join or to one of two other decompositions which each contain an extended star cutset.

In order to prove his decomposition theorem, Seymour used matroid theory. A matroid is *regular* if it is binary and its partial representations can be signed to be totally unimodular (see [21] for relevant definitions in matroid theory). The elementary families in Seymour's decomposition theorem consist of graphic matroids, cographic matroids and a 10-element matroid called

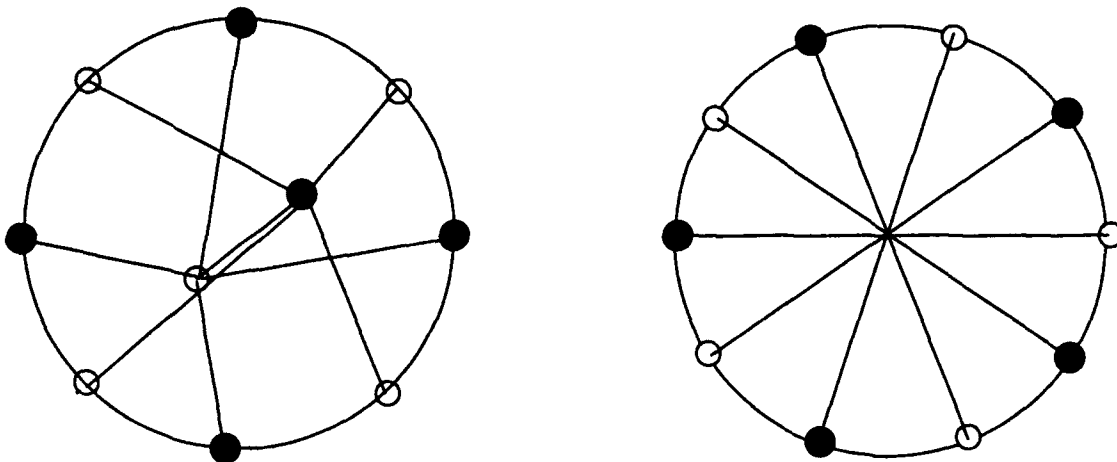


Figure 5: Representations of  $\mathcal{R}_{10}$

$\mathcal{R}_{10}$ .  $\mathcal{R}_{10}$  has exactly two partial representations

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

The bipartite graph representations are shown in Figure 5.

**Theorem 6.1** (Seymour [18]) *A regular matroid is either graphic, cographic, the 10-element matroid  $\mathcal{R}_{10}$ , or it contains a 1-, 2- or 3-separation.*

In order to prove Theorem 6.1, Seymour first showed that a regular matroid which is not graphic or cographic either contains a 1- or 2-separation or contains an  $\mathcal{R}_{10}$  or an  $\mathcal{R}_{12}$  minor, where  $\mathcal{R}_{12}$  is a 12-element matroid having the following matrix as one of its partial representations.

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Note that the bipartite graph representation of this matrix is a connected 6-hole where both sides are fans. So, this first part in Seymour's proof has some similarity with Theorem 5.4 stated above for balanceable bipartite graphs.

Then Seymour showed that, if a regular matroid contains an  $\mathcal{R}_{10}$  minor, either it is  $\mathcal{R}_{10}$  itself or it contains a 1-separation or a 2-separation. We show in Section 8 that if a balanceable bipartite graph contains an  $\mathcal{R}_{10}$  as an induced subgraph, either it is  $\mathcal{R}_{10}$  itself or it contains a biclique cutset.

Seymour completed his proof by showing that, for a regular matroid which contains an  $\mathcal{R}_{12}$  minor, the 3-separation of  $\mathcal{R}_{12}$  induces a 3-separation for the matroid. We show in Sections 9 - 13 that, for a balanceable bipartite graph which contains a connected 6-hole as an induced subgraph, either the 6-join of the connected 6-hole induces a 6-join of the whole graph or there is an extended star cutset.

Our proof differs significantly from Seymour's for the following reason. A regular matroid may have a large number of partial representations which lead to nonisomorphic bipartite graphs. This is the case for  $\mathcal{R}_{12}$ . All these partial representations are related through pivoting. In the case of 0,1 balanceable matrices there is no underlying matroid, so pivoting cannot help reduce the number of cases. Since our proof is broken down differently from Seymour's, we do not consider all these cases explicitly either.

## 7 Definitions and Notation

Let  $G$  be a bipartite graph where the two sides of the bipartition are  $V^r$  and  $V^c$ . We say that  $G$  contains a graph  $\Sigma$  if  $\Sigma$  is an induced subgraph of  $G$ . A node  $v \notin V(\Sigma)$  is *strongly adjacent* to  $\Sigma$  if  $|N(v) \cap V(\Sigma)| \geq 2$ . We say that a strongly adjacent node  $v$  is a *twin* of a node  $x \in V(\Sigma)$  relative to  $\Sigma$  if  $N(v) \cap V(\Sigma) = N(x) \cap V(\Sigma)$ .

A *path*  $P$  is a sequence of distinct nodes  $x_1, x_2, \dots, x_n$ ,  $n \geq 1$  such that  $x_i x_{i+1}$  is an edge, for all  $1 \leq i < n$ . Let  $x_i$  and  $x_l$  be two nodes of  $P$ , where  $l \geq i$ . The path  $x_i, x_{i+1}, \dots, x_l$  is called the  $x_i x_l$ -subpath of  $P$  and is denoted by  $P_{x_i x_l}$ . We write  $P = x_1, \dots, x_{i-1}, P_{x_i x_l}, x_{l+1}, \dots, x_n$  or  $P = x_1, \dots, x_i, P_{x_i x_l}, x_l, \dots, x_n$ . A *cycle*  $C$  is a sequence of nodes  $x_1, x_2, \dots, x_n, x_1$ ,  $n \geq 3$ , such that the nodes  $x_1, x_2, \dots, x_n$  form a path and  $x_1 x_n$  is an edge. The node set of a path or a cycle  $Q$  is denoted by  $V(Q)$ .



Let  $A, B, C$  be three disjoint node sets such that no node of  $A$  is adjacent to a node of  $B$ . A path  $P = x_1, x_2, \dots, x_n$  connects  $A$  and  $B$  if one of the two endnodes of  $P$  is adjacent to at least one node in  $A$  and the other is adjacent to at least one node in  $B$ . The path  $P$  is a *direct connection between  $A$  and  $B$*  if, in the subgraph induced by the node set  $V(P) \cup A \cup B$ , no path connecting  $A$  and  $B$  is shorter than  $P$ . A direct connection  $P$  between  $A$  and  $B$  *avoids*  $C$  if  $V(P) \cap C = \emptyset$ . The direct connection  $P$  is said to be *from  $A$  to  $B$*  if  $x_1$  is adjacent to some node in  $A$  and  $x_n$  to some node in  $B$ .

For  $S \subseteq V(G)$ ,  $N(S)$  denotes the set of nodes in  $V(G) \setminus S$  which are adjacent to at least one node in  $S$ .

## 8 Splitter Theorem for $R_{10}$

An *extended  $R_{10}$*  is a bipartite graph induced by ten nonempty pairwise disjoint node sets  $T_1, \dots, T_{10}$  such that for every  $1 \leq i \leq 10$ , the node sets  $T_i \cup T_{i-1}$ ,  $T_i \cup T_{i+1}$  and  $T_i \cup T_{i+5}$  all induce bicliques and these are the only edges in the graph. Throughout this section all the indices are taken modulo 10.

We consider a balanceable bipartite graph  $G$  which contains a node induced subgraph  $R$  isomorphic to  $R_{10}$ . We denote its node set by  $\{1, \dots, 10\}$  and for each  $i = 1, \dots, 10$ , node  $i$  is adjacent to nodes  $i - 1, i + 1$  and  $i + 5 \pmod{10}$ .

The first step in the proof of the splitter theorem for  $R_{10}$  is to study the structure of the strongly adjacent nodes to  $R$ .

**Theorem 8.1** *Let  $R$  be an  $R_{10}$  of  $G$ . If  $w$  is a strongly adjacent node to  $R$ , then  $w$  is a twin of a node in  $V(R)$  relative to  $R$ .*

*Proof:* First assume that  $w$  has exactly two neighbors in  $R$ . If the neighbors of  $w$  in  $R$  are nodes 1 and 3, the hole  $w, 1, 6, 7, 8, 3, w$  induces an odd wheel with center 2. If the neighbors of  $w$  in  $R$  are nodes 1 and 5, the hole  $w, 1, 2, 7, 8, 9, 4, 5, w$  is an odd wheel with center 10. The other cases where  $w$  has two neighbors in  $R$  are isomorphic.

We now assume that node  $w$  is adjacent to at least three nodes in  $R$ . If node  $w$  is adjacent to nodes  $i, i + 2, i + 4$ , then there exists an odd wheel  $i, i + 1, i + 2, i + 3, i + 4, i + 5, i$  with center  $w$ . So  $w$  is adjacent to exactly three nodes  $i, i + 2, i + 6$ , showing that  $w$  is a twin of  $i + 1$ .  $\square$

**Definition 8.2** Let  $R$  be an  $R_{10}$  of  $G$ . For  $1 \leq i \leq 10$ , let  $T_i(R)$  be the set of nodes comprising node  $i$  in  $R$  and all the twins of node  $i$  relative to  $R$ . Let  $R^*$  be the graph induced by the node set  $\cup_{i=1}^{10} T_i(R)$ .

**Lemma 8.3**  $R^*$  is an extended  $R_{10}$ .

*Proof:* Let  $u \in T_i(R)$  and  $v \in T_j(R)$ , where  $1 \leq i, j \leq 10$ . Let  $R'$  be the  $R_{10}$  obtained from  $R$  by substituting node  $u$  for node  $i$ . Now by Theorem 8.1, node  $v$  is twin of node  $j$  in  $R'$ . Hence nodes  $u$  and  $v$  are adjacent if and only if nodes  $i$  and  $j$  are adjacent.  $\square$

**Theorem 8.4**  $R^*$  satisfies the following two properties:

- (i) If node  $w$  is strongly adjacent to  $R^*$  then for some  $1 \leq i \leq 10$ ,  $N(w) \cap V(R^*) \subseteq T_i(R)$ .
- (ii) If  $R'$  is an  $R_{10}$  induced by the node set  $\{x_1, \dots, x_{10}\}$  where  $x_i \in T_i(R)$  for  $1 \leq i \leq 10$ , then  $T_i(R') = T_i(R)$ .

*Proof:* To prove (i), assume that  $w$  is adjacent to  $w_i \in T_i(R)$  and  $w_j \in T_j(R)$ ,  $i \neq j$ . Let  $R_{w_i w_j}$  be an  $R_{10}$  obtained from  $R$  by replacing node  $i$  with  $w_i$  and node  $j$  with  $w_j$ . Node  $w$  is now strongly adjacent to  $R_{w_i w_j}$ , so by Theorem 8.1 node  $w$  is a twin of a node in  $R_{w_i w_j}$ . Hence  $w$  is adjacent to a node  $k$  of  $R$ . Let  $R_{w_i}$  be an  $R_{10}$  obtained from  $R$  by replacing node  $i$  by  $w_i$ . Since  $w$  is adjacent to  $k$  and  $w_i$ , it is strongly adjacent to  $R_{w_i}$ , hence by Theorem 8.1  $w$  is adjacent to a node  $l \neq k$  of  $R$ . Now  $w$  is a strongly adjacent node of  $R$  and by Theorem 8.1 must be a twin of a node of  $R$ . Hence  $w \in V(R^*)$ , which contradicts our choice of  $w$ .

To prove (ii), note that Lemma 8.3 implies  $T_i(R) \subseteq T_i(R')$ , so it is enough to show that  $T_i(R') \subseteq T_i(R)$ . Let  $u \in T_i(R')$  and suppose that  $u \notin T_i(R)$ . Then node  $u$  is strongly adjacent to  $R^*$  and by (i) we have a contradiction.  $\square$

**Remark 8.5** Considering Theorem 8.4 we can simplify the notation by replacing  $T_i(R)$  by  $T_i$ .

**Definition 8.6** For  $1 \leq i \leq 10$ , let  $K_i$  be the complete bipartite graph induced by the node set  $T_{i-1} \cup T_i \cup T_{i+1} \cup T_{i+5}$ .

We now study the structure of paths between the nodes of  $R^*$ .

**Lemma 8.7** *If  $P = x_1, \dots, x_n$  is a direct connection from  $T_i$  to  $V(R^*) \setminus T_i$  in  $G \setminus E(K_i)$ , then the neighbors of  $x_n$  in  $R^*$  belong to a unique set  $T_j$ , where  $j = i - 1, i + 1$  or  $i + 5$ .*

*Proof:* Assume w.l.o.g. that  $x_1$  is adjacent to node  $i$ . By Theorem 8.4 (i),  $n > 1$  and node  $x_n$  has neighbors in exactly one  $T_j$ . Assume that for some  $j \notin \{i - 1, i + 1, i + 5\}$ ,  $x_n$  is adjacent to a node  $v_j \in T_j$ .

If  $j = i + 2$  then the hole  $i, x_1, P, x_n, v_{i+2}, i + 7, i + 6, i + 5, i$  induces an odd wheel with center  $i + 1$ . If  $j = i + 3$  then the paths  $P_1 = i, x_1, P, x_n, v_{i+3}$ ;  $P_2 = i, i + 1, i + 2, v_{i+3}$  and  $P_3 = i, i - 1, i + 4, v_{i+3}$  induce a  $3PC(i, v_{i+3})$ . If  $j = i + 4$  then the hole  $i, x_1, P, x_n, v_{i+4}, i + 3, i + 8, i + 7, i + 6, i + 1, i$  induces an odd wheel with center  $i + 2$ . This completes the proof since the remaining cases are isomorphic to the above three.  $\square$

**Lemma 8.8** *There cannot exist a path  $P = x_1, \dots, x_n$  with nodes belonging to  $V(G) \setminus V(R^*)$  such that  $x_1$  is adjacent to a node  $v_i \in T_i$  and  $x_n$  is adjacent to a node  $v_j \in T_j$ , where  $i \neq j$  and  $v_i$  and  $v_j$  are not adjacent.*

*Proof:* Let  $P$  be a shortest path contradicting the lemma. Hence  $P$  does not contain an intermediate node adjacent to a node in  $T_i \cup T_j$ . If no node  $x_l$  of  $P$ ,  $2 \leq l \leq n - 1$ , is adjacent to a node in  $V(R^*)$  then  $P$  is a direct connection from  $T_i$  to  $V(R^*) \setminus T_i$  in  $G \setminus E(K_i)$  contradicting Lemma 8.7.

Let  $x_l$  be the node of  $P$ , with the smallest index, adjacent to a node in  $V(R^*) \setminus (T_i \cup T_j)$ , say  $x_l$  is adjacent to  $w \in T_k$ . By Lemma 8.7 and symmetry, we can assume w.l.o.g. that  $k = i + 1$  or  $i + 5$ . No node in  $V(R^*) \setminus T_k$  can be adjacent to an intermediate node of  $P$ , otherwise  $P$  is not a shortest path contradicting the lemma. Let  $x_m$  be the node of  $P$  with highest index which is adjacent to a node  $v_k \in T_k$ .

**Case 1:**  $k = i + 1$ . Lemma 8.7 applied to  $x_m, \dots, x_n$  and the minimality of  $P$  show that  $j = i + 2$  or  $i + 6$ .

**Cases 1.1:**  $j = i + 2$ . Let  $H_1 = v_i, x_1, P, x_n, v_{i+2}, i + 3, i + 4, i - 1, v_i$  and  $H_2 = v_i, x_1, P, x_n, v_{i+2}, i + 7, i + 6, i + 5, v_i$ . Now either  $H_1$  or  $H_2$  induces an odd wheel with center  $i + 1$ .

**Case 1.2:**  $j = i + 6$ . The hole  $v_i, x_1, P, x_n, v_{i+6}, i + 7, i + 2, i + 3, i + 4, i - 1, v_i$  induces an odd wheel with center  $i + 5$ .

**Case 2:**  $k = i+5$ . Lemma 8.7 applied to  $x_m, \dots, x_n$  and the minimality of  $P$  show that  $j = i+4$ . Now the hole  $v_i, x_1, P, x_n, v_{i+4}, i+3, i+8, i+7, i+6, i+1, v_i$  induces an odd wheel with center  $i+2$ .

□

**Theorem 8.9** *If a balanceable bipartite graph  $G$  contains  $R_{10}$  then either  $G$  is  $R_{10}$  itself or  $G$  contains a biclique cutset.*

*Proof:* Let  $R$  be an  $R_{10}$  of  $G$ . By Lemma 8.3,  $R^*$  is an extended  $R_{10}$ . Assume that  $V(G) \neq V(R^*)$ . Let  $w$  be a node in  $V(G) \setminus V(R^*)$  adjacent to a node in  $T_1$ . If the biclique  $K_1$  is not a cutset of  $G$ , separating  $w$  from  $V(R^*)$ , then a path contradicting Lemma 8.8 exists. Hence  $V(G) = V(R^*)$ . If  $G$  is not  $R_{10}$ , then at least one of the node sets  $T_i(R)$  has cardinality greater than one. W.l.o.g. let  $u$  and  $v$  be two nodes in  $T_1(R)$ . Now  $\{u\} \cup N(u)$  is a star cutset separating  $v$  from the rest of the graph. □

## 9 Decomposition of Connected Six-holes

In the remaining sections, we assume that  $G$  is a balanceable bipartite graph and  $\Sigma$  is a connected 6-hole induced by  $T(\Sigma)$  and  $B(\Sigma)$ . We prove that either  $G$  contains an extended star cutset or it has a 6-join which separates the top and the bottom of  $\Sigma$ .

We denote by  $H = h_1, h_2, h_3, h_4, h_5, h_6, h_1$  the 6-hole of  $\Sigma$  induced by the attachments of  $T(\Sigma)$  and  $B(\Sigma)$  and we assume that  $h_1, h_3, h_5 \in T(\Sigma)$  and  $h_2, h_4, h_6 \in B(\Sigma)$ . We also assume  $h_1, h_3, h_5 \in V^c$  and  $h_2, h_4, h_6 \in V^r$ . It will be convenient to define the index of  $h_j$  modulo 6. If  $T(\Sigma)$  is a triad, the three paths defining it are denoted by  $P_1, P_2$  and  $P_3$  and the meet is denoted by  $t$ . For connected 6-hole  $\Sigma', \Sigma''$  and  $\Sigma^k$ , we denote the respective 6-holes by  $H' = h'_1, h'_2, h'_3, h'_4, h'_5, h'_6, h'_1$ ,  $H'' = h''_1, h''_2, h''_3, h''_4, h''_5, h''_6, h''_1$ , and  $H^k = h^k_1, h^k_2, h^k_3, h^k_4, h^k_5, h^k_6, h^k_1$ .

**Remark 9.1** *Let  $X$  be one of the sides of a balanceable connected 6-hole  $\Sigma$ . If  $X$  is a triad, its meet belongs to the same side of the bipartition as its attachments, else  $X$  contains a 3-path configuration. If  $X$  is a fan, its center has a positive even number of neighbors on the path of the fan connecting the other two attachments, else  $X$  contains an odd wheel. Hence  $X$  cannot be both a triad and a fan.*

**Remark 9.2** Let  $h_i$  and  $h_j$  be two distinct attachments of a side  $X$  of  $\Sigma$ . There is a unique chordless path in  $X$ , connecting  $h_i$  and  $h_j$ . This path is denoted by  $P_{ij}$ . For any pair of nodes  $x$  and  $y$  in  $V(\Sigma)$ , there exists a hole containing  $x$  and  $y$  whose node set is included in  $V(\Sigma)$ .

We use the following theorems, proved in [7] Part VI, about the structure of strongly adjacent nodes to an even wheel. We first introduce the relevant notation. Two sectors of a wheel are *adjacent* if they have a common endnode. A *bicoloring* of a wheel is an assignment of colors to the intermediate nodes of its sectors so that the nodes in the same sector have the same color and nodes of adjacent sectors have distinct colors. The endnodes of sectors are left unpainted. Note that a wheel is bicolorable if and only if it is even.

**Theorem 9.3** Let  $(W, v)$ ,  $v \in V^r$ , be an even wheel in a balanceable bipartite graph, and let  $u \in V^c \setminus N(v)$  be a node with neighbors in at least two distinct sectors of the wheel  $(W, v)$ . Then  $u$  satisfies one of the following properties:

**Type a** Node  $u$  has exactly two neighbors in  $W$  and these neighbors belong to two distinct sectors having the same color.

**Type b** There exists one sector, say  $S_j$  with endnodes  $v_i$  and  $v_k$ , such that  $u$  has a positive even number of neighbors in  $S_j$  and has exactly two neighbors in  $V(W) \setminus V(S_j)$ , adjacent to  $v_i$  and  $v_k$  respectively.

**Theorem 9.4** Let  $(W, v)$ ,  $v \in V^r$ , be an even wheel in a balanceable bipartite graph, and let  $u \in V^c \cap N(v)$  be a node which is strongly adjacent to  $(W, v)$ . Then  $u$  satisfies one of the following properties:

**Type a** Node  $u$  has exactly one neighbor in  $W$ .

**Type b** Node  $u$  is not of Type a and in each sector of  $(W, v)$ ,  $u$  has either 0 or an odd number of neighbors. It follows that  $u$  has neighbors in an even number of sectors and that the number of consecutive sectors without neighbors of  $u$ , between two sectors with neighbors of  $u$ , is even.

**Classification 9.5** A node  $u \in V^r$ , strongly adjacent to an even wheel  $(W, v)$ ,  $v \in V^r$ , is classified as follows:

**Type a** There exists a sector of  $(W, v)$  containing all the nodes of  $N(u) \cap W$ .

**Type b** Node  $u$  is not of Type a and all its neighbors in  $W$  are unpainted.  
*Note that, in particular, the center  $v$  of the wheel is of Type b.*

**Type c** Node  $u$  is not of Types a or b and all its painted neighbors in  $W$  have the same color.

**Type d** Node  $u$  has painted neighbors of both colors.

## 10 Strongly Adjacent Nodes to a Connected 6-Hole

The first step in our decomposition of a connected 6-hole  $\Sigma$  is the study of the strongly adjacent nodes. We use notation introduced in Section 9.

**Lemma 10.1** *If  $T(\Sigma)$  is a triad and  $w$  is adjacent to its meet  $t$ , then all nodes of  $N(w) \cap T(\Sigma)$  are contained in a unique path  $P_j$  of the triad, where  $j = 1, 3$  or  $5$ .*

*Proof:* Assume not. Then w.l.o.g.  $w$  has neighbors in  $P_1 \setminus \{t\}$  and  $P_3 \setminus \{t\}$ . Since the hole  $h_2, P_1, P_3, h_2$  induces a wheel with center  $w$ , the node  $w$  has a positive even number of neighbors in one of the paths  $P_1, P_3$  and an odd number (greater than one) of neighbors in the other. Let  $H_1 = h_6, P_1, P_5, h_6$  and  $H_2 = h_4, P_3, P_5, h_4$ . Now either  $(H_1, w)$  or  $(H_2, w)$  induces an odd wheel.  $\square$

### 10.1 Strongly Adjacent Nodes Having Neighbors Both in $T(\Sigma)$ and $B(\Sigma)$

In this section,  $w$  denotes a strongly adjacent node to  $\Sigma$  and we assume w.l.o.g. that  $w$  is in  $V^r$ .

**Theorem 10.2** *If  $w \in V^r$  has neighbors both in  $T(\Sigma)$  and  $B(\Sigma)$ , then  $N(w) \cap T(\Sigma) = \{h_i, h_j\}$ ,  $i \neq j$ ,  $i, j = 1, 3$  or  $5$ .*

To prove this theorem, we need the following lemmas:

**Lemma 10.3** *If  $T(\Sigma)$  is a triad and  $w \in V^r$  has neighbors both in  $T(\Sigma)$  and  $B(\Sigma)$ , then  $N(w) \cap T(\Sigma) = \{h_i, h_j\}$ ,  $i \neq j$ , where  $i, j = 1, 3$  or  $5$ .*

*Proof:* First we show that the neighbors of  $w$  in  $T(\Sigma)$  cannot all be contained in the same path of  $T(\Sigma)$ . Assume the contrary i.e. assume that for some  $j = 1, 3$  or  $5$ ,  $N(w) \cap T(\Sigma) \subseteq P_j$ . Then since  $w \in V^r$  is not adjacent to  $h_2, h_4, h_6 \in V^r$  but has at least one neighbor in  $B(\Sigma)$ , there is a  $3PC(t, h_{j+3})$  where  $t$  is the meet of  $T(\Sigma)$ . Node  $w$  is not adjacent to the meet  $t$ , since otherwise by Lemma 10.1 the neighbors of  $w$  in  $T(\Sigma)$  would all be contained in the same path of  $T(\Sigma)$ . Then node  $w$  has neighbors in at most two paths of  $T(\Sigma)$ , since otherwise there is a  $3PC(w, t)$ . Therefore node  $w$  has neighbors in exactly two distinct paths of  $T(\Sigma)$ , say  $P_1$  and  $P_3$ . Let  $w_1 \in P_1$  and  $w_3 \in P_3$  be neighbors of  $w$ . Assume w.l.o.g. that  $w_3 \neq h_3$ . Now there is a  $3PC(w, t)$  where the intermediate nodes of the three paths are included respectively in  $V(P_1), V(P_3)$  and  $V(P_5) \cup (B(\Sigma) \setminus \{h_2, h_6\})$ . Therefore  $N(w) \cap T(\Sigma) = \{h_1, h_3\}$ .  $\square$

We now study the case where  $T(\Sigma)$  is a fan and we assume w.l.o.g. that  $h_3$  is the center node of the fan.

**Lemma 10.4** *If  $T(\Sigma)$  is a fan and  $w \in V^r$  has neighbors both in  $T(\Sigma)$  and  $B(\Sigma)$  but  $w$  is not adjacent to  $h_3$ , then  $N(w) \cap T(\Sigma) = \{h_1, h_5\}$ .*

*Proof:* Let  $H_{15}$  be the hole induced by the paths  $P_{15}$  in  $T(\Sigma)$  and  $P_{24}$  in  $B(\Sigma)$ . We first show the following claim:

**Claim 1:** Node  $w$  has more than one neighbor in  $T(\Sigma)$ .

*Proof of Claim 1:* Assume not and let  $w_1$  be the unique neighbor of  $w$  in  $T(\Sigma)$ . If  $w_1$  belongs to a sector of  $(H_{15}, h_3)$  having either  $h_2$  or  $h_4$  as endnode, there is an odd wheel with center  $h_3$ . Otherwise there is a  $3PC(w_1, h_6)$ . This proves Claim 1.

So  $w$  is not adjacent to  $h_3$  and is strongly adjacent to  $H_{15}$  and therefore  $w$  is of Type a or b[9.3] relative to  $(H_{15}, h_3)$ .

If  $w$  is of Type a[9.3] with neighbors  $w_1$  and  $w_2$  in  $H_{15}$ , Claim 1 shows  $w_1, w_2 \in T(\Sigma)$ . Hence  $w$  has a neighbor in  $B(\Sigma) \setminus V(H_{15})$ . Now  $w_1, w_2$  must coincide with  $h_1, h_5$ , else there is a  $3PC(w, h_3)$ .

So  $w$  is of Type b[9.3]. If all the neighbors of  $w$  in  $H_{15}$  belong to  $T(\Sigma)$ , there is a  $3PC(w, h_3)$ . If all but one of the neighbors of  $w$  belong to  $T(\Sigma)$ , there is an odd wheel with center  $w$ . The structure of a Type b[9.3] node shows that the only remaining possibility is that the neighbors in  $T(\Sigma)$  of  $w$  are  $h_1, h_5$ , completing the proof of the lemma.  $\square$

**Lemma 10.5** *If  $T(\Sigma)$  is a fan,  $w \in V^r$  has neighbors both in  $T(\Sigma)$  and  $B(\Sigma)$  and  $w$  is adjacent to  $h_3$ , then  $N(w) \cap T(\Sigma) = \{h_1, h_3\}$  or  $\{h_3, h_5\}$ .*

*Proof:* If  $w$  has no neighbor in  $T(\Sigma) \setminus \{h_3\}$  then, since  $w$  has a neighbor in  $B(\Sigma)$ , there is a  $3PC(h_3, h_6)$ .

So  $w$  is strongly adjacent to  $(H_{15}, h_3)$  and satisfies Theorem 9.4, where  $H_{15}$  denotes the hole induced by the paths  $P_{15}$  in  $T(\Sigma)$  and  $P_{24}$  in  $B(\Sigma)$ . We first show that  $w$  has a unique neighbor in  $T(\Sigma) \setminus \{h_3\}$ .

This is clearly the case if node  $w$  is of Type a[9.4], so assume node  $w$  is of Type b[9.4]. If  $w$  is adjacent to a node in the sector  $B(\Sigma) \cap V(H_{15})$  of  $(H_{15}, h_3)$ , then Theorem 9.4 shows that  $w$  has an odd number of neighbors in  $T(\Sigma) \setminus \{h_3\}$ . Hence  $w$  has exactly one neighbor in  $T(\Sigma) \setminus \{h_3\}$ , else this node set together with node  $h_6$  induces an odd wheel with center  $w$ . If  $w$  is not adjacent to  $B(\Sigma) \cap V(H_{15})$  and it has a unique neighbor  $w_1$  in  $B(\Sigma) \setminus V(H_{15})$ , then there is a  $3PC(w_1, h_2)$  or a  $3PC(w_1, h_4)$ . Finally, if  $w$  is not adjacent to  $B(\Sigma) \cap V(H_{15})$  and it has at least two neighbors in  $B(\Sigma) \setminus V(H_{15})$ , then there is a  $3PC(w, h_1)$  or a  $3PC(w, h_5)$ .

Let  $w_1$  be the unique neighbor of  $w$  in  $T(\Sigma) \setminus \{h_3\}$ . If  $w_1$  is distinct from  $h_1$  and  $h_5$ , then there is a  $3PC(w_1, h_6)$ .  $\square$

*Proof of Theorem 10.2:* The proof of the theorem follows from Lemmas 10.3, 10.4 and 10.5.  $\square$

## 10.2 Strongly Adjacent Nodes Having Neighbors Only in One Side of $\Sigma$

In this section we assume w.l.o.g. that the strongly adjacent node  $w$  has no neighbor in  $B(\Sigma)$ .

**Theorem 10.6** *If  $T(\Sigma)$  is a triad, then  $w$  is one of the following types, see Figure 6 :*

**Type a**  $N(w) \cap T(\Sigma) \subset V(P_i)$  for  $i = 1, 3$  or  $5$ .

**Type b**  $w \in V^c$  has at least one neighbor in each path  $P_1$ ,  $P_3$  and  $P_5$ .

**Type c**  $w \in V^c$  has neighbors in exactly two of the paths  $P_1$ ,  $P_3$  and  $P_5$ .

*Furthermore  $w$  either has an even number of neighbors in each of the two paths or has one neighbor in each path and both neighbors are adjacent to the meet  $t$ .*



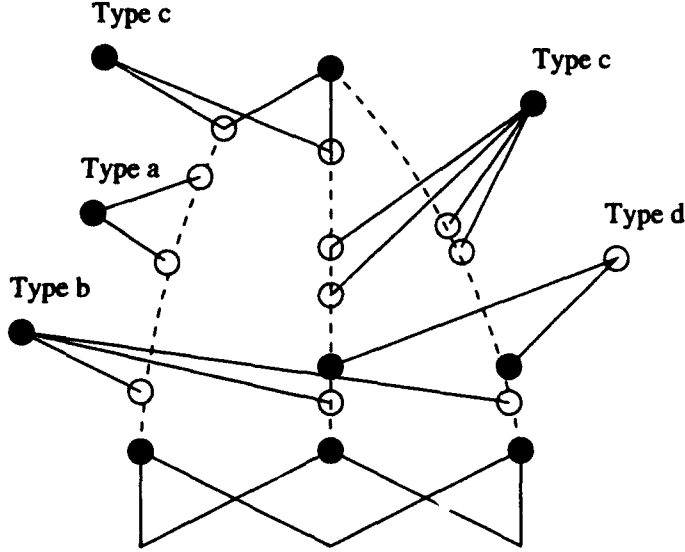


Figure 6: Strongly adjacent nodes with all neighbors in a triad  $T(\Sigma)$

**Type d**  $w \in V^r$  is not adjacent to the meet  $t$  and has two neighbors in  $T(\Sigma)$  which belong to distinct paths of  $T(\Sigma)$ .

*Proof:* If some path of  $T(\Sigma)$  contains all the nodes in  $N(w) \cap T(\Sigma)$ , then we have Type a. If  $w \in V^c$  has neighbors in all three paths, we have Type b.

Assume now that  $w \in V^c$  has neighbors in exactly two paths, say  $P_1$  and  $P_3$ . Then  $w$  cannot have an even number of neighbors in one path and an odd number in the other, else there is an odd wheel with center  $w$ . If  $w$  has an odd number, greater than one, of neighbors in one of the paths, then  $P_5$  closes an odd wheel with center  $w$ . Let  $w_1$  be the unique neighbor of  $w$  in  $P_1$  and let  $w_2$  be the unique neighbor of  $w$  in  $P_3$ . Then  $w_1$  is adjacent to  $t$ , else there is a  $3PC(w_1, t)$ . Similarly  $w_2$  is adjacent to  $t$ . This yields Type c.

Finally assume that  $w \in V^r$  is not of Type a. Lemma 10.1 shows that  $w$  is not adjacent to  $t$ . If  $w$  has neighbors in all three paths of  $T(\Sigma)$ , there is a  $3PC(w, t)$ . So  $w$  has neighbors in exactly two paths and if  $w$  has more than two neighbors in  $T(\Sigma)$  there is a  $3PC(w, t)$ .  $\square$

**Remark 10.7** Let  $\Sigma$  be a connected 6-hole whose top is a fan with center  $h_3$  and let  $H_{15}$  be the hole induced by paths  $P_{15}$  and  $P_{24}$ . A node  $w$  strongly

adjacent to  $\Sigma$  but with no neighbor in  $B(\Sigma)$  can be of any of the types described in Theorems 9.3, 9.4, 9.5 relative to  $(H_{15}, h_3)$ .

**Theorem 10.8** *If  $w$  is a strongly adjacent node to  $\Sigma$ , with no neighbor in  $B(\Sigma)$ , then either  $w$  belongs to a connected 6-hole with top contained in  $T(\Sigma) \cup \{w\}$ , bottom  $B(\Sigma)$  and 6-hole  $H$  or one of the following holds:*

- *$T(\Sigma)$  is a triad and  $w$  is of Type c[10.6] with exactly two neighbors in  $T(\Sigma)$ .*
- *$T(\Sigma)$  is a triad and  $w$  is of Type d[10.6] adjacent to two nodes of the 6-hole.*
- *$T(\Sigma)$  is a fan, say with center  $h_3$ , and  $w$  is of Type a[9.4] relative to  $(H_{15}, h_3)$ .*

*Proof:* If  $T(\Sigma)$  is a triad, the proof follows from Theorem 10.6 by inspection. Now assume  $T(\Sigma)$  is a fan with center  $h_3$  and let  $H_{15}$  be the hole induced by paths  $P_{15}$  and  $P_{24}$ . If  $w$  is adjacent to  $h_3$ , then  $w$  is strongly adjacent to the wheel  $(H_{15}, h_3)$  and the theorem follows from Remark 10.7. If  $w$  is not adjacent to  $h_3$ , let  $Q$  be the shortest path between  $h_1$  and  $h_5$  containing  $w$ , in  $T(\Sigma) \cup \{w\} \setminus \{h_3\}$ .

If  $h_3$  is adjacent to a node of  $Q$ , then  $V(Q) \cup \{h_3\}$  induces a fan with attachments  $h_1, h_3, h_5$ .

If  $h_3$  is not adjacent to a node of  $Q$ , let  $R$  be a direct connection between  $h_3$  and  $V(Q)$ , using nodes of  $T(\Sigma)$ . Then  $V(Q) \cup V(R) \cup \{h_3\}$  induces a triad with attachments  $h_1, h_3$  and  $h_5$  which, together with  $B(\Sigma)$ , induces a connected 6-hole.  $\square$

**Classification 10.9** *Theorems 10.2 and 10.8 partition the strongly adjacent nodes  $w$  to  $\Sigma$  into the following classes:*

**Type a** *Node  $w$  belongs to a connected 6-hole with nodes in  $V(\Sigma) \cup \{w\}$ .*

**Type b** *Node  $w$  is adjacent to exactly two nodes of  $\Sigma$  and these two nodes belong to the 6-hole of  $\Sigma$ . Such a node  $w$  is called a fork.*

**Type c** *Node  $w$  has exactly two neighbors in  $\Sigma$ , both belonging to the same side which is a triad and both neighbors are adjacent to the meet of the triad.*

**Type d** Node  $w$  has exactly two neighbors in  $\Sigma$ , both belonging to the same side which is a fan, say with center  $h_i$ , and  $w$  is adjacent to  $h_i$  and one other node which is not an attachment of the fan.

## 11 Direct Connections from Top to Bottom

**Lemma 11.1** Every direct connection  $P = x_1, \dots, x_n$  between  $T(\Sigma)$  and  $B(\Sigma)$  in  $G \setminus E(H)$  is of one of the following types:

- $n = 1$  and  $x_1$  is a strongly adjacent node satisfying Theorem 10.2.
- One endnode of  $P$  is a fork, adjacent to  $h_{i-1}$  and  $h_{i+1}$  and the other endnode of  $P$  is adjacent to a node of  $V(\Sigma) \setminus V(H)$
- **Bridge of Type a** Nodes  $x_1$  and  $x_n$  are not strongly adjacent to  $\Sigma$  and their unique neighbors in  $\Sigma$  are two adjacent nodes of the 6-hole of  $\Sigma$ .

**Bridge of Type b1** One endnode of  $P$  is a fork, say  $x_1$  is adjacent to  $h_1$  and  $h_3$ , and  $x_n$  has a unique neighbor in  $\Sigma$  which is  $h_2$ .

**Bridge of Type c1** Node  $x_1$  is a fork, say adjacent to  $h_1$  and  $h_3$ , and  $x_n$  is also a fork, adjacent to  $h_2$  and either  $h_4$  or  $h_6$ .

*Proof:* If  $n = 1$ ,  $x_1$  is a strongly adjacent node with neighbors both in  $T(\Sigma)$  and  $B(\Sigma)$  and this possibility is described in Theorem 10.2. So we assume  $n > 1$ ,  $x_1$  has no neighbors in  $B(\Sigma)$  and  $x_n$  has no neighbors in  $T(\Sigma)$ .

**Case 1:** Neither  $x_1$  nor  $x_n$  is a fork of  $\Sigma$ .

**Case 1.1:** Nodes  $x_1$  and  $x_n$  are either not strongly adjacent to  $\Sigma$  or they are of Type a[10.9].

Assume  $x_1$  is a strongly adjacent node. Let  $\Sigma'$  be a connected 6-hole containing  $x_1$  and having node set included in  $V(\Sigma) \cup \{x_1\}$ . Node  $x_1$  does not belong to the 6-hole of  $\Sigma'$ , since  $x_1$  has no neighbor in  $B(\Sigma)$ . This shows  $n > 2$ , otherwise  $x_n$  is a strongly adjacent node with neighbors both in the top and bottom of  $\Sigma'$ , and since  $x_n$  is not a fork of  $\Sigma$  this violates Theorem 10.2. Therefore, after possibly modifying  $P$  and  $\Sigma$  appropriately, we can assume w.l.o.g. that both  $x_1$  and  $x_n$  are not strongly adjacent to  $\Sigma$ . Let  $y$  and  $z$  be the unique neighbors of  $x_1$  and  $x_n$  in  $\Sigma$ , respectively.

If  $y$  and  $z$  belong to the same side of the bipartition, assume w.l.o.g. that  $y \in V^r$ ,  $y \in V(P_{15})$  and  $y$  is not adjacent to  $h_1$ . There exists a  $3PC(h_1, y)$  using  $P$  and the hole induced by  $V(P_{15}) \cup \{h_6\}$ , unless  $z$  coincides with  $h_4$  or  $h_6$ . Assume  $z = h_6$ . Then there is a  $3PC(h_3, h_6)$  unless  $y$  is adjacent to  $h_5$ . But then there is an odd wheel with center  $h_5$  and hole induced by  $V(P) \cup V(P_{35} \setminus \{h_5\}) \cup V(P_{46})$ . Assume  $z = h_4$ . Then there is a  $3PC(y, h_5)$  unless  $y$  is adjacent to  $h_5$ . But then there is an odd wheel with center  $h_5$  and hole induced by  $V(P) \cup V(P_{15} \setminus \{h_5\}) \cup V(P_{46})$ .

By Remark 9.2,  $y$  and  $z$  belong to a hole  $H$  with node set included in  $V(\Sigma)$ . If  $y$  and  $z$  belong to opposite sides of the bipartition and they are not adjacent, the path  $P$  together with  $H$  induces a  $3PC(y, z)$ . If  $y$  and  $z$  are adjacent, then they belong to the 6-hole of  $\Sigma$  and  $P$  is a bridge of Type a. Note that, in this case,  $P$  and  $\Sigma$  were not modified.

**Case 1.2:** Node  $x_1$  is of Type c[10.9].

Then  $T(\Sigma)$  is a triad. Assume w.l.o.g. that the neighbors of  $x_1$  belong to the paths  $P_1$  and  $P_3$ . If  $x_n$  is adjacent to a node in  $B(\Sigma) \setminus \{h_4, h_6\}$ , there is a  $3PC(x_1, h_2)$ . If  $x_n$  is adjacent to  $h_6$  only, there is a  $3PC(x_1, h_6)$ . If  $x_n$  is adjacent to  $h_4$  only, there is a  $3PC(x_1, h_4)$ . Since  $x_n$  is not a fork, Case 1.2 cannot occur.

**Case 1.3:** Node  $x_1$  is of Type d[10.9].

Then  $T(\Sigma)$  is a fan, say with center  $h_3$  and  $x_1$  is adjacent to  $h_3$  and one other node of the fan, say  $y$ , distinct from  $h_1$  and  $h_5$ . If  $x_n$  is adjacent to a node in  $B(\Sigma) \setminus \{h_2, h_4\}$ , there is a  $3PC(y, h_6)$ . If  $x_n$  is adjacent to  $h_2$  only, there is a  $3PC(y, h_2)$ . If  $x_n$  is adjacent to  $h_4$  only, there is a  $3PC(y, h_4)$ . Since  $x_n$  is not a fork, Case 1.3 cannot occur.

**Case 2:** Either  $x_1$  or  $x_n$  is a fork of  $\Sigma$ , but not both.

W.l.o.g. assume  $x_1$  is a fork adjacent to  $h_1$  and  $h_3$ . If  $x_n$  is not adjacent to a node of  $V(\Sigma) \setminus V(H)$  then  $x_n$  has a unique neighbor  $y$  in  $\Sigma$ , where  $y = h_2, h_4$  or  $h_6$ . If  $y = h_2$ , we have a bridge of Type b1. If  $y = h_4$  or  $h_6$ , say  $h_4$ , the hole induced by  $V(P) \cup V(P_{24}) \cup \{h_1\}$  forms an odd wheel with center  $h_3$ .

**Case 3:** Both  $x_1$  and  $x_n$  are forks of  $\Sigma$ .

We have a bridge of Type c1, unless  $x_1$  is adjacent to, say  $h_1$  and  $h_3$ , and  $x_n$  is adjacent to  $h_4$  and  $h_6$ . But, in this case, there is a  $3PC(x_1, x_n)$  if  $x_1$  is not adjacent to  $x_n$ , and an odd wheel with center  $x_1$  if  $x_1$  is adjacent to  $x_n$ .

□

**Lemma 11.2** *Every direct connection  $P = x_1, \dots, x_n$  from  $T(\Sigma) \setminus \{h_1\}$  to  $B(\Sigma)$  avoiding  $\{h_1\}$  in  $G \setminus E(H)$  is either one of the types described in Lemma 11.1 or  $n > 1$ , there exists a node  $x_i$ ,  $1 < i < n$ , adjacent to  $h_1$  and  $P$  satisfies one of the following alternatives:*

- *Node  $x_1$  is adjacent to at least one node in  $T(\Sigma) \setminus \{h_1, h_3, h_5\}$  and  $x_n$  is a fork adjacent to  $h_2$  and  $h_6$ .*
- **Bridge of Type b2** *Node  $x_n$  is adjacent either to  $h_2$  or  $h_6$ , say  $h_2$  and to no other node of  $\Sigma$ . Node  $x_1$  is adjacent to  $h_3$ , possibly  $h_1$  and to no other node of  $\Sigma$ .*

**Bridge of Type c2** *Node  $x_n$  is a fork adjacent to  $h_2$  and  $h_6$ . Node  $x_1$  is adjacent to either  $h_3$  or  $h_5$  but not both, possibly  $h_1$  and to no other node of  $\Sigma$ .*

*Proof:* If no node  $x_i$ ,  $1 < i < n$ , is adjacent to  $h_1$ , then  $P$  is also a direct connection from  $T(\Sigma)$  to  $B(\Sigma)$  in  $G \setminus E(H)$ . Hence  $P$  satisfies Lemma 11.1.

Let  $x_j$ ,  $1 < j < n$ , be the node of highest index which is adjacent to  $h_1$ . The subpath  $P_{x_j, x_n}$  of  $P$  is a direct connection from  $T(\Sigma)$  to  $B(\Sigma)$  satisfying Lemma 11.1. Since  $x_j$  is adjacent to  $h_1$  only,  $P_{x_j, x_n}$  is a bridge of Type a or b [11.1].

Assume that  $P_{x_j, x_n}$  is a bridge of Type a [11.1]. Then  $x_n$  is adjacent to either  $h_2$  or  $h_6$ , say  $h_2$ . If  $x_1$  has a neighbor in  $T(\Sigma) \setminus \{h_1, h_3\}$ , then there exists a chordless path  $Q$  connecting  $x_1$  to  $h_5$  whose intermediate nodes belong to  $T(\Sigma) \setminus \{h_1, h_3\}$ . Now one of the two holes formed by the nodes of  $P$ ,  $Q$  and either  $P_{26}$  or  $P_{24}$  contains an odd number of neighbors of  $h_1$ . So the neighbors of  $x_1$  in  $T(\Sigma)$  are contained in  $\{h_1, h_3\}$  and  $x_1$  is adjacent to  $h_3$ . This yields a bridge of Type b2 [11.2].

Assume now that  $P_{x_j, x_n}$  is a bridge of Type b1 [11.1]. Then  $x_n$  is adjacent to  $h_2$  and  $h_6$ . If  $x_1$  has a neighbor in  $T(\Sigma) \setminus \{h_1, h_3, h_5\}$ , then the first possibility of Lemma 11.2 holds. So the neighbors of  $x_1$  in  $T(\Sigma)$  are contained in  $\{h_1, h_3, h_5\}$ . If  $x_1$  is a fork, adjacent to  $h_3$  and  $h_5$ , then there is a  $3PC(x_1, x_n)$ . This yields a bridge of Type c2 [11.2].  $\square$

**Lemma 11.3** *Every direct connection  $P = x_1, \dots, x_n$  from  $T(\Sigma) \setminus \{h_1, h_3\}$  to  $B(\Sigma)$  avoiding  $\{h_1, h_3\}$  in  $G \setminus E(H)$  is either described in Lemma 11.2, or  $P$  satisfies the following two conditions:*

- There exist nodes  $x_j$  and  $x_k$ ,  $1 < j, k < n$ , of  $P$  adjacent to  $h_1$  and  $h_3$  respectively (possibly  $j = k$ ).
- Let  $x_j$ ,  $j < n$  be the node of highest index adjacent to  $h_1$  or  $h_3$ , say  $h_i$ . Then  $x_n$  is a fork of  $\Sigma$  adjacent to  $h_{i-1}$  and  $h_{i+1}$ .

*Proof:* If no node  $x_j$  of  $P$ ,  $1 < j < n$  is adjacent to  $h_3$ , then  $P$  is also a direct connection from  $T(\Sigma) \setminus \{h_1\}$  to  $B(\Sigma)$  avoiding  $\{h_1\}$  in  $G \setminus E(H)$  and is described in Lemma 11.2. By symmetry, a similar conclusion holds if no node  $x_j$ ,  $1 < j < n$  is adjacent to  $h_1$ . Hence the first condition of the lemma holds. Let  $x_j$  be the node of highest index adjacent to  $h_1$  or  $h_3$ , say  $h_3$ , such that there exists at least one  $x_k$ ,  $k \geq j$  adjacent to  $h_1$  but no node  $x_l$ ,  $l > j$  is adjacent to  $h_3$ . Then the subpath  $P_{x_j x_n}$  of  $P$  is a direct connection from  $T(\Sigma) \setminus \{h_1\}$  to  $B(\Sigma)$  avoiding  $\{h_1\}$  in  $G \setminus E(H)$ .

**Claim 1:** Node  $x_n$  has at most two neighbors in  $\Sigma$ , which are  $h_2$  and possibly either  $h_4$  or  $h_6$ .

*Proof of Claim 1:* Let  $x_l$  be a node of  $P$  adjacent to  $h_1$  and having highest index. (Obviously  $l \geq j$ ). Then the subpath  $P_{x_l x_n}$  of  $P$  is a direct connection from  $T(\Sigma)$  to  $B(\Sigma)$  in  $G \setminus E(H)$ . If  $l > j$  then  $x_l$  has  $h_1$  as unique neighbor in  $\Sigma$  and by Lemma 11.1 the claim holds. If  $l = j$  then the neighbors of  $x_l$  in  $\Sigma$  are  $h_1$  and  $h_3$  and by Lemma 11.1,  $P_{x_l x_n}$  is either a bridge of Type b1 or c1, in which case the claim holds, or  $x_n$  has a neighbor in  $B(\Sigma) \setminus \{h_2, h_4, h_6\}$ . Let  $P'$  be a direct connection using nodes of  $B(\Sigma)$  between  $x_n$  and  $h_6$  and avoiding  $\{h_2, h_4\}$  and  $P''$  be a direct connection using nodes of  $B(\Sigma)$  between  $x_n$  and  $h_4$  and avoiding  $\{h_2, h_6\}$ . Let  $P^*$  be a direct connection using nodes of  $T(\Sigma)$  between  $x_1$  and  $h_5$  and avoiding  $\{h_1, h_3\}$  and consider the holes  $H' = x_1, P^*, h_5, h_6, P', x_n, P, x_1$  and  $H'' = x_1, P^*, h_5, h_4, P'', x_n, P, x_1$ . Then if  $P$  has more than one neighbor of  $h_1$ , either  $(H', h_1)$  or  $(H'', h_1)$  is an odd wheel. Otherwise, if  $h_1$  has a unique neighbor, say  $h^*$  in  $P$ , there is a  $3PC(h^*, h_5)$ . This completes the proof of Claim 1.

Finally assume that  $x_n$  has  $h_2$  as unique neighbor in  $\Sigma$ . Let  $Q$  be a direct connection between  $h_5$  and  $h_2$  avoiding  $\{h_1, h_3\}$  and using nodes of  $T(\Sigma) \cup V(P)$  and let  $C' = h_5, Q, h_2, P_{26}, h_6, h_5$ ,  $C'' = h_5, Q, h_2, P_{24}, h_4, h_5$ . Then either  $(C', h_1)$  or  $(C'', h_1)$  is an odd wheel.  $\square$

## 12 Expanding the 6-Hole

**Definition 12.1** A fork  $z_i$  of  $\Sigma$  adjacent to nodes  $h_{i-1}$  and  $h_{i+1}$ ,  $i$  odd, is attached in  $\Sigma$  if there exists a direct connection  $P = x_1, \dots, x_n, z_i$  from  $T(\Sigma) \setminus \{h_i, h_j\}$  to  $B(\Sigma)$  avoiding  $\{h_i, h_j\}$  in  $G \setminus E(H)$  where  $j = i - 2$  or  $i + 2$  and  $x_1$  is adjacent to at least one node in  $T(\Sigma) \setminus \{h_1, h_3, h_5\}$ . The path  $x_1, \dots, x_n$  is an attachment of  $z_i$  to  $\Sigma$ . If  $i$  is even, an attached fork is defined accordingly.

We will maintain the convention that if  $P = x_1, \dots, x_n$  is an attachment of  $z_i$  to  $\Sigma$ , then  $x_n$  is adjacent to  $z_i$  and  $x_1$  to at least one node in  $\Sigma \setminus \{h_1, \dots, h_6\}$ .

**Definition 12.2** Let  $H_i(\Sigma)$  be the set of attached forks, adjacent to  $h_{i-1}$  and  $h_{i+1}$  together with the nodes adjacent to  $h_{i-1}$  and  $h_{i+1}$  and having neighbors in both sides of  $\Sigma$ . Note that  $h_i \in H_i(\Sigma)$ .

**Lemma 12.3** For every node  $z_i \in H_i(\Sigma)$ , say  $i$  odd, there exists a connected 6-hole  $\Sigma'$  having the following properties:

- $B(\Sigma') = B(\Sigma)$
- If  $z_i$  is a strongly adjacent node, with neighbors in both sides of  $\Sigma$ ,  $T(\Sigma') \subseteq (T(\Sigma) \cup \{z_i\}) \setminus \{h_i\}$ .  
If  $z_i$  is a fork of  $\Sigma$ , with attachment  $P$ ,  $T(\Sigma') \subseteq (T(\Sigma) \cup \{z_i\} \cup V(P)) \setminus \{h_i\}$  and  $V(P) \subseteq T(\Sigma')$ .

*Proof:* Assume w.l.o.g.  $i = 1$  and let  $P_{35}$  be the path of  $T(\Sigma)$  connecting  $h_3$  and  $h_5$ .

**Case 1:**  $z_1$  is a strongly adjacent node with neighbors in top and bottom of  $\Sigma$ .

If  $z_1$  is adjacent to a node in  $P_{35}$ , then  $\{z_1\} \cup V(P_{35})$  induces a fan top of  $\Sigma'$ . (Hence  $z_1$  has more than one neighbor in  $P_{35}$ ). If  $z_1$  is adjacent to a node in  $T(\Sigma) \setminus V(P_{35})$  but to no node in  $P_{35}$ , let  $Q$  be a direct connection in  $T(\Sigma)$  between  $z_1$  and  $V(P_{35}) \setminus \{h_3, h_5\}$  avoiding  $\{h_1, h_3, h_5\}$ . First note that the endnode of  $Q$  adjacent to  $P_{35}$  has a unique neighbor in  $V(P_{35})$ . Now if  $V(Q) \cup V(P_{35}) \cup \{z_1\}$  does not induce a triad then either  $h_3$  or  $h_5$  must have a neighbor in  $Q$ . By construction not both  $h_3$  and  $h_5$  have a neighbor in  $Q$ . Hence  $\Sigma'$  has a fan top with center  $h_3$  or  $h_5$ .

**Case 2:** Node  $z_1$  is a fork of  $\Sigma$ , with attachment  $P = x_1, \dots, x_m$ .

**Case 2.1:** Either  $h_3$  or  $h_5$  is adjacent to a node in  $V(P) \setminus \{x_1\}$ .

Assume w.l.o.g. that  $h_3$  is adjacent to a node in  $V(P) \setminus \{x_1\}$ . Let  $R$  be a direct connection in  $T(\Sigma) \cup V(P) \cup \{z_1\}$  between  $z_1$  and  $h_5$  avoiding  $\{h_1, h_3\}$ . Then  $V(P) \subseteq V(R)$ , hence  $h_3$  has a neighbor in  $R$ , and so  $R$  induces a fan top with center  $h_3$ .

**Case 2.2:** Neither  $h_3$  nor  $h_5$  is adjacent to a node in  $V(P) \setminus \{x_1\}$ .

Then the path induced by the node set  $\{z_1\} \cup V(P)$  satisfies either the second alternative of Lemma 11.1 or the first alternative of Lemma 11.2. Assume first that  $x_1$  has a neighbor in  $V(P_{35})$ . If  $x_1$  is strongly adjacent to  $P_{35}$ , we can shorten  $P$  and modify  $P_{35}$  accordingly. If  $z_1$  becomes adjacent to a node in  $V(P_{35})$  the argument of Case 1 holds. Now consider the case where  $x_1$  has a unique neighbor  $y$  in  $P_{35}$ . If  $y$  is adjacent to  $h_3$  or  $h_5$ , there is an odd wheel with center  $h_3$  or  $h_5$ . Otherwise  $V(P_{35}) \cup V(P) \cup \{z_1\}$  induces a triad.

If  $x_1$  has no neighbors in  $P_{35}$ , let  $Q$  be a direct connection in  $T(\Sigma) \cup V(P)$  between  $z_1$  and  $V(P_{35}) \setminus \{h_3, h_5\}$  avoiding  $\{h_1, h_3, h_5\}$ . Then by construction  $V(P) \subseteq V(Q)$  and  $Q$  cannot have both a neighbor of  $h_3$  and  $h_5$ . Hence  $V(P_{35}) \cup V(Q)$  induces a fan or a triad top.  $\square$

**Definition 12.4** A connected 6-hole  $\Sigma'$  satisfying Lemma 12.3 is said to be obtained from  $\Sigma$  by substituting node  $z_i$  (with attachment  $P_{z_i}$ ) for  $h_i$ . If  $i$  is even,  $T(\Sigma) = T(\Sigma')$  and  $z_i$  is said to be substituted in the bottom. If  $i$  is odd,  $B(\Sigma) = B(\Sigma')$  and  $z_i$  is said to be substituted in the top.

**Lemma 12.5** Let  $\Sigma'$  be a connected 6-hole obtained from  $\Sigma$  by substituting node  $z_i \in H_i(\Sigma)$  for  $h_i$ . Then  $H_j(\Sigma) = H_j(\Sigma')$  for  $j \in \{i-1, i+1, i+3\}$ .

*Proof:* Assume w.l.o.g.  $i = 1$ . Let  $z_1 \in H_1(\Sigma)$ , and if  $z_1$  is a fork of  $\Sigma$ , let  $P_{z_1} = x_1, \dots, x_n$  be an attachment of  $z_1$  to  $\Sigma$ . Assume w.l.o.g. that  $h_5$  is not adjacent to a node of  $V(P_{z_1}) \setminus \{x_1\}$ . Let  $z_j \in H_j(\Sigma)$ , where  $j$  is even. If  $z_j$  is a fork of  $\Sigma$ , let  $P_{z_j} = y_1, \dots, y_m$  be an attachment of  $z_j$  to  $\Sigma$ . Let  $\Sigma''$  be the connected 6-hole obtained from  $\Sigma$  by substituting node  $z_j$  (with attachment  $P_{z_j}$ ) for node  $h_j$ .

**Claim 1:** No node  $x_k$ ,  $1 \leq k \leq n$ , is adjacent to or coincident with a node in  $V(P_{z_j}) \cup \{z_j\}$ .

*Proof of Claim 1:* Suppose not. Let  $x_k$  be the node of  $P_{z_1}$  with the lowest index adjacent to or coincident with a node of  $V(P_{z_j}) \cup \{z_j\}$ . First



note that  $x_1$  cannot coincide with a node of  $V(P_{z_j}) \cup \{z_j\}$ , because  $x_1$  is adjacent to a node of  $T(\Sigma) \setminus \{h_1, h_3, h_5\}$ . If  $x_k$  is adjacent to a node of  $V(P_{z_j})$  then the path  $x_1, \dots, x_k$  is a direct connection from  $T(\Sigma'') \setminus \{h_1'', h_3''\}$  to  $B(\Sigma'')$  avoiding  $\{h_1'', h_3''\}$  in  $G \setminus E(H'')$ . This path contradicts Lemma 11.3 because both endnodes of this path are adjacent to a node of  $V(\Sigma'') \setminus V(H'')$ . Similarly, if node  $x_k$ ,  $k > 1$ , is coincident with a node of  $V(P_{z_j})$ , then the path  $x_1, \dots, x_{k-1}$  contradicts Lemma 11.3 in  $\Sigma''$ . If  $x_k$  is adjacent to  $z_j$ , then the path  $x_1, \dots, x_k$  contradicts Lemma 11.3 in  $\Sigma''$  since  $x_1$  is adjacent to a node of  $T(\Sigma'') \setminus \{h_1'', h_3'', h_5''\}$  and  $x_k$  is not adjacent to any node of  $\{h_2, h_4, h_6\}$ , so it cannot be a fork of  $\Sigma''$ . If node  $x_k$ ,  $k > 1$ , is coincident with  $y_{m+1}$  then the path  $x_1, \dots, x_{k-1}$  contradicts Lemma 11.3. This completes the proof of Claim 1.

**Claim 2:** Node  $z_1$  is not adjacent to or coincident with a node in  $V(P_{z_j})$ .

*Proof of Claim 2:* Suppose not. Let  $y_l$ ,  $1 \leq l \leq m$ , be the node of the lowest index adjacent to or coincident with the node  $z_1$ . If  $z_1$  is adjacent to  $y_l$  then the path  $x_1, \dots, x_n, z_1$  is a direct connection from  $T(\Sigma'') \setminus \{h_1'', h_3''\}$  to  $B(\Sigma'')$  avoiding  $\{h_1'', h_3''\}$  in  $G \setminus E(H'')$ . This path contradicts Lemma 11.3 because both  $x_1$  and  $z_1$  are adjacent to a node of  $V(\Sigma'') \setminus V(H'')$ . Similarly, if  $z_1$  is a fork of  $\Sigma$  coincident with  $y_l$ , then the path  $x_1, \dots, x_n$  contradicts Lemma 11.3. Finally if  $z_1$  is strongly adjacent to  $\Sigma$  with neighbors in both sides of  $\Sigma$  then it cannot coincide with  $y_l$  because  $\Sigma''$  is a connected 6-hole. This completes the proof of Claim 2.

**Claim 3:** Node  $z_1$  is adjacent to  $z_j$  if and only if  $j = 2$  or  $6$ .

*Proof of Claim 3:* Consider the graph  $G^*$  induced by the nodes in  $T(\Sigma') \cup B(\Sigma'')$ .

If  $j = 4$  and  $z_j$  is adjacent to  $z_1$ , then  $G^*$  is a connected 6-hole plus the additional edge  $z_1 z_4$ . If  $T(\Sigma')$  is a triad with meet  $t$ , then there is a  $3PC(z_4, t)$ . If  $T(\Sigma')$  is a fan  $T(\Sigma') \cup \{z_4\}$  induces an odd wheel.

If  $j = 2$  or  $6$ , say  $j = 2$ , and  $z_j$  is not adjacent to  $z_1$ , then  $G^*$  is a connected 6-hole minus the edge  $z_1 z_2$ . Let  $P'_{13}$  be the chordless path between  $z_1$  and  $h_3$  in  $\Sigma'$  and let  $P''_{26}$  be the chordless path between  $z_2$  and  $h_6$  in  $\Sigma''$ . Then there is a  $3PC(h_3, h_6)$  unless  $h_5$  has a neighbor in  $P'_{13}$  or  $h_4$  has a neighbor in  $P''_{26}$ . However in this case there is an odd wheel with center  $h_5$  or  $h_4$ . This completes the proof of Claim 3.

So Claims 1, 2 and 3 show  $z_j \in H_j(\Sigma')$  completing the proof of the lemma.

□

**Corollary 12.6** Given  $z_i \in H_i(\Sigma)$ ,  $i$  even, let  $\Sigma_{z_i}$  be a connected 6-hole obtained from  $\Sigma$  by substituting  $z_i$  for  $h_i$ . Similarly, given  $z_j \in H_j(\Sigma)$ ,  $j$  odd, let  $\Sigma_{z_j}$  be a connected 6-hole obtained from  $\Sigma$  by substituting  $z_j$  for  $h_j$ . Then  $z_j$  can be substituted for  $h_j$  in  $\Sigma_{z_i}$  and  $z_i$  can be substituted for  $h_i$  in  $\Sigma_{z_j}$ .

**Definition 12.7** Let  $T^*(\Sigma)$  be the set of nodes comprising:

- $T(\Sigma)$
- $\cup_{i \text{ odd}} H_i(\Sigma)$  together with all the attachments of forks in  $\cup_{i \text{ odd}} H_i(\Sigma)$ .

The set  $B^*(\Sigma)$  is defined similarly.

An immediate consequence of Lemma 12.5 is the following:

**Remark 12.8**  $T^*$  and  $B^*$  satisfy the following properties:

- (i) No node of  $T^*(\Sigma)$  coincides with a node of  $B^*(\Sigma)$ .
- (ii) Node  $w \in T^*(\Sigma)$  is adjacent to node  $z \in B^*(\Sigma)$  if and only if  $w \in H_i(\Sigma)$  and  $z \in H_j(\Sigma)$ , for  $j = i - 1$  or  $i + 1$ . Hence for every node set  $\{z_1, \dots, z_6\}$  where  $z_i \in H_i(\Sigma)$ ,  $i = 1, \dots, 6$ ,  $z_1, \dots, z_6, z_1$  is a 6-hole.

**Property 12.9** Given nonempty node sets  $A_1, \dots, A_6$ , that are pairwise disjoint, and node sets  $\Theta_T$  and  $\Theta_B$  such that  $\cup_{i \text{ odd}} A_i \subseteq \Theta_T$  and  $\cup_{i \text{ even}} A_i \subseteq \Theta_B$ , we consider a graph  $\Theta(\Theta_T, \Theta_B, A_1, \dots, A_6)$  induced by the node set  $\Theta_T \cup \Theta_B$  that satisfies the following property:

- (1) Every node  $u$  in  $\Theta_T \cup \Theta_B$  is contained in some connected 6-hole  $\Sigma$ , such that  $T(\Sigma) \subseteq \Theta_T$ ,  $B(\Sigma) \subseteq \Theta_B$  and  $h_i \in A_i$  for  $i = 1, \dots, 6$ . Furthermore if  $u \in A_i$ , then  $u = h_i$ .
- (2) Let  $F^t$  be any triad or fan with attachments  $a_i \in A_i$ ,  $i = 1, 3, 5$  such that  $V(F^t) \subseteq \Theta_T$ . Let  $F^b$  be any triad or fan with attachments  $a_i \in A_i$ ,  $i = 2, 4, 6$  such that  $V(F^b) \subseteq \Theta_B$ . Then  $V(F^t) \cup V(F^b)$  induces a connected 6-hole.

**Remark 12.10** If  $\Theta(\Theta_T, \Theta_B, A_1, \dots, A_6)$  satisfies Property 12.9, then it satisfies the following additional properties:

- (1) Let  $u$  be a node in  $\Theta_T$  and  $v$  be a node in  $\Theta_B$ . Then  $\Theta$  contains a connected 6-hole  $\Sigma$  such that  $u \in T(\Sigma) \subseteq \Theta_T$ ,  $v \in B(\Sigma) \subseteq \Theta_B$  and  $h_i \in A_i$ , for  $i = 1, \dots, 6$ . Furthermore, if  $u \in A_i$  for some odd index  $i$ , then  $u = h_i$ . If  $v \in A_j$  for some even index  $j$ , then  $v = h_j$ .
- (2) For every node set  $\{a_i \in A_i, i = 1, \dots, 6\}$ ,  $a_1, a_2, a_3, a_4, a_5, a_6, a_1$  is a 6-hole.

The following procedure constructs a graph, that we will show satisfies Property 12.9.

**Initialization:** Set  $j = 1$ . Let  $\Sigma^1$  be an arbitrary connected 6-hole of  $G$  with 6-hole  $H^1 = h_1^1, h_2^1, \dots, h_6^1, h_1^1$ . Let  $\Theta_T^1 = T^*(\Sigma^1)$ ,  $\Theta_B^1 = B^*(\Sigma^1)$ ,  $A_i^1 = H_i(\Sigma^1)$  for  $i = 1, \dots, 6$ . Let  $\Theta^1(\Theta_T^1, \Theta_B^1, A_1^1, \dots, A_6^1)$  be the graph induced by the node set  $\Theta_T^1 \cup \Theta_B^1$ . Let  $j = 1$  and repeat the following:

**Iterative Step:** If  $G$  contains no connected 6-hole  $\Sigma$  satisfying:

- $h_i \in A_i^j$  for  $i = 1, \dots, 6$ ,
- $\Sigma$  is distinct from all  $\Sigma^k$ ,  $1 \leq k \leq j$ , and one of the following two conditions holds:
  - (i)  $B(\Sigma) = B(\Sigma^k)$  for some  $1 \leq k \leq j$ , and no node of  $T(\Sigma) \setminus \{h_1, h_3, h_5\}$  is adjacent to or coincident with a node of  $\Theta_B^j$ ,
  - (ii)  $T(\Sigma) = T(\Sigma^k)$  for some  $1 \leq k \leq j$ , and no node of  $B(\Sigma) \setminus \{h_2, h_4, h_6\}$  is adjacent to or coincident with a node of  $\Theta_T^j$ ,

then stop. Otherwise, let  $\Sigma^{j+1}$  be such a connected 6-hole  $\Sigma$ . Denote the 6-hole of  $\Sigma^{j+1}$  by  $H^{j+1} = h_1^{j+1}, h_2^{j+1}, \dots, h_6^{j+1}, h_1^{j+1}$ . Let  $\Theta_T^{j+1} = \Theta_T^j \cup T^*(\Sigma^{j+1})$ ,  $\Theta_B^{j+1} = \Theta_B^j \cup B^*(\Sigma^{j+1})$ ,  $A_i^{j+1} = A_i^j \cup H_i(\Sigma^{j+1})$  for  $i = 1, \dots, 6$ . Let  $\Theta^{j+1}(\Theta_T^{j+1}, \Theta_B^{j+1}, A_1^{j+1}, \dots, A_6^{j+1})$  be the graph induced by the node set  $\Theta_T^{j+1} \cup \Theta_B^{j+1}$ . Increment  $j$  by 1, and repeat the Iterative Step.

Let  $w$  be the index when the above procedure terminates.

To illustrate the procedure, we now apply it to the graph in Figure 7

Let  $\Sigma^1$  be the connected 6-hole induced by the node set  $\{a, \dots, h, 1, \dots, 6\}$ . No node is attached to  $\Sigma^1$ , so  $\Theta_T^1 = \{a, b, c, d, 1, 3, 5\}$ ,  $\Theta_B^1 = \{e, f, g, h, 2, 4, 6\}$ ,  $A_i^1 = \{i\}$  for  $i = 1, \dots, 6$ .

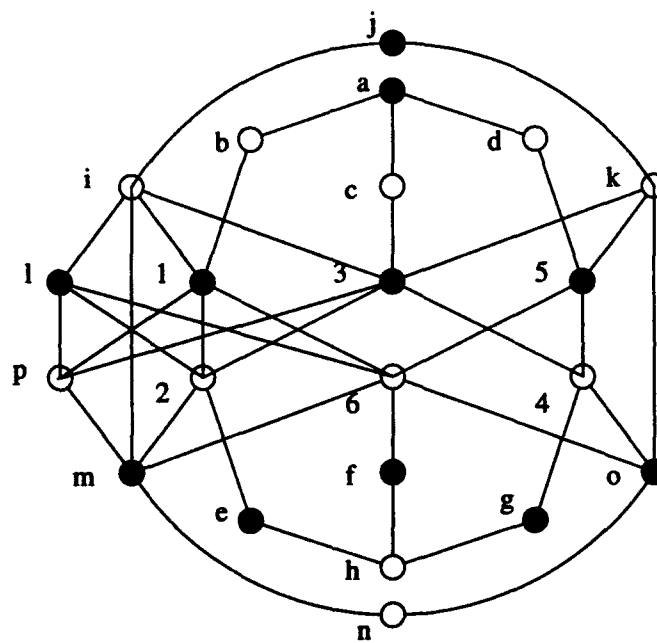


Figure 7: An Example

In the second iteration we can choose  $\Sigma^2$  such that  $B(\Sigma^2) = B(\Sigma^1)$  and  $T(\Sigma^2) = \{i, j, k, 1, 3, 5\}$ .  $\Theta_T^2 = \Theta_T^1 \cup \{i, j, k, l, m, o\}$  and  $\Theta_B^2 = \Theta_B^1$ . The set  $A_1^2$  now becomes  $\{1, l, m\}$ ,  $A_5^2$  is  $\{5, o\}$ .  $A_i^2 = A_i^1$  for  $i = 2, 3, 4, 6$ .

The subsequent iterations will enumerate all distinct connected 6-holes with  $B(\Sigma^1)$  as bottom and top  $\{a_1, 3, a_5, i, j, k\}$  where  $a_i \in A_i^2$  and  $a_5 \in A_5^2$ . The sets  $\Theta_T^2$ ,  $\Theta_B^2$ ,  $A_i^2$ ,  $i = 1, \dots, 6$  remain unchanged in the subsequent iterations. Note that a different choice of  $\Sigma^2$ , namely one having the same top as  $\Sigma^1$ , would yield different sets  $A_i^2$ .

The following lemmas will be used in the proof of the main theorem of this section.

**Theorem 12.11** *The graph  $\Theta^w$  satisfies Property 12.9 .*

**Definition 12.12** *Assume that for some  $1 \leq n \leq w$ ,  $\Theta^n$  satisfies Property 12.9. Then for every  $1 \leq i, j \leq n$ , the graph induced by the node set  $T(\Sigma^i) \cup B(\Sigma^j)$  is a connected 6-hole. We denote this connected 6-hole having top  $T(\Sigma^i)$  and bottom  $B(\Sigma^j)$  by  $\Sigma^{T_i B_j}$ .*

Note that the algorithm labels every possible connected 6-hole  $\Sigma^{T_i B_j}$  as  $\Sigma^k$  for some  $k \leq w$ .

**Lemma 12.13** *Assume that for some  $1 \leq n < w$ ,  $\Theta^n$  satisfies Property 12.9. Then  $\Theta(\Theta_T^n \cup T(\Sigma^{n+1}), \Theta_B^n \cup B(\Sigma^{n+1}), A_1^n, \dots, A_6^n)$  satisfies Property 12.9 and for every  $k \leq n$ , the graphs induced by  $T(\Sigma^{n+1}) \cup B(\Sigma^k)$  and  $T(\Sigma^k) \cup B(\Sigma^{n+1})$  are the connected 6-holes  $\Sigma^{T_{n+1} B_k}$  and  $\Sigma^{T_k B_{n+1}}$ .*

*Proof:* The first statement follows from the conditions imposed on  $\Sigma^{n+1}$  by the procedure. The second statement follows from the first and Remark 12.10.  $\square$

**Lemma 12.14** *Let  $\Sigma$  and  $\Sigma'$  be connected 6-holes such that  $T(\Sigma) = T(\Sigma')$  and  $h_2 = h'_2$ . Let  $z_1 \in H_1(\Sigma)$ .*

- (i) *If  $z_1$  is not a fork of  $\Sigma$ , then  $z_1 \in H_1(\Sigma')$ .*
- (ii) *If  $z_1$  is a fork of  $\Sigma$ , let  $P_{z_1} = x_1, \dots, x_n$  be an attachment of  $z_1$  to  $\Sigma$ . If  $h_5$  is not adjacent to a node of  $V(P_{z_1}) \setminus \{x_1\}$ , then  $z_1 \in H_1(\Sigma')$  and  $P_{z_1}$  is an attachment of  $z_1$  in  $\Sigma'$ .*

*Proof:* Let  $z_1 \in H_1(\Sigma)$  and, if  $z_1$  is a fork of  $\Sigma$  with attachment  $P_{z_1} = x_1, \dots, x_n$ , assume that  $h_5$  is not adjacent to a node of  $V(P_{z_1}) \setminus \{x_1\}$ . We divide the proof into the following two claims:

**Claim 1:** No node of  $P_{z_1}$  is adjacent to or coincident with a node of  $B(\Sigma')$ .

*Proof of Claim 1:* Assume not. Let  $x_k$  be the node of  $P_{z_1}$  with the lowest index that is adjacent to or coincident with a node of  $B(\Sigma')$ . First note that  $x_1$  cannot coincide with a node of  $B(\Sigma') \setminus \{h'_2, h'_4, h'_6\}$  because  $x_1$  is adjacent to a node of  $T(\Sigma) \setminus \{h_1, h_3, h_5\}$ . If node  $x_k$  is adjacent to a node of  $B(\Sigma') \setminus \{h'_2, h'_4, h'_6\}$ , then  $x_1, \dots, x_k$  is a direct connection from  $T(\Sigma') \setminus \{h'_1, h'_3\}$  to  $B(\Sigma')$  avoiding  $\{h'_1, h'_3\}$  in  $G \setminus E(H')$ . This path contradicts Lemma 11.3 because both endnodes of this path are adjacent to a node of  $V(\Sigma') \setminus V(H')$ . Similarly, if node  $x_k, k > 1$ , is coincident with a node of  $B(\Sigma') \setminus \{h'_2, h'_4, h'_6\}$  then the path  $x_1, \dots, x_{k-1}$  contradicts Lemma 11.3. Node  $x_k$  is not adjacent to or coincident with  $h'_2$  since  $h'_2 = h_2$  and  $P_{z_1}$  is an attachment of  $z_1$  to  $\Sigma$ . Node  $x_k$  is not coincident with  $h'_4$  or  $h'_6$ , otherwise  $x_k$  is adjacent to  $h_5$ . Thus node  $x_k$  must be adjacent to a node in  $\{h'_4, h'_6\}$ . But now  $x_1, \dots, x_k$  is a direct connection from  $T(\Sigma') \setminus \{h'_1, h'_3\}$  to  $B(\Sigma')$  avoiding  $\{h'_1, h'_3\}$  in  $G \setminus E(H')$ , so by Lemma 11.3 node  $x_k$  must be a fork of  $\Sigma'$ , adjacent to both  $h'_4$  and  $h'_6$ . Hence  $x_k \in H_5(\Sigma')$ . Let  $\Sigma'_{x_k}$  be the connected 6-hole obtained from  $\Sigma'$  by substituting node  $x_k$  for the node  $h_5$ . If  $h_1$  and  $h_3$  are not adjacent to any node of  $x_{k+1}, \dots, x_n$  then  $x_{k+1}, \dots, x_n, z_1$  is a direct connection between  $T(\Sigma'_{x_k})$  and  $B(\Sigma'_{x_k}) \setminus \{h'_4, h'_6\}$  avoiding  $\{h'_4, h'_6\}$  in  $G \setminus \{h'_1 h'_2, h'_2 h'_3, h'_3 h'_4, h'_4 x_k, x_k h'_6, h'_6 h'_1\}$  and it violates Lemma 11.3. Now assume that  $h_1$  or  $h_3$  is adjacent to some node of  $x_k, \dots, x_n$ , and let  $x_m$  be the node of  $x_k, \dots, x_n$  with the lowest index adjacent to  $h_1$  or  $h_3$ . Assume w.l.o.g. that  $x_m$  is adjacent to  $h_1$ . Let  $x_l, k \leq l < m$  be the node of highest index adjacent to  $h'_4$ . Then  $x_l, \dots, x_m$  is a direct connection from  $T(\Sigma')$  to  $B(\Sigma') \setminus \{h'_6\}$  avoiding  $\{h'_6\}$  in  $G \setminus E(H')$ , violating Lemma 11.2. This completes the proof of Claim 1.

**Claim 2:** Node  $z_1$  is not adjacent to or coincident with a node of  $B(\Sigma') \setminus \{h'_2, h'_6\}$ .

*Proof of Claim 2:* Node  $z_1$  is not adjacent to  $h'_4$ , because otherwise the node set  $\{h_1, h_2, h_3, h'_4, h_5, h_6\}$  induces an odd wheel with center  $z_1$ . If  $z_1$  is strongly adjacent to  $\Sigma$ , with neighbors in  $B(\Sigma)$  and  $T(\Sigma)$ , then  $z_1$  is also strongly adjacent to  $\Sigma'$  with neighbors in  $B(\Sigma')$  and  $T(\Sigma') = T(\Sigma)$ . So by Theorem 10.2  $z_1$  is adjacent to  $h'_2, h'_6$  but no other node of  $B(\Sigma')$ . Now

assume  $z_1$  is a fork of  $\Sigma$ . Then  $z_1$  is not coincident with a node of  $B(\Sigma')$ , else node  $x_n$  contradicts Claim 1. Now assume that  $z_1$  is adjacent to a node of  $B(\Sigma') \setminus \{h'_2, h'_6, h'_4\}$ . Then, by Claim 1,  $x_1, \dots, x_n, z_1$  is a direct connection from  $T(\Sigma') \setminus \{h'_1, h'_3\}$  to  $B(\Sigma')$  avoiding  $\{h'_1, h'_3\}$ , in  $G \setminus E(H')$  which violates Lemma 11.3. This completes the proof of Claim 2.

Now by Claim 1 and Claim 2, if  $z_1$  is not adjacent to  $h'_6$ , then  $z_1 \notin H_1(\Sigma')$ , so the path  $x_1, \dots, x_n, z_1$  contradicts Lemma 11.3 applied to  $\Sigma'$ . Hence  $z_1 \in H_1(\Sigma')$  and if  $z_1$  is a fork of  $\Sigma$ ,  $P_{z_1}$  is an attachment of  $z_1$  to  $\Sigma'$ .  $\square$

**Lemma 12.15** Assume that for some  $m \leq w$ , the graph  $\Theta^{m-1}$  satisfies Property 12.9. Let  $z_i \in H_i(\Sigma^m)$ ,  $i$  odd, and if  $z_i$  is a fork of  $\Sigma^m$ , let  $P_{z_i}$  be an attachment of  $z_i$  to  $\Sigma^m$ . Then  $z_i \in H_i(\Sigma^{T_m B_1})$  and if  $z_i$  is a fork of  $\Sigma^m$ ,  $P_{z_i}$  is an attachment of  $z_i$  to  $\Sigma^{T_m B_1}$ .

*Proof* : Assume w.l.o.g.  $i = 1$ .

**Claim 1:** Assume that for some  $1 < k \leq m$ ,  $z_1 \in H_1(\Sigma^{T_m B_k})$ . If  $z_1$  is a fork of  $\Sigma^{T_m B_k}$ , let  $P_{z_1}$  be an attachment of  $z_1$  to  $\Sigma^{T_m B_k}$ . Then there exists some  $j < k$  such that  $z_1 \in H_1(\Sigma^{T_m B_j})$  and  $P_{z_1}$  is an attachment of  $z_1$  to  $\Sigma^{T_m B_j}$ .

*Proof of Claim 1:* By construction  $h_2^k \in A_2^{k-1}$  and  $h_6^k \in A_6^{k-1}$ , and hence there exists  $i, j < k$  such that  $h_2^k \in H_2(\Sigma^j)$  and  $h_6^k \in H_6(\Sigma^i)$ . Let  $\Sigma^{jk}$  be the connected 6-hole obtained from  $\Sigma^j$  by substituting  $h_2^k$  for  $h_2^j$ . Let  $\Sigma^{ik}$  be the connected 6-hole obtained from  $\Sigma^i$  by substituting  $h_6^k$  for  $h_6^i$ . Since by the definition of attachment, not both  $h_3^m$  and  $h_5^m$  are adjacent to a node of  $V(P_{z_1}) \setminus \{x_1\}$ , applying Lemma 12.14 either with  $\Sigma = \Sigma^{T_m B_k}$  and  $\Sigma' = \Sigma^{T_m B_{jk}}$  or with  $\Sigma = \Sigma^{T_m B_k}$  and  $\Sigma' = \Sigma^{T_m B_{ik}}$  we have that  $z_1 \in H_1(\Sigma^{T_m B_{jk}})$  or  $z_1 \in H_1(\Sigma^{T_m B_{ik}})$  and  $P_{z_1}$  is an attachment of  $z_1$  to one of the two connected 6-holes. Assume w.l.o.g. that  $z_1 \in H_1(\Sigma^{T_m B_{jk}})$  and  $P_{z_1}$  is an attachment of  $z_1$  to  $\Sigma^{T_m B_{jk}}$ . Applying again Lemma 12.14 to  $\Sigma = \Sigma^{T_m B_{jk}}$  and  $\Sigma' = \Sigma^{T_m B_j}$  we have that  $z_1 \in H_1(\Sigma^{T_m B_j})$  and  $P_{z_1}$  is an attachment of  $z_1$  to  $\Sigma^{T_m B_j}$ . This completes the proof of Claim 1.

Now the lemma follows by repeated applications of Claim 1, starting with  $\Sigma^m = \Sigma^{T_m B_m}$ .  $\square$

**Lemma 12.16** Let  $\Sigma$  and  $\Sigma'$  be connected 6-holes such that  $T(\Sigma) = T(\Sigma')$  and  $h_2 = h'_2$ . Let  $z_1 \in H_1(\Sigma)$ .

(i) If  $z_1$  is not a fork of  $\Sigma$  then  $z_1 \in H_1(\Sigma')$ .

(ii) If  $z_1$  is a fork of  $\Sigma$ , let  $P_{z_1}$  be an attachment of  $z_1$  to  $\Sigma$ . If no node of  $P_{z_1}$  is adjacent to or coincident with  $h'_4$  or  $h'_6$  then  $z_1 \in H_1(\Sigma')$  and  $P_{z_1}$  is an attachment of  $z_1$  to  $\Sigma'$ .

*Proof:* Let  $z_1 \in H_1(\Sigma)$  and if  $z_1$  is a fork of  $\Sigma$  then let  $P_{z_1} = x_1, \dots, x_n$  be an attachment of  $z_1$  to  $\Sigma$ . Suppose that  $P_{z_1}$  satisfies the conditions of the lemma. Assume w.l.o.g. that  $h_5$  is not adjacent to a node of  $V(P_{z_1}) \setminus \{x_1\}$ .

**Claim 1:** No node of  $P_{z_1}$  is adjacent to or coincident with a node of  $B(\Sigma')$ .

*Proof of Claim 1:* Assume not. Let  $x_k$  be the node of  $P_{z_1}$  with the lowest index that is adjacent to or coincident with a node of  $B(\Sigma')$ . First note that  $x_1$  cannot coincide with a node of  $B(\Sigma') \setminus \{h'_2, h'_4, h'_6\}$  because  $x_1$  is adjacent to a node of  $T(\Sigma) \setminus \{h_1, h_3, h_5\}$ . If node  $x_k$  is adjacent to a node of  $B(\Sigma') \setminus \{h'_2, h'_4, h'_6\}$ , then  $x_1, \dots, x_k$  is a direct connection from  $T(\Sigma') \setminus \{h'_1, h'_3\}$  to  $B(\Sigma')$  avoiding  $\{h'_1, h'_3\}$  in  $G \setminus E(H')$ . This path contradicts Lemma 11.3 because both endnodes of this path are adjacent to a node of  $V(\Sigma') \setminus V(H')$ . Similarly, if node  $x_k, k > 1$ , is coincident with a node of  $B(\Sigma') \setminus \{h'_2, h'_4, h'_6\}$  then the path  $x_1, \dots, x_{k-1}$  contradicts Lemma 11.3. Node  $x_k$  is not adjacent to or coincident with  $h'_2$  since  $h'_2 = h_2$  and  $P_{z_1}$  is an attachment of  $z_1$  to  $\Sigma$ . This completes the proof of Claim 1.

**Claim 2:** Node  $z_1$  is not adjacent to or coincident with a node of  $B(\Sigma') \setminus \{h'_2, h'_6\}$ .

*Proof of Claim 2:* Identical to proof of Claim 2 in Lemma 12.14.

Now by Claim 1 and Claim 2, if  $z_1$  is not adjacent to  $h'_6$ , then  $z_1 \notin H_1(\Sigma')$ , so the path  $x_1, \dots, x_n, z_1$  contradicts Lemma 11.3 applied to  $\Sigma'$ . Hence  $z_1 \in H_1(\Sigma')$ .  $\square$

*Proof of Theorem 12.11:* Let  $n$  be the smallest index for which  $\Theta^n$  does not satisfy Property 12.9. Lemma 12.5 and Corollary 12.6 show that  $\Theta^1$  satisfies Property 12.9. Hence  $n > 1$ . Furthermore Lemma 12.13 shows that  $\Theta(\Theta_T^{n-1} \cup T(\Sigma^n), \Theta_B^{n-1} \cup B(\Sigma^n), A_i^{n-1}, i = 1 \dots, 6)$  satisfies Property 12.9. By construction of  $\Theta^n$  and by Lemma 12.5 applied to every  $\Sigma^{T^n B^k}, k \leq n$ , Property 12.9 holds for  $\Theta^n$  if the following holds:

Let  $z_i \in H_i(\Sigma^n)$ ,  $i$  odd, and if  $z_i$  is a fork of  $\Sigma^n$ , let  $P_{z_i}$  be an attachment of  $z_i$  to  $\Sigma^n$ . Let  $z_j \in H_j(\Sigma^k)$ , where  $j$  is even and  $k \leq n$ . If  $z_j$  is a fork of  $\Sigma^k$ , let  $P_{z_j}$  be an attachment of  $z_j$  to  $\Sigma^k$ . Then  $z_i \in H_i(\Sigma^{T^n B^k})$  and  $P_{z_i}$  is an



attachment of  $z_i$  to  $\Sigma^{T_n B_k}$  and  $z_j \in H_j(\Sigma^{T_n B^k})$  and  $P_{z_j}$  is an attachment of  $z_j$  to  $\Sigma^{T_n B_k}$ .

The next two claims prove the above statement.

**Claim 1:** Let  $z_j \in H_j(\Sigma^k)$ , where  $j$  is even and  $k \leq n$ . If  $z_j$  is a fork of  $\Sigma^k$ , let  $P_{z_j}$  be an attachment of  $z_j$  to  $\Sigma^k$ . Then  $z_j \in H_j(\Sigma^{T_n B^k})$  and  $P_{z_j}$  is an attachment of  $z_j$  to  $\Sigma^{T_n B_k}$ .

*Proof of Claim 1:* If  $k = n$  the claim follows from Corollary 4.6. If  $k < n$ , then  $z_j$  together with  $V(P_{z_j})$  belongs to  $\Theta_B^{n-1}$ . Hence the claim follows by construction of  $\Sigma^n$ . This completes the proof of Claim 1.

**Claim 2:** Let  $z_i \in H_i(\Sigma^n)$ ,  $i$  odd, and if  $z_i$  is a fork of  $\Sigma^n$ , let  $P_{z_i}$  be an attachment of  $z_i$  to  $\Sigma^n$ . Then for every  $k \leq n$ ,  $z_i \in H_i(\Sigma^{T_n B^k})$  and  $P_{z_i}$  is an attachment of  $z_i$  to  $\Sigma^{T_n B_k}$ .

*Proof of Claim 2:* Assume w.l.o.g. that  $i = 1$ . Let  $k < n$  be the smallest index for which  $z_1 \notin H_1(\Sigma^{T_n B_k})$ . Then by Lemma 12.15,  $k > 1$ . By construction,  $h_2^k$  in  $A_2^{k-1}$ . Hence there exists a  $j \leq k-1$  such that  $h_2^k \in H_2(\Sigma^j)$ . Then, by the choice of  $k$ ,  $z_1 \in H_1(\Sigma^{T_n B_j})$  and by Claim 1,  $h_2^k \in H_2(\Sigma^{T_n B_j})$ . Now Lemma 12.5 applied to  $\Sigma^{T_n B_j}$  shows that  $z_1$  is adjacent to  $h_2^k$  and no node of  $P_{z_1}$  is adjacent to or coincident with  $h_2^k$ . The same argument shows that  $z_1$  is adjacent to  $h_6^k$  but not to  $h_4^k$  and no node of  $P_{z_1}$  is adjacent to or coincident with  $h_4^k$  or  $h_6^k$ . Let  $\Sigma^{jk}$  be the connected 6-hole obtained by substituting  $h_2^k$  for  $h_2^j$  in  $\Sigma^j$ . Consider the connected 6-hole  $\Sigma^{T_n B_{jk}}$ . Now by Lemma 12.16 applied to  $\Sigma = \Sigma^{T_n B_j}$  and  $\Sigma' = \Sigma^{T_n B_{jk}}$ , we have that  $z_1 \in H_1(\Sigma^{T_n B_{jk}})$  and  $P_{z_1}$  is an attachment of  $z_1$  in  $\Sigma^{T_n B_{jk}}$ . Applying again Lemma 12.16 to  $\Sigma = \Sigma^{T_n B_{jk}}$  and  $\Sigma' = \Sigma^{T_n B_k}$ , we have that  $z_1 \in H_1(\Sigma^{T_n B_k})$  and  $P_{z_1}$  is an attachment of  $z_1$  in  $\Sigma^{T_n B_k}$ . This completes the proof of Claim 2.  $\square$

**Corollary 12.17** *If  $\Sigma$  is a connected 6-hole such that  $T(\Sigma) \subseteq \Theta_T^w$ ,  $B(\Sigma) \subseteq \Theta_B^w$ , and  $h_i \in A_i^w$  for  $i = 1, \dots, 6$ , then  $\Sigma$  coincides with  $\Sigma^k$ , for some  $1 \leq k \leq w$ .*

*Proof:* Suppose that  $\Sigma$  does not coincide with any  $\Sigma^i$ ,  $i = 1, \dots, w$ . First we show that for some  $1 \leq k \leq w$ ,  $T(\Sigma) = T(\Sigma^k)$ . Suppose not. Then since by Theorem 12.11  $\Theta^w$  satisfies Property 12.9 (2),  $\Sigma'$  such that  $T(\Sigma') = T(\Sigma)$  and  $B(\Sigma') = B(\Sigma^1)$  is a connected 6-hole satisfying the rules of construction. Hence for some  $1 \leq k \leq w$ ,  $\Sigma'$  coincides with  $\Sigma^k$ . Now  $\Sigma$  is such that  $T(\Sigma) = T(\Sigma^k)$  and no node of  $B(\Sigma) \setminus \{h_2, h_4, h_6\}$  is adjacent to or coincident with a node of  $\Theta_T^w$ . Hence it is possible to define  $\Sigma^{w+1}$ , which contradicts the maximality of  $w$ .  $\square$

**Corollary 12.18** *If  $u \in \Theta_T^\omega$  and  $v \in \Theta_B^\omega$ , then for some  $1 \leq k \leq w$ , nodes  $u$  and  $v$  are contained in  $\Sigma^k$ .*

*Proof:* Follows from Theorem 12.11, Remark 12.10 (1) and Corollary 12.17.  $\square$

### 13 Extended Star Cutset

**Definition 13.1** *Let  $A$  be the graph induced by the node set  $\cup_{i=1}^6 A_i^\omega$ .*

**Lemma 13.2** *If the removal of the edge set  $E(A)$  disconnects the graph  $G$ , then  $G$  contains a 6-join.*

*Proof:* Assume that the edge set  $E(A)$  disconnects the graph  $G$ . By Theorem 12.11,  $\Theta^\omega$  satisfies Property 12.9. Now by Remark 12.10 (2),  $E(A)$  is a 6-join of  $G$ .  $\square$

In this section we prove the following theorem.

**Theorem 13.3**  *$G$  contains a 6-join or an extended star cutset.*

**Lemma 13.4** *If  $P = x_1, \dots, x_n$  is a direct connection from  $\Theta_T^\omega \setminus (A_1^\omega \cup A_3^\omega)$  to  $\Theta_B^\omega$  avoiding  $A_1^\omega \cup A_3^\omega$  in  $G \setminus E(A)$ , then  $N(x_1) \cap V(\Theta^\omega) \subseteq A_1^\omega \cup A_3^\omega \cup A_5^\omega$ ,  $N(x_1) \cap A_5^\omega \neq \emptyset$  and  $N(x_n) \cap V(\Theta^\omega) \subseteq A_2^\omega \cup A_4^\omega \cup A_6^\omega$ .*

*Proof:* If  $x_1$  is adjacent to a node  $v \in \Theta_B^\omega$ , then  $n = 1$ , and let  $u$  be any neighbor of  $x_1$  in  $\Theta_T^\omega$ . By Corollary 12.18, for some  $1 \leq k \leq w$ , nodes  $u$  and  $v$  are contained in  $\Sigma^k$ . Node  $x_1$  is now strongly adjacent to  $\Sigma^k$ , with neighbors in both sides of  $\Sigma^k$ . Hence for some  $i \in \{1, \dots, 6\}$ ,  $x_1 \in H_i(\Sigma^k)$ . By construction of  $\Theta^\omega$ ,  $x_1 \in V(\Theta^\omega)$  which contradicts our choice of  $P$ . Therefore  $N(x_1) \cap V(\Theta^\omega) \subseteq \Theta_T^\omega$ . Now suppose that  $x_1$  is adjacent to a node  $u \in \Theta_T^\omega \setminus (A_1^\omega \cup A_3^\omega \cup A_5^\omega)$ . Let  $v$  be any neighbor of  $x_n$  in  $\Theta_B^\omega$ . By Corollary 12.18, for some  $1 \leq k \leq w$ , nodes  $u$  and  $v$  are contained in  $\Sigma^k$ . Now by Lemma 11.3 and Definition 12.1,  $x_n$  is an attached node with respect to  $\Sigma^k$ . By construction of  $\Theta^\omega$ ,  $x_n \in \Theta_T^\omega$  which contradicts our choice of  $P$ . Therefore  $N(x_1) \cap V(\Theta^\omega) \subseteq A_1^\omega \cup A_3^\omega \cup A_5^\omega$ . Similarly  $N(x_n) \cap V(\Theta^\omega) \subseteq A_2^\omega \cup A_4^\omega \cup A_6^\omega$ .  $\square$

**Definition 13.5** A bridge of Type  $c$  with respect to a connected 6-hole  $\Sigma$ , is a configuration  $C$  satisfying the following properties:

- $C$  is connected.
- There exist nodes  $h_{i-1}, h_i, h_{i+1}, h_{i+2}$  of  $\Sigma$  that are adjacent to at least one node of  $C$ . No other node of  $\Sigma$  is adjacent to a node of  $C$ .
- $C$  is minimal with the above two properties.

Note that bridges of Type  $c_1$  and Type  $c_2$  defined in Lemma 11.1 and Lemma 11.2 are bridges of Type  $c$ .

**Lemma 13.6** Let  $C$  be a bridge of Type  $c$ . Then  $C$  satisfies the following property:

- $C$  induces a path  $P = x_1, \dots, x_n$  which is a direct connection between  $h_{i-1}$  and  $h_{i+2}$  in  $C$ .
- $P$  contains at least one node adjacent to  $h_i$  and at least one node adjacent to  $h_{i+1}$ .

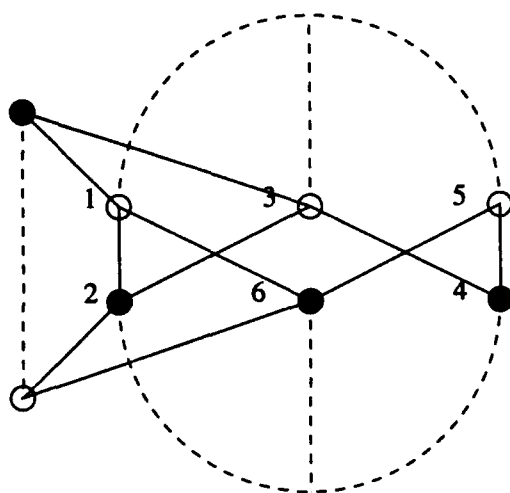
*Proof:* Assume w.l.o.g. that  $i = 1$ . Let  $P$  be a direct connection in  $C$  between  $h_3$  and  $h_6$ . If  $P$  satisfies the second property of the lemma, then by minimality of  $C$ ,  $P$  and  $C$  coincide. If  $P$  has no node adjacent to  $h_2$ , then  $P$  is a direct connection between  $T(\Sigma) \setminus \{h_1\}$  and  $B(\Sigma)$  avoiding  $\{h_1\}$  in  $G \setminus E(H)$  violating Lemma 11.2. If  $P$  has no node adjacent to  $h_1$ , the proof is identical.  $\square$

Figure 8 depicts possible bridges of Type  $c$ .

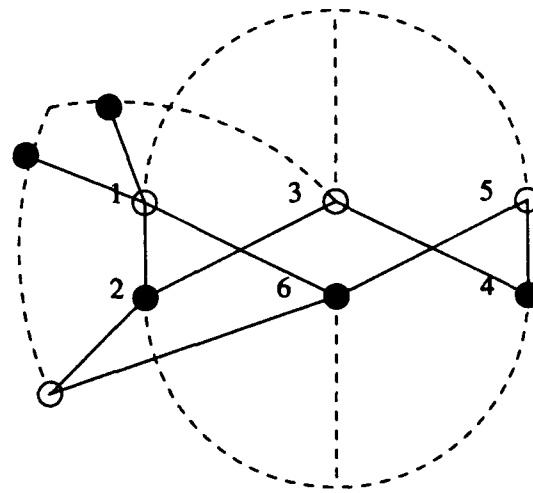
**Lemma 13.7** Let  $P$  be a bridge of Type  $c$  with respect to  $\Sigma^k$ , for some  $1 \leq k \leq w$ , containing nodes adjacent to  $h_1^k, h_2^k, h_3^k, h_6^k$ . If no node of  $P$  is adjacent to any node in  $V(\Theta^w) \setminus V(A)$ , then every node in  $A_1^w \cup A_2^w$  has a neighbor on  $P$ .

*Proof:* Let  $\Sigma^k$  and  $P$  satisfy the conditions of the lemma. It suffices to prove the result for  $A_2^w$ .

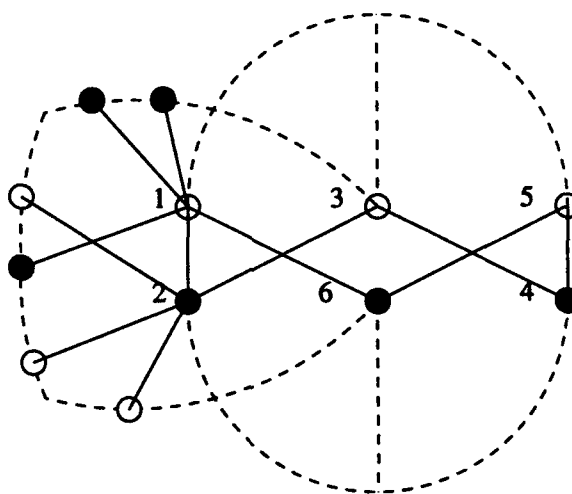
**Claim 1:** For  $1 \leq m \leq w$ , if some node  $x \in H_2(\Sigma^m)$  has a neighbor on  $P$ , then every node in  $H_2(\Sigma^m)$  has a neighbor on  $P$ .



Bridge of Type c1



Bridge of Type c2



Bridge of Type c

Figure 8: Bridges of Type c

*Proof of Claim 1:* Let  $y \in H_2(\Sigma^m)$ , and suppose that  $y$  is not adjacent to any node of  $P$ . Let  $\Sigma_x^m$  (resp.  $\Sigma_y^m$ ) be a connected 6-hole obtained from  $\Sigma^m$  by substituting node  $x$  (resp.  $y$ ) for  $h_2^m$ .

If there exists a chordless path  $Q$  from  $h_6^k$  to  $y$  using only nodes in  $(\Theta_B^w \setminus (A_2^w \cup A_4^w \cup A_6^w)) \cup \{h_6^k, y\}$ , then there exists a  $3PC(h_3^k, h_6^k)$  using  $P$ ,  $Q$  and a path connecting  $h_3^k$  and  $h_5^k$  in  $T(\Sigma^k)$ . So assume that no such path exists. In particular  $h_6^k$  is not adjacent to any node in  $B(\Sigma_x^m) \cup B(\Sigma_y^m)$ . Now note that  $V(P) \cup \{h_6^k, h_1^k, h_3^k, h_5^k\}$  induces a fan top attached to  $h_1^k, h_3^k, h_5^k$ , with center  $h_1^k$ , which is node disjoint from  $B(\Sigma_y^m)$  and has no adjacencies to  $B(\Sigma_y^m) \setminus \{y, h_4^m, h_6^m\}$ . Let  $\Sigma$  be the connected 6-hole induced by the node set  $V(P) \cup \{h_6^k, h_1^k, h_3^k, h_5^k\} \cup B(\Sigma_y^m)$ . The 6-hole of  $\Sigma$  is  $h_1^k, y, h_3^k, h_4^m, h_5^m, h_6^m, h_1^k$ . Since both  $x, y \in H_2(\Sigma^m)$ , by Lemma 12.5  $x \in H_2(\Sigma_y^m)$ . If  $x$  has neighbors in both sides of  $\Sigma_y^m$  then  $x$  is adjacent to a node in  $B(\Sigma_y^m) \setminus \{y, h_6^m, h_4^m\}$ . By assumption  $x$  is adjacent to  $P$ . Hence  $x$  is a strongly adjacent node to  $\Sigma$  which violates Theorem 10.2. If  $x$  is a fork of  $\Sigma_y^m$ , let  $x_1, \dots, x_n$  be an attachment of  $x$  to  $\Sigma_y^m$ . We can assume w.l.o.g. that  $x_1, \dots, x_n, x$  is a direct connection from  $B(\Sigma_y^m) \setminus \{y, h_6^m\}$  to  $T(\Sigma_y^m)$  avoiding  $\{y, h_6^m\}$  in  $G \setminus \{h_1^m y, y h_3^m, h_3^m h_4^m, h_4^m h_5^m, h_5^m h_6^m, h_6^m h_1^m\}$ . Hence  $x_1$  is adjacent to  $B(\Sigma_y^m) \setminus \{y, h_4^m, h_6^m\}$ . By assumption  $x$  has a neighbor on  $P$ , hence  $x_1, \dots, x_n, x$  is a direct connection from  $B(\Sigma) \setminus \{y, h_6^m\}$  to  $T(\Sigma)$  avoiding  $\{y, h_6^m\}$  in  $G \setminus \{h_1^k y, y h_3^k, h_3^k h_4^m, h_4^m h_5^k, h_5^k h_6^m, h_6^m h_1^k\}$ , violating Lemma 11.3. This completes the proof of Claim 1.

**Claim 2:** If for some  $1 < n \leq w$ , every node in  $H_2(\Sigma^n)$  has a neighbor on  $P$ , then there exists  $1 \leq m < n$  such that every node in  $H_2(\Sigma^m)$  has a neighbor on  $P$ .

*Proof of Claim 2:* Assume that  $1 < n \leq w$ , and every node in  $H_2(\Sigma^n)$  has a neighbor on  $P$ . In particular  $h_2^n$  is adjacent to a node of  $P$ . But by construction  $h_2^n \in A_2^{n-1}$ , thus there exists  $1 \leq m < n$  such that  $h_2^n \in H_2(\Sigma^m)$ . Now by Claim 1, every node in  $H_2(\Sigma^m)$  has a neighbor on  $P$ . This completes the proof of Claim 2.

Now we show that for every  $1 \leq n \leq w$ , every node in  $A_2^n$  has a neighbor on  $P$ , by induction on  $n$ . By Claim 1 every node in  $H_2(\Sigma^k)$  has a neighbor on  $P$ , so by repeated application of Claim 2, every node in  $H_2(\Sigma^1) = A_2^1$  has a neighbor on  $P$ , hence the base case holds. Now assume that for  $1 \leq n < w$ , every node in  $A_2^n$  has a neighbor on  $P$ . By construction  $h_2^{n+1} \in A_2^n$ , hence by Claim 1 every node in  $H_2(\Sigma^{n+1})$  has a neighbor on  $P$ . Thus every node in  $A_2^{n+1}$  has a neighbor on  $P$ . This completes the proof of the lemma.  $\square$

**Lemma 13.8** *Let  $P = x_1, \dots, x_n$  be a direct connection from  $\Theta_T^w \setminus (A_1^w \cup A_5^w)$  to  $\Theta_B^w \setminus (A_2^w \cup A_4^w)$  avoiding  $A_1^w \cup A_5^w \cup A_2^w \cup A_4^w$  in  $G \setminus E(A)$ . If for some  $1 \leq k \leq w$ ,  $\Sigma^k$  is such that  $x_1$  is adjacent to a node of  $T(\Sigma^k) \setminus \{h_1^k, h_5^k\}$  and  $x_n$  is adjacent to a node of  $B(\Sigma^k) \setminus \{h_2^k, h_4^k\}$ , then  $P$  is a bridge of Type c with respect to  $\Sigma^k$ , with nodes  $h_1^k, h_2^k, h_3^k, h_6^k$  adjacent to at least one node in  $P$  or nodes  $h_3^k, h_4^k, h_5^k, h_6^k$  adjacent to  $P$ .*

*Proof:* Let  $\Sigma = \Sigma^k$  and  $P$  satisfy the conditions of the lemma. Let  $x_i$  be the node of  $P$  with the lowest index that is adjacent to a node in  $\Theta_B^w$ . Then  $P_{x_1 x_i}$  is a direct connection from  $\Theta_T^w \setminus (A_1^w \cup A_5^w)$  to  $\Theta_B^w$  avoiding  $A_1^w \cup A_5^w$  in  $G \setminus E(A)$ , so by Lemma 13.4  $N(x_1) \cap V(\Theta^w) \subseteq A_1^w \cup A_3^w \cup A_5^w$ . In particular  $x_1$  is adjacent to  $h_3$ . Similarly  $N(x_n) \cap V(\Theta^w) \subseteq A_2^w \cup A_4^w \cup A_6^w$  and  $x_n$  is adjacent to  $h_6$ . Let  $x_j$  be the node of lowest index adjacent to  $B(\Sigma)$ . Then the path  $P_{x_1 x_j}$  is a direct connection from  $T(\Sigma) \setminus \{h_1, h_5\}$  to  $B(\Sigma)$  avoiding  $\{h_1, h_5\}$  in  $G \setminus E(H)$ , so by Lemma 11.3, node  $x_j$  is adjacent to some node in  $\{h_2, h_4\}$ . Similarly some node in  $\{h_1, h_5\}$  is adjacent to a node of  $P$ .

In the following claim we prove the lemma, with the restriction that no node in one of the sets  $A_1^w, A_2^w, A_4^w$  or  $A_5^w$  has a neighbor on  $P$ .

**Claim:** If no node in one of the sets  $A_1^w, A_2^w, A_4^w$  or  $A_5^w$  has a neighbor on  $P$ , then  $P$  is a bridge of Type c with respect to  $\Sigma$ .

*Proof of Claim:* Assume w.l.o.g. that no node in  $A_5^w$  has a neighbor in  $V(P)$ . First suppose that  $V(P)$  does not contain neighbors of both  $h_2$  and  $h_4$ . If  $h_4$  is adjacent to a node of  $P$ , then some subpath of  $P$  is a direct connection from  $B(\Sigma)$  to  $T(\Sigma) \setminus \{h_1\}$  avoiding  $\{h_1\}$  in  $G \setminus E(H)$  which contradicts Lemma 11.2. Thus  $P$  must contain nodes adjacent to  $h_1$  and  $h_2$ , and no node adjacent to  $h_4$  and  $h_5$ . Hence  $P$  is a bridge of Type c with respect to  $\Sigma$ .

We now show that  $P$  cannot have neighbors from both  $h_2$  and  $h_4$ . Assume the contrary. Let  $x_j$  be the node of highest index in  $P$  adjacent to a node in  $A_1^w \cup A_3^w$ . Let  $x_i$  be adjacent to node  $h'$  in  $A_1^w \cup A_3^w$ . By Corollary 12.18, let  $\Sigma'$  be a connected 6-hole of  $\Theta^w$  containing nodes  $h'$  and  $h'_6 = h_6$ . By Corollary 12.17 we can assume that  $B(\Sigma') = B(\Sigma)$ . Node  $h'_4$  must have a neighbor on  $P$  since  $h_4 = h'_4$ . If  $P_{x_1 x_j}$  has a neighbor of  $h'_4$ , then  $h' = h'_1$ , since  $h'_3$  can only be adjacent to node  $x_1$  on  $P$ . Let  $x_p \in V(P_{x_1 x_j})$  be the node of highest index adjacent to  $h'_4$ . Now some subpath of  $P_{x_p x_j}$  is a direct connection between  $T(\Sigma')$  and  $B(\Sigma') \setminus \{h'_2\}$  avoiding  $\{h'_2\}$  in  $G \setminus E(H')$  which violates Lemma 11.2. So  $h'_4$  must have a neighbor on  $P_{x_j x_n}$ . Let  $x_p \in V(P_{x_j x_n})$  be the node of

lowest index adjacent to  $h'_4$ . Now  $P_{x_px_j}$  is a direct connection between  $T(\Sigma')$  and  $B(\Sigma') \setminus \{h'_2\}$  avoiding  $\{h'_2\}$  in  $G \setminus E(H')$ . By Lemma 11.2,  $x_j$  must be a fork of  $\Sigma'$  adjacent to  $h'_1$  and  $h'_3$ . But  $h'_3$  can only be adjacent to  $x_1$  on  $P$ . So  $j = 1$  and no node of  $A_1^\psi$  is adjacent to a node of  $P_{x_2x_n}$ . Now let  $x_k$  be the node of highest index in  $P$  adjacent to a node in  $\{h_2, h_4\}$ . If  $x_k$  is adjacent to  $h_4$  then let  $x_l$  be the node of highest index adjacent to  $h_2$ . The subpath  $P_{x_lx_n}$  has no neighbors of  $A_1^\psi \cup A_5^\psi$ , thus the node set  $\{h_2, h_4, h_6, x_l, \dots, x_n\}$  induces a fan bottom with center  $h_4$ , which contradicts the maximality of  $w$ . Similarly if  $x_k$  is adjacent to  $h_2$  then we can obtain a fan bottom with center  $h_2$ . This completes the proof of Claim.

If one of the node sets  $A_1^\psi, A_2^\psi, A_4^\psi, A_5^\psi$  has no node adjacent to  $P$  then we are done by the Claim above. So assume all four node sets have at least one node adjacent to a node in  $V(P)$ . Let  $x_k$  be the node of highest index adjacent to a node in  $A_1^\psi \cup A_5^\psi$ . Assume w.l.o.g. it is adjacent to  $h'_1 \in A_1^\psi$ . Notice that  $k > 1$  since otherwise  $x_1$  is adjacent to  $a_1 \in A_1^\psi$ ,  $a_3 \in A_3^\psi$  and  $a_5 \in A_5^\psi$  and so we have an odd wheel with center  $x_1$ .

First we show that no node of  $A_4^\psi$  is adjacent to  $P_{x_1x_k}$ . Assume not and let  $x_l$  be the node of  $P_{x_1x_k}$  with the highest index adjacent to a node in  $A_4^\psi$ . Let  $h''_4$  be the node of  $A_4^\psi$  adjacent to  $x_l$ . Let  $x_m$  be the node of  $P_{x_lx_k}$  with the lowest index adjacent to a node in  $A_1^\psi$ , and let  $h''_1$  be that node. By Corollary 12.18 let  $\Sigma''$  be a connected 6-hole of  $\Theta^\psi$  containing nodes  $h''_4$  and  $h''_1$ . Now  $P_{x_lx_m}$  is a direct connection from  $\Theta_B^\psi \setminus (A_2^\psi \cup A_6^\psi)$  to  $\Theta_T^\psi \setminus (A_3^\psi \cup A_5^\psi)$  avoiding  $A_2^\psi \cup A_3^\psi \cup A_5^\psi \cup A_6^\psi$  in  $G \setminus E(A)$ . Also  $\Sigma''$  is such that  $x_m$  is adjacent to  $h''_1$ ,  $x_l$  is adjacent to  $h''_4$  and no node of  $A_3^\psi$  is adjacent to a node in  $V(P_{x_lx_m})$  (since the only node of  $P$  that can have a neighbor in  $A_3^\psi$  is  $x_1$ ). Now by the Claim,  $P_{x_lx_m}$  is a bridge of Type c with respect to  $\Sigma''$ , with neighbors from  $h''_5$  and  $h''_6$ . But the only neighbor  $h''_6$  can have on  $P$  is  $x_n$ , hence we have a contradiction. Therefore no node of  $A_4^\psi$  is adjacent to  $P_{x_1x_k}$ .

Let  $x_l$  be the node of  $P_{x_kx_n}$  with the lowest index adjacent to some node  $A_4^\psi$ , and let  $h'_4$  be that node. By Corollary 12.18 let  $\Sigma'$  be a connected 6-hole of  $\Theta^\psi$  containing nodes  $h'_1$  and  $h'_4$ .  $P_{x_kx_l}$  is a direct connection from  $\Theta_T^\psi \setminus (A_3^\psi \cup A_5^\psi)$  to  $\Theta_B^\psi \setminus (A_2^\psi \cup A_6^\psi)$  avoiding  $A_2^\psi \cup A_3^\psi \cup A_5^\psi \cup A_6^\psi$  in  $G \setminus E(A)$ . Also  $\Sigma'$  is such that  $x_k$  is adjacent to  $h'_1$ ,  $x_l$  is adjacent to  $h'_4$  and no node of  $A_3^\psi$  is adjacent to a node in  $V(P_{x_kx_l})$  (since the only node of  $P$  that can have a neighbor in  $A_3^\psi$  is  $x_1$  and  $k > 1$ ). Now by the Claim,  $P_{x_kx_l}$  must be a bridge of Type c with respect to  $\Sigma'$ , with neighbors from  $h'_5$  and  $h'_6$  and no neighbor of  $h'_2$  and  $h'_3$ . By Lemma 13.7 every node in  $A_5^\psi$  is adjacent to

a node in  $P_{x_k x_l}$ . By our choice of  $x_k$  all nodes in  $A_5^w$  must be adjacent to  $x_k$ . Also since  $h'_6$  is adjacent to  $P_{x_k x_l}$  we must have  $l = n$ . If any node in  $A_2^w$  is adjacent to a node of  $P_{x_k x_n}$  then let  $x_p$  be the node of  $P_{x_k x_n}$  of lowest index adjacent to a node in  $A_2^w$ , say  $h''_2$ . Note that  $p < n$ . By Corollary 12.18 let  $\Sigma''$  be a connected 6-hole of  $\Theta^w$  containing  $h''_2$  and  $h''_1 = h'_1$ . Now  $P_{x_k x_p}$  is a direct connection from  $T(\Sigma'')$  to  $B(\Sigma'')$  in  $G \setminus E(H'')$  which contradicts Lemma 11.1 since  $p < n$  and so  $h'_4$  and  $h'_6$  are not adjacent to  $x_p$ . Thus no node of  $A_2^w$  is adjacent to a node of  $P_{x_k x_n}$ .

Now since some node of  $A_2^w$  must be adjacent to a node of  $P$ , this node must be in  $P_{x_1 x_k}$ . Let  $x_p$  be the node of  $P_{x_1 x_k}$  with the highest index adjacent to some node in  $A_2^w$  and let that node be  $h''_2$ . Let  $x_q$  be the node of lowest index in  $P_{x_p x_k}$  adjacent to a node in  $A_5^w$  and let that node be  $h''_5$ . Notice that such a node must exist since every node in  $A_5^w$  is adjacent to  $x_k$ . By Corollary 12.18, let  $\Sigma''$  be a connected 6-hole of  $\Theta^w$  containing  $h''_5$  and  $h''_2$ . Now  $P_{x_p x_q}$  is a direct connection from  $\Theta_B^w \setminus (A_4^w \cup A_6^w)$  to  $\Theta_T^w \setminus (A_1^w \cup A_3^w)$  avoiding  $A_1^w \cup A_3^w \cup A_4^w \cup A_6^w$  in  $G \setminus E(A)$ . Also  $\Sigma''$  is such that  $x_p$  is adjacent to  $h''_2$ ,  $x_q$  is adjacent to  $h''_5$  and no node of  $A_3^w$  is adjacent to a node of  $P_{x_p x_q}$  (since the only neighbor that a node in  $A_3^w$  can have on  $P$  is  $x_1$ ). But now by the Claim,  $P_{x_p x_q}$  is a bridge of Type c with respect to  $\Sigma''$ , with neighbors from  $h''_1$  and  $h''_6$ . But the only neighbor that  $h''_6$  can have on  $P$  is  $x_n$ , hence we have a contradiction. But then  $A_2^w$  does not have any node adjacent to a node of  $P$ , which contradicts our assumption.  $\square$

**Lemma 13.9** *Let  $P = x_1, \dots, x_n$  be a bridge of Type c with respect to  $\Sigma^k$ , for  $1 \leq k \leq w$ , with adjacencies to  $h_1^k, h_2^k, h_3^k$  and  $h_6^k$ , where  $x_1$  is adjacent to  $h_3^k$ . Then  $A_1^w \cup A_2^w \cup A_3^w \cup N(h_2^k)$  is an extended star cutset separating  $x_1$  from  $\Theta^w$ .*

*Proof:* Let  $P$  and  $\Sigma$  satisfy the conditions of the lemma. Let  $R = A_1^w \cup A_2^w \cup A_3^w \cup N(h_2)$  and suppose that  $R$  is not an extended star cutset. Let  $Q = y_1, \dots, y_m$  be a direct connection from  $x_1$  to  $\Theta^w \setminus R$  in  $G \setminus R$ . By Lemma 13.4,  $y_m$  cannot have neighbors in both  $\Theta_T^w$  and  $\Theta_B^w$ . Let  $Q' = y_0, y_1, \dots, y_m$  where  $y_0 = x_1$ .

**Case 1:**  $N(y_m) \cap V(\Theta^w) \subseteq \Theta_B^w$

Some subpath of  $Q'$  is a direct connection from  $\Theta_T^w$  to  $\Theta_B^w \setminus (A_2^w \cup A_4^w)$  avoiding  $A_2^w \cup A_4^w$  in  $G \setminus E(A)$  or a direct connection from  $\Theta_T^w$  to  $\Theta_B^w \setminus (A_2^w \cup A_6^w)$  avoiding  $A_2^w \cup A_6^w$  in  $G \setminus E(A)$ . In either case, by Lemma 13.4,  $N(y_m) \cap V(\Theta^w) \subseteq A_2^w \cup A_4^w \cup A_6^w$ . Hence  $y_m$  is adjacent to a node in  $A_4^w \cup A_6^w$ .



Suppose that  $y_m$  is adjacent to a node  $x \in A_6^w$ . Let  $y_i$  be the node of  $Q'$  with highest index adjacent to a node in  $A_3^w$ . Then  $Q'_{y_i y_m}$  is a direct connection from  $\Theta_T^w \setminus (A_1^w \cup A_5^w)$  to  $\Theta_B^w \setminus (A_2^w \cup A_4^w)$  avoiding  $A_1^w \cup A_5^w \cup A_2^w \cup A_4^w$  in  $G \setminus E(A)$ . By Corollary 12.18, let  $\Sigma'$  be a connected 6-hole of  $\Theta^w$  containing node  $x$  and a node of  $A_3^w$  that is adjacent to  $y_i$ . By Lemma 13.8 applied to  $Q'_{y_i y_m}$  and  $\Sigma'$ ,  $Q'_{y_i y_m}$  is a bridge of Type c with respect to  $\Sigma'$ . Since  $Q'_{y_i y_m}$  is not adjacent to any node in  $A_5^w$ ,  $Q'_{y_i y_m}$  is adjacent to  $h'_1$  and  $h'_2$ . Now by Lemma 13.7 every node in  $A_2^w$  has a neighbor on  $Q'_{y_i y_m}$ . In particular  $h_2$  is adjacent to  $Q$ , contradicting our choice of  $Q$ . Therefore  $y_m$  is not adjacent to any node in  $A_6^w$ .

Now suppose that  $y_m$  is adjacent to a node  $x \in A_4^w$ . First we will show that no node of  $A_1^w$  is adjacent to a node of  $Q'$ . Assume not and let  $y_i$  be the node of  $Q'$  with the highest index adjacent to a node in  $A_1^w$ . Then  $Q'_{y_i y_m}$  is a direct connection from  $\Theta_T^w \setminus (A_3^w \cup A_5^w)$  to  $\Theta_B^w \setminus (A_2^w \cup A_6^w)$  avoiding  $(A_3^w \cup A_5^w \cup A_2^w \cup A_6^w)$  in  $G \setminus E(A)$ . By Corollary 12.18, let  $\Sigma'$  be a connected 6-hole of  $\Theta^w$  containing node  $x$  and a node of  $A_1^w$  adjacent to  $y_i$ . Then by Lemma 13.8,  $Q'_{y_i y_m}$  is a bridge of Type c with respect to  $\Sigma'$ . Since  $Q$  is not adjacent to any node in  $A_5^w$ ,  $h'_2$  is adjacent to  $Q'_{y_i y_m}$ , and by Lemma 13.7 every node of  $A_2^w$  has a neighbor on  $Q$ . In particular  $h_2$  is adjacent to  $Q$ , which contradicts our choice of  $Q$ . Therefore no node of  $A_1^w$  is adjacent to a node of  $Q'$ .

Now let  $x_j$  be the node of  $P$  with the lowest index adjacent to a node of  $A_1^w$ . Let  $y_i$  be the node of  $Q'$  of highest index adjacent to a node of  $P_{x_1 x_j}$ . Let  $x_l$  be the node of  $P_{x_1 x_j}$  with highest index adjacent to  $y_i$ . By the same argument as above, the path induced by the node set  $V(P_{x_l x_j}) \cup V(Q'_{y_i y_m})$  must have a neighbor of  $h_2$  and a neighbor of  $h_3$  on it. By construction of  $Q$  the neighbor of  $h_2$  is on  $P_{x_l x_j}$ . Let  $x_s$  be the neighbor of  $h_2$  on  $P_{x_l x_j}$  with the lowest index. By construction of  $P$ ,  $h_3$  has no neighbors on  $P_{x_s x_j}$ . By Corollary 12.17, let  $\Sigma'$  be a connected 6-hole of  $\Theta^w$  with  $h'_4 = x$ . Let  $Y$  be the path connecting  $h'_6$  and  $x$  in  $\Sigma'$ . If there exists a chordless path  $X$  from  $x$  to  $h_2$  using nodes in  $\Theta_B^w$  only then there are two wheels with center  $h_3$ :  $x_s, \dots, x_l, y_i, \dots, y_m, X, x_s$  and  $x_j, \dots, x_l, y_i, \dots, y_m, Y, h_1, x_j$ . One of these wheels must be odd, thus we have a contradiction. Otherwise, if no such path  $X$  exists,  $x$  has no neighbors in  $B(\Sigma)$ . Now the path  $x_s, \dots, x_l, y_i, \dots, y_m, x$  is a direct connection from  $B(\Sigma)$  to  $T(\Sigma) \setminus \{h_1, h_3\}$  avoiding  $\{h_1, h_3\}$  in  $G \setminus E(H)$ . Since  $x$  is adjacent to  $h_3$  and  $h_5$ , by Lemma 11.3  $x_s$  is adjacent to  $h_4$ , which contradicts our choice of  $P$ .

**Case 2:**  $N(y_m) \cap V(\Theta^w) \subseteq \Theta_T^w$

If some node of  $A_2^w$  has a neighbor on  $Q$ , let  $y_i$  be the node of highest index adjacent to some node in  $A_2^w$ , and let  $Y = Q_{y_i y_m}$ . Otherwise let  $x_i$  be the node of  $P$  with the lowest index adjacent to some node in  $A_2^w$ , and let  $Y$  be the path induced by the node set  $V(Q) \cup V(P_{x_i x_i})$ . Then  $Y$  is a direct connection from  $\Theta_T^w \setminus (A_1^w \cup A_3^w)$  to  $\Theta_B^w$  avoiding  $A_1^w \cup A_3^w$  in  $G \setminus E(A)$ . By Corollary 13.4  $N(y_m) \cap V(\Theta^w) \subseteq A_1^w \cup A_3^w \cup A_5^w$ . Hence  $y_m$  is adjacent to some node  $x \in A_5^w$ .

Let  $x_j$  be the node of  $P$  with lowest index adjacent to a node of  $A_2^w$ . Let  $y$  be a node of  $A_2^w$  adjacent to  $x_j$ . Path  $X$  induced by the node set  $V(P_{x_i x_j}) \cup V(Q)$  is a direct connection from  $\Theta_T^w \setminus (A_1^w \cup A_3^w)$  to  $\Theta_B^w \setminus (A_4^w \cup A_6^w)$  avoiding  $A_1^w \cup A_3^w \cup A_4^w \cup A_6^w$  in  $G \setminus E(A)$ . By Corollary 12.18, let  $\Sigma'$  be a connected 6-hole of  $\Theta^w$  containing nodes  $x$  and  $y$ . By Lemma 13.8,  $X$  is a bridge of Type c with respect to  $\Sigma'$ . Since no node of  $A_4^w$  is adjacent to any node in  $V(P) \cup V(Q)$ ,  $h'_6$  must be adjacent to  $x_j$ , and by Lemma 13.7 every node in  $A_6^w$  has a neighbor in  $X$ . In particular  $h_6$  is adjacent to  $X$ , hence  $j = n$ . Therefore no node of  $V(P) \cup V(Q) \setminus \{x_n\}$  is adjacent to any node in  $\Theta_B^w$ . Let  $Y$  be a chordless path from  $h_1$  to  $x$  in  $V(P) \cup V(Q) \setminus \{x_n\}$ . If  $h_3$  is adjacent to  $Y$ , then  $Y$  induces a fan top with center  $h_3$  contradicting the maximality of  $w$ . Else let  $X$  be a direct connection from  $h_3$  to  $Y$  in the graph induced by  $V(P) \cup V(Q) \setminus \{x_n\}$ . If  $X$  has a neighbor of  $h_1$  or  $x$ , then there is a fan top with center  $h_1$  or  $x$  contradicting the maximality of  $w$ . Otherwise some subset of  $V(X) \cup V(Y)$  induces a triad top contradicting the maximality of  $w$ .  $\square$

*Proof of Theorem 13.3:* If the edge set  $E(A)$  does not disconnect  $\Theta_T^w$  from  $\Theta_B^w$  then the subgraph  $G$  obtained by removing the nodes  $V(\Theta^w)$  contains a connected component  $S$  having at least one node adjacent to a node in  $\Theta_T^w$  and at least one node adjacent to a node in  $\Theta_B^w$ . Let  $N(S)$  be the set of nodes of  $V(\Theta^w)$  adjacent to at least one node in  $S$ .

**Claim:**  $N(S) \cap (A_1^w \cup A_3^w \cup A_5^w) \neq \emptyset$  and  $N(S) \cap (A_2^w \cup A_4^w \cup A_6^w) \neq \emptyset$ .

*Proof of Claim:* Suppose that  $N(S) \cap (A_1^w \cup A_3^w \cup A_5^w) = \emptyset$ . Then  $S$  contains a path  $P = x_1, \dots, x_n$  which is a direct connection from  $\Theta_T^w \setminus (A_1^w \cup A_3^w)$  to  $\Theta_B^w$  avoiding  $A_1^w \cup A_3^w$  in  $G \setminus E(A)$ , such that  $x_1$  is adjacent to a node of  $\Theta_T^w \setminus (A_1^w \cup A_3^w \cup A_5^w)$ . This contradicts Lemma 13.4. Hence  $N(S) \cap (A_1^w \cup A_3^w \cup A_5^w) \neq \emptyset$ . Similarly  $N(S) \cap (A_2^w \cup A_4^w \cup A_6^w) \neq \emptyset$ . This completes the proof of the Claim.

So we only need to consider the following two cases.

**Case 1:** For every  $u \in A_1^\omega \cup A_3^\omega \cup A_5^\omega$  and  $v \in A_2^\omega \cup A_4^\omega \cup A_6^\omega$  such that  $u, v \in N(S)$ ,  $uv$  is an edge.

Then for some  $i \in \{1, \dots, 6\}$ ,  $V(A) \cap N(S) \subseteq A_{i-1}^\omega \cup A_i^\omega \cup A_{i+1}^\omega$ . W.l.o.g. assume  $i = 2$ . By Theorem 12.11 the node set  $K = A_1^\omega \cup A_2^\omega \cup A_3^\omega$  induces a biclique. Now we show that  $K$  is a biclique articulation separating  $S$  from  $\Theta^\omega$ . Suppose not. Then  $S$  contains a path  $P = x_1, \dots, x_n$  such that either  $P$  is a direct connection between  $\Theta_T^\omega \setminus (A_1^\omega \cup A_3^\omega)$  and  $\Theta_B^\omega$  avoiding  $A_1^\omega \cup A_3^\omega$  in  $G \setminus E(A)$  and  $x_1$  is adjacent to a node of  $\Theta_T^\omega \setminus (A_1^\omega \cup A_3^\omega \cup A_5^\omega)$ , or  $P$  is a direct connection between  $\Theta_B^\omega \setminus (A_2^\omega \cup A_6^\omega)$  and  $\Theta_T^\omega$  avoiding  $A_2^\omega \cup A_6^\omega$  in  $G \setminus E(A)$  and  $x_n$  is adjacent to a node of  $\Theta_B^\omega \setminus (A_2^\omega \cup A_4^\omega \cup A_6^\omega)$ . In either case  $P$  contradicts Lemma 13.4.

**Case 2:** There are nodes  $u \in A_1^\omega \cup A_3^\omega \cup A_5^\omega$  and  $v \in A_2^\omega \cup A_4^\omega \cup A_6^\omega$  such that  $u, v \in N(S)$  and  $uv$  is not an edge.

W.l.o.g. assume that  $N(S) \cap A_3^\omega \neq \emptyset$  and  $N(S) \cap A_6^\omega \neq \emptyset$ . Then there exists a path  $P = x_1, \dots, x_n$  which is a direct connection from  $\Theta_T^\omega \setminus (A_1^\omega \cup A_5^\omega)$  to  $\Theta_B^\omega \setminus (A_2^\omega \cup A_4^\omega)$  avoiding  $A_1^\omega \cup A_5^\omega \cup A_2^\omega \cup A_4^\omega$  in  $G \setminus E(A)$ . Let  $p \in \Theta_T^\omega \setminus (A_1^\omega \cup A_5^\omega)$  be adjacent to  $x_1$  and  $q \in \Theta_B^\omega \setminus (A_2^\omega \cup A_4^\omega)$  be adjacent to  $x_n$ . By Corollary 12.18, let  $\Sigma$  be a connected 6-hole of  $\Theta^\omega$  containing  $p$  and  $q$ . By Lemma 13.8  $P$  is a bridge of Type c with respect to  $\Sigma$ . W.l.o.g. assume  $P$  is adjacent to nodes  $h_1, h_2, h_3$  and  $h_6$ . Now by Lemma 13.9  $A_1^\omega \cup A_2^\omega \cup A_3^\omega \cup N(h_2)$  is an extended star cutset separating  $x_1$  from  $\Theta^\omega$ .  $\square$

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