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SOUND WAVE SCATTERING FROM A RIGID SPHERE

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Abstract

The Helmholtz integral equation may be formed by combining the scalar wave equation with Euler's equation for motion within a fluid. The solution of this integral equation yields the radiated pressure from a submerged, vibrating body and may be used to characterize the scattering of incident sound waves from bodies.

In this report the scattering from underwater rigid spheres is investigated and results are presented for $ka \leq 8.0$.

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Contents

Sound Wave Scattering from a Rigid Sphere

1. *Introduction*

Anechoic coatings when applied to objects have the ability to absorb and modify the scattering behaviour of incident sound waves. The first step in the investigation of anechoic coatings is to consider their effect when applied to simple objects such as a sphere or spherical shell. Before an analysis of this is possible it is necessary to have a firm grasp of the scattering characteristics of the body prior to the coating being applied.

The purpose of this report is to investigate the scattering characteristics of a rigid sphere. The mathematical derivation of the scattered pressure using the Helmholtz integral equation essentially follows that given in the earlier work of Junger and Feit [1]. When this is combined with various scattering parameters found in the literature [2-5], such as target strength, reflection factor and intensity field around the body, the scattering behaviour for the sphere, may be characterized. It will be necessary to extend this work later to consider simple elastic spheres and shells as well as those having complete or partial viscoelastic (anechoic) coatings attached to their outer surface.

2. *Helmholtz Integral Equation*

2.1 *Introduction*

A sound wave can be thought of as a time dependent pressure fluctuation P around the static pressure P_s in a compressible fluid, such as water. A sound field is generated by a vibrating elastic structure in contact with the fluid, and a sound field can be modified by the presence of an object that reflects or scatters the incident sound waves. The temporal and spatial variations of the pressure fluctuation are governed by the wave equation

$$\nabla^2 p = \frac{1}{c^2} \frac{\partial P^2}{\partial t^2} \tag{2.1}$$

where c is the wave propagation speed through the fluid. If we restrict ourselves to steady-state situations associated with pressures varying periodically with time, then any function $\Phi(t)$ which is periodic with period T, can be represented in the form of a Fourier series

$$\begin{aligned} \Phi(t) &= \frac{1}{2} a_o + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)] \\ &= \sum_{n=-\infty}^{\infty} [c_n (e^{-i\omega t})^n] \end{aligned} \tag{2.2}$$

where $\varpi = 2\pi/T$ is the fundamental frequency. The coefficients are computed using the relations

$$a_n = \frac{2}{T} \int_o^T \Phi(t) \cos(n\omega t)\, dt, \quad b_n = \frac{2}{T} \int_o^T \Phi(t) \sin(n\omega t)\, dt \tag{2.3a}$$

$$c_n = \frac{1}{T} \int_o^T \Phi(t) (e^{i\omega t})^n\, dt \tag{2.3b}$$

In this report, only linear systems are considered, where the n^{th} term of the response is associated exclusively with the n^{th} term of the excitation. Therefore, no restriction is imposed on the range of applicability of the solution if one considers a harmonic excitation involving a single frequency only. The time dependence of the wave equation may be factored out by inserting into the equation a solution of the form

$$P(\underset{\sim}{r}, t) = P(\underset{\sim}{r}) e^{-i\omega t} \tag{2.4}$$

to yield

$$(\nabla^2 + k^2)P(\underset{\sim}{r}) = 0 \tag{2.5}$$

This equation is referred to as either the "Helmholtz equation" or "steady-state wave equation", where the acoustic wavenumber is given by $k = (\varpi/c)$. Euler's equation of motion within a non-viscous, homogeneous fluid is given by

$$\nabla P = -\rho \frac{\partial \underset{\sim}{v}}{\partial t} \tag{2.6}$$

where ρ and $\underset{\sim}{v}$ are the density and velocity respectively. For a non-viscous fluid, pressure can only exert forces in a direction normal to a surface boundary S_o, and the above equation becomes

$$\nabla P \cdot \underset{\sim}{n} = -\rho \frac{\partial}{\partial t}(\underset{\sim}{v} \cdot \underset{\sim}{n}) \qquad \text{on } S_o \tag{2.7}$$

where $\mathbf{n}$ is the unit outward normal to S_o. Letting the normal component of velocity be given by

$$\dot{w}(\mathbf{r}_o, t) = \dot{w}(\underset{\sim}{\mathbf{r}}_o)e^{-i\omega t}$$

where r_o defines the surface S_o, results in

$$\ddot{w} = \frac{\partial \dot{w}}{\partial t} = (-i\omega)\dot{w}$$

and the above boundary condition becomes

$$\frac{\partial P}{\partial n} = (i\omega\rho)\dot{w} \qquad \text{on } S_o \tag{2.8}$$

An elastic body vibrating with a surface velocity $\dot{w}$ will produce a pressure field of the form and magnitude given by the homogeneous equation (2.5), and the non-homogeneous boundary condition (2.8). Therefore, equations (2.5) and (2.8) may be regarded as defining a pressure radiation problem. Instead of solving these directly, it will be easier to combine these two equations into one expression, to form the Helmholtz integral equation, which may then be solved using the Green's function technique.

2.2 *Theory of Green's Functions*

The Green's function technique may be explained as follows. The equation for a pressure field in the presence of a source is an inhomogeneous partial differential equation of the form

$$(\nabla + k^2)P = \xi \tag{2.9}$$

where ξ is the source density, i.e. the strength of the source per unit volume. Conversely, ξ is set to zero in the above equation for a field with no sources present. Boundary conditions requiring the field to be zero at the surface are termed homogeneous boundary conditions. For non-homogeneous boundary conditions, the boundary values may be said to be caused by a surface layer of sources, corresponding to an imhomogeneous equation. Therefore in summary, when either the equation or boundary conditions are inhomogeneous, sources may be said to be present. When both are homogeneous, no sources are present.

Green's method is obvious enough physically. To obtain the field caused by a distributed source, we calculate the effects of each part of the source and add them all. If $G(\mathbf{r}\,|\,\mathbf{r}_0)$ is the field at the observer's point $\mathbf{r}$ caused by a unit point source at $\mathbf{r}_0$, then the field at $\mathbf{r}$ caused by a source distribution $\xi(\mathbf{r}_0)$ is the volume integral of $G(\mathbf{r}\,|\,\mathbf{r}_0)\xi(\mathbf{r}_0)$ over the whole range of $\mathbf{r}_0$ occupied by the source. G is called Green's function and is a solution of the inhomogeneous Helmholtz equation

$$(\nabla^2 + k^2)G(\underset{\sim}{\mathbf{r}}|\underset{\sim}{\mathbf{r}}_0) = \delta(\underset{\sim}{\mathbf{r}} - \underset{\sim}{\mathbf{r}}_0) \tag{2.10}$$

where $\delta(\underset{\sim}{\mathbf{r}} - \underset{\sim}{\mathbf{r}}_0)$ is the Dirac delta function. $G(\underset{\sim}{\mathbf{r}}|\underset{\sim}{\mathbf{r}}_0)$ satisfies the reciprocity principle

$$G(\underset{\sim}{\mathbf{r}}|\underset{\sim}{\mathbf{r}}_0) = G(\underset{\sim}{\mathbf{r}}_0|\underset{\sim}{\mathbf{r}}) \tag{2.11}$$

Boundary conditions can be satisfied in the same way. We compute the field at $\mathbf{r}$ for the boundary value P_0 (or normal gradient $\partial P/\partial n_0$) as zero at every point on the surface except for $\underset{\sim}{\mathbf{r}}_0^s$ (which is on the surface). At $\underset{\sim}{\mathbf{r}}_0^s$ the boundary value has a delta function behaviour, so that its integral over a small surface area near $\mathbf{r}_0^s$ is unity. This field at $\mathbf{r}$ (not on the boundary) we can call $G(\underset{\sim}{\mathbf{r}}|\underset{\sim}{\mathbf{r}}_0^s)P_0$ (or $G(\underset{\sim}{\mathbf{r}}|\underset{\sim}{\mathbf{r}}_0^s)\ \partial P/\partial n_0$) over the boundary. G here is also Green's function.

One can solve the "inhomogeneous equation" for a field caused by a source distribution ξ, by means of a product of the source density with a Green's function integrated over a volume. The solution of a homogeneous equation having specified values on a surface can be obtained in terms of another

Green's function, integrated over the boundary surface. This Green's function happens to be identical to the first. Therefore, Green's function for a radiation problem governed by the equations

$$(\nabla^2 + k^2)P = 0, \quad \frac{\partial P}{\partial n} = (i\omega\rho)\dot{w} \quad \text{on } S_o \tag{2.12}$$

may be determined from

$$(\nabla^2 + k^2)G(\underset{\sim}{r}|\underset{\sim}{r}_o) = \delta(\underset{\sim}{r} - \underset{\sim}{r}_o), \quad \frac{\partial G}{\partial n} = 0 \quad \text{on } S_o \tag{2.13}$$

Before beginning to formulate the Helmholtz integral equation, there remains the task of investigating the behaviour of Green's function $G(\mathbf{r}\,|\,\mathbf{r}_o)$ for the situation where observation point $\mathbf{r}$ and the source point $\mathbf{r}_o$, are such that the magnitude $R = |\mathbf{r} - \mathbf{r}_o|$ is small compared with the distance of either point from a surface boundary. In this case, G is just a function of R, i.e. the source is completely symmetrical, so that G cannot depend on the direction of R, only on its magnitude. This Green's function is termed the free space Green's function g, and has a singularity at $R = 0$. Therefore, for a general problem involving a surface boundary, $G(\mathbf{r}\,|\,\mathbf{r}_o)$ may be separated into two parts: a part Γ which depends on the boundary conditions at S_o, and a part g, which is continuous everywhere except at $\mathbf{r} = \mathbf{r}_o$ and which is a function of R alone. In the next section, the free space Green's function for a three dimensional situation is derived.

2.3 *The Free Space Green's Function*

To begin with, the free space Green's function must be a solution to the inhomogeneous Helmholtz equation

$$(\nabla + k^2)g(|\underset{\sim}{r} - \underset{\sim}{r}_o|) = \delta(\underset{\sim}{r} - \underset{\sim}{r}_o) \tag{2.14}$$

To construct a solution, a spherical system of coordinates is chosen where the origin coincides with the singularity in equation (2.14). Thus the distance $|\underset{\sim}{r} - \underset{\sim}{r}_o|$ reduces to the radial coordinate R. In this spherical system, the solution is a function of R only and not of the angular coordinates. Therefore, except at the origin, Green's function satisfies the homogeneous Helmholtz equation

$$\left(\frac{\partial^2}{\partial R^2} + \frac{2\partial}{R\partial R} + k^2\right)g(R) = 0, \quad R > 0 \tag{2.15}$$

which has the general solution

$$g(R) = (1/R)(Ae^{ikR} + Be^{-ikR}) \tag{2.16}$$

Earlier in this report, the Helmholtz equation was obtained from the wave equation by specifying the time dependence as simple harmonic, i.e. $P(\underset{\sim}{r},t) = P(\underset{\sim}{r})e^{-i\omega t}$. If g is to be used as a solution of the wave equation, then the complete solution is

$$g(R)e^{-i\omega t} = (1/R)[Ae^{i(kR-\omega t)} + Be^{-i(kR+\omega t)}]$$

To obtain waves travelling outward from the source point, we must set B to zero, such that

$$g(R)e^{-i\omega t} = (A/R)e^{i(kR-\omega t)}$$

represents an outgoing wave. The coefficient A may be determined by integrating equation (2.14) over a spherical volume element of radius a concentric with the singular point $R = 0$, and then taking the limit as $a \to 0$. This process yields

$$\lim_{a\to 0} \iiint \nabla^2 g \, dV = 1$$

From Gauss' divergence theorem, this may be rewritten as

$$\lim_{a\to 0} \iint \nabla g \cdot \underset{\sim}{n} \, dS = -1$$

where $\underset{\sim}{n}$ is a unit vector that points into the volume V. Now

$$\nabla g \cdot \underset{\sim}{\mathbf{n}} = \frac{-\partial g}{\partial R} \rightarrow \frac{A}{a^2} \quad \text{as } a \rightarrow 0, \; R = a$$

Since this formulation is independent of the angular coordinates, the surface integral is obtained simply by multiplying this result by the area of the sphere, $4\pi a^2$, which yields $A = -1/4\pi$. Substituting for A and replacing the spherical coordinate R by $|\underset{\sim}{\mathbf{r}} - \underset{\sim}{\mathbf{r}}_o|$, the free space Green's function is obtained

$$g(|\underset{\sim}{\mathbf{r}} - \underset{\sim}{\mathbf{r}}_o|) = \frac{-e^{ik|\underset{\sim}{\mathbf{r}} - \underset{\sim}{\mathbf{r}}_o|}}{4\pi|\underset{\sim}{\mathbf{r}} - \underset{\sim}{\mathbf{r}}_o|} \tag{2.17}$$

2.4 *Formulation of the Integral Equation*

In order to formulate the Helmholtz integral equation, we will require the use of Green's theorem which may be derived as follows. From Gauss' divergence theorem

$$\iiint \nabla \cdot \underset{\sim}{\mathbf{F}} \, dV = -\iint \underset{\sim}{\mathbf{F}} \cdot \underset{\sim}{\mathbf{n}} \, dS, \quad \underset{\sim}{\mathbf{n}} \text{ into the volume } V \tag{2.18}$$

Consider two scalar functions U and V and let the function F be given by

$$\underset{\sim}{\mathbf{F}} = U\nabla V - V\nabla U \tag{2.19}$$

Substituting for F in the above integral expression yields Green's relation between volume and surface integrals, i.e.

$$\iiint [U\nabla^2 V - V\nabla^2 U] \, dV = -\iint [U\nabla V - V\nabla U] \cdot d\underset{\sim}{S} \tag{2.20}$$

To formulate the Helmholtz integral equation, consider the two inhomogeneous partial differential equations

$$(\nabla^2 + k^2)P(\underset{\sim}{r}) = \xi(\underset{\sim}{r}) \tag{2.21}$$

$$(\nabla^2 + k^2)G(\underset{\sim}{r}|\underset{\sim}{r}_o) = \delta(\underset{\sim}{r} - \underset{\sim}{r}_o) \tag{2.22}$$

Multiplying equation (2.22) by $P(\underset{\sim}{r})$ and equation (2.21) by $G(\underset{\sim}{r}|\underset{\sim}{r}_o)$ and subtracting, exchanging $\underset{\sim}{r}$ for $\underset{\sim}{r}_o$ at the same time, yields

$$P(\underset{\sim}{r}_o)\nabla^2 G(\underset{\sim}{r}_o|\underset{\sim}{r}) - G(\underset{\sim}{r}_o|\underset{\sim}{r})\nabla^2 P(\underset{\sim}{r}_o) = [P(\underset{\sim}{r}_o)\delta(\underset{\sim}{r} - \underset{\sim}{r}_o) - G(\underset{\sim}{r}|\underset{\sim}{r}_o)\xi(\underset{\sim}{r}_o)]$$

Integrating over the volume of S_o, and using the property of the delta function, namely

$$\iiint [P(\underset{\sim}{r}_o)\delta(\underset{\sim}{r} - \underset{\sim}{r}_o)]\, dV = \begin{cases} P(\underset{\sim}{r}) & \underset{\sim}{r} \text{ inside } V \\ 0 & \underset{\sim}{r} \text{ outside } V \end{cases} \tag{2.23}$$

yields

$$P(\underset{\sim}{r}) = \iiint [P(\underset{\sim}{r}_o)\nabla^2 G(\underset{\sim}{r}|\underset{\sim}{r}_o) - G(\underset{\sim}{r}|\mathbf{r}_o)\nabla^2 P(\underset{\sim}{r}_o)]\, dV + \iiint [G(\underset{\sim}{r}|\underset{\sim}{r}_o)\xi(\underset{\sim}{r}_o)]\, dV \tag{2.24}$$

The first term may be expressed as a surface integral by using equation (2.20) with the substitutions $U = P(\underset{\sim}{r}_o)$, $V = G(\underset{\sim}{r}|\underset{\sim}{r}_o)$, to yield

$$P(\underset{\sim}{r}) = -\iint [P(\underset{\sim}{r}_o)\nabla G(\underset{\sim}{r}|\underset{\sim}{r}_o) - G(\underset{\sim}{r}|\underset{\sim}{r}_o)\nabla P(\underset{\sim}{r}_o)] \cdot dS + \iiint [G(\underset{\sim}{r}|\underset{\sim}{r}_o)\xi(\underset{\sim}{r}_o)]\, dV \tag{2.25}$$

Earlier in the report, we had the equations for the radiation problem as being

$$(\nabla^2 + k^2)P = 0, \quad \frac{\partial P}{\partial n} = (i\omega\rho)\dot{w} \quad \text{on } S_o, \quad \xi = 0 \tag{2.26}$$

Substituting equation (2.26) into (2.25), yields

$$P(\underset{\sim}{r}) = -\iint [P(\underset{\sim}{r}_o)\frac{\partial G}{\partial n} - (i\omega\rho)G(\underset{\sim}{r}|\underset{\sim}{r}_o)\dot{w}(\underset{\sim}{r}_o)]dS,\ S = S_o,\ \underset{\sim}{r} \text{ inside } V \tag{2.27}$$

We have now come up against a problem, the solution to the above expression represents the pressure fluctuation $P(r)$ inside a boundary surface S_o, yet we are trying to formulate an equation for the radiated pressure outside a vibrating body. To overcome this problem, the volume V inside the boundary S is defined as the volume V between the vibrating surface S_o and that of an infinite sphere of radius r_1. The expression for the pressure field then becomes

$$P(\underset{\sim}{r}) = -\iint [P(\underset{\sim}{r}_o)\frac{\partial G}{\partial n} - (i\omega\rho)G(\underset{\sim}{r}|\underset{\sim}{r}_o)\dot{w}(\underset{\sim}{r}_o)]dS,\ \underset{\sim}{r} \text{ inside } S = S_o + S_1$$

where S_1 is the surfa.e of the infinite sphere. Unless the surface integral at infinity in the above equation vanishes, one reaches the conclusion that the pressure field is not uniquely determined by the boundary conditions over the radiating surface S_o. The surface integral over S_1 may be written as

$$-\iint [P(\underset{\sim}{r}_1)\frac{\partial G}{\partial n} - G(\underset{\sim}{r}|\underset{\sim}{r}_1)\frac{\partial P}{\partial n}]\, dS$$

$$= \lim_{\underset{\sim}{r}_1\to\infty} -(4\pi r_1^2)[P(\underset{\sim}{r}_1)\frac{\partial G}{\partial r_1} - G(\underset{\sim}{r}|\underset{\sim}{r}_1)\frac{\partial P}{\partial r_1}]$$

Because there are no boundary conditions on the infinite sphere, the Green's function G may be replaced by the free space Green's function, i.e.

$$g(|\underset{\sim}{r} - \underset{\sim}{r}_1|) \approx -\frac{e^{ik|\underset{\sim}{r}-\underset{\sim}{r}_1|}}{4\pi|\underset{\sim}{r} - \underset{\sim}{r}_1|} \approx \frac{-e^{ikr_1}}{4\pi r_1}$$

The surface integral becomes

$$\lim_{\underset{\sim}{r}_1\to\infty} r_1[ikP(\underset{\sim}{r}_1) - \frac{\partial P}{\partial r_1}]e^{ikr_1} = 0$$

To satisfy this condition, the pressure must decrease with increasing r_1 as $|r_1 - r_o|^{-1}$ or faster. According to Jung・ and Feit [1], this is the case for all sources of finite extent, such as the sphere. Therefore, the integral equation can be written as

$$P(\underset{\sim}{r}) = -\iint [P(\underset{\sim}{r_o})\frac{\partial G}{\partial n} - (i\omega\rho)G(\underset{\sim}{r}|\underset{\sim}{r_o})\dot{w}(\underset{\sim}{r_o})]dS(r_o), \quad \underset{\sim}{r} \text{ outside } S_o \tag{2.28}$$

which is typically known as the Helmholtz integral equation. This integral may be reduced to a simple expression involving only Green's function and the known surface velocity on the boundary, providing that Green's function can be constructed in a way that satisfies the Neumann boundary condition

$$\frac{\partial G}{\partial n} = 0 \quad \text{on the boundary } S_o \tag{2.29}$$

then $P(r)$ reduces to

$$P(\underset{\sim}{r}) = (i\omega\rho)\iint [G(\underset{\sim}{r}|\underset{\sim}{r_o})\dot{w}]\,dS_o \tag{2.30}$$

i.e. the field is just the surface integral of Green's function multiplied by the boundary values as mentioned in section 2.2. If instead of a surface layer of sources, a distributed source ξ is the only source present, then from equation (2.25), the pressure field reduces to the volume integral

$$P(\underset{\sim}{r}) = \iiint [G(\underset{\sim}{r}|\underset{\sim}{r_o})\xi(\underset{\sim}{r})]\,dV \tag{2.31}$$

as mentioned previously, provided that G satisfies the Neumann condition on the boundary for this case also.

3. *Pressure Field for a Spherical Radiator*

3.1 *Green's Function for a Vibrating Sphere*

In order to obtain a solution to the Helmholtz integral equation for a vibrating source, Green's function G must be obtained. As mentioned earlier in section 2.2, a general Green's function may be constructed by adding a term Γ to the free space Green's function, where G satisfies the boundary condition

$$\frac{\partial G}{\partial n} = 0 \quad \text{on } S_o \tag{3.1}$$

This additional term contains the natural eigenfunctions for a sphere (see Appendix A for eigenfunction derivation). The first step in the construction of Green's function is to express the free space Green's function in spherical coordinates. To do this, we must first separate g as

$$g(|\underset{\sim}{r}-\underset{\sim}{r}_o|) = -\frac{e^{ik|\underset{\sim}{r}-\underset{\sim}{r}_o|}}{4\pi|\underset{\sim}{r}-\underset{\sim}{r}_o|} = \frac{-k}{4\pi}\left[\frac{\cos(k|\underset{\sim}{r}-\underset{\sim}{r}_o|)}{k|\underset{\sim}{r}-\underset{\sim}{r}_o|} + \frac{i\sin(k|\underset{\sim}{r}-\underset{\sim}{r}_o|)}{k|\underset{\sim}{r}-\underset{\sim}{r}_o|}\right] \tag{3.2}$$

From trigonometry, the distance between the field point $\underset{\sim}{r}$ and the source point $\underset{\sim}{r}_o$ is

$$|\underset{\sim}{r}-\underset{\sim}{r}_o| = (r^2 + r_o^2 - 2rr_o\cos\psi)^{1/2} \tag{3.3}$$

where ψ is the angle between $\underset{\sim}{r}$ and $\underset{\sim}{r}_o$. The components of g can be represented in terms of ψ by the expressions [6]

$$\begin{aligned} \cos(k|\underset{\sim}{r}-\underset{\sim}{r}_o|)/k|\underset{\sim}{r}-\underset{\sim}{r}_o| &= -\sum_{n=o}^{\infty} y_n(kr)(2n+1)j_n(kr_o)P_n(\cos\psi) \\ \sin(k|\underset{\sim}{r}-\underset{\sim}{r}_o|)/k|\underset{\sim}{r}-\underset{\sim}{r}_o| &= \sum_{n=o}^{\infty} j_n(kr)(2n+1)j_n(kr_o)P_n(\cos\psi) \end{aligned} \tag{3.4}$$

where $j_n(kr)$, $y_n(kr_o)$ are spherical Bessel functions and $P_n(\cos\,\psi)$ is the Legendre polynomial. Thus

$$g(|\underset{\sim}{r}-\underset{\sim}{r}_o|) = -(ik/4\pi)\sum_{n=o}^{\infty}(2n+1)j_n(kr_o)h_n(kr)P_n(\cos\psi) \tag{3.5}$$

where

$$h_n(kr) = j_n(kr) + iy_n(kr) \tag{3.6}$$

Due to symmetry about the $\vartheta = 0$ axis, the angle ψ may be written as $\psi = \vartheta - \vartheta_o$, and

$$P_n(\cos\psi) = P_n(\cos\vartheta)P_n(\cos\vartheta_o)$$

Thus, the free space Green's function in terms of (r, ϑ) and (r_o, ϑ_o) instead of $\underset{\sim}{r}$ and $\underset{\sim}{r}_o$ is

$$g(|\underset{\sim}{r}-\underset{\sim}{r}_o|) = -(ik/4\pi)\sum_{n=o}^{\infty}(2n+1)P_n(\cos\vartheta)P_n(\cos\vartheta_o)j_n(kr_o)h_n(kr) \quad (3.7)$$

The complete Green's function may be represented as

$$G(\underset{\sim}{r}|\underset{\sim}{r}_o) = g(|\underset{\sim}{r}-\underset{\sim}{r}_o|) + \Gamma(\underset{\sim}{r}|\underset{\sim}{r}_o) \quad (3.8)$$

and must satisfy the boundary condition

$$\frac{\partial\Gamma}{\partial r_o} + \frac{\partial g}{\partial r_o} = 0 \quad \text{at } r_o = a \quad \text{(Sphere surface)} \quad (3.9)$$

Let the (r, r_o) modal components of g and Γ be represented by g_n and Γ_n respectively. Assuming that Γ_n is made up of the eigenfunctions of the sphere and must also satisfy the reciprocity condition $\Gamma_n(r\,|\,r_o) = \Gamma_n\,(r_o\,|\,r)$ we then can expect Γ_n to have the form

$$\hat{\Gamma}_n(r,r_o) = Ah_n(kr_o)h_n(kr) \quad (3.10)$$

where from equations (3.9) and (3.10)

$$\partial\hat{\Gamma}_n/\partial r_o = kAh_n'(kr_o)h_n(kr) = -kj_n'(ka)h_n(kr)$$

Thus

$$\hat{\Gamma}_n(r,r_o) = -h_n(kr)h_n(kr_o)j_n'(ka)/h_n'(ka) \quad (3.11)$$

Adding g_n and Γ_n and using the relation

$$j_n(ka)y_n'(ka) - y_n(ka)j_n'(ka) = (ka)^{-2} \tag{3.12}$$

yields

$$\hat{g}_n + \hat{\Gamma}_n = \frac{ih_n(kr)}{(ka)^2 h_n'(ka)} \quad \text{at } r_o = a \tag{3.13}$$

The function Γ has the same (ϑ, ϑ_o) dependence as g. Therefore, the resulting Green's function for a spherical source may be written as

$$G(r,\vartheta\,|\,a,\vartheta_o) = \frac{1}{4\pi ka^2}\sum_{n=o}^{\infty}(2n+1)P_n(\cos\vartheta)P_n(\cos\vartheta_o)\left[\frac{h_n(kr)}{h_n'(ka)}\right] \tag{3.14}$$

3.2 Solution to Integral Equation

Now that Green's function has been determined, the radiated pressure field may be obtained from the Helmholtz integral equation (2.30), i.e.

$$\begin{aligned} P(\mathbf{r}) &= (i\omega\rho)\iint[G(\underline{r}\,|\,\underline{r}_o)\dot{w}(\underline{r}_o)]\,dS \\ &= (i\omega\rho)(2\pi a^2)\int_o^{\pi}[G(r,\vartheta\,|\,a,\vartheta_o)\dot{w}(\vartheta_o)]\sin\vartheta_o\,d\vartheta_o \end{aligned} \tag{3.15}$$

To solve this integral, it is helpful to express the surface velocity as

$$\dot{w}(\vartheta) = \sum_{n=o}^{\infty} W_n P_n(\cos\vartheta) \tag{3.16}$$

where the coefficients of the series are given by

$$\dot{W}_n = \frac{(2n+1)}{2} \int_{-1}^{1} P_n(\eta)\dot{w}(\eta)\,d\eta, \quad \eta = \cos\vartheta \tag{3.17}$$

Using the relation

$$\int_{-1}^{1} P_n(\eta)P_m(\eta)\,d\eta = (2\delta_{nm})/(2n+1) \tag{3.18}$$

results in the final expression for the pressure field, i.e.

$$P(r,\vartheta) = i(\rho c) \sum_{n=o}^{\infty} \dot{W}_n P_n(\cos\vartheta) \left[\frac{h_n(kr)}{h_n'(ka)}\right] \tag{3.19}$$

4. *Scattering of a Plane Wave from a Rigid Sphere*

Consider a distant point sound source, which generates a continuous sound wave. Far away from this source and over suitably restricted regions, these waves may be said to approximate plane waves. Consider these plane waves to be incident upon a stationary rigid sphere, whose surface S_o is given by the position vector $\mathbf{r}_o$, shown in Figure 1. If we define a wavenumber vector **k**, whose magnitude is k and which lies in the direction of wave propagation, then the incident pressure wave may be written as

$$P_i(\underline{r},t) = P_o \exp i(\underline{k}\cdot\underline{r} - \omega t)$$

In spherical coordinates, a wave incident from the ϑ = 180° direction is represented by

$$P_i(r,\vartheta) = P_o \exp i(kr\cos\vartheta) \tag{4.1}$$

The introduction of the rigid boundary S_o, produces a disturbance in the pressure field P_i, which is termed the scattered pressure $P_{s\infty}$, where the ∞ symbol refers to the infinite impedance of the scattering surface. The resultant pressure in the presence of the scatterer is defined as

$$P(\underset{\sim}{r}) = P_i(\underset{\sim}{r}) + P_{s\infty}(\underset{\sim}{r}) \quad (4.2)$$

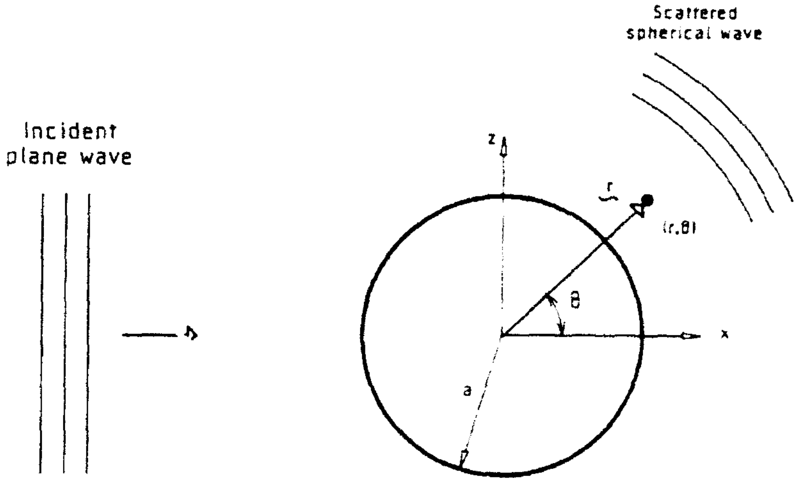

Figure 1: Sound scattering from a rigid sphere.

In order to calculate the scattered pressure and hence the resultant pressure field, we need to make use of the idea of re-radiation. This amounts to regarding the boundary surface as having a distribution of virtual sources, whose nature, strengths and phases are determined by the incident wave and by the properties of the water and the boundary medium. The radiation from these virtual sources then determines the behaviour of the reflected or scattered wave. To use this method, we need to investigate the components that make up the velocity distribution on the surface of the sphere. Since the boundary of the sphere is rigid, the resultant velocity on S_0 normal to the surface must be zero, i.e.

$$\dot{w}(\underset{\sim}{r}) = \dot{w}_i(\underset{\sim}{r}) + \dot{w}_{s\infty}(\underset{\sim}{r}) = 0 \quad (4.3)$$

The resultant surface velocity may be related to the pressure gradient via

$$\frac{\partial P}{\partial n} = (i\omega\rho)\dot{w} \quad \text{on } S_o \tag{4.4}$$

which may be rewritten as

$$\dot{w}_{s\infty} = -\dot{w}_i = \frac{i}{\omega\rho}\frac{\partial P_i}{\partial n} = \frac{-1}{\omega\rho}(\mathbf{k}\cdot\mathbf{n})P_i \quad \text{on } S_o \tag{4.5}$$

Therefore, $P_{s\infty}$ may be regarded as the radiated pressure field due to a vibrating spherical source with the velocity distribution $w_{s\infty}$, and hence the solution to the Helmholtz integral equation for a spherical radiator may be used. In order to apply the Helmholtz solution, the velocity $w_{s\infty}$ must first be expressed in terms of Legendre polynomials as

$$\dot{w}_{s\infty} = \sum_{n=o}^{\infty} \dot{W}_n P_n(\cos\vartheta) \tag{4.6}$$

To do this, we proceed as follows. The expression for a plane wave

$$P_i(r,\vartheta) = P_o \exp i(kr\cos\vartheta)$$

can be written in terms of a series of concentric spherical waves as

$$P_i(r,\vartheta) = P_o \sum_{n=o}^{\infty} (2n+1)(i)^n P_n(\cos\vartheta)\, j_n(kr) \tag{4.7}$$

The coefficients for $w_{s\infty}$ then become

$$\dot{W}_n = (iP_o/\rho c)(2n+1)(i)^n j_n'(ka) \tag{4.8}$$

From the solution for a spherical radiator, the radiated pressure field is represented by

$$P(r,\vartheta) = (i\rho c) \sum_{n=0}^{\infty} \dot{W}_n P_n(\cos\vartheta) h_n(kr) / h_n'(ka) \quad (4.9)$$

Substituting equation (4.8) for W_n into equation (4.9), yields the rigid scattered pressure field

$$P_{s\infty}(r,\vartheta) = -P_o \sum_{n=0}^{\infty} (2n+1)(i)^n P_n(\cos\vartheta) j_n'(ka) h_n(kr) / h_n'(ka) \quad (4.10)$$

The above series may be simplified somewhat if we restrict ourselves to large distances r from the boundary such that $kr >> 1$. Then the Hankel function $h_n(kr)$ may be approximated by

$$(1/kr) e^{ikr} (i)^{-(n+1)} \qquad kr >> n^2 + 1 \quad (4.11)$$

Substituting $h_n(kr)$ into $P_{s\infty}$ yields what is termed in this report as the "far-field" scattered pressure

$$P_{s\infty}(r,\vartheta) = (iP_o/kr)\exp(ikr) \sum_{n=0}^{\infty} (2n+1) P_n(\cos\vartheta) j_n'(ka) / h_n'(ka) \quad (4.12)$$

"Near-field" in this report, means that equation (4.10) is used for the rigid scattered pressure, while for "far-field" (i.e. $kr >> 1$), equation (4.12) is applied. The first ten terms of the series have been found to be sufficient to determine the scattered pressure in the ka range under investigation.

5. Methods of Presenting the Scattering Solutions

In this chapter mathematical expressions are developed for a number of parameters which are useful in examining the scattering behaviour of the rigid sphere. These parameters include (i) the resultant pressure field magnitude in the backscatter direction, (ii) the target strength of the scatterer, (iii) the reflection factor around the scattering body and (iv) the intensity field around the scatterer. To determine these various parameters, the following values for the propagation medium will be used.

Water

Density of medium	$\rho = 1000.0$	(kg/m^3)
Speed of sound	$c = 1500.0$	(m/s)

5.1 Resultant Pressure Field in Front of Scatterer

Here, the resultant pressure magnitude as a function of distance from the scatterer is computed, in the direction towards the incident waves. The normalized resultant pressure field for the rigid sphere may be written as

$$\hat{P} = |P|/P_o = |(P_i/P_o) + (P_{s\infty}/P_o)| \tag{5.1}$$

where the incident and rigid scattered pressures close to the sphere have been determined previously to be

$$P_i/P_o \approx P_i^N(r,\vartheta)/P_o = \exp(i\hat{z}\tau\cos\vartheta) \tag{5.2}$$

and

$$P_{s\infty}^N/P_o = P_{s\infty}^N(r,\vartheta)/P_o \approx -\sum_{n=0}^{\infty} (2n+1)(i)^n P_n(\cos\vartheta)\left[\frac{j_n'(\tau)}{h_n'(\tau)}\right]h_n(\tau\hat{r}) \tag{5.3}$$

where

$$\tau = ka, \quad \hat{r} = r/a \tag{5.4}$$

5.2 Target Strength

Target strength is a far-field parameter which for convenience is defined as the ratio of the scattered intensity considered at one metre to the incident intensity expressed in decibels, i.e.

$$TS = 10\log_{10}\frac{I_{s\infty}(r,\vartheta)}{I_o}\Big|_{r=1} \tag{5.5}$$

where I_o is the intensity of the incident (plane) wave as defined in equation (5.13). The scattered intensity is proportional to the square of the pressure, and in the far-field $P_{s\infty}$ is proportional to $1/r$. Therefore, the target strength may be rewritten in the form

$$TS = 20 \log_{10} \left[\left| \frac{r P^F_{s\infty}(r,\vartheta)}{P_o} \right| \right]_{\vartheta=180°} \tag{5.6}$$

where $P^F_{s\infty}$ is the scattered pressure in the far-field and

$$(r/P_o) P^F_{s\infty}(r,\vartheta) = i(a/\tau)\exp(i\tau\hat{r}) \sum_{n=o}^{\infty} (2n+1) P_n(\cos\vartheta) [j_n'(\tau)/h_n'(\tau)] \tag{5.7}$$

Results obtained for different *ka* values, will be compared with the commonly used expression for the target strength of a sphere, i.e.

$$TS = 10 \log_{10} (a^2/4) \tag{5.8}$$

5.3 *Reflection Factor*

Using the expressions in the previous section, the ratio of scattered to incident pressure may be written as

$$\frac{P^F_{s\infty}}{P_i} = \left(\frac{a}{r}\right) e^{i\tau\hat{r}(1-\cos\vartheta)} \left((i/\tau) \sum_{n=o}^{\infty} (2n+1) P_n(\cos\vartheta) \left[\frac{j_n'(\tau)}{h_n'(\tau)} \right] \right) \tag{5.9}$$

The $1/r$ term characterizes the spherical spreading of the scattered wave, whilst the exponential term takes into account the phases of the incident and scattered waves. The reflection factor can be made independent of these terms by defining it as

$$R = (r/a) \left| e^{-i\tau\hat{r}(1-\cos\vartheta)} [P^F_{s\infty}/P_i] \right|$$

to yield

$$R = \left| (i/\tau) \sum_{n=0}^{\infty} (2n+1) P_n(\cos\vartheta) \left[\frac{j_n'(\tau)}{h_n'(\tau)} \right] \right| \tag{5.10}$$

The reflection factor will be computed as a function of direction around the body, for various *ka* values.

5.4 *Intensity Field Around Scatterer*

Using the solution to the resultant pressure field, the construction of two different types of time-averaged intensity plots is possible.

(A) Plots of the intensity magnitude around the body (both near-field and far-field)

(B) Plots of the intensity vector field around the body (near-field only)

For both types A and type B plots, the intensities for the scattered and resultant pressure fields have been computed. The radial and transverse components of the intensity vector (averaged over one period of the wave) are defined as

$$I_r(r,\vartheta) = \frac{1}{2} Re[P(r,\vartheta)\dot{u}_r^*(r,\vartheta)] \tag{5.11a}$$

$$I_\vartheta(r,\vartheta) = \frac{1}{2} Re[P(r,\vartheta)\dot{u}_\vartheta^*(r,\vartheta)] \tag{5.11b}$$

where *Re* denotes the real part, * the complex conjugate and u_r, u_ϑ the velocity components corresponding to the pressure field *P*, whether it be a scattered or resultant pressure. The magnitude *I* and direction ϕ of the intensity vector are given by the formulae

$$I = [I_r^2(r,\vartheta) + I_\vartheta^2(r,\vartheta)]^{1/2}$$
$$\phi = \vartheta + \arctan[I_\vartheta(r,\vartheta)/I_r(r,\vartheta)] \tag{5.12}$$

where

$$\vartheta = \arctan[z/x], \qquad r = [x^2 + z^2]^{1/2}$$

and x, z are shown in Figure 1. For the polar intensity plots of type A, the intensity value is first divided by the incident intensity

$$I_o = \frac{P_o^2}{2\rho c} \tag{5.13}$$

and then normalized with respect to the maximum value that it obtains over the range of ϑ varying from 0° to 360°. For the type B intensity vector plots, each vector has first been divided by I_o, and then normalized with respect to the magnitude of the largest vector in the plot. The square root of the magnitude is then calculated and multiplied by the grid spacing. The vectors are plotted over the coordinate range $-2 \le x/a \le 2$ and $0 \le z/a \le 2$, where a is the radius of the spherical scatterer.

In order to compute the intensity, the velocity field associated with the pressure field must be found. This may be determined from Euler's equation $\nabla P = (i\omega\rho)\mathbf{u}$, which in component form becomes

$$\dot{u}_r = \left(\frac{1}{\rho c}\right)\left(\frac{-i}{k}\right)\frac{\partial P}{\partial r}, \quad \dot{u}_\vartheta = \left(\frac{1}{\rho c}\right)\left(\frac{-i}{kr}\right)\frac{\partial P}{\partial \vartheta} \tag{5.14}$$

Using the near-field and far-field pressure expressions given in sections 5.1 and 5.2, the following velocity components may be obtained.

Velocity due to incident pressure P_i

$$(\rho c/P_o)\,[\dot{u}_r]_i = \cos\vartheta\, e^{ikr\cos\vartheta} \tag{5.15a}$$

$$(\rho c/P_o)\,[\dot{u}_\vartheta]_i = -\sin\vartheta\, e^{ikr\cos\vartheta} \tag{5.15b}$$

Velocity due to rigid scattered pressure $P_{s\infty}$

Near-field:

$$(\rho c/P_o)[\dot{u}_r^N]_{s\infty} = \sum_{n=o}^{\infty} (2n+1)(i)^{n+1} P_n(\cos\vartheta) \frac{j_n'(\tau)}{h_n'(\tau)} h_n'(\tau\hat{r}) \qquad (5.16a)$$

$$(\rho c/P_o)[\dot{u}_\vartheta^N]_{s\infty} = -(\sin\vartheta/\tau\hat{r}) \sum_{n=o}^{\infty} (2n+1)(i)^{n+1} P_n'(\cos\vartheta) \frac{j_n'(\tau)}{h_n'(\tau)} h_n(\tau\hat{r})$$

(5.16b)

Far-field:

$$(\rho c/P_o)[\dot{u}_r^F]_{s\infty} = (i/\tau\hat{r}) e^{i\tau\hat{r}} \sum_{n=o}^{\infty} (2n+1) P_n(\cos\vartheta) \frac{j_n'(\tau)}{h_n'(\tau)} \qquad (5.17a)$$

$$(\rho c/P_o)[\dot{u}_\vartheta^F]_{s\infty} = (-\sin\vartheta/(\tau\hat{r})^2) e^{i\tau\hat{r}} \sum_{n=o}^{\infty} (2n+1) P_n'(\cos\vartheta) \frac{j_n'(\tau)}{h_n'(\tau)} \qquad (5.17b)$$

Therefore, by inserting the pressure and velocity formulae given respectively by equations (5.2), (5.3), (5.7) and (5.15), (5.16), (5.17) into equation (5.11), the scattered and resultant intensity expressions may be obtained.

6. *Results and Discussion*

6.1 *Resultant Pressure Field*

Figures 2(a) to 2(d) show the normalized resultant pressure as a function of r/a for the ka values 1, 2, 4 and 8. In these figures, the reflected wave combines with the incident wave to form a standing wave whose scattered component decreases proportionally to $1/r$ where r is the distance from the scatterer. As the frequency increases, (ka increasing), the pressure at the surface approaches twice the incident pressure amplitude, which is the case for a plane wave incident upon a rigid wall.

6.2 *Target Strength*

Figure 3 shows the far-field target strength for the rigid sphere over the *ka* range from 0 to 8. From this figure, it is evident that for frequencies corresponding to *ka* greater than 2.0 and within the range considered, the target strength is approximately constant. These results compare favourably with the usual expression for the target strength (due to specular reflection) of a rigid sphere, i.e.

$$TS = 10 \log_{10} (a^2/4) \approx -6 \text{ (dB)} \quad \text{for sphere of radius } a = 1$$

which is shown by the dashed line in Figure 3.

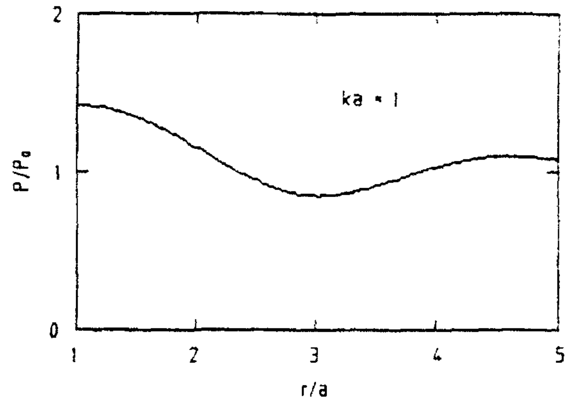

Figure 2(a): Resultant Pressure in front of Rigid Sphere.

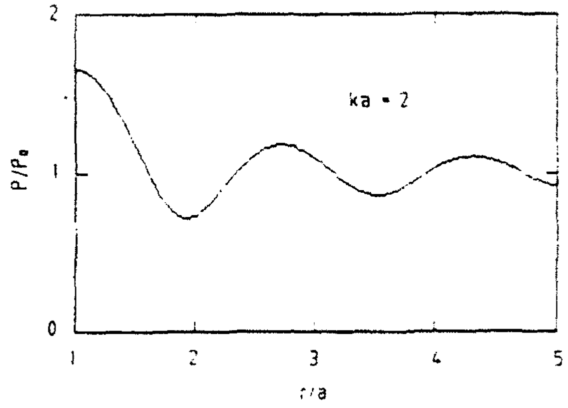

Figure 2(b): Resultant Pressure in front of Rigid Sphere.

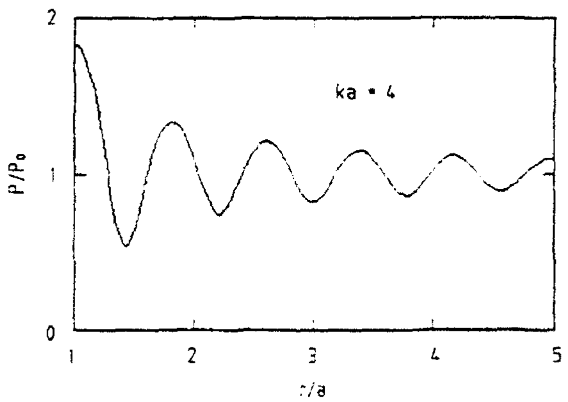

Figure 2(c): Resultant Pressure in front of Rigid Sphere.

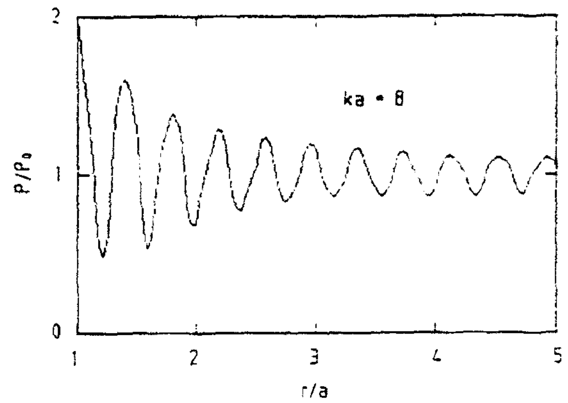

Figure 2(d): Resultant Pressure in front of Rigid Sphere.

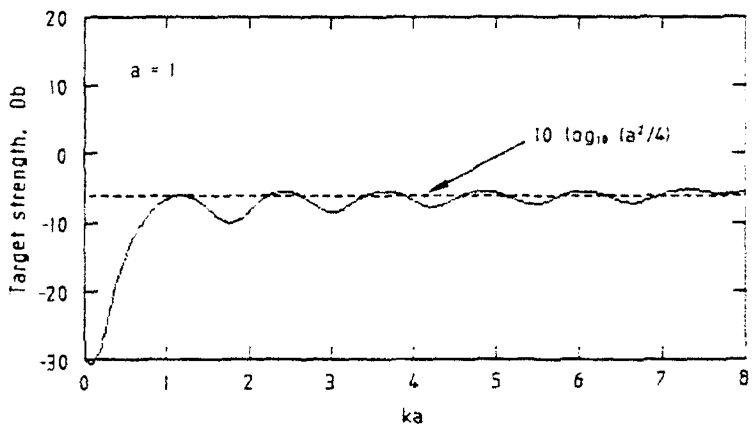

Figure 3: Monostatic Target Strength of a Rigid Sphere. (Dashed line represents a rigid sphere of radius a = 1).

6.3 *Reflection Factor*

Figures 4(a) to 4(d) show the bistatic reflection factor around the scatterer for the *ka* values 1, 2, 4 and 8. From these figures, we can see that for small *ka* values (where the sphere is small compared with the wavelength), the wave propagates in all directions. For large *ka* values, part of the scattered wave spreads out more or less uniformly in all directions from the scatterer, and the rest is concentrated in the 0° direction. When the resultant field is computed by adding the incident and scattered components, the 0° beam of the scattered component interferes destructively with the unchanged plane wave behind the sphere, creating a sharp edged shadow.

6.4 *Intensity*

Figures 5 and 6 show the scattered intensity magnitude around the body in the near and far-fields for *ka* values 1, 2, 4 and 8. These figures exhibit essentially the same behaviour as observed with the reflection factor. Figures 7(a) to 7(d) show the resultant intensity close to the body. The large side-lobes evident in Figures 7(a) and 7(d) arise from the constructive interference of the incident and scattered waves.

Figures 8(a) to 8(d) show the scattered intensity vector field around the body, and illustrates more clearly the polar intensity plots in Figure 5. Figures 9(a) to 9(d) represent the resultant intensity field around the body and show clearly the reduction in intensity behind the sphere as the frequency increases.

7. *Conclusion*

An analysis of the plane wave scattering from a rigid sphere of radius *a* submerged in water has been carried out and results are presented for *ka* values up to 8. Various scattering parameters such as target strength, reflection factor and the intensity field around the body have been computed to illustrate the solutions. Resultant intensity vector plots show clearly the reduction in intensity behind the sphere as the frequency increases.

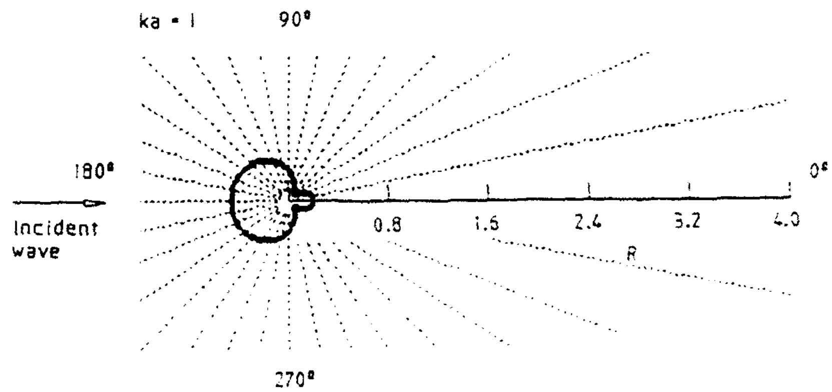

Figure 4(a): Reflection Factor for a Rigid Sphere.

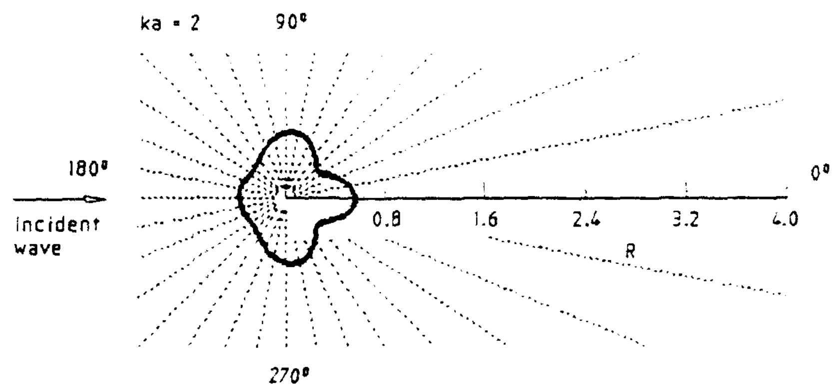

Figure 4(b): Reflection Factor for a Rigid Sphere.

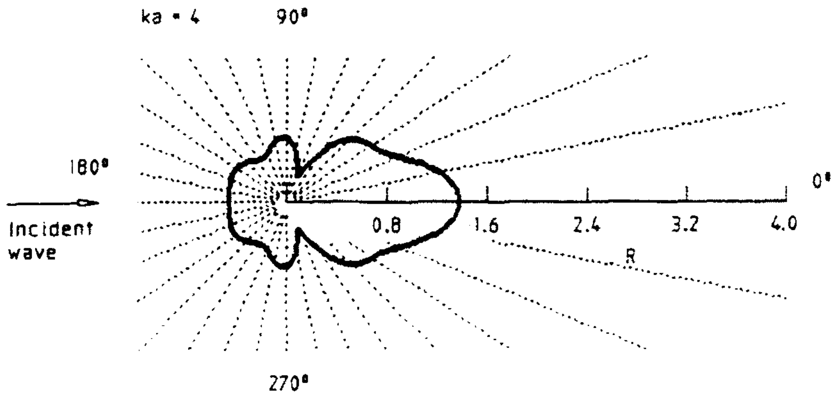

Figure 4(c): Reflection Factor for a Rigid Sphere.

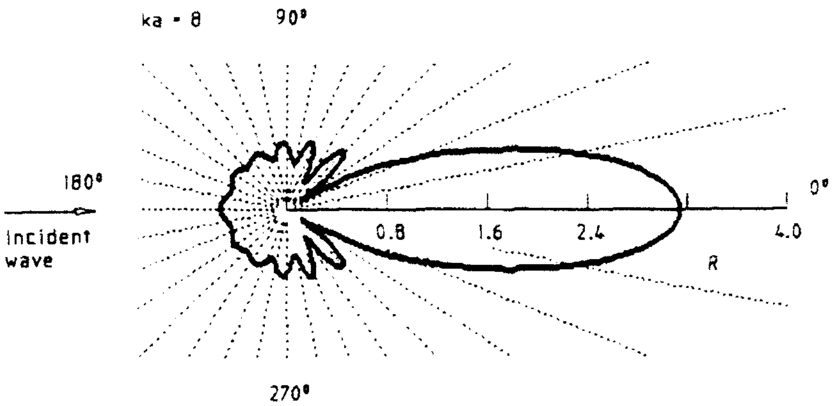

Figure 4(d): Reflection Factor for a Rigid Sphere.

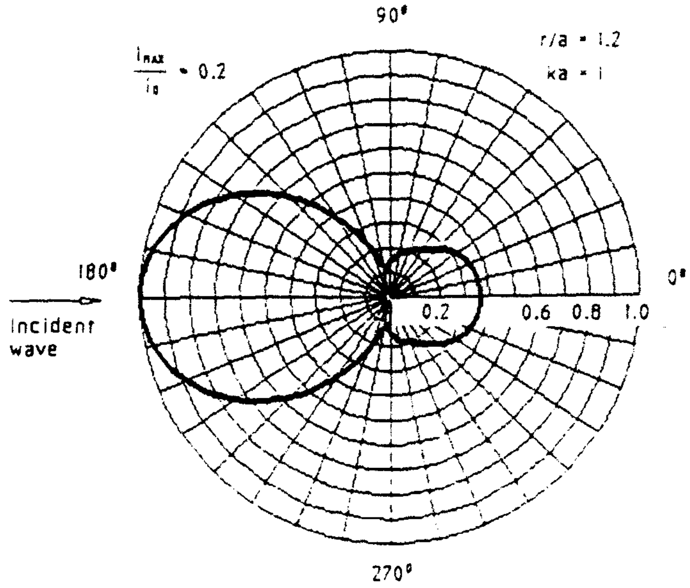

Figure 5(a): Scattered Intensity I/Imax around a Rigid Sphere.

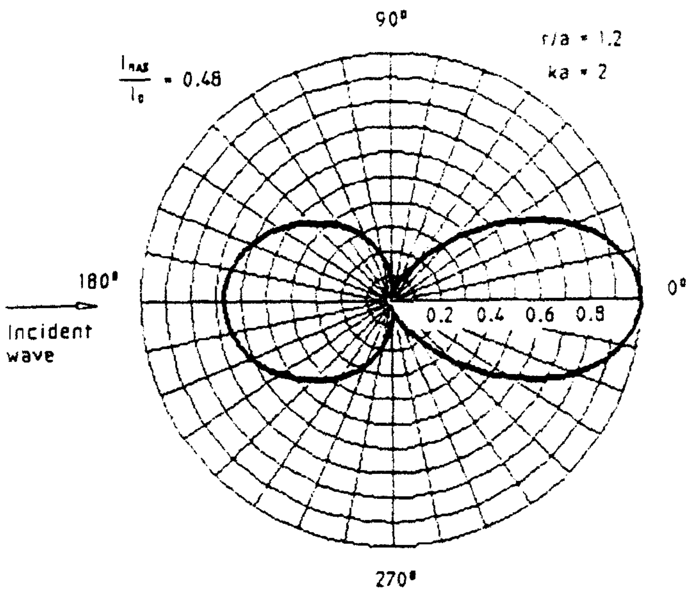

Figure 5(b): Scattered Intensity I/Imax around a Rigid Sphere.

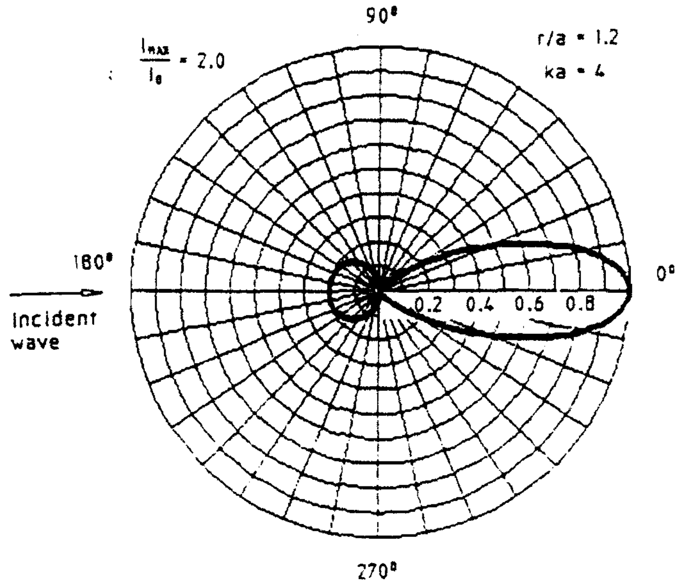

Figure 5(c): Scattered Intensity I/Imax around a Rigid Sphere.

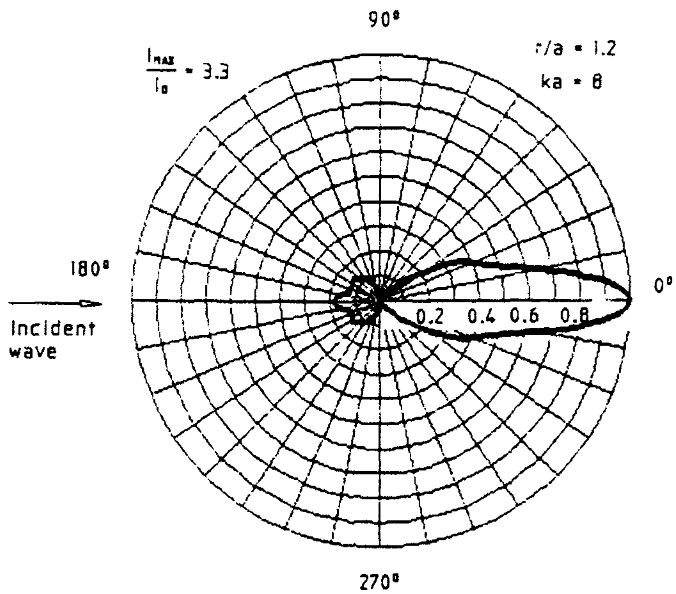

Figure 5(d): Scattered Intensity I/Imax around a Rigid Sphere.

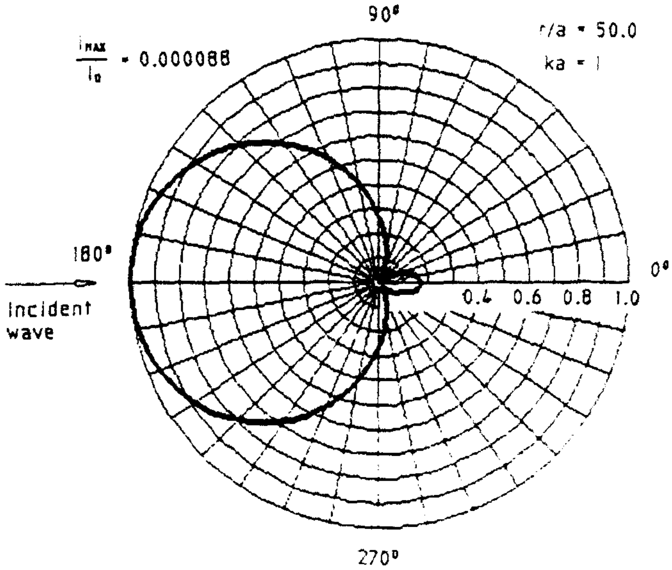

Figure 6(a): Scattered Intensity I/Imax around a Rigid Sphere.

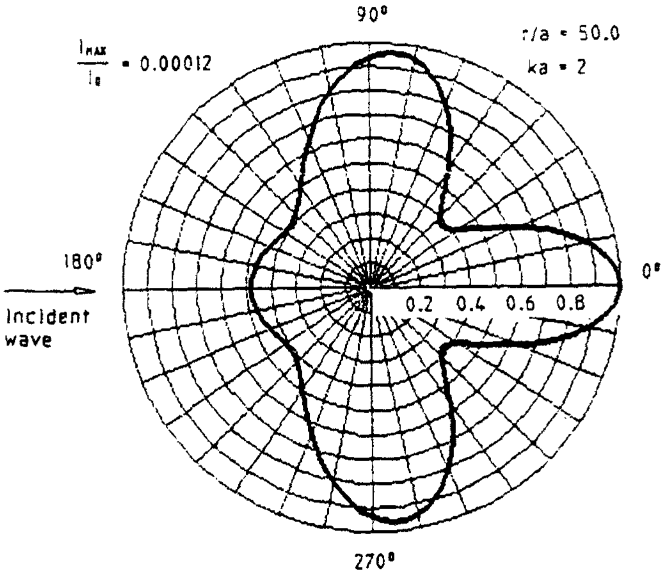

Figure 6(b): Scattered Intensity I/Imax around a Rigid Sphere.

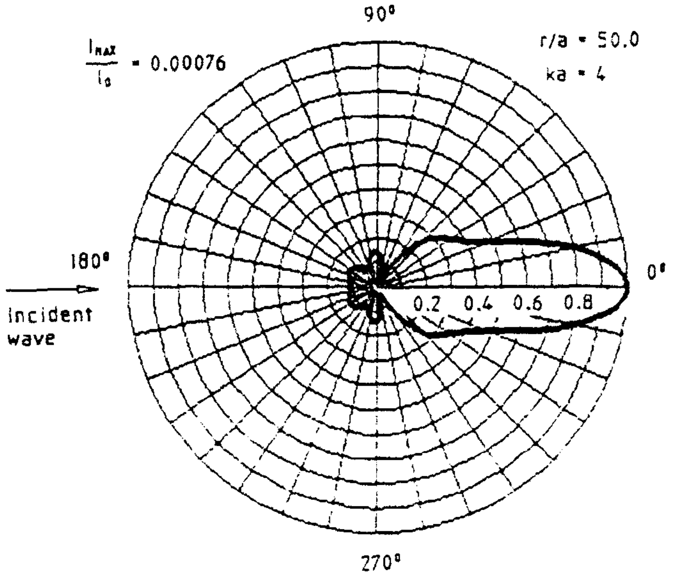

Figure 6(c): Scattered Intensity I/Imax around a Rigid Sphere.

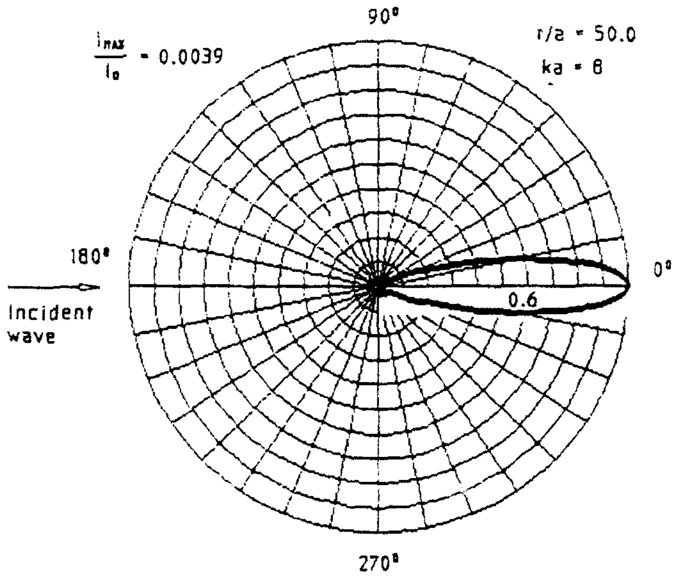

Figure 6(d): Scattered Intensity I/Imax around a Rigid Sphere.

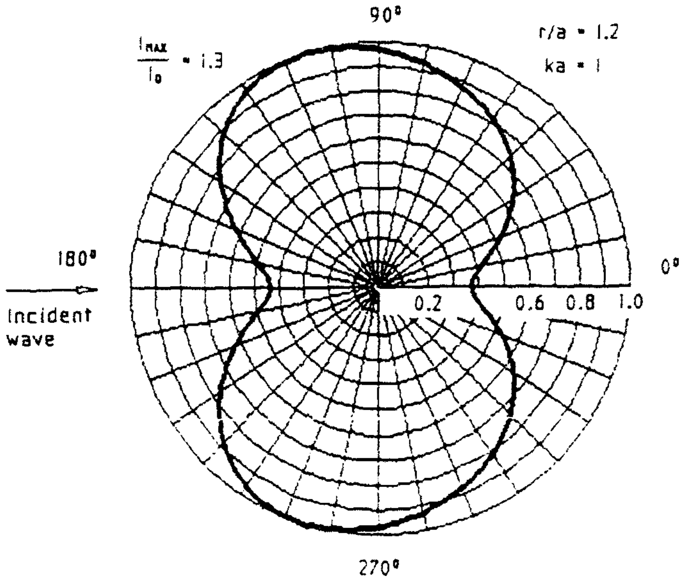

Figure 7(a): Resultant Intensity I/Imax around a Rigid Sphere.

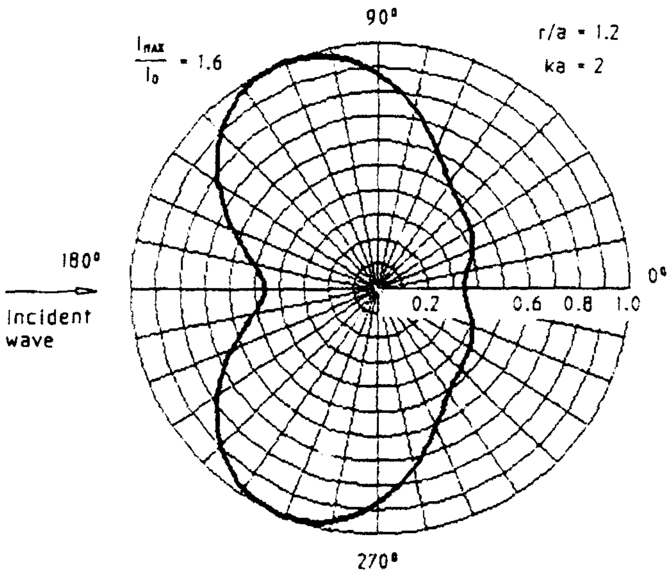

Figure 7(b): Resultant Intensity I/Imax around a Rigid Sphere.

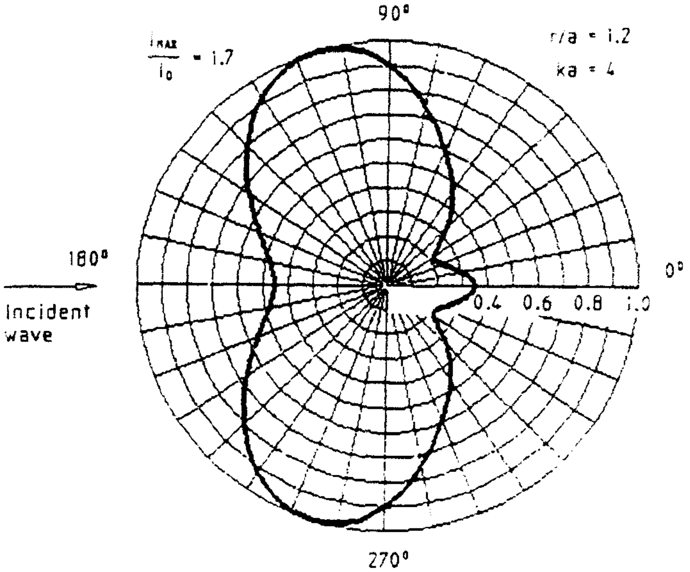

Figure 7(c): Resultant Intensity I/Imax around a Rigid Sphere.

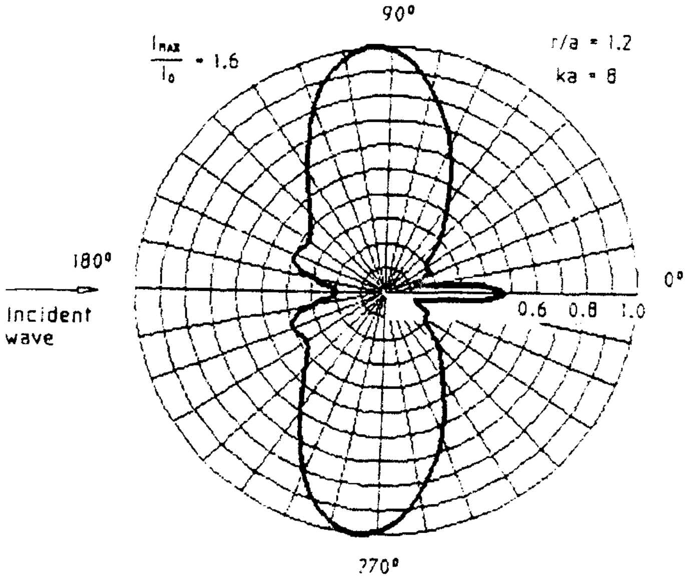

Figure 7(d): Resultant Intensity I/Imax around a Rigid Sphere.

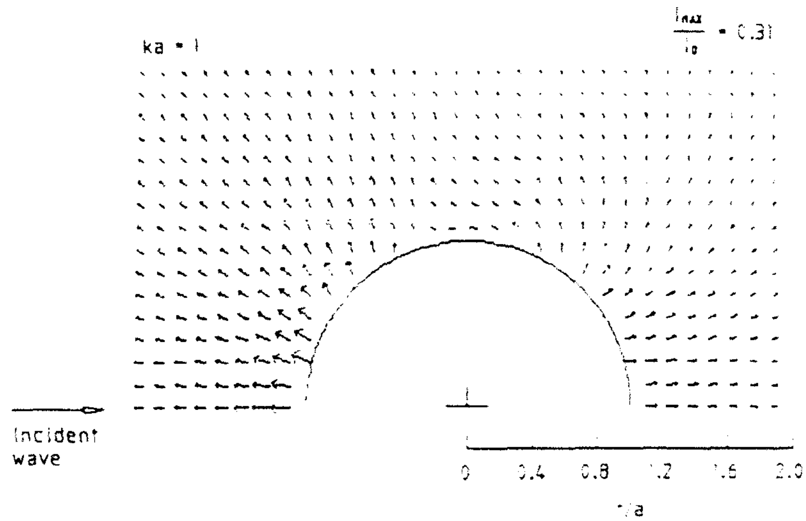

Figure 8(a): Scattered Intensity $[I/Imax]^{1/2}$ around a Rigid Sphere.

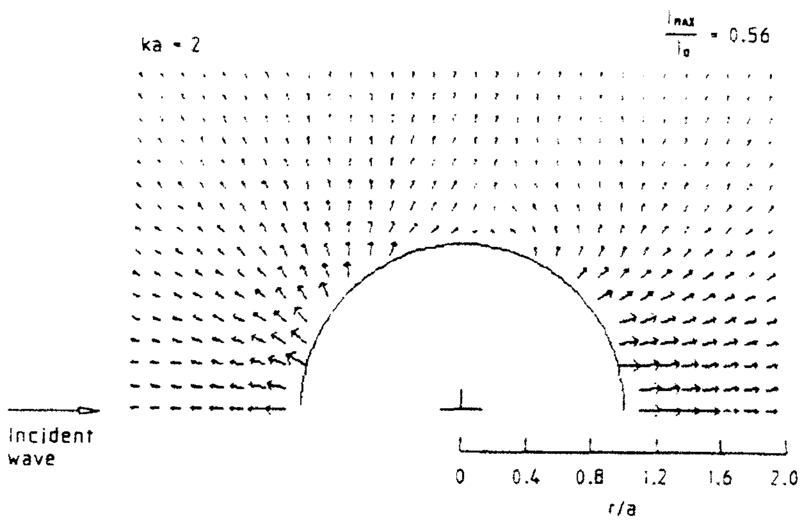

Figure 8(b): Scattered Intensity $[I/Imax]^{1/2}$ around a Rigid Sphere.

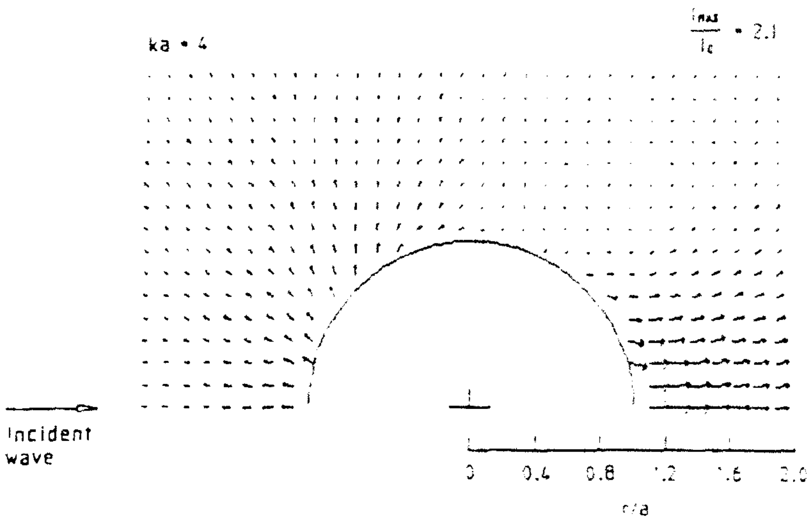

Figure 8(c): Scattered Intensity $[I/Imax]^{1/2}$ around a Rigid Sphere.

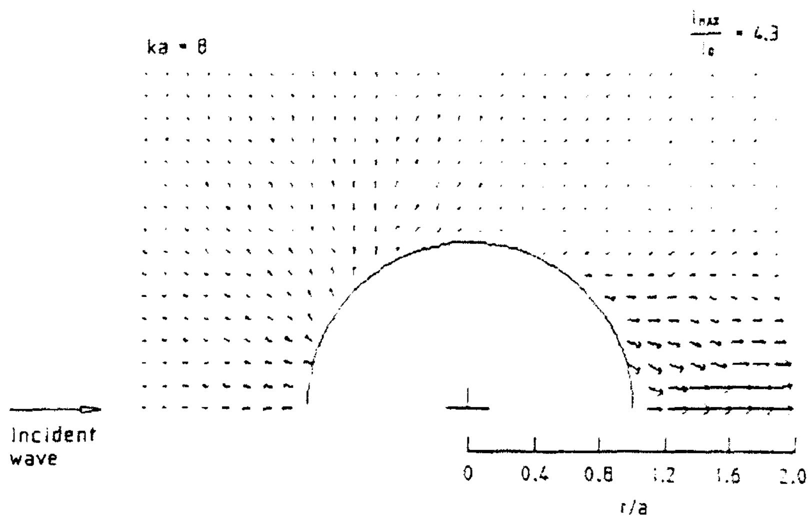

Figure 8(d): Scattered Intensity $[I/Imax]^{1/2}$ around a Rigid Sphere.

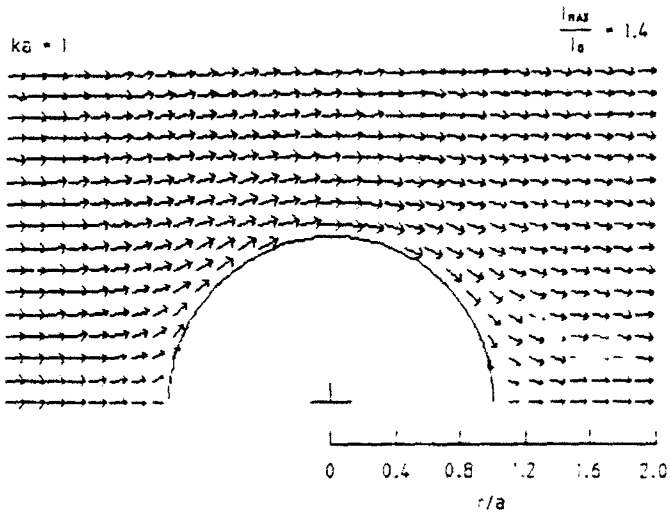

Figure 9(a): Resultant Intensity $[I/Imax]^{1/2}$ around a Rigid Sphere.

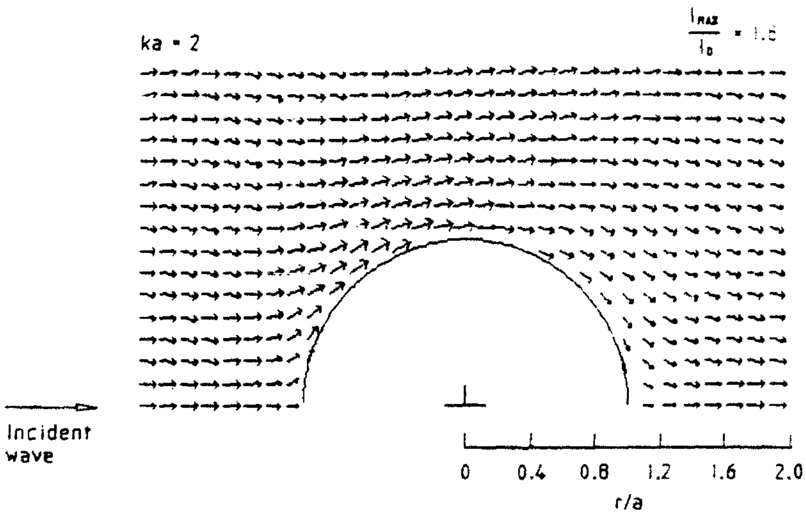

Figure 9(b): Resultant Intensity $[I/Imax]^{1/2}$ around a Rigid Sphere.

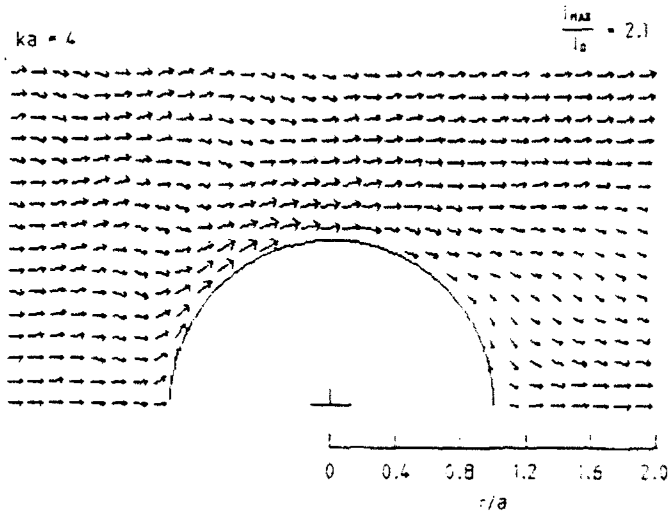

Figure 9(c): Resultant Intensity $[I/Imax]^{1/2}$ around a Rigid Sphere.

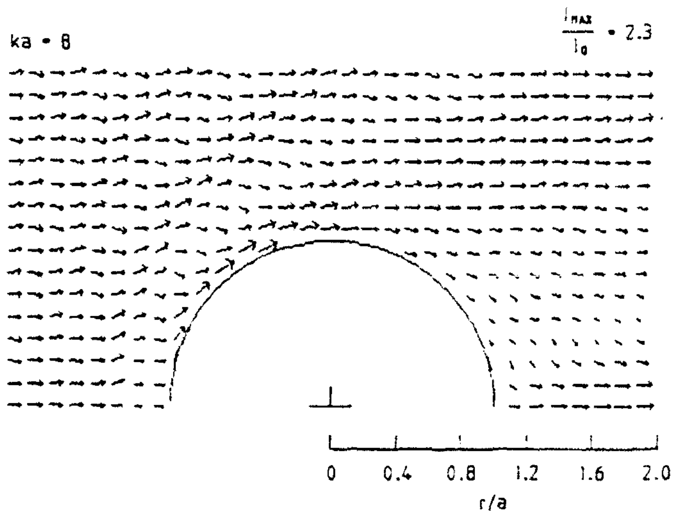

Figure 9(d): Resultant Intensity $[I/Imax]^{1/2}$ around a Rigid Sphere.

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9. Glossary of Symbols

p	Resultant pressure
p_o	Incident pressure wave amplitude
p_i	Incident pressure
$p_{s\infty}$	Scattered pressure from a rigid body (infinite impedance)
w	Normal velocity component of resultant pressure at S_o
w_i	Normal velocity component of incident pressure at S_o
$w_{s\infty}$	Normal velocity component of rigid scattered pressure at S_o
I_o	Average intensity magnitude of the incident pressure
$I_{s\infty}$	Average intensity magnitude of the scattered pressure (rigid body)
I	Average intensity magnitude of the resultant pressure
$\underline{r}, \underline{r}_o$	Position vectors of the field and source points
$G(\underline{r}\|\underline{r}_o)$	Green's function
$g(\|\underline{r}-\underline{r}_o\|)$	Free space Green's function
r, ϑ, ϕ	Spherical coordinates
a	Radius of sphere
S_o	Boundary of sphere ($r = a$)
$\underline{n}$	Unit outward normal to boundary S_o
k	Acoustic wavenumber
ka	Dimensionless frequency parameter
ω	Angular frequency
c	Speed of sound in water
ρ	Density of water
R	Reflection factor

Appendix A

Eigenfunctions of a Sphere

In this section the eigenfunctions of a sphere, which are required in order to construct a suitable Green's function for a spherical radiator, are derived. The Helmholtz equation

$$(\nabla^2 + k^2)P = 0 \tag{A.1}$$

may be expressed in spherical coordinates (r, ϑ, Φ) as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial P}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial P}{\partial r}\right) + k^2P = 0 \tag{A.2}$$

where it is assumed that the sphere in question is symmetrical about the $\vartheta = 0$ axis, such that the ϕ dependence may be omitted. Letting the pressure be represented by $P(r, \vartheta) = R(r)T(\vartheta)$, modifies the Helmholtz equation to

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + k^2r^2 = \frac{-1}{T}\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta\frac{dT}{d\vartheta}\right) \tag{A.3}$$

Each side of this equation must equal the same constant C, i.e.

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left(k^2 - \frac{C}{r^2}\right)R = 0 \tag{A.4}$$

$$\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta\frac{dT}{d\vartheta}\right) + CT = 0 \tag{A.5}$$

Letting $\eta = \cos\vartheta$, $C = n(n+1)$, Legendre's differential equation is obtained

$$(1-\eta^2)\frac{d^2T}{d\eta^2} - 2\eta\frac{dT}{d\eta} + n(n+1)T = 0 \tag{A.6}$$

which has the Legendre polynomial solution

$$T_n(\vartheta) = P_n(\cos\vartheta) \tag{A.7}$$

The equation for the radial factor R can be modified by substituting $z = kr$, to yield the equation for the spherical Bessel function

$$z^2\frac{d^2R}{dz^2} + 2z\frac{dR}{dz} + [z^2 - n(n+1)]R = 0 \tag{A.8}$$

from which the possible solutions are $j_n(kr)$, $y_n(kr)$, $h_n^{(1)}(kr)$, and $h_n^{(2)}(kr)$, where $j_n(kr)$ and $y_n(kr)$ are the spherical Bessel functions, and $h_n^{(1)}(kr)$, $h_n^{(2)}(kr)$ are the Hankel functions of the first and second kind respectively. The chosen solution is

$$R(r) = h_n^{(1)}(kr) \tag{A.9}$$

due to the fact that the spherical Hankel function $h_n^{(1)}(kr)$ corresponds to an outgoing wave, appropriate for the situation where acoustic energy is being radiated outward into an unbounded medium. Therefore in summary, the eigenfunctions of a sphere are $P_n(\cos\vartheta)$ and $h_n^{(1)}(kr)$, and the pressure at an arbitrary point may be written in terms of these eigenfunctions as

$$P(r,\vartheta) = \sum_{n=0}^{\infty} A_n P_n(\cos\vartheta) h_n^{(1)}(kr) \tag{A.10}$$

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ABSTRACT

The Helmholtz integral equation may be formed by combining the scalar wave equation with Euler's equation for motion within a fluid. The solution of this integral equation yields the radiated pressure from a submerged, vibrating body and may be used to characterize the scattering of incident sound waves from bodies. In this report the scattering from underwater rigid spheres is investigated and results are presented for $ka \leq 8.0$.

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Sound Wave Scattering from a Rigid Sphere

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