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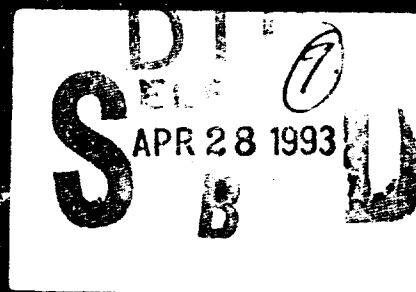
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# CLASSICAL AND QUANTUM SYSTEMS

## Foundations and Symmetries

Proceedings of the II International Wigner Symposium



Edited by  
H. D. Doebner  
W. Scherer  
F. Schroeck, Jr.

FOUNDATIONS AND SYMMETRIES  
CLASSICAL AND  
QUANTUM SYSTEMS

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Dear Sirs:

This letter refers to Grant Number N00014-91-J-1679.

Enclosed is one copy of the publication "Classical and Quantum Symmetries, Foundations and Symmetries, Proceedings of the II International Wigner Symposium". These proceedings are sent to you in order to complete my obligation with respect to my position as Principal Investigator for this grant.

Prof. Franklin E. Schroeck, Jr.

Principal Investigator

cc: Michael Karp

cc: Thomas Roberts

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**CLASSICAL AND  
QUANTUM SYSTEMS**  

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**Foundations and Symmetries**

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# CLASSICAL AND QUANTUM SYSTEMS

## Foundations and Symmetries

Proceedings of the II International Wigner Symposium

Goslar, Germany July 16 - 20, 1991

organized by: *Arnold Sommerfeld Institute for Mathematical Physics,  
Technical University Clausthal and  
Department of Mathematics, Florida Atlantic University*

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## Preface

The Wigner Symposia series emerged from an International Symposium "On Space Time Symmetries", held in College Park, Maryland, 1988 organized by Y.S. Kim and W. Zachary and commemorating the 50<sup>th</sup> anniversary of Wigner's fundamental paper "On the Unitary Representations of the Inhomogenous Lorentz Group". That symposium covered various areas of physics, of theoretical and mathematical physics to which Eugene Paul Wigner has contributed. It has shown how new and fruitful ideas connected with symmetries for a mathematical modelling of complex physical systems developed under the influence of Wigners work. The meeting displayed the unity of physics and it was this impression which gave the idea to have a Wigner Symposium series (WIGSYM series) a remarkable momentum.

The II. International Wigner Symposium, July 16-20, 1992, was organized jointly by the Arnold Sommerfeld Institute, Technical University of Clausthal and the Department of Mathematics, Florida Atlantic University, Boca Raton, with H.D. Doebner, Clausthal, and F. Schroeck, Jr., Boca Raton, as organizers and W. Scherer, Clausthal, as scientific secretary.

Following the idea of the symposia to discuss new developements in physics centered around fundamental questions topics with emphasis on theoretical and mathematical physics were chosen and new frontiers like neuronal networks and quantum holography, whose origins can be traced back to Wigner's work, where included. The aim was to show the unity of different parts in physics and to demonstrate the power of arguments which are directly or indirectly related with mathematical realizations of symmetries. In this spirit the topics discussed in the symposium and appearing in this volume where

- Foundations of Quantum Mechanics
- General group Theoretical and Quantization Methods
- Coherent States
- Berry Phases
- Phase Space and Wigner Distributions
- Applications of Quantum Mechanics: From Channel Space to Wigner Crystal
- Quantum Fields and Particles
- $C^*$ -Algebraic and Methods
- Differential Geometric Methods
- Nonlinear Partial Differential Equations, Dynamical Systems and Neural Networks.

The volume collects plenary contributions and furthermore 132 research articles. Because of the limited page number of the volume we were not able to include all the material presented at the symposium. We regret this. For the selection of the articles, there was a refereeing procedure through the members of the advisory committee.

The symposium was made possible, through support of the following institutions and agencies

International Union for Pure and Applied Physics

International Association for Mathematical Physics

The United States Office of Naval Research.

The Department of Mathematics at Florida Atlantic University

The Division of Sponsored Research at Florida Atlantic University

Niedersächsisches Ministerium für Wissenschaft und Kultur

Deutsche Forschungsgemeinschaft

Alexander von Humboldt Stiftung

Deutscher Akademischer Austauschdienst

Fritz-Thyssen Stiftung

Stifterverband der Deutschen Wissenschaft

Technical University of Clausthal

Arnold Sommerfeld Institut at the Technical University of Clausthal

Zentrum für Technologietransfer und Weiterbildung der TU Clausthal

We are very grateful for this support which in particular made it possible to have among the participants many physicists and mathematicians from different parts of Eastern Europe. The timing of the symposium placed it in a period of breathtaking historic changes in Europe and this gave the II. Wigner Symposium a unique flavour.

Last but not least we would like to thank members and coworkers and especially the students of the Arnold Sommerfeld Institute for their help in organizing this conference.

Special thanks are due to E.M. Herms for his diligent work before and during the symposium, to T. Müller for his invaluable help in preparing these proceedings and to the conference secretary Susanne Gottschlich whose patience, reliability and efficiency deserve a good deal of credit for the success of the II. International Wigner Symposium.

H.D. Doebner  
F. Schroeck, Jr.  
W. Scherer

Eugene P. Wigner

Princeton, NJ  
USA

July 7, 1991

**To the Participants of  
the Symposium in Goslar**

Greetings from Princeton!

I am grateful to the Organizers of the Second International Wigner Symposium for inviting me to come to Germany. I would love to go there and to be among the participants. I feel fine and strong, and, most of all, I am eager to discuss physics with you. However, my physician tells me that I must not travel to Europe. I asked him why. His explanation was that I might get sick there. How many of you have become sick there? I disagree with him, and I would still like to join you.

I also disagree with the Organizers of the Symposium on the title of the meeting. It is quite remarkable that we now have a conference series in which fundamental problems in physics are discussed irrespective of which branch of physics one pursues. However, this issue should not be associated with any particular person. Physics belongs to all physicists, not to any single individual.

As some of you know, I was born and raised in Hungary and studied in Germany. I learned German before English. I would therefore love to write this letter in German, but am writing in English at the request of the Symposium Organizers. Indeed, when I was spending my post-doctoral years in Berlin in the 1920s, part of my job was to visit Goettingen regularly. I used to be on the steam-driven train going through the region where the Symposium is being held. Goslar at that time was known as a town with clean air. Arnold Sommerfeld used to work at Clausthal. I assume that the rail service between Goettingen and Berlin is now available after a long pause. I would really like to be on that train again.

I read the Symposium poster very carefully. The topics of the Symposium appear to include many different subjects. I assume that they are discussed at one scientific meeting, because they are based on the same set of fundamental principles. It is quite possible that not everybody realizes this point. However, the direction of one's research effort should be toward the view that there is only one physics, not toward further division of the subject.

If you are not able to appreciate this point, you do not have to disagree with me too much. There is yet another avenue for working toward the same goal. Fortunately, the same set of theoretical methods can be applied to many different branches of physics. For instance, group theory is applicable to atomic, nuclear, particle, and condensed matter

physics. My younger colleagues are telling me that group theory plays an important role in modern optics. Those group theoretical methods are based on the same group theory. If the theoretical methods for different branches of physics are the same, and if physicists are able to appreciate this, then it would be easier for them to see that there is only one physics.

While I was able to build my intellectual background first in Hungary and then in Germany, I would say that I had the most productive years in the United States of America. Indeed, I am grateful to Princeton University for providing such a wonderful environment in which to live and work. During my years here, I have met many outstanding students and colleagues. However, I should also mention two of my fellow Hungarian-born physicists with whom I maintained very close contact while in the United States. They are John von Neumann and Edward Teller. Their contributions to science are well known. I am grateful to them for their unfailing friendship. I still call Edward Teller whenever I have problems which I cannot solve alone.

As for von Neumann, I have a story to tell. It is my understanding that this Symposium was generated from a topical meeting held at the University of Maryland in 1988 to commemorate the 50th anniversary of the paper which I wrote on the inhomogeneous Lorentz group in 1937 and which was published in 1939 in the *Annals of Mathematics*. I originally submitted this article to one of the prestigious journals in mathematics (not the *Annals*), which rejected my paper. At that time, von Neumann was the editor of the *Annals of Mathematics*. Knowing that my article was rejected, von Neumann invited me to publish the paper in the *Annals*. This means that your Symposium owes a great deal to von Neumann.

I am very happy to hear that the Organizers of this Symposium made a special effort to invite many physicists from the Eastern European countries. There must be many Hungarians. For the reasons mentioned above, Hungary should be given an opportunity to host one of the future meetings of this Symposium series.

Even though I cannot go there, I feel as if I am there. Enjoy the meeting, and enjoy Germany. My thanks to the Organizers of the Symposium, and my thanks to all. I am with you!

Yours truly,

Eugene Wigner

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## **I. Plenary Contributions**



## THE PHILOSOPHY OF EUGENE P. WIGNER

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To the richly diverse audience I am to address this evening I must say, from the onset, that this lecture cannot be the authorized summary, or even a critical account, of a philosophical treatise systematically articulated by Wigner himself: unfortunately, no such sum exists at this time. Consequently, this lecture will rather be an attempt to collect and capture the common themes that run through the vast composition offered by Wigner's contributions to our science.

Almost by its very nature, any such attempt will be colored by one's own personal encounters. I first met Wigner at the NATO 1962 Summer School in Istanbul where Josef Jauch, my thesis adviser, had sent me "to learn the World". Feza Gursey had assembled there a most impressive collection of lecturers: several of them were subsequently elected members of national Academies and/or awarded Nobel prizes: more importantly, they all made themselves immediately accessible. Wigner was one of them. In my first conversation with him, I started by trying to describe to him the place of superselection rules in the Geneva programme on the foundations of quantum mechanics, freely using the language of projective geometry and lattice theory I had just learned. Wigner however soon interrupted me with one of his inimitable remarks: "You know, I am a *very ignorant man*: you must start at the beginning"; he then proceeded, that very afternoon, to administer to me a 3-hour oral qualifying examination! Needless to say, I came out wishing I was that "ignorant": it must be said also that Wigner had brought me, patiently but firmly, to what should have been the center of my argument. The next year, the Committee of my doctoral defense at the University of Geneva comprised Wigner's long-time collaborator Valentine Bargmann. Like Wigner, he too first gave me his own private examination to make sure that the mathematical results I had obtained were not hiding the physics of the problem. As he then offered me to spend a year as a postdoctoral research associate, at Princeton, close also to the group of another of Wigner's long-time colleagues, Arthur Wightman, I was thoroughly trained, at first only as a subject, to the gentle art of the *maieutic* which Wigner practiced with such mastery. Just recently, I had another encounter with Wigner, this time with about 80 of his papers which I was asked to annotate for the forthcoming edition of Wigner's Complete Works Arthur Wightman and Jagdish Mehra are preparing for Springer: most of the papers I read at this occasion appeared in print after the 1950's.

From these perspectives I want to propose, as a sort of working hypothesis for this evening, that *three* main threads are braided in the development of Wigner's thoughts: (1) the guidance provided by *invariance principles*; (2) the elusive role of *consciousness* in physics; and (3) the uncomfortable *responsibilities* of the scientists.

One way to gain entry in what will ultimately be known as "Wigner's philosophy" is to train one's sight first to his contributions to mathematical physics, noting immediately that in the course of Wigner's life, the very territory of mathematical physics has drastically changed; and this, to no small part, due to Wigner's own influence.

In the experience of most students in my generation, Wigner's name was first associated to the extraordinary success met by the systematic exploitation of invariance principles for the theory of elementary particles, first the atoms, and then ever more subtle subatomic particles. An early, but archetypical, representative of this approach is Wigner's 1931 book: *Gruppentheorie und Ihre Anwendungen in der Quantenmechanik der Atomspektren*. Soon afterwards, another seminal contribution appeared: his 1939 paper *On Unitary Representations of the Inhomogeneous Lorentz Group*.

It would be tempting to dwell on their specific physical or mathematical merits. Resisting this temptation here, I will focus along the following lines my commentaries on these two pioneering works: (a) the novelty of their *technical* sophistication; (b) their *philosophical* background; and (c) some of the events that moved these ideas to the foreground of the *foundations* of our science.

My first commentary focuses on what Wigner later called the "unreasonable effectiveness of mathematics in the natural sciences" (1960). Other creative scientists reflected also on this puzzling effectiveness: Einstein, in his lecture *On the Methods of Theoretical Physics* (1933); von Neumann, in his essay *The Mathematician* (1947); or Bohnenberger, in his book *The Role of Mathematics in the Rise of Science* (1966). As Wigner phrased it, one could marvel on how it came about that "we got something out of the equations that we did not put in". Lest we may be led to think that this runs *away from the positivist* tenets that the Founders of quantum mechanics often acknowledged were prevalent in their motivations, and that this renewed emphasis on the power of abstract thought could open to a reincarnation of a *panlogism* à la Leibnitz, we should note three twists which Wigner suggested we put to this overly simple idea. The first of these twists was that we should not view this success as a manifestation of the *power of abstract thought* but of the *power of abstraction* inherent to science, and most importantly to its transmission: the equations are a manifestation of a deeper substrate, namely the invariance *principles* which Wigner characterized as "the laws that the laws of nature must obey". Even though the contents of Noether's theorem in classical mechanics, or of Stone's theorem in quantum mechanics, fitted in this characterization, it should nevertheless be remembered that many in the physics community - and among them, some otherwise well enlightened spectroscopists - were so unprepared for such a systemic onslaught, that they referred to the disciples of this view as the "Gruppenpest". The second twist was that we often conveniently forget how we *chose* the domain of applicability of our theories: the roulette wheel does not violate the laws of mechanics but, as Hadamard pointed out already at the end of the nineteenth century, the serene predictability of mechanics evaporates when one considers non-linear systems with sensitive dependence on initial conditions. The third twist was that any discussion on the internal beauty and self-consistency of a physical theory is to be checked, not only against the *experimental data* the collection of which is suggested by the theory, but also against those data that may seem to lay outside its immediate purview. Wigner's comments were not put forward as mere speculations on the essence of the process of theorizing: they reflected his own sustained interest in the competing models of nuclear physics as well

as of his repeated attempts to look for a world picture that would encompass physics in harmony with other sciences, such as psychology.

Let me also recall that while many physicists may have been neither ready for, nor comfortable with, the new mathematization of their science heralded by Wigner's pre-eminent use of symmetries, it is true that neither were most mathematicians inclined to look in that corner - fortunately some of the best were, e.g. von Neumann. Wigner's 1939 paper on the Lorentz group had undoubtedly its roots in physics, specifically in the work of Dirac and of Majorana, but Wigner himself warned that "the difference ... lies ... mainly in its greater rigor", a concern that one hears more often from mathematicians. The paper, published in the *Annals of Mathematics*, showed a sure professional knowledge of some of the exciting mathematics of the time, in particular of the lines of research opened, just about then, by Murray and von Neumann and by Haar, to mention only two of the most difficult, and mathematically pregnant, works cited in the paper. In his paper, Wigner did work out from scratch two main problems, both of which needed quite some time before they could be integrated in the standard body of mathematics. The first was the theory of projective representations, which was later systematically developed by Bargmann (1954), and can now be seen as an early example of the theory of equivalence of (group) extensions, as formalized for instance in the book of Cartan and Eilenberg (1956). The second was the genuine generalization to a Lie group of the theory of induced representations of discrete groups which had been elaborated originally at the turn of this century by Frobenius and Schur. Wigner's extension of this theory found its general setting thanks to the work of Mackey in the 1950's: that general setting in turn was used, both by Wigner and by Gelfand, to systemize the study of the special functions of classical analysis. This progression provides a concrete case study in which Wigner's aphorism could be reversed: one could indeed equally well marvel at the amount of nice mathematics that came out from so specific a physical problem as the enumeration of all systems that are elementary relatively to *one* very particular group of invariance. Upon looking at these developments, one could almost be tempted to conclude once again that Nature had chosen the *best possible* group for the purpose of eliciting the imagination of mathematicians!

To continue my discussion of Wigner seminal contributions to the use of invariance principles, namely their philosophical background, I will try to rely on Wigner's own testimony, even though it came as an *a posteriori* reflection.

I think it is fair to say that Wigner's original works, and his later comments about the positions to which they have led him, placed him squarely among the Natural Philosophers, a lineage that started in the 17th century, gained wide currency in the Enlightenment, and then led - in part as a reaction to the 19th century's sometimes extreme enthusiasms bordering on arrogance - to the painstaking vigilance of the positivist school. Some of the positivist methodology was certainly part and parcel of the intellectual training of quantum mechanics' Founders; however, we sometimes tend to forget that they had to readjust seriously some of the positivists' conclusions. Listen to Wigner: "The first physics book I read said: *Atoms and molecules may exist, but this is irrelevant from the point of view of physics*; and this was entirely correct - at that time physics dealt only with macroscopic phenomena and Brownian motion was sort of a miracle". While classical mechanics also stood for some drastic revisions as well, Wigner credited its methodology with one major achievement: the separation between

the accidental - *the initial conditions* - and the expression of regularities - *the laws of motion*. He traced back the implementation of this basic distinction to Newton, whom he contrasted with Kepler, noticing about the latter that although "we owe [him] the three precise laws of planetary motion, [he] tried to explain also the [individual] sizes of the planetary orbits and their periods". As for the other beacons Wigner recognized, he gave us a list in his paper on *Two Kinds of Reality* (1964): seminal: Freud, Poincaré and Hadamard; authoritative: Heisenberg, Schrödinger, von Neumann, London & Bauer, and Einstein; challenging: Bridgman and Margenau. I am prepared to accept that the importance of this list is contextual: it has however the value to be as explicit and concise a testimony as we have: there is thus some sense in trying to understand why these names are so singled out: and not others: individually, or even as they are aggregated in these three groups. If you do not want to do this entirely on your own, I can assure you that the paper makes good reading.

Finally, Wigner's 1939 paper on the Lorentz group was much more than a mere *tour de force*: it prompted a new line of enquiries in theoretical physics, especially visible after 1955: the spectacularly successful formulation of conservation laws for generalized charges; the repercussions of the discovery of parity violation; and the exploration of an axiomatic theory of quantum fields in interaction.

More could be said on this first thread in Wigner's thoughts: the synthetic power of invariance principles; but we need to turn now to the second of these threads: the elusive role of consciousness in physics.

The late 1950's and early 1960's mark the time when Wigner, upon revisiting the basic structures of quantum mechanics, started to express publicly his doubts on the *philosophical underpinnings* of the enterprise. From an epistemological point of view, we should distinguish three problems: (a) the limitations of the *formalism* of quantum mechanics; (b) the limitations to its *interpretation*; and (c) the limitations on its *purview*.

I happen to think that the understanding of why it is so hard nowadays to question the foundations of quantum mechanics requires some awareness to the history of its development. In a very real sense, quantum mechanics is *a solution which came before the problem*. In the first quarter of our century, solutions like the *Bohr atom*; the *photon* (a particle in Einstein's photoelectric effect; both a particle and a wave in Einstein's fluctuation formula for the electromagnetic radiation); the *phonon* (explaining the temperature dependence of the specific heat in solids): all came as their own answers, widely separate and certainly very pragmatic. The crisis was around the corner, and it came to a head in the late 1920's, when an apparently general and self-consistent theory came into existence through the labors of Heisenberg, of Schrödinger, and of Born. The theory could commend itself to the physicists by its good predictive power. The mathematicians soon could recognize a definite structure in the presentation von Neumann made of it in his *Mathematische Grundlagen der Quantenmechanik* (1932). And still, some of the fundamental features of the theory were in direct conflict with the tenets of the rest of physics, what is called today classical physics. Wigner, who was born in 1902, came of age in the midst of these events. We already discussed this evening his contribution to some of the successful developments of quantum mechanics in the second quarter of the century. It is now time to address what he increasingly saw as some fundamental shortcomings of its formalism.

Back in his Göttingen days, Wigner had questioned some of the pragmatic bases of quantum mechanics when he worked with Jordan and von Neumann *On an Algebraic Generalization of the Quantum Mechanical Formalism* (1934). That paper however was mathematically too much ahead of its time, and it came to life only in the work of Segal (1947), and of Haag & Kastler (1964). This line of research was prompted in part by the indirect meaning of the Hilbert space in which the wave functions, or state vectors, are supposed to "live": this gap has bothered Wigner much beyond this early paper. He even entertained some questions as to whether the Hilbert space of quantum mechanics should be constructed on the complex numbers, rather than on some other field: that question was largely laid to rest by the Geneva school in the 1960's. Wigner's later criticisms of the formalism expressed his uneasiness with the ambiguities tolerated in the objective status of the wave function. They evolved in part around the heretofore unquestioned validity of the superposition principle, a concept borrowed directly from the classical theory of differential equations. His answer, obtained in collaboration with Wick and Wightman (1952), was that the superposition principle is *not* unconditionally valid in quantum mechanics, and fails to hold in the presence of *superselection rules*. This led also to the new concept of *essential observables*, i.e. non-trivial observables that are compatible with all other observables, and behave therefore as classical observables do. These are specific examples of the observables Wigner had singled out, in a paper of the same year (1952), as those that are precisely measurable: "no observable which does not commute with the additive conserved quantities (such as linear and angular momentum, or electric charge) can be measured precisely". Referring to the observables that do not satisfy this stringent requirement, he adds: "in order to increase the accuracy of the measurement one has to use a very large apparatus". This argument was later refined by Araki & Yanase (1960) and repeatedly revisited by Wigner to emphasize the *positivist* requirement that the elements of the formalism must have direct empirical meaning.

Wigner found it most disturbing that a reconciliation seemed so elusive between the quantum measuring process and the Born statistical interpretation of the wave-function, especially so since both were integral parts of the "orthodox" theory, as presented in the books of von Neumann (1932) and of London & Bauer (1939). "This interpretation states that the wave-function does not describe reality but is merely a tool to determine the statistical relations between successive observations. I must admit, however, that I am not satisfied with this interpretation".

Wigner did not seem to have entertained seriously any theory of *hidden variables* postulating a classical, as yet unattainable, subquantum description. As far as I can make up, such constructions did not appeal to him as proposals that would address squarely his problem. The experimental evidence later obtained by Aspect (1983), on the basis of Bell's inequalities (1965 & 6), have born out Wigner's presumption.

His problem was then with the collapse of the wave-packet, and the infinite regression in von Neumann's description of the quantum measuring process. These led him to conjure up the archetypical *Wigner's friend* in order to illustrate the "dreadful" solipsisms to which one would be cornered by a strict adherence to the "orthodox" interpretation. The difficulties to which Wigner drew attention were of three kinds. Firstly, the word "reality" was used too lightly. Secondly, the active participation of the observer was not properly taken into account. Thirdly, the orthodox faith appeared inadequate to

explain explicitly how an intrinsically statistical description could apply to a single system or particle. Together with these three difficulties, he later listed a fourth, when he learned of the diligent, and yet original, remark made by Zeh (1970) on how quantum fluctuations made illusory any abstraction involving supposedly isolated systems.

All these difficulties seemed to have conspired to turn Wigner towards demands that the phenomena of *consciousness* be taken into account more genuinely when the foundations of the natural sciences are discussed. This increasingly became for him a preeminent theme of personal and public reflections. I like to illustrate the direction taken by his speculations by two quotes from his *Remarks on the Mind-Body Question* (1961): "regions of enquiry, which were long considered to be outside the province of science, were drawn into this province ... The best known example is the interior of the atom, which was considered to be a metaphysical subject ... When the province of physical theory was extended to encompass microscopic phenomena, through the creation of quantum mechanics, the concept of consciousness came to the fore again: it was not possible to formulate the laws of quantum mechanics in a fully consistent way without reference to consciousness". For Wigner, there was an unresolved ideological jump between classical and quantum theory. The shadows on the wall were not affected by whether the platonic philosopher observed them or not: the classical physicist thought that he could, at least in principle, rig his measuring devices in such a manner as to make arbitrarily small any influence his measurements might have on the system under investigation: a clean separation between the observer and the observed was a justifiable abstraction. Contrast this with Wigner's own words concerning the situation encountered in quantum mechanics: "even though the dividing line between the observer ... and the observed ... can be shifted ... it cannot be eliminated." Wigner rebelled against the idea that the description of the quantum measuring process involved, in the "orthodox" theory, a uniquely different scheme than that provided by the usual evolution equations written to describe an isolated quantum system: such a theory had to be incomplete, and it was condemned to be incomplete as long as the role of consciousness was neglected.

The last batch of philosophical criticisms Wigner directed to quantum mechanics concerned its purview, especially its connection with the theory of relativity. Here again the first problems on which he would draw our attention are the empirical basis for the most primary concepts: the existence of a position operator, the subject of his early work with Newton (1949); or the limitations his work with Salecker (1958) had shown to be imposed by quantum theory on the measurement of space-time distances on small scales, necessary to a direct experimental determination of metrics and curvatures. He also repeatedly protested (at least as late as 1986) that a relativistic formulation of the quantum measurement process would need some serious readjustements since, in order to be satisfactory, it would in particular have to clarify how two observers in relative motion could agree on the meaning of the collapse of the state vector. On such developments as the relativistic theory of quantum fields, he professed a cautious optimism: while he agreed that there was a "great difference between the relation of special relativity and quantum theory on the one hand, and general relativity and quantum theory on the other", he was pleased to note that "it is at least possible to formulate the requirements of special relativistic invariance for quantum theories and to ascertain whether these requirements are met. The fact that the answer is more nearly

*no* than *yes*, that quantum mechanics has not yet been fully adjusted to the postulates of the special theory [of relativity], is perhaps irritating. It does not alter the fact that the question of consistency of the two theories ... by now has more nearly the aspect of a puzzle than that of a problem" (1957). My point in this long quotation is not to argue whether that situation has fundamentally changed in the intervening 35 years, but rather to illustrate how Wigner, the philosopher of science, was perennially looking for the full consistency of the broadest picture and the general sense one had to find beyond specific computations and particular schemes.

This last remark brings me to the third thread that runs through Wigner's thoughts: the responsibility of the scientist. He saw it both as a task internal to the scientific community, and one that reaches outside this community.

Wigner's active scientific life ran through a period that saw extraordinary changes in the way science was done. One of these changes, he denounced as the *balkanization of science* that accompanied its growth. Several of his public addresses in the 1970's were admonitions pleading for a reaction. Reflecting about the nature of science when he entered the arena, he commented: "It was possible at the time to know physics; today it is difficult to know nuclear physics ... During my work on nuclear chain reactions, I already became scared by the increasing specialization"; he had indeed noted one dreaded result of this specialization: "The problem of communication is not only external to physics; it threatens also to develop into an internal one". With others, such as Alvin Weinberg, he then advocated a remedy that is worth thinking about again: "to write our articles for a less specialized readership, to devote more time and energy to the composition of reviews and to reading of more of the reviews covering the results of sister sciences", lest we fail to recognize that "science is an edifice, not a pile of bricks".

The growth of science has not been only in a diversification of interests; it has also involved increasing public funding of large scientific projects - the so-called *big science* of national or international laboratories - as well as the appropriations of vast resources to harness or control its applications. Both needed some explaining to the public and some listening to the politicians. Even though our purpose this evening was to focus on Wigner the philosopher in science, we should want to remember that the scientific enterprise has known also another Wigner, the policymaker for science. While I am not sure that this is an entirely different story, it is probably not one you still want to hear tonight.

## MEASUREMENT AND OBJECTIVITY IN QUANTUM MECHANICS

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## THE PROBLEM

In quantum mechanics, not all physical quantities can be simultaneously 'objective,' i.e., for a system in a given quantum mechanical pure state, not all physical quantities of the system can have determinate values in the state - no assignment of values to all quantities is consistent with the quantum mechanical functional relationships holding between these quantities. Quantities that are not objective in a given state, i.e., quantities whose values are indeterminate (not merely uncertain or unknown), somehow become objective, or obtain (or manifest) determinate values, when measured, while other quantities, correspondingly, become nonobjective.

A classical system is described in terms of a commutative algebra of physical quantities. These quantities are all real-valued functions on the phase space of the system and take values at all times, even when the system is interacting with other systems. A physical quantity is a magnitude associated with a set of possible values. An idempotent quantity has only two possible values, 0 or 1. The algebra of physical quantities can be generated from the subalgebra of idempotent quantities, roughly because each quantity, e.g., position, corresponds to a set of idempotents, in this case the set of characteristic functions on phase space that assign 0, 1 values to subsets associated with different ranges of position values. To assign a value to the position of a particle is equivalent to assigning a 1 to every range of positions containing the value and a 0 to every other range of positions, or assigning the truth values 'true' and 'false' to the corresponding propositions.

I shall refer to the algebra of idempotent physical quantities of a system as the 'property structure' of the system. The property structure of a classical system is a Boolean algebra. It represents, through its ultrafilters, all possible ways in which the system can manifest its properties, or all possible ways in which the properties of the system can fit together as simultaneously determinate.

The transition from classical to quantum mechanics involves the transition from a commutative algebra of physical quantities to a noncommutative algebra, equivalently the transition from a Boolean to a non-Boolean algebra of idempotent quantities or properties. This is, in effect,



the formal significance of quantization. The non-Boolean property structure of a quantum mechanical system can be represented as a partial Boolean algebra - a family of Boolean algebras that are 'pasted together' in a certain way by identifying common elements.

Consider, for example, a quantum mechanical system associated with a 3-dimensional Hilbert space,  $\mathfrak{H}_3$ . Each set of three orthogonal 1-dimensional subspaces defines an 8-element Boolean algebra generated by these subspaces as atoms of the algebra. The algebra contains the three atoms, the three planes spanned by these atoms pairwise, together with the zero element (0) corresponding to the null subspace, and the unit element (1) corresponding to the whole 3-dimensional space. It is isomorphic to the corresponding algebra of projection operators, representing the idempotent magnitudes of the system. Evidently, some of these 8-element Boolean algebras have elements in common. For example, if we fix a 1-dimensional subspace  $\mathfrak{K}_1$ , and consider two choices for the remaining pair of orthogonal lines - any initial pair  $\mathfrak{K}_2, \mathfrak{K}_3$  orthogonal to  $\mathfrak{K}_1$ , and any other pair  $\mathfrak{K}_2', \mathfrak{K}_3'$  orthogonal to  $\mathfrak{K}_1$  - then the two 8-element Boolean algebras are pasted together at the elements 0,  $\mathfrak{K}_1, \mathfrak{K}_1^\perp, 1$ , where  $\mathfrak{K}_1^\perp$  represents the plane orthogonal to  $\mathfrak{K}_1$ .

There are two notions of state in classical mechanics: (1) the state  $s$  as a point in phase space, assigning values to all dynamical quantities, and (2) the state  $w$  as a probability measure on phase space. The first notion of state (call this a 'property state') selects an ultrafilter of properties in the Boolean property structure  $\mathfrak{B}$  (isomorphic to the set of Borel subsets of phase space), associated with propositions (assigning ranges of values to the dynamical quantities) that are true in  $s$ . This ultrafilter corresponds to the properties or propositions represented by subsets of phase space containing  $s$  - these are the properties possessed by the system in the state  $s$ . Properties represented by subsets of phase space not containing  $s$  are not properties of the system in the state  $s$ . Equivalently,  $s$  defines a 2-valued homomorphism on  $\mathfrak{B}$ , with 1 corresponding to 'true' or 'possessed,' and 0 corresponding to 'false' or 'not possessed.' The second notion of state (call this a 'statistical state') assigns probabilities to elements of  $\mathfrak{B}$  or, equivalently, to the ultrafilters or 2-valued homomorphisms defined by the property states (hence, to the different possible ways in which the properties of the system can fit together as simultaneously determinate).

The property structure of a quantum mechanical system,  $\mathfrak{L}$ , is not embeddable into any Boolean algebra (except in the special case of system with a 2-dimensional Hilbert space). This means that there are no 2-valued homomorphisms on  $\mathfrak{L}$  in the general case, and the quantum state is defined as a generalized statistical state assigning probabilities to elements of  $\mathfrak{L}$ . We can express the difference between  $\mathfrak{B}$  and  $\mathfrak{L}$  this way: All the elements of  $\mathfrak{B}$  can be *determinate* or *objective* simultaneously, i.e., every element can have

a definite truth value at all times, and the different possible sets of truth values correspond to the different classical property states. This is not the case for a quantum mechanical property structure  $\mathcal{L}$  that is not embeddable into a Boolean algebra - only a proper subset of elements of  $\mathcal{L}$  can be objective at any particular time.

The question I want to consider here is this: What principles govern the selection of a subset of properties as objective or determinate, given that the properties of a quantum mechanical system can not all be determinate simultaneously (on pain of contradiction)? In effect, different interpretations of quantum mechanics involve different principles of objectivity, different proposals for defining the objective or determinate physical quantities for a system in a given quantum state.

#### A PROPOSAL

The proposal sketched below is presented as capturing certain minimal aspects of what we require of a notion of objectivity, however we understand this notion in a metaphysical sense. Just as classical mechanics does not select what propositions are true and what propositions are false, or what the values of physical quantities are - the theory only tells us how given values are transformed dynamically, or places restrictions on the values of certain quantities given the values of other quantities, which ultimately come from outside the theory - so quantum mechanics cannot be expected to select what quantities are objective, i.e., what quantities have determinate values. What the theory ought to be able to do is show how objectivity is transformed, i.e., given that certain (Boolean algebras of) properties are objective (and this information ultimately has to be stipulated or come from outside the theory), the theory ought to yield information about what other properties are objective, or place certain limits on what can be taken as objective, or show how the set of objective properties is transformed dynamically.

If the quantum state of an isolated system  $S$  is represented by a vector  $\psi \in \mathcal{H}_S$ , and no properties are designated as objective, the proposal is to take the objective properties of the system as the partial Boolean subalgebra in  $\mathcal{L}$  consisting of the set of Boolean subalgebras  $\{\mathcal{B}_\psi\}$  that intersect in the Boolean subalgebra generated by the atoms  $P_\psi$  and  $P_\psi^\perp$ , where  $P_\psi$  is the projection operator onto the 1-dimensional subspace spanned by  $\psi$  and  $P_\psi^\perp$  is the projection operator onto the orthogonal complement of this subspace in  $\mathcal{H}_S$ . I shall refer to the set of Boolean subalgebras  $\{\mathcal{B}_\psi\}$  generated by a vector  $\psi$  in this way as the 'fan' defined by  $\psi$ . Take the property state of the system as the ultrafilter generated by  $\psi$  in the fan.

It is evident that (i) any two algebras in the fan intersect in an algebra in the fan, (ii) all the algebras in the fan have the same minimum and maximum elements (corresponding to the null space and the whole Hilbert space, respectively), (iii) for any element or pair of elements in the fan that

belong to the intersection of two algebras in the fan, the orthocomplements and the infima or suprema (Boolean meets or joins) taken with respect to the two algebras coincide, and finally ( $\vee$ ) for any  $n$  elements in the fan, such that for every pair of these elements there is an algebra in the fan containing the pair, there exists an algebra in the fan containing all  $n$  elements. These conditions define the 'pasting' required for a set of Boolean algebras to form a partial Boolean algebra.

Notice that all the elements of the Boolean algebras in the fan are assigned 0,1 probabilities by  $\psi$ , and no other elements in  $\mathcal{L}$  are assigned 0,1 probabilities by  $\psi$ . So  $\psi$  defines a 2-valued homomorphism on the fan, even though the elements in the fan are not all mutually compatible in the technical sense of quantum mechanics - the different compatible subsets form the different Boolean subalgebras in the fan. In other words, a fan is a partial Boolean algebra of properties that can all be taken as mutually objective or determinate without violating the quantum mechanical functional relationships holding between the projection operators representing these properties. Notice, also, that every Boolean algebra in the fan that is nonmaximal in  $\mathcal{L}$  is a subalgebra of a Boolean algebra in the fan that is maximal in  $\mathcal{L}$ . So the fan is completely specified by the set of maximal Boolean algebras in the fan.

If the quantum state of the system is  $\psi$  and some Boolean subalgebra  $\mathcal{B}$  in  $\mathcal{L}$  is designated as objective, then the proposal is to take the objective properties of the system as the properties in the set of Boolean subalgebras  $\{\mathcal{B}_\psi\}/\mathcal{B}$ , where  $\mathcal{B}_\psi/\mathcal{B}$  is defined, for each  $\mathcal{B}_\psi$ , as the extension of  $\mathcal{B}$  obtained by completion from  $\mathcal{B}$  and the elements in  $\mathcal{B}_\psi$  that are compatible (in the technical sense of quantum mechanics) with  $\mathcal{B}$ . That is, for each  $\mathcal{B}_\psi$  in  $\{\mathcal{B}_\psi\}$ , we add to  $\mathcal{B}$  everything in  $\mathcal{B}_\psi$  that is compatible with  $\mathcal{B}$ , together with meets, joins, and complements, and reject the remaining elements in  $\mathcal{B}_\psi$ . (So the operation '/' is a kind of conditionalization of  $\{\mathcal{B}_\psi\}$  with respect to  $\mathcal{B}$ .)

If  $\mathcal{B}$  is generated by  $n$  atoms  $P_i$ , where the  $P_i$  are projection operators onto  $n$  mutually orthogonal subspaces  $\mathcal{K}_i$  that span the Hilbert space of the system, then (as I shall demonstrate below) the maximal Boolean algebras in the set  $\{\mathcal{B}_\psi\}/\mathcal{B}$  all intersect in the Boolean algebra generated by the atoms  $\{O_i, P_i-O_i\}$ ,  $i = 1, \dots, n$ , where  $O_i$  is the projection operator onto the 1-dimensional subspace or ray that is the projection of  $\psi$  onto  $\mathcal{K}_i$ , and  $P_i-O_i$  is the projection operator onto the relative orthocomplement of this ray in  $\mathcal{K}_i$ . It follows that  $\{\mathcal{B}_\psi\}/\mathcal{B}$  is also a partial Boolean algebra, a (generalized) fan of properties of  $\mathcal{L}$ . Again, every Boolean algebra in the fan that is nonmaximal in  $\mathcal{L}$  is a subalgebra of a Boolean algebra in the fan that is maximal in  $\mathcal{L}$ , so the generalized fan is also completely specified by the set of maximal Boolean algebras in the fan.

Take the property state of the system as one of the ultrafilters in  $\{\mathbf{B}_\psi\}/\mathbf{B}$  generated by the  $O_i$ , for some  $i = 1, \dots, n$ . This is consistent with the property state defined by  $\psi$  in the simple fan  $\{\mathbf{B}_\psi\}$ , in the sense that properties that belong to the intersection of an algebra in  $\{\mathbf{B}_\psi\}/\mathbf{B}$  and an algebra in  $\{\mathbf{B}_\psi\}$  retain their status of belonging, or not belonging, to the property state (i.e., elements that survive the transition from  $\{\mathbf{B}_\psi\}$  to  $\{\mathbf{B}_\psi\}/\mathbf{B}$  retain their truth values). So, as in the simple fan, the property states define 2-valued homomorphisms on the generalized fan, even though the elements in the fan are not all mutually compatible, i.e., the generalized fan is also a partial Boolean algebra of properties than can be taken as mutually objective or determinate, without violating the quantum mechanical functional relationships. The Born rule for the probabilities defined by  $\psi$  can now be interpreted as specifying probabilities in the classical (Kolmogorov) sense for the different property states (ultrafilters) in  $\{\mathbf{B}_\psi\}/\mathbf{B}$ .

In the following section, I shall show that this proposal leads to a reformulation and solution of the measurement problem.

#### MEASUREMENT

Consider a measurement as an interaction between two quantum systems, the object system  $S$  and the measuring instrument  $M$ . Suppose the indicator quantity  $R$  ('pointer reading') of  $M$  is designated as objective (by requiring, say, that superselection rules apply to  $M$ , which we can represent formally by stipulating that  $R$  is in the center of the algebra of physical quantities of  $M$  and so commutes with all physical quantities of  $M$ ). Then it follows from the above proposal that the Boolean subalgebra of measured properties of  $S$  and indicator properties of  $M$  is selected as objective in the property structure  $\mathbf{L}$  of the composite system  $S+M$ . So if the indicator quantity of the measuring instrument  $M$  is objective, it follows that the measured quantity of  $S$  becomes objective in virtue of the correlations induced by the measurement interaction.

As usual, assume that the measurement interaction between  $S$  and  $M$  results in the state transition  $\psi \otimes \rho_0 = \sum c_i \alpha_i \otimes \rho_0 \rightarrow \sigma = \sum c_i \alpha_i \otimes \rho_i$ , where  $\psi = \sum c_i \alpha_i \in \mathcal{H}_S$  is the initial (pure) state of  $S$ ,  $\rho_0 \in \mathcal{H}_M$  represents an eigenvector of the zero value of the indicator quantity  $R$  of  $M$ ,  $\alpha_i$  are eigenstates of the measured quantity  $A$  of  $S$ , and  $\rho_i$  are eigenstates of  $R$ . Then the objective Boolean algebras are those in the set  $\{\mathbf{B}_\sigma\}/\mathbf{B}_R$ , where  $\mathbf{B}_R$  is the Boolean algebra generated by the indicator properties  $I \otimes P_i$ ,  $i = 1, \dots, n$  as atoms, the  $P_i$  being projection operators onto the  $n$  subspaces of  $\mathcal{H}_M$  corresponding to the  $n$  distinct eigenvalues of the indicator quantity  $R$ .

In general,  $\mathcal{H}_M$  and the subspaces corresponding to the projection operators  $P_i$  will be  $\infty$ -dimensional. To illustrate the construction  $\{\mathbf{B}_\sigma\}/\mathbf{B}_R$ , it

suffices to take  $\mathfrak{H}_S$  and  $\mathfrak{H}_M$  as 3-dimensional. All significant formal features of the general case are present in the 3x3-dimensional case.

Consider, for the moment, a maximal measurement, i.e., three indicator values for the three eigenvalues of  $A$ .  $\mathfrak{B}_R$  is the 8-element nonmaximal Boolean subalgebra in  $\mathcal{L}$  generated by the three atoms  $I \otimes P_i$  (where each projection operator  $P_i$ ,  $i = 1, 2, 3$ , projects onto a 1-dimensional subspace or ray in  $\mathfrak{H}_M$  spanned by the vector  $\rho_i$ ). The final state  $\sigma = \sum c_i \alpha_i \otimes \rho_i$  defines a fan of Boolean subalgebras  $\{\mathfrak{B}_\sigma\}$  in  $\mathcal{L}$  that intersect in the Boolean subalgebra generated by the atoms  $P_\sigma, P_\sigma^\perp$ .

Some of the Boolean algebras in the fan are generated by the atoms corresponding to the six orthogonal vectors  $\alpha_i \otimes \rho_j$ ,  $i \neq j$ , that span a 6-dimensional subspace  $\mathfrak{K} \in \mathfrak{H}_{S+M}$  orthogonal to  $\sigma$ , and three orthogonal vectors  $\sigma, \phi, \chi$ , where  $\phi, \chi$  are any two vectors orthogonal to  $\sigma$  in the 3-dimensional subspace  $\mathfrak{K}^\perp$  orthogonal to  $\mathfrak{K}$ . Consider any one of these algebras. Call it  $\mathfrak{B}_\sigma^*$ . What is  $\mathfrak{B}_\sigma^*/\mathfrak{B}_R$ ?

The six rays (spanned by the vectors)  $\alpha_i \otimes \rho_j$  in  $\mathfrak{K}$  are all compatible with  $\mathfrak{B}_R$  - taking the Boolean meet of a ray with an atom in  $\mathfrak{B}_R$  yields the ray or the null element. So these elements all belong to  $\mathfrak{B}_\sigma^*/\mathfrak{B}_R$ , i.e., the atoms  $P_{ij}$ ,  $i \neq j$ , belong to  $\mathfrak{B}_\sigma^*/\mathfrak{B}_R$ , where  $P_{ij}$  is the projection operator onto the ray spanned by  $\alpha_i \otimes \rho_j$ . None of the rays  $\sigma, \phi, \chi$  are compatible with  $\mathfrak{B}_R$ . But  $P_\sigma + P_\phi + P_\chi = P_{11} + P_{22} + P_{33} = P_{\mathfrak{K}^\perp}$ , where  $P_{ii}$  is the projection operator onto the ray  $\alpha_i \otimes \rho_i$ , is compatible with  $I \otimes P_i$ ,  $i = 1, 2, 3$ , hence with  $\mathfrak{B}_R$ . Taking the meet of this element with  $I \otimes P_i$ ,  $i = 1, 2, 3$ , yields the three remaining atoms of the maximal Boolean subalgebra  $\mathfrak{B}_{A,R}$ , generated by the atoms  $P_{ij}$ ,  $i = 1, 2, 3; j = 1, 2, 3$ .

So  $\{\mathfrak{B}_\sigma\}/\mathfrak{B}_R$  contains  $\mathfrak{B}_{A,R}$ . The question now is whether there are other Boolean subalgebras in  $\{\mathfrak{B}_\sigma\}/\mathfrak{B}_R$ . I shall show that:

(1) No other maximal Boolean subalgebra,  $\mathfrak{B}_{B,R}$ , generated by the atoms  $Q_{ij}$ , where  $Q_{ij}$  is the projection operator onto the ray  $\beta_i \otimes \rho_j$ ,  $i = 1, 2, 3; j = 1, 2, 3$ , with  $\beta_i$  derived via a unitary transformation from  $\{\alpha_i\}$ , belongs to  $\{\mathfrak{B}_\sigma\}/\mathfrak{B}_R$ .

(2) There are other maximal Boolean subalgebras in  $\{\mathfrak{B}_\sigma\}/\mathfrak{B}_R$ , but while they all contain the subalgebra of indicator properties of  $M$  (by construction), none contain any nontrivial subalgebra of properties of  $S$ . I shall refer to such subalgebras as 'nonseparable.'

(3) The only nonmaximal Boolean subalgebras in  $\{\mathfrak{B}_\sigma\}/\mathfrak{B}_R$  are all subalgebras of  $\mathfrak{B}_{A,R}$ , or subalgebras of one of the nonseparable algebras.

(4) All the maximal Boolean subalgebras in  $\{\mathfrak{B}_\sigma\}/\mathfrak{B}_R$  intersect in the nonmaximal Boolean subalgebra generated by the atoms  $\{P_{11}, P_{21} \vee P_{31}, P_{22}, P_{12} \vee P_{32}, P_{33}, P_{13} \vee P_{23}\}$ , where the  $P_{ii}$  are the projection operators onto the

1-dimensional subspaces or rays that are the projections of  $\sigma$  onto the subspaces defined by  $I \otimes P_i$ ,  $i = 1, 2, 3$ , and the  $P_{ji} \vee P_{ki}$  are the projection operators onto the relative orthocomplements of these rays in the corresponding subspaces.

To see this, consider the special case where  $\mathbf{B}_{B,R}$  is generated by the atoms  $Q_{ij}$ , where  $Q_{ij}$  is the projection operator onto the ray  $\beta_i \otimes \rho_j$ ,  $i = 1, 2, 3$ ;  $j = 1, 2, 3$ , with  $\beta_i$  defined by the transformation in  $\mathcal{H}_S$ :

$$\beta_1 = \alpha_1 \quad \beta_2 = 1/\sqrt{2}(\alpha_2 + \alpha_3) \quad \beta_3 = 1/\sqrt{2}(\alpha_2 - \alpha_3)$$

Can we generate the required nine atoms  $\beta_i \otimes \rho_j$ ,  $i = 1, 2, 3$ ;  $j = 1, 2, 3$ , from some (maximal) Boolean subalgebra in the fan  $\{\mathbf{B}_\sigma\}$  via  $\mathbf{B}_R$ ? Call this hypothetical algebra  $\mathbf{B}_{\sigma'}$ . It must be related to  $\mathbf{B}_{\sigma^*}$  (any one of the algebras labelled  $\mathbf{B}_{\sigma^*}$  above) by unitarily transforming ('rotating') some or all of the vectors  $\alpha_i \otimes \rho_j$ ,  $i \neq j$ ,  $\phi, \chi$ , about  $\sigma$ , i.e., the atoms of  $\mathbf{B}_{\sigma'}$  correspond to vectors that are linear superpositions of  $\alpha_i \otimes \rho_j$ ,  $i \neq j$ ,  $\phi, \chi$ .

The nine atoms of  $\mathbf{B}_{B,R}$  we want correspond to the vectors:

$$\begin{aligned} \beta_1 \otimes \rho_1 &= \alpha_1 \otimes \rho_1 & \beta_1 \otimes \rho_2 &= \alpha_1 \otimes \rho_2 & \beta_1 \otimes \rho_3 &= \alpha_1 \otimes \rho_3 \\ \beta_2 \otimes \rho_1 &= 1/\sqrt{2}(\alpha_2 + \alpha_3) \otimes \rho_1 & \beta_2 \otimes \rho_2 &= 1/\sqrt{2}(\alpha_2 + \alpha_3) \otimes \rho_2 & \beta_2 \otimes \rho_3 &= 1/\sqrt{2}(\alpha_2 + \alpha_3) \otimes \rho_3 \\ \beta_3 \otimes \rho_1 &= 1/\sqrt{2}(\alpha_2 - \alpha_3) \otimes \rho_1 & \beta_3 \otimes \rho_2 &= 1/\sqrt{2}(\alpha_2 - \alpha_3) \otimes \rho_2 & \beta_3 \otimes \rho_3 &= 1/\sqrt{2}(\alpha_2 - \alpha_3) \otimes \rho_3 \end{aligned}$$

The only atoms in this list that could belong to an algebra  $\mathbf{B}_{\sigma'}$  derivable in this way from some  $\mathbf{B}_{\sigma^*}$  are the four atoms corresponding to the vectors  $\beta_1 \otimes \rho_2$ ,  $\beta_1 \otimes \rho_3$  (these belong to  $\mathbf{B}_{\sigma^*}$ ) and  $\beta_2 \otimes \rho_1$ ,  $\beta_3 \otimes \rho_1$  (these can be obtained from  $\mathbf{B}_{\sigma^*}$  by rotating  $\alpha_2 \otimes \rho_1$  and  $\alpha_3 \otimes \rho_1$  through  $45^\circ$  in the plane spanned by  $\alpha_2 \otimes \rho_1$  and  $\alpha_3 \otimes \rho_1$ ). These atoms are all compatible with the atoms  $I \otimes P_i$  of  $\mathbf{B}_R$ .

The remaining five atoms correspond to vectors that all involve a term of the form  $\alpha_i \otimes \rho_i$  and so cannot be derived by a unitary transformation of the eight vectors  $\alpha_i \otimes \rho_j$ ,  $i \neq j$ ,  $\phi, \chi$  in the plane orthogonal to  $\sigma$ , since  $\alpha_i \otimes \rho_i$  does not lie in this plane (assuming none of the coefficients of  $\sigma$  are zero). These atoms would have to be generated from nonatomic elements of  $\mathbf{B}_{\sigma'}$  corresponding to multidimensional subspaces that are orthogonal to the span of  $\beta_1 \otimes \rho_2$ ,  $\beta_1 \otimes \rho_3$ ,  $\beta_2 \otimes \rho_1$ ,  $\beta_3 \otimes \rho_1$ , and also compatible with the atoms  $I \otimes P_i$  of  $\mathbf{B}_R$ , by taking the Boolean meets of these elements with  $I \otimes P_i$ ,  $i = 1, 2, 3$  (i.e., by intersecting the corresponding subspaces). But the only element of  $\mathbf{B}_{\sigma'}$  satisfying this condition is the element corresponding to the 5-dimensional subspace orthogonal to the span of  $\beta_1 \otimes \rho_2$ ,  $\beta_1 \otimes \rho_3$ ,  $\beta_2 \otimes \rho_1$ ,  $\beta_3 \otimes \rho_1$ . Since  $\beta_1 = \alpha_1$  and the span of  $\beta_2 \otimes \rho_1$ ,  $\beta_3 \otimes \rho_1$  is the span of  $\alpha_2 \otimes \rho_1$ ,  $\alpha_3 \otimes \rho_1$ , this is the subspace spanned by  $\alpha_1 \otimes \rho_2$ ,  $\alpha_1 \otimes \rho_3$ ,  $\alpha_2 \otimes \rho_1$ ,  $\alpha_3 \otimes \rho_1$ . So the orthogonal subspace is spanned by the vectors  $\alpha_2 \otimes \rho_3$ ,  $\alpha_3 \otimes \rho_2$ ,  $\sigma$ ,  $\phi$ ,  $\chi$ , equivalently by  $\alpha_2 \otimes \rho_3$ ,  $\alpha_3 \otimes \rho_2$ ,  $\alpha_1 \otimes \rho_1$ ,  $\alpha_2 \otimes \rho_2$ ,  $\alpha_3 \otimes \rho_3$ . Intersecting this subspace with the subspaces corresponding to  $I \otimes P_i$ ,  $i = 1, 2, 3$ , yields the ray  $\alpha_1 \otimes \rho_1$  ( $= \beta_1 \otimes \rho_1$ ) and the

elements corresponding to the two planes spanned by  $\alpha_2 \otimes \rho_2, \alpha_3 \otimes \rho_2$  (equivalently  $\beta_2 \otimes \rho_2, \beta_3 \otimes \rho_2$ ) and  $\alpha_2 \otimes \rho_3, \alpha_3 \otimes \rho_3$  (equivalently  $\beta_2 \otimes \rho_3, \beta_3 \otimes \rho_3$ ). So we generate a nonmaximal, 7-atom Boolean subalgebra of  $\mathfrak{B}_{B,R}$ , with 5 atoms corresponding to the rays  $\beta_1 \otimes \rho_1, \beta_1 \otimes \rho_2, \beta_1 \otimes \rho_3, \beta_2 \otimes \rho_1, \beta_3 \otimes \rho_1$  and 2 atoms corresponding to the planes spanned by  $\beta_2 \otimes \rho_2, \beta_3 \otimes \rho_2$  and  $\beta_2 \otimes \rho_3, \beta_3 \otimes \rho_3$ .

If we keep no vector fixed in the transformation from  $\{\alpha_i\}$  to  $\{\beta_j\}$ , then none of the atoms in  $\mathfrak{B}_{B,R}$  could belong to an algebra  $\mathfrak{B}_{\sigma'}$  derivable from  $\mathfrak{B}_{\sigma^*}$  by unitarily transforming the vectors orthogonal to  $\sigma$ , and no nontrivial subalgebra of such a maximal Boolean subalgebra is in  $\{\mathfrak{B}_{\sigma}\}/\mathfrak{B}_R$ .

In the  $n \times n$ -dimensional case, we can keep more than one vector fixed in the transformation (i.e., either one vector fixed, or two fixed, etc.). Consider a unitary transformation from  $\{\alpha_i\}$  to  $\{\beta_j\}$  where (i)  $\alpha_1 = \beta_1$ , or (ii)  $\alpha_1 = \beta_1, \alpha_2 = \beta_2$ , etc. In case (i), the  $n-1$  vectors  $\beta_1 \otimes \rho_j$  ( $j = 1, \dots, n$ ) and the  $n-1$  vectors  $\beta_i \otimes \rho_1$  ( $i = 1, \dots, n$ ) can all be derived from  $\mathfrak{B}_{\sigma^*}$  by a unitary transformation in the plane orthogonal to  $\sigma$ . This yields  $2(n-1)$  vectors. The remaining atoms of  $\mathfrak{B}_{B,R}$  all correspond to vectors that are expressible as linear superpositions that include a term of the form  $\alpha_i \otimes \rho_i$ , which does not belong to the subspace orthogonal to  $\sigma$  and so cannot be derived in this way. In case (ii), the same  $2(n-1)$  vectors can be derived by unitary transformation in the plane orthogonal to  $\sigma$ , but now the  $n-2$  vectors  $\beta_2 \otimes \rho_j$  ( $j = 3, \dots, n$ ) and the  $n-2$  vectors  $\beta_i \otimes \rho_2$  ( $i = 3, \dots, n$ ) can also be derived in this way, yielding  $2(n-1) + 2(n-2)$  vectors. And so on. In each case, the remaining atoms of  $\mathfrak{B}_{B,R}$  all correspond to vectors that are expressible as linear superpositions that include a term of the form  $\alpha_i \otimes \rho_i$  and so cannot be derived in this way. We can generate the atoms corresponding to (i)  $\beta_1 \otimes \rho_1$ , or (ii)  $\beta_1 \otimes \rho_1, \beta_2 \otimes \rho_2$ , or ... by intersecting the subspace orthogonal to the span of the (i)  $2(n-1)$ , or (ii)  $2(n-1) + 2(n-2)$ , or ... vectors with the subspaces corresponding to the atoms (i)  $I \otimes P_1$ , or (ii)  $I \otimes P_1, I \otimes P_2$ , or ... of  $\mathfrak{B}_R$ . But intersecting this subspace with the subspaces corresponding to the remaining atoms of  $\mathfrak{B}_R$  in each case yields elements corresponding to multidimensional subspaces, i.e., to the Boolean join of atoms in  $\mathfrak{B}_{B,R}$ . So, while  $\{\mathfrak{B}_{\sigma}\}/\mathfrak{B}_R$  contains the maximal Boolean subalgebra  $\mathfrak{B}_{A,R}$ , it does not contain any other maximal Boolean subalgebra  $\mathfrak{B}_{B,R}$ , where  $B$  is incompatible with  $A$ .

There are, clearly, other maximal Boolean subalgebras in  $\{\mathfrak{B}_{\sigma}\}/\mathfrak{B}_R$ . The nonmaximal Boolean subalgebras considered above were all derived from maximal Boolean subalgebras  $\mathfrak{B}_{\sigma'}$  by  $/\mathfrak{B}_R$ , where  $\mathfrak{B}_{\sigma'}$  differs from  $\mathfrak{B}_{\sigma^*}$  by a unitary transformation in the plane orthogonal to  $\sigma$  in which (for the case of a 3-dimensional Hilbert space) (a) the vectors  $\alpha_1 \otimes \rho_2$  and  $\alpha_1 \otimes \rho_3$  are unaltered by the transformation, (b) the vectors  $\alpha_2 \otimes \rho_1$  and  $\alpha_3 \otimes \rho_1$  are rotated in their plane, and (c) the remaining vectors  $\alpha_2 \otimes \rho_3, \alpha_3 \otimes \rho_2, \phi, \chi$  are transformed in

some way (or some permutation of the transformation described in (a), (b), (c), corresponding to which vector is kept constant in the transformation from  $\{\alpha_i\}$  to  $\{\beta_j\}$ ).

Consider, now, the maximal Boolean subalgebra  $\mathbf{B}_\sigma''$  in the fan  $\{\mathbf{B}_\sigma\}$  that differs from (any one of the)  $\mathbf{B}_\sigma^*$  by the transformation in  $\mathcal{H}_S \otimes \mathcal{H}_M$ :

$$\beta_2 \otimes \rho_1 = 1/\sqrt{2}(\alpha_2 \otimes \rho_1 + \alpha_3 \otimes \rho_1) \quad \beta_3 \otimes \rho_1 = 1/\sqrt{2}(\alpha_2 \otimes \rho_1 - \alpha_3 \otimes \rho_1)$$

In other words,  $\mathbf{B}_\sigma''$  is generated by the atoms  $\alpha_1 \otimes \rho_2, \alpha_1 \otimes \rho_3, \beta_2 \otimes \rho_1, \alpha_2 \otimes \rho_3, \beta_3 \otimes \rho_1, \alpha_3 \otimes \rho_2, \sigma, \phi, \chi$ . The atoms corresponding to the six vectors  $\alpha_1 \otimes \rho_2, \alpha_1 \otimes \rho_3, \beta_2 \otimes \rho_1, \alpha_2 \otimes \rho_3, \beta_3 \otimes \rho_1, \alpha_3 \otimes \rho_2$  are all compatible with  $I \otimes P_i, i = 1, 2, 3$ . As before,  $P_\sigma + P_\phi + P_\chi = P_{11} + P_{22} + P_{33} = P_{\chi^\perp}$  is compatible with  $I \otimes P_i, i = 1, 2, 3$ , hence with  $\mathbf{B}_R$ . Taking the meet with  $I \otimes P_i, i = 1, 2, 3$ , yields the three atoms  $P_{11}, P_{22}, P_{33}$  associated with the vectors  $\alpha_1 \otimes \rho_1, \alpha_2 \otimes \rho_2, \alpha_3 \otimes \rho_3$ . So  $\mathbf{B}_\sigma''/\mathbf{B}_R$  is the maximal Boolean subalgebra  $\mathbf{B}^\#$  generated by the atoms corresponding to the vectors  $\alpha_1 \otimes \rho_1, \alpha_1 \otimes \rho_2, \alpha_1 \otimes \rho_3, \beta_2 \otimes \rho_1, \alpha_2 \otimes \rho_2, \alpha_2 \otimes \rho_3, \beta_3 \otimes \rho_1, \alpha_3 \otimes \rho_2, \alpha_3 \otimes \rho_3$ .

Algebras like  $\mathbf{B}^\#$  contain the subalgebra of indicator properties of M by construction, but do not contain any nontrivial subalgebra of properties of S. Evidently, the nonmaximal Boolean subalgebras in  $\{\mathbf{B}_\sigma\}/\mathbf{B}_R$  are all subalgebras of  $\mathbf{B}_{A,R}$ , or subalgebras of nonseparable algebras like  $\mathbf{B}^\#$ . For example, the nonmaximal Boolean subalgebra with five atoms corresponding to the rays  $\beta_1 \otimes \rho_1, \beta_1 \otimes \rho_2, \beta_1 \otimes \rho_3, \beta_2 \otimes \rho_1, \beta_3 \otimes \rho_1$ , and two atoms corresponding to the planes spanned by  $\beta_2 \otimes \rho_2, \beta_3 \otimes \rho_2$ , and  $\beta_2 \otimes \rho_3, \beta_3 \otimes \rho_3$ , where  $\beta_1 = \alpha_1, \beta_2 = 1/\sqrt{2}(\alpha_2 + \alpha_3), \beta_3 = 1/\sqrt{2}(\alpha_2 - \alpha_3)$ , is a subalgebra of  $\mathbf{B}^\#$ , because the plane spanned by  $\beta_2 \otimes \rho_2, \beta_3 \otimes \rho_2$  is spanned by  $\alpha_2 \otimes \rho_2, \alpha_3 \otimes \rho_2$  and the plane spanned by  $\beta_2 \otimes \rho_3, \beta_3 \otimes \rho_3$  is spanned by  $\alpha_2 \otimes \rho_3, \alpha_3 \otimes \rho_3$ . It should now be clear that the different maximal Boolean subalgebras in  $\{\mathbf{B}_\sigma\}/\mathbf{B}_R$  all contain the three atoms  $P_{ii}, i = 1, 2, 3$ , corresponding to the projections of the state  $\sigma$  onto the three 3-dimensional subspaces defined by the projection operators  $I \otimes P_i$ , with the remaining six atoms corresponding to projection operators onto (mutually orthogonal) 1-dimensional subspaces that span the planes orthogonal to the  $P_{ii}$  in these 3-dimensional subspaces.

Similar considerations apply to nonmaximal measurements. To illustrate, it suffices to consider a 3-dimensional Hilbert space for S and a 2-dimensional Hilbert space for M. The measurement interaction correlates, say, the first and second eigenvalues of the measured quantity A of S with the first indicator value (+), and the third eigenvalue of A with the second indicator value (-), resulting in the final state  $\sigma = (c_1 \alpha_1 + c_2 \alpha_2) \otimes \rho_+ + c_3 \alpha_3 \otimes \rho_-$ . The maximal Boolean subalgebras in  $\{\mathbf{B}_\sigma\}/\mathbf{B}_R$  are 6-atom Boolean subalgebras that all coincide on the 4-atom Boolean subalgebra generated by the two atoms corresponding to the two rays  $\alpha \otimes \rho_+, \alpha_3 \otimes \rho_-$ , where  $\alpha$  is the normalized projection of  $\psi$  onto the plane spanned by  $\alpha_1$  and  $\alpha_2$ , and the two planes spanned by  $\alpha_3 \otimes \rho_+, \alpha' \otimes \rho_+$  and by  $\alpha_1 \otimes \rho_-, \alpha_2 \otimes \rho_-$  (equivalently, by  $\alpha \otimes \rho_-$ ,



$\alpha' \otimes \rho$ ), where  $\alpha'$  is orthogonal to  $\alpha$  and  $\alpha_3$  in  $\mathcal{H}_S$ . The different 6-atom algebras correspond to different decompositions of the two planes as the span of orthogonal rays.

Quantum mechanics is conceptually puzzling because the theory introduces statistical states without explicitly relating these states to property states. The core difficulty in the way of defining property states for a quantum mechanical system is the objectivity problem. An interpretation of quantum mechanics should say something about property states, either by introducing new structures (as in a hidden variable theory), or by proposing some way in which quantum mechanics can be understood as sustaining a conception of property states that resolves the objectivity problem.

The proposal sketched above is presented as a minimal interpretation that suggests a reformulation and solution of the measurement problem. It is traditional to see the measurement problem as that of explaining how Schrodinger's cat can avoid the embarrassment of having its properties of being alive and being dead become nonobjective after an interaction that correlates these properties with certain properties of a microsystem. The problem arises because the assumption that the cat is determinately alive or determinately dead (i.e., that these properties are objective) is inconsistent with the assumption that the only objective properties of the cat+microsystem are those assigned probability 1 or 0 by the state of the composite system (i.e., the properties in the simple fan defined by the state). But if we take the objective properties of the cat+microsystem as those in the generalized fan  $\{\mathcal{B}_\sigma\}/\mathcal{B}_{cat}$ , where  $\sigma$  is the final state of the cat+microsystem after the interaction, then we can consistently maintain that the state of the composite system is a linear superposition of tensor product states over alive and dead states of the the cat after the interaction, and that the cat is determinately alive or determinately dead.

What has been shown is that quantum mechanics can sustain a consistent notion of objectivity. If we assume that some designated subsystems in a quantum mechanical universe, including cats and systems that can function as measuring instruments, are characterized by nontrivial Boolean subalgebras of properties that are always objective (i.e., compatible with all other subalgebras in  $\mathcal{L}$ ), then we see how properties (of other subsystems) that are nonobjective at a particular time can become objective in interactions that correlate these properties with the objective properties of the designated subsystems. There remains something like a problem of boundary conditions or initial conditions: providing an account of those physical systems that are characterized by nontrivial Boolean algebras of properties that are always objective. But this latter problem, surely, is not the measurement problem.

# The Quantum Theory of Unsharp Measurements

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Quantum mechanics allows for an observable concept more general than selfadjoint operators: both theoretical inconsistencies as well as new experimental developments require an analysis in terms of general POV measures rather than only spectral (PV) measures. Some of the ensuing achievements in the quantum theory of measurement of such *unsharp observables* are reviewed with particular emphasis put on new experimental possibilities.

## 1. Introduction

Ordinary quantum mechanics is incomplete in that it is based on too narrow a concept of observable. Due to this limitation the theory had been facing conceptual and interpretational problems. For the same reason, appropriate tools were lacking for an adequate description of various physical phenomena. About twenty years ago several lines of research have independently induced the consideration of the most general notion of observables which is compatible with the probabilistic structure of quantum mechanics. Observables are to be described as *effect valued*, or *positive operator valued* (POV) measures, representing generally *unsharp* observables and containing ordinary (*sharp* observables as special cases (spectral measures). Some of the arguments will be reviewed which show the physically compelling nature of this generalization. A survey of applications will be given which have been worked out in various fields of quantum physics, like measurement theory, stochastic processes, quantum optics, signal detection, stochastic quantum mechanics, and others. In particular, the existence of *phase space representations* of quantum mechanics, of *statistically complete observables* and of *joint measurements* of noncommuting quantities, as well as further unexpected results, provide a clear illustration of the idea of unsharpness inherent in the new conception of quantum observables.

## 2. Unsharp Observables in Quantum Physics

The most remarkable point about unsharp observables in quantum mechanics is that they have not been introduced in order to account for measurement inaccuracies within the theory: on the contrary, the term unsharpness is intended to reflect an ingenious exploitation of genuine quantum indeterminacies, which has led to unexpected resolutions of longstanding theoretical problems as well as new applications of quantum mechanics.

### 2.1 Definition of subject.

The quantum theory of unsharp measurements shall be understood just as Hilbert space quantum mechanics based on general observables, including measurement theory. Observables are to be represented generally as positive operator valued (POV), or effect valued measures. The formal difference between projection valued (PV), or spectral measures and others corresponds to the distinction between "sharp" (ordinary) and "unsharp" observables.

### 2.2 What is an observable.

The formal representation of an observable should account for the statistics of a given experiment. In other words, it should allow one to predict the probability distributions of measurement outcomes for any given state. Together with the linear structure of quantum mechanics, this operational requirement fixes the notion of observable in the sense of a POV measure [Lud83, Bus87, BLM91]:

$$(1) \quad E : \Sigma \rightarrow \mathcal{E}(\mathcal{H}), \quad X \mapsto E(X),$$

with  $(\Omega, \Sigma)$  denoting a measurable space, the value space of the measurement under consideration,  $\mathcal{E}(\mathcal{H}) = \{a \in \mathcal{L}(\mathcal{H}) : 0 \leq a \leq I\}$  the set of effects, and  $E$  satisfying the usual measure postulates (positivity, normalization,  $\sigma$ -additivity). The probability for an outcome in  $X \in \Sigma$  in state  $\rho$  is given by the familiar trace formula:

$$(2) \quad E_\rho(X) = \text{Tr}[\rho \cdot E(X)].$$

As emphasized above, unsharpness of observables does not merely account for a kind of classical measurement inaccuracy; rather it is intended to cover also quantum mechanical indeterminacies of measurement outcomes which arise from the quantum features of measuring devices. Thus, there are typically two cases: First, an unsharp observable may be a kind of approximation to some sharp observable; in that case the interpretation of unsharpness as inaccuracy or indeterminacy depends on the construction of the apparatus. Secondly, there are unsharp observables with no sharp counterpart. It is this class of general observables which opens up new theoretical and experimental possibilities. Among them one finds joint observables for noncommuting sets of observables, such as the phase space observable reviewed below.

### 2.3 Measurement theory.

Both, the theoretical analysis of experiments as well as the operational characterization of theoretical concepts, require the machinery of quantum measurement theory [BLM91]. In the quantum theory of measurement the measuring device, or part of it, is treated as a quantum mechanical system. A measurement consists of an interaction of finite duration between the object system  $\mathcal{S}$  (Hilbert space  $\mathcal{H}_{\mathcal{S}}$ ) and the apparatus  $\mathcal{A}$  (Hilbert space  $\mathcal{H}_{\mathcal{A}}$ ), followed by a registration of the pointer observable  $P_{\mathcal{A}}$  of  $\mathcal{A}$ . The measurement coupling  $V$  [a linear state transformation on  $\mathcal{T}(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{A}})$ ], the state space of the compound system  $\mathcal{S} + \mathcal{A}$ ] is supposed to establish correlations between the observable  $E$  to be measured and the pointer observable such that the *probability reproducibility condition* [BLM91] holds for  $\rho, \rho_{\mathcal{A}}$  being the initial states of  $\mathcal{S}$  and  $\mathcal{A}$ ,  $X \in \Sigma$ :

$$(3) \quad E_\rho(X) = \text{Tr}[\rho E(X)] = \text{Tr}[V(\rho \otimes \rho_{\mathcal{A}}) \cdot I \otimes P_{\mathcal{A}}(X)].$$

Here it is assumed that the value spaces of  $E$  and  $P_{\mathcal{A}}$  coincide. An  $E$ -measurement may be summarized as the quadruple  $\mathcal{M} = (\mathcal{H}_{\mathcal{A}}, P_{\mathcal{A}}, \rho_{\mathcal{A}}, V)$ . Given such an  $\mathcal{M}$ , the measured observable  $E$  is uniquely determined by (1). The physical problem in devising a measurement of a given observable  $E$  consists of constructing a suitable apparatus  $(\mathcal{H}_{\mathcal{A}}, P_{\mathcal{A}}, \rho_{\mathcal{A}})$  and finding a suitable interaction  $(V)$  such that (3) is satisfied.

Besides the probabilistic aspects of a measurement, which are related to the determination of the state *prior to measurement*, it will be of interest to find out, or control, the influence of the measurement on the system, that is, the state change due to measurement. Again, it turns out that the final object state is fixed by  $\mathcal{M}$ . In fact, for any  $X \in \Sigma$  there exists a unique state transformation  $\mathcal{I}(X)$  satisfying the following for all states  $\rho$ , all effects  $a \in \mathcal{E}(\mathcal{H}_S)$ :

$$(4) \quad \text{Tr}[a \cdot \mathcal{I}(X)(\rho)] = \text{Tr}[V(\rho \otimes \rho_{\mathcal{A}}) \cdot a \otimes P_{\mathcal{A}}(X)].$$

This formula follows from the fact that after normalization by the factor  $\text{Tr}[\mathcal{I}(X)(\rho)]$ , it gives rise to the appropriate post-measurement conditional probabilities for the object system. The mapping  $X \mapsto \mathcal{I}(X)$  constitutes a state transformation (or operation) valued measure, in short: the *instrument* induced by the measurement  $\mathcal{M}$ . The measured observable  $E$  is associated to the instrument  $\mathcal{I}$  in a unique way via the condition

$$(5) \quad \text{Tr}[\rho \cdot E(X)] = \text{Tr}[\mathcal{I}(X)(\rho)],$$

which is stipulated to hold for arbitrary states  $\rho$ , all sets  $X$ .

#### 2.4 Three levels of measurement theory.

The quantum theory of unsharp measurement can and has been applied on three levels of abstraction. On *level one*, the abstract scheme of measurement theory is employed to yield an operational characterization of the basic concepts of quantum mechanics [Kra83]. In particular, two ways of interpreting unsharp observables have been spelled out in a systematic and rigorous manner: Either one may refer to unsharp measurement values such that the corresponding unsharp observable constitutes a coarse grained version of some sharp observable. This type of quantum unsharpness must necessarily be built into position and momentum in order to achieve their joint measurability (Section 3). Thus, for example, unsharp position  $E_f^Q$  is obtained by convolution of sharp position  $E^Q$  with a confidence function  $f$ .

$$(6) \quad \begin{aligned} E_f^Q(X) &= \chi_X * f(Q) = \int E^Q(X+x)f(x) dx, \\ Q &= \int x E^Q(dx) = \int x E_f^Q(dx). \end{aligned}$$

The underlying idea of coarse graining of observables has been analyzed, for instance, in [Dav76, Lud83, MM90a, Pru86, Schr92]. The second interpretation of unsharp observables leads to a relaxation of the well-known measurement theoretical notions of ideality, predictability, and repeatability [Dav76]. In this way one obtains a generalization of the famous Einstein-Podolsky-Rosen criterion of reality [EPR35] into a criterion of unsharp reality which allows for a realistic individual interpretation of quantum mechanics pertaining to general observables [Bus85a, BL89].

The "classical" topics of the quantum theory of measurement are the fundamental problems arising with a quantum mechanical formulation of the measurement process. This part of measurement theory typically entails both level-one as well as level-two questions. On the abstract level one has found deep incompatibilities between the dynamical description [BCL90] and the *objectification* of measurement outcomes, which can be formalized in terms of rigorous no-go theorems; for a systematic review, cf. [BCL91]. It seems that the only way out of these difficulties *within* quantum mechanics is an approximate one which makes use of features of *concrete* quantum mechanics. The basic task consists of understanding the quasi-classical level of quantum mechanics, which again has been seen to require general — namely macroscopically unsharp — observables [Omn90, BLM91].

Further topics of *level-two* measurement theory are concrete physical problems and model considerations. Investigations are concerned with limitations and conceptual problems of conventional quantum mechanics [BCL90, Oza91, BLM91], but also with analysis of new theoretical and experimental possibilities. In both areas considerable progress has been made on the basis of general observables [BGL89]. An interesting recent example is the operational definition of phase observables [Gra89.91a]. It is well known that in ordinary quantum mechanics there exists no selfadjoint operator corresponding to a phase quantity conjugate to some number, or spin observable. Remarkably a phase can be defined in the sense of a POV measure, which therefore must constitute an intrinsically unsharp observable. In the same way the known obstacles against defining time observables fall away if POV measures are taken into account [Hol82].

Finally, *level-three* measurement theory is devoted to establishing the connection with real experimentation. Concerning the analysis of actual experiments, it is important to note that a complete definition of a measurement is to be based on equations (3)-(5); in fact, these conditions provide a complete specification of the observable measured as well as the ensuing state changes. In the quantum optics literature one usually finds a weaker condition, namely, equality of first moments of  $E$  and  $P_A$  instead of the full probability reproducibility (3). However, this procedure does not specify a unique POV measure but rather a whole class of them. In order to draw as much information as possible from the statistics of a given experiment, one must pursue the theoretical analysis far enough so as to determine the full observable measured. This programme has been applied, for instance, to optical homodyne and heterodyne detection [Bar91, Gra91b,c, MM91], or new polarization and interferometry experiments [Bus87, BS89, MM90b]. Specification of the scheme given by the quadruple  $\mathcal{M}$  not only allows one to determine the observable actually measured in a certain experimental setup, but also shows how to modify the setup in order to approach the intended observable. In this way one may determine precisely the kind of information which a real Stern-Gerlach device yields with respect to some spin component: the actual magnetic field interaction allowed by Maxwell's equations and the irreducible spreading of wave packets allow only for the definition of some unsharp spin observable [BS89, SM91].

In the sequel (Section 3) some samples of level-two and level-three activities and achievements are briefly revisited.

### 3. Some Applications of General (Unsharp) Observables

#### 3.1 Recent developments in abstract theory.

General observables have been introduced and applied independently in various branches of quantum physics. A rather comprehensive survey of the literature up to 1988 can be found in [BGL89]. Therefore, apart from citing the relevant monographs, main attention is focussed on the more recent publications in the following account. One may distinguish at least three broad groups of activities.

*Operational quantum mechanics* comprises various approaches towards an axiomatic reconstruction of the theory, such as quantum logic [Mit78, Pir76], manual approach [FS90] or operational and statistical foundations [Dav76, Hol82, Lud83]. In the past decade some progress has been made in clarifying the connections between these approaches, which started out from fairly distinct points of view. For instance, the preconditions of quantum logic can be formalized within the operational approach [FPR83, LB85]. The manual approach has been applied to yield a motivation of general observables in the spirit stochastic quantum mechanics [FS90]. As a last example, the quantum logical question-proposition system has been given a new realization in terms of the set of quantum mechanical effects [CN90].

*Stochastic quantum mechanics* originated from an attempt to formulate quantum mechanics on phase space, which amounts to introducing a quantization necessarily involving the idea of fuzzy, or stochastic localization, hence general observables [Ali85, Pru86, Schr92]. The ensuing technique of defining concrete observables as systems of covariance (generalized systems of imprimitivity) of some kinematical group applies not only for the Galilei group, but also in the case of the Poincaré group [Bro91]. In recent years it was found that also other quantization procedures can be formulated and generalized in a natural way on the basis of general POV measures [Ali91, AD87,90, AE86]. Finally, new efforts have been made to reconcile quantum physics and gravity on the basis of the conception of stochastic localization in quantum space time [Pru90].

*Quantum statistics* summarizes efforts arising from practical needs of generalizing classical probability theory to quantum mechanics. The resulting methods — quantum stochastic processes, theory of repeated and continuous measurements, quantum stochastic calculus — have been applied, among others, to photon statistics, quantum detection [Bar91, Gra91b, Hol91], signal processing [Schr91a,b] and relativistic quantum mechanics [Den91]. Among the classical monographs devoted to these subjects are [Dav76], [Hel76] and [Hol82].

After this brief literature survey a few issues shall be reviewed in some detail, illustrating some of the basic new features of the quantum theory of unsharp measurements.

#### 3.2 Coexistence, informational completeness.

An observable  $E$  may be called coarser than another one  $F$  if the range of  $E$  is contained in the range of  $F$ .  $F$  can be called a refinement of  $E$ . If a collection of observables  $E_i$  possesses a common refinement  $F$  then these observables are called *coexistent*.  $F$  constituting a *joint observable* for them. In fact,  $F$  yields a joint probability distribution for the  $E_i$  in any state, so that a measurement of  $F$  must be regarded as a joint measurement of the  $E_i$ . If the  $E_i$  are ordinary observables (spectral measures), then their coexistence is equivalent to commutativity [Kra83]. Remarkably, there exist

coexistent noncommuting families of POV measures. In general, for a pair of noncommuting ordinary observables there exist unsharp versions which are coexistent. This will be illustrated for the position-momentum case below.

Joint observables yield more probabilistic information than their marginals alone would. The extreme case occurs if an observable  $F$  allows a complete specification of the state prior to measurement. That is, states are separated by their probability distributions. Such observables are called *informationally*, or *statistically complete*. They play an important role in introducing classical representations of quantum mechanics.

It is well known that no ordinary observable is statistically complete. Thus, statistically complete observables are necessarily unsharp observables. Moreover, they do not admit repeatable (that is, preparatory) measurements [BL89]. These facts can be particularly clearly illustrated in the case of spin- $\frac{1}{2}$  observables. First of all, the statistics of a complete measurement in two-dimensional Hilbert space must yield three independent numbers in order to determine the state. Hence such an experiment must have at least four outcomes (taking into account one normalization condition). A four-valued observable in a two-dimensional Hilbert space cannot be a spectral measure since there are no four orthogonal projections available.

Far from being merely theoretical ideas, coexistence and informational completeness are well accessible possibilities in present day technology. Joint measurement schemes providing complete statistics have been proposed for polarization observables and other complementary pairs [Bus87, BS89]. Simultaneous measurements of path and interference observables have been performed in a photon split-beam device [MPSS7, Bus87] and similarly in neutron interferometry [MM90, RSZ90]. Using the tools of quantum measurement theory, the determinative and preparatory abilities of *quantum non-demolition* measurements of quantum optical phase space observables have been shown to be optimally balanced [BL90]. There are a lot of quantum optical experimental setups awaiting a closer measurement theoretical analysis, some being done with very satisfactory results (as cited earlier).

### 3.3 Classical representations of quantum mechanics.

The striking differences between quantum and classical mechanics have induced several interesting attempts to reformulate quantum mechanics in a classical language. Among these approaches are the famous Wigner distribution [Wig32], the path-integral method, various hidden-variable theories, or stochastic quantization (references can be found in [BLM91]). Formally, also the starting point of *stochastic quantum mechanics* is the same as that of Wigner's approach: the linear embedding of quantum states into a space of phase space distribution functions. This procedure can be systematically generalized in the following way (see also [Bug91] and [Stu91]).

Any observable  $E : \Sigma \rightarrow \mathcal{E}(\mathcal{H})$  on  $(\Omega, \Sigma)$  induces a mapping from the quantum states to classical probability measures on  $\Sigma$ ,

$$(7) \quad \mathcal{V}_E : \rho \mapsto E_\rho, \quad E_\rho(X) = \text{Tr}[\rho \cdot E(X)],$$

which extends uniquely to a linear map from the trace class into the space of  $\sigma$ -additive set functions on the measurable space  $(\Omega, \Sigma)$ . The dual map of  $\mathcal{V}_E$  associates quantum

mechanical operators  $a_f$  to classical observables  $f : \Omega \rightarrow \mathcal{R}$ :

$$(8) \quad \text{Tr}[\rho \cdot a_f] = \int f(x) E_\rho(dx).$$

Hence,

$$(9) \quad a_f = \int f(x) E(dx).$$

(Conditions under which these integrals are well defined can be found in [Stu91] or [Schr92].) Thus, the map  $\mathcal{V}_E$  affords a linear embedding of quantum mechanics into a classical framework. Moreover, this embedding is injective — and hence a *classical representation* — exactly when the inducing observable  $E$  is statistically complete.

While equation (7) is commonly used in the valuation of experimental data already in ordinary quantum mechanics, its extension to general, and in particular to informationally complete observables, brings about a vast variety of new representations of quantum mechanics of the Wigner type. A significant advantage of this new general method lies in the fact that quantum states are represented by proper probability distributions; furthermore, the underlying measurement theory allows for a straightforward physical interpretation of these probability distributions.

### 3.4 Phase space quantum mechanics.

Localization in phase space is characterized by covariance with respect to space translations and (Galilei) boosts. Taking the Weyl operators on  $\mathcal{H} = \mathcal{L}^2(\mathcal{R})$ ,

$$(10) \quad W(q,p) = \exp\{i(qP + pQ)\},$$

covariance of a POV measure  $A : \mathcal{B}(\Gamma) \rightarrow \mathcal{E}(\mathcal{H})$  on phase space  $\Gamma = \mathcal{R}_{config} \times \mathcal{R}_{mom}$  reads:

$$(11) \quad W(-q,p)A(\Delta)W(-q,p)^+ = A(\Delta + (q,p)).$$

This can be easily realized by the following: let  $\sigma$  be a positive trace-one operator, then define

$$(12) \quad \Delta \mapsto A(\Delta) = \int_{\Delta} T^\sigma(q,p) d\mu(q,p),$$

where

$$(13) \quad T^\sigma(q,p) = W(-q,p)\sigma W(-q,p)^+.$$

and  $d\mu(q,p) = (2\pi\hbar)^{-1}dqdp$ . This phase space observable constitutes a joint observable for unsharp position  $Q_f$  and momentum  $P_g$  observables as defined in (6). A full measurement theoretical analysis has been given for this type of phase space observables, showing that the confidence distributions  $f, g$  obey the uncertainty relations in the sense of a limitation of the sharpness of phase space localization [Bus85b]. In spite of the irreducible unsharpness, it has been shown that phase space localization may still have quite a controllable degree of reproducibility [BL89], which turns out to be



decisive for the quasi-classical nature of quantum mechanical phenomena such as bubble chamber trajectories [Bus85, Omn90].

As is well known, the relation between the phase space representation

$$(14) \quad \rho \mapsto p_\rho^\sigma(q, p) := (2\pi\hbar)^{-1} \text{Tr}[\rho \cdot T^\sigma(q, p)]$$

induced by the phase space POV measure  $\mathcal{A}$  and the Wigner distribution

$$(15) \quad \rho \mapsto \mathcal{V}_{Wig}(\rho) := (\pi\hbar)^{-1} \text{Tr}[\rho \cdot \mathcal{P} \cdot W(2q, 2p)]$$

( $\mathcal{P}$  denoting space inversion) is given by convolution [Dav76]:

$$(16) \quad p_\rho^\sigma(q, p) = \int \mathcal{V}_{Wig}\rho(q', p') \mathcal{V}_{Wig}\sigma(q - q', p - p') dq' dp'.$$

Performing the Fourier transform, one obtains first

$$(17) \quad \int \epsilon^{i(qy - px)} \mathcal{V}_{Wig}\rho(q, p) dq dp = \text{Tr}[\rho \cdot W(x, y)],$$

and then

$$(18) \quad \int_{\Gamma} \epsilon^{i(qy + px)} p_\rho^\sigma(q, p) dq dp = \text{Tr}[\sigma \cdot W(x, y)^+] \cdot \text{Tr}[\rho \cdot W(x, y)].$$

The informational completeness of  $\mathcal{A}$  is equivalent to the fact that  $\text{Tr}[\sigma \cdot W(x, y)]$  is nonzero (almost everywhere) [Schr92]. Hence (16)-(18) show that in this case one can reconstruct either one of  $\rho$ ,  $p_\rho^\sigma$  and  $\mathcal{V}_{Wig}\rho$  from any one of the remaining two entities.

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## SURVEY OF QUANTIZATION METHODS

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**ABSTRACT.** A quick overview is presented of the different types of quantization methods currently used for making the transition from a classical to a quantum theory. Particular attention has been given to the theories of Borel, geometric and Berezin or prime quantization. The question of dequantization is also mentioned in connection with quantization via deformations.

### 1. Introduction

*Quantization* is the method by which one makes the transition:

$$\textit{Classical Mechanics} \longrightarrow \textit{Quantum Mechanics}$$

There is a general belief, mostly for historical reasons, that a classical theory contains enough information within its geometric and algebraic structure for possible quantum theories - of which the given classical theory is the limit - to be built. The opposite process

$$\textit{Quantum Mechanics} \xrightarrow[\hbar \rightarrow 0]{} \textit{Classical Mechanics}$$

by which, starting with the (in a sense more fundamental) quantum theory, one arrives at its classical limit, is called *dequantization*. To use a medieval metaphor, one believes that the ghost of quantum theory hovers over its classical remains. *Quantizing then amounts to incarnating this spirit while dequantizing involves exorcising the ghost!* The present survey will consider mainly the problem of quantization, looking at the more commonly used methods. A few remarks on the dequantization problem will be made at the end, in conjunction with quantization via deformations.

### 2. Canonical quantization

The originators of quantum theory used the following simple technique for quantizing a classical system: Let  $q^i, p_j$ ,  $i, j = 1, 2, \dots, n$  be the canonical position and momenta, respectively, of a classical system with  $n$  degrees of freedom. Then their quantized counterparts,  $Q^i, P_j$ , are realized as operators on the Hilbert space  $\mathfrak{H} = L^2(\mathbb{R}^n, d\mathbf{x})$ , by the prescription:

$$Q^i \psi(\mathbf{x}) = x^i \psi(\mathbf{x}) \quad P_j \psi(\mathbf{x}) = -i\hbar \frac{\partial}{\partial x^j} \psi(\mathbf{x}).$$

for a suitable set of vectors  $\psi$  in  $\mathfrak{H}$ . Von Neumann's *uniqueness theorem* [1] then states that, up to unitary equivalence, this is the only representation which yields the canonical commutation relations (CCR):

$$[Q^k, P_j] = i\hbar \delta_{kj}, \quad k, j = 1, 2, \dots, n. \quad (1)$$

It was also realized that this quantization process amounted to a replacement of the classical *Poisson bracket* by the quantum *commutator bracket*:

$$\{q^i, p_j\} \longrightarrow \frac{1}{i\hbar} [Q^i, P_j],$$

where for any two well behaved phase space functions  $f$  and  $g$ ,

$$\{f, g\} = \sum_{j=1}^n \left( \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q^j} - \frac{\partial f}{\partial q^j} \frac{\partial g}{\partial p_j} \right). \quad (2)$$

A number of questions immediately arise:

(i) Let  $M$  be the position space manifold of the classical system and  $q$  any point in it. Geometrically, the phase space of the system is the cotangent bundle  $\Gamma = T^*M$  of  $M$ . Its points will be denoted by  $(q^i, p_i)$  in some local coordinate chart. If  $M$  is linear,  $M \simeq \mathbb{R}^n$ , then the replacement  $q^i \rightarrow x_i$ ,  $p_j \rightarrow -i\hbar \frac{\partial}{\partial x_j}$  works fine. But what if  $M$  is not a linear space?

(ii) How do we quantize observables which involve powers of  $q^i, p_j$ , such as for example  $f(q, p) = (q^i)^n (p_j)^m$ ?

The general aim of a quantization programme is to attempt a systematic answer to these questions. There are usually two points of departure:

(i) Start with the system localized in the position space  $M$  and then proceed to quantize the theory.

(ii) Start with the system localized on the phase space  $\Gamma$ , or more generally, on an arbitrary symplectic manifold and then make the transition to a quantum theory.

In either case one exploits the *geometry* and/or the *Borel structure* of these underlying spaces. Additionally, a kinematical symmetry at the classical level is expected to manifest itself at the quantum level as well – a fact which is also to be exploited.

### 3. Quantization starting from position space

Originally proposed by Segal [2], this method is a generalization of canonical quantization and very much within the same spirit. A group theoretical method was suggested by Mackey [3], within the context of the theory of *induced representations* of finite dimensional groups. A much more general method, combining the Segal and Mackey approaches, was later developed by Doebner, Tolar, Pasemann, Angermann and Müller [4,5] under the name of *Borel quantization*, while a method using infinite dimensional diffeomorphism groups was suggested by Goldin [6].

The technique is as follows: The position space  $M$  is an  $n$ -dimensional  $C^\infty$ -manifold. The quantum observables of position are to arise from the smooth functions  $f: M \rightarrow \mathbb{R}$

while the observables of momentum have to be built out of the vector fields  $X$  of the manifold  $M$ . For  $C^\infty$ -functions  $a^i : M \rightarrow \mathbb{R}$ , one has (in local coordinates  $q^i$ )

$$X = \sum_{i=1}^n a^i(q) \frac{\partial}{\partial q^i}.$$

Quantization involves first choosing a Borel measure  $\mu$  on  $M$ , locally equivalent to the Lebesgue measure on  $\mathbb{R}^n$ , and the Hilbert space  $\mathfrak{H} = L^2(M, d\mu)$ . The quantum observables of position and momentum are then given by the mappings,  $f \mapsto Q(f)$  and  $X \mapsto P(X)$ , respectively, where for suitable vectors  $\psi \in \mathfrak{H}$ ,

$$\begin{aligned} (Q(f)\psi)(q) &= f(q)\psi(q), & P(X) &= -i\hbar(X + K_X) \\ (X\psi)(q) &= \sum_{i=1}^n a^i(q) \frac{\partial}{\partial q^i} \psi(q) & (K_X\psi)(q) &= \frac{1}{2}(Xw)(q)\psi(q) + \frac{1}{2}l(X)(q)\psi(q) \end{aligned}$$

where  $l(X) = \sum_{i=1}^n \frac{\partial a^i}{\partial q^i}$ , and locally  $d\mu(q) = w(q) dq^1 dq^2 \dots dq^n$ .

The additional term  $K_X$  is needed to ensure self-adjointness of the operator  $P(X)$ . In terms of the Lie bracket  $[X, Y] = X \circ Y - Y \circ X$  of the vector fields, one obtains for the quantized operators the following commutation relations, which clearly generalize (1):

$$[P(X), P(Y)] = -i\hbar P([X, Y]) \quad [P(X), Q(f)] = -i\hbar Q(Xf) \quad [Q(f), Q(g)] = 0. \quad (3)$$

Actually Segal suggested going over to the group of diffeomorphisms of  $\mathbb{R}^n$  and its unitary representations to attend to domain questions associated to  $Q(f), P(X)$  and then gave a classification of possible unitarily inequivalent quantizations in these terms.

But at this stage it is better to move to the more general Borel quantization technique. Let  $\mathcal{B}(M)$  be the Borel sets of  $M$  and  $\mathfrak{X}_c(M)$  the complete vector fields on  $M$ . If  $X \in \mathfrak{X}_c(M)$  then  $\exists$  a unique one-parameter group of diffeomorphisms of  $M$ , called a flow,  $\phi_t^X : M \rightarrow M$ ,  $t \in \mathbb{R}$  satisfying

$$\begin{aligned} \phi_{t_1}^X \circ \phi_{t_2}^X &= \phi_{t_1+t_2}^X \\ \frac{\partial}{\partial t} \phi_t^X(q) &= X(\phi_t^X(q)), & \phi_{t=0}^X(q) &= q, \quad x \in M. \end{aligned}$$

Let  $Diff(M)$  be the group of all  $C^\infty$  diffeomorphisms of  $M$  which become rapidly trivial at infinity. Each  $\phi_t^X$ ,  $t \in \mathbb{R}$  defines a one-parameter subgroup of  $Diff(M)$ . If  $\mathfrak{H}$  carries a unitary representation  $U$  of  $Diff(M)$ , then (by Stone's theorem) the unitary subgroup  $U(\phi_t^X)$ ,  $t \in \mathbb{R}$  has the generators

$$P(X) = \lim_{t \rightarrow 0} \frac{U(\phi_t^X) - I}{it}, \quad \text{with } U(\phi_t^X) = \exp\left\{-\frac{i}{\hbar} P(X)t\right\}.$$

and  $P(X)$  is a densely defined self adjoint operator.

A *classical Borel kinematics* is now defined to be a pair  $(\mathcal{B}(M), \mathfrak{X}_c(M))$  together with a flow model

$$\Delta \mapsto \phi_t^X(\Delta) = \{q \in M \mid \phi_{-t}^X(q) \in \Delta\}.$$

Suppose that on the Hilbert space  $\mathfrak{H}$ , there exists a projection-valued (PV) measure,  $\Delta \mapsto E(\Delta)$ ,  $\Delta \in \mathcal{B}(M)$ ,  $E(\Delta)^2 = E(\Delta) = E(\Delta)^*$  for which

$$U(\phi_t^X)E(\Delta)U(\phi_{-t}^X) = E(\phi_t^X(\Delta)). \quad (4)$$

This expression is a generalization of Mackey's imprimitivity condition for finite dimensional groups. A quantization of the classical Borel kinematics is then defined as the pair  $U, E$ , satisfying the imprimitivity condition (4). Equivalently, classical observables of position,  $f : M \mapsto \mathbb{R}$ , and of momentum,  $X \in \mathfrak{X}_c$ , are mapped to the quantum position and momentum operators

$$Q(f) = \int_X f(q) dE(q) \quad \text{and} \quad P(X).$$

respectively.

The question immediately arises as to how many inequivalent quantizations one could now build, starting with a given manifold  $M$ . In other words, what replaces the Von Neumann uniqueness theorem here? Some general answers are available in the case in which the PV-measure has fixed multiplicity  $n$ , so that the Hilbert space  $\mathfrak{H}$  is naturally isomorphic to  $L^2(M, \mu; \mathbb{C}^n)$ . In this case  $\mathfrak{H}$  may be thought of as arising from the sections of a Hermitian  $\mathbb{C}^n$ -bundle, whose Hermitian structure is compatible with the usual scalar product in  $\mathbb{C}^n$ , and which has a *flat* connection  $\nabla$ . With a further technical restriction on the nature of the allowed  $P(X)$ , the inequivalent quantizations are isomorphic to  $\text{Hom}(\pi_1(M), U(n))$ , i.e., to the set of inequivalent representations of the first fundamental group  $\pi_1(M)$  of the manifold  $M$  by unitary  $n \times n$  matrices (the group  $U(n)$ ).

#### 4. Quantization starting from phase space

The phase space is looked upon as a *symplectic manifold*  $\Gamma$  of dimension  $2n$ , i.e., it is a manifold equipped with a non-degenerate, antisymmetric two form  $\omega$ , which is closed ( $d\omega = 0$ ) and which in local coordinates  $q^i, p_j$ ,  $i, j = 1, 2, \dots, n$  has the form  $\omega = \sum_{i=1}^n dq^i \wedge dp_i$ . Usually  $\Gamma = T^*M$ , i.e., it is the cotangent bundle of a manifold  $M$  of dimension  $n$ . In that case there exists a *symplectic potential*  $\theta$ , which locally has the form  $\theta = \sum_{i=1}^n p_i dq^i$  and for which  $\omega = d\theta$ . The classical algebra of observables is then the algebra of all  $C^\infty$ -functions  $f : \Gamma \rightarrow \mathbb{R}$ , under the Poisson bracket product of eq.(2). Quantization now involves constructing a linear map  $f \mapsto \hat{f}$  into the set of self-adjoint operators on some Hilbert space  $\mathfrak{H}$ , satisfying,

- (i) the set  $\{\hat{f}\}$  acts irreducibly on  $\mathfrak{H}$ ;
- (ii)  $\mathbb{I} \mapsto I$ , where  $\forall (q, p) \in \Gamma$ ,  $\mathbb{I}(q, p) = 1$ , and  $I$  is the identity operator on  $\mathfrak{H}$ ;
- (iii) for a chosen set of functions  $f, g$  (which cannot include them all without relaxing (i)),

$$\{f, g\} \longrightarrow \frac{1}{i\hbar}[f, g].$$

The Kostant-Souriau method of *geometric quantization* [7] has so far been one of the most successful attempts in this direction. It proceeds in two stages: (i) prequantization; (ii) choosing a polarization. One starts by defining the *Hamiltonian vector fields*  $X_f$  on  $\Gamma$ , coming from the functions  $f \in C^\infty$  as:

$$X_f \lrcorner \omega = -df \quad \text{which implies} \quad \omega(X_f, X_g) = \{f, g\}.$$

Next one assumes a certain integrality condition for  $\omega$ , namely that its integral over any closed surface be an integer multiple of  $2\pi$ . The starting point for (i) is the construction of a Hermitian line bundle  $\mathbb{L}$ , with base space  $\Gamma$  and a connection  $\nabla$ , having the connection 1-form  $\alpha$ , such that  $\omega = -d\alpha$  (i.e., the symplectic 2-form gives rise to the curvature tensor of the connection). For  $X \in \mathfrak{X}(M)$ ,  $\nabla_X$  acts on the sections  $s$  of the line bundle  $\mathbb{L}$ . Next one constructs the Hilbert space  $L^2(\Gamma, \omega^n)$  of all square integrable (with respect to the Liouville form  $\omega^n = \omega \wedge \omega \wedge \dots \wedge \omega$ ) sections of  $\mathbb{L}$ . On this Hilbert space, the prequantized operators  $\hat{f}$  corresponding to the classical observables  $f$  are given by

$$\hat{f} = -i\hbar \nabla_{X_f} + f, \quad (5)$$

and hence at this stage for *all* classical observables  $f, g$ , one gets  $\{f, g\} \rightarrow \frac{1}{i\hbar} [\hat{f}, \hat{g}]$ . For example, in the case where  $\Gamma = \mathbb{R}^{2n}$ ,  $\mathbb{L} = \mathbb{C} \times \Gamma$  and

$$\hat{p}_j = -i\hbar \frac{\partial}{\partial q^j}, \quad \hat{q}^j = i\hbar \frac{\partial}{\partial p_j} + q^j, \quad j = 1, 2, \dots, n.$$

However the system is in general highly reducible. To obtain an irreducible representation, it is necessary to reduce the size of the Hilbert space and the number of observables for which the classical Poisson bracket goes over to the quantum commutator bracket. This involves the choice of a polarization. In other words one tries to find a *Lagrangian submanifold*, i.e., a submanifold the tangent spaces of which are generated by vector fields  $X_j$  such that

- (i)  $[X_i, X_j] = c_{ijk} X_k$ , where the  $c$ 's are numerical constants;
- (ii)  $\omega(X_i, X_j) = 0$ ;
- (iii) the dimension of the submanifold is  $n$ .

One then considers only those sections  $s$  which are covariant constants along the leaves of this polarization, i.e.,  $\nabla_{X_i} s = 0$ . To change the measure  $\omega^n$  appropriately, in order to get a reduced Hilbert space, one could make use of half-forms [7].

The efficacy of the geometric quantization scheme is limited by the fact that

- (i) the integrality condition has to be fulfilled;
- (ii) without further extending the theory, only first order differential operators are obtained for the quantized observables;
- (iii) more physical questions, involving the ordering of operators, for example, cannot be addressed within its framework.

On the other hand, if there is a symmetry group available for the system under study, the orbits of this group under the coadjoint action have natural symplectic structures, which can then be exploited using the Kirillov theory [8] to perform a geometric quantization.



### 5. A general scheme exploiting phase space localization

The quantization scheme we are about to present subsumes the techniques proposed by Berezin [9] and later independently by Prugovečki [10], in the context of a relativistic system (in fact all previous work had been for non-relativistic systems only). A refinement of the technique, encompassing the question of ordering of operators, has recently been proposed by Ali and Doebner [11] under the name of *prime quantization*. The method has also been applied to quantization on certain coadjoint orbits of diffeomorphism groups [12]. Similar quantization procedures, particularly when the underlying phase space has a Kählerian structure, have been studied by Odziejewicz [13], who also clarified its connection with the geometric and Berezin quantizations. The present method also generalizes the technique of quantization using the coherent states of locally compact groups [14].

Suppose we have a quantum system on a Hilbert space  $\mathfrak{H}$ . Let  $\Gamma$  be its classical phase space. We would like to study the localizability properties of this system on  $\Gamma$ , possibly reinterpreted appropriately [15]. For probabilistic reasons this implies the existence of a *positive operator valued* (POV) measure on phase space,  $\Delta \mapsto a(\Delta)$ , where  $\Delta \subset \Gamma$  are the Borel sets, and  $a(\Delta)$  are bounded positive operators on  $\mathfrak{H}$ , satisfying

$$a(\emptyset) = 0, \quad a(\Gamma) = I, \quad a\left(\bigcup_{i \in J} \Delta_i\right) = \sum_{j \in J} a(\Delta_j), \text{ for disjoint sets;}$$

$$a(\Delta) = \int_{\Delta} F(q, p) d\nu(q, p), \quad \nu = \text{some natural measure on } \Gamma.$$

Assuming that only a finite number of particles can be accommodated in a phase space cell of volume  $\hbar$ ,

$$F(q, p) = \sum_{i=1}^n |\eta_{q,p}^i\rangle \langle \eta_{q,p}^i|, \quad n < \infty, \quad \eta_{q,p}^i \in \mathfrak{H}.$$

The condition  $a(\Gamma) = I$  then implies

$$\sum_{i=1}^n \int_{\Gamma} |\eta_{q,p}^i\rangle \langle \eta_{q,p}^i| d\nu(q, p) = I \quad (6)$$

The  $\eta_{q,p}^i$  form an *overcomplete family of states*. When they arise from a group action, they are called *coherent states*.

Using  $a$ , one can map  $\mathfrak{H}$  isometrically onto a subspace of  $L^2(\Gamma, \nu; \mathbb{C}^n)$  as

$$(W\psi)_i(q, p) = \Psi_i(q, p) = \langle \eta_{q,p}^i | \psi \rangle.$$

Let  $W\mathfrak{H} = \mathfrak{H}_K \subset L^2(\Gamma, \nu; \mathbb{C}^n)$ . Then  $\mathfrak{H}_K$  is a *reproducing kernel Hilbert space*, with kernel  $K$ , having the properties:

$$K_{ij}(q, p; q', p') = \langle \eta_{q,p}^i | \eta_{q',p'}^j \rangle$$

$$K_{ij}(q, p; q, p) > 0, \quad K_{ij}(q, p; q', p') = \overline{K_{ji}(q', p'; q, p)}$$

$$\sum_{j=1}^n \int_{\Gamma} K_{ij}(q, p; q', p') \Psi_j(q', p') d\nu(q', p') = \Psi_i(q, p), \quad \forall \Psi \in \mathfrak{H}_K.$$

Using  $K$  we can define, for each  $(q, p) \in \Gamma$ , a bounded, linear *evaluation map*  $E_K(q, p) : \mathfrak{H}_K \rightarrow \mathbb{C}^n$  for which

$$E_K(q, p)\Psi = \Psi(q, p) = \int_{\Gamma} K(q, p; q', p')\Psi(q', p')d\nu(q', p'), \quad \Psi \in \mathfrak{H}_K$$

Also, let  $E_K(q, p)^* : \mathbb{C}^n \rightarrow \mathfrak{H}_K$  be the dual map. Then,

$$\begin{aligned} K(q, p; q', p') &= E_K(q, p)E_K(q', p')^* \\ F_K(q, p) &= W^*F(q, p)W^{-1} = E_K(q, p)^*E_K(q, p). \end{aligned}$$

A classical observable  $f \in C^\infty(\Gamma)$  is then associated to an operator  $\hat{f}$  on  $\mathfrak{H}_K$  by

$$\hat{f} = \int_{\Gamma} f(q, p)F_K(q, p)d\nu(q, p). \quad (7)$$

Thus, calling  $f \mapsto \hat{f}$  a *quantization*, we see that the method of prime quantization involves:

- (i) starting with a  $\mathbb{C}^n$ -bundle  $\mathbb{E}$ , with base space  $\Gamma$ ;
- (ii) constructing the Hilbert space  $\mathfrak{H}_{\mathbb{E}}$  of sections of  $\mathbb{E}$  which are square integrable with respect to  $\nu$ . This space is isomorphic to  $L^2(\Gamma, \nu; \mathbb{C}^n)$ ;
- (iii) finding subspaces  $\mathfrak{H}_K \subset \mathfrak{H}_{\mathbb{E}}$  which are reproducing kernel Hilbert spaces and then prime quantizing via eq.(7).

The technique of quantization just outlined is very general and clearly further restrictions are necessary to make it physically meaningful. One possible restriction, implicitly used by Berezin [9,13] is the following: Consider, for simplicity, the case  $n = 1$ , so that eq.(6) becomes  $\int_{\Gamma} |\eta_{q,p}\rangle\langle\eta_{q,p}|d\nu(q, p) = I$ . The phase space  $\Gamma$  carries a symplectic form  $\omega$ . On the other hand  $\mathfrak{H}_K$ , being a Hilbert space, also admits a natural  $\mathbb{C}^1$ -bundle structure, over its projective space  $\mathbf{P}(\mathfrak{H}_K)$  (consisting of one-dimensional projection operators,  $|\Psi\rangle\langle\Psi|/\|\Psi\|^2$ ,  $\Psi \in \mathfrak{H}_K$ ). Indeed, for each  $\Psi \in \mathfrak{H}_K \setminus \{0\}$  denote by  $[\Psi]$  the complex line  $\mathbb{C}\Psi$ , passing through it. Then  $\mathbf{P}(\mathfrak{H}_K)$  has a unique structure of a Hilbert manifold [16] such that the map

$$\pi : \mathfrak{H}_K \setminus \{0\} \longrightarrow \mathbf{P}(\mathfrak{H}_K), \quad \Psi \longmapsto [\Psi]$$

is a holomorphic submersion. The projective space  $\mathbf{P}(\mathfrak{H}_K)$  also carries a natural Kähler structure and a symplectic form  $\omega_K$  which is consistent with a curvature two-form defined naturally on the sections of the above mentioned line bundle. In other words, we have here a prequantization in the sense of the previous section. Let  $\xi_{q,p} = W\eta_{q,p}$  be the coherent states, arising as images in  $\mathfrak{H}_K$  of the coherent states  $\eta_{q,p}$  effecting a prime quantization, and denote by  $\mathfrak{S}_K$  the set of all such  $\xi_{q,p}$ . Then the prime quantization leads to a map  $(q, p) \mapsto \xi_{q,p}$  of  $\mathfrak{S}_K$  into  $\mathbf{P}(\mathfrak{H}_K)$ . As a restriction on the class of prime quantizations, one could require that, if this map is smooth, the pull-back of  $\omega_K$  to  $\mathbf{P}(\mathfrak{H}_K)$  under it should coincide with  $\omega$ . Whenever this condition is satisfied, we have a quantization in the sense of Berezin. In the case where the  $\eta_{q,p}$  arise from the action of a symmetry group [10-12] we get coherent state quantization, a method which has been applied extensively to the *generalized square integrable representations* of the Galilei, Poincaré and diffeomorphism groups.

## 6. Quantization by deformation

In the above scheme, let us associate to any bounded operator  $\hat{A}$  on  $\mathfrak{H}_K$  the phase space function  $A(q, p) = \langle \eta_{q,p} | \hat{A} \eta_{q,p} \rangle$ . This function is called the *symbol* [9,17] of the operator  $\hat{A}$  and the function  $A(q, p; q', p') = \langle \eta_{q,p} | \hat{A} \eta_{q',p'} \rangle$  is an extension of the symbol to  $\Gamma \times \Gamma$ . This extended function determines the operator  $\hat{A}$  completely, but the symbol  $A(q, p)$  need not do so in general. However, in a large number of interesting cases (e.g., when  $\Gamma$  is a complex domain arising from the homogeneous space of a group such as  $Sp(2n, \mathbb{R})$ ,  $U(n)$ , etc.), the function  $A(q, p)$  turns out to be holomorphic in the variable  $z = \frac{1}{\sqrt{2}}(q - ip)$  and then  $A(q, p)$  determines  $\hat{A}$  uniquely. In such cases, putting in appropriate factors of  $\hbar$  we can introduce a product, denoted  $*_{\hbar}$ , in the space of symbols, under which if  $\hat{A}\hat{B}$  has the symbol  $A *_{\hbar} B$ , then

$$A *_{\hbar} B(q, p) = \int_{\Gamma} K(q, p; q', p') A(q', p'; q, p) B(q, p; q', p') K(q', p'; q, p) d\nu(q', p').$$

The algebraic structure of the set  $\mathcal{L}(\mathfrak{H}_K)$  of bounded operators on  $\mathfrak{H}_K$  is thereby transferred to the algebraic structure (under this  $*_{\hbar}$ -product) on the space of symbols. The Lie bracket  $[\hat{A}, \hat{B}]$  is now replaced by the *symbol Lie bracket*:

$$\mu_{\lambda}(A, B) = A *_{\hbar} B - B *_{\hbar} A, \quad \lambda = \frac{i\hbar}{2}, \quad (8)$$

and  $\mu_{\lambda}(A, B)$  is called the *deformation of the classical Poisson bracket*  $\{A, B\}$ . One then proves that

$$\mu_{\lambda}(A, B) = \sum_{j=1}^{\infty} \lambda^j C_j(A, B) + \{A, B\}.$$

$C_j(A, B) =$  higher order brackets in  $A, B$ . Consequently, as  $\lambda \rightarrow 0$ , we see that  $\mu_{\lambda} \rightarrow \{A, B\}$ .

Quantization by deformation starts with the algebra of  $C^{\infty}(\Gamma)$  under the Poisson bracket, satisfying the *Jacobi identity*, and then tries to deform it to obtain an algebra of symbols under a  $\mu_{\lambda}$ -product and again obeying the Jacobi identity [18,19]. In this context dequantization is the passage to the opposite limit, [20,21] i.e., starting with an appropriate twisted algebra for a given quantum system one tries to obtain the underlying classical system by taking the limit  $\lambda \rightarrow 0$ .

## 7. Conclusion

In the above pages, we have tried to touch on the essential features of some of the more common techniques currently employed for obtaining a quantum theory, starting from its classical counterpart. As is certainly clear, from the wide variety of not necessarily overlapping possibilities available at the moment, that the problem of quantization is far from being solved. Indeed there exist other suggestions that we have not had the opportunity to discuss. For example, nothing has been said of field quantization or more esoteric problems, such as the quantization of strings. Neither did we have the occasion to mention quantization using path integrals. However, many of these other quantization procedures rely, in one form or another, on techniques similar to the ones described here. Still, the plethora of techniques, developed in the past two decades or so to attack the problem, prompts us to conclude with the observation that quantization procedures are about as diverse and colourful as the gods in any pagan pantheon!

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**FROM 'ANYONS' TO NONLINEAR  
QUANTUM MECHANICS: PHYSICAL PREDICTIONS FROM  
DIFFEOMORPHISM GROUP REPRESENTATIONS**

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**ABSTRACT.** Diffeomorphism groups are intrinsic to quantum mechanics, and their unitary representations predict physical possibilities otherwise difficult to recognize. Many properties of the particles obeying what is now called "anyon statistics" were first obtained this way, in joint work by the author with R. Menikoff and D. H. Sharp. More recently, in collaboration with H.-D. Doebner, a fundamental nonlinear Schrödinger equation admitting diffusion currents was proposed that we regard as a serious candidate for generalizing the usual quantum mechanics to accommodate dissipation. This equation, introducing an "arrow of time" at the quantum level, was obtained too from diffeomorphism group representations. The method, and the physical results that follow, are described briefly.

## 1. Introduction

For over twenty years I have been working with some other physicists, especially David Sharp and Ralph Menikoff at Los Alamos National Laboratory, to develop a foundation for quantum theory based on Lie groups of diffeomorphisms and their associated Lie algebras of vector fields [1-22]. We commenced between 1968 and 1974 with a study of local, equal-time current algebra representations in nonrelativistic and relativistic physics. Our work evolved during subsequent years into a mathematical framework for quantum mechanics based on gauge-invariant quantities, incorporating ideas from unitary group representations, differential geometry, and topology. The resulting theory unifies the description of an extraordinary variety of quantum systems. Our method is to obtain systems with different particle numbers [1-7], spins [14,16-18], statistics and parastatistics [12-15,20], and internal structures of tightly-bound components [14,19], as well as certain quantum thermodynamical systems such as infinite Bose and Fermi gases [4,8-10], as distinct unitary representations of a single type of infinite-dimensional group—the group of diffeomorphisms of physical space [14,18,22]. Representations of the Lie algebra for this group, the algebra of vector fields, were subsequently obtained by Heinz-Dietrich Doebner, Jiri Tolar, and Bernd Angermann [23-25] in the context of kinematical quantization methods (quantum Borel kinematics); see the accompanying talk by S. Twareque Ali [26]. This fortunate confluence of two approaches led to my present collaborations with Doebner and with Ali.

Our theoretical study generated some physical predictions difficult to reach by other means. One of these was the possibility of "anyon" statistics in two-dimensional space.

i.e. phases under particle exchange intermediate between Bose (+1) and Fermi (-1), which Menikoff, Sharp, and I obtained in 1980-81 independently of their earlier derivation by Jon Leinaas and Jan Myrheim [27-29]. Our results included the now well-known "shift" in the angular momentum and energy spectra, and the relation to the Aharonov-Bohm effect [13]. The idea was subsequently rediscovered by Frank Wilczek (who named such objects "anyons") in his work on fractional quantum numbers [30-31], and it was developed further by many people [32]. In 1983 our methods led us to the role of the braid group  $B_N$  in theories of anyons [14-15], which I believe we were the first to obtain. We showed the admissibility of quantum particles associated with representations of  $B_N$  of dimension greater than one, despite an argument to the contrary by Yong-Shi Wu that paralleled closely the much earlier ideas of Michael Laidlaw and Cécile Morette-DeWitt [33-36]. An important theoretical insight was that the topological properties of the two-dimensional,  $N$ -particle configuration space from which Leinaas and Myrheim first conjectured anyon statistics, are a *result* of the local symmetry of the diffeomorphism group. One need not arbitrarily exclude the so-called "diagonal" points, where two or more particles occupy the same position in space, to obtain anyon statistics [14-15,20]. It also follows that fractional statistics can occur for *distinguishable* particles [18,20,34], based on a subgroup of  $B_N$  (the "colored braids").

A second prediction concerns quantized vortices. Motivated by earlier work of Mario Rasetti and Tullio Regge [37-38], Menikoff, Sharp, and I obtained a possibly surprising result: in an ideal, incompressible quantum fluid in two-space, whose configurations are described by the group of area-preserving diffeomorphisms of  $\mathbf{R}^2$ , there cannot exist pure *point* vortices; but quantum vortex *filaments* (e.g., loops) are permitted [39-40]. Analogously, in  $\mathbf{R}^3$  one-dimensional filaments of vorticity cannot occur, but two-dimensional *surfaces* (e.g., quantum vortex tubes and ribbons) are possible [40]. In this work, we made extensive use of geometric quantization methods.

A third consequence of the theory, obtained just this year in my joint work with Doebner [42], is a general, complex nonlinear Schrödinger equation outside the classes most often studied. Our equation follows from adding to the usual quantum current a diffusion current, so that the mass and momentum densities satisfy an equation of a Fokker-Planck type in place of the standard equation of continuity. The diffusion coefficient  $D$  yields observable effects dependent on a dimensionless constant  $\Gamma = mD/\hbar$ . The diffusion current stems from our interpretation of a certain class of representations of the Lie algebra of vector fields (and correspondingly, of the group of diffeomorphisms), that were obtained independently by Menikoff, Sharp, and me in 1980, and by Angermann, Doebner, and Tolar in 1983 [24,43]. In 1987, I reported on their link with the Fokker-Planck equation [44]. The idea of substituting a Fokker-Planck equation for the continuity equation had by then already been proposed, in a remarkable series of papers by Dieter Schuch and his collaborators in the field of quantum chemistry [45-49]. They did not, however, go quite so far as to obtain the general nonlinear Schrödinger equation discussed below and in Ref. 42.

## 2. Diffeomorphism Groups and their Unitary Representations

Imagine a fluid in a region  $X$  of space; a configuration  $C_1$  is given if we label each fluid element by its position  $\mathbf{x} \in X$ . Let  $\phi: X \rightarrow X$  be an invertible map, displacing each element to a new position  $\mathbf{y} = \phi(\mathbf{x})$ ; then  $\phi$  gives a new fluid configuration  $C_2$ . If  $\phi$  and  $\phi^{-1}$  are  $C^\infty$ , we call  $\phi$  a diffeomorphism. Let  $\phi'$  similarly transform  $C_2$  to  $C_3$ , with  $\mathbf{z} = \phi'(\mathbf{y})$ ; the composite transformation from  $C_1$  to  $C_3$  is a single diffeomorphism  $\mathbf{z} = (\phi' \circ \phi)(\mathbf{x})$ . The diffeomorphisms of  $X$  form a *group* under composition, whose identity is  $\epsilon(\mathbf{x}) \equiv \mathbf{x}$ ; we have  $\phi \circ \phi^{-1} = \phi^{-1} \circ \phi = \epsilon$ . This group is infinite-dimensional as a Lie group; its elements can deform small regions of  $X$  independently—turning in different directions, stretching, shrinking, and/or shearing by varying amounts. We are interested mainly in the infinite-dimensional subgroup  $Diff(X)$ , of diffeomorphisms that can be reached continuously from  $\epsilon(\mathbf{x})$ . One can think of  $\phi \in Diff(X)$  as *transforming*  $X$ , or as a global, general coordinate map *relabeling*  $X$ . Since a smooth relabeling should not affect the physics, we interpret  $Diff(X)$  as a *local symmetry group*. More generally,  $X$  can be a  $C^\infty$  manifold; it is the *physical space* for quantum theories based on  $Diff(X)$ . Now a  $C^\infty$  vector field  $\mathbf{g}$  on  $X$  can be regarded as a (fixed) velocity field for streamline flow of a fluid in  $X$ . For technical reasons, we require  $\mathbf{g}$  to vanish faster than any polynomial, with all derivatives, when  $|\mathbf{x}| \rightarrow \infty$  (or, alternatively, to have compact support). For any such  $\mathbf{g}$  there is a unique one-parameter subgroup  $\phi_s^{\mathbf{g}}$  of  $Diff(X)$ , describing a *flow* (under the velocity field  $\mathbf{g}$ , for time  $s$ ); i.e.,  $\phi_s^{\mathbf{g}}(\mathbf{x})$  solves  $\partial_s \phi_s^{\mathbf{g}}(\mathbf{x}) = \mathbf{g}(\phi_s^{\mathbf{g}}(\mathbf{x}))$ , with the boundary condition  $\phi_{s=0}^{\mathbf{g}}(\mathbf{x}) = \mathbf{x}$ . Requiring that  $\mathbf{g} \rightarrow 0$  (rapidly, with all derivatives) at infinity ensures not only that  $\phi_s^{\mathbf{g}}$  exists, but that the diffeomorphisms satisfy  $\phi(\mathbf{x}) \rightarrow \mathbf{x}$  (rapidly, with all derivatives) at infinity. If  $\mathbf{g}$  has compact support, then for all  $s$ ,  $\phi_s^{\mathbf{g}}$  has compact support.

The relation between vector fields and flows lets us exponentiate the self-adjoint operators for quantum-mechanical current densities and obtain a unitary group. Let  $\psi_{op}(\mathbf{x}, t)$  be a second-quantized, nonrelativistic field in a Fock representation, satisfying equal-time canonical commutation (−) or anticommutation (+) relations  $[\psi_{op}(\mathbf{x}, t), \psi_{op}^*(\mathbf{y}, t)]_{\pm} = \delta(\mathbf{x} - \mathbf{y})$ . In terms of  $\psi_{op}(\mathbf{x}, t)$ , the momentum density is (in local coordinates)

$$\mathbf{J}(\mathbf{x}, t) = (\hbar/2i) \{ \psi_{op}^*(\mathbf{x}, t) [\nabla \psi_{op}(\mathbf{x}, t)] - [\nabla \psi_{op}^*(\mathbf{x}, t)] \psi_{op}(\mathbf{x}, t) \}. \quad (1)$$

Define the spatially averaged operator  $J(\mathbf{g}, t) = \int \mathbf{J}(\mathbf{x}, t) \cdot \mathbf{g}(\mathbf{x}, t) dx$ . From (1), together with either canonical bracket, follows the fixed-time *local current algebra*

$$[J(\mathbf{g}_1), J(\mathbf{g}_2)] = -i\hbar J([\mathbf{g}_1, \mathbf{g}_2]), \quad (2)$$

where  $[\mathbf{g}_1, \mathbf{g}_2] = \mathbf{g}_1 \cdot \nabla \mathbf{g}_2 - \mathbf{g}_2 \cdot \nabla \mathbf{g}_1$  is the *Lie bracket* of  $\mathbf{g}_1$  and  $\mathbf{g}_2$ . Thus (2) is a self-adjoint representation of the Lie algebra  $Vect(X)$  of vector fields on  $X$ . Exponentiating (2) yields a continuous unitary representation (CUR) of  $Diff(X)$ : to each  $J(\mathbf{g})$  corresponds the unitary group  $V(\phi_s^{\mathbf{g}}) = \exp[i(s/\hbar)J(\mathbf{g})]$  in the Fock Hilbert space.

Thus we already see that  $Diff(X)$  is not being introduced arbitrarily: it occurs intrinsically in the standard quantum mechanics of bosons and fermions. To construct representations it is convenient to adjoin to (2) the number density operator,

$\rho_{op}(\mathbf{x}, t) = \psi_{op}^*(\mathbf{x}, t)\psi_{op}(\mathbf{x}, t)$ . For  $f$  a real-valued  $C^\infty$  function on  $X$  vanishing (rapidly, with all derivatives) at infinity, set  $\rho(f, t) = \int \rho_{op}(\mathbf{x}, t)f(\mathbf{x}, t) d\mathbf{x}$ . There then follow the additional equal-time commutators

$$[\rho(f_1), \rho(f_2)] = 0. \quad (3)$$

$$[\rho(f), J(\mathbf{g})] = i\hbar\rho(\mathbf{g} \cdot \nabla f), \quad (4)$$

where  $\mathbf{g} \cdot \nabla f$  is the Lie derivative. Eqs. (2)-(4) represent (by self-adjoint operators) a *semidirect product* of two Lie algebras—the commutative algebra  $S$  of smooth scalar functions on  $X$ , and the algebra of vector fields. Exponentiating (2)-(4), with  $U(f) = \exp[i\rho(f)]$  and  $V$  representing  $Diff(X)$ , we obtain the unitary group

$$U(f_1)V(\phi_1)U(f_2)V(\phi_2) = U(f_1 + f_2 \circ \phi_1)V(\phi_2 \circ \phi_1). \quad (5)$$

The framework for quantum theory that Sharp and I developed takes  $Vect(X)$  and  $Diff(X)$  as fundamental structures. The *unitarily inequivalent, irreducible representations* of (2)-(5) describe possible, physically distinct quantum systems—including systems different from the canonical, second-quantized Fock fields. In this sense, we regard the diffeomorphism group as a “universal group” for quantum theory. Given any CUR obeying (5), the above provides its physical interpretation: for each flow subgroup  $\phi_t^{\mathbf{g}}$  in  $Diff(X)$ , recover  $J(\mathbf{g})$  as the (unique) self-adjoint generator of  $V(\phi_t^{\mathbf{g}})$ ; then  $J(\mathbf{g})$  is the momentum density averaged with the vector field  $\mathbf{g}$ .

One success of this program was the characterization of particle statistics using the operators  $\rho$ ,  $J$ ,  $U$ , and  $V$ , without introducing anticommutators or other brackets. The main idea is that in more than one space dimension, diffeomorphisms (even constrained to be trivial at infinity) can always implement the physical exchange of  $N$  particles with arbitrary given positions. The consequences of such a permutation are thus fully described by the CUR of the diffeomorphism group and its algebra. Furthermore, obtaining a diffeomorphism that implements the exchange as a succession of flows provides a *path* for the exchange. In two space dimensions, we represent  $Diff(\mathbf{R}^2)$ ; the same considerations then lead to fractional statistics, and were the basis of our 1981 results that included many of the important physical properties of anyons.

Another result was our ability to describe particle spin, despite the absence of explicit spin operators in the algebra or the group, without inserting them arbitrarily. We note that every diffeomorphism  $\phi$  of  $\mathbf{R}^3$  (for example) has at the point  $\mathbf{x}$  a matrix of derivatives  $D_\phi(\mathbf{x})$ , with entries  $(\partial\phi^k/\partial x^j)(\mathbf{x})$  for  $j, k = 1, 2, 3$ . Such a  $3 \times 3$ , nonsingular matrix belongs to the *general linear group*  $GL(3, \mathbf{R})$ , whose maximal compact subgroup is the rotation group  $SO(3)$ . A flow subgroup of  $Diff(\mathbf{R}^3)$ , holding  $\mathbf{x}$  fixed, then yields a path through the identity in  $GL(3, \mathbf{R})$ , corresponding (up to homotopy equivalence) to an element of the universal covering group. This idea results in representations of (5) that describe particles with different spins, either integer and half-integer—but because we have the general linear group and not just the rotation group, an irreducible CUR of  $Diff(\mathbf{R}^3)$  describes a *tower* of spins and not just one fixed spin. Representations of  $GL(3, \mathbf{R})$  and its covering had been applied earlier in quantum mechanics by L. Weaver, L. C. Biedenharn, and R. Y. Cusson [50], e.g. to excitations of nuclei: the results they



obtained thus fit nicely into our general study of diffeomorphism group representations. A similar procedure in  $\mathbf{R}^2$  leads to the fractional spin of anyons.

To see how such results are obtained, let us look briefly at one way to write CUR's of the diffeomorphism group (or its semidirect product with the scalar functions). Take the Hilbert space  $\mathfrak{H} = L^2_\mu(\Delta, \mathfrak{M})$  to be a space of square-integrable functions  $\Psi(\gamma)$ , for  $\gamma$  in a quantum configuration-space  $\Delta$ ;  $\Psi$  takes values in a complete inner product space  $\mathfrak{M}$  that may be  $n$ -dimensional ( $n \geq 1$ ) or infinite-dimensional. The norm  $(\Psi, \Psi)$  is defined from the inner product in  $\mathfrak{M}$  by  $(\Psi, \Psi) = \int \langle \Psi(\gamma), \Psi(\gamma) \rangle_{\mathfrak{M}} d\mu(\gamma)$ , where  $\mu$  is a suitable measure on  $\Delta$ . The space  $\Delta$  is a  $G$ -space for the diffeomorphism group—i.e.,  $\phi : \Delta \rightarrow \Delta$  continuously, in a way that respects the group law. We write  $(\phi, \gamma) \rightarrow \phi\gamma$  for this action. Then  $\mu$  is required to be *quasi-invariant*, in that the class of  $\mu$ -measure zero sets in  $\Delta$  is preserved when transformed by any  $\phi \in \text{Diff}(X)$ . The quasi-invariance of  $\mu$  implies existence of the Radon-Nikodym derivative  $(d\mu_\phi/d\mu)(\gamma)$  of the transformed measure  $\mu_\phi$  with respect to  $\mu$ . Now, a CUR of  $\text{Diff}(X)$  in  $\mathfrak{H}$  is given by

$$[V(\phi)\Psi](\gamma) = \chi_\phi(\gamma)\Psi(\phi\gamma)\sqrt{\frac{d\mu_\phi}{d\mu}(\gamma)}, \quad (6)$$

where the unitary operators  $\chi_\phi(\gamma)$  act in  $\mathfrak{M}$  so as to satisfy, for  $\phi_1, \phi_2 \in \text{Diff}(X)$ ,

$$\chi_{\phi_1}(\gamma)\chi_{\phi_2}(\phi_1\gamma) = \chi_{\phi_2 \circ \phi_1}(\gamma) \quad (7)$$

almost everywhere. Such a system of operators  $\chi$  is called a unitary cocycle;  $\chi$  must also satisfy appropriate technical conditions, which I shall not discuss here. Equivalent (i.e., cohomologous) cocycles on the same configuration-space turn out to describe unitarily, and therefore physically, equivalent representations; while mutually inequivalent cocycles on  $N$ -particle configuration spaces describe the inequivalent quantum statistics. When  $\mathfrak{M}$  is the field of complex numbers, the particle statistics is described by scalar phases; while the higher-dimensional case corresponds to "paraparticles". The operators  $U(f)$  also act on  $\mathfrak{H}$ : they do so as multiplication operators, completing the representation of (5).

George Mackey's "method of semidirect products" for finite-dimensional Lie groups [51] extends partially to the infinite-dimensional case [2.7.52], and allows quantum configuration spaces  $\Delta$  to be constructed systematically. In this formalism *irreducible* CUR's satisfying (5) are associated with *ergodic* measures on the continuous dual to the space  $S$  of scalar functions on  $X$ —i.e., on the space  $S'$  of tempered distributions on  $X$ . An ergodic measure is a normalized probability measure  $\mu$  for which all sets invariant under diffeomorphisms are of measure 0 or 1;  $\mu$  may be concentrated on a single orbit  $\Delta \subset S'$ , or it may be that every orbit has measure 0 and  $\Delta$  is the union of uncountably many orbits. On a single-orbit configuration space for  $\text{Diff}(X)$ , a systematic, physically motivated construction of cocycles is possible. For fixed  $\gamma \in \Delta$ , consider the *stability subgroup*  $K_\gamma = \{\phi \mid \phi\gamma = \gamma\}$ . For  $\phi_1$  and  $\phi_2$  in  $K_\gamma$ , the cocycle equation (7) reduces to a unitary representation of  $K_\gamma$  (acting in  $\mathfrak{M}$ ). Conversely, CUR's of  $K_\gamma$  lead to cocycles by extension of the method of induced representations [12.13.20]. One way to obtain representations of  $K_\gamma$  is from a homomorphism mapping  $K_\gamma$  to the *fundamental group* of the configuration space,  $\pi_1(\Delta)$ , which is a discrete group (see below). Then a unitary

representation of  $\pi_1(\Delta)$  automatically gives a CUR of  $K_\gamma$ , which induces the CUR of (5) having the appropriate cocycle. More generally,  $\pi_1(\Delta)$  can be replaced here by a finite-dimensional Lie group, which serves as a *gauge group* for the theory.

Another way to construct configuration spaces and cocycles systematically, which I shall not review here, is the method of coadjoint orbits [21,38-40,53], where the orbits are (reduced) phase spaces; their *foliations*, obtained through geometric quantization, become the quantum configuration spaces for representing the group.

The above glosses over some mathematical difficulties, especially problems associated with the absence of Haar measure for infinite-dimensional Lie groups. These have, however, been overcome for  $Diff(X)$  in the cases of greatest physical interest—including the description of anyons and the dissipative nonlinear quantum theory highlighted here.

### 3. Diffeomorphism Group Representations and Anyons

Ordinary  $N$ -particle quantum mechanics fits naturally into our framework. A configuration  $\gamma = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$  occurs in  $S'$  as the sum of  $N$  distinct evaluation functionals,

$$\gamma = \sum_{j=1}^N \delta_{\mathbf{x}_j}, \quad (\mathbf{x}_i \neq \mathbf{x}_j \text{ for } i \neq j). \quad (8)$$

where  $\langle \delta_{\mathbf{x}}, f \rangle = f(\mathbf{x})$  for  $f \in S$ . A diffeomorphism acts on a term  $\delta_{\mathbf{x}}$  of  $\gamma$  by transforming  $\mathbf{x}$ , keeping (of course) any two distinct points distinct. Thus the orbit  $\Delta$  containing  $\gamma$  is exactly the  $N$ -particle configuration space of unordered  $N$ -tuples in  $X$ , *without* the "diagonal" points mentioned earlier: the latter are excluded *automatically*, and  $\Delta$  has nontrivial homotopy. Its stability group  $K_\gamma \subset Diff(X)$  consists of all diffeomorphisms which leave the *set*  $\{\mathbf{x}_1, \dots, \mathbf{x}_N\}$  fixed, including not just those keeping each individual  $\mathbf{x}_j$  fixed, but also those which *permute* the points; hence, we have a natural group homomorphism from  $K_\gamma$  onto the symmetric group  $S_N$ . Any unitary representation of  $S_N$  in a vector space  $\mathfrak{M}$  provides a CUR of  $K_\gamma$ , inducing a CUR of  $Diff(X)$  described by a unitary cocycle  $\chi$ , and giving us a consistent quantum theory. The symmetric and antisymmetric one-dimensional representations of  $S_N$  lead to bosons and fermions respectively, while higher-dimensional irreducible representations of  $S_N$  (associated with the usual Young diagrams) induce CUR's of  $Diff(X)$  describing parastatistics [54-56]. The action of  $V(\phi)$  is given by (6), where  $\mu$  is locally equivalent to Lebesgue measure and  $\chi$  is a cocycle reducing on  $K_\gamma$  to the indicated representation, and the action of  $U(f)$  is just to multiply the wave function  $\Psi(\gamma)$  by  $\exp[i\langle \gamma, f \rangle]$ . In this way Bose, Fermi, and parastatistics all occur as special cases of the general theory.

Note that in these examples, the symmetric group acts on the *values* rather than the *indices* of the  $\mathbf{x}_j$ . This distinction has no physical consequences for bosons or fermions, but it becomes important in the description of paraparticles [20,57]. "Index permutations" commute with all observables in the theory, while "value permutations" do not. The group of diffeomorphisms compels us to define the statistics from value permutations, but without giving up the indistinguishability of the particles built into the elements  $\gamma$  of the configuration space  $\Delta$ . The *unitary inequivalence* of Bose, Fermi, and "para" representations of  $Diff(X)$  means that a system of  $N$  paraparticles,  $N \geq 3$ ,

is (at this level of description) *physically distinguishable* from a system of  $N$  bosons or fermions, even when the latter are equipped with additional quantum numbers.

In two space dimensions, though, there are still other possibilities. When  $X$  is  $\mathbf{R}^3$  and  $\Delta$  is the  $N$ -particle configuration space, the group  $S_N$  is the fundamental group  $\pi_1(\Delta)$ . But when  $X$  is  $\mathbf{R}^2$ ,  $\pi_1(\Delta)$  is larger than  $S_N$ , since it is possible in the plane to distinguish the *path* whereby a set of particles have been exchanged; in fact, the fundamental group is the *braid group*  $B_N$ . And for a configuration  $\gamma$  of  $N$  indistinguishable particles in the plane, the stability group  $K_\gamma$  in  $\text{Diff}(\mathbf{R}^2)$  maps onto  $B_N$ , not merely onto  $S_N$ —because a diffeomorphism  $\phi$  that implements an exchange, written as a succession of flows, provides a path for the exchange that is well-defined (up to homotopy equivalence)! Here the fact that diffeomorphisms act on values rather than on indices is essential for anyon statistics (as is the natural exclusion of the diagonal points from the orbit).

A one-dimensional representation of  $B_N$ , where a simple counterclockwise exchange of a pair is represented by  $\exp[i\theta]$ , induces a CUR of  $\text{Diff}(\mathbf{R}^2)$  describing anyons with the intermediate phase shift  $\theta$ . The physical interpretation of  $J(\mathbf{g})$  as the momentum density averaged by  $\mathbf{g}$  allows the determination of the properties of the quantum particles described by such a representation, including the shifted angular momentum spectrum. We also have, as an immediate consequence, the consistent possibility of quantum mechanics based on the higher-dimensional representations of  $B_N$ .

To describe distinguishable anyons in our framework, configurations must be *ordered*  $N$ -tuples of points in  $\mathbf{R}^2$ . Such configurations also occur in  $S^1$ ; writing  $\gamma = \sum_{j=1}^N \lambda_j \delta_{\mathbf{x}_j}$ , where  $\mathbf{x}_i \neq \mathbf{x}_j$  for  $i \neq j$ , and where the real coefficients  $\lambda_j$  are all different, we obtain a new orbit. The  $\lambda_j$  correspond to distinct outcomes for measurements of  $\rho$  on each particle. The fundamental group for this orbit is the group of "colored braids"—the kernel of the natural homomorphism from  $B_N$  onto  $S_N$ . The stability group  $K_\gamma$  is smaller than its counterpart on the identical-particle configuration space, and *different* phase shifts can occur when different pairs of particles circle each other in the plane.

In the preceding I have sought to convey how anyons arise kinematically, through certain diffeomorphism group representations. Next I would like to describe a different family of representations, and offer a physical interpretation for them.

#### 4. A Nonlinear Schrödinger Equation for a Dissipative Quantum Theory

Let  $D$  be a real parameter. A self-adjoint representation of the Lie algebra (2)-(4) in  $\mathfrak{h} = L^2(\mathbf{R}^3)$ , describing a single quantum particle in three dimensions, is:

$$\left. \begin{aligned} \rho(f)\Psi(\mathbf{x}) &= mf(\mathbf{x})\Psi(\mathbf{x}), \\ J^D(\mathbf{g})\Psi(\mathbf{x}) &= \frac{\hbar}{2i} \{ \mathbf{g}(\mathbf{x}) \cdot \nabla v(\mathbf{x}) + \nabla \cdot [\mathbf{g}(\mathbf{x})\Psi(\mathbf{x})] \} + mD[\text{div } \mathbf{g}(\mathbf{x})]\Psi(\mathbf{x}). \end{aligned} \right\} \quad (9)$$

For distinct  $D$ , we have mutually inequivalent one-particle representations of the current algebra;  $D$  may be viewed as a quantum number deriving from the algebraic properties of  $\text{Vect}(\mathbf{R}^3)$ . The usual Fock representation corresponds to  $D = 0$ .

Since (9) is linear in  $f$  and  $\mathbf{g}$ , we can define the (singular) operator-valued mass density  $\rho_{op}(\mathbf{x}, t)$  and momentum density  $\mathbf{J}(\mathbf{x}, t)$ , so that  $\rho(f) = \int \rho_{op}(\mathbf{x}, t)f(\mathbf{x})d\mathbf{x}$  and  $J(\mathbf{g}) = \int \mathbf{J}(\mathbf{x}, t) \cdot \mathbf{g}(\mathbf{x})d\mathbf{x}$ . Then the representation (9) can be obtained from

the  $D = 0$  case by the transformation  $\mathbf{J}^D(\mathbf{x}, t) = \mathbf{J}^{D=0}(\mathbf{x}, t) - D\nabla\rho_{op}(\mathbf{x}, t)$ . To relate  $D$  to a dynamical situation, one makes the generic assumption of a continuity equation relating the mass and momentum density operators, and expressing the conservation of mass,  $\partial_t\rho_{op} = -\nabla \cdot \mathbf{J}^D$ . This yields a Fokker-Planck type of equation for  $\mathbf{J}^{D=0}$ , including a term describing diffusion of  $\rho$ :  $\partial_t\rho_{op} = -\nabla \cdot \mathbf{J} + D\nabla^2\rho_{op}$ , where  $D$  serves as a diffusion coefficient. The dimensions of  $D$  are length<sup>2</sup>/time. Thus I proposed earlier [44] to regard  $-D\nabla\rho_{op}$  as a diffusion current, and to base the time evolution of  $\rho_{op}$  on the Fokker-Planck equation. We recover the scalar functions  $\rho(\mathbf{x}, t)$  (the probability density) and  $\mathbf{j}(\mathbf{x}, t)$  (the probability current) as expectation values of  $\rho_{op}(\mathbf{x}, t)$  and  $\mathbf{J}(\mathbf{x}, t)$  respectively, by setting  $\rho(\mathbf{x}, t) = \int \overline{\Psi(\mathbf{x}')} m^{-1} \rho_{op}(\mathbf{x}, t) \Psi(\mathbf{x}') d\mathbf{x}'$  and  $\mathbf{j}(\mathbf{x}, t) = \int \overline{\Psi(\mathbf{x}')} m^{-1} \mathbf{J}(\mathbf{x}, t) \Psi(\mathbf{x}') d\mathbf{x}'$ . Then they, too, obey a Fokker-Planck equation,

$$\partial_t\rho = -\nabla \cdot \mathbf{j} + D\nabla^2\rho. \quad (10)$$

The idea advanced in [45-48] is thus rediscovered from fundamental considerations of local symmetry. With  $U(f) = \exp[i\rho(f)]$  and  $V(\phi_{\mathbf{x}}) = \exp[i(s/\hbar)J(\mathbf{g})]$ , we obtain

$$V^D(\phi)\Psi(\mathbf{x}) = e^{iD\ln J_{\phi}(\mathbf{x})}\Psi(\phi(\mathbf{x}))\sqrt{J_{\phi}(\mathbf{x})}, \quad (11)$$

where  $J_{\phi}(\mathbf{x})$  is the Jacobian of  $\phi$  at  $\mathbf{x}$ . Compare (11) with (6), and note the nontrivial, complex cocycle which is the group-theoretical origin of the diffusion current.

But what wave equation derives from (10)? The usual linear Schrödinger equation, or a nonlinear equation with a real nonlinear "potential", is compatible with the continuity equation by standard arguments [58-59], assuming that  $\rho = \psi\bar{\psi}$  and  $\mathbf{j} = (\hbar/2mi)(\psi\nabla\bar{\psi} - \bar{\psi}\nabla\psi)$ . Under the same assumptions, Doebner and I obtained from (10) the nonlinear Schrödinger equation

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + F(\psi, \bar{\psi})\psi + iD\hbar\nabla^2\psi + iD\hbar\frac{|\nabla\psi|^2}{|\psi|^2}\psi, \quad (12)$$

where  $F$  is a real functional. We shall take  $F$  to be just a multiplicative potential  $V(\mathbf{x}, t)$ . With the notation  $H_0 = -(\hbar^2/2m)\nabla^2 + V(\mathbf{x}, t)$  and  $G(\psi) = \nabla^2\psi + (|\nabla\psi|^2/|\psi|^2)\psi$ , we abbreviate (12) as  $i\hbar\partial_t\psi = H_0\psi + iD\hbar G(\psi)$ . The imaginary nonlinear potential is uniquely determined by (10), since  $\nabla^2\rho/\rho = 2\text{Re}[G(\psi)/\psi]$ . Note the *linear* as well as the nonlinear term with imaginary coefficient; they must occur together for (10) to hold. There is no linear Schrödinger equation consistent with (10), for  $D \neq 0$ .

Despite considerable effort, we have thus far been unable to find a discussion of (12) in the literature. Treating  $\rho(\mathbf{x}, t)$  as a fluid is somewhat in the spirit of the hydrodynamical and stochastic reinterpretations of quantum mechanics [60]; but here, we obtain something different by allowing  $\rho$  itself to diffuse. Schuch *et al.* study a Schrödinger equation with a logarithmic nonlinearity,  $i\hbar\partial_t\psi = H_0\psi - i\hbar\gamma(\ln\psi - \langle\ln\psi\rangle)\psi$ . This equation is independent of (12); it is compatible with it only under the constraint  $D\nabla^2\rho/\rho = -\gamma(\ln\rho - \langle\ln\rho\rangle)$ . Common solutions of the logarithmic equation and the constraint equation must also obey (12). A real, nonlinear potential  $F = \lambda|\nabla\psi|^2/|\psi|^2$  was considered but rejected by Kibble [61].

Some interesting properties of (12) are: (a) The probability is conserved. (b) The equation is homogeneous, in the sense that if  $\psi$  is a solution, then  $\alpha\psi$  is a solution for any complex constant  $\alpha$ . (c) With  $V = 0$ , plane waves  $\psi(\mathbf{x}, t) = \exp [i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$  with  $\omega = E/\hbar$  and  $|\mathbf{k}|^2 = 2mE/\hbar$ , are solutions. (d) The equation is Euclidean- and time-translation invariant (for  $V = 0$ ). If  $\psi$  is a solution, then  $\bar{\psi}$  solves the time-reversed equation with the additional substitution  $D \rightarrow -D$ , as would be expected for a diffusion process. The sign of  $D$  thus gives a directionality to the flow of time in the quantum theory. (e) Noninteracting particle subsystems remain uncorrelated (separation property). Distinct values of  $D$  can occur for different particle species. (f) Extra, "dissipative" terms occur in the time-rate of change of  $\langle \mathbf{p} \rangle$  and  $\langle i\hbar\partial_t \rangle$ . While  $(d/dt)\langle \mathbf{x} \rangle = m^{-1}\langle \mathbf{p} \rangle$ , we also have

$$\left. \begin{aligned} (d/dt)\langle \mathbf{p} \rangle &= -\langle \nabla V \rangle + 2D \operatorname{Re} \int \bar{\psi}(-i\hbar\nabla)G(\psi) d\mathbf{x}, \\ (d/dt)\langle i\hbar\partial_t \rangle &= \langle \partial_t V \rangle + 2D \operatorname{Re} \int \bar{\psi}H_0G(\psi) d\mathbf{x}. \end{aligned} \right\} \quad (13)$$

The dissipative terms in (13) can be interpreted in terms of diffusion currents; but there exist as well classes of solutions to (12) that are non-dissipative.

I shall close with mention of a family of such solutions, with stationary density; i.e.,  $\partial_t \rho(\mathbf{x}, t) \equiv 0$ . Writing  $\psi = \chi \exp [iK]$  with  $\chi(\mathbf{x})$  and  $K(\mathbf{x}, t)$  real, we have  $\mathbf{j} = (\hbar/m)\rho\nabla K$ , whence  $K = (Dm/\hbar) \ln \rho - \Omega(t)$ ;  $\Gamma = Dm/\hbar$  is a dimensionless constant. Then  $\psi = \chi \exp [i\Gamma \ln |\chi|^2 - i\Omega(t)]$  solves (12) when  $\chi$  obeys the *linear* equation

$$-(\hbar^2/2m)(1 + 4\Gamma^2)\nabla^2\chi + V\chi = E\chi, \quad (14)$$

where  $E = \hbar(d\Omega/dt)$  is the energy (constant if  $V$  is time-independent). We see that the energy spectrum is shifted from ordinary quantum mechanics: precision experiments could thus give a bound on  $\Gamma$ . For these solutions  $\langle \mathbf{p} \rangle = 0$  and the dissipative terms in (13) vanish. Singularities in the phase of  $\psi$  occur at its nodes; but  $\mathbf{j}$  is nonsingular.

Consistent with the interpretation of  $\text{Diff}(\mathbf{R}^3)$  as a local symmetry group, our perspective is to take all of its representations seriously as kinematically allowed possibilities for quantum physics; and there is no *a priori* reason to exclude those with  $D \neq 0$ . One interpretation of (12) is that  $D$  is a small, fundamental constant of nature associated with particular particle species, whose measurement is an empirical matter. Alternatively, physical situations in the context of the linear Schrödinger equation in which statistical "Brownian" fluctuations occur, permitting dissipative effects, may be modeled phenomenologically by (12). Eq. (12) is perhaps the most elementary way in which an "arrow of time" can be introduced into ordinary Schrödinger quantum mechanics.

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## Quantum Chaos - Towards the Formulation of an Alternate Quantum Theory

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### Abstract

A summary of our recent work in large Poincaré systems (LPS) is presented. The most striking property of LPS is that unity transformations leading to the diagonalization of the Hamilton operator analytic in the coupling constant diverge. We have shown that to cure these Poincaré divergences we have to introduce a complex spectral theory. The theory has in general to be formulated in the space of density matrices. The representation of density matrices we obtain is irreducible. Here irreducibility means that the density matrices cannot be expressed in terms of products of wave functions as is the case in conventional quantum mechanics. As a result, statistical features appear which are in addition to those which arise from the usual uncertainty relations. This is the situation which we associate to "quantum chaos." The consideration of LPS leads to a new formulation of quantum theory which from the point of view of its time structure is similar to classical dynamics while still keeping quantum effects.

### 1. Introduction

We are very happy to participate in this symposium honoring Professor E. Wigner. One of us (I.P.) had the privilege to meet regularly with Prof. Wigner at the famous Solvay meetings. He remembers vividly one of the questions Prof. Wigner used to bring up.<sup>1)2)</sup> How to define unstable particles? How to deal with the epistemological problems of quantum theory. Wigner's friend has become a mythical figure somewhat like Schrödinger's cat.

We would like to show in this paper that we can now achieve a better understanding of these questions once we incorporate instability and chaos in the frame of quantum theory. It is well-known that these concepts have drastically modified our views on classical mechanics. They force us to go beyond the description in terms of trajectories and to use a statistical approach involving ensembles.<sup>3)4)</sup> Several authors have studied the effects which occur in quantum theory when applied to systems which display chaotic behavior in the classical limit.<sup>5)</sup> However, when we speak about quantum chaos here we have in mind a quite different situation. We shall summarize here our work on dynamical unstable quantum systems (*Large Poincaré Systems*, in short LPS). Then the consistent incorporation of instability forces us to go beyond the Schrödinger description in terms of wave function and to formulate quantum theory in terms of density matrices.



Indeed, the most striking property of LPS is that unitary transformations leading to the diagonalization of the Hamilton operator analytic in the coupling constant diverge.<sup>6)</sup> We have shown that to cure these Poincaré divergences we have to introduce a complex spectral theory.<sup>7)8)9)</sup> The theory has in general to be formulated in the space of density matrices. The representation of density matrices we obtain is irreducible. Here irreducibility means that the density matrices cannot be expressed in terms of products of wave functions as is the case in conventional quantum mechanics. As a result, statistical features appear which are in addition to those which arise from the usual uncertainty relations. This is the situation which we associate to "quantum chaos."

The aim of this paper is to clarify the physical meaning of irreducible representations associated with a collapse of the wave function. The basic ingredients are "non-integrability" in the Poincaré sense (see section 2) and persistent interactions. In other words, we have to go beyond the idealization involved in the use of  $S$ -matrix formalism which eliminates time from the physical description (see the very interesting comments by G. Källén<sup>10)</sup>).

As a concrete example, we shall discuss two- and three-body scattering. The standard application of  $S$ -matrix approach is to two-body scattering. Both "in-states" and "out-states" are described by free-moving wave packets. In this case there exists no irreducible representation in our sense. However, the situation changes when we consider persistent interactions (for example, starting from a plane wave). The asymptotic limit of the wave function (as well as of the density matrix) then becomes ill-defined. The usual relation between density matrices and wave functions is then lost as we shall show in detail in section 3. This is the simplest situation we may associate with *quantum chaos*. The analogy with classical chaos is obvious. Classical chaos does not imply that Newton's equations are "wrong", but that the trajectory description becomes an over-idealization. Similarly quantum chaos does not mean that Schrödinger's equation becomes wrong, but that it leads to ill-defined results. Our theory goes beyond this qualitative statement and permits to express the density matrix as a sum of terms each of which has for all times  $t$  a well-defined meaning in terms of *test functions*. In other words, the average of an operator  $A$  which acts as a test function for  $\rho$  is given by  $A$  is given by

$$\langle A \rangle = \text{tr}[A^\dagger \rho_{Sch}(t)] = \text{tr}[A^\dagger \rho_{irr}(t)], \quad (1.1)$$

where  $\rho_{Sch}$  is a density matrix as calculated by the time-dependent Schrödinger equation, and  $\rho_{irr}$  corresponds to our irreducible representation (see sections 3 and 4).

In addition to providing a systematic approach for the calculation of averages (as our complex spectral theory puts the use of perturbation techniques in spite of Poincaré's divergences), our theory leads to the introduction of a number of new concepts such as time symmetry breaking, microscopic expression of entropy, definition of unstable particles, and so on.<sup>7)8)9)</sup>

A simple example which illustrates the need to formulate an alternative generalized quantum theory is three-body scattering for free incident particles. As is well-known,  $S$ -matrix theory leads to divergent results as a consequence of repeated two-body scattering.<sup>11)</sup> (These divergences cannot be resolved by the Fadeev type of expansion of the  $T$ -matrix, since the divergence appears in processes involving all three particles). These repeated two-body scattering can occur at arbitrarily large space separations. This makes the three-body scattering a persistent process whatever the initial preparation. We shall show that the singularities are a manifestation of the non-integrability of the three-body scattering system and therefore of Poincaré's divergences. The wave function (and the density matrix) obtained by the time-dependent Schrödinger theory are highly singular, in spite of this fact our approach leads

unambiguously to a well-defined value of the three-body scattering cross-section. As the main aim of quantum theory is to predict values of physical observables, this is a simple example where it is necessary to generalize quantum theory and to replace the usual spectral theory in the Hilbert space by a generalized (complex) spectral theory in the Liouville space.

In conclusion, we see that in general we cannot isolate interactions and decompose physical evolution into successions of events separated by free motion. Our extension of quantum mechanics leads to a number of epistemological consequences which we briefly discuss in the concluding section.

## 2. The complex spectral theory

For Hamiltonian systems the distinction between stable and unstable ones is based on Poincaré's celebrated classification into "integrable" and "non-integrable" systems.<sup>6)</sup> This classification applies to quantum systems with continuous spectrum as well as to classical systems. Consider a Hamiltonian of the form for a quantum system

$$H = H_0 + \lambda V, \quad (2.1)$$

where  $\lambda$  is a coupling constant. We suppose that the eigenfunction  $|\alpha\rangle$  and the eigenvalues of  $H_0$  are known

$$H_0|\alpha\rangle = \omega_\alpha|\alpha\rangle. \quad (2.2)$$

How to use this knowledge to construct the eigenfunction and the eigenvalues of  $H$

$$H|\varphi_\alpha\rangle = E_\alpha|\varphi_\alpha\rangle. \quad (2.3)$$

We would like to find solutions we could expand in powers of  $\lambda$  to apply perturbative techniques. However, Poincaré's result shows that this is impossible as the result of divergences which result from resonances between the unperturbed frequencies  $\omega_\alpha$ . Poincaré's non-integrable systems with continuous spectrum and continuous sets of resonances are quite common in physics. These are the systems we call Large Poincaré Systems (LPS).

The fact that we cannot diagonalize the Hamiltonian of LPS through expansion in powers of the coupling constant is of course especially important in field theory as it casts doubts on some basic assumptions of field theory.<sup>12)</sup> We shall come back to this point in our concluding section.

A simple example of LPS is the well-known Friedrichs model in which a discrete state is coupled to a field with a continuous spectrum, as well as scattering problems. The Friedrichs model has been treated in detail in recent publications.<sup>7)13)</sup> We shall not go back to it here. We only note that Poincaré's divergences are eliminated by an appropriate time-ordering in the Hilbert space. (Transitions from the excited state to the ground state are "future"-oriented, while transitions from the ground state to the excited state are "past"-oriented. This leads to an appropriate analytic continuation).

Another example of LPS is potential (or two-body) scattering. The Hamiltonian is

$$H = \sum_k \omega_k |k\rangle\langle k| + \lambda \sum_{kk'} V_{kk'} |k\rangle\langle k'|. \quad (2.4)$$

The difference with the Friedrichs model is there is no natural time-ordering in the Hilbert space to eliminate Poincaré's divergences. We therefore turn to a statistical description in terms of density matrices. As is well-known  $\rho$  satisfies the Liouville-von Neumann equation

$$i \frac{\partial \rho}{\partial t} = L_H \rho \quad \text{with} \quad L_H = H \times 1 - 1 \times H, \quad (2.5)$$

where the  $L_H$  is a "superoperator" acting on  $\rho$  (it is the commutator with  $\rho$ ). We look then for the spectral representation of  $L_H$ . For integrable systems this is of course trivial.<sup>8)9)</sup> However, for non-integrable systems we recover the usual Poincaré divergences. Now as the consequence of (2.1) we can also decompose the Liouville operator

$$L_H = L_0 + \lambda L_V. \quad (2.6)$$

For  $L_0$  we can construct a complete set of spectral projectors which satisfy the condition

$$P L_0 = L_0 P, \quad \sum_{\nu} P = 1, \quad P P^{(\nu')} = P \delta_{\nu\nu'}, \quad P = P^\dagger. \quad (2.7)$$

However, the corresponding projectors for  $L_H$  cannot be obtained through expansion in  $\lambda$ .<sup>14)</sup> But, we can obtain a complete set of projectors  $\Pi^{(\nu)}$  for  $L_H$  giving up the hermiticity conditions and using an appropriate analytic continuation (or time-ordering). These  $\Pi$  satisfy the conditions<sup>9)14)</sup>

$$\Pi L_H = L_H \Pi, \quad \sum_{\nu} \Pi = 1, \quad \Pi \Pi^{(\nu')} = \Pi \delta_{\nu\nu'}, \quad \Pi \neq \Pi^\dagger. \quad (2.8)$$

Our rule of analytic continuation is the natural extension of the rule used for the Friedrichs model but now in the Liouville space. It can easily be shown that the dynamics (2.5) associated with the Liouville operator can be expressed in terms of a "flow of correlations". Consider, for example, an  $N$ -body system such as studied in kinetic theory. Collisions between uncorrelated particles ("vacuum of correlations") lead to two-body correlations, subsequent collisions transfer them into 3-body, 4-body ... correlations. Our rule is then: transitions to the higher-order correlations are future-oriented, while transitions to lower-order correlations are past-oriented. As shown elsewhere,<sup>14)</sup> Poincaré's divergences are eliminated and we obtain well-defined expressions for the projection operators  $\Pi^{(\nu)}$  (we have called subdynamics this approach). In this way the density matrix  $\rho$  is decomposed in a sum of independent contributions.

$$\rho(t) = \sum_{\nu} \Pi^{(\nu)} \rho(t) = \sum_{\nu} (P + C) \epsilon^{-i\theta t} A (P + D) \rho(0). \quad (2.9)$$

We shall not go into the definition of these quantities which are given elsewhere.<sup>8)9)14)</sup> Note only that  $\theta$  and  $A$  are diagonal on  $\nu$ , while  $C$  corresponds to the creation of correlations  $\nu'$  out of  $\nu$  (its matrix elements are of the form  $C_{\nu\nu'}^{(\nu)}$ ), while  $D$  corresponds to the destruction of correlation  $\nu'$  (its matrix elements are  $D_{\nu\nu'}^{(\nu)}$ ).

Each component  $\overset{(\nu)}{H} \rho$  is a particular solution of the Liouville equation. (This can be verified by straightforward derivation). The sum (2.9) provides us with a *complete* set of solutions.

The whole time-dependence in (2.9) is in the generators of motion  $\overset{(\nu)}{\theta}$ , which we call the "collision operators". Of special importance is the contribution for  $\nu = 0$ , the "vacuum of correlation":  $\overset{(\alpha)}{H} \rightarrow \overset{(\alpha)}{P}$  for  $\lambda \rightarrow 0$  and corresponds then to the evolution of the diagonal element of  $\rho$ . The usual kinetic description (e.g. Fokker-Planck equation or Pauli master equation) is limited to  $\overset{(\alpha)}{H}$  space. Moreover this space contains the asymptotic contribution to  $\rho$  for time  $t \rightarrow \infty$ .

Once we have derived the decomposition of  $\rho$  into subdynamics, it is easy to go one step further and to obtain the complex spectral representation of  $L_H$  with right eigenstate  $|\overset{(\nu)}{F}_\alpha\rangle$  and left eigenstates  $\langle\langle \overset{(\nu)}{F}_\alpha | \text{8)9)$

$$L_H |\overset{(\nu)}{F}_\alpha\rangle = Z_\alpha |\overset{(\nu)}{F}_\alpha\rangle, \quad \langle\langle \overset{(\nu)}{F}_\alpha | L_H = Z_\alpha \langle\langle \overset{(\nu)}{F}_\alpha |. \quad (2.10a)$$

and

$$\overset{(\nu)}{H} = \sum_\alpha |\overset{(\nu)}{F}_\alpha\rangle \langle\langle \overset{(\nu)}{F}_\alpha |. \quad (2.10b)$$

where  $|\rho\rangle\rangle$  denotes a "superstate" in the density matrix space.

As the result we have therefore

$$\rho(t) = e^{-iL_H t} \rho(0) = \sum_{\nu, \alpha} e^{-iZ_\alpha t} |\overset{(\nu)}{F}_\alpha\rangle \langle\langle \overset{(\nu)}{F}_\alpha | \rho(0)\rangle\rangle. \quad (2.11)$$

The index  $\alpha$  refers to possible degeneracy in each subspace  $\nu$ .

Note that the eigenstates  $|\overset{(\nu)}{F}_\alpha\rangle$  and  $\langle\langle \overset{(\nu)}{F}_\alpha |$  are now density matrices and not wave functions; moreover,  $\alpha$  is an index corresponding to possible degeneracy in subdynamics. Moreover, we can show that the eigenvalue  $Z_\alpha$  is just the same as the eigenvalue of the collision operator  $\overset{(\nu)}{\theta}$  in (2.9) which is in general complex number. <sup>8)9)</sup>

Formula (2.11) corresponds to our complex spectral theory in the Liouville space. As mentioned in section 1, it leads to an irreducible representation of the density matrix as the complex eigenvalue  $Z_\alpha$  and the eigenfunctions cannot be reduced to the expressions they would have for wave functions (then  $Z_\alpha$  would be the difference between two eigenvalues and  $|\overset{(\nu)}{F}_\alpha\rangle$  products of two wave functions). As in the case of the Friedrichs model, the eigenfunctions are complex distribution, and (2.11) has to be used with suitable test functions.

Our spectral representation (2.11) describes the approach to equilibrium, and give a microscopic meaning to entropy. <sup>8)9)</sup> It also disentangles various contributions known as "non-Markovian" effects. We shall however not go into these questions here <sup>13)</sup> and concentrate on the problem: What is the relation between our irreducible representation in terms of density matrices and the Schrödinger equation? To consider this problem, let us first go back to the simple example of scattering as described by the Hamiltonian (2.4).

### 3. Two-body scattering

Let us first consider two-body scattering. The Hamiltonian is given by (2.4). We make the usual assumption (short-range force  $V_{kk'} \sim O(L^{-3})$ ,  $V_{kk} = 0$ , where  $L^3$  is the volume of the system). The usual theory is based on the Lippman-Schwinger equation which lead to two complete sets of eigenfunctions  $|\Psi_\alpha^\pm\rangle$  which diagonalize the Hamiltonian<sup>11)</sup>,

$$|\Psi_\alpha^\pm\rangle = |\alpha\rangle + \frac{1}{\omega_\alpha - H_0 \pm i\epsilon} \lambda V |\Psi_\alpha^\pm\rangle. \quad (3.1)$$

We may follow the evolution of the wave function in the interaction representation. Using standard notation, we have ( $\Psi_I(t)$  is the wave function in the interaction representation)

$$|\Psi_I(t)\rangle = U_I(t, t_0) |\Psi_I(t_0)\rangle. \quad (3.2)$$

We can distinguish two cases. In the first case the initial wave function is a localized wave packet centered around  $k_0$ . We then obtain for the transition probability in the limit  $t \rightarrow +\infty, t_0 \rightarrow -\infty$

$$\lim_{\substack{t \rightarrow +\infty \\ t_0 \rightarrow -\infty}} |\langle k_1 | \Psi_I(t) \rangle|^2 = 2\pi |T_{k_1, k_0}^+(\omega_{k_1})|^2 \delta(\omega_{k_1} - \omega_{k_0}) t_c. \quad (3.3)$$

where  $t_c$  is the duration of the collision. As expected this expression is time-independent. This is the well-known result obtained by  $S$ -matrix theory. The application of our complex spectral representation (2.9) or (2.11) leads (fortunately!) to identical results.<sup>16)</sup>

We next consider initial conditions corresponding to a plane wave and leading therefore to persistent interaction. It is well-known that this can lead to difficulties.<sup>11)</sup> We now obtain (we take  $t_0 = 0$ )

$$\begin{aligned} \rho_{k_1 k_1}(t) = |\langle k_1 | \Psi_I(t) \rangle|^2 = & 4 |T_{k_1, k_0}^+(\omega_{k_1})|^2 \frac{\sin^2(\omega_{k_1} - \omega_{k_0})t/2}{(\omega_{k_1} - \omega_{k_0})^2} \\ & + (\text{asymptotically vanished oscillating contributions}) \end{aligned} \quad (3.4)$$

For  $t \rightarrow +\infty$  this expression is ill-defined. However, in conjunction with test functions we obtain the secular contribution (for  $t \rightarrow +\infty$ )

$$\rho_{k_1 k_1}(t) \rightarrow 2\pi t |T_{k_1, k_0}^+(\omega_{k_1})|^2 \delta(\omega_{k_1} - \omega_{k_0}) t. \quad (3.5)$$

This is in contrast with the off-diagonal elements of the density matrix which are time-independent (in conjunction with test functions). Note that the secular term in (3.5) comes from a resonant interference between a ket-state and bra-state. These resonances do not appear in the Schrödinger equation and cannot be associated to the Hilbert space. We have therefore the alternatives: either we stay with (3.4) and conclude that this problem has not a well-defined solution. But this is difficult to accept as nothing in quantum theory prevents us from taking the initial condition as close as we want to a plane wave.

Another way out is to generalize quantum mechanics to deal with the situations where the fundamental quantity is now an "irreducible" density matrix. That is the situation which we may associate with "quantum chaos" and irreducible representations in Liouville space.

We now compare the wave function formalism with our irreducible Liouville representation for persistent interaction. After a few elementary calculations we obtain for the diagonal elements of the density matrix (to lowest order in  $\lambda$  and for  $k_1 \neq k_0$ )<sup>16)</sup>

$$\rho_{k_1 k_1}^{(irr)} - |\langle k_1 | \Psi_I(t) \rangle|^2 = \pi \lambda^2 |V_{k_1, k_0}|^2 \left[ 2\delta(\omega_{k_1} - \omega_{k_0}) t + i\delta'(\omega_{k_1} - \omega_{k_0})(e^{-i(\omega_{k_1} - \omega_{k_0})t} - e^{i(\omega_{k_1} - \omega_{k_0})t}) \right]. \quad (3.6)$$

The difference is not identically zero. However in conjunction with integrations with test functions the right-hand side vanishes. Equation (1.1) is satisfied; average values of test functions are identical.

We see that our approach goes beyond standard quantum mechanics as it permits to retain secular terms and to decompose the time-dependent evolution into independent modes. This is the decisive step to formulate a complex spectral theory for the Liouville operator.

Let us now consider three-body scattering. The failure of standard quantum mechanics then becomes quite obvious.

#### 4. Three-body scattering

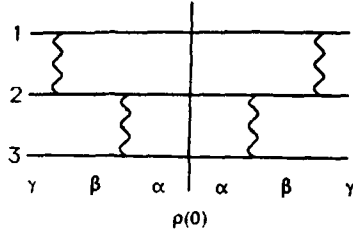
For the three-body scattering, the Hamiltonian is (cf. (2.11))

$$H = \sum_i \frac{k_i^2}{2m_i} + \lambda \sum_{i>j} V^{(ij)}. \quad (4.1)$$

As in the binary collision,  $V_{\alpha\beta} \sim O(L^{-3})$  and  $V_{\alpha\alpha} = 0$ , where  $|\alpha\rangle \equiv |k_1, k_2, k_3\rangle$ . For free incident particles the Lippman-Schwinger equation (3.1) leads here to divergences due to rescattering.<sup>11)</sup> For example, the contribution to the solution  $\langle \beta | \Psi_{\alpha}^{\pm} \rangle$  by any processes involving intermediate diagonal transitions  $|\alpha\rangle \leftarrow |\alpha\rangle$  diverges as  $\epsilon^{-1}(L^3)^{-2}\delta(\omega_{\alpha} - \omega_{\gamma})$  with  $\epsilon \rightarrow 0+$ . This is a typical form of the divergence due to resonances which lead to Poincaré's catastrophe.<sup>6)</sup> There is no time-independent description of the three-body scattering. There is no consistent  $S$ -matrix approach as there is no asymptotic free state for  $t \rightarrow +\infty$  as the result of the rescattering.

We therefore turn first to the time-dependent description in terms of wave functions. But here we come to another difficulty due to the appearance of ill-defined wave functions. As an illustration, we consider the second order transitions in  $\lambda$  from  $|\alpha\rangle$  to  $|\gamma\rangle$  through  $|\beta\rangle$ , where the interaction between (23) is followed by the interaction (12). The contribution to the density matrix is schematically given in Fig. 1.

Figure 1. Diagrammatic representation of a typical rescattering process.



This process leads to

$$|\langle \gamma | \overline{U_I(t, 0)} | \alpha \rangle|^2 = 4\lambda^4 |V_{\gamma\beta}|^2 |V_{\beta\alpha}|^2 \left[ \frac{1}{(\omega_\beta - \omega_\alpha)^2} \left( \frac{\sin(\omega_\gamma - \omega_\alpha)t/2}{(\omega_\gamma - \omega_\alpha)} - \frac{\sin(\omega_\gamma - \omega_\beta)t/2}{(\omega_\gamma - \omega_\beta)} \right)^2 + 4 \frac{\sin(\omega_\gamma - \omega_\alpha)t/2}{(\omega_\gamma - \omega_\alpha)} \frac{\sin(\omega_\gamma - \omega_\beta)t/2}{(\omega_\gamma - \omega_\beta)} \frac{\sin^2(\omega_\beta - \omega_\alpha)t/4}{(\omega_\beta - \omega_\alpha)^2} \right] \quad (4.2)$$

where the bar denotes the particular term we are looking at. Again in the asymptotic limit the right-hand sides of (4.2) is ill-defined. A simple replacement by  $\delta$ -functions as we did going from (3.4) to (3.5) leads to a divergence due to the square of the  $\delta$ -function. The evaluation of the dominant secular terms  $\sim t^2$  is already complicated. But in addition we need a precise evaluation of the secular terms  $\sim t$ , since they give the "genuine" three-body scattering transition rate. The situation is even worse as only a part of the secular term in  $t$  is related to the three-body cross-section (see (4.3)).

In contrast, our theory gives us an unambiguous expression for the density matrix. For the expression to (4.2) we have (see (2.9))

$$\begin{aligned} \overline{\varrho}_{\gamma\gamma}^{(irr)}(t) &= \frac{1}{2!} (-i\lambda^2 \theta_{2t}^{(\alpha)})^2 + (-i\lambda^2 \theta_{2t}^{(\alpha)}) \lambda^2 A_2^{(\alpha)} \\ &\quad + (-i\lambda^4 \theta_{4t}^{(\alpha)}) + \lambda^4 A_4^{(\alpha)} + \sum_{\nu(\neq 0)}^{(\nu)} \varrho_{\gamma\gamma}^{(\nu)}(t) \\ &= \lambda^4 |V_{\gamma\beta}|^2 |V_{\beta\alpha}|^2 \left[ (2\pi)^2 \delta(\omega_\beta - \omega_\gamma) \delta(\omega_\gamma - \omega_\alpha) \frac{t^2}{2} \right. \\ &\quad + 2\pi \delta(\omega_\beta - \omega_\gamma) \left( \frac{1}{(\omega_\gamma - \omega_\alpha - i\epsilon)^2} + c.c. \right) t \\ &\quad + 2\pi \delta(\omega_\gamma - \omega_\alpha) \left( \frac{1}{(\omega_\alpha - \omega_\beta - i\epsilon)^2} + c.c. \right) t \\ &\quad \left. - i \left( \frac{1}{i\epsilon - \omega_\beta + \omega_\gamma} \frac{1}{i\epsilon - \omega_\beta + \omega_\alpha} \frac{1}{i\epsilon - \omega_\gamma + \omega_\alpha} - c.c. \right) t \right] \\ &\quad + \lambda^4 A_4^{(\alpha)} + \sum_{\nu(\neq 0)}^{(\nu)} \varrho_{\gamma\gamma}^{(\nu)}(t), \end{aligned} \quad (4.3)$$

where all terms except for the last one contributions come from the  $\overset{(\circ)}{II}$  subspace. The last term comes from  $\overset{(\nu)}{II}$  subspace with  $\nu \neq 0$  and vanishes asymptotically. The first term  $\sim t^2$  in the square bracket of the right-hand side comes from the rescattering process. The next term comes from the contribution  $(-i\lambda^2 \overset{(\circ)}{\theta}_2 t)\lambda^2 \overset{(\circ)}{A}_2$ . The last two terms in the square bracket are the genuine three-body scattering terms which come from the contribution  $-i\lambda^4 \overset{(\circ)}{\theta}_4 t$ .

Note that it is not sufficient to determine the secular term in  $t$  as only a part from it contributes to the scattering cross-section. In addition to the process described in Fig. 1, we have other diagrams. They are also give finite contribution to the three-body cross-section. We shall indicate the total result in a separate paper.<sup>16)</sup>

Note that the three-body cross-section is not an "on-energy-shell" quantity. This is due to the fact that the initial state is not an eigenstate of the total Hamiltonian (the three-body problem comes in this sense closer to the bound state problem.)

Because of the great complexity of the time-dependent wave function, we could derive a formula such as (3.5). But we know from general arguments that the condition (1.1) is here also satisfied.

## 5. Concluding remarks

Dynamical instability as manifest in LPS forces us to proceed with an extension of quantum theory. Unstable dynamical systems are characterized by supplementary physical quantities, which are associated to the eigenfunctions and eigenvalues of our complex spectral theory.

These conclusions have wide-ranging consequences. The basic postulate of conventional field theory is the existence of a simple correspondence between "in" and "out" states through the  $S$ -matrix theory.<sup>12)</sup> This implies an idealization - which is not applicable to persistent interaction.

To come back to the problems we mentioned in the introduction, we see that our complex spectral theory incorporates irreversibility as it deals with a "larger space" (the rigged Hilbert space<sup>17)</sup>). Moreover, most of the epistemological problems are avoided. The consideration of instability and of quantum chaos eliminates the *dual* structure of conventional quantum mechanics. Bohr emphasized that we cannot describe a measurement device as a quantum object in terms of Schrödinger's equation.<sup>1)</sup> In our approach this means that measurement must involve unstable dynamical systems whose description is in terms of irreducible density matrices and therefore in terms of quantities which have a classical analogue. In short the two time description involving the Hilbert space (bra and ket) is replaced by a single time description as in the case in classical mechanics (where the Liouville operator is also irreducible). It is not astonishing that to speak about "measurement" we need a "classical" time ordering as provided by our theory. The consideration of LPS leads to a new formulation of quantum theory which form the point of view of its time structure is similar to classical dynamics while still keeping quantum effects. We therefore conclude that our approach is in line with the general ideas put forward by the Copenhagen school.

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## The Geometric Phase in Quantum Physics

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Berry's phase has been fashionable in many areas of physics and in chemistry and also among mathematicians and mathematical physicists. The mathematical people are attracted to this area because it is related to the beautiful mathematics of fibre bundles which underlay gauge theories. In fact this is the most accessible example of a gauge theory for people who know just the elementary facts of nonrelativistic quantum mechanics.

The chemists and physicists are interested because the geometric or Berry phase has observable consequences, which could not be explained before. It is this aspect which distinguishes the Berry phase from the many other fashions of mathematical physics. Though the Berry phase may turn out not to be as important as its present popularity suggests, it is a discovery which will remain forever. The surprising thing about it is that its importance has been realized 60 years too late. The reason for this was that many people (including myself) thought that phases are unimportant in quantum mechanics, because a quantum mechanical state is not described by a vector  $\psi$  but by a ray or a projection operator  $|\psi\rangle\langle\psi|$  and that phase factors could always be removed by a suitable phase or gauge transformation.

That this is not always possible and under which conditions this is not possible was shown by Berry in his famous 1984 paper.<sup>1)</sup> And the Berry phase fashion started when Simon<sup>2)</sup> explained that Berry's phase is the holonomy (element of the holonomy group) for a fibre bundle with a particular connection, the adiabatic connection.

But the physical effect of the Berry phase had been known for quite some time. It was observed as some anomalies in the spectra of molecules<sup>3)</sup> and then, (1978), explained in a series of remarkable papers by C.A. Mead and Truhlar<sup>4)</sup> by the introduction of a gauge potential, which is identical with the one derived by Berry and which is now called Berry connection.<sup>5)</sup> This gauge potential emerges naturally from the Born-Oppenheimer procedure in molecular physics<sup>6)</sup> if one does not make the drastic Born-Oppenheimer approximation.<sup>7)</sup>

The Born-Oppenheimer method is concerned with the study of complicated molecules by dividing them into two parts: the electronic motion described by a set of "fast" variables, and the collective motion described by a set of "slow" variables. Berry connection and Berry phase, therefore, arise in the dissection of complicated quantum physical systems into simpler subsystems. This reduction to the simpler is

the basic meaning of understanding in science. In the old, drastic approximation the dissection results in the trivial direct product of the states for the two subsystems. In the less-drastic, adiabatic approximation or in the exact theory, the motion of one subsystem (the "fast" subsystem) alters the dynamics of the "slow" subsystem by inducing in it a Berry gauge potential. Thus, the parts of a complicated quantum physical system turn out to be different from what we naively expected. Our discussions here will be mainly concerned with these aspects of the Berry connection.

The Hamiltonian for a (diatomic) molecule is,

$$H = \frac{\mathbf{P}^2}{2\mu} + \frac{\hat{\mathbf{p}}^2}{2m} + V(\mathbf{X}, \hat{\mathbf{r}}) \quad (1)$$

where  $\hat{\mathbf{p}}$ ,  $\hat{\mathbf{r}}$  stand for the observables of the fast electrons and  $\mathbf{P}$ ,  $\mathbf{X}$  stand for the observables of the slow nuclei. Since the light electrons instantaneously follow the motion of the heavy nuclei, the slow variables can also be understood as being the variables of the molecule as a whole, i.e. the collective variables. In particular, for the diatomic molecule  $\mathbf{X}$  will be the vector along the internuclear axis and  $\mathbf{P}$  its conjugate momentum (in addition there are the center of mass position and momenta which are, as always in a non-relativistic theory, ignored). The potential  $V(\mathbf{X}, \hat{\mathbf{r}})$  is a complicated function of the operators  $\mathbf{X}, \hat{\mathbf{r}}$  and possibly some other operators like spin.

The Hamiltonian  $H$  of (1) is split into two parts,

$$H = \frac{\mathbf{P}^2}{2\mu} + h(\mathbf{X}) \quad (2)$$

$$h(\mathbf{X}) = \frac{\hat{\mathbf{p}}^2}{2m} + V(\mathbf{X}, \hat{\mathbf{r}}) \quad (3)$$

where  $h(\mathbf{X})$  denotes the "fast" or electronic Hamiltonian that depends upon the "slow" operator  $\mathbf{X}$ . The eigenvalue problem,

$$H | \psi^E \rangle = E | \psi^E \rangle \quad (4)$$

is solved in the Born-Oppenheimer procedure by first solving the eigenvalue problem for the operator  $h(\mathbf{X})$ . In the drastic Born-Oppenheimer approximation  $\mathbf{X}$  is considered as a classical parameter  $\mathbf{x}$  which is fixed. With  $\mathbf{X} = \mathbf{x} = \text{fixed}$ ,  $h(\mathbf{X})$  commutes with  $\mathbf{P}$  and thus  $h(\mathbf{X})$  and  $H$  can be diagonalized together (which amounts to ignoring the effect of the kinetic energy of the slow variables in (1)):

$$| N, n; \mathbf{x} \rangle = | N \rangle \odot | n(\mathbf{x}) \rangle \quad (5)$$

$$h(\mathbf{x}) | n(\mathbf{x}) \rangle = \varepsilon_n(x) | n(\mathbf{x}) \rangle \quad (6)$$

$$H | N, n; \mathbf{x} \rangle = \left( \frac{\mathbf{P}^2}{2\mu} + \varepsilon_n(x) \right) | N, n; \mathbf{x} \rangle = E_{N,n} | N, n; \mathbf{x} \rangle \quad (7)$$

One first solves (6) for every value of the *fixed* parameter  $\mathbf{x}$ , and obtains the electronic energy values  $\varepsilon_n(x_e)$  where  $x_e$  is the minimum (equilibrium) value of the "potential

curve"  $\varepsilon_n(x)$ . The eigenvectors  $\{|n(\mathbf{x})\} \{n = 1, 2, \dots\}$  for every value of  $\mathbf{x}$  form a complete system of basis vectors for the space of physical states  $\mathcal{H}^{fast}$  for the fast

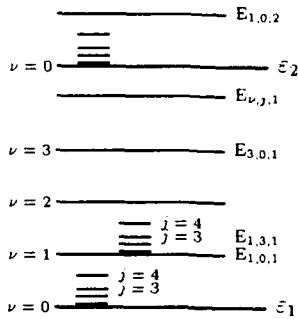


Fig.1 Schematics of typical molecular spectra

subsystem. After  $\varepsilon_n(x)$  has been obtained from (6) for every value of the fixed parameter  $\mathbf{x}$ , one inserts it into the right-hand side of (7) as an "induced scalar potential" and solves (7) for a given value of the electronic quantum number  $n$ . As  $\varepsilon_n(x)$  is (often) approximately an oscillator potential,  $\varepsilon_n(x_e)$  splits into vibrational excitations with quantum number  $\nu$ . And as the diatomic molecule (dumbbell) also rotates about its center of mass, each vibrational excitation splits into rotational bands with quantum number  $j$ . The collective quantum numbers  $N$  are thus the vibrational quantum number  $\nu$  and the angular momentum  $j$ :  $N = \nu, j$ ; and one obtains the typical spectrum of molecules, Fig. 1.

The time evolution of the fast system is described by the Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = h(X) |\psi(t)\rangle \quad (8)$$

and if initially the state vector is an electronic energy eigenstate,

$$\psi(0) = |n(\mathbf{x})\rangle, \quad (9)$$

then the solution of (8) is

$$\psi(t) = e^{-\frac{i}{\hbar} \varepsilon_n(x)t} |n(\mathbf{x})\rangle = e^{-\frac{i}{\hbar} \int_0^t dt' \varepsilon_n(x(t'))} |n(\mathbf{x})\rangle \quad (10)$$

iff  $\mathbf{x}$  = "fixed" parameter.

We will now consider the less drastic, adiabatic approximation.<sup>8)</sup> If the internuclear distance and direction  $\mathbf{X}$  is considered a classical parameter  $\mathbf{x}(t)$  which changes slowly in time (fast quantum system in a slowly changing classical environment) then an initial eigenstate of  $h(\mathbf{X}(t))$ ;  $\psi(0) = |n(\mathbf{x}(0))\rangle$  may "jump" into a state which also has different electronic quantum numbers  $n' \neq n$ . The adiabatic approximation is an evolution in which  $\mathbf{x}(t)$  changes so slowly that an eigenstate of  $h(\mathbf{X}(t))$  always remains in the same eigenstate.<sup>9)</sup>

$$|\psi(t)\rangle \langle \psi(t)| = |n(\mathbf{x}(t))\rangle \langle n(\mathbf{x}(t))| \quad (11)$$

The solution of (8) for an initial eigenstate (9) with time-dependent  $\mathbf{x}(t)$  is then given by

$$\psi(t) = e^{-\frac{i}{\hbar} \int_0^t dt' \varepsilon_n(x(t'))} e^{i\gamma_n(t)} |n(\mathbf{x}(t))\rangle \quad (12)$$

In addition to the dynamical phase factor of (10), there appears another phase factor  $e^{i\gamma_n(t)}$ . This phase factor had always been omitted in the old adiabatic approximation because it was believed that it can always be absorbed into the eigenvector  $|n(\mathbf{x})\rangle$  by a phase (gauge) transformation:

$$|n(\mathbf{x})\rangle \rightarrow |n(\mathbf{x})\rangle' = e^{i\zeta_n(\mathbf{x})} |n(\mathbf{x})\rangle \quad (13)$$

where  $|n(\mathbf{x})\rangle'$  is again a normalized eigenvector in (6).

For cyclic time evolution

$$\mathcal{C} : \mathbf{x}(0) \rightarrow \mathbf{x}(t) \rightarrow \mathbf{x}(T) = \mathbf{x}(0), \quad (14)$$

when the internuclear axis returns to its original position after a period  $T$ , the solution of the Schrödinger equation (8) for the vector of the state  $|\psi(T)\rangle$  ( $\langle\psi(T)| = \langle n(\mathbf{x}(T))|$ ) is

$$|\psi(T)\rangle = e^{-\frac{i}{\hbar} \int_0^T dt' \epsilon_n(\mathbf{x}(t'))} e^{i\gamma_n(T)} |n(\mathbf{x}(0))\rangle \quad (15)$$

where

$$\gamma_n(T) = \int_{\mathcal{C}} d\mathbf{x} \mathbf{A}_n(\mathbf{x}) = \int_S \int d\mathbf{S} \cdot \mathbf{B}_n \quad (16)$$

with

$$\mathbf{A}_n \equiv i \langle n(\mathbf{x}) | \nabla_{\mathbf{x}} | n(\mathbf{x}) \rangle \quad ; \quad \mathbf{B}_n = \nabla_{\mathbf{x}} \wedge \mathbf{A}_n \quad (17)$$

and where  $\mathcal{C}$  is the closed path in the parameter space (14) and  $S$  is a surface spanned by  $\mathcal{C}$ . It is convenient and a standard convention to choose eigenvectors which are single valued functions of the parameter  $\mathbf{x}$  in the region that contains  $\mathcal{C}$ :

$$|n(\mathbf{x}(T))\rangle = |n(\mathbf{x}(0))\rangle \quad (18)$$

Under this convention  $\mathbf{A}_n(\mathbf{x})$  is called Berry connection or Berry gauge potential,  $\mathbf{B}_n$  is called Berry curvature and  $\gamma_n(T)$  is called the Berry phase angle.

It can be shown that  $e^{i\gamma_n(T)}$  is an invariant with respect to the transformation (13) (gauge invariant) which may be different from unity.<sup>1)</sup> It can therefore not be transformed away by (13).

An example of a system where the Berry phase  $\gamma_n(T)$  and Berry connection are non-trivial (*i.e.* not removable by a gauge transformation) is the quantum magnetic moment  $\mathbf{m} = -\frac{e}{2mc} g \mathbf{j}$  (of the electrons with spin  $\mathbf{j}$ ) in a slowly rotating magnetic field  $\mathbf{B}^{mag} = B \hat{\mathbf{X}}(t)$  (along the internuclear axis of the molecule caused by the rapidly orbiting electrons). The fast Hamiltonian for this case is

$$h(t) = h(\mathbf{X}(t)) = -\mathbf{m} \cdot \mathbf{B}^{mag}(t) = -\frac{e}{2mc} g B \hat{\mathbf{X}}(t) \cdot \mathbf{j} = b \hat{\mathbf{X}}(t) \cdot \mathbf{j} \quad (19)$$

The eigenvectors  $|n(\mathbf{x})\rangle = |k(x, \theta(t), \varphi(t))\rangle$  depend upon the polar coordinates ( $\theta(t)$ ,  $\varphi(t)$ ) of the unit vector  $\hat{\mathbf{X}}(t)$ , and  $k$  denotes the component of angular momentum of the fast system along the internuclear axis

$$\mathbf{x}(t) \cdot \mathbf{j} |k(\theta, \varphi)\rangle = k |k(\theta, \varphi)\rangle \quad ; \quad \varepsilon_k = \hbar b k \quad (20)$$

By a straightforward calculation using

$$|k(\theta, \varphi)\rangle = \epsilon^{-i\varphi J_3} \epsilon^{-i\theta J_2} \epsilon^{i\varphi J_3} |k(\theta = 0, \varphi = 0)\rangle \quad \text{for } \theta < \pi \quad (21)$$

and

$$|k(\theta, \varphi)\rangle' = \epsilon^{-i\varphi J_3} \epsilon^{-i\theta J_2} \epsilon^{-i\varphi J_3} |k(\theta = 0, \varphi = 0)\rangle \quad \text{for } \theta > 0 \quad (22)$$

one obtains for the Berry connection

$$\mathbf{A}^{k'k} = i\langle k'(\theta, \varphi) | \nabla | k(\theta, \varphi)\rangle = \mathbf{e}_r \hat{A}_r^{k'k} + \mathbf{e}_\theta \hat{A}_\theta^{k'k} + \mathbf{e}_\varphi \hat{A}_\varphi^{k'k} : \quad (23)$$

the following results

$$\mathbf{A}_r = 0 \quad \hat{A}_\theta^{k'k} = 0, \quad \hat{A}_\varphi^{k'k} = -\frac{\hbar(1 - \cos \theta)}{r \sin \theta} \quad \theta < \pi \quad (24)$$

$$\hat{A}_\varphi^{k'k} = \frac{k(1 + \cos \theta)}{r \sin \theta} \quad \theta > 0 \quad (24')$$

The two vectors  $|k(\theta, \varphi)\rangle$  and  $|k(\theta, \varphi)\rangle'$  (and the corresponding two vector potentials  $\mathbf{A}$  and  $\mathbf{A}'$ ) had to be chosen differently for the domain ( $\theta < \pi$ ) and the domain ( $\theta > 0$ ) so that they are single-valued in each domain.

The Berry curvature (17) calculated from (24) and (24') is

$$\mathbf{B}^k = -\frac{k}{r^2} \hat{\mathbf{X}} \quad (25)$$

And the Berry phase (16) calculated from (24) (or (24')) is given by the standard result

$$\gamma_k(\mathcal{C}) = -k(\text{solid angle subtended by } \mathcal{C}) = -k\Omega. \quad (26)$$

The result (25) is identical with the field strength of Dirac's magnetic monopoles<sup>10)</sup>  $\epsilon \mathbf{B} = \frac{e\hbar}{4\pi} \frac{\hat{\mathbf{X}}}{X^2}$ , except that the electromagnetic constant  $\frac{e\hbar}{4\pi}$  is replaced by the motion constant  $k$ , the component of angular momentum along the internuclear axis, which in this approximation is a fixed number. From this result we already suspect that something like a magnetic monopole must be part of the diatomic molecule (except if  $k = 0$ ). This motional or mechanical "monopole" will remain uncovered if one uses the drastic Born-Oppenheimer approximation.<sup>11)</sup>

If the eigenvalues of the fast Hamiltonian  $\varepsilon_n(x_e)$  are degenerate or close to each other (compared with the splitting between  $E_{N',n}$  and  $E_{N'',n}$ ) then the adiabatic approximation (11) is apparently not good and one may consider in place of the  $U(1)$  gauge transformation (13) an  $U(\mathcal{N})$  gauge transformation (where  $\mathcal{N}$  is the degeneracy of  $\varepsilon_n$ ) and obtain a non-abelian Berry connection<sup>12)</sup>  $\mathbf{A}^{mn}(\mathbf{x})$  in place of the abelian  $\mathbf{A}_n(\mathbf{x}) = \mathbf{A}^n(\mathbf{x})$  of (17). Alternatively one can treat the slow variables  $\mathbf{P}$  and  $\mathbf{X}$  as operators and solve the problem quantum mechanically by the Born-Oppenheimer method.<sup>13)</sup> Then these non-abelian Berry connections will emerge naturally. This we will discuss now:

The space of physical states is, according to the basic principles of quantum mechanics, the direct product of the space for the fast motion  $\mathcal{H}^{fast}$  and the space

for the slow motion  $\mathcal{H}^{slow}: \mathcal{H} = \mathcal{H}^{slow} \otimes \mathcal{H}^{fast}$ . The slow variables  $\mathbf{P}$  and  $\mathbf{X}$  are the operators  $\mathbf{P} = \mathbf{P} \otimes 1^{fast}$  and  $\mathbf{X} = \mathbf{X} \otimes 1^{fast}$  and the operator  $h(\mathbf{X}) = h(\hat{\mathbf{p}}, \hat{\mathbf{r}}; \mathbf{X})$  acts in both factors of  $\mathcal{H}$ . As basis of  $\mathcal{H}^{fast}$  one takes the  $|n(\mathbf{x})\rangle$ , as basis of  $\mathcal{H}^{slow}$  one takes a basis of generalized eigenvectors of  $\mathbf{X}$  ( $|x \dots\rangle$ ) and the basis system of  $\mathcal{H}$  is something like the direct product basis  $|x \dots n\rangle = |x \dots\rangle \otimes |n(\mathbf{x})\rangle$ .

A straightforward<sup>6,13)</sup> but lengthy calculation shows that the effective Hamiltonian  $H^{eff}$  for the slow motion in  $\mathcal{H}^{slow}$  is not given as in the drastic approximation by (7) but by

$$H^{eff} = \frac{1}{2\mu} \mathbf{\Pi}^2 + \varepsilon_n(X) \quad (27)$$

where

$$\mathbf{\Pi}^{mn}(\mathbf{x}) = \mathbf{P} \delta^{mn} - \mathbf{A}^{mn}(\mathbf{X}) \quad (28)$$

and where

$$\mathbf{A}^{mn}(\mathbf{x}) = i \langle m(\mathbf{x}) | \nabla | n(\mathbf{x}) \rangle = (\mathbf{A}^{mn})^\dagger \quad (29)$$

$H^{eff}$ ,  $\mathbf{\Pi}$  and  $\mathbf{A}$  are  $\mathcal{N} \times \mathcal{N}$  matrices and  $\mathbf{\Pi}^2$  in (27) means matrix multiplication  $\sum_m \mathbf{\Pi}^{n'm} \cdot \mathbf{\Pi}^{mn}$ . Thus the fast motion induces in the dynamics of the slow motion not only a scalar potential  $\varepsilon_n(X)$  as in the drastic approximation (7) but also a non-abelian vector potential  $\mathbf{A}^{mn}(\mathbf{X})$  and in place of the canonical momentum  $\mathbf{P}$  of (7) one has the gauge-covariant momentum (28).

If one makes no approximation then  $m, n$  in (29) range over all values of the electronic quantum numbers ( $\mathcal{N} = \infty$ ) and the Berry connection  $\mathbf{A}^{mn}$  is an infinite matrix. But this  $\mathbf{A}^{mn}$  is trivial (can be gauged away by a  $U(\infty)$  gauge transformation). The eigenvalue equation of  $H^{eff}$  is then an infinite system of coupled differential equations, which is useless for practical calculations. One obtains a workable eigenvalue equation only if one can restrict oneself to a small number  $\mathcal{N}$  of eigenvalues  $\varepsilon_n$  and eigenvectors  $|n(\mathbf{x})\rangle$   $n = 1, 2, \dots, \mathcal{N}$  (Born-Huang approximation).<sup>14)</sup>  $\mathbf{A}^{mn}(\mathbf{x})$  of (29) is then a connection of a  $U(\mathcal{N})$  gauge theory, which is in general non-trivial.

As a special case we consider the doubly degenerate  $\Lambda$ -levels of a diatomic molecule for which  $\mathcal{N} = 2$  and  $m, n$  takes the two values  $k = \pm\Lambda$  where  $k$  is again the component of angular momentum along the internuclear axis and the  $|n(\mathbf{x})\rangle$  in (29) are given by (21) (or (22)).

The space of physical states of the slow system is

$$\mathcal{H}^{slow} = \mathcal{H}^{k=1} \oplus \mathcal{H}^{k=-1} \quad (30)$$

and (27), (28), (29) are  $2 \times 2$  operator matrices:

$$H^{k'k} = \frac{1}{2\mu} \sum_{k''=-1}^{+1} \mathbf{\Pi}^{k'k''} \cdot \mathbf{\Pi}^{k''k} + \varepsilon(x) \quad (31)$$

$$\mathbf{\Pi}^{k'k} = \delta^{k'k} \mathbf{P} - \mathbf{A}^{k'k} \quad (32)$$

By a straightforward calculation one obtains as in (24) for the spherical components of the Berry connection  $\mathbf{A}^{k'k}$ :

$$A_\theta^{k'k} = \frac{\Lambda}{x \sin \theta} \begin{pmatrix} -(1 - \cos \theta) & 0 \\ 0 & (1 - \cos \theta) \end{pmatrix} ; \quad A_\varphi^{k'k} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} . \quad (33)$$

The rapidly orbiting and spinning motion of the electrons about the internuclear axis of a diatomic molecule thus leads to an induced vector potential in the dynamics of the slow (collective) motion, which is the same as that of a pair of magnetic monopoles with the monopole strength  $g$  given by  $\frac{eg}{4\pi} = \pm\Lambda$ . The components of the gauge covariant momentum operator do no more commute but fulfill the commutation relation

$$[\Pi_i^{k'k''} , \Pi_j^{k''k}] = -i\varepsilon_{ij\ell} \frac{X_\ell}{X^3} \Lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = iB_{ij}^{k'k} , \quad (34)$$

where  $B_{ij}^{k'k} = \varepsilon_{ij\ell} B_\ell^{k'k}$ , is the Berry curvature (17) for the  $\mathbf{A}^{k'k}$  given by (33).

The commutation relations (34) are the well known *c.r.* for a charge-monopole system and the first term in the Hamiltonian (31) is the monopole Hamiltonian with free radial motion,  $H_{monopole} = \frac{1}{2\mu} \Pi^2$ . Due to the induced scalar potential, which is approximately a radial oscillator potential  $\varepsilon(x) \approx \frac{1}{2}(x - x_e)^2$ , the radial motion of (31) is not free. To dissect the system described by (31) into a radial part and an angular part, we use the radial momentum operator<sup>15)</sup>

$$P_{rad} = \frac{1}{2} \left\{ \frac{X_i}{X} , P_i \right\} \quad (35)$$

and the angular momentum operator of a monopole:<sup>16)</sup>

$$J_i = \varepsilon_{ij\ell} X_j \Pi_\ell - X_i \frac{1}{2} \varepsilon_{mn\ell} X_m B_{n\ell} = \varepsilon_{ij\ell} X_j \Pi_\ell + k \frac{X_i}{X} . \quad (36)$$

Then we obtain after a straightforward calculation for  $H$  of (31):

$$H = \frac{1}{2\mu} \tilde{\Pi}^2 + \varepsilon(X) = \underbrace{\frac{1}{2\mu X^2} (\mathbf{J}^2 - k^2)} + \underbrace{\frac{1}{2\mu} P_{rad}^2 + \varepsilon(X)}$$

or

$$H = H_{monopole} + \varepsilon(X) = H_{rot} + H_{radial\ oscillator} . \quad (37)$$

$H_{rot}$  is the Hamiltonian of a rotating dumbbell with flywheel on its axis<sup>17)</sup> whose doubly degenerate rotator spectrum  $\frac{1}{2\mu x^2} j(j+1)$  starts at  $j = k$ . Thus the spectrum is again something like shown in Fig. 1 except that the rotational levels do not start at  $j = 0$  but, as observed for diatomic molecules, at  $j = k$  (because from (36) follows that  $\frac{1}{X} X_i J_i = k$ ). Therewith we have obtained the standard result in a way, which shows that it is caused by the monopole dynamics induced by the fast motion in the slow collective motion.

Although we used the diatomic molecule as an example, the same arguments should hold for all kinds of quantum systems in which the fast subsystem is a rapid



rotation about a slowly moving axis like a spinning quark about the axis of a thin flux tube.

Fifty years after Dirac conceived the idea of magnetic monopoles he wrote (in a letter to Abdus Salam): "I am inclined now to believe that monopoles do not exist." Though magnetic monopoles of the electromagnetic kind may not exist, physical systems with the same dynamics as that of monopoles do exist. These monopoles are "parts" of complicated physical systems, like the "part" which performs the collective motion of molecules or the collective motion of a flux tube.

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## WIGNER FUNCTIONS AND LORENTZ TRANSFORMATIONS

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In the Wigner phase-space picture of quantum mechanics, the uncertainty relation is stated in terms of the minimal finite volume in phase space. The relation remains invariant under volume-preserving transformations. For a single pair of position and momentum variables, the volume is an area element in two-dimensional phase space. It is shown that the Lorentz transformation in four-dimensional space-time is a volume preserving transformation in eight-dimensional phase space. This is illustrated in terms of the covariant harmonic oscillator formalism, which is an effective language for the basic phenomenological features of relativistic hadrons.

### 1. Introduction

There are several different representations of quantum mechanics. The Schrödinger picture is convenient for solving problems in atomic and nuclear physics. The Heisenberg and interaction pictures are useful in quantum field theory. The Wigner phase-space picture [1] serves useful purposes in many branches of physics. In particular, it gives a detailed description of the uncertainty relation [2].

As the wave function plays the central role in the Schrödinger picture, the phase-space distribution function is the starting point in the phase-space picture of quantum mechanics. This distribution function is widely known as the Wigner function. The Wigner function is constructed from the Schrödinger wave function through the density matrix, and is a function of both position and momentum variables. Since, in quantum mechanics, it is not possible to determine the position and momentum variables simultaneously, it is not possible to define a point in the coordinate system of position and momentum. We have to settle with an uncertainty region in phase space [2]. The size of this region determines the amount of uncertainty. In the case of the two-dimensional space consisting of one pair of position and momentum variables, the size of the region is measured by its area.

In this note, we are interested in the question of whether the size of the uncertainty region remains invariant under Lorentz transformations. For this purpose, we note first that the Wigner function takes a very simple form for the ground-state harmonic oscillator wave function which is the traditional instrument for quantifying the uncertainty relations. The question then is whether it is possible to perform Lorentz transformations of the ground-state harmonic oscillator wave function.

The mathematics of Lorentz transformations is the Lorentz group [3]. The shortest and safest way to Lorentz-transform the oscillator wave function is to construct a representation of the Lorentz group using the harmonic oscillators. Indeed, it was Dirac [4]

who attempted to construct a set of normalizable harmonic oscillator wave functions in four-dimensional space-time, which we call the covariant harmonic oscillator formalism.

Thus the mathematics for the present problem is straight-forward. The question then is what physics we expect to learn from this formalism. In Sec. 2, we start with the Wigner function for the ground-state oscillator wave function. In Sec. 3, the physical interpretation is given for Wigner's little-group representation of the Poincaré group which governs the internal space-time symmetries of relativistic particles. In Sec. 4, the observable effects of the covariant oscillator formalism is discussed. In Sec. 5, we discuss the uncertainty-preserving deformation of the Wigner function under Lorentz boosts.

## 2. Wigner Functions and Harmonic Oscillators

If  $\psi(x)$  is the Schrödinger wave function in the one-dimensional world, the Wigner function is defined as

$$W(x, p) = \frac{1}{\pi} \int e^{2ipy} \psi^*(x+y) \psi(x-y) dy. \quad (1)$$

This form and its properties have been widely discussed in the literature [2]. In the Wigner phase-space picture of quantum mechanics, both the position and momentum variables are  $c$ -numbers. On the other hand, it is not possible to define a point in phase space. The uncertainty relation is stated in terms of the minimum area element in phase space. The quantum probability distribution in the  $x$  position space can be recovered from the Wigner function through

$$\rho(x) = \int W(x, p) dp. \quad (2)$$

Consequently, the Wigner function is normalized as

$$\int W(x, p) dx dp = 1. \quad (3)$$

Indeed, the Wigner function has many interesting properties, including its relation to the density matrix and applications to statistical mechanics. They have been extensively discussed in the literature. The generalization to higher dimensions is straightforward. If the wave function is separable in the Cartesian coordinate variables, the Wigner function is also separable.

For the ground state of the harmonic oscillator with unit frequency, the wave function is

$$\psi(x) = \left(\frac{1}{\pi}\right)^{1/4} \exp(-x^2/2), \quad (4)$$

and the Wigner function takes the form [2]

$$W(x, p) = \left(\frac{1}{\pi}\right) \exp\{-(x^2 + p^2)\}. \quad (5)$$

Since the ground-state wave function is separable in the Cartesian coordinate system, the generalization of the above form to the three-dimensional space is trivial. For the two-dimensional harmonic oscillator, the wave function is

$$\psi(x_1)\psi(x_2) = \left(\frac{1}{\pi}\right)^{1/2} \exp\{-(x_1^2 + x_2^2)\}. \quad (6)$$

and the Wigner function is

$$W(x_1, p_1)W(x_2, p_2) = \left(\frac{1}{\pi}\right) \exp\{-(x_1^2 + x_2^2 + p_1^2 + p_2^2)\}, \quad (7)$$

which is separable.

In Sec. 4 and Sec. 5, we shall use this two-dimensional oscillator formalism to study Lorentz transformation properties of relativistic extended hadrons. During the process of Lorentz boost, the transverse components are not affected. For the ground-state oscillator wave function, these components can be separated from the longitudinal component. Thus, we only have to consider the longitudinal and time-like components. This is how the problem becomes that of the two-dimensional harmonic oscillator.

Furthermore, we are interested in maintaining the separability of the coordinate system in order to establish the connection between the wave function and the Wigner function. As we shall see in Sec. 4 and Sec. 5, Dirac's light cone-coordinate system [5] is very useful for this purpose.

### 3. Wigner's Little Groups of the Poincaré Group

The internal space-time symmetries of elementary particles are governed by the little groups of the Poincaré group [3,6]. The little groups for massive and massless particles are locally isomorphic to  $O(3)$  and  $E(2)$  (two-dimensional Euclidean group) respectively. The  $E(2)$ -like little group for massless particles is an infinite-momentum/zero-mass limit of the  $O(3)$ -like little group [2,6,7]. The role of the little groups is illustrated in the second row of Fig. 1.

Wigner's little group is the maximal subgroup of the Lorentz group whose transformations leave the four-momentum of a given particle invariant [3]. For a massive point particle, there is a Lorentz frame in which the particle is at rest. In this frame, the little group is the three-dimensional rotation group. The internal space-time symmetry of massless particles is governed by the cylindrical group which is locally isomorphic to  $E(2)$  [7]. In this case, we can visualize a circular cylinder whose axis is parallel to the momentum. On the surface of this cylinder, we can rotate a point around the axis or translate along the direction of the axis. The rotational degree of freedom is associated with the helicity, while the translation corresponds to a gauge transformation in the case of photons [7].

This translational degree of freedom is shared by all massless particles, including neutrinos and gravitons [6]. Indeed, the requirement of invariance under this symmetry leads to the polarization of neutrinos [6,8]. Since this translational degree of freedom is a gauge degree of freedom for photons, we can extend the concept of gauge transformations to all massless particles [6] and massive particles in the infinite-momentum limit [9].

The group of Lorentz transformations is generated by three rotation and three boost generators. These generators are readily available in the literature. If  $J_i$  and  $K_i$  are the generators of rotations and boosts,  $J_3$  and  $K_3$  take the form

$$J_3 = -i \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right), \quad K_3 = -i \left( t \frac{\partial}{\partial z} + z \frac{\partial}{\partial t} \right), \quad (8)$$

applicable to functions of  $x, y, z$  and  $t$  which are the coordinate variables for the internal space-time degrees of freedom.

The  $O(3)$ -like little group for a particle at rest is generated by  $J_1, J_2$ , and  $J_3$ . If the particle is boosted along the  $z$  direction with the boost operator  $B(\eta) = \exp(-i\eta K_3)$ , the little group is generated by

$$J'_i = B(\eta) J_i B(-\eta). \quad (9)$$

Since  $J_3$  commutes with  $K_3$ ,  $J_3$  remains invariant under this boost.  $J'_1$  and  $J'_2$  take the form

$$\begin{aligned} J'_1 &= (\cosh \eta) J_1 + (\sinh \eta) K_2, \\ J'_2 &= (\cosh \eta) J_2 - (\sinh \eta) K_1. \end{aligned} \quad (10)$$

For large values of  $\eta$ , we can consider  $N_1$  and  $N_2$  defined as

$$N_1 = -(\cosh \eta)^{-1} J'_2, \quad N_2 = (\cosh \eta)^{-1} J'_1. \quad (11)$$

respectively. Then, in the infinite- $\eta$  limit [6,10],

$$N_1 = K_1 - J_2, \quad N_2 = K_2 + J_1. \quad (12)$$

These operators satisfy the commutation relations:

$$[J_3, N_1] = iN_2, \quad [J_3, N_2] = -iN_1, \quad [N_1, N_2] = 0. \quad (13)$$

$N_1$  and  $N_2$  are the generators of the  $E(2)$ -like little group for massless particles [8]. For normalizable functions of the space-time variables, the little group is generated by  $J_3$  of Eq.(8), and by [9]

$$\begin{aligned} N_1 &= -i \left\{ x \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial z} \right) - (z - t) \frac{\partial}{\partial x} \right\}, \\ N_2 &= -i \left\{ y \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial z} \right) - (z - t) \frac{\partial}{\partial y} \right\}. \end{aligned} \quad (14)$$

If Lorentz boosts are made along the  $z$  direction, it is more convenient to use the light-cone coordinate system [5], in which  $z$  and  $t$  variables are replaced by  $u$  and  $v$ . These are

$$u = (t + z)/\sqrt{2}, \quad v = (z - t)/\sqrt{2}. \quad (15)$$

When boosted along the  $z$  axis,  $u$  and  $v$  are multiplied by  $e^\eta$  and  $e^{-\eta}$  respectively. As we shall see in Sec. 4, the separability of the ground-state oscillator wave function is maintained in this system. In terms of these light-cone variables,

$$\begin{aligned} N_1 &= -(i/\sqrt{2}) \left\{ x \frac{\partial}{\partial u} - v \frac{\partial}{\partial x} \right\}, \\ N_2 &= -(i/\sqrt{2}) \left\{ y \frac{\partial}{\partial u} - v \frac{\partial}{\partial y} \right\}. \end{aligned} \quad (16)$$

When  $\eta$  is very large, the  $v$  variable may be dropped when the operators are applied to the function with a narrow  $v$  distribution. Consequently, except for the factor  $\sqrt{2}$ , the  $N_1$  and  $N_2$  operators may be written as

$$N_1 = -ix \frac{\partial}{\partial u}, \quad N_2 = -iy \frac{\partial}{\partial u}. \quad (17)$$

We shall discuss in the following section a possible physical application of this limiting process.

|                         |                      |                                |                       |
|-------------------------|----------------------|--------------------------------|-----------------------|
|                         | Massive<br>Slow      | ← between →                    | Massless<br>Fast      |
| Energy<br>Momentum      | $E = \frac{p^2}{2m}$ | $E = \sqrt{m^2 + p^2}$         | $E = p$               |
| Spin, Gauge<br>Helicity | $S_3$<br>$S_1$ $S_2$ | ← Little Groups →              | $S_3$<br>Gauge Trans. |
| Quarks<br>Partons       | Quark Model          | ← (Covariant<br>Phase Space) → | Parton Model          |

FIG. 1. Slow and fast particles. Einstein's  $E = (P^2 + m^2)^{1/2}$  unifies the energy-momentum relations for massive (nonrelativistic) particles and for massless particles. The second row indicates that the little group of the Poincaré group unifies the internal space-time symmetries of massive and massless particles. The third row states that the covariant phase-space picture of quantum mechanics forms the physical basis for the covariant harmonic oscillator formalism which has been shown to give a unified picture of the quark model and the parton picture.

#### 4. Covariant Harmonic Oscillator Formalism

It is not difficult to associate the symmetry of a point particle with that of a composite particle if they are massive and at rest, because both of them are governed by the three-dimensional rotation group. The story is quite different for rapidly moving

composite particles or hadrons. Does a rapidly moving hadron have the same set of space-time degrees of freedom as that of photons? We can study this problem by constructing a cylindrical symmetry for a hadron with infinite momentum [9]. Then, is this symmetry consistent with Feynman's parton picture [11]? We shall study this problem using the covariant harmonic oscillator formalism [2,6,12,13].

This oscillator formalism has been shown to be effective in describing many of the basic phenomenological properties of relativistic hadrons, including hadronic mass spectra, form factors and the parton phenomenon [2,6]. It also is a representation of the Poincaré group for relativistic composite particles [6].

If the space-time position of two quarks bound together inside a hadron are specified by  $x_a$  and  $x_b$ , the system can be described the variables [13]:

$$X = (x_a + x_b)/2, \quad x = (x_a - x_b)/2\sqrt{2}. \quad (18)$$

The four-vector  $X$  specifies where the hadron is located in space and time, while the variable  $x$  measures the space-time separation between the quarks. As for the four-momenta of the quarks  $p_a$  and  $p_b$ , we can combine them into the total four-momentum and momentum-energy separation between the quarks [13]:

$$P = p_a + p_b, \quad q = \sqrt{2}(p_a - p_b), \quad (19)$$

where  $P$  is the hadronic four-momentum conjugate to  $X$ . The internal momentum-energy separation  $q$  is conjugate to  $x$ .

The covariant oscillator wave functions are Hermite polynomials multiplied by a Gaussian factor [2,4,6], which dictates the localization property of the wave function. The Gaussian factor takes the form [4]

$$\exp\left\{-\frac{\Omega}{2}(x^2 + y^2 + z^2 + t^2)\right\}. \quad (20)$$

This expression is localized in the four-dimensional space-time. Since the  $x$  and  $y$  components are invariant under Lorentz boosts along the  $z$  direction, and since the oscillator wave functions are separable in the Cartesian coordinate system, the  $x$  and  $y$  variables can be dropped from the above expression, and they may be restored whenever necessary. The ground-state wave function can then be written as

$$v_\eta(z, t) = \left(\frac{\Omega}{\pi}\right)^{1/2} \exp\left\{-\left(\frac{\Omega}{2}\right)(e^{-2\eta}u^2 + e^{2\eta}v^2)\right\}. \quad (21)$$

Figure 2 illustrates the Lorentz-deformation property of this form.

As  $\eta$  becomes very large, the distribution in  $v$  becomes very narrow. Since  $v = (z - t)/\sqrt{2}$ , the terms containing  $(z - t)$  in Eq.(14) will produce a factor like  $v\delta(v)$  when applied to the Lorentz-deformed function of Eq.(21), and can therefore be dropped. The operators  $N_1$  and  $N_2$  then become those given in Eq.(17). Since they satisfy the commutation relations of Eq.(13),  $N_1$  and  $N_2$  of Eq.(17) together with  $J_3$  of Eq.(8) can be chosen for the generators of the little group for relativistic extended particles in the infinite-momentum limit.



The distribution becomes narrower in  $v$ , while it becomes wider along the  $u$  axis in the manner described for the oscillator case in Fig. 2. This property is shared by all other normalizable functions [9]. The generators of the little group do not depend on the shape of wave functions. Therefore, in the infinite-momentum limit, the above conclusion is valid for all distribution functions localized in space and time.

## QUARKS $\longrightarrow$ PARTONS

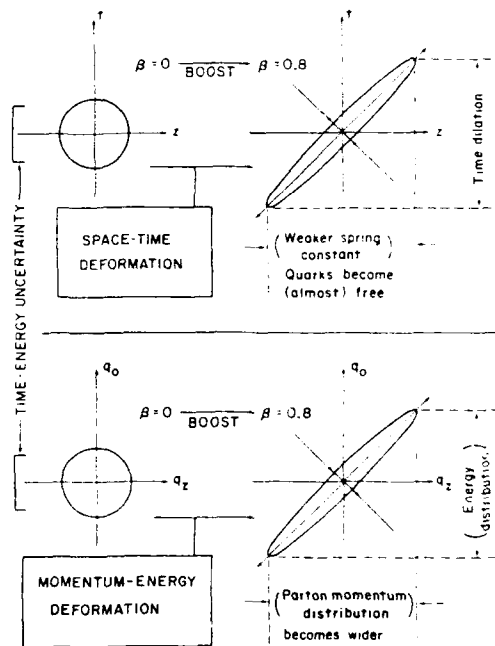


FIG. 2. Lorentz deformations of space-time and momentum-energy wave functions. Both of them have the same Lorentz-deformation property. The major (minor) axis in the space-time ellipse is conjugate to the minor (major) axis of the momentum-energy ellipse. This figure explains why a hadron appears as a tightly bound state of quarks to an observer in the Lorentz frame where the hadron is at rest, while it appears as a collection of free partons with a wide-spread momentum distribution to an observer in the frame where the hadron moves very rapidly. This figure is reprinted from Refs. 2, 6, and 9.

The same reasoning can be carried out for the momentum-energy space, with

$$q_u = (q_z - q_0)/\sqrt{2}, \quad q_v = (q_z + q_0)/\sqrt{2}. \quad (22)$$

The momentum-energy wave function is

$$\phi_\eta(q_z, q_0) = \left(\frac{1}{\pi\Omega}\right)^{1/2} \exp\left\{-\left(\frac{1}{2\Omega}\right)(e^{2\eta}q_u^2 + e^{-2\eta}q_v^2)\right\}. \quad (23)$$

It has a narrow distribution in  $q_u$  or  $(q_z - q_0)/\sqrt{2}$ , and the  $N_1$  and  $N_2$  operators take the form

$$N_1 = -iq_x \frac{\partial}{\partial q_v}, \quad N_2 = -iq_y \frac{\partial}{\partial q_v}. \quad (24)$$

These are also the generators of gauge transformations, as in the case of all massless particles.

In the infinite-momentum limit with  $t = z$  and  $q_z = q_0$ ,  $u$  and  $q_v$  become  $\sqrt{2}z$  and  $\sqrt{2}q_z$  respectively. Both the  $z$  and  $q_z$  distributions become wide-spread. Furthermore, the momentum of each quark can be parameterized as

$$p_{az} = \xi P_z, \quad (25)$$

where the parameter  $\xi$  ranges approximately between zero and 1. This type of distribution was postulated by Feynman in his parton picture of hadrons in the infinite-momentum limit [11].

We can now integrate  $|\phi_\eta(q_z, q_0)|^2$  over  $q_u$  to get the momentum distribution function. There are three quarks in the proton, and the generalization to the three-quark system is straight-forward. In the large- $\eta$  limit, the momentum distribution function becomes [6]

$$\rho(\xi) = 3M(1/2\pi\Omega)^{1/2} \exp\{-M^2(3\xi - 1)^2/2\}. \quad (26)$$

This form of the parton distribution function can be compared with the experimental data [14].

The parameter  $\xi$  is essentially Feynman's  $x$  variable [11] whose variation produces observable effects. It is linear in the  $q_v$  variable, whose variation is generated by  $N_1$  and  $N_2$  of Eq.(17). Feynman's  $x$  variable is therefore a gauge transformation parameter in the hadronic system with an infinite momentum.

### 5. Wigner Function and Uncertainty Relations

Let us go back to the Wigner function. Because of the separability of the ground-state harmonic oscillator, the evaluation of the Wigner function for the formalism of Sec. 4 is straightforward. In the unit system of  $\Omega = 1$ , the Wigner function is [2]

$$W_\eta(z, t, q_z, q_0) = \left(\frac{1}{\pi}\right)^2 \exp\left\{-(e^{-2\eta}u^2 + e^{2\eta}q_u^2 + e^{2\eta}v^2 + e^{-2\eta}q_v^2)\right\}. \quad (27)$$

This Wigner function is separable in the light-cone coordinate system. The above expression is a product of two Wigner functions:

$$W_\eta(u, q_u) = \frac{1}{\pi} \exp\left\{-(e^{-2\eta}u^2 + e^{2\eta}q_u^2)\right\} \quad (28)$$

and

$$W_{eta}(v, q_v) = \frac{1}{\pi} \exp \{ -(e^{2\eta} v^2 + e^{-2\eta} q_v^2) \}. \quad (29)$$

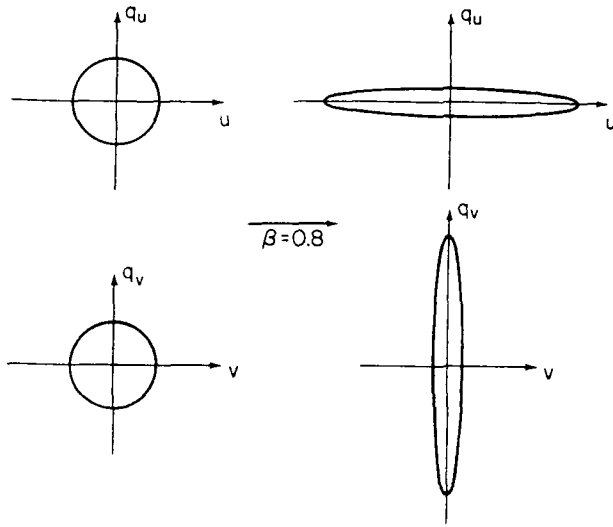


FIG. 3. Lorentz deformations in the light-cone phase space consisting of two pairs of conjugate variables. The uncertainty products  $\Delta u \Delta q_u$  and  $\Delta v \Delta q_v$  are preserved. The Lorentz boost is an area-preserving canonical transformation in each coordinate system. This figure is from Ref. 2.

As is illustrated in Fig. 3, the Lorentz boost deforms the Wigner functions of Eq.(28) and Eq.(29). They are deformed in such a way that the area of the uncertainty region for each Wigner function is preserved. The Lorentz transformation indeed conserves the uncertainty relations. Feynman's parton picture is another indication that quantum mechanics is compatible with special relativity.

Let us go back to Fig. 1. Wigner's little group unifies the  $O(3)$ -like internal space-time symmetry of massive particles and the  $E(2)$ -like symmetry for massless particles. The covariant harmonic oscillator gives a unified picture for the quark model for massive hadrons and the parton model for rapidly moving hadrons. The phase-space picture of quantum mechanics provides the physical basis for the covariant oscillator formalism.

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## Wigner Function of Nonstationary Quantum and Optical Systems

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The aim of this talk is to present the new approach to construct Wigner function of nonstationary quantum systems based on the existence of time-dependent integrals of the motion. The time-dependent linear integrals of the motion for the one-dimensional forced parametric oscillator has been found in Ref. [0]. The with respect to the position to the position and momentum operators linear integrals of motion for multidimensional forced parametric oscillator have been obtained in Ref. [0]. These integrals of motion have been used to find the propagator of the multidimensional forced nonstationary oscillator and to construct the density matrix of the system with quadratic Hamiltonians in Ref. [0, 0].

Let us consider the quantum system with the Hamiltonian which is a nonstationary quadratic form with respect to the position and momentum operators  $x_k, p_k = -i\partial/\partial x_k$ ,  $k = 1, 2, \dots, N$   $\hbar = 1$

$$\mathbb{H} = \frac{1}{2} \sum_{\alpha, \beta=1}^N q_\alpha B_{\alpha\beta}(t) q_\beta + \sum_{\alpha=1}^N c(t) q_\alpha \equiv \frac{1}{2} q B(t) q + c(t) q . \quad (1)$$

Here the  $2N$ -vector  $q = (q_1, q_2, \dots, q_{2N}) = (p_1, \dots, p_N, x_1, \dots, x_N)$  has the projections  $q_\alpha$  ( $\alpha = 1, 2, \dots, 2N$ ) and the  $2N$ -vector  $c(t)$  and  $2N \times 2N$ -matrix  $B(t)$  depend on time. The Hamiltonian (1) may describe the behaviour of the parametric multidimensional forced harmonic oscillator and the behaviour of  $N$  interacting optical modes in a resonator with moving walls or with the time-dependent refraction index. This case corresponds to the existence of the nonstationary Casimir effect when the energy of an external source is converted to the energy of the electromagnetic field due to the work against the Casimir forces. The time-dependence of the parameters of the system  $B(t)$  and  $c(t)$  may be fast. Thus, we can have the kicked system when the matrix  $B(t) = B_0 + B_k \sum_{n=-\infty}^{\infty} \delta(t - nT)$  and the vector  $c(t) = c_0 + c_k \sum_{n=-\infty}^{\infty} \delta(t - nT)$  have constant components and the components corresponding to very short periodic pulses. The model for these pulses may be chosen as the model of  $\delta$ -kicks. On the other hand if the parameters of the system  $B(t)$  and  $c(t)$  vary slowly we can have the adiabatic behaviour of the multidimensional forced parametric oscillator. In this case the Berry phase may be calculated as the phase in the explicit solution of the Schrödinger

equation found in Ref. [0, 0]. The Hamiltonian (1) has the symplectic group  $\text{Isp}(2N, \mathbb{R})$  structure. In fact, the  $N$  position operators  $x_k$ ,  $N$  momentum operators  $p_k$ , all the quadratic operators  $x_k x_l$ ,  $p_k p_l$ ,  $x_k p_l$  and the unit operator form the basis  $\mathbb{L}_i$  of Lie algebra of the inhomogeneous symplectic group. Thus, the Hamiltonian (1) may be considered as the Lie algebra generator

$$\mathbb{H} = \sum_i c_i(t) \mathbb{L}_i \quad (2)$$

with nonstationary coefficients  $c_i(t)$ . The operators  $\mathbb{L}_i$  obey the commutation relations

$$[\mathbb{L}_i, \mathbb{L}_j] = c_{ij}^m \mathbb{L}_m \quad (3)$$

with the structure constants  $c_{ij}^m$  of the inhomogeneous symplectic group Lie algebra. Due to that the evolution operator  $\mathbb{U}(t)$  of the quantum and optical nonstationary systems with Hamiltonian (1) is the representation operator of the symplectic group. The propagator of the considered system is the matrix element of the symplectic group representation in a corresponding basis. If the parameters of the system  $B(t)$  and  $c(t)$  are the random functions of the time we have to integrate the propagator using the given distribution function for these random parameters. Then the evolution operator may become the operator which does not belong to the representation of the symplectic group but it belongs to the enveloping algebra of the Lie algebra of the inhomogeneous symplectic group. For the periodic dependence of the quantum and optical system parameters  $B(t)$  and  $c(t)$  the important role is played by the Floquet operator  $\mathbb{U}(T)$  where  $T$  is the period of the system and  $\mathbb{U}(t)$  is the evolution operator. For hermitian Hamiltonians the evolution operator is the unitary operator and its eigenvalues  $e^{iE_n T}$  where  $E_n$  are the quasienergies of the system may be found as the eigenvalues of an effective Hamiltonian  $h_0$ . The Floquet operator is connected with this effective Hamiltonian  $\mathbb{U}(T) = \exp(-iT h_0)$ . On the other hand the Berry phase is the phase of the eigenvalue  $e^{-i\varphi_n}$  of the unitary evolution operator  $\mathbb{U}(t_0)$  where  $t_0$  is the time at which the parameters of the system return to their initial values, i.e.  $B(t_0) = B(0)$  and  $c(t_0) = c(0)$  in the process of the adiabatic change. Mathematically the problem of the diagonalization of the Floquet operator  $\mathbb{U}(T)$  and of the diagonalization of the evolution operator  $\mathbb{U}(t_0)$  in the case of determining the Berry phases are identical. Thus, the quasienergy spectra of the system and the geometrical phases have identical behaviour. Knowing the quasienergy spectra of the system we know the Berry phases and vice versa. The continuous quasienergy spectrum of the kicked system may be interpreted as the property of quantum chaos in the behaviour of the system [0]. Consequently the geometrical phases of the quantum system are connected with its chaotic or regular behaviour.

The system with the Hamiltonian (1) has the following  $2N$ -vector integral of motion

$$Q(t) = \Lambda(t)q + \Delta(t) \quad (4)$$

The  $2N \times 2N$  real symplectic matrix  $\Lambda(t)$  and  $2N$ -vector  $\Delta(t)$  obey the system of equations of motion for classical trajectory of the multidimensional parametric forced oscillator in the phase space

$$\dot{\Lambda}(t) = \Lambda(t) \Sigma B(t), \quad \dot{\Delta}(t) = \Lambda(t) \Sigma c(t) . \quad (5)$$

where the  $2N \times 2N$ -matrix  $\Sigma$  has the form

$$\Sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \quad (6)$$

We will consider the matrix elements of the evolution operator  $\mathbb{U}(t)$  in Wigner representation [0]. It means that we will use Weyl symbol [0] for the evolution operator. The kernel of the evolution operator in the coordinate representation  $G(x, x', t)$  is connected with its Weyl symbol or Wigner function  $G(x, p, t) \equiv G(q, t)$ ,  $q = (p, x)$  by the relation

$$G(x, p, t) = \int G\left(x + \frac{1}{2}r, x - \frac{1}{2}r, t\right) \exp(-ipr) dr \quad (7)$$

Since the quantum system propagator satisfies the system of equations which connects the time-dependent integrals of the motion with Green function [0] the symbol of the evolution operator  $\mathbb{G}(q, t)$  obeys the system of equations

$$\left( \Lambda q - \frac{i}{2} \Lambda \Sigma \frac{\partial}{\partial q} + \Delta \right) G(q, t) = \left( q + \frac{i}{2} \Sigma \frac{\partial}{\partial q} \right) G(q, t) . \quad (8)$$

The solution to this system of equations has the form [0]

$$\begin{aligned} G(q, t) &= 2^N \{ \det [\Lambda + E] \}^{-1/2} \exp \left\{ iqA(t)q \right. \\ &= \quad \left. + 2iqI(t) + \frac{i}{4} \Delta \Sigma (\Lambda + E)^{-1} (\Lambda - E) \Delta \right. \\ &= \quad \left. + \frac{i}{2} \int_0^t \dot{\Delta} \Sigma \Delta d\tau \right\} . \end{aligned} \quad (9)$$

The initial condition for the propagator has the form

$$G(q, 0) = 1 . \quad (10)$$

The  $2N$ -matrix  $E$  is the unit matrix and the matrix  $A$  and the vector  $I(t)$  are expressed through the classical trajectory of the system

$$A(t) = \Sigma (\Lambda + E)^{-1} (\Lambda - E), \quad I(t) = \Sigma (\Lambda + E)^{-1} \Delta . \quad (11)$$

If the multidimensional oscillator parameters  $B$  and  $c$  do not depend on time the propagator (10) may be expressed in terms of the Hamiltonian parameters explicitly

$$\begin{aligned} G(q, t) &= \left[ \det \cosh \left( \Sigma B \frac{t}{2} \right) \right]^{-1/2} \exp \left\{ i(q + B^{-1}c) \Sigma \right. \\ &= \quad \left. \times \tanh \left( \Sigma B \frac{t}{2} \right) \left( q + B^{-1}c \right) + \frac{it}{2} c B^{-1}c \right\} \end{aligned} \quad (12)$$

In this case the equilibrium Wigner function  $W(q, \beta)$  may be obtained from the propagator (12) by the change of variables  $t = -i\beta$  and it has the form

$$\begin{aligned} W(q, \beta) &= \left[ \det \cos \left( \frac{1}{2} \beta \Sigma B \right) \right]^{-1/2} \exp \left\{ \frac{1}{2} \beta c B^{-1} c \right. \\ &= \left. + (q + B^{-1} c) \Sigma \left( \frac{1}{2} \beta \Sigma B \right) (q + B^{-1} c) \right\} \end{aligned} \quad (13)$$

The partition function  $Z(\beta)$  is connected with the Wigner function by the relation

$$Z(\beta) = \int W(q, \beta) (2\pi)^{-N} dq . \quad (14)$$

The integral (14) may be easily calculated

$$Z(\beta) = 2^{-N} \left[ \det \sin \left( \frac{1}{2} \beta \Sigma B \right) \right]^{-1/2} \exp \left( \frac{1}{2} \beta c B^{-1} c \right) . \quad (15)$$

For the usual one-dimensional oscillator  $N = 1$ ,  $\hbar = m = \omega = 1$  we have

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad c = 0 . \quad (16)$$

For the oscillator the Wigner function has the form

$$W(p, x, \beta) = \left( \cosh \frac{\beta}{2} \right)^{-1} \exp \left[ - \tanh \frac{\beta}{2} (p^2 + x^2) \right] . \quad (17)$$

The formula (14) for the partition function gives the usual result for the one-dimensional oscillator

$$z(\beta) = \left( 2 \sin \frac{\beta}{2} \right)^{-1} . \quad (18)$$

One can introduce the integrals of the motion  $A(t)$  and  $A^+(t)$  with boson commutation relations. Let the real  $2N \times 2N$ -matrix  $\Lambda(t)$  and the  $2N$ -vector  $\Delta(t)$  have the  $N$ -block form

$$\Lambda(t) = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix}, \quad \Delta(t) = (\delta_1(t), \delta_2(t)) . \quad (19)$$

If we introduce the complex  $N \times N$ -matrices

$$\lambda_p = \frac{1}{\sqrt{2}} (\lambda_3 + i\lambda_1), \quad \lambda_q = \frac{1}{\sqrt{2}} (\lambda_4 + i\lambda_2) . \quad (20)$$

and the complex  $N$ -vector

$$\delta = \frac{1}{\sqrt{2}} (\delta_2 + i\delta_1) . \quad (21)$$



the operator

$$A(t) = \lambda_p p + \lambda_q x + \delta(t) \quad (22)$$

will be the vector-integral of the motion. The commutation relations for the components of the integrals of the motion  $A_i(t)$ ,  $A_k^+(t)$  are

$$[A_i(t), A_k^+(t)] = \delta_{ik}, \quad [A_i(t), A_k(t)] = 0, \quad i, k = 1, 2, \dots, N. \quad (23)$$

The normalized eigenstates  $|\alpha\rangle$  of the "annihilation" integral of the motion  $A(t)$ , with complex eigenvalue  $\alpha$  has the wave function in the coordinate representation

$$\begin{aligned} \Psi_\alpha(x, t) = & (2\pi)^{-N/4} (\det \lambda_p)^{-1/2} \exp \left\{ -\frac{i}{2} x \lambda_p^{-1} \lambda_q x \right. \\ & + i x \lambda_p^{-1} (\alpha - \delta) + \frac{1}{2} \alpha \lambda_p^* \lambda_p^{-1} \alpha + \alpha (\delta^* - \lambda_p^* \lambda_p^{-1} \delta) \\ & \left. + \frac{1}{2} \delta \lambda_p^* \lambda_p^{-1} \delta - \frac{1}{2} |\delta|^2 + i \int^t (\delta \delta^*) d\tau - \frac{1}{2} |\alpha|^2 \right\} \quad (24) \end{aligned}$$

This function describes the polymode squeezed and correlated state of the electromagnetic field in a resonator with moving walls. If one takes the Wigner function  $W_{\alpha, \beta}(q, t)$  of the state with the density operator  $\rho_{\alpha, \beta} = |\alpha\rangle \langle \beta|$  it obeys the evolution equation of the type

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial q} [\Sigma (Bq + c) W] \quad (25)$$

The equation (25) follows from the evolution equation for the density operator

$$i\dot{\rho} = [H, \rho] \quad (26)$$

The advantage of the Wigner representation is that the evolution equation (25) contains the first derivatives with respect to the coordinate of the Wigner function  $q$ . The evolution equation for the density matrix in other representations than the coordinate representation or the coherent state representation contains the second derivative terms. Since only first derivatives with respect to  $q$  variables are present in the evolution equation (25), the propagator for this equation may be expressed in terms of  $\delta$ -function of the classical trajectory of the system in its phase space.

Due to equation (26) the density operator  $\rho$  is the integral of motion. In fact, the density operator  $\rho(t)$  at the moment  $t$  is connected with the initial density operator  $\rho(0)$  by means of the evolution operator  $\mathbb{U}(t)$

$$\rho(t) = \mathbb{U}(t) \rho(0) \mathbb{U}^{-1}(t) \quad (27)$$

The operator of the form (27) is the integral of the motion. Due to that the density operator  $\rho(t)$  may be expressed in terms of the other integrals of the motion.

For example, if the initial density operator  $\rho(0)$  is a function of the coordinate and momentum operators  $x$  and  $p$

$$\rho(0) = \rho(x, p) = \rho(q) \quad (28)$$

the density operator  $\rho(t)$  is the same function of the integrals of the motion  $Q(t)$

$$\rho(t) = \rho(Q(t)) \quad (29)$$

Let us denote the Weyl symbol of the integral of the motion  $A(t)$  (22) as the complex variable

$$z = \lambda_p(t)p + \lambda_q(t)x + \delta(t) \quad (30)$$

where the real  $N$ -vectors  $p$  and  $x$  are the Weyl symbols of the momentum and position operators. Thus, the  $c$ -number vector  $z(t)$  is the integral of the motion for classical trajectories of the oscillator.

The function  $W_{\alpha\beta}(q, t)$  may easily be found as the solution of the evolution equation (25), and this function is expressed in terms of the symbols  $z$  of the time-dependent linear integrals of the motion  $A(t)$  in the following manner [0]

$$W_{\alpha\beta} = 2^N \exp \left\{ -2zz^* + 2\alpha z^* + 2\beta^* z - \alpha\beta^* - \frac{1}{2} (|\alpha|^2 + |\beta|^2) \right\} \quad (31)$$

Since the Wigner function  $W_{\alpha\beta}$  is the generating function for the symbol  $W_{mn}$  of the operator  $\rho_{mn} = |m\rangle\langle n|$  where the state  $|m\rangle$  is the eigenstate of the quadratic integral of the motion  $A^+(t)A(t)$ , i.e.

$$A_i^+(t)A_i(t) |m\rangle = m_i |m\rangle, \quad m_i = 0, 1, 2, \dots, i = 1, 2, \dots, N \quad (32)$$

the Wigner function of the state  $\rho_{mn}$  has the completely factorized form [0]

$$W_{mn} = 2^N \prod_{i=1}^N 2^{m_i - n_i} (-1)^{n_i} \left( \frac{n_i!}{m_i!} \right)^{1/2} (z_i^*)^{m_i - n_i} \times e^{-2z_i z_i^*} L_{n_i}^{m_i - n_i} (4z_i z_i^*) \quad (33)$$

Thus, the geometrical phase of the analogue of stationary state is the phase of the complex classical trajectory which is contained in the complex variables  $z_i$ . The function  $L_N^M$  is the Laguerre polynomial. Thus, in Wigner representation we have complete factorization of the scattering matrix for polydimensional nonstationary forced harmonic oscillator. The result about the factorization of the evolution operator in Wigner representation may be generalized in WKB-approximation for any potential. In fact, if one uses path integral representation for the propagator of the quantum system, the

quasiclassical approximation means, that we extract the term connected with the action under classical trajectory for the system and then we calculate two correction terms, which gives us formally the path integral for nonstationary quadratic quantum system. This part of the propagator may be factorized completely. Thus, the result may be formulated as follows. The propagator of an arbitrary N-dimensional system may be completely factorized in Wigner representation in quasiclassical approximation in the sense that the part of the propagator connected with the correction to the classical action is equivalent to the propagator of the multidimensional forced nonstationary oscillator which is the factorizable system.

Now let us discuss the Wigner function for the anharmonic oscillator. The Hamiltonian for it may be taken as follows:

$$\mathbb{H} = \left( a^+ a + \frac{1}{2} \right) + \lambda \left( a^+ a + \frac{1}{2} \right)^2 \quad (34)$$

The partition function of the quadratic anharmonic oscillator may be expressed as follows:

$$z(\beta) = \exp \left( -\frac{\beta}{2} - \frac{\lambda\beta}{4} \right) \sum_{n=0}^{\infty} \epsilon^{-\lambda\beta n^2 - \beta n(1+\lambda)} \quad (35)$$

The series in the partition function is the well known Jacobi  $\Theta_3$ -function. The density operator for the anharmonic oscillator may be represented by means of the formula

$$\epsilon^{A^2} = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} dy \epsilon^{-\frac{y^2}{4}} \epsilon^{yA} \quad (36)$$

in the following form

$$\epsilon^{-\beta\mathbb{H}} = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} dy \exp \left\{ -\frac{y^2}{4} - \beta \left( a^+ a + \frac{1}{2} \right) + \sqrt{-\beta\lambda} y \left( a^+ a + \frac{1}{2} \right) \right\} . \quad (37)$$

Due to that the Wigner function  $W_4$  of the quadratic anharmonic oscillator may be obtained from the Wigner function of the harmonic oscillator (17). We have

$$W_4(p, x, \beta) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} dy \left( \cosh \left[ \left( \beta - \sqrt{-\beta\lambda} y \right) / 2 \right] \right)^{-1} \exp \left\{ -\frac{y^2}{4} - (p^2 + x^2) \tanh \left[ \left( \beta - \sqrt{-\beta\lambda} y \right) / 2 \right] \right\} \quad (38)$$

Since the Wigner function (38) and the partition function (35) are connected by the relation

$$z(\beta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp dx W_4(p, x, \beta) . \quad (39)$$

we have the integral formula for  $\Theta_3$ -function

$$\begin{aligned} & \pi^{-1/2} \int_{-\infty}^{\infty} dy \left\{ \exp \left( \frac{\beta}{2} - \sqrt{-\beta\lambda} y \right) - \exp \left( -\frac{\beta}{2} + \sqrt{-\beta\lambda} y \right) \right\}^{-1} \epsilon^{-y^2} \\ = & \left\{ \exp \left[ -\frac{\beta}{2} - \frac{\lambda\beta}{4} \right] \right\} \sum_{n=0}^{\infty} \epsilon^{-\lambda\beta n^2 - \beta n(1+\lambda)} . \end{aligned} \quad (40)$$

If one considers the normalized Wigner function of a general quadratic system with the Hamiltonian (1) and if this Wigner function has Gaussian form the Wigner function may always be expressed as follows

$$W(q, t) = (\det m(t))^{-1/2} \exp \left\{ -\frac{1}{2} (q - \langle q(t) \rangle) m^{-1}(t) (q - \langle q(t) \rangle) \right\} \quad (41)$$

Here the parameters  $\langle q(t) \rangle$  and  $m(t)$  are the mean values of the position and momentum and the dispersion matrix of the parametric oscillator, i.e.

$$\langle q_\alpha(t) \rangle = \text{Tr } \rho(t) q_\alpha, \quad \alpha = 1, 2, \dots, 2N, \quad (42)$$

and

$$m_{\alpha\beta}(t) = \text{Tr} \left[ \frac{\rho}{2} (q_\alpha q_\beta + q_\beta q_\alpha) \right] - [\text{Tr}(q_\alpha \rho)] [\text{Tr}(q_\beta \rho)] \quad (43)$$

For the density matrix of the stationary quadratic system with the Wigner function (13) and partition function (14) the characteristic function

$$\langle \epsilon^{qt} \rangle = z^{-1}(\beta) \int \epsilon^{qt} W(q, \beta) (2\pi)^{-N} dq \quad (44)$$

may be calculated [0]

$$\langle \epsilon^{qt} \rangle = \exp \left( -\frac{1}{2} t D t + t D b \right) \quad (45)$$

Here the  $2N \times 2N$ -matrix  $D$  and  $2N$ -vector  $b$  are given by the formulae

$$D = -\frac{1}{2} \left[ \cot \left( \frac{1}{2} \beta \Sigma B \right) \right] \Sigma \quad (46)$$

and

$$b = -2 \left[ \tan \left( \frac{1}{2} \beta \Sigma B \right) \right] B^{-1} c \quad (47)$$

We have for the stationary quadratic system

$$\langle q \rangle = -B^{-1} c \quad (48)$$

and the dispersion matrix  $D$  is given by the formula (46). The mean values of the products of the  $q_\alpha$ -variables may be calculated expanding the characteristic function (45) into power series with respect to the parameters  $t_\alpha$ ,  $\alpha = 1, 2, \dots, 2N$ . We have

$$\langle q_1^{n_1} q_2^{n_2} \dots q_{2N}^{n_{2N}} \rangle = H_{n_1 n_2 \dots n_{2N}}^{(D)}(b) \quad (49)$$

Here the functions  $H_{n_1 n_2 \dots n_{2N}}^{(D)}(b)$  are the Hermite polynomials of  $2N$  variables. All the highest momenta (49) depend, in fact, on the second momenta matrix  $D$  and the first momenta  $\langle a \rangle$ . It is the property of Gaussian distributions. The same takes place for the correlation functions. If we have the time-dependent density matrix with the

Gaussian Wigner function the highest correlation functions of the position and momentum may be expressed in terms of the second-order correlation functions. The Wigner function of the nonstationary quadratic system may be used for calculating the transition probabilities between the energy levels connected with the parametric excitations of the nonstationary forced oscillator. Thus, if one has the states  $|n, t_1\rangle \langle n, t_1| = \rho_{nn}(t_1)$  and  $|m, t_2\rangle \langle m, t_2| = \rho_{mm}(t_2)$ , where  $n = (n_1 n_2 \dots n_N)$ ,  $m = (m_1 m_2 \dots m_N)$ ,  $n_i, m_i = 0, 1, 2, \dots$ , the transition probabilities between the states with the density matrices  $\rho_{nn}(t_1)$  and  $\rho_{mm}(t_2)$  are given by the formula

$$P_n^m(t_1, t_2) = \int W_{nn}(t_1) W_{mm}(t_2) \frac{dq}{(2\pi)^N} \quad (50)$$

Taking the explicit form (33) of the Wigner functions of the states  $\rho_{nn}$  and  $\rho_{mm}$ , we can express the integral (50) in terms of the Hermite polynomials of  $2N$  variables. For one-dimensional unforced parametric oscillator the Hermite polynomial of 2 variables, in the case when its argument is equal to zero, may be reduced to the Legendre polynomial. Then the transition probability looks as follows

$$P_n^m = \left(E + \frac{1}{2}\right)^{-\frac{1}{2}} \frac{n!}{m!} \left( P_{\frac{m-n}{2}}^{\frac{m+n}{2}} \left( \left(E + \frac{1}{2}\right)^{-\frac{1}{2}} \right) \right)^2, \quad (51)$$

$$m \geq n; \quad (52)$$

and

$$P_n^m = \left(E + \frac{1}{2}\right)^{-\frac{1}{2}} \frac{m!}{n!} \left( P_{\frac{n-m}{2}}^{\frac{n+m}{2}} \left( \left(E + \frac{1}{2}\right)^{-\frac{1}{2}} \right) \right)^2, \quad (53)$$

$$n > m. \quad (54)$$

Here the dimensionless energy of classical motion

$$E = \frac{|\epsilon|^2}{2} + \frac{|\dot{\epsilon}|^2}{2} \quad (55)$$

is expressed in terms of the complex solution of the classical equation of motion of the parametric oscillator

$$\ddot{\epsilon}(t) + \Omega^2(t)\epsilon(t) = 0. \quad (56)$$

The initial conditions for this equation are

$$\Omega(0) = \epsilon(0) = 1, \quad \dot{\epsilon}(0) = i. \quad (57)$$

The normalized state  $|n, t\rangle$  of the parametric oscillator has the following wave function in the coordinate representation

$$\Psi_n(x, t) = \pi^{-1/4} \epsilon^{-1/2} \left( \frac{\epsilon^*}{2\epsilon} \right)^{n/2} (n!)^{-1/2} \epsilon^{i\frac{\epsilon}{2\epsilon} x^2} H_n \left( \frac{x}{|\epsilon|} \right) \quad (58)$$

The coordinate independent phase of the wave function is expressed through the phase of the complex classical trajectory

$$\varphi(t) = \left( n + \frac{1}{2} \right) \arg \epsilon(t) \quad (59)$$

In the case of the regular behaviour of the parametric oscillator with periodically kicked frequency  $\Omega^2(t) = 1 + T\kappa \sum_{n=-\infty}^{\infty} \delta(t - nT)$  the quasienergy state of the parametric oscillator has the wave function (58) where  $\epsilon(t + T) = e^{i\arg(T)} \epsilon(t)$ . The spectrum of the monodromy operator  $\mathbf{V}(T)$  is determined completely by the argument of the classical solution  $\epsilon(T)$ . Thus, the quasienergy spectrum for that case may be determined in terms of the properties of the classical trajectory of the parametric oscillator. The geometrical phase (Berry phase) is the phase of the classical trajectory  $\epsilon(T)$  too but the time  $T$  is the time after which the frequency  $\Omega(t)$  takes the initial value  $\Omega(T) = \Omega(0) = 1$ . The motion in this case may be a not periodical one. The squeezing phenomena is connected with the parametric excitation of the harmonic oscillator. So, the ratios of the coordinate and momentum dispersions to their value at the ground state of the stationary oscillator are given by the formulae

$$k_x = \frac{\delta x}{1/\sqrt{2}} = |\epsilon|; \quad \frac{\delta p}{1/\sqrt{2}} = |\epsilon| = k_p \quad (60)$$

Thus, the squeezing parameters are determined by the properties of the classical trajectory  $\epsilon(t)$ . Due to Wronskian conservation

$$i\dot{\epsilon}^* - \dot{\epsilon}^* \epsilon = 2; \quad (61)$$

the phase of the complex trajectory  $\epsilon(t)$  is connected with the modulus of this function

$$\arg \epsilon(t) = \int_0^t \frac{d\tau}{|\epsilon(\tau)|^2} \quad (62)$$

It means that the geometrical phase is the integral over the squeezing coefficient time variation. Thus, two phenomena, squeezing and the geometrical phase, are related. Since the squeezing coefficient time-dependence determines the geometrical phase which is connected, in turn, with the properties of the quasienergy spectrum (it means that it is connected with quantum chaos) the variation of the squeezing with time determines the instability (or chaoticity) of the parametric oscillator. The squeezing phenomena may be connected with the area overlap concept in phase space of the oscillator [0, 0]. The squeezing phenomena is related to the correlation phenomena of the position and momentum of the parametric oscillator. The correlation coefficient  $r$  of these two conjugate variables is expressed in terms of the squeezing coefficients of the position and the momentum [0]

$$r = |\epsilon\dot{\epsilon}|^{-1} \left( |\epsilon\dot{\epsilon}|^2 - 1 \right)^{1/2} \quad (63)$$

The state which is created from the ground state of the usual oscillator due to the parametric excitation is the squeezed and correlated [0] state which minimizes the Schrödinger uncertainty relation [0]

$$\delta n \delta x \geq \frac{\hbar}{2} \frac{1}{\sqrt{1 - r^2}} \quad (64)$$

This uncertainty relation generalizes the Heisenberg uncertainty relation [0] to the cases when the correlation coefficient is not equal to zero. If one knows the time-dependence of the squeezing coefficient  $k_x(t)$  all the other parameters are known, too, since the function  $k_p(t)$  is connected with the function  $k_x(t)$  by the relation

$$k_p(t) = \sqrt{[k_x(t)]^2 - k_x^{-2}(t)} \quad (65)$$

Due to that the correlation coefficient may also be connected with the time dependent squeezing coefficient  $k_x(t)$ . If one considers the photon distribution function is squeezed and correlated polymode light the distribution function is given by the phase space overlap integral of the Wigner functions

$$P_n = \int \frac{dq}{2\pi} W_{n_1} W_{n_2} \quad (66)$$

Where the function  $W_{n_1}$  is given by the relation (30) and the function  $W_{n_2}$  is given by the relation (33). The integral (66) may be expressed in terms of the Hermite polynomials of several variables. The photon distribution function (66) has very way behaviour since the Hermite polynomial of several variables has this very way behaviour. Thus, the property of the photon distribution function of the squeezed and correlated polymode light is similar to the properties of one mode squeezed [0] and correlated light [0]. For two-mode case the Hermite polynomial of two variable, the generating function for which

$$\begin{aligned} & \exp \left[ -\frac{1}{2} \sum_{i=1}^2 \sum_{k=1}^2 t R_{ik} x_k + \sum_i t R_{i1} x_i \right] \\ & = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{t_1^{n_1} t_2^{n_2}}{n_1! n_2!} H_{(n_1, n_2)}^{(R)}(x_1, x_2) \end{aligned} \quad (67)$$

is determined by the  $2 \times 2$ -matrix  $R$ , has three different canonical forms [0]. One canonical form corresponds to the matrix  $R$  of the type

$$R_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (68)$$

In this case the Hermite polynomial of 2 variables may be expressed in terms of Laguerre polynomials

$$\begin{aligned} H_{m,n}^{(R_1)}(x_1, x_2) & = (-1)^{m+n} \mu_{m,n}! x_1^{m+n} x_2^{m+n} \\ & \times x_2^{(m+n+|m-n|)/2} L_{\mu_{m,n}}^{(m+n)}(x_1 x_2) \end{aligned} \quad (69)$$

Here the numbers  $\mu_{mn}$  are

$$\mu_{mn} = \min(m, n) . \quad (70)$$

From the formula (69) we have

$$H_{mn}^{(00)}(x_1, x_2) = (-1)^n n! L_n(x_1, x_2) . \quad (71)$$

The second canonical form is connected with the matrix  $R$  of the type

$$R_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} . \quad (72)$$

In this case the Hermite polynomial  $H_{mn}^{(10)}$  of two variables may be expressed in terms of the usual Hermite polynomials of one variable.

$$H_{mn}^{(10)}(x_1, x_2) = m!n! \sum_{k=0}^{\mu_{mn}} \frac{(-1)^k 2^{(k-n)/2}}{k!(m-k)!(n-k)!} x_1^{n-k} H_{m-k} \left( \frac{x_1 + x_2}{\sqrt{2}} \right) . \quad (73)$$

The third canonical form is connected with the matrix  $R$  of the type

$$R_1 = \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} . \quad (74)$$

In this case the Hermite polynomial of two variables may be expressed through the products of Hermite polynomials of one variable

$$\begin{aligned} H_{mn}^{(\beta\beta)}(x_1, x_2) &= m!n! 2^{-(m+n)/2} \sum_{k=0}^{\mu_{mn}} \frac{(-2\beta)^k}{k!(m-k)!(n-k)!} \\ &= \times H_{m-k} \left( \frac{x_1 + \beta x_2}{\sqrt{2}} \right) H_{n-k} \left( \frac{x_2 + \beta x_1}{\sqrt{2}} \right) . \end{aligned} \quad (75)$$

Thus, two-mode squeezed and correlated light described by the wave function (24) with photon distribution function given by the formulae (66), (30) and (33) demonstrates the different regime of the wavy variation of the photon distribution function connected with the different time variation of parameters  $B(t)$  and  $c(t)$  in the Hamiltonian (1). The different cases are described in terms of three different canonical forms for Hermite polynomials of two variables.

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# From the One-Atom Maser to Schrödinger Cats

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## 1 Introduction

We join the larger community to salute to Prof. Eugene Paul Wigner on the occasion of the II International Wigner Symposium and celebrate his many contributions to physics summarized so beautifully in the after dinner speech of Prof. G. Emch. We do not intend to repeat this wonderful list of achievements but want to emphasize in this article how the field of quantum optics has profited so much from the pioneering work of Prof. Wigner.

From his early days in physics Prof. Wigner has been interested in the quantum properties of light. His work with V. Weisskopf on the spontaneous emission of an atom lies at the foundations of quantum electrodynamics. In the sixties he pushed for a deeper understanding of the measurement process in quantum mechanics best summarized by the problem of *Wigner's friend*. The one-atom maser — the subject of Sec. 2 — is closely related to both topics: The use of Rydberg atoms as the masing material inhibits the spontaneous emission. Moreover, this amazing maser allows intriguing probes of the complementarity principle.

The superposition principle of quantum mechanics — on first sight a rather innocent principle — leads to quite unusual *Gedankenexperiments* such as the paradox of *Schrödinger's cat*. We in Sec. 3 bring out most vividly the properties of these nonclassical states using the example of a quantum mechanical superposition of two coherent states. A phase space approach based on a pseudo phase space distribution such as the Wigner distribution makes the phenomena of sub-Poisson statistics c

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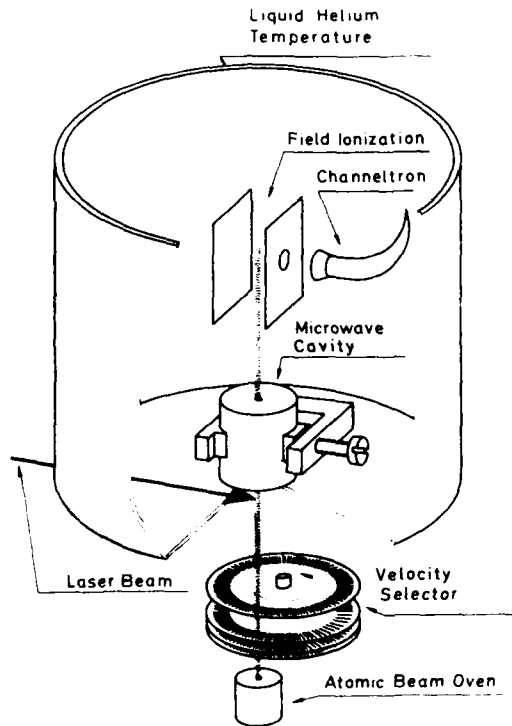


Figure 1: Experimental setup of one-atom maser.

squeezing arising in such a state, transparent. An intuitive argument for the Wigner function based on the notion of a quantum jump constitutes the last section of our salute to Prof. Wigner.

## 2 The One-Atom Maser – a Bridge from Quantum Light to Quantum Measurement

Send a stream of excited atoms through a superconducting cavity with a single field mode coupled to a single transition of the atom. An atom may deposit a photon in the cavity. The next atom entering the cavity interacts with the field created by all earlier atoms and so on. This summarizes in brief the underlying mechanism of the one-atom maser depicted in Fig. 1.

A maser operation driven by a single atom in the cavity – just another example of a Gedankenexperiment? Just another toy from the playground of a quantum optics theoretician? No, this device has been promoted from the wonderland of

mathematics and theory to the real world of hardware and experiment [1]. But what are the statistical properties of the light created in this maser? It is *nonclassical!* What does it mean, *nonclassical!*

Electromagnetic radiation can indeed show nonclassical properties [2, 3], that is, properties that cannot be explained by classical probability theory. Loosely speaking we need to invoke "negative probabilities" such as appearing in the Wigner function to get deeper insight into these features. We know of essentially three phenomena which demonstrate the nonclassical character of light: photon antibunching [4], sub-Poissonian photon statistics [5] and squeezing [6]. Mostly methods of nonlinear optics are employed to generate nonclassical radiation. However, also the fluorescence light from a single atom caught in a trap exhibits some nonclassical features [7, 8].

Yet another nonclassical light generator is the one-atom maser. We recall that the Fizeau velocity selector shown in Fig. 1 preselects the velocity of the atoms: Hence the interaction time is well-defined which leads to conditions usually not achievable in standard masers. Under appropriate conditions the number distribution of the photons in the cavity is sub-Poissonian [9, 10] that is, narrower than a Poisson distribution. Even a number state, that is, a state of well-defined photon number can be generated [11, 12]. What is the use of such a number state? Test of quantum theory of measurement? Yes! The field of the one-atom maser as a "which way" detector, that is, test of complementarity [13] serves as one motivation for number state production.

But what are the conditions we have to satisfy in order to achieve such an ambitious goal? (1) We have to use a cavity with a sufficiently high enough quality factor and (2) we cannot admit any thermal photons in the cavity. Both conditions can be fulfilled when the superconducting cavity is operated at very low temperatures, that is, at temperatures smaller than about 0.5 K. Recently a new maser in Garching operated at a temperature below 0.1 K. At such low temperatures more interesting features such as trapping states of the cavity [14] make their appearance.

Unfortunately, the measurement of the nonclassical photon statistics in the cavity is not that straightforward. It invokes the coupling to a measuring device whereby losses lead inevitably to a destruction of the nonclassical properties. The ultimate technique to obtain information about the field employs the Rydberg atoms themselves: Measure the statistics via the dynamical behavior of the atoms in the radiation field -- via the collapse and the revivals of the Rabi oscillations -- that is one possibility [15, 16, 17]. However, a much more conclusive approach probes the population of the atoms in the upper and lower maser levels when they leave the cavity. In this case, the interaction time is kept constant. Moreover, this measurement is relatively easy since electric fields can be used to selectively ionize the atoms. The detection sensitivity is sufficient to investigate the atomic statistics. This technique maps the photon statistics of the field inside the cavity onto the atomic statistics. Experiments carried out along these lines have shown a variance of the photon number distribution,  $\langle n^2 \rangle - \langle n \rangle^2 = 0.3\langle n \rangle$ , which is 70% below the shot noise level [18]. Another method proposed to measure the photon statistics of the maser relies on

a Ramsey-type setup together with a quantum non-demolition scheme [19]. This technique has not been realized experimentally yet.

The experiments on the steady state photon statistics involve only the diagonal elements of the density matrix of the maser field. The maser spectrum [20] however invokes the off-diagonal elements as well. A multiple microwave field method similar to the Ramsey fringe technique allows to measure a quantity closely related to the maser linewidth. Another method [21] to investigate the phase dynamics of the maser field rests on the state reduction of the combined atom-field system after applying a classical coherent field to the atoms leaving the cavity. Their population statistics allows to deduce the field statistics. This method is based on the back action due to the measurement process.

We conclude this section by emphasizing that the one-atom maser is a unique device, and that for three reasons: (1) it is the first maser which sustains oscillations with less than one atom on average; (2) this setup allows to study in detail the conditions necessary to obtain nonclassical radiation especially sub-Poisson light; and (3) it allows intriguing probes of the quantum measurement process.

### 3 Nonclassical State from two Pseudo-Classical States

A key point in the discussion of quantum theory of measurement is the production of *Schrödinger cats*. It has been proposed [22] to use the one-atom maser to create such states. Here we do not want to address the question of how to produce them using the one-atom maser [23] nor do we want to elaborate on various other schemes [24], nor do we want to elucidate the subtleties involved in their detection [25], but we want to identify interference as the central ingredient that makes these Schrödinger cats states so nonclassical [26].

At the heart of quantum mechanics lies the superposition principle - to quote from the first chapter of Dirac's classical treatise [27] - "... any two or more states may be superposed to give a new state ". Insight into the far reaching consequences of this principle is offered by the most elementary example of superposing two coherent states [28, 29],  $|\alpha e^{i\varphi/2}\rangle$  and  $|\alpha e^{-i\varphi/2}\rangle$ , of identical mean number of photons  $\langle m \rangle = \alpha^2$  but with a phase difference  $\varphi$  as shown in Fig. 2.

A coherent state of an electromagnetic field mode, or in the language of its mechanical analogue, of a harmonic oscillator with dynamically conjugate variables  $x$  and  $p$  minimizes the uncertainty product with identical uncertainties  $(\Delta x)^2 = (\Delta p)^2 = 1/2$  [30]. Thus they are quantum states closest to classical states - pseudo-classical states [28, 29]. In contrast, the quantum mechanical superposition of two such coherent states forming the state

$$|\psi\rangle = \mathcal{N} \frac{1}{\sqrt{2}} (|\alpha e^{i\varphi/2}\rangle + |\alpha e^{-i\varphi/2}\rangle) \quad (1)$$

exhibits highly nonclassical features [31, 32] such as sub-Poissonian [5] and oscilla-

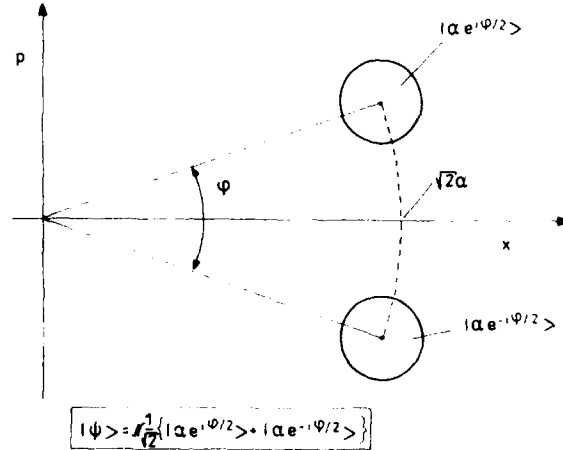


Figure 2: In its most elementary version the quantum mechanical superposition of two coherent states of mean photon number  $\langle m \rangle = \alpha^2$  and phase difference  $\varphi$  can be visualized by two circles of radius unity displaced by an amount  $\sqrt{2} \alpha$  from the origin and having the central angle  $\varphi$  between them.

tory photon statistics [33] as well as squeezing [34] of the  $x$ -variable [35], as discussed in Figs. 4 and 5.

In Fig. 3 we analyze the photon statistics, that is, the probability  $W_m$  of finding  $m$  photons in the state  $|\psi\rangle$ , in its dependence on the phase difference  $\varphi$ . Figure 3 shows  $\varphi$ -domains in which the photon count probability curve gets narrower than the Poisson distribution of a single coherent state; that is, we find sub-Poissonian photon statistics - an indicator of a nonclassical state. These domains are separated from each other by zones in which  $W_m$  is broader than a Poisson distribution, that is super-Poissonian. The resulting oscillations in the normalized variance  $\sigma^2 \equiv \langle m^2 \rangle / \langle m \rangle - \langle m \rangle$ , displayed in Fig. 4 and similar to those in the photon statistics of the micromaser [9], die when the two coherent states are distinguishable. As a consequence  $W_m$  shows rapid oscillations with the familiar Poisson envelope. The analogous effect arises in the photon statistics of a highly squeezed state [33]. Figure 5 deals with the question of possible squeezing in  $|\psi\rangle$ . A single coherent state shows identical uncertainties  $(\Delta x)^2$  and  $(\Delta p)^2$  in the conjugate variables  $x$  and  $p$  equal to  $1/2$ . In contrast in the state  $|\psi\rangle$  the uncertainty  $(\Delta x)^2$  can fall below this coherent state value provided the phase difference  $\varphi$  lies appropriately in the domain of sub-Poissonian photon statistics as indicated in Fig. 5. Thus,  $|\psi\rangle$  exhibits squeezing. Moreover,  $(\Delta p)^2$  increases while preserving (approximately) the minimum uncertainty product  $(\Delta x)^2(\Delta p)^2 = 1/4$ . No squeezing in  $x$  is found in the region of super-Poissonian or oscillatory photon statistics.

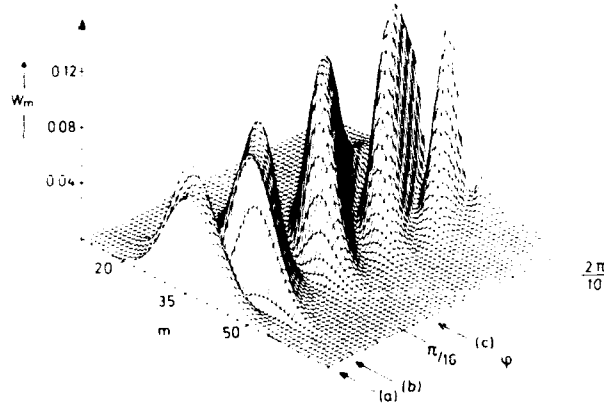


Figure 3: The photon count probability  $W_m$  of the quantum mechanical superposition state  $|\psi\rangle$  in its dependence on the relative phase difference  $\varphi$ . For increasing  $\varphi$  the Poissonian distribution for the case  $\varphi = 0$  (a), narrows while its maximum moves toward smaller  $m$  values, (b). This curved wave front suddenly breaks off to start a new front and yields a distribution broader than a Poissonian with more than one maximum (c). (Here we have chosen  $\alpha^2 = 36$ .)

This example of the superposition of two coherent states displaying nonclassical effects such as sub-Poissonian statistics and squeezing in contrast to a single coherent state identifies once more in a striking way the principle of superposition as the main actor behind the scenes. In the same spirit is the interesting observation [36] that the quantum mechanical superposition of two number states, such as the vacuum,  $|0\rangle$ , and a one-photon state,  $|1\rangle$ , shows squeezing, whereas neither  $|0\rangle$  nor  $|1\rangle$  exhibits any squeezing by itself.

More insight into these nonclassical effects is offered by phase space considerations within the framework of the Wigner function [37],

$$P_{|\psi\rangle}^W(x, p) = \pi^{-1} \int e^{2ipy} \psi^*(x+y) \psi(x-y) dy, \quad (2)$$

of the state  $|\psi\rangle$ , displayed in Fig. 6, which is not just the sum of the two Gaussian bell Wigner functions of the two coherent states, but involves an interference term [38, 39]. The phase difference,  $\varphi$ , can sharpen this peak and can even force it to assume negative values. It is this Wigner-interference term that is the deeper origin for the nonclassical features [26] of the photon count probability,  $W_m$ , and the squeezing.

The striking consequences of the superposition principle of quantum mechanics - a single coherent state, a quasi-classical state, the quantum mechanical superposition of two coherent states of identical average number of photons but well-defined phase difference, a highly nonclassical state that exhibits sub-Poissonian and oscillatory photon statistics - are the central result of the present section.

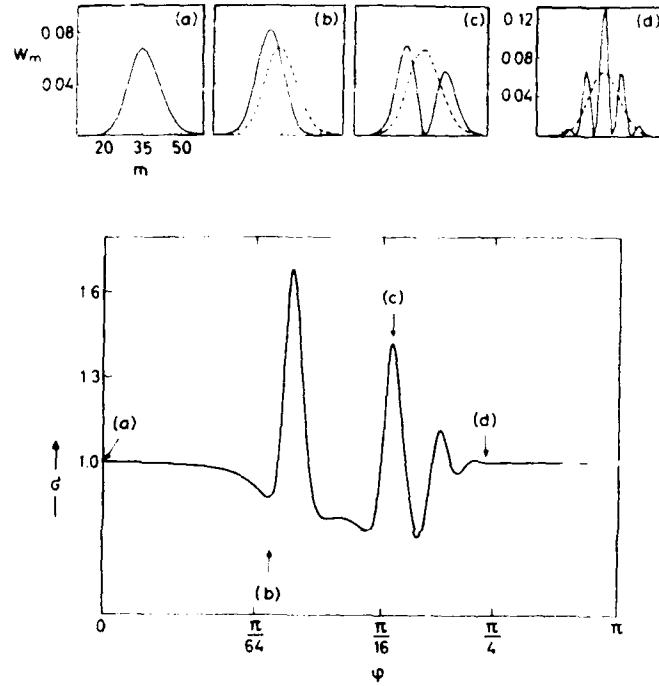


Figure 4: The photon count distribution  $W_m$  for the quantum mechanical superposition of two coherent states, is Poissonian (a), sub-Poissonian (b), super-Poissonian (c) or oscillatory (d) depending on the relative phase  $\varphi$  between the two coherent states as expressed by the normalized variance  $\sigma$ . The Poissonian distribution for  $\varphi = 0$  is plotted for comparison by a dashed line. To emphasize the oscillations in  $\sigma$  we have chosen a logarithmic scale for  $\varphi$ . (Here we have chosen  $\alpha^2 = 36$ .)



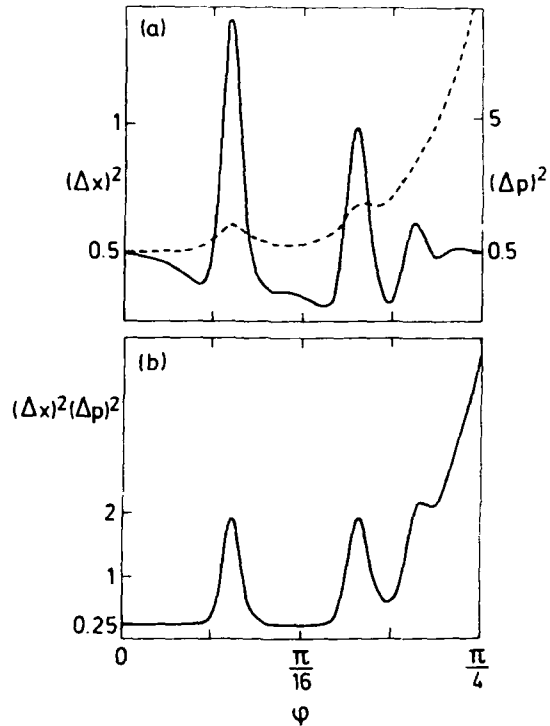


Figure 5: In the  $\varphi$ -domains of sub-Poissonian photon statistics (compare to Fig. 4) the  $x$ -variable exhibits squeezing, that is, the variance  $(\Delta x)^2$  depicted in (a) by a solid line falls below the coherent state value of  $(\Delta x)^2 = 0.5$ . The variance  $(\Delta p)^2$  shown in (a) by a dashed curve, however never falls below 0.5, but increases so as to satisfy the uncertainty product  $(\Delta x)^2 (\Delta p)^2 \geq 1/4$  indicated in (b). In the first two domains of squeezing the state  $|\psi\rangle$  is approximately a minimum uncertainty state whereas for  $\varphi \geq \pi/8$  the  $x$  variable is still squeezed, but not in a minimum uncertainty state. (Here we have chosen  $\alpha^2 = 36$ .)

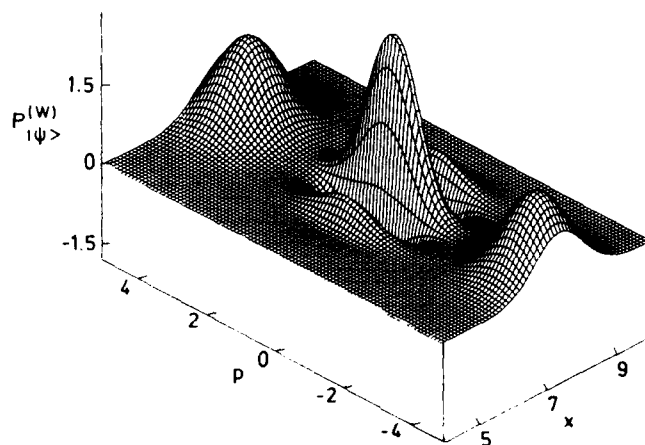


Figure 6: The Wigner function,  $P_{|\psi\rangle}^{(W)}$ , of the quantum mechanical superposition of two coherent states does not consist of only two Gaussian bells located in  $x$ - $p$  oscillator phase space at  $x = \sqrt{2} \alpha \cos \varphi/2$  and  $p = \pm \sqrt{2} \alpha \sin \varphi/2$  corresponding to two individual coherent states  $|\alpha e^{i\varphi/2}\rangle$  and  $|\alpha e^{-i\varphi/2}\rangle$  but involves an interference term located on the  $x$ -axis. This contribution originates from the quantum mechanical superposition of the two coherent states and the bilinearity of the Wigner distribution, Eq. (2), in the wave function. This interference hill can be narrower in the  $x$ -direction than the individual Gaussian bells giving rise to squeezing in the  $x$ -variable (Fig. 5a); or it can even take on negative values to create an oscillatory photon count probability,  $W_m$ , Fig. 4d. (Here we have chosen  $\alpha^2 = 36$  and  $\varphi = \pi/3$ .)

## 4 A Jump Shot at the Wigner Distribution

We conclude our salute to Prof. Eugene Paul Wigner by presenting an intuitive, physical argument to motivate the standard definition of the Wigner phase space distribution which we have already depicted for the case of *Schrödinger cats* in Sec. 3. The central ingredient of our approach [40] is the notion of a quantum jump.

Consider a quantum particle at position  $x$  moving in one dimension with momentum  $p$ . Here the uncertainty relation allows for a pseudo-probability only. In the spirit of Heisenberg's matrix mechanics, we replace the single position  $x$  by a quantum jump from an initial position  $x'$  to a final position  $x''$ . It is reasonable to identify  $x$  with the geometric center of these two positions of a single particle:  $x = (x' + x'')/2$ . But how to incorporate velocity or momentum into this picture of a particle hopping by an increment  $\xi \equiv x'' - x'$ ? The physics of de Broglie together with the mathematics of Fourier provides the immediate answer: transformation from  $\xi$  to  $k = p/\hbar$ . But what is the function we have to Fourier transform in this way? Heisenberg guides us in finding the answer: He represents an atomic Bohr transition — a quantum jump from an orbital of quantum number  $n'$  into one of quantum number  $n''$  — by a matrix element  $A_{n''n'} = \langle n''|A|n' \rangle$ . Here  $A$  stands for any dynamical variable, such as the dipole moment. Similarly we now consider the density operator  $\hat{\rho} = |\psi\rangle\langle\psi|$  for a pure state  $|\psi\rangle$  and its matrix elements

$$\rho(x'', x') \equiv \langle x''|\hat{\rho}|x' \rangle = \langle x''|\psi\rangle\langle\psi|x' \rangle = \psi(x'')\psi^*(x') \quad (3)$$

in position representation. This accounts for our quantum jump from  $x'$  to  $x''$ .

To bring out the structure of this jump we express the function  $\rho$  in terms of the mean position  $x$  and the increment  $\xi$ , which leads to

$$\sigma(x, \xi) \equiv \psi(x + \xi/2)\psi^*(x - \xi/2). \quad (4)$$

This is the quantity we want to Fourier transform with respect to the jump increment  $\xi$ . Thus we arrive at

$$\begin{aligned} P_{|\psi\rangle}^W(x, p) &= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \sigma(x, \xi) e^{-i\xi p/\hbar} d\xi \\ &= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \psi(x + \xi/2)\psi^*(x - \xi/2) e^{-i\xi p/\hbar} d\xi. \end{aligned} \quad (5)$$

With the integration variable  $y \equiv -\xi/2$  and  $\hbar = 1$  this expression becomes identical with the standard definition of the Wigner pseudo-distribution [37] of Eq. (2).

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## Ensemble or Individual System, Collapse or no Collapse: A Description of a Single Radiating Atom

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### Abstract

We use ordinary quantum mechanics to analyze a gedanken experiment of repeated photon measurements on an atom. The measurements are a short, but finite, time  $\Delta t$  apart. This leads to a coarse-grained time scale and to a description of photon counts from a single atom by a sample path of a classical stochastic process governed by quantum mechanics. It is shown that a collapse, or reduction, of the state vector at a no-photon ("null") measurement is not needed but may be used as a very convenient technical tool. We also show that within the coarse-grained time scale the axiomatic theory of continuous measurements of Davies and Srinivas can in the case of a radiating atom be obtained from ordinary quantum mechanics. Applications to macroscopic dark phases and quantum beats are indicated.

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### 1. Introduction

In recent years there have been exciting experiments on single atoms in Paul traps. Particularly interesting theoretical questions arise in connection with macroscopic dark periods or macroscopic quantum jumps of a single atom. Macroscopic dark periods of a single fluorescent atom were predicted by Dehmelt [1] for a system with two excited states, one rapidly decaying and the other metastable. Driving such a system by two lasers one intuitively expects frequent transitions from the ground state to the nonmetastable excited state with the subsequent emission of a spontaneous photon ("light period"). Once in a while there will be a transition to the metastable state, where the electron will stay for an extended period, and there will be no photons ("dark period", "electron shelving"). Quantum mechanically the situation is less clear than semiclassically because by the time development the atom will in general be in a coherent superposition of all three states, with an admixture of the rapidly decaying state always present, so that one may wonder if the dark periods really exist. These ideas have been analyzed semiclassically by the telegraph process [2] as well as quantum mechanically [3, 4, 5, 6]. Macroscopic dark periods were indeed found experimentally for single atoms in a Paul trap [7], confirming a spectacular quantum effect.

In the following we use only ordinary quantum mechanics. By this we mean the statistical interpretation as well as the reduction of the state vector after a measurement [8]. According to the statistical interpretation, as understood here, one deals with probability statements which are experimentally verified as frequencies, or relative numbers of events, in an ensemble. Instead of considering many systems one may also measure on an individual system, then prepare it again in the same way, measure again, and so on. In the case of interest here this would mean an ensemble of many atoms each with its own radiation field and laser, or a single atom with radiation field and laser, observed infinitely long, then prepared again as before, observed again, and so on.

For reductions we take the von Neumann-Lüders rule [9]. For example, if one measures the energy of a system in the state  $|\psi\rangle$  and finds a particular eigenvalue of the energy which is degenerate, then the state immediately after the measurement is given by the projection of the state  $|\psi\rangle$  onto the eigenspace of the respective eigenvalue, with ensuing normalization. Such a reduction is surely an idealization of the measurement process and only a substitute for a detailed theory of the measuring apparatus.

Such a change of the state vector will, of course, influence the results of subsequent measurements. As an example we consider a spin  $\frac{1}{2}$  in a magnetic field  $B = (B, 0, 0)$  in x-direction. The Hamiltonian is given by

$$H = \frac{1}{2} \hbar B \cdot \sigma = \frac{1}{2} \hbar B \sigma_1$$

As initial state  $|\varphi_0\rangle$  we take the one with spin up,  $\sigma_3 |\varphi_0\rangle = |\varphi_0\rangle$ . Then, at time  $t$ ,

$$|\varphi_t\rangle = \exp\{-i\sigma_1 B t/2\} |\varphi_0\rangle.$$

We now imagine the 3-component of the spin measured in two different ways.

a) We measure at time  $T = \pi/B$  only. This gives

$$\langle \sigma_3/2 \rangle_T = -1/2$$

b) We measure first at time  $T/2$ , then perform a reduction of the wave function, and measure again at time  $T$ . This gives

$$\langle \sigma_3/2 \rangle_T = 0$$

which differs radically from the previous outcome.

We now consider an atom which radiates photons. One then has the problem that if the photons are detected by a counter or seen by the eye of an observer one performs, in principle, a measurement, with all the consequences of the theory. Moreover, due to the stochastic nature of the emission times, how does one know *when* to measure?

## 2. Null Measurements

We imagine a general N-level atom possibly illuminated by one or several lasers and use an ideal detector of efficiency 1 to measure the photons emitted. We start at  $t = 0$  and



assume the first photon to be detected at time  $t_1$ . This is obviously a photon measurement, and one may try to describe it quantum mechanically as follows. At time  $t_0 = 0$  one has a no-photon state  $|0_{ph}\rangle$  together with an atomic state  $|\psi_A\rangle$ . Until time  $t_1$  one uses the unitary time development,

$$U(t_1, t_0) |0_{ph}\rangle |\psi_A\rangle,$$

and then one would have a reduction at time  $t_1$ . The probability  $P_0(t)$  to have found no photon at time  $t$ ,  $t \leq t_1$ , is then

$$P_0(t) = \sum_{j_A} |\langle j_A | \langle 0_{ph} | U(t, t_0) | 0_{ph} \rangle |\psi_A\rangle|^2$$

where the sum is over all atomic states. Defining the projector  $P_0$  by

$$\begin{aligned} P_0 &\equiv \sum_{j_A} |0_{ph}\rangle \langle j_A| \langle j_A| \langle 0_{ph}| \\ &\equiv |0_{ph}\rangle \langle 0_{ph}| \end{aligned} \quad (1)$$

one can write  $P_0(t)$  as

$$P_0(t) = \| P_0 U(t, t_0) |0_{ph}\rangle |\psi_A\rangle \|^2 \quad (2)$$

The right-hand side of this has been calculated by Porrati and Putterman [6]. The probability  $P_0(t)$  is important for the determination of photon rates and for dark periods, as first pointed out in Ref. [3].

There is an objection to this. In order to *know* that there had been no photon before  $t_1$ , one would have had to open the detector between 0 and  $t_1$  without detecting a photon. Opening the detector and *not* finding a photon, however, is also a measurement which may be called a "null measurement" [10]. Should each of these null measurements not also be accompanied by a reduction? In the above procedure leading to (2) they are not manifestly taken into account.

How many of these null measurements does one need? Ideally, infinitely many, and ultimately this line of reasoning would lead to "continuous measurements" [11]. However, it is well-known that the von Neumann-Lüders rule leads to difficulties with continuous measurements since in the idealized limit of measurements repeated infinitely fast it leads to a freezing of the state, the so-called quantum Zenon effect [12].

By an axiomatic extension of quantum mechanics Davies and Srinivas [13] have constructed a theory of continuous measurement which is adapted to counting rates. But for any particular situation one needs a phenomenological input or some intuition to obtain the explicit form. We will come back to this theory in the Section 5.

### 3. From Ensemble to Single System

We return to the N-level system of the last section and consider a *gedanken* experiment. To avoid freezing of the state due to the quantum Zenon effect we open the detector at

instances a very short – but finite – time  $\Delta t$  apart and perform, at each null measurement, an explicit reduction of the state vector. To obtain limits on  $\Delta t$  we require that  $\Delta t$  should be

- (i) much shorter than the life time of a level, i.e., much less than  $10^{-8}$ s;
- (ii) large compared to the time it takes a photon to travel the distance of an atomic diameter (essentially the correlation time in quantum optics).

We thus arrive at

$$\Delta t \cong 10^{-16} - 10^{-12} \text{ s} .$$

Therefore we have up to  $10^8$  additional reductions per second on top of the  $10^8$  photons or so.

*Reductions and sample paths.* We assume that the single atom from above, with its radiation field, is a member of an ensemble  $\mathcal{E}$  described by the initial state  $|0_{ph}\rangle |\psi_A\rangle$ . The atoms may or may not be driven by external pumping. At times  $\Delta t$  apart we imagine a measurement on each system of  $\mathcal{E}$ . By  $\mathcal{E}_0^{(\Delta t)}$  we denote the subensemble of all systems for which no photon was found at time  $\Delta t$ . Similarly,  $\mathcal{E}_0^{(n\Delta t)}$  denotes the subensemble of systems for which at times  $\Delta t, 2\Delta t, \dots, n\Delta t$  no photon was found. Clearly one has  $\mathcal{E}_0^{(\Delta t)} \supset \dots \supset \mathcal{E}_0^{(n\Delta t)}$ . According to the von Neumann-Lüders rule the subensemble  $\mathcal{E}_0^{(\Delta t)}$  is described by the state vector

$$P_0 U(\Delta t, 0) |0_{ph}\rangle |\psi_A\rangle / \|\cdot\|$$

where the projector  $P_0$  is given by (1). The  $n$ -th subensemble  $\mathcal{E}_0^{(n\Delta t)}$  is described by

$$P_0 U(n\Delta t, (n-1)\Delta t) P_0 \dots P_0 U(\Delta t, 0) |0_{ph}\rangle |\psi_A\rangle / \|\cdot\| \quad (3)$$

which shows the intermittent unitary time development interrupted by repeated reductions in between. The first subensemble  $\mathcal{E}_0^{(\Delta t)}$  has relative magnitude

$$\| P_0 U(\Delta t, 0) |0_{ph}\rangle |\psi_A\rangle \|^2$$

which in the statistical interpretation is the probability of finding no photon at time  $\Delta t$ . The relative magnitude of subensemble  $\mathcal{E}_0^{(n\Delta t)}$  compared to  $\mathcal{E}$  is

$$P_0(n\Delta t) \equiv \| P_0 U(n\Delta t, (n-1)\Delta t) P_0 U((n-1)\Delta t, (n-2)\Delta t) \dots \dots P_0 U(2\Delta t, \Delta t) P_0 U(\Delta t, 0) |0_{ph}\rangle |\psi_A\rangle \|^2 \quad (4)$$

which gives the probability of finding no photon at the times  $\Delta t, \dots, n\Delta t$ . This expression is quite different from the previous one in (2).

At each measurement on the individual system under consideration chance decides according to the probabilities  $P_0(\Delta t), \dots, P_0(n\Delta t), \dots$  whether or not a photon is detected. This behavior can be simulated by flipping a coin weighted with the conditional

probabilities for each measurement. Once a photon is detected and absorbed our individual system becomes a member of a new ensemble  $\mathcal{E}'$ . How to describe the new ensemble is in general quite a subtle question and will be discussed elsewhere. Here we will assume for simplicity that the system is reset to  $|0_{ph}\rangle|0_A\rangle$  where  $|0_A\rangle$  is the atomic ground state [14]. With no external pumping the state remains constant after  $t_1$ . If a driven atom starts out from the ground state and if at time  $t_1 = (n+1)\Delta t$ , say, the first photon is detected then the procedure starts again, with a time  $t_2$  for the next photon detection, and so forth.

Thus in the above approach the photon and no-photon detection times for a single atom form a sample path of a classical stochastic process which is governed by quantum mechanics. Without pumping the sample path terminates. In the simple case of a driven atom which is reset to the ground state after a photon detection one obtains a path of a renewal process since after each photon detection the memory is lost. In general, however, this need not be so.

*Evaluation of  $P_0(n\Delta t)$ .* We can rewrite the state vector of the  $n$ -th subensemble in (3) as

$$\begin{aligned} \langle 0_{ph} | \langle 0_{ph} | U(n\Delta t, (n-1)\Delta t) | 0_{ph} \rangle \cdots \langle 0_{ph} | U(\Delta t, 0) | 0_{ph} \rangle | \psi_A \rangle / \|\cdot\| \\ \equiv |0_{ph}\rangle | \psi_A(n\Delta t) \rangle / \|\cdot\| \end{aligned} \quad (5)$$

where the expressions  $\langle 0_{ph} | U(n\Delta t, (n-1)\Delta t) | 0_{ph} \rangle$  are now purely atomic operators and where  $|\psi_A(n\Delta t)\rangle$  is a vector in the atomic space of norm less than 1, due to the repeated reductions. One has

$$P_0(n\Delta t) = \| \psi_A(n\Delta t) \|^2. \quad (6)$$

Since  $\Delta t$  is small one can use ordinary perturbation theory to evaluate (5). A standard Hamiltonian is [15]

$$H = H_A + H_F + D \cdot (E + E_L(t)) \quad (7)$$

where  $H_A$  is the purely atomic part,  $H_F$  the radiation field part,  $E$  the quantized field,  $E_L(t)$  a possible classical field of lasers, and  $D$  the atomic dipole operator.

$$D = e \sum |i_A\rangle \langle i_A| X |j_A\rangle \langle j_A|.$$

Going over to the interaction picture with respect  $H_0 = H_A + H_F$  one obtains in second order

$$\begin{aligned} \langle 0_{ph} | U(m\Delta t, (m-1)\Delta t) | 0_{ph} \rangle \\ = e^{-iH_A m\Delta t} \left\{ 1_A - i\hbar^{-1} \int_{(m-1)\Delta t}^{m\Delta t} dt' \langle 0_{ph} | H_I(t') | 0_{ph} \rangle \right. \\ \left. - \hbar^{-2} \int_{(m-1)\Delta t}^{m\Delta t} dt' \int_{(m-1)\Delta t}^{t'} dt'' \langle 0_{ph} | H_I(t') H_I(t'') | 0_{ph} \rangle \right\} e^{iH_A(m-1)\Delta t} \end{aligned}$$

The expression in the curly brackets is easy to evaluate as indicated in footnote [16] where for  $\Delta t \rightarrow 0$  the quantum Zenon effect appears automatically; a particular case is

evaluated in [17]. E.g., for a three-level V-system with states  $|0\rangle$ ,  $|1\rangle$ ,  $|2\rangle$  and with transition frequencies  $\omega_1$ ,  $\omega_2$  and laser frequencies  $\omega_{L1}$ ,  $\omega_{L2}$  the curly bracket can be written as [18]

$$1_A = i\hbar^{-1} \int_{(m-1)\Delta t}^{m\Delta t} dt' \left[ \sum_j \left( 1/2\Omega_j |j\rangle\langle 0| e^{i(\omega_j - \omega_L)t'} + h.c. - \frac{1}{2} i A_j |j\rangle\langle j| \right) \right. \\ \left. - i \gamma_{12} |1\rangle\langle 2| e^{-i\omega_{21}t'} - i \gamma_{21} |2\rangle\langle 1| e^{i\omega_{21}t'} \right] \\ \cong \exp \left\{ -i \hbar^{-1} \int_{(m-1)\Delta t}^{m\Delta t} dt' [\dots] \right\}$$

where

$$\gamma_{ij} = \langle 0 | X | i \rangle \cdot \langle j | X | 0 \rangle \omega_j^3 \epsilon^2 / 6\pi\epsilon_0 \hbar c^3 \quad (8)$$

and where  $A_j = 2\gamma_{jj}$  are the Einstein coefficients and  $\Omega_j$  the Rabi frequencies, which are proportional to the laser amplitudes. Now the product over  $m$  from 1 to  $n$  can be performed, leading to a time-ordered expression of the form

$$|\psi_A(n\Delta t)\rangle = \mathcal{T} \exp \left\{ -i \hbar^{-1} \int_0^{n\Delta t} dt' H_{red}(t') \right\} |\psi_A(0)\rangle \quad (9)$$

The "reduced Hamiltonian"  $H_{red}$  is nonhermitian. In particular, for the V-system one obtains for  $H_{red}$  in matrix form with respect to the basis  $|0\rangle$ ,  $|1\rangle$ ,  $|2\rangle$

$$\hbar H_{red} = \begin{pmatrix} 0 & \Omega_1/2 e^{i\omega_{L1}t} & \Omega_2/2 e^{i\omega_{L2}t} \\ \Omega_1/2 e^{-i\omega_{L1}t} & E_1 - i A_1/2 & -i \gamma_{12} \\ \Omega_2/2 e^{-i\omega_{L2}t} & -i \gamma_{21} & E_2 - i A_2/2 \end{pmatrix} \quad (10)$$

We now introduce a "coarse-grained" time,

$$t = n\Delta t, \quad n = 0, 1, 2, \dots \quad (11)$$

Then (9) can be written as

$$\frac{d}{dt} |\psi_A(t)\rangle = -i \hbar^{-1} H_{red} |\psi_A(t)\rangle \quad (12)$$

and one has

$$P_0(t) = \|\psi_A(t)\|^2. \quad (13)$$

*Collapse or no collapse:* How does this result compare with that calculated from (2)? Surprisingly, within the *coarse-grained* time scale our  $P_0(t)$ , obtained by  $n$ -fold reductions, coincides completely with that obtained with no reductions [19]. The  $n$ -fold repeated reductions do not seem to have an effect except for the temporal coarse-graining! The

approach of this section, however, has the advantage of mathematical simplicity since straightforward perturbation theory can be used.

*Ensemble or individual system:* The coarse-grained time can, for all questions relating to time differences much larger than  $\Delta t$ , be considered as practically continuous. On the coarse-grained time scale the detection of photons from an atom can thus be described by a sample path of a classical stochastic process with continuous time, a process which is governed by quantum mechanics. Without external pumping these paths terminate and then it is clear that one can make no definite statements about an individual system. With external pumping, however, this is possible due to the ergodic property of the process. Ergodicity allows one to replace time averages over a sample path by ensemble averages which in turn can be calculated by probability theory. For a renewal process this is more or less evident, and for the general case ergodicity is physically expected. This explains that although in the statistical interpretation quantum mechanics deals with ensembles it can make certain definite predictions for a *single* driven atom.

#### 4. Applications

*Macroscopic dark periods* [3, 4, 6, 17]. For the Dehmelt V-system the off-diagonal elements  $\gamma_{ij}$  in  $H_{red}$  can be neglected. This is most easily seen by going to an interaction picture with respect to an auxiliary Hamiltonian  $H'_0 = \omega_{L1} |1\rangle\langle 1| + \omega_{L2} |2\rangle\langle 2|$  which removes the time dependence from the  $\Omega_i$ -terms and adds an  $\exp\{\pm i(\omega_{L1} - \omega_{L2})t\}$  to the  $\gamma_{ij}$ -terms. This produces rapid oscillations which lead to negligible contributions. With their neglect one obtains a  $P_0(t)$  identical to that of Cohen-Tannoudji and Dalibard [3], and their analysis applies. Due to the fact that one level is metastable, level 2 say, one has  $A_2 \ll A_1$  and this leads  $P_0(t)$  to split into the sum of two parts, one rapidly decaying roughly like  $\exp\{-A_1 t\}$ , the other very slowly decaying, roughly like  $\exp\{-A_2 t\}$ , and with a very small factor in front. There is thus a small probability to reach this region where the second term prevails — i.e. very many photon detections will be needed —, but once this region is reached one has to wait a very long time for the next photon since its probability density is  $w_1(t) = -P'_0(t)$ . During this dark period,  $|\psi_A(t)\rangle$  is not completely in the metastable state  $|2\rangle$  but has a  $|1\rangle$ -component. Hence in contradiction to the semi-classical electron-shelving picture there is a finite probability — in fact it can be approximately 1/2 — that the next photon does not originate from the transition metastable to ground state [6, 17].

*Quantum beats.* We consider a three-level V-system whose upper levels have only a very small energy difference  $\hbar\delta\omega$  and no laser ( $\Omega_i = 0$ ). We consider the decay from one of the excited states. In this case the off-diagonal terms  $\gamma_{ij}$  in (10) become important.  $P_0(t)$  will now contain oscillating terms which leads to a non-exponential decay, the well-known quantum beats. Here it turns out that these beats also occur for the decays of levels 1

and 2 separately, not only for coherent superpositions as required in some textbooks [20].

*Macroscopic dark periods without metastable state* [21]. Again we consider a V-system with very small upper-level separation  $\hbar\delta\omega$  and irradiate it with a *single* laser tuned to the vicinity of the upper levels. The Rabi frequency is denoted by  $\Omega$ . We now assume in addition that the transition dipole moments are *parallel*. For  $\delta\omega \ll \Omega$  light and dark periods are predicted. Their mean duration  $T_L$  and  $T_D$  can be explicitly calculated for arbitrary laser detuning. In particular, if the transition dipole moments are equal and if the laser is tuned to the 0 - 1 or 0 - 2 transition one finds

$$\begin{aligned} T_L &= 4 \Omega^2 / A_1 (\delta\omega)^2 \\ T_D / T_L &= \Omega^2 / 2 (\delta\omega)^2 . \end{aligned}$$

If the laser is tuned exactly halfway between the upper levels the surprising phenomenon is predicted, for any  $\delta\omega$ , that after the emission of a number of photons the atom will stop fluorescing completely ( $T_D \rightarrow \infty$ ). This is related to a nonabsorption resonance in gases [22, 23].

### 5. Connection with the Continuous Measurement Theory of Davies and Srinivas

Davies and Srinivas [13] have extended the axiomatics of quantum mechanics by postulates for 'homogeneous quantum counting processes'. In particular, their postulates imply the existence of two 'superoperators'  $J$  and  $S_t$  which map trace class operators to trace class operators and satisfy certain properties. For an individual system of an ensemble described by a density matrix  $\rho$  their meaning is as follows.  $Tr(S_t \rho)$  is the probability of finding no counting event in  $[0, t]$ , and the probability density  $w(t_1, \dots, t_n; [0, t])$  for finding a counting event exactly at the times  $t_1, \dots, t_n$  in  $[0, t]$  is given by

$$w(t_1, \dots, t_n; [0, t]) = Tr(S_{t-t_n} J S_{t_n-t_{n-1}} J \dots J S_{t_2-t_1} J S_{t_1} \rho) \quad (14)$$

For a particular system  $J$  and  $S_t$  have to be determined phenomenologically or by intuition.

Since we have an explicit expression for  $P_0(t)$  we can *derive* the form of the superoperators  $J$  and  $S_t$ . Only ordinary quantum mechanics is used and no additional postulates are required. We illustrate this for the three-level V-system, two excited states coupled to a common ground state, with two lasers. Let the initial atomic state be  $\rho$ . The corresponding  $P_0(t; \rho)$ , valid until the detection of the first photon, is obtained from (9) and (10) by carrying these equations over to density matrices in an obvious way. Then, after the detection of the  $i$ -th photon the atom is each time reset to the ground state, and the corresponding  $P_0(t - t_i; |0\rangle)$  is obtained from (10) with  $|\psi_A\rangle = |0\rangle$ . By standard arguments of classical probability theory one then finds for the  $n$ -photon probability density  $w(t_1, \dots, t_n; [0, t])$  [17]

$$w(t_1, \dots, t_n; [0, t]) = P_0(t - t_n; |0\rangle) w_1(t_n - t_{n-1}; |0\rangle) \dots w_1(t_2 - t_1; |0\rangle) w_1(t_1; \rho) \quad (15)$$

where  $w_1 = -P'_0$ .

We now define superoperators  $J$  and  $S_t$  by

$$J \rho := |0\rangle\langle 0| \text{Tr}\{i(H_{red} - H_{red}^*)\rho\} \quad (16)$$

$$S_t \rho := \mathcal{T} \exp\left\{-i\hbar^{-1} \int_0^t dt' H_{red}(t')\right\} \rho \left[\mathcal{T} \exp\left\{-i\hbar^{-1} \int_0^t dt' H_{red}(t')\right\}\right]^* \quad (17)$$

We note that  $J$  is time independent, by (10). Now, first of all it is apparent that  $\text{Tr}(S_t \rho)$  coincides with  $P_0(t; \rho)$ . Inserting (16) and (17) on the right-hand side of (14) one finds by a calculation similar to one in Ref. [17] that this agrees with (15).

Using only ordinary quantum mechanics we have in this way exhibited operators that satisfy the requirements of the quantum counting process of Ref. [13]. There is, however, a severe conceptual *proviso*. In our approach we are dealing with a coarse-grained time, and the above seemingly continuous variables  $t_1, \dots, t_n$  in (15) are, truly speaking, discrete. We thus arrive at the conclusion that the "continuous" measurement theory of Ref. [13] can, at least in the case considered here, be derived from ordinary quantum mechanics if one relaxes the "continuous" and goes over to a coarse-grained time scale.

## 6. Discussion

Our intuitive idea is that it should make no difference for the photon statistics whether or not all photons are actually observed once they are sufficiently far away from the atom and do no longer interact with it. In a cavity with reflecting walls this would evidently not be true. Therefore we think that the results of Section 3, together with ordinary probability theory, can also be applied to situations where only a part of the photons are actually detected. This is substantiated by the result that for the photon statistics it makes no difference whether or not reductions are performed at no-photon ("null") measurements. With these reductions, however, elementary perturbations theory can be used since  $\Delta t$  is very small, and this simplifies the analysis considerably. In this sense the null reductions may be considered as technical tool.

No attempt has been made to give a detailed theory of the measurements, but at each measurement and depending on its outcome a straightforward reduction of the state vector according to the von Neumann-Lüders rule is carried out. These reductions put an individual system each time into a particular subensemble, and this branching into subensembles may depend on the system under consideration. In the simple case considered in Section 3 the atom is reset to the ground state after a photon detection, and for a driven atom one then obtains a sample path of a renewal process. In general, however, the resetting will not always be to the same state and might in fact even be time dependent. In this way we arrive at the result that on a time scale much coarser than  $\Delta t$  the photon emissions of an atom can be regarded as a sample path of a classical stochastic process obtained from ordinary quantum mechanics. Ergodicity allows one to replace time averages over a sample path by ensemble averages, and such quantities can thus be calculated for a *single* radiating atom.

It should be pointed out, however, that our *gedanken* experiment with its repeated reductions and temporal coarse-graining and its reduced description of the atomic state is not applicable to all questions encountered for a single radiating atom. The term 'emission' of a photon appears to be imprecise and should be replaced by 'detection' since it is doubtful whether it makes sense quantum mechanically to speak about emission without an actual observation. If one is interested in spectral distributions of the emitted light repeated observations would cause changes, as is clear from the time-energy uncertainty relation. For such observables like the spectrum, which in a sense is complementary to photon counting, one will need other facets of the complete wave function of atom plus radiation field. The true wave function contains all information and gives a 'holistic' description of all aspects of the system while some partial aspects, as the photon statistics, may be amenable to a simplified description.

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$$\int_{(m-1)\Delta t}^{\Delta t} dt' \int_{(m-1)\Delta t}^{t'} dt'' \sum_{k\lambda} g_{ijk\lambda} g_{lmk\lambda} \exp\{-i(\omega_k - \omega_{ij})t'\} \exp\{i(\omega_k - \omega_{lm})t''\} |i\rangle\langle j|m\rangle\langle l|$$

where  $\hbar \omega_{ij} = E_i - E_j$  and where the coupling constants  $g_{ijk\lambda}$  contain the transition dipole moments for the  $i$ - and  $j$ -levels. The exponentials can be written as

$$\exp\{i(\omega_{ij} - \omega_{lm})t'\} \exp\{-i(\omega_k - \omega_{lm})(t' - t'')\}.$$

The integral over  $\tau = t' - t''$  gives approximately  $\pi \delta(\omega_k - \omega_{lm})$ , up to a line shift which can be neglected or incorporated in the levels. Alternatively one can use that

$$\sum_{k\lambda} g_{ijk\lambda} g_{lmk\lambda} \exp\{-i(\omega_k - \omega_{lm})\tau\}$$

is sharply peaked at  $\tau = 0$  and essentially vanishes if  $\tau$  is much larger than the correlation time. This gives the lower limit for  $\Delta t$  required in (ii). The relevance of the correlation time was pointed out to us by R. Reibold (private communication). If, however,  $\Delta t$  is much smaller than the correlation time, in particular if  $\Delta t \rightarrow 0$ , then the double integral behaves as  $(\Delta t)^2$  and does not contribute. Only the purely atomic part remains and there is no photon emission, in accordance with the quantum Zenon effect.

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## Magnetotransport in Two-Dimensional Mesoscopic Superlattices: Experimental Consequences of Commensurability Effects

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Electrons restricted to a plane perpendicular to a constant, homogeneous magnetic field  $B$  and subject to a weak lateral superlattice potential have unusual properties. The energy spectrum plotted versus  $1/B$  shows a self-similar fractal structure known as "Hofstadter's butterfly". The origin of this interesting behavior is the interplay of two independent length scales, the period  $a$  of the two-dimensional superlattice and the magnetic length  $l = (ch/eB)^{1/2}$ . Mathematical aspects of this problem have been discussed during the last four decades. But only with modern microstructuring techniques it becomes possible to fabricate samples for which measurable consequences of these commensurability effects can be expected in an accessible range of magnetic field values and temperature. Recent experiments related to this mathematics will be discussed.

### 1 Introduction

With the method of molecular-beam-epitaxy, GaAl - Al<sub>x</sub>Ga<sub>1-x</sub>As heterostructures (with  $x \sim 0.3$ ) can be fabricated which contain a nearly ideal two-dimensional electron gas (2DEG), with a mobility as high as  $\mu \geq 10^6 \text{ cm}^2/\text{Vs}$ . The 2DEG is defined by the lowest energy eigenstate in the triangular quantum well, formed by the conduction band discontinuity at the interface and the self-consistent electrostatic potential in the GaAs [1]. At low temperatures ( $T \leq 5\text{K}$ ), the energy gap ( $\sim 20\text{meV}$ ) to the next state in the well prohibits motion of the electrons in the  $z$ -direction perpendicular to the interface, whereas the electrons move freely, with the effective mass  $m = 0.067m_0$  of GaAs, in the  $x$ - $y$ -plane parallel to the interface. In a strong homogeneous magnetic field  $\vec{B} = B\hat{z}$  the cyclotron motion of the electrons is quantized, and the single-electron energy spectrum is purely discrete but highly degenerate. Of course, in real systems this degeneracy is lifted by random impurity scattering, and the discrete Landau levels (LLs) are collision-broadened. This system shows many interesting properties, such as the integer and the fractional quantum Hall effect [2], and at still higher  $B$ -values, eventually a magnetic-field induced Wigner crystallization.

In the following we will discuss properties of a *periodically modulated* 2DEG in a perpendicular magnetic field, but we will consider a situation relevant to experiments, where a lateral superlattice is imposed on the 2DEG by external electrostatic fields, and where it is not necessary to take the Coulomb interaction explicitly into account. A 2DEG subject to such conditions has unusual properties due to the interplay of two length scales, the period  $a$  of the superlattice and the magnetic length  $l = (ch/eB)^{1/2}$ . The single-particle energy spectrum for such a situation has been investigated theoretically by many authors [3-6]. For a two-dimensional (2D) super-

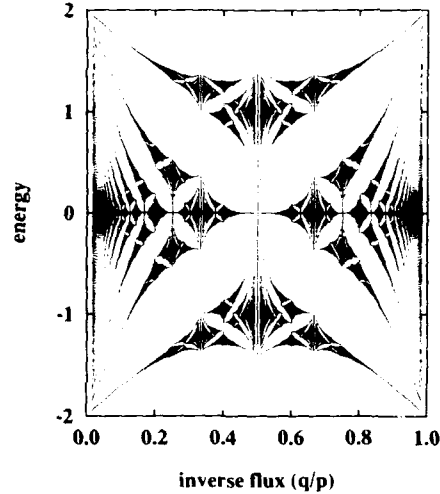


Figure 1: Hofstadter's butterfly [5]. The scaled-energy eigenvalue spectrum  $\epsilon_n$  of Eq. (7) was calculated for  $V_y = V_x$  and all rational values  $p/q$  of the flux ratio, Eq.(1), with relative prime integers  $p$  and  $q$  satisfying  $1 \leq q < p \leq 60$ . The allowed energy intervals (subbands) are plotted, in units of  $V_x$ , as vertical lines over the inverse flux ratio  $q/p$ .

lattice, the energy spectrum as a function of the magnetic field  $B$  is determined by the complicated self-similar, fractal structure, which is shown in Fig. (1) and known as Hofstadter's "butterfly" [5]. This result is obtained in the two complementary, but mathematically equivalent limits [4] of, first, a strong lattice potential and a weak magnetic field in the tight-binding approximation [3,4,5], and of, second, a weak periodic perturbation in a Landau quantized 2DEG [4,6]. In the second case, which we will consider in the following, one finds that each LL splits into  $p$  subbands if

$$Ba^2/\Phi_0 \equiv a^2/2\pi l^2 = p/q, \quad (1)$$

i.e., if the flux  $Ba^2$  per unit cell is a rational multiple of the flux quantum  $\Phi_0 = hc/e$ . Some important results on the single-electron spectrum for this situation will be recalled in sect. 2.

On the basis of this energy spectrum the Hall conductivity has been predicted to assume quantized values, integer multiples of  $e^2/h$ , if the Fermi energy falls into a gap between such subbands [7,8]. Also the diagonal components of the conductivity tensor for a 2DEG in a one-dimensional (1D) superlattice have been calculated before, and a pronounced, strongly anisotropic dependence on the filling fraction within a LL was predicted [9]. Up to now, none of these theoretical predictions could be verified experimentally, since flux ratios in a suitable range ( $0.1 \leq a^2/2\pi l^2 \leq 10$ ) could not be achieved. With the modern techniques of microstructuring, however,

it becomes possible to approach this situation. Lateral superlattices with periods  $a \sim 100nm$  and of different modulation strength have been produced, and the resulting effects on the transport properties of the 2DEG have been reviewed recently [10,11]. We keep the following discussion short and refer the interested reader to these reviews. In sect. 3 we present transport experiments on samples with weak lateral superlattice potentials, with distinctly different results for 1D and for 2D superlattices, respectively [12-16]. In sect. 4 we sketch a transport theory based on Kubo's formulas, which explains these differences as resulting from the subband splitting of the Hofstadter-type spectrum. This provides the first indication for the realization of the Hofstadter-type spectrum in these semiconductor systems. An explicit resolution of this spectrum is, however, not yet possible and requires superlattices with still smaller periods.

## 2 Hofstadter-type energy spectrum

Since, in the experiments, the period of the superlattice ( $a \sim 300nm$ ) is much smaller than the electronic mean free path ( $\lambda_{free} \geq 10\mu m$ ), we include the effect of the modulation potential, which we assume to be of the simple form

$$V(x, y) = V_x \cos(Kx) + V_y \cos(Ky), \quad (2)$$

with the same period  $a = 2\pi/K$  in the x- and the y-direction, in the single-electron energy-spectrum. We consider the model Hamiltonian

$$H = \frac{1}{2m} \left[ \vec{p} + \frac{e}{c} \vec{A}(\vec{r}) \right]^2 + V(\vec{r}), \quad (3)$$

where  $-e$  is the charge of an electron,  $\vec{A}(\vec{r}) = (0, xB)$  is the vector potential in Landau gauge, and all the vectors are in the x-y-plane. Classically, this is, for  $V_x V_y \neq 0$  a non integrable Hamiltonian system and leads to chaotic motion in certain energy regions [16]. The quantum treatment starts from the unmodulated ( $V(\vec{r}) \equiv 0$ ) system which has energy eigenvalues  $E_n = \hbar\omega_c(n + \frac{1}{2})$  and eigenstates  $|n, k_y\rangle$  with wavefunctions  $\psi_{nk_y}(x, y) = L_y^{-1/2} \exp(ik_y y) \phi_n(x - x_0)$ , where  $x_0 = -l^2 k_y$  and  $\phi_n$  is an oscillator function. The modulation potential  $V(x, y)$  is assumed to be so weak, that it does not couple different Landau levels and can be taken into account in first order perturbation theory. Then the relevant matrix elements of the superlattice potential follow from

$$\langle nk'_y | \exp(i\vec{q} \cdot \vec{r}) | n, k_y \rangle = \delta_{k'_y, k_y + q_y} \exp\left[-\frac{i}{2} l^2 q_x (k'_y + k_y)\right] \mathcal{L}_n(Q), \quad (4)$$

with  $Q = \frac{1}{2} l^2 q^2$ ,  $q = (q_x^2 + q_y^2)^{1/2}$ , and  $\mathcal{L}_n(Q) = \exp(-\frac{1}{2}Q) L_n(Q)$ , where  $L_n(Q)$  is a Laguerre polynomial.

For a 1D modulation potential  $V(\vec{r}) = V_x \cos(Kx)$ , the Hamiltonian (3) is, in the Landau gauge, translation-invariant in the y-direction. Thus,  $x_0 = -l^2 k_y$  is still a good quantum number. One obtains the energy eigenvalues

$$E_n(x_0) = \hbar\omega_c \left(n + \frac{1}{2}\right) + \mathcal{L}_n V_x \cos(Kx_0), \quad (5)$$

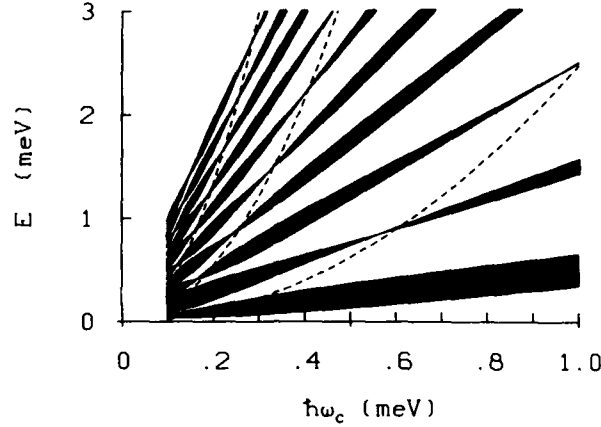


Figure 2: Sketch of the modulation-broadened Landau fan diagram, calculated from Eq. (5) with  $V_x = 0.2\text{meV}$  and  $2\pi/K = 200\text{nm}$ . Allowed energy regions in the Landau bands with  $n = 0, 1, \dots, 9$  are shown black. The flat band energies,  $\varepsilon_\lambda$  for  $\lambda = 1, 2, 3$ , are indicated by dashed lines.

where  $\mathcal{L}_n = \mathcal{L}_n(\frac{1}{2}l^2K^2)$ . It was checked [17] that this first-order result is, under the conditions of the experiment [12,13], an excellent approximation for  $B > 0.1T$ , but breaks down for  $B \rightarrow 0$ . The Laguerre polynomials in  $\mathcal{L}_n$  lead to an oscillatory bandwidth of the modulation-broadened Landau levels, as is illustrated in Fig. (2). The bandwidth becomes zero, i.e., the Landau bands become flat, at the zeroes of the Laguerre polynomials. For large  $n$ , this flat band condition can be expressed in terms of the cyclotron radius  $R_n = l\sqrt{2n+1}$  as

$$2R_n = a\left(\lambda - \frac{1}{4}\right), \quad \lambda = 1, 2, \dots \quad (6)$$

The corresponding "flat band energies"  $\varepsilon_\lambda = E_{n_\lambda} = \frac{1}{8}m\omega_c^2 a^2(\lambda - \frac{1}{4})^2$  appear in Fig. (2) as (dashed) parabolas. The eigenstates carry current in  $y$ -direction,  $\langle v_y \rangle = \hbar^{-1}dE_n(-l^2k_y)/dk_y$ , the quantum analogue of the classical Hall drift.

For the 2D modulation (2), the modulations in the  $x$ - and in the  $y$ -direction have the same period and lead to the same factors  $\mathcal{L}_n$  in Eq. (4). Thus, the energy eigenvalues can be written in the form

$$E_{n;\alpha} = \hbar\omega_c\left(n + \frac{1}{2}\right) + \mathcal{L}_n \varepsilon_\alpha, \quad (7)$$

i.e., all LLs have the same internal structure, which is represented by the  $n$ -independent  $\varepsilon_\alpha$ . (Here we make use of the fact that a weak modulation does not mix the LLs. The case of arbitrary LL mixing has also been treated [18].) The factor  $\mathcal{L}_n(Q)$ , which has an oscillatory  $n$ -dependence and determines the width of the  $n$ -th Landau level, is the same as in the corresponding 1D case and illustrated in Fig. 2. The modulation in  $y$ -direction couples Landau states with center coordinates differing by

by integer multiples of  $l^2K$ , so that the eigenstates can be written in the form  $|n; \alpha\rangle = \sum_{\lambda=-\infty}^{\infty} c_{\lambda}(\alpha) |n, k_y + \lambda K\rangle$ . Considering the magnetic translation group [19] it can be shown that, if the commensurability condition (1) holds, the quantum numbers  $\alpha = (\vec{k}, j)$  can be chosen as a wavevector  $\vec{k}$  defined on the magnetic Brillouin zone (MBZ)  $|k_x| \leq \pi/aq$ ,  $|k_y| \leq \pi/a$  [6,15] and a discrete number  $j = 1, \dots, p$  that labels the eigenstates of an effective  $p \times p$  Hamiltonian matrix  $h^{(p)}(\vec{k})$  with eigenvalues  $\epsilon(\vec{k}; j)$ , which determine  $p$  energy subbands  $E_{n,j}(\vec{k}) = E_n + \mathcal{L}_n \epsilon(\vec{k}; j)$  per LL. The allowed  $\epsilon(\vec{k}; j)$  plotted versus the ratio  $q/p$  of Eq.(1) yield the self-similar [20,21] pattern shown in Fig. 1.

Contrary to the unmodulated case, the velocity operator now has nonzero intra-LL matrix elements which are diagonal in  $\vec{k}$  but not in the subband quantum number  $j$ . It can be shown that the matrix elements of the velocity operators can be calculated from the  $p \times p$  matrices [6,15]

$$v_{\mu}^{(p)}(\vec{k}) = \hbar^{-1} \mathcal{L}_n \partial h^{(p)}(\vec{k}) / \partial k_{\mu}, \quad (8)$$

where  $\mu = x, y$ , and the matrix  $v_{\mu}^{(p)}(\vec{k})$  is independent of the modulation amplitude  $V_{\mu}$  in  $\mu$ -direction and depends only on the modulation in the other direction, just as the classical drift velocity.

### 3 Experiments

Typical results for the magnetoresistivity obtained by Weiss et al. [12] on samples with a 1D superlattice are shown in Fig. 3. For the simultaneous measurement of the resistance parallel ( $\rho_{\parallel} = \rho_{yy}$ ) and perpendicular ( $\rho_{\perp} = \rho_{xx}$ ) to the equipotential lines, L-shaped samples were used as sketched in the inset of Fig. 3. In addition to the usual Shubnikov-de Haas (SdH) oscillations, which occur for  $B > 0.5T$ , pronounced novel oscillations appear in  $\rho_{\perp}$  and weaker oscillations with a phase shift of  $180^{\circ}$  are seen in  $\rho_{\parallel}$ , whereas the Hall resistance shows no sizable oscillations. Similar to the SdH oscillations, the novel oscillations are periodic in  $1/B$ . The period depends, however, on the period  $a$  of the modulation, with minima of  $\rho_{\perp}$  and maxima of  $\rho_{\parallel}$  if the cyclotron radius  $R_c = l^2 k_F$  at the Fermi level  $E_F = \hbar^2 k_F^2 / 2m$  satisfies the commensurability condition (6) with  $R_n = R_c$ .

The first theoretical explanation of these findings was based on the quantum mechanical picture [13,22]: the group velocity ( $v_y$ ) according to Eq. (5) leads a "band conductivity" [22]  $\sigma_{yy}$ , which determines the novel oscillations of  $\rho_{xx}$ . However, this "band conductivity" can also be obtained from a quasi-classical calculation [23] emphasizing the importance of the guiding center drift of the cyclotron orbits in the weak electric field of the 1D superlattice.

The weaker antiphase oscillations of  $\rho_{\parallel}$ , which are observed in all the high-mobility samples [12,10], can neither be explained by the quasi-classical approach [23], nor by the quantum mechanical approach assuming *ad hoc* a constant relaxation time [13]. These oscillations reflect the quantum oscillations of the electron-impurity scattering rate [17], which is determined by the thermal average of the square of the density of states (DOS) [1]. As an immediate consequence of the band-width

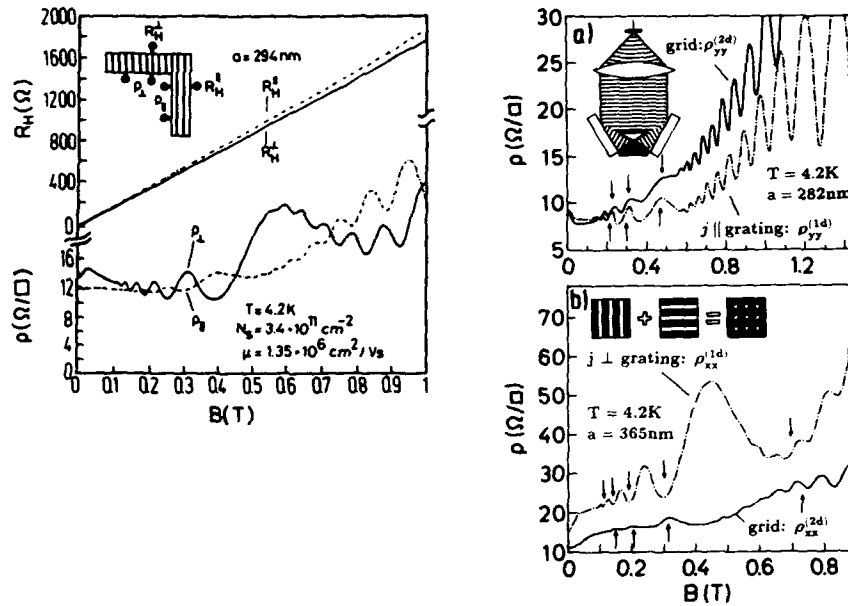


Figure 3: Magnetoresistivity  $\rho$  and Hall resistance  $R_H$  parallel and perpendicular to the interference fringes (equipotential lines). The inset defines the use of the indices  $\perp$  and  $\parallel$  with respect to the latter. From Ref. [12].

Figure 4: Magnetoresistance in a grating (with modulation in x-direction) and a grid potential for two periods and samples. The insets sketch the creation of the potential (a) by *in situ* holographic illumination, and (b) the resulting pattern. The arrows indicate the flat band situation defined by Eq.(6) (the second illumination always increases the electron density). The grid potential, created as superposition of two gratings at right angles, suppresses the bandconductivity in high-mobility samples, and the oscillations due to the scattering rate (with maxima at the arrow positions) dominate. From Ref. [14].

oscillations, the Landau DOS shows an  $B^{-1}$ -periodic amplitude modulation. This results in envelope oscillations of the magnetocapacitance signal [24].

Recently, also the effects of *two-dimensional* superlattices on a 2DEG have been studied. Figure 4 summarizes typical results of a series of experiments in which a grid modulation with square-lattice symmetry was created in two steps by holographic illumination [12,13]. In a first step a split laser beam reflected from two mirrors, as sketched in the inset of Fig. 4a, produced an interference line pattern. Thus, a grating potential with modulation in x-direction was created and the anisotropic resistivity components  $\rho_{xx}^{(1d)}$  (Fig. 4b) and  $\rho_{yy}^{(1d)}$  (Fig. 4a) for the unidirectionally modulated (1D) systems were measured. The second illumination, with the sample rotated by  $90^\circ$ , results in a grid potential with modulation in x- and in y-direction. As demonstrated in Fig. 4, the resistivities  $\rho_{xx}^{(2d)} = \rho_{yy}^{(2d)}$  of the bidirectionally modulated (2D) samples show oscillations which, at small magnetic fields ( $B \leq 0.6T$ ),



are similar to and in phase with the weak oscillations of  $\rho_{yy}^{(1d)}$  (Fig. 4a), but smaller than and  $180^\circ$  out of phase with the large-amplitude oscillations of  $\rho_{xx}^{(1d)}$  in the corresponding 1D situation. The data shown in Fig. 4a were obtained from a sample with mobility  $1.4 \cdot 10^6 \text{ cm}^2/\text{Vs}$  and electron density  $N_s = 5.1 \cdot 10^{11} \text{ cm}^{-2}$  after the second illumination, those of Fig. 4b from a sample with mobility  $1.2 \cdot 10^6 \text{ cm}^2/\text{Vs}$  and  $N_s = 3.7 \cdot 10^{11} \text{ cm}^{-2}$ .

Figure 4 shows that a square superlattice leads to magnetoconductance oscillations with the same period in  $B^{-1}$  as a 1D superlattice with the same lattice constant  $a$ . This is immediately understood if the oscillations result from the bandwidth oscillations of the modulation-broadened LLs, since the  $\mathcal{L}_n$  factors are the same for the 1D and the 2D superlattice. Phase and amplitudes of the oscillations clearly show that the band conductivity is strongly suppressed by the 2D superlattice in these high-mobility samples. This behavior is not expected within the quasi-classical approach [11]. A quantum mechanical approach, using the simplest approximations for both modulation potential and collision broadening, explains this suppression as a consequence of the subband splitting of the Hofstadter-type spectrum, cf. Fig. 1.

#### 4 Collision broadening and conductivities

We [17,14,11] describe collision broadening effects by a quantum-number independent self-energy (a cut off  $n \leq 2E_F/\hbar\omega_c$  is implied)

$$\Sigma^-(E) = \Gamma_0^2 \sum_{n,j} (l^2/2\pi) \int d^2k G_{n,\vec{k};j}^-(E), \quad (9)$$

with the Green's function  $G_{n\alpha}^-(E) = [E - E_{n\alpha} - \Sigma^-(E)]^{-1}$ . In the absence of modulation ( $V_x = V_y = 0$ ) this reduces to the self-consistent Born approximation (SCBA) for randomly distributed short-range scatterers [1], with  $\Gamma_0^2 = \frac{1}{2\pi} \hbar\omega_c \cdot \hbar/\tau$  and  $\tau$  the corresponding life time for zero magnetic field. With the spectral function  $A_{n\alpha}(E) = \frac{1}{\pi} \text{Im} G_{n\alpha}^-(E)$  this yields for the DOS  $D(E) = 2 \sum_n D_n(E) = \text{Im}[\Sigma^-(E)/(\pi l \Gamma_0)^2]$ , where  $D_n(E) = (2\pi)^{-2} \sum_j \int d^2k A_{n\alpha}(E)$  is the DOS of the  $n$ -th LL and one spin direction. For comparison of the 1D ( $V_y = 0$ ) and the 2D ( $V_y = V_x$ ) case, Fig. 5a shows  $D_n(E)$  for  $B = 0.23T$  and  $a = 300 \text{ nm}$ , which means  $p = 5$  and  $q = 1$ , and for both small ( $\Gamma_0 \ll V_x \mathcal{L}_n$ ) and large collision broadening.

Our calculation of conductivities is based on Kubo's formulas [1,17], which for the diagonal components and in the approximation consistent with Eq.(9) read [17]  $\sigma_{\mu\mu} = \int dE [-f'(E)] \sigma_{\mu\mu}(E)$ , with  $f'$  the derivative of the Fermi function and

$$\sigma_{\mu\mu}(E) = \frac{e^2 \hbar}{2\pi} \int d^2k \sum_{n,n'} \sum_{j,j'} |(n'; \alpha' | v_\mu | n; \alpha)|^2 A_{n\alpha}(E) A_{n'\alpha'}(E), \quad (10)$$

where  $\alpha = (\vec{k}; j)$  and  $\alpha' = (\vec{k}; j')$ . For the 1D case, Eqs. (10) and the corresponding equation for the Hall conductivity have been evaluated numerically on the basis of Eqs. (5) and (9) [17]. A typical result is shown in Fig. 6. It reproduces all the characteristic features of the experiment: the pronounced Weiss oscillations of  $\rho_{xx}$  and, with opposite phase,  $\rho_{yy}$  superimposed on the SdH oscillations, which are

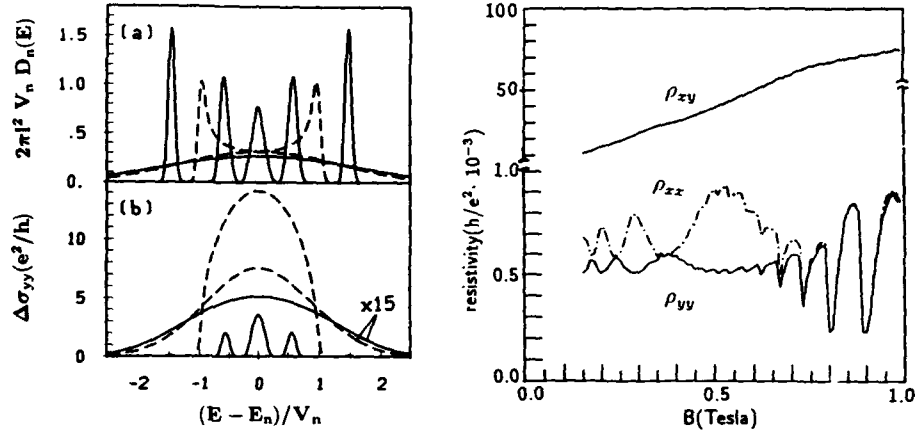


Figure 5: (a) Calculated density of states  $D_n(E)$  and (b) band conductivity  $\Delta\sigma_{yy}(E)$  for one Landau level and two values of the collision broadening,  $\Gamma_0/V_n=1.0$  and  $0.05$ . Solid (dashed) curves are for a grid (grating) potential with  $V_x\mathcal{L}_n=V_y\mathcal{L}_n=V_n$  ( $V_x\mathcal{L}_n=V_n$ ,  $V_y=0$ ) and  $p/q=5$ . For  $\Gamma_0/V_n=1.0$  the internal bandstructure is not resolved,  $D_n(E)$  and  $\Delta\sigma_{yy}(E)$  [here  $15 \times \Delta\sigma_{yy}(E)$  is plotted] are similar for grid and grating. For  $\Gamma_0/V_n=0.05$ , the resolved subband splitting dramatically reduces  $\Delta\sigma_{yy}$  for the grid. From Ref. [14]

Figure 6: Calculated resistivity in units of  $10^{-3}h/e^2=25.8\Omega$  for a 2DEG with mean density  $N_s=3.4 \cdot 10^{11}cm^{-2}$ , collision broadening  $\Gamma_0=0.056\sqrt{B[T]}meV$  and an one-dimensional potential modulation in x-direction with  $V_x=0.25meV$  and period  $a=294nm$  at temperature  $T=4.2K$ . From Ref. [17]

resolved only at higher magnetic fields, whereas the Weiss oscillations extend to much lower  $B$ -values, and no sizable oscillations of the Hall resistance.

To gain a deeper understanding of these effects [14,15] it is important to distinguish two contributions to Eq. (10),  $\sigma_{\mu\mu}(E) = \sigma_{\mu\mu}^{sc}(E) + \Delta\sigma_{\mu\mu}(E)$ , a band conductivity  $\Delta\sigma_{\mu\mu}(E)$  arising from intra-LL contributions ( $n' = n$ ) which diverges in the absence of random scatterers and vanishes for the unmodulated system, and an inter-LL ( $n' \neq n$ ) contribution  $\sigma_{\mu\mu}^{sc}(E)$ , which arises from scattering and is the only contribution in the unmodulated case. These contributions are further investigated to lowest order in the modulation ( $V_x\mathcal{L}_n, V_y\mathcal{L}_n \ll \hbar\omega_c$ ) and in the collision broadening ( $\omega_c\tau \gg 1$ ). Then, for the Hall conductivity the free electron result  $\sigma_{yx} \sim e^2N_s/m\omega_c$  is sufficient, and the resistivity components are given by the conductivities according to  $\rho_{xx} \approx \sigma_{yy}/\sigma_{yx}^2$ ,  $\rho_{yy} \approx \sigma_{xx}/\sigma_{yx}^2$ , and  $\rho_{xy} = 1/\sigma_{yx}$ .

Since the intra-LL ( $n' = n$ ) velocity matrix elements are proportional to the modulation potential, one might calculate  $\Delta\sigma_{\mu\mu}(E)$  to lowest order in the modulation using the approximation  $A_{n\alpha}(E)A_{n\alpha'}(E) \approx [A_n(E)]^2$ . The resulting  $\Delta\sigma_{yy}$  is independent of  $V_y$  and equals exactly the result for a unidirectional modulation in

x-direction [15]. At flat bands the velocity matrix elements and thus  $\Delta\sigma_{\mu\mu}$  vanish. In the interesting range of temperatures, where  $k_B T$  is larger than  $\hbar\omega_c$  but smaller than the energy separation  $\Delta_\lambda \approx \frac{1}{4}m\omega_c^2 a^2(\lambda - \frac{1}{4})$  of adjacent flat bands, this reduces to the result of the quasi-classical approach, but it can not explain the suppression of the band conductivity observed in experiment.

To understand this suppression, we must take the peculiar subband splitting of the Hofstadter-type energy spectrum seriously. From the mobility at zero magnetic field, one can estimate [17] that, in the experiments shown in Fig. 4, the collision broadening is indeed small enough to resolve the gross features of the Hofstadter spectrum (if one is not near a flat band situation). If the splitting of the subbands  $j$  and  $j'$  is resolved, the corresponding spectral functions in Eq. (10) do not overlap, and thus the non-diagonal matrix elements of the velocity between these subbands do not contribute to  $\Delta\sigma_{\mu\mu}(E)$ . Then the band conductivity of the system with a 2D superlattice is considerably smaller than that of same system with a 1D superlattice, as is visualized for a typical situation by the numerical results in Fig. 5b.

In the inter-LL contribution  $\sigma_{\mu\mu}^{sc}$  to the conductivity, one may neglect the effect of the modulation on the velocity matrix elements [17]. Then, the modulation affects  $\sigma_{\mu\mu}^{sc}(E)$  only via the self-energy and one gets  $\sigma_{\mu\mu}^{sc}(E) = (e^2/\hbar) \sum_n (2n + 1) [2\pi l^2 \Gamma_0 D_n(E)]^2$ , the same analytical form as for zero modulation [1]. In the interesting temperature range,  $\hbar\omega_c < k_B T < \Delta_\lambda$ ,  $\sigma_{\mu\mu}^{sc}$  is then proportional to an effective scattering rate  $\tilde{\Gamma}_n/\hbar$  defined by

$$\tilde{\Gamma}_n = 2\pi \int dE [2\pi l^2 \Gamma_0 D_n(E)]^2, \quad (11)$$

which oscillates as a function of  $n$  with maxima for flat bands. It is obvious from Eq. (11) that  $\sigma_{\mu\mu}^{sc}$  becomes maximum if the Landau bands at the Fermi energy become flat (and the peaks of the DOS become high near  $E = E_F$ ). In this situation the *band conductivity* becomes minimum, since the intra-LL velocity matrix elements approach zero.

## 5 Summary

We have sketched a straightforward quantum transport theory which explains all the novel magnetoresistance oscillations observed experimentally [12,15,11]. Among these, only the large-amplitude 'band conductivity' oscillations, observed in unidirectionally modulated systems when the current flows in the modulation direction, can be understood within a quasi-classical approach [23]. The weaker antiphase oscillations, observed when the current flows in the other direction and determined by the 'scattering conductivity', as well as the suppression of the band conductivity in bidirectionally modulated systems reflect properties of the peculiar, quantized energy spectrum, which persist at elevated temperatures where the individual SdH oscillations are not resolved. The key for the understanding is the oscillatory width of the modulation-broadened Landau bands. If these become narrow, the density of states and the scattering conductivity become large, whereas the group velocity and the band conductivity become small. The experiment reflects these antiphase oscillations at the Fermi energy.

Our theory predicts that at very low temperatures the novel oscillations appear as amplitude modulations of the SdH oscillations. For a 2D grid modulation, the magnitude of the band conductivity depends on the values of  $V_x$  and  $\Gamma_0$  in a complicated manner and may dominate the scattering conductivity or not [11]. Experimentally both situations are possible, too, as has been shown recently for a gated high-mobility sample in which the modulation strength could be tuned over a wide range [15].

In summary, the existing experiments are well explained by our quantum theory, and, in particular, the observed suppression of the band conductivity indicates a subband splitting of the Landau levels. For a direct resolution of such a Hofstadter-type energy spectrum, however, still smaller nanostructures are required. In the experimental situation of Fig. 4, with a period  $a \sim 300\text{nm}$ , the flux ratio of Eq. (1) is at  $B = 0.2T$  about  $Ba^2/\Phi_0 = 5$ , corresponding to  $q/p = 0.2$  in Fig. 1. The Fermi energy ( $E_F \sim 12\text{meV}$ ) is then in a high Landau band ( $n \sim 35$ ). Due to the Laguerre polynomial factors [cf. Eq. (7)], these bands are very narrow [cf. Fig. 2] and internal structure can not be resolved. At higher magnetic fields ( $B \sim 10T$ ),  $E_F$  is in the lowest Landau band. But then  $q/p$  is so small and the subbands in Fig. 1 are so dense, that the subband structure again can not be resolved. It is not only the temperature that prohibits to resolve these structures, but also mesoscopic fluctuations of the electrostatic potential across the sample. For superlattices with  $a \sim 60\text{nm}$  the situation would be much better. Experimental progress in this direction is expected for the near future. Thus we may hope that a fascinating area of physics, which so far was a reserved playground for theorists, will soon become accessible to experimental investigation.

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**Symmetries, Points, Superselection**

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It was in 1952 in Copenhagen that I was first drawn into the spell of Wignerism. Arthur Wightman, then a fervent missionary of this creed told me that I absolutely must read the 1939 paper by Wigner on the irreducible, unitary representations of the inhomogeneous Lorentz group. Returning to my home base, Munich, after the conference I sat down, read it and indeed found it a revelation which dominated the direction of my work for many years. Here was a rational approach, leading from a few simple principles to important physical consequences. I list a few.

1) *Pure states correspond to rays rather than vectors in Hilbert space. Therefore a symmetry in quantum physics is described by either a unitary or an antiunitary operator. In the case of a continuous symmetry group we need a unitary representation up to a phase factor, a projective representation. This is equivalent to a true representation of the covering group.*

2) *In the context of special relativity the geometric symmetry group is the inhomogeneous Lorentz group, nowadays called the Poincare group. The first task is therefore to classify the irreducible, unitary representations of its covering group.*

3) *This task being done one recognizes that among the resulting representations there are some which correspond to the simplest physical systems, namely to single particles. These are the representations in which the generator of time translations, the energy, has positive spectrum. There are two classification parameters, the mass and the spin (or in the case of zero mass the helicity). To conserve the positivity of the energy the operator corresponding to time reversal must be antiunitary.*

This 1939 paper, its results but perhaps even more its spirit, provided the ignition spark for the development of axiomatic (general) quantum field theory in the fifties. In fact, many of the ingredients were there. The main one yet missing was the principle of locality. Wigner addressed this in his beautiful study of the localization properties of single particle states [1]. The conclusion: no Lorentz covariant definition of the position operator of a particle is possible. Localization of a particle is a qualitative concept, meaningful up to length scales of the order of the Compton wave length (for zero mass particles not at all). On the other hand we have the tremendous impact of the locality principle in classical physics (Faraday-Maxwell-Einstein) and its transcription to quantum physics in quantum field theory. One regards a 4-dimensional space-time continuum as the basic arena of physics and, as long as we do not go beyond special relativity, this continuum is equipped with a known causal and metric structure. Strict locality is implemented in the quantum theory by the requirement of commutativity between any two observables which are attached to mutually space-like regions.

Between 1957 and 1959 I had the benefit of many personal discussions with Wigner. I was a little disappointed by noting that he did not believe in quantum field theory. When I once mentioned this he replied: "That is an understatement". Here it may be perhaps appropriate to recall a few other of his remarks, unforgettable because of their unique combination of penetrating perception with courteousness and benevolent irony. Talking once about a colleague who had spent a few years in Germany he said: "...and when he came back he was transformed into a German physicist". Puzzled, I asked "What is a German physicist?" The answer: "Well, an American physicist, if he has no ideas, he makes himself useful, perhaps he calculates something. A German physicist, if he has no ideas, he just does nothing". Or once, I complained that it was a shame that nobody had properly thought about collision theory in quantum physics. Reply: "Surely somebody must have thought of this. Perhaps it was Mott. But that was a few years before you were born".

Coming back to the question of locality we may recall that in the decades between 1940 and 1970 there was a strong undercurrent of dissatisfaction with our uncritical acceptance of Minkowski space, with strict locality and the excessive formalism of quantum field theory. The ideas about a fundamental length, pure S-matrix theory where space-time enters only on the macroscopic level, are examples. We can say now that the pendulum has swung back and that the principle of locality has proved its worth to a much finer level of accuracy than one had any reason to expect. This implied a dethronization of the concept of particles as the basic building blocks of the theory and the realization that the concepts of particles and fields are not closely related. Fields are the vehicle to implement the principle of locality. Particles are states of special interest, related in general in a complicated way to the basic fields. The distinction between elementary and composite particles is not fundamental; no sharp definition is known.

Let us come now to the third item in the title. Again it was Wigner who recognized first that the superposition principle could not have unrestricted validity. In [2] it was suggested that the Hilbert space of state vectors we usually consider decomposes into a direct sum of mutually incoherent subspaces, the *superselection sectors*. The search for a natural understanding of this feature was the seed for the so called "algebraic approach to quantum field theory". The essential point here is that the algebras generated by local observables must be regarded as abstract algebras, not primarily as algebras of operators acting in a Hilbert space. The latter result by representations of the abstract algebra and, if there are inequivalent representations then each equivalence class gives a superselection sector. In particle physics the superselection quantum numbers correspond to generalized charges. The abstract point of view leads to an understanding of the possible types of exchange symmetry compatible with the locality principle (the Bose-Fermi-alternative and its generalizations, up to braid group statistics in low dimensional models). It relates the composition and conjugation rules of the charges



to global gauge groups. But the range of superselection rules is not exhausted by this. One may, for instance, consider thermal equilibrium states of an infinitely extended medium. Then the temperature and chemical potentials appear as superselection parameters.

Let us turn back once more to the problem of locality. Again I take very seriously the question which Wigner put at the end of a talk by me a few years ago: "There are those of us who think there are no points. What do you think?" Somewhat later he elaborated on this by pointing out that if we consider space-time points as basic in the theory then we must, at least in principle, give a physical method to mark them. If particles are not adequate for this, what else? We should bear this in mind, in particular in view of the challenge to quantum field theory posed by general relativity.

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## SU(2/1), SUPERCONNECTIONS AND GEOMETRIC HIGGS FIELDS

by

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## Abstract

It has been suggested that the Standard Model  $SU(2) \times U(1)$  be embedded in the supergroup  $SU(2/1)$ . The successful features relate mostly to the composition of the spectrum of leptons and quarks and to the Higgs field assignment. The result  $\sin^2\theta = .25$  involves normalization of the algebra by traces rather than by supertraces. We derive this result from the "superconnection" geometry and discuss the present state of the theory.

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1. SU(2/1). The Weinberg-Sala. theory involves a large number of independent assumptions: 6 choices of SU(2)xU(1) multiplets for leptons and quarks  $[(\nu_L, e_L), e_R, u_R, (u_L, d_L), d_R]$  and for the Higgs Goldstone scalar field  $\Phi$ , 5 couplings  $[g, \sin^2\theta, -m^2, \lambda, g_H]$ . It was suggested [1,2] in 1979 that this apparent arbitrariness be reduced by embedding SU(2)xU(1) in the supergroup SU(2/1). The fermion multiplets are grouped in the  $\underline{3}$  (leptons) and  $\underline{4}$  (quarks), the fundamental representations of SU(2/1) [3,4]; moreover, the  $\underline{4}$  reduces into  $\underline{3+1}$  for integer charges! The assignment for  $\Phi$  is forced by the superalgebra structure. All couplings except for  $-m^2$  are related; the eleven independent choices are thus reduced to four. About twenty papers on SU(2/1) were published between 1979 and 1982; two basic difficulties were encountered: the apparent loss of the spin-statistics correlation in representations and the fact that  $\sin^2\theta = .25$  involves normalization of the algebra's  $\mu$  matrices by  $\text{tr}(\mu^2) = N$  (while supertraces yield no result). The answer was given in 1982 [5], in the form of a method for the gauging of an internal supergroup, further developed in 1990 [6]. An alternative approach was suggested in 1986 [7]. The theory has recently been "rediscovered", causing a new flurry of excitement [8]. It has been linked with the methods of non-commutative geometry [9] - related to the conceptual advances we introduced in ref. [5]. We review the status of the theory in this study.

The possibility explored in ref. [7] assumes that SU(2/1) is an accidental low energy "flavour-type" symmetry (like the eightfold way). The emergence of SU(2/1) may just be due [10] to the fact that, to the extent that we correlate gradings with chirality assignments for the lepton and quark multiplets, and since the electric charges  $Q$  are the same for left and right chiralities of the same particle,  $\text{str}(Q) = 0$ . Since weak isospin  $I_W$  is confined to left-chiral states,  $\text{tr}(I_W) = 0$  implies  $\text{str}(I_W) = 0$ . Applying the Gell-Mann / Nishijima rule, we then get  $\text{str}(Y_W) = 0$  too. Our postulate would thus amount to the embedding of the "naturally" supertraceless SU(2)xU(1) within the smallest supergroup with that property. We predict [7]  $M(\Phi) = 2 M(W) \sim 175$  GeV; should the mass formula for mesons be written for  $M^2$  instead of  $M$ , as in SU(3), we would have  $M^2(\Phi) = 2 M^2(W)$ , i.e.  $M(\Phi) \sim 125$  GeV. Note that SU(2/1) was extended so as to include SU(3)<sub>colour</sub>; the embedding supergroup is SU(7/1), with 4 generations of quark and leptons [11] and vanishing anomalies [12]. Since the  $Z^0$  decays limit the number of generations to 3, the fourth would presumably involve a very heavy neutrino.

Now to the spin-statistics problem. In a lepton or quark multiplet, if the left- (or right-) chiral states are fermionic, are the right- (or left-) chiral states to be considered as bosonic? This should be so within Lagrangian field theory, as applied to the superalgebra generators in a Noether theorem, according to the spin-statistics constraints. The answer is that the odd generators of  $SU(2/1)$  contain the Parity operator ( $\gamma^0$  matrix), as they relate left and right chiralities. They operate on a direct sum  $(1/2,0)+(0,1/2)$ ; the odd operators are in  $(1/2,1/2)$ . Acting on the  $SL(2,C)_L$  representation, for instance, they connect  $1/2$  to  $0$  (and similarly in  $SL(2,C)_R$ ). The R state is thus a spinless boson under  $SL(2,C)_L$  and vice versa. The lesson is then to use doubled representations [4]. This interpretation still does not tell us what are the  $K^*$ -like "fermionic" vector-mesons accompanying the  $W^\pm, Z, \gamma$ , in the vector ( $j = 1$ ) octet, though we understand how they arise. We shall return to that point in what follows. Note, however, that in the case of the Higgs field  $\phi$ , the statistics do fit "natural" field assignments: its companions in the spin 0 meson multiplet are identifiable with the (fermionic) ghosts of the  $W^\pm, Z, \gamma$  vector mesons (of course, the entire issue does not arise if we assume  $SU(2/1)$  to merely represent a set of "accidental" algebraic constraints, as treated in refs. [7,10]). An answer was provided in ref.[5], in which the method of gauging an internal supergroup was developed, including the construction of "superconnections", later independently suggested in mathematics [9]. The new gauge ghosts fit into the appropriate BRST equations [5,13,14], but the situation is not entirely clear for the lepton and quark ghost-like states. Another open question relates to the " $-m^2\phi^2$ " term triggering the spontaneous symmetry breakdown. In [5] it was put in "by hand"; can it have a dynamical or geometrical derivation? A tentative geometric answer has been suggested in the new work [8], based on the addition of a discrete piece to the exterior derivative operator; an alternative dynamical triggering has been tried in ref.[14] but appears to yield too small a mass for the Higgs field.

In what follows, we shall discuss the resolution of the  $\sin^2\theta = .25$  paradox. This result is derived with a normalization of the superalgebra's  $\mu$  matrices by the traces, as in  $SU(3)$ , whereas the Killing metric of  $SU(2/1)$  would have involved normalization by  $\text{str}(\mu_A \mu^A) = N$ , which in this case would have yielded no result; similarly, the (gauge) geometric Lagrangian with the  $\lambda\phi^4$  coupling involves the compact metric of  $SU(3)$ , rather than that of  $SU(2/1)$  with its unwanted minus sign.

2. The Supergroup.  $G$  is a supergroup gauged internally,  $\Gamma$  its Lie superalgebra  $\Gamma = \Gamma^+ + \Gamma^-$ ,  $\Gamma^+$  the even and  $\Gamma^-$  the odd generators, with the Lie superbrackets  $[\Gamma_{\hat{a}}, \Gamma_{\hat{b}}] = if_{\hat{a}\hat{b}}^{\hat{c}} \Gamma_{\hat{c}}$ ,  $[\Gamma_{\hat{a}}, \Gamma_{\hat{i}}] = if_{\hat{a}\hat{i}}^{\hat{j}} \Gamma_{\hat{j}}$ ,  $\{\Gamma_{\hat{i}}, \Gamma_{\hat{j}}\} = d_{\hat{i}\hat{j}}^{\hat{a}} \Gamma_{\hat{a}}$ , with  $\Gamma_{\hat{a}, \hat{b}} \in \Gamma^+$  and  $\Gamma_{\hat{i}, \hat{j}} \in \Gamma^-$ . Selecting a section over the superbundle,  $x$  is the (spacetime) horizontal coordinate,  $y$  the (internal) vertical. We construct the Grassmann algebra  $\Lambda(x, y)$  over both, with  $\Lambda = \Lambda^+ + \Lambda^-$ , the even and odd pieces  $(\Lambda^0, \Lambda^2, \dots) \subset \Lambda^+$ ,  $(\Lambda^1, \Lambda^3, \dots) \subset \Lambda^-$ ; for forms in  $x$ ,  $\Lambda^n$ :  $n \leq 4$ , whereas  $n \leq N$ ,  $N = \dim \Lambda^2(y)$  for forms in  $y$ . Notice that the  $y$  coordinate itself is commutative, like  $x$ ; to supply an anticommuting parameter for  $\Gamma_{\hat{i}}$  we simply use odd order forms, e.g.  $\Gamma_{\hat{i}} \cdot f_m^{\hat{i}}(x, y) dy^m = \Gamma_{\hat{i}} \cdot F^{\hat{i}}$ . The anticommutativity property of the parameter  $F^{\hat{i}}$  derives from its being a one-form in the  $y$  variable. The carets  $\hat{\phantom{a}}$  over the indices indicate the presence of a superalgebra, i.e. the caret over the  $\hat{i}$  index contains the information about the anticommutation property, which is not yet present in the function  $f^{\hat{i}}(x, y)$ . Here the  $\hat{i}$  index stands for the same subalgebraic quantum numbers, but taken within the related (even) Lie algebra within the "Hermitian Lie algebra" [15,16]. Such a Hermitian Lie algebra occurs when the same basic set of generators can close either as a Lie algebra, or by selecting a subalgebra  $\Gamma^+$  and imposing a  $Z(2)$  grading and anticommutation rules for  $\Gamma^- = \Gamma/\Gamma^+$ , they close on a superalgebra. For our application, we use  $U(3) = U(2/1)$ . For the  $\hat{i} \rightarrow i$  transition, the transformation relates only to the change in the generalized Lie bracket with  $\hat{j} \rightarrow j$ , from  $\{\hat{i}, \hat{j}\}$  to  $[i, j]$ ; for the  $\hat{a} \rightarrow a$  transition, the change is only in the decomposition over the  $(\mu^{\hat{B}}, \mu^0)$  versus  $(\lambda^{\hat{B}}, \lambda^0)$  vectors in the  $(8, 0)$  plane, since  $\text{str}(\mu^{\hat{B}}) = 0$ , but  $\text{tr}(\lambda^{\hat{B}}) = 0$ . In ref. [5],  $F^{\hat{i}} = f^{\hat{i}}_{\mu} dx^{\mu}$  was used explicitly, the forms being taken over  $\Lambda(x)$ , i.e. only over space time. As a result, gauging involved higher tensors as gauge fields (such as the Kalb-Ramond  $B^1_{\mu\nu}$ ). However, we shall now show that the identification of the Higgs field used  $\Lambda(y)$ , an important point in the understanding of the emergence of the  $U(3)$  metric.

3. Connections and gauge, ghost and Higgs fields. Connections  $C$  over a Principal Fibre Bundle (= a Yang-Mills gauge theory) enter  $D = \Delta + C$  and when acting on the forms over that bundle (coordinates  $x, y$ ) increase their

degree by one, with  $D = D^+ + D^-$  (vertical and horizontal - projecting over a selected section)  $\Delta = d + s$  with  $d$  the horizontal exterior derivative  $d = dx^\mu \partial/\partial x^\mu$  and  $s$  the vertical exterior derivative  $s = dy^m \partial/\partial y^m$ . In the geometric interpretation [17-19]  $s$  is the BRST operator,  $C = A + \chi$  where  $A$  is the horizontal one-form  $A = A^a_\mu \lambda_a dx^\mu$ , with  $A^a_\mu(x,y)$  the Yang-Mills field for the algebra  $\lambda_a$ . Here  $\chi = \chi^a(x,y) \lambda_a$  and  $\chi^a = \chi^a_m dy^m$  is the ghost field. Note that physics texts write  $A^a_\mu(x)$  and  $\chi^a(x)$ , making no mention of the  $y$  dependence, but this is just the selection of a gauge, since the usual local gauge transformation will induce such a dependence. Note that this geometric interpretation of  $s$  and  $\chi^a(x,y)$  and the identification of the anticommutative property of ghost fields (originally derived from the Feynman diagram analysis) with the characteristic feature of odd degree forms was resisted for a number of years but is now accepted by the scientific consensus [20]. Note that there is no nilpotence problem [21] because we are working in quantum field theory,  $A$  and  $\chi$  are fields, i.e. distributions and their products in the Lagrangian, etc are taken at different points  $x$ .

Changing now to a supergroup, but staying still with the Yang-Mills picture in which  $y$  (and not yet  $\Lambda$ ) spans the fibre, we can immediately identify the Higgs field. The connection  $\underline{C}$  is still an anticommutative one-form and so are  $\underline{A}$  and  $\underline{\chi}$ . However,  $\underline{A} = A^{\hat{a}}_\nu \mu_{\hat{a}} dx^\nu + A^{\hat{1}}_\nu \mu_{\hat{1}} dx^\nu$  and  $\underline{\chi} = \chi^{\hat{a}}_m \mu_{\hat{a}} dy^m + \chi^{\hat{1}}_m \mu_{\hat{1}} dy^m$ . We have underlined in each term the factors inducing fermion behaviour, whether it is the odd degree of the form or the odd part in the superalgebra. We note that whereas  $\underline{\chi}^{\hat{a}} = \chi^{\hat{a}}_m dy^m$  is indeed a fermionic scalar ghost field,  $\chi^{\hat{1}} = \chi^{\hat{1}}_m dy^m$  has boson statistics (the fermi feature in  $\underline{\chi}$  is supplied here by the  $\mu_{\hat{1}}$ ). This is how  $SU(2/1)$  occurred to Ne'eman [1] (Fairlie arrived at the supergroup from dimensional reduction arguments). Here the Higgs is a one-form valued in the odd part of the superalgebra. We now change to the "superconnection" geometry of refs. [5,6]. The fibre is no more spanned by  $y$ , it is spanned by  $\Lambda(y)$ . The parameters  $F^{\hat{a}}(x,y) \in \Lambda^+(x,y)$  and  $F^{\hat{1}}(x,y) \in \Lambda^-(x,y)$ . The gauge field has the quantum numbers of the parameters it gauges (this is the essence of the Yang-Mills idea) and a degree higher by one, in the variable of the relevant parallel-transport (covariant derivative) it spans. For  $F^{\hat{a}}(x,y)$  nothing has changed if it is taken from  $\Lambda^0(x,y)$ ; should it, however, involve a higher even degree, such as  $\Lambda^2(x,y)$ , the relevant piece of the connection is the gauge totally antisymmetric tensor  $C^{\hat{a}}_{\mu\nu\sigma}$  of [5].

Now take the action of the odd generator  $\Gamma_{\bar{y}}$ , the  $F_{\bar{y}}^1 \in \Lambda_{\bar{y}}^1(x,y)$  parameters, one-forms in  $dy^m$ . For the vertical projection ("the ghost") of the connection  $H^i(x,y) = \chi_{\bar{y}}^1 dy^{\bar{n}} = h_{mn}^i(x,y) dy^m \wedge dy^{\bar{n}}$ .  $H^i$  (our former  $\chi^i$ ) is now an even-order form in  $dy^m$ . In ref. [5], using a symmetric (Curci-Ferrari) BRST with  $\Lambda(x,y,\bar{y})$ ,  $\bar{y}$  the complex conjugate of  $y$ , we found a two-form  $\Phi^i = h_{mn}^i(x,y,\bar{y}) dy^m \wedge dy^{\bar{n}}$ . The forms  $dy$  have ghost number 1, those in  $d\bar{y}$  have -1. Thus the  $\Phi^i$  has ghost number 0 and appears accordingly in the horizontal (and thus non-vanishing) part of the "generalized" curvature of the superconnection, i.e. in the relevant BRST equations [14]. The transition from the Principal Fibre Bundle (Yang-Mills) [22] geometry for an internal supergauge to that of the Superconnection [5,6] thus puts the  $\Phi^i$  field in the even part of the Grassmann algebra and with the relevant  $U(3)$  index instead of  $U(2/1)$ . As a matter of fact, we can generalize the result and directly identify  $\Phi^i$  with a zero-form [6]. The supercurvature involves  $(d^a_{ij} \Phi^i \Phi^j)$ , which is then squared, yielding the  $\lambda \Phi^a$  term. Since the geometry of the superconnection makes the transition from  $U(2/1)$  to  $U(3)$  for the entire algebra (the "super" feature having been relegated to the  $dy$  parameter field), we get the  $\sin^2\theta = .25$ . However,  $U(3)$  is not simple, and we should still check whether or not this coupling is preserved in the renormalization procedure.

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## W-Algebras and Conformally Reduced WZNW Theories

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**Abstract:** W-algebras are defined as polynomial extensions of the Virasoro algebra by primary fields, and their occurrence in the context of Kac-Moody (KM) algebras and integrable systems is recalled. It is shown that their occurrence in 2-dimensional Toda theories is explained by the fact that the Toda theories can be regarded as Wess-Zumino-Novikov-Witten (WZNW) theories which are reduced by a set of first-class conformal-invariant constraints. The general form of such constraints, which leads to other 2-dimensional integrable systems with W-algebras as symmetry algebras, is sketched.

### 1 Introduction.

In recent years two-dimensional conformally-invariant field theories have attracted an enormous amount of attention. This is due to the fact they they span a number of hitherto unrelated physical and mathematical disciplines such as statistical mechanics, string theory and the theories of integrable systems, Riemann surfaces and Hamiltonian reductions. All these properties are based on the fact that the two-dimensional conformal group is much less trivial than its higher-dimensional counterparts, consisting of *all* analytic transformations  $x_+ \rightarrow f(x_+)$  and  $x_- \rightarrow g(x_-)$ , where  $x_{\pm} = x \pm t$  (or  $x \pm it$  in the Euclidean case), where  $x, t$  are the conventional space-time coordinates. Correspondingly, its Lie algebra is the direct sum of two Virasoro algebras [1] of the form

$$\{L(y), L(y')\} = \partial_y L(y)\delta(y - y') + L(y)\partial_y \delta(y - y') + c(\partial_y)^3 \delta(y - y'), \quad (1.1)$$

where  $y = x_{\pm}$ , the bracket is either Poisson or commutator and  $c$  is a constant that characterizes the one-parameter central extension. The physical meaning of the generators  $L(y) = L(x_{\pm})$  is that they are the components  $T_{++}(x_+)$  and  $T_{--}(x_-)$  of the energy-momentum tensor  $T_{\mu\nu}$  (which are chiral because of translational invariance and the vanishing of the trace,  $T_{+-}(x, t) = 0$ ). Thus the generators of the Virasoro algebra carry all of the physical information contained in the energy-momentum tensor.

The continuous unitary irreducible representations of the 2-dimensional conformal group can be classified using essentially the same methods as were used by Wigner for the 4-dimensional Poincaré group in 1939, and can be realized in terms of local tensor fields. These fields, called *primary* fields, transform according to

$$\phi(x_+, x_-) \rightarrow \left(\frac{\partial f(x_+)}{\partial x_+}\right)^{(-s_+)} \phi(f(x_+), x_-), \quad (1.2)$$

and similarly for  $x_-$ , where the indices  $s_{\pm}$  are called the conformal weights. The infinitesimal form of (1.2) is easily seen to be

$$\{L(y), \phi(y', \tilde{y}')\} \rightarrow \partial_y \phi(y, \tilde{y}) \delta(y - y') + (1 - s) \phi(y, \tilde{y}) \partial_y \delta(y - y') \quad (1.3)$$

where  $L(y)$  are the Virasoro generators and  $y = x_{\pm}$  and  $\tilde{y} = x_{\mp}$ .

In 1984 Zamolodchikov [2] considered the possibility that given a Virasoro algebra with generators  $L(x)$  and a finite set of primary fields  $\phi_k(x)$  the Poisson brackets or commutators of the primary fields with themselves might close to yield a polynomial in the Virasoro operator, the primary fields and their derivatives. If the space-time coordinates are assigned a conformal weight  $(-1)$ , in which case the delta-functions would have unit conformal weight, then the polynomial has to be homogeneous and the brackets would be of the general form

$$\{\phi_s(y) \phi_t(y')\} = \sum (\partial_y)^a \phi_u(x) (\partial_y)^b \phi_v(x) (\partial_y)^c \phi_w \dots \delta^n(x - y), \quad (1.4)$$

where  $a + b + c + \dots + u + v + w + \dots + n + 1 = s + t$ . Such algebras are called *W-algebras* and since their first proposal have been realized in a number of different situations.

The most straightforward realization of *W-algebras* is in the context of *KM algebras*, which [1] are current-algebras of the form

$$\{J^a(y), J^b(y')\} = f_c^{ab} J^c(y) \delta(y - y') + \kappa g^{ab} \partial_y \delta(y - y'), \quad (1.5)$$

where the  $f_c^{ab}$  and the  $g^{ab}$  are the structure constants and Cartan metric of a semi-simple Lie algebra  $G$  and  $\kappa$  is a constant. For these a (Poisson-bracket)  $W$ -algebra is generated by the Sugawara-Virasoro operator

$$L(y) = g_{ab} J^a(y) J^b(y), \quad (1.6)$$

(suitably normal-ordered in the quantum case) and the set of primary fields

$$W_s(y) = d_{abc\dots} J^a(y) J^b(y) J^c(y) \dots \quad \text{where} \quad C_s = d_{abc\dots} X^a X^b X^c \dots \quad (1.8)$$

are the Casimir operators of order  $s$  for the generators  $X^a$  of a simple Lie group  $G$ . This was first shown by Zamolodchikov himself [2] for the  $SU(3)$  case. (Whether the Poisson brackets can be generalized to commutator brackets for all representations is not yet clear [3]).

Shortly afterwards it was found that a set of Poisson-bracket algebras already considered by the mathematicians in connection with KdV hierarchies [4] were  $W$ -algebras. In a further development it was found that  $W$ -algebras were realized in a variety of Lax-pair systems [5] and in particular in Toda systems [6]. It is the Toda realization (and its generalizations) that I wish to consider in the present talk.

The main point is that this and other aspects of Toda theory (such as their integrability) can be very easily understood by the observation [7] that Toda theory is nothing but a Wess-Zumino-Novikov-Witten (WZNW) theory which is reduced by a set of first-class linear constraints. Using these constraints the general solution of the Toda field equations is easily deduced from the (trivial) general WZNW solution and the  $W$ -algebras emerge as the canonical symmetry algebras of the Toda system. They are also seen to be the algebras of gauge-invariant polynomials of the constrained KM currents and to be the Dirac star-algebras of the second-class constraints produced by gauge-fixing.

## 2 Standard Two-Dimensional Conformal Field Theories.

For scalar fields the general two dimensional conformal action is [8] of the form

$$I(\phi_k) = \sum_{i,k} \int d^2x \{ C_{ij} (\partial\phi_i \partial\phi_k) + g_i e^{K_{i,k} \phi_k} \}, \quad (2.1)$$

where  $C$ ,  $K$  and  $g$  are constants i.e. it has exponential-type potentials. The special case in which there is only one field is the well-known Liouville theory which occurs

in a variety of situations. In particular it is just the action for the two-dimensional gravity theory induced by renormalization in standard string theory, expressed in the conformal gauge [9]. The Liouville theory is integrable but for more than one scalar field the system (2.1) is in general not integrable. On the other hand it is well-known that for any number of fields it becomes integrable if the  $g$ 's are unity and  $C$  and  $K$  are the Coxeter and Killing matrices of any simple Lie algebra of rank  $l$  with fundamental roots  $\alpha_i$  i.e.

$$g_i = 1, \quad C_{ik} = \frac{4(\alpha_i, \alpha_k)}{(\alpha_i)^2(\alpha_k)^2} \quad \text{and} \quad K_{ik} = \frac{2(\alpha_i, \alpha_k)}{(\alpha_i)^2}. \quad (2.2)$$

Equations (2.1) and (2.2) define the Toda theories [10]. It is not clear at this level why the association of couplings in (2.1) with the C-K matrices of a simple Lie group makes the system integrable, but this is one of the questions which is clarified by the WZNW reduction.

The natural generalization of the above examples to non-abelian groups is the WZNW action [1], for which the fields  $g(x)$  take their values in a simple Lie group  $G$  and the action takes the form

$$I(g) = \kappa \int d^2x \text{tr}(J_+(x)J_-(x)) + \frac{2\kappa}{3} \int d^3x \epsilon_{rst} \text{tr}(J_r(x)J_s(x)J_t(x)), \quad (2.3)$$

where

$$J_+(x) = g(x)\partial_+g^{-1}(x) \quad \text{and} \quad J_-(x) = (\partial_-g^{-1}(x))g(x). \quad (2.4)$$

Here the 3-dimensional integral is topological in the sense that its *variation* is a pure divergence and thus reduces to an integral over its boundary, which is assumed to be the 2-dimensional space under consideration. The insertion of this term has the consequence that the field equations take the simple form

$$\partial_-J_+(x) = 0 \quad \text{and} \quad \partial_+J_-(x) = 0, \quad (2.5)$$

which simply state that the currents  $J(x)$  are chiral i.e. are functions of  $x_{\pm}$  only.

The WZNW action (2.3) is invariant with respect to the global transformations

$$g(x) \rightarrow g(x_+)g(x) \quad \text{and} \quad g(x) \rightarrow g(x)g(x_-), \quad (2.6)$$

and the Noether currents for these transformations are just the chiral currents  $J_{\pm}(x_{\pm})$ . As a result each of these currents satisfies a KM algebra of the form (1.5) and commutes with the other one. Thus the WZNW theories provide a natural Lagrangian realization of the KM algebras.

Since the WZNW theory is conformally-invariant the trace  $T_{+-}(x, t)$  of the WZNW energy-momentum tensor is zero and the remaining two components are the Virasoro operators. These are actually quadratic in the currents and take the form

$$L(y) = \frac{1}{(2K)} \text{tr}(J(y))^2, \quad \text{where } L(y) = T_{++}(x_+) \text{ or } T_{--}(x_-), \quad (2.7)$$

and  $2K = 2\kappa$  and  $(2\kappa + g)$  where  $g$  is the Coxeter number of  $G$  in the classical and quantum cases respectively.

### 3 Conformal Reduction.

In this section we wish to show that the WZNW theories can be reduced to the Toda theories by means of first class constraints. The form of the first-class constraints can be expressed very simply at the KM level as follows: Let the KM currents  $J(y)$  of (1.5) be those in the Cartan basis i.e.  $\{J_{-\alpha}(y), J_i(y), J_\alpha(y)\}$  in conventional notation. Then the reduction is simply to let

$$J_{-\alpha_i}(y) = 1 \quad \text{and} \quad J_{-\alpha}(y) = 0, \quad (3.1)$$

according as the roots are fundamental or not fundamental. This reduction is first class since from (1.5) the commutation relations of any two negative components has no central term and no fundamental root. Of course, this reduction is only possible for those Lie algebras which are the *real* linear spans of the Cartan generators, the so-called *split* Lie algebras. These Lie algebras are highly non-compact and for each series of Lie algebras there is just one. For example, for the A and D series of Lie algebras they are the Lie algebras of  $SL(N, R)$  and  $SO(N, N)$  respectively.

To obtain an intuitive feeling for the meaning of the reduction (3.1) it is useful to consider the  $SL(N, R)$  case, for which the reduced current takes the form

$$J^{constr.}(z) = \begin{pmatrix} j_{11}(y) & j_{12}(y) & j_{13}(y) & \dots & j_{1n}(y) \\ 1 & j_{22}(y) & j_{23}(y) & \dots & j_{2n}(y) \\ 0 & 1 & j_{33}(y) & \dots & j_{3n}(y) \\ 0 & 0 & 1 & \dots & j_{4n}(y) \\ 0 & 0 & 0 & \dots & j_{5n}(y) \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 1 & j_{nn}(y) \end{pmatrix}, \quad (3.2)$$

Although the first-class nature of the reduction (3.1) is obvious the conformal invariance is not, since KM currents have conformal spins  $(\pm 1)$  and hence to put

some of them equal to constants breaks the conformal invariance generated by the WZNW Virasoro algebra (2.9). So how is the conformal invariance preserved? The answer is that the Virasoro generators can be modified so that the components of the currents which are set equal to constants become scalars. The modification is

$$L(\mathbf{y}) \rightarrow \Lambda(\mathbf{y}) = L(\mathbf{y}) + \partial_{\mathbf{y}} H(\mathbf{y}), \quad \text{where } H(\mathbf{y}) = (H, J(\mathbf{x})), \quad (3.3)$$

and  $H$  is the (unique) element of the Cartan subalgebra for which all the fundamental roots have weight unity,  $[H, E^{\alpha}] = E^{\alpha}$ . It is easy to verify from the KM algebra that with respect to the conformal group generated by  $\Lambda(\mathbf{y})$  the conformal spins of the KM current-components become  $(1 + h)$  where the  $h$  are their weights with respect to  $H$ . Thus in particular the components corresponding to the negative fundamental roots  $E^{-\alpha}$  become scalars. Setting them equal to constants then preserves the conformal invariance.

Of course, since there are two chiral sectors a similar procedure must be carried out for each one. So far the procedures have been chosen to be dual in the sense that for the respective chiral sectors it is the  $J_{\pm\alpha}(\mathbf{y})$  that are constrained, and  $L(\mathbf{y}) \rightarrow L(\mathbf{y}) \pm \partial_{\mathbf{y}} H(\mathbf{y})$  that are the modifications.

The physical meaning of the field  $H(\mathbf{y})$  is two-fold. First, in Toda theories (and their generalizations) the field  $H(\mathbf{x}_-) + H(\mathbf{x}_+)$  can be interpreted as a 2-dimensional gravitational connection [11]. Second the modified Virasoro generators  $\Lambda(\mathbf{x}_{\pm})$  turn out to be the components of the improved energy momentum tensor in the reduced theory.

#### 4 Reduction of the WZNW Action.

For any set of first-class constraints there is a standard strategy for obtaining the reduced action. This is to gauge the original action with respect to the group generated by the constraints, omitting kinetic terms for the gauge fields which then appear as Lagrange-multipliers, and then to eliminate the gauge-fields means of their Euler-Lagrange equations (or by functional integration in the quantum case).

Applying this general strategy to our case we see that the gauge groups for our constraints are the KM transformations generated by the current components  $J^{\pm\alpha}(\mathbf{x}_{\pm})$  and hence the gauge fields are simply

$$A_+(\mathbf{x}_+) = a_{\alpha}(\mathbf{x}_+)E^{\alpha} \quad \text{and} \quad A_-(\mathbf{x}_-) = a_{-\alpha}(\mathbf{x}_-)E^{-\alpha}, \quad (4.1)$$

respectively. Accordingly, the gauged WZNW action is

$$I_{WZ}(g) + \int d^2x \text{tr} \{ A_+(J - M_-) + A_-(J - M_+) + A_+gA_-g^{-1} \}, \quad (4.2)$$

and the Euler-Lagrange field equations for the Lagrange-multiplier fields  $A_{\pm}$  are

$$A_+^{\alpha} = (E^{\alpha}, g(J - M_+)g^{-1}) \quad \text{and} \quad A_-^{-\alpha} = (E^{-\alpha}, g^{-1}(J - M_-)g). \quad (4.3)$$

If one re-inserts these values of  $A_{\pm}$  in (4.2) and makes the (Gauss) decomposition

$$g = e^{\epsilon_{\alpha}(x,t)E^{\alpha}} e^{\phi_i(x,t)H^i} e^{\epsilon_{-\alpha}(x,t)E^{-\alpha}}, \quad (4.4)$$

of  $g$  one finds that the fields  $\epsilon_{\pm}(x,t)$  drop out and (4.2) reduces to exactly the Toda action (2.3) for the fields  $\phi_k(x,t)$ . This derivation of the Toda theory explains why that theory is associated with the C-K matrices of a semi-simple Lie group. It also explains the integrability of the Toda theory. Indeed the general solution of the Toda field equations can be obtained directly from the well-known general solution  $g(x,t) = g_+(x_+)g_-(x_-)$  of the WZNW field equations [7].

## 5 W-algebras and their Interpretation.

A KM algebra such as (1.5) may be thought of as defining a closed symplectic form and hence a phase space for the current components  $J^a(y) = (a, J(y))$ . In such a phase space the canonical transformations generated by functionals  $F(J)$  of the currents would be then of the form

$$\delta J^a(y) = \{F, J^a(y)\} = \int d^2y' \frac{\delta F}{\delta J^b(y')} \{J^b(y'), J^a(y)\}. \quad (5.1)$$

Let us now consider those functionals which preserve the constrained form of the current i.e. such that

$$(E^{\alpha}, \delta J^a(y)) = 0, \quad (5.2)$$

for the chiral sector with positive  $\alpha$  (and similarly for the other sector). This property is evidently preserved under the Poisson bracket operation and hence the set of all such functionals, which will be denoted by  $W(J)$ , forms a closed algebra (the little algebra of the constrained currents within the canonical algebra) with respect to Poisson brackets. It will be seen below that for the constraints of section 3 it is a W-algebra in the sense of Zamolodchikov.

Because the W-algebras as just defined are chiral they preserve the WZNW field equations (2.7) and since by definition they respect the constraints on the currents

it follows that they preserve also the Toda equations. Thus the  $W$ -algebras emerge as symmetry algebras of the Toda system and their Noether charges are conserved by the Toda field equations. Furthermore, it turns out that there are as many independent generators of the  $W$ -algebras as there are independent components of the Toda fields, so in a certain sense they give a complete description of the Toda system.

A second interpretation of the  $W$ -algebras can be obtained if one recalls that the constrained components of the KM currents generate a gauge group. Then the fact that the generators  $W$  of the  $W$ -algebras commute (weakly) with the constraints means that the  $W$  are gauge-invariant functions of the currents. (And, conversely, every gauge invariant functional of the currents qualifies as a  $W$ ). Hence an alternative definition of the  $W$ -algebras is as the algebras of gauge-invariant functions of the constrained currents.

Although for general reductions the bases for such algebras would not be a set of polynomial functions, the present reduction is such that they are polynomials and thus the  $W$ -algebra is a polynomial algebra as specified by Zamolodchikov. To see this one first notes that the gauge transformations of the currents are of the form

$$J(y) \rightarrow J^g(y) = e^{a(y)E^+} (J(y) + \partial_y) e^{-a(y)E^+}, \quad \text{where } J(y) = j(y) + M, \quad (5.3)$$

the  $j(y)$  are zero on the negative root sector and  $(M_-)_{rs} = \delta_{r,s+1}$ . With respect to the grading operator  $H$  the current components are non-negative and the parameters are strictly positive and the crucial point is that there exist a set of gauges (the so-called Drinfeld-Sokolov (DS) gauges [12]) in which the current has no zero grade components and only one component for each positive grade. Furthermore these gauge-fixings are complete, so the current-components in these gauges constitute a complete set of gauge-invariant functions. Their polynomiality then follows from the fact that according to (5.3) the  $J^g(y)$  are polynomials in the parameters  $a(y)$  and their derivatives, and according to (5.3), using  $J^g(y) = J^{DS}(y)$  and iterating in the grades, that the parameters themselves are polynomials in the original currents and their derivatives. The explicit details are given in [6].

A final interpretation of the  $W$ -algebras may be obtained by noting that total set of constraints consisting of the original first-class constraints and the DS gauge-fixing form a second-class system of constraints in the sense of Dirac. But since the ordinary and Dirac star-brackets for the functionals  $W(J)$  coincide (because they



respect the first-class constraints) and because the  $W(J)$ 's reduce to the current components in the DS gauges we have

$$\{W(J(y)), W(J(y'))\} = \{W(J(y)), W(J(y'))\}^* = \{J^{DS}(y), J^{DS}(y')\}^*. \quad (5.4)$$

Thus we obtain a final interpretation of the W-algebras as the Dirac star-algebras of the gauge-fixed currents.

## 6 General Structure of Reduction and Generalizations.

More recent work [13] [14] [15] concerns the analysis of the WZNW  $\rightarrow$  Toda reduction with a view to simplifying and generalizing it. As the general structure is actually quite simple (in some respects simpler than the specific Toda example) I should like to conclude by sketching this structure. The general idea is to impose *linear* constraints of the form

$$J(y) = j(y) + M, \quad \text{where } (\gamma, j(y)) = 0, \quad \text{for } \gamma \in \Gamma, \quad (6.1)$$

on a KM algebra (1.5), where  $\Gamma$  is a subalgebra of the Lie algebra  $G$ , and  $M$  is a constant element of the Lie algebra which is not zero and not in  $\Gamma$ .

The conditions that the constraints described by (6.1) be *first-class* are two-fold, namely,

$$(\alpha, \beta) = 0 \quad \text{and} \quad \omega(\alpha, \beta) = (M, [\alpha, \beta]) = 0, \quad \alpha, \beta \in \Gamma. \quad (6.2)$$

and follow from the fact that the KM centre  $\kappa$  and the constant component  $M$  of the current are not zero, respectively. The anti-symmetric form  $\omega$  will be recognized as the Kostant-Kirilov (KK) form for  $M$  evaluated at the origin. It plays a central role and can be used to simplify the definition of the DS gauges as follows: The extension of  $\omega$  to the whole Lie algebra  $G$  vanishes on the kernel  $K$  of the operator  $M$ , but on any subspace of  $G$  complementary to  $K$  it is non-degenerate. Hence if we assume that  $\Gamma$  does not intersect  $K$  we can choose a complementary space which contains  $\Gamma$  and an  $\omega$ -dual space  $\Theta$ . The DS gauges are then simply the gauges in which  $(\theta, j(y)) = 0$  for all  $\theta \in \Theta$ .

The condition that the constraints (6.1) be *conformally-invariant* is that there should exist some grading element  $H$  in the Lie algebra  $G$  such that

$$[H, \gamma] \in \Gamma, \quad (H, \gamma) = 0 \quad \text{and} \quad [H, M] = M. \quad (6.3)$$

The first condition in (6.3) implies that  $H$  should be a grading operator for  $\Gamma$  as well as  $G$ . The most important condition is the third one which gives a specific relation between  $H$  and  $M$ . In particular it implies that the generator  $M$  is nilpotent.

Using the constraints (6.1) satisfying the first-class and conformal invariant conditions (6.2) and (6.3) one obtains a conformal invariant reduction of the KM system and hence (using the dual conditions for the opposite chiral sector) of any concomitant field theory, such as the WZNW theory. The reduced theory will have symmetry algebras corresponding to the W-algebras of the Toda theory, and, as before, these will be the algebras of gauge-invariant functions of the constrained currents, or, equivalently, the Dirac star-algebras of gauge-fixed currents. The only difference will be that, in general, the gauge-invariant functions will *not be polynomials* in the constrained KM currents and their derivatives.

A fairly general *sufficient* condition for the algebras to be polynomial can be found and can be expressed quite simply in terms of the form  $\omega$  and, using this condition, a class of WZNW reductions which generalize the Toda reduction can be constructed. The condition is that if  $\{\gamma_i, \theta_j\}$  is an  $H$ -graded basis for the complementary space spanned by  $\{\Theta, \Gamma\}$  such that

$$\omega(\gamma_i, \theta_j) = \delta_{ij}, \quad (6.4)$$

then the corresponding W-algebra will be polynomial if

$$[\gamma_i, \theta_j] \in \Gamma \quad \text{for} \quad h(\gamma_i) \geq h(\theta_j), \quad (6.5)$$

where  $h$  are the  $H$ -grades. This condition is automatically satisfied for the Toda reduction. But it can be satisfied in a variety of other cases and in these cases provides new conformal reductions of the WZNW theories to integrable systems with polynomial W-algebras. For example it provides a generalization of the Toda system to one which consists of WZNW fields interacting in a nearest-neighbour fashion. More precisely it provides an action of the form

$$I(g_p) = \sum_p I_p(g_p) + \int d^2x \text{tr}(g_p^{-1} M_{p,p-1} g_{p-1} M_{p-1,p}), \quad (6.6)$$

where the  $g_p$  are WZNW fields belonging to diagonal blocks in the original WZNW algebra and the  $M_{p,p-1}$  and  $M_{p-1,p}$  are constant matrices that connect neighbouring blocks. This system reduces to the original Toda one when the blocks are 1-dimensional. It also produces the systems discussed recently in [13] and [14].

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## The Wave Equation on Symmetric Spaces

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### 1. Introduction. Hadamard's Problem.

To begin with let us consider the wave equation in Euclidean space,

$$\frac{\partial u^2}{\partial t^2} = \frac{\partial u^2}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}, \quad u(x, 0) = 0, \quad u_t(x, 0) = f(x)$$

for which one has the classical solution formula of Poisson and Tedone (cf. [7])

$$(1.1) \quad u(x, t) = \frac{1}{(n-2)!} \frac{\partial^{n-2}}{\partial t^{n-2}} \int_0^t (M^r f)(x) r (t^2 - r^2)^{(n-3)/2} dr,$$

where  $(M^r f)(x)$  denotes the spherical mean value

$$(M^r f)(x) = \frac{1}{A(r)} \int_{|y-x|=r} f(y) dw(y)$$

with  $A(r)$  denoting the area of the sphere  $S_r(x)$  of radius  $r$ . Denoting by  $B_r(x)$  the corresponding open ball formula (1.1) implies

$$(1.2) \quad u(x, t) \text{ is determined by } f|_{B_t(x)},$$

the vertical bar denoting restriction. If  $n$  is odd the right hand side of (1.1) can be differentiated out and we obtain for certain constants  $a_k$ ,

$$(1.3) \quad u(x, t) = \sum_{k=0}^{(n-3)/2} a_k t^{k+1} \frac{d^k}{dt^k} (M^t f)(x).$$

This implies that for each  $\epsilon > 0$ :

$$(1.4) \quad u(x, t) \text{ is determined by } f|(B_{t+\epsilon}(x) - B_{t-\epsilon}(x)).$$

In other words,  $u(x, t)$  is determined by the initial data in an arbitrarily thin shell around  $S_t(x)$ .

This is an instance of the well known Huygens' principle which for general Riemannian manifolds can be formulated as follows. Let  $X$  be a Riemannian manifold

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with distance function  $d$  and let  $L_X$  denote the Laplace-Beltrami operator. Consider the equation with initial data

$$(1.5) \quad \frac{\partial^2 u}{\partial t^2} = (L_X + c)u, \quad u(x, 0) = 0, \quad u_t(x, 0) = f(x),$$

$c$  being some constant. *Huygens' principle* is said to hold for (1.5) if property (1.4) holds for each  $\epsilon > 0$ . Hadamard proved that if  $\dim X$  is even, Huygens' principle does not hold; he posed the problem of finding all  $X$  such that Huygens' principle holds for (1.5). In spite of important work on this problem, particularly by Hadamard, Stellmacher, Günther, Ásgeirsson and others, the problem cannot be said to be satisfactorily solved and it appears doubtful that there are *simple* geometric conditions which are both necessary and sufficient. For a quick orientation and a deep study of the problem see Günther [5c] and [5b], respectively.

## 2. The Case of a Symmetric Space. The Fourier Transform and the Radon Transform.

In the group-theoretic spirit of this conference we shall now investigate Hadamard's problem in the case of a symmetric space  $X = G/K$  where  $G$  is a connected semisimple Lie group with finite center and  $K$  a maximal compact subgroup. The tools we employ are a certain *Fourier transform* and a *Radon transform* on  $X$ . To describe these consider the Iwasawa decomposition of  $G$  and its Lie algebra  $\mathfrak{g}$ ,

$$(2.1) \quad G = NAK, \quad \mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k},$$

where  $N$  and  $A$  are nilpotent and abelian, respectively. The Haar measure  $dg$  on  $G$  decomposes accordingly.  $dg = e^{2\rho(\log a)} dn da dk$  where  $\rho \in \mathfrak{a}^*$ , the dual space of  $\mathfrak{a}$ . By a *horocycle* in  $X$  is meant an orbit of a point in  $X$  under a subgroup of  $G$  of the form  $gNg^{-1}$ . If  $M$  denotes the centralizer of  $A$  in  $K$  and  $o = \{K\}$ ,  $\xi_o = N \cdot o$  then each horocycle  $\xi$  has the form

$$(2.2) \quad \xi = ka \cdot \xi_o,$$

where  $kM \in K/M$  and  $a \in A$  are unique. Thus we have the identification

$$(2.3) \quad K/M \times A \approx \Xi,$$

where  $\Xi$  denotes the space of all horocycles. The coset  $kM$  is called the *normal* to the horocycle  $\xi$  in (2.2) and  $a$  is called its *composite distance* from the origin  $o$ . If  $|\cdot|$  is the norm on  $\mathfrak{a}$  corresponding to the Killing form metric and  $d$  the distance function on  $X$  then  $d(o, \xi) = |\log a|$ . Given  $x \in X$  and  $b \in B = K/M$  there exists a unique horocycle  $\xi(x, b)$  through  $x$  with normal  $b$ . If  $b = kM$  we determine  $A(x, b) \in \mathfrak{a}$  by

$$(2.4) \quad \xi(x, kM) = k \exp A(x, kM) \cdot \xi_o.$$

These notions are easily visualized in the case of the hyperbolic plane  $\mathbf{H}^2$ ; in the unit disk model the horocycles are the circles tangential to the unit circle.

**Definition.** Given a function  $f$  on  $X$  its *Fourier transform*  $\tilde{f}(\lambda, b)$  is defined by

$$(2.5) \quad \tilde{f}(\lambda, b) = \int_X f(x) e^{(-i\lambda + \rho)(A(x, b))} dx, \quad \lambda \in \mathfrak{a}^*, b \in B,$$

and its *Radon transform*  $\hat{f}(\xi)$  by

$$(2.6) \quad \hat{f}(\xi) = \int_{\xi} f(x) dm(x) \quad \xi \in \Xi.$$

Here  $dx$  is the  $G$ -invariant volume element on  $X$  and  $dm(x)$  is the induced measure on  $\xi$ .

These definitions are motivated by analogies with  $\mathbf{R}^n$ , the horocycles corresponding to hyperplanes. In fact, if  $F$  is a function on  $\mathbf{R}^n$  we can write its Fourier transform as

$$(2.7) \quad \tilde{F}(\lambda\omega) = \int_{\mathbf{R}^n} F(x) e^{-i\lambda(x, \omega)} dx \quad \lambda \in \mathbf{R}, |\omega| = 1$$

and then  $(x, \omega)$  is the distance from 0 to the plane through  $x$  with normal  $\omega$ .

While one has inversion- and Plancherel type formulas for the transforms (2.5), (2.6), (cf. [6a], [6b]), results about their ranges are more useful for applications to differential equations.

**Definition.** A  $C^\infty$  function  $\psi$  on  $\mathfrak{a}^* \times B$  is said to be of *uniform exponential type* if  
 (i)  $\lambda \rightarrow \psi(\lambda, b)$  extends to a holomorphic function on the complex space  $\mathfrak{a}_c^* = \mathfrak{a}^* + i\mathfrak{a}^*$ ;  
 (ii) There exists an  $R > 0$  such that for each  $N \geq 0$

$$(2.8) \quad \sup_{(\lambda, b) \in \mathfrak{a}_c^* \times B} (1 + |\lambda|)^N e^{-R|\operatorname{Im} \lambda|} |\psi(\lambda, b)| < \infty.$$

**Theorem 2.1.** The Fourier transform  $f \rightarrow \tilde{f}$  is a bijection of  $C_c^\infty(X)$  onto the space of functions  $\psi$  of uniform exponential type satisfying the functional equations

$$\int_B \psi(s\lambda, b) e^{(is\lambda + \rho)(A(x, b))} db \equiv \int_B \psi(\lambda, b) e^{(i\lambda + \rho)(A(x, b))} db$$

for each  $s$  in the Weyl group  $W = W(\mathfrak{g}, \mathfrak{a})$ .

The proof ([6d]) gives at the same time a description of the range of  $C_c^\infty(X)$  under the Radon transform; this is because of the connection

$$(2.9) \quad \tilde{f}(\lambda, kM) = \int_A \hat{f}(ka \cdot \xi_0) e^{(-i\lambda + \rho)(\log a)} da.$$

The following corollary is however more suitable for the applications.

**Corollary 2.2.** Let  $R > 0$ . Suppose  $f \in C_c^\infty(X)$  satisfies

$$\hat{f}(\xi) = 0 \quad \text{for} \quad d(o, \xi) > R.$$

Then

$$f(x) = 0 \quad \text{for} \quad d(o, x) > R.$$

These results imply the following solvability result for an arbitrary  $G$ -invariant differential operator  $D \neq 0$  on  $X$ :  $DC^\infty(X) = C^\infty(X)$ ; in other words, the equation  $Du = f$  is always globally solvable.

For applications to the wave equation we need to extend these results to distributions  $T$  of compact support, that is, elements of  $\mathcal{E}'(X)$ . The definition of the Fourier transform  $\tilde{T}(\lambda, b)$  is obvious: just replace  $f(x)dx$  in (2.5) by  $dT(x)$ . The characterization in Theorem 2.1 extends to a similar description of  $\mathcal{E}'(X)^\sim$ : the condition (2.8) is then just required to hold for one  $R > 0$  and one  $N < 0$  (cf. [6d], proof of Theorem 8.5 or [2]). This done, defining the polynomial  $P_D(\lambda)$  by

$$D_x e^{(i\lambda + \rho)(A(x, b))} = P_D(\lambda) e^{(i\lambda + \rho)(A(x, b))},$$

one proves that for  $T \in \mathcal{E}'(X)$  the equation  $DS = T$  has a solution  $S \in \mathcal{E}'(X)$  if and only if  $\tilde{T}(\lambda, b)/P_D(\lambda)$  is holomorphic on  $\mathfrak{a}_c^*$ .

For a definition of the Radon transform  $\hat{T}$  formula (2.6) is unsuitable since the restriction of a distribution to a submanifold is not defined. Thus we proceed indirectly: Given a function  $\phi$  on the horocycle space  $\Xi$  we define the point function  $\check{\phi}$  on  $X$  by

$$(2.10) \quad \check{\phi}(x) = \int_{\xi \ni x} \phi(\xi) d\mu(\xi),$$

the average of  $\phi$  over the set of horocycles  $\xi$  passing through  $x$ . The transform  $\phi \rightarrow \check{\phi}$  is geometrically dual to  $f \rightarrow \hat{f}$  but in addition these transforms are adjoint as operators, in fact (cf. [6c])

$$(2.11) \quad \int_X f(x) \check{\phi}(x) dx = \int_{\Xi} \hat{f}(\xi) \phi(\xi) d\xi$$

where  $d\xi$  is a  $G$ -invariant measure on  $\Xi$ . This relation suggests defining  $\hat{T}$  by

$$(2.12) \quad \hat{T}(\phi) = T(\check{\phi}), \quad \phi \in C^\infty(\Xi).$$

Corollary 2.2 now extends as follows ([6g], p. 119, or [6h]) where we put

$$\beta_R = \{\xi \in \Xi : d(o, \xi) < R\}.$$

**Theorem 2.3.** Let  $T \in \mathcal{E}'(X)$  and suppose

$$\text{supp}(\hat{T}) \subset \bar{J}_R.$$

Then

$$\text{supp}(T) \subset \overline{B_R(o)}.$$

Here  $\text{supp}$  denotes support. We shall also need the inversion formula alluded to earlier.

**Theorem 2.4.** There exists a specific pseudodifferential operator  $\Lambda$  on  $\Xi$  such that

$$(2.13) \quad f = (\Lambda \hat{f})^\vee \quad \text{for} \quad f \in \mathcal{C}_c^\infty(X).$$

Moreover,  $\Lambda$  is a differential operator exactly when  $G$  has all its Cartan subgroups conjugate.

For the Radon transform on  $\mathbf{R}^n$  (with integration over hyperplanes) one has a formula like (2.13); then  $\Lambda$  is a differential operator if and only if  $n$  is odd.

### 3. Huygens' Principle for $X = G/K$ .

Using tools from §2 one can prove the following result about Huygens' principle.

**Theorem 3.1.** Suppose  $G$  has all its Cartan subgroups conjugate and that  $\dim X$  is odd. Then the modified wave equation on  $X$ ,

$$(3.1) \quad \frac{\partial^2 u}{\partial t^2} = (L_X + |\rho|^2)u \quad u(x, 0) = 0, \quad u_t(x, 0) = f(x)$$

satisfies Huygens' principle.

**Remarks.** The irreducible symmetric spaces  $X$  satisfying the conditions of the theorem are the following:

- (i)  $X = G/K$  with  $G$  simple, complex of odd dimension.
- (ii)  $X = \mathbf{H}^{2n+1}$ , the  $(2n+1)$ -dimensional hyperbolic space. Here  $G = \mathbf{SO}_o(2n+1, 1)$ .
- (iii)  $X = \mathbf{SU}^*(4n)/\mathbf{Sp}(2n)$ .

For cases (i) the result is given in [6e], [6f]. For (ii) an explicit solution formula can be obtained in various ways ([6a], [9a], [8], [6f]) implying in particular the validity of Huygens' principle. More explicitly, the space  $\mathbf{H}^n = \mathbf{SO}_o(n, 1)/\mathbf{SO}(n)$  has curvature  $-1$  in the metric  $g = B/2(n-1)$ ,  $B$  being the Killing form. Here  $\rho = \frac{1}{2}(n-1)\alpha$  where  $\alpha$  is the single positive restricted root so  $|\rho|^2 = (n-1)/8$ . Thus

$$(3.2) \quad L_X + |\rho|^2 = \frac{1}{2(n-1)} \left[ L + \frac{(n-1)^2}{4} \right],$$

where  $L$  is the Laplace-Beltrami operator for  $\mathbf{H}^n$  relative to  $g$ . The equation

$$(3.3) \quad \frac{\partial^2 u}{\partial t^2} = \left( L + \frac{(n-1)^2}{4} \right) u \quad u(x, 0) = 0, \quad u_t(x, 0) = f(x)$$



has the solution

$$(3.4) \quad u(x, t) = c \left( \frac{\partial}{\partial(2cht)} \right)^{(n-3)/2} \{sh^{n-2}t(M^t f)(x)\}$$

implying Huygens' principle. We remark that the solution formula in [6f] implies that the solution  $v$  to

$$(3.5) \quad \frac{\partial^2 v}{\partial t^2} = (L + (\frac{n-1}{2})^2) v, \quad v(x, 0) = f(x), \quad v_t(x, 0) = 0$$

is related to  $u$  by

$$(3.6) \quad v = u_t.$$

While the implication (3.3)  $\implies$  (3.5) is obvious the converse is less so

Theorem 3.1 in general (even with  $G$  reductive) was proved by Ólafsson and Schlichtkrull in [10] using the Radon transform on  $X$  (Cor. 2.2 and Theorem 2.4). Such a proof had been indicated without proof by Solomatina [11] and resembles the proof of Lax-Phillips [9a] for  $X = \mathbf{H}^3$ .

My independent proof proceeds via the Fourier transform and was inspired by the proof of Branson-Ólafsson [1] on the energy equipartition which I shall describe later.

Taking the Fourier transform of (3.1) one obtains

$$\ddot{u}_t(t, \lambda, b) + |\lambda|^2 \ddot{u}(t, \lambda, b) = 0$$

whence

$$(3.7) \quad \ddot{u}(t, \lambda, b) = \tilde{f}(\lambda, b) \frac{\sin |\lambda| t}{|\lambda|}$$

(cf. [1]). By a generalization of the classical Paley-Wiener theorem (cf. [3], p. 145).

$$(3.8) \quad \frac{\sin |\lambda| t}{|\lambda|} = \int_A e^{-i\lambda(\log a)} dT_t(a),$$

where  $T_t \in \mathcal{E}'(A)$  is unique. On the other hand, the indicated characterization of  $\mathcal{E}'(X)$  shows that

$$(3.9) \quad \frac{\sin |\lambda| t}{|\lambda|} = \int_X e^{(-i\lambda + \rho)(A(x, b))} d\tau_t(x)$$

where  $\tau_t \in \mathcal{E}'(X)$ . From an analog of (2.9) to distributions we deduce from (3.8)-(3.9) that

$$(3.10) \quad \dot{\tau}_t = 1 \otimes e^\rho T_t,$$

the tensor product referring to (2.3). For  $R > 0$  let

$$\sigma_R = \{\xi \in \Xi : d(o, \xi) = R\}.$$

The following lemma, which follows from Theorems 2.3-2.4, is crucial.

**Lemma 3.2.** Suppose  $G$  has all its Cartan subgroups conjugate and let  $R > 0$ . Then if  $T \in \mathcal{E}'(X)$ ,

$$\text{supp}(\hat{T}) \subset \sigma_R \implies \text{supp}(T) \subset S_R(o).$$

Now the convolution on  $G$  induces a convolution  $*$  on  $X$ . Moreover, if  $f \in C_c^\infty(X)$  and if  $T \in \mathcal{E}'(X)$  is  $K$ -invariant we have

$$(3.11) \quad (f * T)^\sim(\lambda, b) = \hat{f}(\lambda, b)\tilde{T}(\lambda, b).$$

In particular, (3.7) implies

$$(3.12) \quad u(x, t) = (f * \tau_t)(x).$$

Note that by (3.8)  $T_t$  is invariant under  $a \rightarrow a^{-1}$ . Thus  $\tau_\lambda(g^{-1} \cdot o) = \tau_\lambda(g \cdot o)$  and (3.12) becomes

$$(3.13) \quad u(g \cdot o, t) = \int_X f(g \cdot x) d\tau_t(x).$$

For  $\dim A > 1$  it is well known from Euclidean Fourier analysis (cf. e. g. [4], Ch. II, §3) that  $\text{supp}(T_t) \subset S_t(e)$ , the sphere of radius  $t$  in  $A$ . Thus by (3.10) and Lemma 3.2 we have  $\text{supp}(\tau_t) \subset S_t(o)$  and Huygens' principle (1.4) follows at once from (3.13).

In the case  $\dim A = 1$  the spaces  $X$  in the theorem are just the hyperbolic spaces  $X = \mathbf{H}^{2n+1}$  of odd dimension. Now (3.8) shows that  $T_t$  is a constant multiple of the characteristic function of the interval  $|\log a| \leq t$ . Taking the derivative with respect to  $t$  we get

$$(3.14) \quad \cos |\lambda| t = \int_A e^{-i\lambda(\log a)} dT'_t(a) = \int_X e^{(-i\lambda + \rho)(A(x, b))} d\tau'_t(x)$$

$$(3.15) \quad u_t(g \cdot o, t) = \int_X f(g \cdot x) d\tau'_t(x).$$

Now we have  $\text{supp}(T'_t) \subset S_t(e)$  so by (3.10) and Lemma 3.2,  $\text{supp}(\tau'_t) \subset S_t(o)$ . Thus by (3.6) Huygens' principle holds for equation (3.5). Formula (3.4) shows the same for (3.3).

#### 4. Huygens' Principle for a Compact Lie Group.

Let  $K$  be a simply connected semisimple Lie group with the bi-invariant Riemannian metric given by the negative of the Killing form. Determine  $R > 0$  such that the exponential mapping is a diffeomorphism of  $B_R(0)$  onto  $B_R(e)$  (balls in the Lie algebra and in  $K$ ).

**Theorem 4.1.** Suppose  $K$  above has odd dimension and let  $f \in C_c^\infty(B_R(e))$ . Then the modified wave equation

$$(4.1) \quad \frac{\partial^2 u}{\partial t^2} = \left( L_K - \frac{\dim K}{24} \right) u \quad u(k, 0) = 0, \quad u_t(k, 0) = f(x)$$

satisfies Huygens' principle.

This is stated in [6e] with a proof in [6f]. For  $\dim K = 3$  we have  $K = \mathbf{S}^3$  so the wave equation on the 3-sphere satisfies Huygens' principle. This is true more generally on  $\mathbf{S}^{2n+1}$  since the sphere is conformally flat [12]; it can also be proved directly [9b].

Since the spaces (i), (ii) listed after Theorem 3.1 are the symmetric space duals to the compact groups and the spheres respectively, Ólafsson and Schlichtkrull [10] raise the question whether the modified wave equation on the space  $\mathbf{SU}(4n)/\mathbf{Sp}(2n)$  (which is dual to  $\mathbf{SU}^*(4n)/\mathbf{Sp}(2n)$ ) satisfies Huygens' principle. As Ólafsson and I have observed, the equations correspond formally under the substitution  $x_j \rightarrow ix_j$ , and since the operators are analytic, Günther's necessary and sufficient moment conditions ([5b], Ch. VII, Theorem 5.10) apply. This makes a positive answer plausible but a genuine proof is required.

### 5. Equipartition of Energy.

Consider again the modified wave equation (1.5),  $X$  being again an arbitrary Riemannian manifold. The energy is then defined as the integral

$$(5.1) \quad \mathcal{E}_c(u) = \int_X \frac{1}{2} (u_t^2 + |du|^2 - cu^2)(t, x) dx \quad (u \text{ real}).$$

Here  $|du|$  is the norm of the 1-form  $du$  relative to the Riemannian structure. It is known ([1], Lemma 1.1) that for  $f$  of compact support,  $\mathcal{E}_c(u)$  is finite and independent of  $t$ . The two terms

$$\mathcal{K}(u)(t) = \frac{1}{2} \int_X u_t^2(t, x) dx$$

$$\mathcal{P}(u)(t) = \frac{1}{2} \int_X (|du|^2 - cu^2)(t, x) dx$$

are called the kinetic and potential energy, respectively.

**Theorem 5.1.** With  $X$  as in Theorem 3.1 and  $c = |\rho|^2$  the solutions to the modified wave equation (3.1) satisfy

$$\mathcal{K}(u)(t) \equiv \mathcal{P}(u)(t)$$

for  $t \geq R$ , if  $\text{supp}(f) \subset \overline{B_R(o)}$ .

This is proved in Branson-Ólafsson [1], using, among other things, some of the tools which were applied in the proof of Theorem 3.1.

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# Lie Point Symmetries and Exact Solutions of Nonlinear Differential Equations

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## Abstract

A review is given of the method of symmetry reduction for partial differential equations. Two physical applications are presented. One is a solution of the stimulated Raman scattering equations, the other is a classification of variable coefficient KdV equations according to their Lie point symmetry groups.

## Introduction

The purpose of this presentation is to first review a general method for obtaining exact analytic solutions of nonlinear partial differential equations (PDEs) and then to apply the method to two physical systems. The method is that of symmetry reduction and it goes back to Sophus Lie. There are new twists to the method and it has been turned into an efficient algorithm. This is due to developments in group theory, the theory of integrable dynamical systems and

in computer science. Moreover, the ever increasing importance of nonlinear phenomena in physics has emphasized the need for mathematical methods, providing solutions of nonlinear equations.

The applications to be discussed come from nonlinear optics, namely stimulated Raman scattering and from hydrodynamics, namely the variable coefficient Korteweg-de Vries equation.

## 1 The Algorithm for Symmetry Reduction

The basic idea is that one looks for solutions of a given systems of *PDEs* that are invariant under some group  $G_0$ , compatible with the equations. The invariance condition reduces the number of independent variables in the equation. This amounts to obtaining solutions, corresponding to boundary conditions with particular symmetries.

More specifically the algorithm consists of the following steps, applicable to an arbitrary system of differential equations:

$$\begin{aligned} \Delta^\mu(x, u, u_x, u_{xx}, \dots) &= 0 \\ \mu &= 1, \dots, n, x \in R^p, u \in R^q, \end{aligned} \quad (1)$$

where for instance  $u_{xx}$  denotes all second derivatives, and the order of the equation is  $N$ , with  $p, q, n$  and  $N$  arbitrary.

*Step 1.* Find the symmetry group  $G$  of local point transformations

$$\tilde{u} = \Omega(x, u, g), \tilde{x} = \Lambda(x, u, g), \quad (2)$$

such that whenever  $u(x)$  is a solution, then  $\tilde{u}(\tilde{x})$  is also a solution, as long as the functions  $\Omega$  and  $\Lambda$  are defined. Notice that we restrict ourselves to point transformations:  $\tilde{u}$  and  $\tilde{x}$  do not depend on derivatives of  $u$ . The algorithm for finding the Lie group  $G$ , or rather its Lie algebra  $L$  is well known and described e.g. in Ref. [1,2]. Moreover, computer programs using REDUCE [3], MAC SYMA [4] and other symbolic languages exist, that realize it. The essence is that one constructs a vector field

$$\hat{X} = \xi_i(x, u) \partial x_i + \phi_a(x, u) \partial u_a, \quad (3)$$

such that its  $N$ -th prolongation annihilates the equations (1) on their solution set

$$pr \hat{X} \cdot \Delta^\mu|_{\Delta^\mu=0} = 0. \quad (4)$$

Eq.(4) provides a system of linear *PDEs* for the function  $\xi_i(x, u)$  and  $\phi_a(x, u)$ . This system of "determining equations" is usually overdetermined and its only solution may be  $\xi_i = \phi_a = 0$ . In this case the method is not applicable. In interesting cases the solution may depend on  $k$  significative integration constants, and we obtain a  $k$ -dimensional symmetry algebra, or the solution may depend on arbitrary functions of  $x$  or  $u$ , and we obtain an infinite dimensional symmetry algebra.

*Step 2.* Identify the Lie algebra  $L$  as an abstract Lie algebra by transforming it to a "canonical" basis. If it is decomposable, it will be decomposed into a direct sum of Lie algebras. Each component will be visibly simple, solvable, or its Levi decomposition into a semidirect sum with a solvable ideal (the radical) will be

manifest. Algorithms performing this identification exist[5] and have to some degree been computerized.

*Step 3.* Classify the subalgebras of  $L$  into conjugacy classes under the action of the symmetry group  $G$ . Algorithmic methods [6,7] for doing this exist, different for simple Lie algebras, direct sums, or Lie algebras with nontrivial ideals (see Ref.2 for a review).

*Step 4.* Once the subalgebras  $L_i \subset L$  are known, we find the invariants of the corresponding subgroup  $G_i \subset G$  in the space  $X \times U$  of independent and dependent variables. In the optimal case we can find  $k$  invariants  $\xi_1(x), \dots, \xi_k(x)$ ,  $0 \leq k < p$  depending only on the independent variables  $x$  and  $q$  invariants  $I_1(x, u), \dots, I_q(x, u)$  such that the Jacobian of the transformation  $(u_1, \dots, u_q) \rightarrow (I_1, \dots, I_q)$  is nonsingular. We then set

$$I_i(x, u) = F_i(\xi_1, \dots, \xi_k), \quad i = 1, \dots, q \quad (5)$$

and solve for  $u_i$  to obtain

$$u_i = U_i(x_1, \dots, x_p, F_i(\xi)). \quad i = 1, \dots, q. \quad (6)$$

Substituting (6) into the original system (1) we obtain a reduced system

$$\hat{\Delta}(\xi, F, F_\xi, F_{\xi\xi}, \dots) = 0. \quad (7)$$

Since we have  $k < q$ , we obtain a dimensional reduction ("symmetry reduction"), i.e. fewer independent variables. For  $k = 0$  we obtain algebraic equations, for  $k = 1$  a system of ordinary differential equations.

*Step 5.* Solve the reduced equations (7) (if possible). They may be linearizable, or integrable by some linear technique, like the inverse scattering transform [8]. A singularity analysis (Painlevé analysis) may help to solve them. Alternatively, more group theory can be used to reduce the number of variables further, or to reduce the order of the equations for ODEs.

*Step 6.* Do physics with the solutions, i.e. analyse their stability, their asymptotics, calculate various observable quantities, etc. The last step is of course the least algorithmic one.

We mention that group theory can provide solutions even if conditions used in Step 4 are not satisfied. Thus, if the variables  $\xi_i$  depend on some of the dependent variables  $u_i$ , we obtain implicit solutions. If the transformation  $u \rightarrow I$  is not invertible, we may obtain "partially invariant solutions" [9-11].

## 2 Solutions of the Stimulated Raman Scattering Equations

The application to be considered here is joint work with D.Levi and C.Menyuk [12]. The SRS equations describe the nonlinear interaction of three colinear complex waves, for instance when pulses are periodically boosted by an optical pump [13,14]. In a simplified notation the SRS equations are

$$\begin{aligned} v_{1,x} + ia_1 v_2^* v_3^*, \quad v_{2,x} + ia_2 v_1^* v_3^*, \quad v_{3,t} + ia_3 v_1^* v_2^* \\ a_i = \pm 1, \quad v_k = \rho_k e^{i\phi_k}, \quad \rho_k \geq 0, \quad 0 \leq \phi_k < 2\pi. \end{aligned} \quad (8)$$

We shall identify  $v_1, v_2$  and  $v_3$  with the Stokes wave (the signal), the pump wave and the material excitation, respectively. We use the algorithm described above to obtain exact solutions. In particular, we are interested in a "physical"

solution in which  $v_1$  for  $t$  fixed goes to a constant nonzero value for  $x \rightarrow +\infty$ , where  $v_2$  and  $v_3$  go to zero in the same limit.

The symmetry algebra for eq.(8) is infinite dimensional. A basis can be written as

$$\begin{aligned} P &= \partial_x, \quad D = x\partial_x - \frac{1}{2}(\varrho_1\partial_{\varrho_1} + \varrho_2\partial_{\varrho_2} + 2\varrho_3\partial_{\varrho_3}) \\ V &= -\partial_{\varphi_2} + \partial_{\varphi_3}, \quad U(h) = h(t)(-\partial_{\varphi_1} + \partial_{\varphi_3}) \\ T(f) &= f(t)\partial_t - \frac{1}{2}\dot{f}(t)(\varrho_1\partial_{\varrho_1} + (\varrho_2\partial_{\varrho_2})), \end{aligned} \quad (9)$$

where  $f(t)$  and  $h(t)$  are arbitrary functions of time. We obtain translations  $P, T(1)$ , dilations  $D$  and  $T(t)$ , constant and time dependent changes of phase  $V$  and  $U(h)$ . More generally  $T(f)$  corresponds to a reparametrization of time and the corresponding Lie algebra  $\{T(f)\}$  is isomorphic to the Virasoro algebra (with no central extension).

The Lie algebra  $L$  is the direct sum of three Lie algebras

$$L = \{P_1, D\} \oplus \{V\} \oplus \{T(f), U(h)\}, \quad (10)$$

the last one being a  $\hat{u}(1)$  Kac-Moody Virasoro algebra [2].

The third step is to classify subalgebras of the symmetry algebra. The SRS equations involve only two independent variables. We wish to reduce to an ODE, we hence only need one-dimensional subalgebras (and subgroups). Ten classes of them exist [12]. For instance the algebra  $\{P_1 + aT(1) + bV\}$  ( $a, b$  are constants) leads to solutions in terms of Jacobi elliptic functions, or to solitons. The subalgebra  $\{P_1 + aU(1) + bV\}$  leads to solutions known as "phase wave solutions" in optics.

We shall concentrate on solutions invariant under the subgroup generated by the subalgebra  $\{D + aT(t) + bV\}$ . The corresponding seven invariants will be denoted  $\xi, M_i$  and  $\alpha_i$  and the complex wave amplitudes are expressed as

$$\begin{aligned} v_1 &= t^{-(a+1)/2a} M_1(\xi) e^{i\alpha_1(\xi)} & \xi &= xt^{-1/a} \\ v_2 &= t^{-(a+1)/2a} e^{-i(b/a)\ln t} M_2(\xi) e^{i\alpha_2(\xi)} & a &\neq 0 \\ v_3 &= \frac{1}{x} e^{i b \ln x} M_3(\xi) e^{i\alpha_3(\xi)}. \end{aligned} \quad (11)$$

We fix the constants  $a_i$  in (8) to be  $a_3 = a_2 = -a_1 = 1$ , as corresponds to the usual physical situation. We substitute eq.(11) into eq.(8), separate the real and imaginary parts and obtain six coupled real ODEs for amplitudes  $M_i$  and phases  $\alpha_i$ . We make use of two first integrals

$$\begin{aligned} I_1 &= M_1^2 + M_2^2, \quad I_2 = M_1 M_2 M_3 \cos \phi - \frac{b}{2} M_1^2 \\ \phi &= \alpha_1 + \alpha_2 + \alpha_3 + b \ln(\xi) \end{aligned} \quad (12)$$

to decouple the equations. For  $M_1$  we obtain a second order nonlinear ODE:

$$\begin{aligned} \xi \ddot{M}_1 + \dot{M}_1 &= a_2 \xi \frac{M_1 \dot{M}_1^2}{a_2 M_1^2 - I_1} + \frac{a_2}{\xi M_1} \frac{(a_1 I_2 - \frac{b}{2} M_1^2)^2}{a_2 M_1^2 - I_1} \\ &+ (I_1 a a_3 - \frac{b^2}{4\xi}) M_1 - a a_2 a_3 M_1^3 + \frac{I_2^2 a_1^2}{\xi M_1^3}. \end{aligned} \quad (13)$$



Eq.(13) looks formidable, but it is linear in  $\ddot{M}_1$ , rational in  $M_1$  and  $\dot{M}_1$ , and analytic (actually also rational) in  $\xi$ . Such equations were studied by Painlevé and Gambier (see Ince [15] for a summary of results). They classified all equations of this type that have what is now called the "Painlevé property". This means that their solutions have no movable critical points, i.e. no singularities, other than poles, the position of which depends on the initial conditions. An algorithmic necessary test for the Painlevé property exists [16] and has been turned into a MACSYMA routine [17]. Eq.(13) passes the test after the transformation  $M_1 = H^{\frac{1}{2}}$ . A further transformation takes it into a standard form. Thus, we put

$$M_1 = \frac{I_1 W}{W - 1} \quad (14)$$

and obtain the equation for a particular case of the fifth Painlevé transcendent

$$\begin{aligned} W &= P_V(\alpha, \beta, \gamma, \delta; \xi) \\ \ddot{W} &= \left(\frac{1}{2W} + \frac{1}{W-1}\right)W^2 - \frac{1}{\xi} \dot{W} + \\ &\frac{(W-1)}{\xi^2} \left[-\frac{1}{2}\left(\frac{2I_2}{I_1} + b\right)^2 W + \frac{2I_2^2}{I_1^2} \frac{1}{W}\right] + \frac{2I_1 a}{\xi} W \\ \alpha &= -\frac{1}{2}\left(\frac{2I_2}{I_1} + b\right)^2, \beta = \frac{2I_2^2}{I_1^2}, \gamma = 2I_1 a, \delta = 0. \end{aligned} \quad (15)$$

Thus, we have an exact solution for  $M_1(\xi)$ , valid for all  $\xi, 0 \leq \xi < \infty$ . The other amplitudes  $M_2, M_3$  and phases  $\alpha_i$  are then all expressed in terms of  $M_1$  and the integrals  $I_2, I_3$ .

From the point of view of physics, the important question is: given values of  $M_1$  and  $\alpha_1$  for  $\xi = 0$ , how will they develop for  $\xi \rightarrow \infty$ . This is the "connection problem" for Painlevé transcendents and a large literature exists on this topic [18].

Here we shall address a simpler problem, namely give an asymptotic formula for the waves  $v_i$ . Skipping all details [12], concerning the derivation and the absence of secular terms, we just present the final result, valid for  $a = -1, \xi \rightarrow +\infty$ :

$$\begin{aligned} v_1 &\sim \frac{1}{(xt)^{1/4}} e^{i\alpha_1} \left[ \sqrt{\beta^2 - \frac{I_2^2}{I_1}} \cos 4\sqrt{I_1} (\sqrt{\xi} - \sqrt{\xi_0}) + \beta \right]^{1/2} \\ v_2 &\sim e^{ib\ln t} e^{i\alpha_2} \sqrt{I_1} \\ v_3 &\sim \frac{t^{1/4}}{x^{3/4}} e^{ib\ln x} e^{i\alpha_3} \left[ \beta - \sqrt{\beta^2 - \frac{I_2^2}{I_1}} \cos 4\sqrt{I_1} (\sqrt{\xi} - \sqrt{\xi_0}) \right]^{1/2} \\ \xi &= xt. \end{aligned} \quad (16)$$

For  $t = t_0$  fixed,  $x \rightarrow \infty$  we have

$$v_1 \rightarrow 0, \quad v_2 \rightarrow \text{const}, \quad v_3 \rightarrow 0 \quad (17)$$

where  $v_2$  is the Stokes wave. Eq.(17) describes the behaviour of the wave envelopes, the waves themselves oscillate within these envelopes. The experimentally observed behaviour [19] is reproduced very well by eq.(16).

### 3 Symmetries of the Variable Coefficient Korteweg-de Vries Equation

In this section we shall discuss the symmetries of an equation containing arbitrary functions, namely

$$\begin{aligned} u_t + f(x, t)uu_x + g(x, t)u_{xxx} &= 0 \\ f \neq 0, \quad g \neq 0 \end{aligned} \quad (18)$$

which we shall call the variable coefficient Korteweg-de Vries equation (VCKdV). For  $f$  and  $g$  constant we obtain the original KdV equation. This is the prototype of a "soliton equation". It is integrable by inverse scattering techniques, corresponds to a completely integrable Hamiltonian system with infinitely many integrals of motion in involution, allows for soliton and multisoliton solutions, as well as periodic and multiperiodic ones (see e.g. ref.8). Physically the KdV describes long small amplitude waves in shallow water and can be derived from the basic equations of hydrodynamics in the corresponding approximation.

Eq.(18) is obtained under less restrictive conditions, allowing for a variable density of the fluid, variable depth, the presence of vorticity and other space and time dependent effects. Its symmetries for various classes of functions  $f$  and  $g$  were studied in collaboration with J.P.Gazeau [20].

The KdV equation itself has a four-dimensional symmetry group of local point transformations, generated by space and time translations ( $P_1, P_0$ ), dilations  $D$ , and Galilei boosts  $B$ . It also has infinitely many "higher symmetries", but we are not concerned with those here.

For the VC KdV equation our aim is twofold:

1. Classify these equations into equivalence classes under the action of fiber preserving local point transformations

$$\begin{aligned} u &= U(\tilde{x}, \tilde{t}, \tilde{u}) \quad x = X(\tilde{x}, \tilde{t}), \quad t = T(\tilde{x}, \tilde{t}) \\ \frac{\partial U}{\partial \tilde{u}} &\neq 0, \quad \frac{\partial(X, T)}{\partial(\tilde{x}, \tilde{t})} \neq 0. \end{aligned} \quad (19)$$

2. Find the Lie point symmetry group for each class.

The first task, a preliminary classification of equations, is achieved using global methods, rather than infinitesimal ones. The result is that "allowed transformations" taking eq.(18) into a VC KdV equation, not necessarily with the same functions  $f$  and  $g$ , have the form:

$$\begin{aligned} u(x, t) &= A(t)\tilde{u}(\tilde{x}, \tilde{t}) + B(x, t) \\ \tilde{x} &= \alpha(t)x + \beta(t), \quad \tilde{t} = \Theta(t), \end{aligned} \quad (20)$$

where the functions involved must satisfy

$$\begin{aligned} \dot{\alpha}x + \dot{\beta} + fB\alpha &= 0, \quad \dot{A} + fB_x A = 0, \\ B_t + fBB_x + gB_{xx} &= 0, \end{aligned} \quad (21)$$

and the dots signify time derivatives.

Analyzing the system (21) that determines the "allowed transformations", we find that we must distinguish the following classes of VC KdV equations.

1. The generic case, when  $f(x, t)$  and  $g(x, t)$  are arbitrary. Allowed transformations are dilations, translations and the reparametrization of time

$$B = 0, \quad A = A_0, \quad \alpha = \alpha_0, \quad \beta = \beta_0, \quad \Theta = \Theta(t). \quad (22)$$

Additional allowed transformations, and hence additional symmetries, exist only in the following special cases.

2.  $f = f(t), g(x, t)$  arbitrary

3.

$$f(x, t) = p(t)e^{xq(t)}, \quad g(x, t) = -\frac{\dot{q}}{q^3}x + h(t) \quad (23)$$

4.

$$f(x, t) = p(t)[x + q(t)]^{r(t)}$$

$$g(x, t) = \frac{(x + q)^2}{r(1 - r^2)} \{ -(x + q)h(t) + (r - 1)\dot{q} + r(x + q)\ln(x + q) \} \quad (24)$$

The Lie point symmetries were obtained for each class of VC KdV equations and the results on the symmetry algebra  $L$  can be summed up as follows.

1.  $\dim L \leq 4$

2.  $\dim L = 4$  if and only if the VC KdV equation can be transformed into the KdV:

$$f_x = g_x = 0, \quad g(t) = f(t)[c_1 \int_0^t f(s)ds + c_2] \quad (25)$$

The same conditions have been shown to be necessary and sufficient for the VC KdV equation to have the Painlevé property [21,22].

3.  $\dim L = 3$  occurs in precisely the following mutually inequivalent cases:

$L = sl(2, \mathbf{R})$  :

$$u_t + uu_x + xu_{xxx} = 0 \quad (26)$$

$$u_t + \frac{1}{x}uu_x + \frac{1}{x}u_{xxx} = 0 \quad (27)$$

$$u_t + (x + t)^{1/2}uu_x - \frac{4}{3}(x + t)^2 u_{xxx} = 0 \quad (28)$$

$L$  decomposable, but not abelian

$$u_t + xuu_x + \frac{x^3}{t}u_{xxx} = 0 \quad (29)$$

$$u_t + uu_x + t^2 u_{xxx} = 0 \quad (30)$$

$L$  nilpotent:

$$u_t + xuu_x + x^3 u_{xxx} = 0 \quad (31)$$

$L$  solvable (with diagonal action on nilradical):

$$u_t + uu_x + t^\alpha u_{xxx} = 0, \quad \alpha \neq 2 \quad (32)$$

$L$  solvable (with complex action on nilradical)

$$u_t + uu_x + (1 + t^2)^{1/2} e^{3\alpha \arctan t} u_{xxx} = 0 \quad (33)$$

$L$  solvable (with Jordan action on nilradical)

$$u_t + uu_x + e^{3t} u_{xxx} = 0 \quad (34)$$

For a classification of real Lie algebras  $L$  with  $\dim L = 3$ , see ref.23.

4.  $\dim L = 2$  and  $L = 1$  occur for infinitely many different VC KdV equations that in some cases involve arbitrary functions of one variable [20].

#### 4 Conclusions

Lie group theory started out as a tool for solving differential equations. At present, for most nonlinear PDEs, Lie group theory is the only tool available for obtaining analytic exact solutions. Moreover, it is a powerful tool. The "similarity" solutions provided by group theory tend to be particularly "robust": large classes of solutions, starting out from quite general initial conditions, approach the similarity solutions asymptotically. An important part of the symmetry reduction algorithm is a classification of subgroups of the symmetry group. From the physical point of view this is equivalent to a classification of different types of symmetry breaking by boundary conditions.

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# BOHR'S INDETERMINACY PRINCIPLE IN QUANTUM HOLOGRAPHY, ADAPTIVE NEURAL NETWORKS, MOLECULAR COMPUTERS, AND CORTICAL SELF-ORGANIZATION

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*Some ten years after writing his fundamental papers on optics, Hamilton made a startling observation: that the same formalism applies to mechanics of point particles. Replace the optical axis by the time. Then the transformation from initial position and momenta to final position and momenta is always symplectic. This discovery led to remarkable progress in the nineteenth century. In the 1920s - almost a century later - Hamilton's analogy between optics and mechanics served as one of the major clues in the discovery of quantum mechanics.*

Shlomo Sternberg (1988)

*Great progress was possible thanks to the wonderful tool in atomic physics which is the laser.*

Alain Aspect (1986)

*The basic principles of cortical organization are similar at different processing levels.*

Wolf Singer (1990)

## Abstract

A rigorous proof of quantum parallelism cannot be based on the Heisenberg inequality because the standard deviation of self-adjoint operators is insensitive to fine structures of the interference distribution generated by a Mach-Zehnder interferometer. Actually Niels Bohr's indeterminacy principle of spatio-temporal quantum electrodynamics cannot be based on any of the known uncertainty principles.

It is shown how the holographic transform allows to circumvent the difficulties with the standard deviation by using a group theoretical implementation

of the canonical commutation relation of quantum electrodynamics. The geometric quantization approach combined with the emitter-absorber transaction model of quantum dynamics on the whole real line  $\mathbf{R}$  allows to describe by a Liouville density the flow and counterflow of optical photons in split fan-in/fan-out coherent optical channels. It makes the heuristic arguments concerning quantum parallelism rigorous by considering wavepackets as symplectic spinors on the hologram plane. Moreover, it implies the existence of single-photon holograms and includes the standard uncertainty inequality as a special case.

As an application of quantum holography to the area of quantum computers, we study adaptive neural networks and the photonic implementation of cortical self-organization principles as suggested by recent experimental results in brain research. These findings establish that synchronization reflects global properties of the stimulus. High resolution spectroscopy leads to the concept of magnetic resonance imaging (MRI) and, combined with the imaging capabilities of optical holography, to the concept of molecular computer. As another result of the geometric quantization approach to optical holography, the uncertainty minimizing Gabor wavelets arise. This kind of wavelets forms excellent approximations to the receptive field profiles and provides useful wavelet expansions for image analysis, segmentation, and compression.

*Es ist charakteristisch für das Hologramm, daß man bei der Beleuchtung eines seiner Teile stets Information über das ganze Bild erhält, wenn auch weniger detailliert und aus weniger Blickwinkeln gesehen. Je mehr vom Hologramm man beleuchtet, desto detaillierter und umfassender wird die Information. Subjekt oder Objekt der Information ist jedoch stets dieses eine Ganze. Die verschiedenen Teile des Hologramms sind nicht Entsprechungen von verschiedenen Teilen des Objekts. Vielmehr vermittelt jeder einzelne Teil etwas von dem Ganzen.*

David Bohm (1986)

*Each photon bears information about the entire system.*

H. John Caulfield (1991)

*The Schrödinger equation preserves the norm of the wave function and thus the number of particles. Now it is well known that in the presence of sources, photons can be absorbed or emitted. Thus one cannot introduce a Schrödinger equation for a single photon in the presence of sources. In fact, the electromagnetic field itself must be quantized, and photons then occur as elementary excitations of the quantized field.*

Claude Cohen-Tannoudji (1989)

*The basic element of the transactional interpretation is an emitter-absorber transaction through the exchange of advanced and retarded waves, as first described by Wheeler and Feynman.*

John G. Cramer (1986)

## 1 Holographic Imaging

Optical holography is a two-step coherent imaging process. In the first processing step, the recording or write-in step, the three-dimensional image is spatio-temporally coherently encoded in both amplitude and phase information by an interference distribution in the hologram plane. The interference distribution arises by coherently mixing the diffuse reference beam at optical frequency  $\nu$  and the beam coherently scattered by the object. Thus the hologram forms a linear superposition of interference fringe patterns which are generated by phase-shifted beam splitter interferometer experiments. The result of these interferometric experiments are photonic excitation distributions recording the raw optical data that are transmitted in terms of both amplitude and relative phase information by the flow and counterflow of optical photons in split coherent optical channels. In terms of geometric quantization, the symplectic hologram plane ( $\mathbf{R} \pm \mathbf{R}, \Omega_\nu$ ) which is formed by quantum-sensitive detectors carries the structure of a metaplectic manifold.

In the second processing step, the read-out step, the raw optical data spatio-temporally, coherently encoded by the hologram as an excitation distribution are spatially coherently decoded by an application of the adaptive resonance procedure. After slice selection by tuning the laser frequency  $\nu$  or an application of the first order Bragg diffraction condition

$$2\nu \sin \beta = 1$$

( $\beta$  = diffraction angle of incidence measured from the normal of the slice) similar to the spin slice selection procedure in MRI ([20]), the three-dimensional image of the object is simultaneously reconstructed in both amplitude and phase by coherently diffracting the diffuse reading light illumination. According to the geometric quantization procedure, the holographic information retrieval is performed by adaptive resonance within the Hilbert space bundle associated with the metaplectic manifold structure of the symplectic hologram plane ( $\mathbf{R} \pm \mathbf{R}, \Omega_\nu$ ).

The most important difference between optical holography and ordinary photography is the interferometric recording of the relative phase between the reference and the object beam during the write-in processing step. Due to this property mechanically stable conditions are needed and the quality and the reproducibility of optical holograms strongly depend on the full-fulfillment of this stability requirement. It is the inclusion of the relative phase information which performs the adaptability of the holographic code.

The fundamental information processing steps of optical holography are the adaptive encoding and decoding steps. Because these processing steps are concerned with the flow and counterflow of photons, both the write-in and read-out steps are of a quantum theoretical character. The quantum parallelism according to which different alternatives at the quantum level are allowed to coexist in linear superposition, irrespective of how different from one another the quantum states might be, is equivalent to Niels Bohr's indeterminacy principle of spatio-temporal quantum dynamics which says that in the phase shifted beam splitter interferometer experiment the



interference distribution appears if and only if we cannot determine the pathways of the optical photons. Bohr who established his conclusion by a heuristic application of the uncertainty principle did not realize that the indeterminacy principle allows to generate single-photon holograms.

The uncertainty principle occupies a peculiar position in physics and in signal processing. Since the fundamental 1927 paper by Werner Heisenberg ([21]), it is often considered as the hallmark of quantum mechanics. In his Chicago lectures of spring 1929, he regarded the inequality

$$(\Delta q)_\psi \cdot (\Delta p)_\psi \geq \frac{1}{2} \hbar$$

as the precise mathematical expression of the uncertainty principle within the formalism of quantum theory ([22]). On the other hand, although it is easy to derive it in a rigorous mathematical manner, there is still a great deal of discussion about its correct interpretation in physics, information theory, and in the fusion of both areas ([8], [23], [24], [42], [44]). It turns out that the preceding Cauchy-Schwarz type inequality has a number of weaknesses, however, particularly related to the fact that the standard deviations  $(\Delta q)_\psi$  and  $(\Delta p)_\psi$  only give very general information about the spreads of the probability density functions of position and momentum, respectively. Moreover, in quantum holography we find that  $(\Delta q)_\psi$  or  $(\Delta p)_\psi$  are divergent. Therefore the inequality is not appropriate for the purposes of quantum holography. In fact, it does not conclusively establish Niels Bohr's indeterminacy principle which states that in the Mach-Zehnder interferometer "the interference distribution appears if and only if we cannot determine the pathways of the optical photons."

*The uncertainty principle of Heisenberg is one of the most important aspects of the Copenhagen interpretation.*

John G. Cramer (1986)

*The correct way to understand the uncertainty relations is to see that they represent an inherent limitation on the set of states which can be produced for quantum mechanical systems, but that they cannot be 'explained' or 'deduced' by the naive disturbance argument.*

Michael Redhead (1987)

## 2 The Uncertainty Inequality

Let  $\psi$  denote the state vector of a one-dimensional quantum-mechanical system. Thus  $\psi \in L^2(\mathbf{R}; dt)$  satisfies the normalization condition

$$\|\psi\|_2 = \left( \int_{\mathbf{R}} |\psi(t)|^2 dt \right)^{1/2} = 1.$$

In the emitter-absorber transaction model of quantum dynamics ([8]) on the whole real line  $\mathbf{R}$ , the state vector  $\psi$  forms a wavepacket density  $\psi(t)dt$  of normalized

energy  $\| \psi \|_2 = 1$ . The position and momentum translation operators are defined by the  $\mathbf{C}$ -linear mappings

$$q : \psi \mapsto x.\psi(x),$$

and

$$p : \psi \mapsto -i\hbar\psi'$$

where  $\psi'$  denotes the distributional derivative of  $\psi$  with respect to the real variable  $x$ ,  $\hbar$  denotes Planck's constant, and as usual in quantum theory,  $\hbar = h/2\pi$ . Suppose

$$\int_{\mathbf{R}} |x|^2 |\psi(x)|^2 dx < +\infty,$$

and

$$\int_{\mathbf{R}} |\xi|^2 |\mathcal{F}\psi(\xi)|^2 d\xi < +\infty$$

where  $\mathcal{F}\psi$  denotes the Fourier transform of  $\psi$  in the real variable  $\xi$  which is dual to the real variable  $x$ . Hence

$$\mathcal{F}\psi(\xi) = \int_{\mathbf{R}} \psi(x) e^{-2\pi i \xi x} dx$$

in the dual state space  $L^2(\mathbf{R}; dx)$ . Then the position and momentum operators are self-adjoint operators  $q$  and  $p$ , respectively, in the complex Hilbert space  $L^2(\mathbf{R}; dx)$  and the expected values of position and momentum are given by

$$\langle q \rangle_{\psi} = \int_{\mathbf{R}} x |\psi(x)|^2 dx,$$

$$\langle p \rangle_{\psi} = \int_{\mathbf{R}} \frac{\xi}{h} |\mathcal{F}\psi(\frac{\xi}{h})|^2 d\xi.$$

Their standard deviations are the expressions

$$(\Delta q)_{\psi} = \left( \int_{\mathbf{R}} |x - \langle q \rangle_{\psi}|^2 |\psi(x)|^2 dx \right)^{1/2},$$

$$(\Delta p)_{\psi} = \left( \int_{\mathbf{R}} |\xi - \langle p \rangle_{\psi}|^2 \frac{1}{h} |\mathcal{F}\psi(\frac{\xi}{h})|^2 d\xi \right)^{1/2}.$$

The Heisenberg inequality

$$\int_{\mathbf{R}} x^2 |\psi(x)|^2 dx \cdot \int_{\mathbf{R}} \xi^2 |\mathcal{F}\psi(\xi)|^2 d\xi \geq \frac{1}{16\pi^2} \left( \int_{\mathbf{R}} |\psi(x)|^2 dx \right)^2$$

which is a direct consequence of the Plancherel-Fourier theorem combined with the Cauchy-Schwarz inequality implies the uncertainty inequality

$$(\Delta q)_{\psi} \cdot (\Delta p)_{\psi} \geq \frac{1}{2}\hbar,$$

mentioned in Section 1 supra. It follows from the Cauchy-Schwarz inequality that in the preceding inequality identity holds if and only if

$$\psi'(x) = c.x\bar{\psi}(x) \quad (c \in \mathbf{C})$$

holds in  $L^2(\mathbf{R}; dx)$ , i.e., if and only if  $\psi \in L^2(\mathbf{R}; dx)$  is a vacuum coherent state vector

$$\psi(x) = C \cdot e^{-|x|^2/2} \quad (x \in \mathbf{R})$$

where the constant  $C \in \mathbf{C}$  in front of the Gaussian wavepacket is determined by the condition  $\|\psi\|_2 = 1$  up to one overall phase factor. The uncertainty minimizing property of the coherent states is an important ingredient of the Gabor wavelet expansions ([9], [10]) for image analysis, segmentation and compression; see Section 8 infra.

*An advantage of the Heisenberg point of view is that it allows one to define 'two-time averages', that is to say, mean values in the state of the system of a product of two operators taken at two different times  $t$  and  $t'$ . Important examples of two-time averages are the symmetric correlation functions and the linear response functions.*

Claude Cohen-Tannoudji (1989)

*Du siehst, mein Sohn, zum Raum wird hier die Zeit.*  
Richard Wagner, Parsifal

### 3 Quantum Holography

The main idea to avoid the difficulties involved in the standard deviation  $\Delta$  of self-adjoint operators is to start off with the quantum mechanical commutation relation

$$q \circ p - p \circ q = i\hbar$$

and to put it in a group theoretical context.

Let  $G$  denote the connected Lie group of unipotent matrices ([50])

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & \xi \\ 0 & 0 & 1 \end{pmatrix} \mid x, \xi, z \in \mathbf{R} \right\}$$

with the real Lie algebra  $\mathfrak{g}$  spanned by the set  $\{P, Q, Z\}$  of matrices

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which satisfy the commutator condition

$$[P, Q] = PQ - QP = Z.$$

In order to analyze the geometry of  $G$  one can naturally identify  $G$  with the set  $\mathbf{C} \times \mathbf{R}$  equipped with the non-commutative group law

$$(v, r) \cdot (w, s) = (v + w, r + s - \operatorname{Im}(v \cdot w)).$$

Then the boundary of the compact unit ball of  $\mathbf{C}^2$  can naturally be identified with the one-point compactification of  $G$ . The geometry that the group  $G$  inherits from the biholomorphic geometry of the unit ball has been called Heisenberg geometry ([14], [15]). For the purposes of quantum holography, the unitary dual of  $G$  and its multilayered coadjoint orbit geometry is of great importance.

Harmonic analysis on the Heisenberg nilpotent Lie group  $G$  governs the emitter-absorber transaction model of quantum dynamics ([8]) on the whole real line  $\mathbf{R}$  and therefore the flow and counterflow of optical photons in the photonic implementation of artificial neural networks. The argument concerning photonic flows and synapses goes as follows. For any optical frequency  $\nu \neq 0$ ,  $G$  admits an irreducible unitary linear representation  $U_\nu$ , unique up to unitary isomorphy, acting on  $\psi \in L^2(\mathbf{R}; dt)$  according to the prescription

$$U_\nu \left( \begin{pmatrix} 1 & x & z \\ 0 & 1 & \xi \\ 0 & 0 & 1 \end{pmatrix} \right) \psi(t) = e^{2\pi\nu(z+t\xi)} \psi(t+x) \quad (t \in \mathbf{R}).$$

It is instructive to interpret  $U_\nu$  as an induced representation of  $G$  from a normal subgroup. Based on the representation  $U_\nu$ , the Heisenberg inequality takes the form of a Robertson relation ([48], [44])

$$\Delta U_\nu(P) \cdot \Delta U_\nu(Q) \geq \frac{1}{2} |U_\nu(Z)|.$$

The mapping defined by the assignment

$$\psi(t') \cdot \nu dt' \otimes \varphi(t) \cdot \nu dt \mapsto H_\nu(\psi, \varphi; x, \xi) \cdot \nu dx \wedge d\xi$$

is called the holographic transform of the temporally coherent two-mixing wavepacket densities  $\psi(t') \cdot \nu dt'$  and  $\varphi(t) \cdot \nu dt$  of optical frequency  $\nu \neq 0$ . It describes by complex linear superposition of spatio-temporally coherent wavepackets of time difference  $t' - t = x$  and relative phase  $e^{2\pi\nu\xi t}$  the interference distribution in the symplectic hologram plane  $(\mathbf{R} \dot{\div} \mathbf{R}, \Omega_\nu)$ . Directional derivatives identify the layer  $(\mathbf{R} \dot{\div} \mathbf{R}, \Omega_\nu)$  with each tangent plane to the natural linear symplectic manifold modelled on  $(\mathbf{R} \dot{\div} \mathbf{R}, \Omega_\nu)$ . For each point  $(x, \xi) \in \mathbf{R} \dot{\div} \mathbf{R}$ , a canonical isomorphism

$$\mathbf{R} \dot{\div} \mathbf{R} \ni v \mapsto v_{(x,\xi)} \in T_{(x,\xi)}(\mathbf{R} \dot{\div} \mathbf{R})$$

is defined by

$$v_{(x,\xi)} f = \frac{d}{dt} f((x, \xi) + tv)_{t=0}$$

for smooth complex-valued functions  $f$  on  $\mathbf{R} \dot{\div} \mathbf{R}$ . In the  $\nu$  layer theory ([2]), the standard symplectic form

$$\Omega_\nu = \nu \cdot dx \wedge d\xi \quad (\nu \neq 0)$$

carries the optical frequency  $\nu$  as its scaling factor and determines the diffraction angle  $\beta$  of incidence of Bragg diffraction. It is determined by the polarized primitive

$$\theta_\nu = \nu \cdot (x, d\xi).$$

The density  $H_\nu(\psi, \varphi; x, \xi)$  with respect to the Liouville two-form  $\Omega_\nu = d\theta_\nu$  on the hologram plane  $(\mathbf{R} \dot{\pm} \mathbf{R}, \Omega_\nu)$  involved in the holographic transform represents a phased cross-correlation function. It takes the form of an entry function of the representation  $U_\nu$ :

$$H_\nu(\psi, \varphi; x, \xi) = \int_{\mathbf{R}} \psi(t+x) \bar{\varphi}(t) e^{2\pi i \nu \xi t} dt = \langle U_\nu \left( \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & \xi \\ 0 & 0 & 1 \end{pmatrix} \right) \psi \mid \varphi \rangle$$

Therefore the Liouville density  $H_\nu(\psi, \varphi; \dots)$  admits the interpretation of a wavelet transform. The integration along the whole real line  $\mathbf{R}$  "freezes up" the time  $t$  of the advanced and retarded wavepackets in order to convert it into the spatial variables  $(x, \xi)$  of the hologram plane  $(\mathbf{R} \dot{\pm} \mathbf{R}, \Omega_\nu)$ . It supposes temporal coherence of the mixing wavepacket densities. Because in the emitter-absorber transaction model of quantum dynamics ([8]) on the whole real line  $\mathbf{R}$  the expression

$$|H_\nu(\psi, \varphi; x, \xi)|^2 / \|\psi\|_2^2 \|\varphi\|_2^2$$

provides the overlap quantum probability in the hologram plane  $(\mathbf{R} \dot{\pm} \mathbf{R}, \Omega_\nu)$ , spatial encoding of the relative phase by means of the coordinate  $\xi$  avoids that phase information gets lost under wavepacket reduction or collapse of the state vector during the read-out process of optical holography. The overall phase factor  $e^{2\pi i \nu z}$  of

$U_\nu \left( \begin{pmatrix} 1 & x & z \\ 0 & 1 & \xi \\ 0 & 0 & 1 \end{pmatrix} \right) \psi \in L^2(\mathbf{R})$  has been normalized by setting  $z = 0$ . According to the

Stone-von Neumann uniqueness theorem,  $U_\nu$  ( $\nu \neq 0$ ) is determined up to unitary isomorphy by the central character

$$\chi_\nu : \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto e^{2\pi i \nu z}$$

of  $G$  ([50]). Thus the Liouville densities associated with different frequencies  $\nu \neq 0$ ,  $\nu' \neq 0$  satisfy the orthogonality condition

$$\langle H_\nu(\psi, \varphi; \dots) \mid H_{\nu'}(\psi', \varphi'; \dots) \rangle = 0 \quad (\nu \neq \nu')$$

for all amplitudes  $\psi, \varphi, \psi', \varphi'$  belonging to the complex Hilbert space  $L^2(\mathbf{R}; dt)$ . It follows that there is no cross talk between asynchronous excitation distributions located in different hologram planes  $(\mathbf{R} \dot{\pm} \mathbf{R}, \Omega_\nu)$  and  $(\mathbf{R} \dot{\pm} \mathbf{R}, \Omega_{\nu'})$ , respectively. For  $\nu = \nu'$ , however, the Frobenius-Schur-Godement identity

$$\langle H_\nu(\psi, \varphi; \dots) \mid H_\nu(\psi', \varphi'; \dots) \rangle = \langle \psi \odot \varphi \mid \psi' \odot \varphi' \rangle \quad (\nu \neq 0)$$

shows that the Liouville density  $H_\nu(\psi, \varphi; \dots)$  forms a learning kernel of Hebbian synaptic plasticity ([4]). By an extension, for all  $f \in L^2(\mathbf{R} \oplus \mathbf{R})$  the kernel

$$k_\nu(f) : (x, u) \mapsto (\tilde{\mathcal{F}}_\cdot f)(x + u, \nu u)$$

( $\tilde{\mathcal{F}}_\cdot$  = Fourier cotransform acting on the second variable) associated to the Hilbert-Schmidt operator  $U_\nu(f)$  on  $L^2(\mathbf{R}; dt)$  forms a natural generalization of Steinbuch's notion of learning matrix ([4]). Adopting the terminology of adaptive resonance theory (ART), the integrated form  $U_1(f)$  of the linear Schrödinger representation  $U_1$  of  $G$  acts as a bottom-up adaptive filter in response to input distributions ([5]).

*Something stronger is needed to uphold Bohr's conclusion.*

John F. Price (1985)

*The most surprising findings (and the most difficult to explain) are those concerning the large spatial separations across which stimulus-evoked synchrony may occur.*

Stephen Grossberg (1991)

## 4 Adaptive Wavelet Decomposition and Quantum Parallelism

It is well known that the Poisson summation procedure plays an important rôle in the global wavelet decomposition of wavepackets ([58]). In the context of  $L^2$  harmonic analysis on the Heisenberg group  $G$ , for any optical frequency  $\nu \neq 0$  the Poisson summation type operator

$$w_\nu : \psi \mapsto \left( \begin{array}{ccc} 1 & x & z \\ 0 & 1 & \xi \\ 0 & 0 & 1 \end{array} \right) \mapsto e^{2\pi i \nu z} \sum_{m \in \mathbf{Z}} e^{2\pi i \nu m \xi} \psi(m + x)$$

performs the adaptive wavelet decomposition  $w_\nu(\psi)$  of  $\psi \in L^2(\mathbf{R}; dt)$ . It maps for  $\nu = 1$  the linear Schrödinger representation  $U_1$  of  $G$  onto the right translations  $\delta_1$  of  $G$ . The linear lattice representation  $\delta_1$  of  $G$  generates the global wavelet decomposition

$$H_1(\psi, \varphi; x, \xi) = \left\langle \delta_1 \left( \begin{array}{ccc} 1 & x & 0 \\ 0 & 1 & \xi \\ 0 & 0 & 1 \end{array} \right) w_1(\psi) \mid w_1(\varphi) \right\rangle$$

The quantum geometric background behind this periodization construction is the fact that  $G$  can be projected onto the compact Heisenberg nilmanifold which forms a principal circle bundle over the two-dimensional flat torus  $\mathbf{T}^2$ . Because the compact oriented surface  $\mathbf{T}^2$  of  $\mathbf{R}^3$  has the topological genus  $g = 1$ , the symplectic hologram plane  $(\mathbf{R} \oplus \mathbf{R}, \Omega_\nu)$  can be identified with the cohomology group  $H^1(\mathbf{T}^2, \mathbf{R})$  of real dimension  $2g$ , see [2].

The global wavelet decomposition identity gives rise to a "distributed" form of the holographic transform

$$\psi(t') \cdot \nu dt' \otimes \varphi(t) \cdot \nu dt \mapsto H_\nu(\psi, \varphi; x, \xi) \cdot \Omega_\nu.$$

The quadratic lattice

$$\mathbf{Z} \oplus \mathbf{Z}$$

which is at the basis of the global wavelet decomposition has unit cells which are cooperatively coupled by adaptive filtering. It gives rise to  $k$ -fold periodic tilings ( $k \in \{2, 3, 4, 6\}$ ) of the hologram plane and allows to introduce the notion of holographic fractal ([51], [52]). These fractals reflect for each specific tiling with the translational and rotational symmetries of the kernel  $k_1(f)$  associated with the Hilbert-Schmidt operator  $U_1(f)$  on  $L^2(\mathbf{R})$ . Perturbations breaking the planar symmetry of the holographic fractals lead to the speckle phenomenon of coherent optics. This result is in accord with the fact that self-organizing systems react extremely sensitively to symmetry-breaking perturbations of self-similar structures.

As another application, the same global wavelet decomposition forms the mathematical basis to rigorously establish Niels Bohr's indeterminacy principle of spatio-temporal quantum dynamics ([7]). Indeed, unfolding of the flat torus  $\mathbf{T}^2$  establishes that the hologram appears if and only if the photons cannot be localized on their pathways for an arbitrary short time interval. Experimental evidence of this mathematical result are the single-photon holograms which can be generated by the commercially available Photon-counting Image Acquisition System (PIAS) of Hamamatsu Photonics.

As a final important consequence of this reasoning one concludes that it is impossible to construct an operator corresponding within the Heisenberg picture to the position of an individual optical photon. The non-existence of a position operator which describes the trajectory of an optical photons within the photonic flow and counterflow implies the quantum parallelism by global synchronization. It plays a rôle in bosonic string theory ([35]), too. However, more than Bohr's conclusion about quantum parallelism in split transmission channels actually is true. Electrophysiological experiments have shown that quantum parallelism by global synchronization of different cortical areas plays an important rôle as a self-organization principle of the visual cortex ([11], [16], [17], [29], [54], [19]). Indeed, simultaneous multi-electrode extracellular recordings established large spatial separations across which stimulus-evoked synchrony of firing activity may occur. The nonlocality of quantum dynamics described by the emitter-absorber transaction model ([8]) on the whole real line  $\mathbf{R}$  provides a natural explanation for these most surprising findings.

*The anatomical substrates of different parts of the cortex appear similar, whether these parts be devoted to various sensory, motor, or "associational" tasks.*

Richard Miles (1991)

*Wir haben entdeckt, daß die Antworten von Merkmalsdetektoren rhythmisch sind und mit einer mittleren Frequenz von etwa 40 Hertz oszillieren. Wir haben ferner*

beobachtet, daß räumlich verteilt liegende Gruppen von Merkmalsdetektoren ihre rhythmischen Aktivitäten synchronisieren können und dann in Phase schwingen. Solche Synchronisationen traten besonders häufig auf zwischen Zellgruppen, die ähnliche Merkmale codieren, also zum Beispiel zwischen Neuronen, die ähnliche Richtungs- und Orientierungspräferenzen aufweisen. Die oszillierenden Antworten räumlich verteilter Merkmalsdetektoren beginnen in Phase zu schwingen, wenn im Bereich ihrer rezeptiven Felder Konturen angeboten werden, die sich mit gleicher Geschwindigkeit in die gleiche Richtung bewegen.

Wolf Singer (1990)

*Multilayer optical neural network architectures based on storing weights as holographic gratings in photorefractive crystals have been proposed which are capable of implementing such neural network paradigms as backward propagation and simulated annealing.*

Yuri Owechko (1989)

## 5 Adaptive Neural Network Organization

The organization of neural networks is based on

- complex-valued signals

and

- synaptic interconnections.

The signals simultaneously transmit both amplitude and relative phase information whereas the neurons act as coherence detectors ([60]). As a consequence, the rôle of the field  $\mathbf{C}$  of complex numbers and its anti-automorphism  $\psi \mapsto \bar{\psi}$  of complex conjugation is in the theory of neural network organization as crucial as in the description of quantum dynamical phenomena. Notice, however, that the Schrödinger equation does not have advanced solutions  $\bar{\psi}$  ([8]).

The interconnections transmit the signal flows and counterflows and are characterized by their synaptic weights. The simplest module of a neural network is formed by a McCulloch-Pitts neuron which acts as an adaptive filter. A neural network generates by adaptive resonance certain excitation distributions which are determined by the structure of the network and by the input distribution. Due to synaptic plasticity, interconnections are modified in strength in response to signals in the excitation distribution. The competition rule for synaptic connections says that only the more successful synapses can grow, the less successful ones weaken and eventually disappear.

For the holographic implementation of neural networks, the identity of the preceding section establishes that the information distributed by the holographic encoding procedure on the natural linear symplectic manifold structure of the symplectic hologram plane  $(\mathbf{R} \oplus \mathbf{R}, \Omega_\nu)$  is a nonlocal one. The term nonlocality implies phased correlations across spatial distances. Similar experimental results in brain research concerning locally and globally synchronized connections in neural oscillator layers



are supporting the holographic hypothesis of memory structure in brain function and perception ([3], [39], [40], [41]): Cell assemblies adaptively coding for coherent features in visual scenes may not be distinguished by the fact that the constituting neurons are particularly active. Rather it appears that such cell assemblies are characterized by global synchronization of oscillatory responses over considerable tangential distances across spatially separate functional columns and even between spatially separate regions of the cortex. In particular, these studies present experimental evidence that temporal coherence and resonance form a basic principle of the dynamics of cortical coding and the cortical self-organization ([11], [16], [17], [29], [54], [57], [60], [61]).

From the theory of symplectic spinors ([30]) it is known that the symplectic hologram plane  $(\mathbf{R} \oplus \mathbf{R}, \Omega_\nu)$  forms the base of a fiber bundle, the fiber at each of its points  $(x_0, \xi_0)$  being a copy of the three-dimensional Heisenberg nilpotent group  $G$ . More precisely, a choice of a symplectic frame in a tangent plane at  $(x_0, \xi_0)$  to the natural symplectic manifold modelled on  $(\mathbf{R} \oplus \mathbf{R}, \Omega_\nu)$  determines a definite Lie group isomorphism of a central extension ([50]) of the tangent plane onto  $G$ . This definite isomorphism models the information redundancy of the holographic encoding procedure: Each part of the optical hologram contains all the information stored by the whole optical hologram. In view of Niels Bohr's indeterminacy principle, the excitation distribution spatio-temporally coherently encoded by the hologram generates a synchronized neural network modelled by the associated Hilbert space bundle. The associative holographic memory of this adaptive optical neural network can be spatially coherently decoded by the fundamental read-out formula

$$\int \int_{\mathbf{R} \oplus \mathbf{R}} H_\nu(\nu, \varphi; x, \xi) \epsilon^{-2\pi i \nu \xi t} \bar{\psi}(t-x) \cdot \Omega_\nu = \bar{\varphi}(t) \cdot \nu dt \quad (\|\psi\|_2 = 1).$$

The proof follows from the Frobenius-Schur-Godement identity of Section 3 supra.

The read-out formula which is written for a fiber at  $x_0 = 0$ ,  $\xi_0 = 0$  represents the locally synchronized response by adaptive resonance of the neural network and therefore forms the associative recall ([6]). It is this adaptive resonance read-out which performs the plasticity in the self-organization of the holographic neural network. In this context, self-organization means the spontaneous generation of complexity represented by the information bearing object beam which is exchanged by the no information bearing reference beam. In the visual system, the self-organization is performed by an exchange of the retinal input signals and internally generated gating signals.

*The transactional interpretation is so apparent in the Schrödinger-Dirac form of the quantum-mechanical formalism, which its combinations of normal and time-reversed waves, that one might fairly ask why this obvious interpretation of the formalism had not been made previously. No one can, of course, explain why something did not occur in the history of the development of quantum physics.*

John G. Cramer (1986)

## 6 Counterflow

The read-out formula shows that timing is crucial in neural network dynamics. The phase conjugation involved allows to implement the feedback learning mode of the counterpropagation neural network by degenerate four-wave mixing. In this way, the Owechko-Soffer resonator ([34], [55], [56]), for instance, efficiently uses the associative holographic memory function of volume holograms stored in photorefractive crystals ([12], [13], [37]) in the fields of pattern recognition and pattern completion. A disadvantage of this design is the surprisingly slow speed by which the plasticity in the self-organization is performed. Currently available photorefractive crystals operate as phase conjugate mirrors with time constants measured in hundreds of milliseconds. Therefore considerable advancement in materials and techniques must be made before such devices become suitable for practical applications. However, the Owechko-Soffer resonator proves that all-optical neurocomputers actually can be built and that their performance can be improved by the application of hybrid architectures.

In spatio-temporal quantum dynamics, Wigner's theorem ([53]) establishes the linear transformation of quantum state vectors in time. It says the transformation from one time scale  $t$  to a shifted time scale  $t' = t + x$  can be performed by an operator that is either linear or antilinear. Both operators acting on  $L^2(\mathbf{R}; dt)$  are uniquely determined up to one overall phase factor. The contragredient representation  $\bar{U}_\nu$  to  $U_\nu$  of  $G$  which is determined up to unitary isomorphy by the central character

$$\bar{U}_\nu : \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \epsilon^{-2\pi i \nu z}$$

describes the counterflow in neural network dynamics. Obviously the identity

$$\langle \bar{U}_\nu \left( \begin{pmatrix} 1 & x & z \\ 0 & 1 & \xi \\ 0 & 0 & 1 \end{pmatrix} \right) \bar{\psi} | \bar{\varphi} \rangle = \langle \varphi | U_\nu \left( \begin{pmatrix} 1 & x & z \\ 0 & 1 & \xi \\ 0 & 0 & 1 \end{pmatrix} \right) \psi \rangle$$

holds for all elements  $\begin{pmatrix} 1 & x & z \\ 0 & 1 & \xi \\ 0 & 0 & 1 \end{pmatrix} \in G$ . The Liouville density associated with  $\bar{U}_\nu$  in the hologram plane  $(\mathbf{R} \oplus \mathbf{R}, \Omega_{-\nu})$  gives rise by an application of the adaptive

resonance procedure ([4], [5]) to a similar read-out formula for the holographic information retrieval from the counterpropagating neural network. The learning kernel associated with the Hilbert-Schmidt operator  $\bar{U}_1(f)$  on  $L^2(\mathbf{R}; dt)$  is given by  $k_1(\bar{f})$ . In ART architectures ([5]), it acts as a top-down adaptive filter that leads to code self-stabilization.

*Spectral hole burning is not only a powerful method for the investigation of molecular properties and guest-host interactions but is also of considerable interest in the field of high density optical storage. The frequency and electric field multiplexing properties of spectral hole burning have been combined with the imaging capability of holography, leading to image-storing devices.*

Urs P. Wild (1989)

## 7 Molecular Computers

In the concept of Next Generation Computer architectures, the distinction between hardware and software and between memory and processor will become superfluous. Data processing will occur directly in the memory and neither transportation nor processing in separate devices will be needed. Besides photorefractive crystals other recording media for optical holograms are certain specific dye-doped polymer films like oxazine-4 in polyvinylbutyral (PVB) immersed in superfluid liquid helium between transparent electrodes in a bath cryostat at extremely low temperatures of  $1^\circ - 2^\circ$  K. Using a tunable single-mode dye laser source, these recording materials for spectrally narrow optical holograms allow to implement molecular computers by the spectral hole-burning method of high resolution laser spectroscopy ([32], [25], [31], [43], [45], [46], [47], [49], [62], [63], [64], [65], [20]). The interference distribution generated by coherent two-wave mixing is stored as a spectral hole burned by the laser in the absorption spectrum of the dye molecules. Similar to the holographic time-reversal by volume holograms stored in photorefractive crystals ([12], [13], [37]), adaptive resonance read-out of the burned holograms by a coherent beam which is antiparallel to the reference beam generates the phase conjugate object beam ([43]) of the counterpropagating neural network. The experiments performed with molecular computers ([62]) suggest that up to 20.000 holograms could be stored at a temperature of  $1.7^\circ$  K. As in the neurophysiology of simple cells of the visual cortex ([38]) cyclic phase changes between adjacent holograms can decrease the cross-talk.

The development of molecular computers based on the spectral hole-burning method is hampered by such major limitations as cryogenic operating conditions, lack of a recording threshold which limits the number of read-out cycles, and the special need for a wavelength-tunable laser. But because such processors can be considered as quantum computers, quantum holography provides a new approach to that highly active area of quantum computation which exploits the high temporal bandwidth of optics and the massive parallelism inherent in arrays of photonic devices. Because of continued progress, such as the development of new recording materials, molecular computers continue to be promising designs for the future.

*Gabor proposed a new method for analyzing arbitrary signals: a type of local-in-time frequency analysis. The primary message of Gabor's paper was that the optimal set of basis functions for analyzing signals consists of sinusoidal functions of time multiplied by a Gaussian function of time. The sinusoidal portion of this signal introduces a "waviness," whereas the Gaussian portion of the signal localizes it primarily to a region in time surrounding the time corresponding to the "mean" of the Gaussian. Gabor showed that, for a signal of finite duration, the use of such basis functions minimizes our joint uncertainty regarding the product of the effective time duration of the signal times its effective bandwidth. No other set of basis functions has this property. Such Gaussian-weighted sinusoids were dubbed logons by Gabor.*

Robert Hecht-Nielsen (1990)

*Compressed data save memory space, it is true, but entail more computational work.*

Tommaso Toffoli (1990)

## 8 The Gabor Filter

In a two-dimensional real symplectic vector space, any line is totally isotropic, hence a Lagrangian vector subspace. If  $V$  and  $W$  denote two transversal lines of the tangent space at the point  $(x_0, \xi_0)$  to the natural symplectic manifold modelled on  $(\mathbf{R} \hat{\oplus} \mathbf{R}, \Omega_\nu)$ , the one-dimensional complex vector space of tempered distributions on the plane  $\mathbf{R} \hat{\oplus} \mathbf{R}$  annihilated by the vacuum state  $U_1|V \hat{\oplus} W$  for the Lagrangian vector subspace  $V \hat{\oplus} W$  of  $\mathfrak{g}$  is spanned by the Gaussian envelope at  $(x_0, \xi_0)$ :

$$(x, \xi) \mapsto e^{-\pi(a^2(x-x_0)^2 + b^2(\xi-\xi_0)^2)}$$

where the real coefficients  $a, b$  determine a positive semi-definite quadratic form at the point  $(x_0, \xi_0)$ . Notice that it forms the generating function of the family of orthogonal wavelet transforms

$$(H_1(h_m, h_m; x - x_0, \xi - \xi_0))_{m \geq 0}$$

where  $h_m$  denotes the normalized Hermite function based on the Gaussian weight and the classical Hermite polynomial of degree  $m \geq 0$  ([50], [18]). The interpretation of the learning kernel  $H_1(h_m, h_n; x - x_0, \xi - \xi_0)$  as a matching polynomial of the complete bichromatic graph

$$K_{m,n}(x_0, \xi_0) \quad (m \geq n \geq 0)$$

admitting  $m + n$  vertices in the tangent plane at the point  $(x_0, \xi_0)$  shows that the wavelet transform performs the local Hebbian synaptic plasticity on clusters of postsynaptic neurons in a competitive neural network. Lifting from the polarized cross-section to the central extension ([50]) at  $(x_0, \xi_0) \in \mathbf{R} \hat{\oplus} \mathbf{R}$  yields the resolution generator or analyzing Gabor wavelet

$$(x, \xi) \mapsto e^{2\pi i(u(x-x_0) + v(\xi-\xi_0))} \cdot e^{-\pi(a^2(x-x_0)^2 + b^2(\xi-\xi_0)^2)}$$

where the real modulation parameters  $u, v$  determine a phase factor at the position  $(x_0, \xi_0)$ . This family of complex-valued functions on  $\mathbf{R} \mp \mathbf{R}$  for which identity holds in the Heisenberg inequality for  $L^2(\mathbf{R} \mp \mathbf{R})$  give rise to expansions which provide excellent fits to the empirical simple cell neural receptive field profiles ([26], [27], [28]). Its action by convolution is called the Gabor filter.

*One might speculate that somewhere deep in the brain, cells are to be found of single quantum sensitivity. If this proves to be the case, then quantum mechanics will be significantly involved in brain activity.*

Roger Penrose (1989)

*The assembly of detectors which respond to coherent features of a scene becomes distinguished from detectors responding to non-coherent features by the fact that the oscillatory responses of the cells in the assembly coding for coherent features are synchronous and in-phase ... The circuits responsible for synchronization have not been identified with certainty but the tangential intracortical connections are good candidates. How synchronization is achieved is yet unresolved.*

Wolf Singer (1990)

*The source of the 40-60 Hz oscillations that have been reported has yet to be identified.*

Stephen Grossberg (1991)

*Until a few years ago the developed glial tissue was mainly and generally thought of as a pure support and supply medium for the neuron network. But during the last decade several bioscientists have been led by accurate experiments to state that peculiar ion transport processes, affecting the bioelectrical activity of the neurons and of the cerebral cortex as a whole, certainly take place inside it. The sodium and potassium ion displacements cooperate in order to generate a self-sustaining linear electrochemical wave which propagates throughout the glial tissue.*

Renato Nobili (1985)

## 9 Quantum Dynamics and Brain Activity

Using the basis  $\{P, Q, Z\}$  of the Heisenberg Lie algebra  $\mathfrak{g}$  let

$$R = \frac{1}{2}(P + iQ), \quad R^+ = \frac{1}{2}(P - iQ).$$

Then the boson annihilation and creation operators, respectively, of quantum field theory are defined by

$$a = U_1(R), \quad a^+ = U_1(R^+)$$

and satisfy the boson commutation relation

$$[a^+, a] = \pi.$$

It follows for the harmonic oscillator wave functions  $(h_m)_{m \geq 0}$  of degree  $m$ :

$$a(h_m) = -(\pi m)^{1/2} h_{m-1} \quad (m \geq 1), \quad a^+(h_m) = (\pi(m+1))^{1/2} h_{m+1} \quad (m \geq 0).$$

In the Bargmann-Fock-Segal model the standard symplectic form  $\Omega_\nu = \nu dx \wedge d\xi$  ( $\nu \neq 0$ ) is determined by the isotropic primitive

$$\theta_\nu = \frac{\nu}{2} \cdot (x \cdot d\xi - \xi \cdot dx).$$

The Bargmann-Fock-Segal representation ([50]) of  $G$  acts by left and right translations on the Fock space of entire holomorphic functions on  $\mathbf{C}$  square-integrable with respect to the Gaussian measure on  $\mathbf{C}$ . By standard facts about square integrable unitary group representations, the linear Schrödinger representation  $U_1$  of  $G$  is isomorphic to the action of the right representation restricted to the Hilbert space spanned by the total family of entry functions

$$\mathbf{R} \oplus \mathbf{R} \ni (q, p) \mapsto (\pi^m m!)^{-1/2} \langle (a^+)^m (h_0) | U_1(\exp(qQ + pP))(h_0) \rangle.$$

From this group representational reasoning the single quantum resolution of the read-out formula follows. As a consequence the global synchronization of oscillatory responses over tangential distances across spatially separate cortical areas is mathematically equivalent to single quantum sensitivity of postsynaptic neurons. It is well known that the vertebrate's retina morphologically forms part of the brain. Electrophysiological experiments have shown that cells with single-photon sensitivity are present in the retina: the absorption of a single photon impinging on the dark-adapted retina can be sufficient to trigger a macroscopic nerve signal by a retinal rod. Thus the synchronization as the basic neurobiological signaling mechanism and fundamental cortical self-organization principle corresponds to the ultra high sensitivity of the detectors.

The Bargmann-Fock-Segal form of the Liouville density in the hologram plane ( $\mathbf{R} \oplus \mathbf{R}, \Omega_\nu$ ) again is illustrated by single-photon holograms. The quantum fluctuations which are implicit in this formula have been observed as vacuum state holograms or daydreaming phenomena in a holographic ring resonator memory ([1]).

*Ich habe vor mehr als zwölf Jahren auf der Grundlage bekannter Fakten der Neuroanatomie und Neurophysiologie eine präzise Theorie formuliert, die die ausgedehnte Speicherung von Gedächtnis im Gehirn nach dem Prinzip des Hologramms erklären kann. Seither haben viele eigene und fremde Experimente diese These gestützt.*

Karl. H. Pribram (1986)

*There is considerable controversy as to whether holography can serve as a good model for certain aspects of brain function. The roots of this controversy are often to be found in misunderstanding of what holography is and what the proponents of a holographic hypothesis are claiming.*

Karl H. Pribram (1987)

*Ich bin der Ansicht, daß das Hologramm ein Beispiel für die eingefaltete oder implizite Ordnung ist.*

David Bohm (1986)

## 10 Neural Network Simulations

The quantum holographic model does not intend to represent in detail actual networks of neurons within the brain nor the neuroanatomical or neurophysiological details, but rather to shed light on the basic cortical self-organizational principles by a functional photonic implementation which works massively parallel. The non-locality of quantum holography in particular establishes the global synchronization of oscillatory responses over tangential distances across spatially separate columns and even between cortical areas. In a specific computer simulation of a sample neural network, the details of the simulation depend on the connectivity pattern of the model network as it exists in the computer hardware. But in a conventional digital computer the information is not available to make sure that the model network matches the details of the sample neural network. It is impossible to predict the firing rate of a particular cluster of neurons at a given time. In view of this fundamental restriction, hard wired computer simulations of the plasticity in the cortical self-organization actually are of a limited value ([36], [59]). In a quantum holographic computer implementation, however, the gate coefficients  $c_{m,n}$  of a sample distribution  $f \in L^2(\mathbf{R} \div \mathbf{R})$  are automatically and massively parallelly computed according to the prescription

$$c_{m,n} = \langle f | H_1(h_m, h_n; \dots) \rangle \quad (m \geq 0, n \geq 0).$$

Then the orthogonal projection on the Gabor wavelet

$$c_{m,n} H_1(h_m, h_n; \dots)$$

defines the synaptic weights of the complete bichromatic graph  $K_{m,n}(0,0)$  ([51]). By wavepacket reduction or collapse of the state vector ([8]), the numbers

$$|c_{m,n}|^2 / \|f\|_2^2$$

represent the transition probabilities of the competitive neural network. Notice that these probabilities are determined by the input distribution  $f \neq 0$  in  $L^2(\mathbf{R} \div \mathbf{R})$  and photonically generate by the threshold conditions

$$|c_{m,n}|^2 \geq \nu \|f\|_2^2 \quad (\nu > 0)$$

the synaptic interconnections of the local model neural network  $K_{m,n}(0,0)$  inside the quantum holographic computer. The self-organization is performed by the phase information via adaptive resonance ([5]) and linear superposition.

*Ein Computer ist eine Organisationsform elementarer funktionaler Komponenten, an denen für das Verhalten des ganzen Systems nur ihre Funktion relevant ist.*  
Herbert A. Simon (1990)

*The use-dependent modifications of synaptic transmission which presumably mediate learning in the adult brain depend on similar mechanisms as experience-dependent self-organization of neuronal connectivity during development.*

Wolf Singer (1990)

## 11 Conclusions

Information technology plays an increasingly important rôle in modern life. New and faster computers are presently used in many areas and the need of even greater computing power is accompanied by the need to store even larger quantities of information. Looking beyond the intrinsic limits of today's information systems - integration density, interconnection speed, storage density - alternative concepts ([18]) have to be developed. Keywords such as self-organization, massive parallel information processors, photonic computers, adaptive neural networks, neurocomputer architectures, molecular electronics, and even quantum computers appear in the discussions of future high-performance information processing technologies. Neural network models form a dramatic departure from the conventional digital computing paradigm. The massive parallelism of photonic and molecular computation cannot be based on any known version of the uncertainty principle. The aim of this paper, however, is to establish that quantum holography provides a mathematically rigorous approach to describe by means of a Liouville density the flow and counterflow of optical photons within split fan-in/fan-out optical channels. In particular this quantum field approach proves Niels Bohr's fundamental indeterminacy principle and the non-existence of a position operator describing the trajectory of an individual optical photon. Finally, an application of the

- linear Schrödinger representation
- linear lattice representation
- Bargmann-Fock-Segal representation

of the Heisenberg group  $G$  to quantum holography provides mathematical insights into the recently discovered principles of

- cortical self-organization by adaptive resonance
- global synchronization and quantum parallelism
- single-photon sensitivity

which are shown to be mathematically equivalent.

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NEURAL NETWORKS :  
FROM SPIN GLASSES  
TO CONSCIOUSNESS

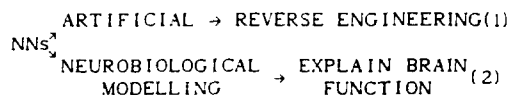
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1. Introduction

Neural networks is presently in an explosive phase. Over the last 5 years or so there has been an exponential increase in activity in the subject, as evidenced by the numbers of people entering the field and the numbers of papers published (and by the numbers of new journals!) [1]. We must remember that this is a second revolution in the field; there was an earlier one, starting in the late 50's and the 60's and associated with the attack on the subject made by Minsky and Papert in 1969 [2], which was effective due not only to the intrinsic shortcomings of the perceptron and its variants but also due to at least two other key features: (i) the hype that had accompanied the subject, with claims of having plumbed the 'secrets of the brain', and (ii) the remoteness of the subject from related areas, in particular neurobiological modelling and information theory. The successor to neural networks in the 70's and early 80's, artificial intelligence (or AI), has apparently also fallen prey to the first of these diseases. The second time round there seems to have been more care taken by researchers in neural networks to avoid the earlier exaggerated claims (although see [1]). At the same time an interdisciplinary character has emerged for the subject, with not only neurobiology but psychology, philosophy, linguistics, mathematics physics and numerous other subjects involved. Moreover a vast range of successful applications of neural networks have emerged, so that the engineering community are also heavily involved (as evinced by the new Neural Networks chapter of the IEEE).

We can recognise that there are effectively two separate (although related) motivations for neural network activity, which can be

displayed by the following diagram:



The first of these tries to use simplified models of neurons and architectures ('artificial') to construct systems able to solve information processing problems, possibly even by hardware devices ('reverse engineering'). The second approach is to be regarded as a branch of biology, trying to model brain function by successively more sophisticated models. There is a strong relationship between the two motivations ; more effective models of brain function can be expected to lead to better 'reverse engineering' solutions of information processing tasks, whilst conversely a more powerful understanding of the general nature of information handling should lead to more effective models of brain activity.

The ultimate goal of constructing information processing systems is to be able to create autonomous devices with intelligence. Consciousness and attention are clearly important concomitants of our own ability to solve tasks in an intelligent manner. It is therefore important to understand consciousness ; that is one of the areas of increasing growth in neural network research. It is appropriate to consider this topic at this Wigner Symposium. Eugene Wigner pointed two theses [3] :

" ... if certain physico-chemical conditions are satisfied, a consciousness, ... , arises" (I)

"Does, conversely, the consciousness influence the physico-chemical conditions?" (II)

I wish to add a further thesis of my own :

"Consciousness is explicable without quantum mechanics directly but quanta, oscillations and phases may be important" (III)

The purpose of this contribution is to explore these theses through recent theoretical and experimental work on neural networks, the theoretical side involving statistical mechanics, temporal sequence storage and coupled oscillators, with neurobiology and coupled channel magneto-encephalography (MEG) included in the experimental one. The contribution begins with an introduction to neural nets, reviews the statistical mechanical approach, and then turns to the problem of 'thinking' neural nets.

## 2. Introduction to Neural Networks

A model neuron is considered as an multi input-single output device

$$u_1, u_2, \dots, u_n \rightarrow u \quad (1)$$

where the inputs  $u_1, \dots, u_n$  are evaluated at some time  $t$ , and the output  $u$  of the neuron will be assumed to arise at some time delay  $\tau$  afterwards ( $\tau$  arising from various axonal and synaptic delay times in the neuron). Of the many functions which may be described by (1) it is usual to choose the simplest, involving a linear weighted sum of the inputs  $u_1 \dots u_n$ , the weights  $a_i$  being termed the connection weights. Then (1) becomes

$$u(t + \tau) = f(\sum_j a_j u_j(t) - t_o) \quad (2)$$

where  $t_o$  is a threshold value. The inputs  $u_i$  and output  $u$  may all be binary, with  $u=0(1)$  corresponding to no activity (activity) on the corresponding line ;  $f$  is then the unit step function. On the other hand  $u_i$  and  $u$  may be taken to belong to  $\underline{R}$ , denoting the mean firing activity on the corresponding lines. In that case  $f$  is often taken to be the sigmoidal function  $f(x) = [1 + e^{-\beta x}]^{-1}$ , for suitable  $\beta$ .

The above equation (2) becomes that for a network if an additional suffix  $i$  is added to both sides, so becoming

$$u_i(t + \tau) = f(\sum_j a_{ij} u_j(t) - t_i + \text{input}_i) \quad (3)$$

In (3) the  $\text{input}_i$  denotes input from outside the net apposing on neuron  $i$ , whereas the summation term in  $f$  is the total weighted activity from the other neurons (including the  $i^{\text{th}}$  if  $a_{ii} \neq 0$ ). As before  $t_i$  is a threshold. The net dynamics is determined by (3), together with initial values of activity and external inputs given at all the times considered.

It was already known in the 40's that neural nets could perform logical functions on the inputs in the binary case. Regarding the set  $(a_{i1}, \dots, a_{iN})$  as a vector  $\underline{a}_i$  ( $N$  being the number of neurons in the net), the activity in  $f$  in (3) is of form  $\underline{a}_i \cdot \underline{u}$ , where  $\underline{u} = (u_1, \dots, u_N)$ . Response is thus determined by geometry, where this latter is of the hyperplanes with normals  $\underline{a}_i$ . Thus for one binary neuron two classes arise, one with  $\underline{a}_i \cdot \underline{u} < s$ , the other with  $\underline{a}_i \cdot \underline{u} > s$ .

The perceptron learning algorithm changes the weights  $a_i$  to achieve a correct classification if it had been incorrect : increase  $a_i$  if  $\underline{a}_i \cdot \underline{u} < t$  but one should have  $\underline{a}_i \cdot \underline{u} > t$ , for a given input  $\underline{u}$ , or decrease it



if vice versa (and otherwise leave  $\underline{a}$  unchanged). This learning algorithm is termed supervised, since the desired classification of the various feature vectors  $\underline{u}$  is supposed known. It has been extended to the case when there are 'hidden' units, whose desired output is not known directly, by the back-error propagation method. This learning method, whilst slow in training, has proved effective in a broad range of applications [1]. Other learning paradigms have also been proposed for training NNs, these being of either supervised, unsupervised (with no teacher) or reinforcement (with only a single evaluation of the NN response) types. All of these are being developed, compared to each other and non-NN techniques in solving specific problems, and applied to an enormous range of real-world problems in vision, time-series, control, finance, etc [1].

### 3. Hopfield Nets and Statistical Mechanics

One of the important contributions to NNs in the 80's, and a prime factor in the emergence of the second revolution in the subject, was Hopfield's method of setting up a relaxation net with given centres of attraction; the Hopfield net could act as a content-addressible memory, functioning very effectively in pattern completion.

Under the conditions

$$a_{ii} = 0, \quad a_{ij} = a_{ji} \quad (4)$$

the energy function

$$E = - \sum_{i < j} a_{ij} u_i u_j + \sum_j u_j t_j \quad (5)$$

may be shown not to increase if single neurons are updated as binary decision neurons (3) :

$$\sum a_{ij} u_j > t_i \text{ at time } t \Leftrightarrow u_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ at time } (t+1). \quad (6)$$

Patterns  $\underline{u}_i^{(s)}$  ( $s=1, \dots, P$ ) may be stored by using an outer product form

$$a_{ij} = \sum_{s=1}^P (2u_i^{(s)} - 1)(2u_j^{(s)} - 1) \quad (7)$$

where on average the pattern  $\underline{u}^{(s)}$  reproduces itself with few errors if  $P/N = \alpha < \alpha_c \sim 0.14$  [4] otherwise cross-talk noise between patterns becomes large.

The system can be analysed by statistical mechanics using the energy function  $E$  of (5), and the spin variables  $s_i = (2u_i - 1) = \pm 1$ .

Thus the NN becomes an interacting spin system with local field

$$h_i = \sum_j a_{ij} s_j \text{ at neuron } i.$$

At non-zero temperature  $\beta^{-1}$  the partition function is

$$Z = \sum_{\underline{s}} \exp[-\beta H(\underline{s})] \quad (8)$$

where

$$H(\underline{s}) = - \sum_{i,j} a_{ij} s_i s_j + \underline{t} \cdot \underline{s} \quad (9)$$

Statistical concepts, such as disorder, frustration, competition, etc can now be used. Also the powerful mean field approximation can be employed. In the case of (7), and defining the overlaps, for  $\ell=1, \dots, P$ ,

$$m^{(\ell)} = N^{-1} \sum_i s_i^{(\ell)} \langle s_i \rangle \quad (10)$$

then one obtains the mean field equations

$$m^{(\ell)} = \langle\langle s_i^{(\ell)} \tanh(\beta \underline{m} \cdot \underline{s}_i) \rangle\rangle \quad (11)$$

where the double brackets denotes averaging over uncorrelated stored patterns. For  $T = \beta^{-1} > 1$  there are no solutions, but for  $T < 1$  there are a range of solutions. In the range  $0.4598 < T < 1$  the only stable states are those corresponding to non-zero overlap with a single pattern, but for lower  $T$  there are other states, which are termed spurious, and degrade the recall. The general phase diagram in the  $(T, \alpha)$  plane has been calculated, and the interested reader is referred to [5] for details.

This statistical mechanical analysis has been extended to asymmetric weights, to dilute nets, to graded activity, to synaptic weights with decay, and to a variety of other modifications. One important modification is to introduce biological realism into the system other than by the temperature  $\beta^{-1}$  of (8). Living neurons have synaptic noise in all nerve impulse transmission due to the probabilistic release of chemical transmitter vesicles at the synapse both on stimulation by arrival of a nerve impulse and spontaneously. The amount of chemical transmitter in the synaptic cleft is thus a random variable with some distribution functions  $p(q)$ , which may be binomial or multi-binomial; the most popular is the single vesicle case:

$$p(q) = p\delta(q - q_0) + (1 - p)\delta(q) \quad (12)$$

where  $p$  is the probability of release of one vesicle of amount  $q_0$ . The

connection weight  $a$  will be the product of  $q$  and the efficiency  $e$  of post-synaptic up take of transmitter.

This framework leads to the random iterative net (PIN) approach [6], based on the earlier work [7] of modelling noisy synaptic transmission. RINs arise from (3) (with  $f$  the unit step function  $Y$ ) by replacing  $a_{ij}$  by  $q_{ij} e_{ij}$ , so leading to the probability  $p(i|\underline{u})$  of the  $i^{\text{th}}$  neuron firing one time step after the net activity is  $\underline{u}$  as

$$p(i|\underline{u}) = \int \prod q_{ij} dt_i p(q_{ij}) p(t_i) Y(\sum q_{ij} e_{ij} u_j - t_i) \quad (13)$$

Note that threshold noise has been included in (13), so incorporating the spin-glass approach above by suitable choices of the  $p$ 's [6]. A Markov chain then ensues on the states  $\underline{u}$  of the net, and functional techniques to analyse recall can then be developed and applied to obtain recall in various dilution limits. The RIN framework can thus be recognised as biologically realistic, and incorporating the statistical mechanical approach.

At the same time a hardware realisation of synaptically noisy neurons has been achieved in terms of probabilistic RAMs (pRAMs), [8] in which the content at any address in a RAM is a probability. This is used to generate spike trains. Hardware version of the pRAM have been built [9] and hardware-realizable learning algorithms developed. By the end of '91 a chip with about 1300 pRAM neurons should be available.

#### 4. Towards Thinking Neural Nets

To achieve more powerful information processing, ever more powerful forms of neurons (leaky integrators, compartmental, active membranes, noisy synapses, etc) and architectures (hidden layers, recurrent connections, time delays, etc) are being extracted from living neural systems and employed in artificial systems. But whilst these lead to better classification and generalisation systems they have not led to any clear advance towards modelling higher brain function, especially that associated with reasoning. There are the beginnings of ideas associated with temporal sequence storage (TSS) [10], where concatenation of suitable TSSs allows a neural system to determine if a suitable set of activities lead to a desired goal. This is related to finding sub-goals whose achievement had already been learnt by the system. But this still seems to leave untouched the question of attention and consciousness. There have been very recent results which indicate that a new paradigm may be appropriate here, and

that new data over the next few years should lead to much greater clarity about how to solve that problem.

The new data have arisen from the deployment of multi-channel MEG (Squid) detectors of the low-level magnetic fields concomitant with neural activity. The main advance that MEG results make over EEGs is that the former allow detection of activity at deep as well as superficial levels; magnetic fields do not lead to polarisation effects in intervening cortex. This has allowed inverse problem programs to determine simultaneous cortical and brain stem neural activity whilst a subject is awake. In particular it has been found [11] that there appears to be a sweep of neural activity at about 40Hz from front to back of the head, with brain stem activity leading that in cortex by about 3msec. Such a correlated sweep may be functioning as a phase difference detector for controlling attention in the following manner [12]. Activity from the hippocampus or nearby cortex excited by slightly earlier experience, but also determined by considerably earlier experience (episodic memory), is sent to frontal and/or parietal lobes. It functions therefore as an 'expectancy' wave. It is compared with new activity from primary sensory cortex. If the expectancy is met (the new activity is highly correlated with that expected) attention is not directed to the new stimulus for further processing; if the expectancy is not met then attention is switched and the new stimulus examined further. There is good evidence that such attention disengagement occurs in parietal and frontal lobes (with brain stem causing movement and re-engagement of attention). The model can be made more precise by using coupled oscillatory neurons, and taking account of the detailed architecture of the brain stem [12]. It also appears that the model fits naturally into the general framework for 'thinking' machines given in [13], in which meaning and conscious content are determined by the overlap thesis: the meaning of an experience is given by the relations between that experience and past (episodic) memories related to the experience. It was necessary to have a discriminator unit to be able to determine the degree of this overlap; that may well be provided by the phase comparisons in frontal/parietal lobe.

This model is explicated in more detail in [12], to which the reader is recommended.

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## II. Foundations of Quantum Mechanics

**The Problem of Objectification in Quantum Mechanics  
and the Ignorance Interpretation (\*)**

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**1. Introduction**

Let  $S$  be a quantum system in a pure state  $W = P[\varphi]$  and  $A$  a discrete observable with eigenvalues  $a_i$  such that  $\varphi$  is not an eigenstate of  $A$ . The pair  $(W, A)$  then defines a probability distribution  $p(W, a_i)$ , the experimental meaning of which is given by the statistical interpretation of quantum mechanics: The number  $p(W, a_i)$  is the probability to obtain the result  $a_i$  if the observable  $A$  is measured on the system  $S$  in the preparation state  $W$ . This "minimal interpretation" does not necessarily mean that after the measuring process by means of which the result  $a_i$  was obtained, the object system  $S$  actually possesses the value  $a_i$ . This is the case only for ideal premeasurements, whereas in the general case the object is disturbed in some way. Hence in the most general situation one knows merely that the measuring apparatus possesses the value  $Z_i$  of the pointer which shows that  $a_i$  was measured.<sup>(1)</sup>

In addition to the minimal interpretation one could tentatively assume that a certain value  $a_i$  of  $A$  pertains *objectively* to the system  $S$  before the measurement, i. e. in state  $W$ , but that this value is *subjectively* unknown to the observer who knows only the probability  $p(W, a_i)$  of the value  $a_i$ . The hypothetical attribution of a certain  $A$ -value  $a_i$  to the system  $S$  will be called *weak objectification*. It corresponds to the "ignorance interpretation" of the probability distribution  $p(W, a_i)$ .

It is obvious that one could also make the stronger assumption that the system  $S$  does not only possess a value  $a_j$  but that  $S$  is actually in an eigenstate  $\varphi^{a_j}$  of  $A$ . The state  $W$  would then merely describe the observer's incomplete knowledge about the actual state  $\varphi^{a_j}$  of the system. The hypothetical attribution of a certain eigenstate of  $A$  to the system  $S$  as its actual state will be called *strong objectification*. It corresponds to a strengthening of the "ignorance interpretation" of the probability distribution.

## 2. Nonobjectification theorems for pure states

The question whether the strong or weak objectification hypothesis can be applied to a system in a pure state can be answered by the following two theorems which make use of the same way of reasoning: From the assumption of the strong ( or weak ) objectification hypothesis probabilistic relations are derived which are known to be violated experimentally in agreement with quantum mechanics. (2)

*Theorem I* : Let  $S$  be a proper quantum system which is prepared in a pure state  $W$ , and let  $A$  be a discrete observable with eigenvalues  $a_j$  such that  $W$  is not an eigenstate of  $A$ . Then it is not possible to attribute an eigenstate  $\varphi^{a_j}$  of  $A$  which belongs to the value  $a_j$  to the system such that  $S$  is actually in one of the states  $\varphi^{a_j}$  but the observer knows only its probability  $p(W, a_j)$ .

It is obvious that theorem I does not exclude automatically the hypothesis of weak objectification. In fact one could imagine that for the system in the state  $W$ , the values  $a_j$  of  $A$  can be attributed to the system in a hypothetical way. This possibility is excluded by

*Theorem II* : Let  $S$  be a proper quantum system in a pure state  $W$  and let  $A$  be a discrete observable with eigenvalues  $a_j$  such that  $W$  is not an



eigenstate of  $A$ . Then it is not possible to assign a value  $a_i$  to the system such that  $a_i$  pertains objectively to the system but that this value is subjectively unknown to the observer.

### 3. Nonobjectification theorems for mixed states

The nonobjectification *theorems I* and *II* are of particular importance for the interpretation of quantum mechanics. They show that for the system  $S$  with state  $W$  the value  $a_i$  of  $A$  is not only subjectively unknown to the observer but objectively undecided. The same argument applies even more to the eigenstates  $\varphi^{a_i}$  of  $A$ . On the basis of this interpretation the following problem arises. If  $A$  is not objective in  $W$  and if  $A$  can be measured such that after the measuring process the system is in an eigenstate of  $A$  and hence possesses a value  $a_i$ , then the measuring process must provide the objectification of  $A$ . Hence one could expect that the mixed state  $W_S'$  of  $S$  after the premeasurement<sup>(1)</sup> admits the strong or weak ignorance interpretation i.e.  $S$  would be in an eigenstate  $\varphi^{a_i}$  of  $A$  or it would at least possess one of the values  $a_i$ . In case of non-ideal measurements one would at least expect that the mixed state  $W_M'$  of the measuring apparatus admits the strong or the weak ignorance interpretation, i.e. that  $M$  is in an eigenstate  $\Phi^{Z_i}$  of  $Z$  or that a value  $Z_i$  of  $Z$  pertains to  $M$ .

In order to discuss this question we consider the compound system  $S + M$  in the pure state  $\Psi'(S+M) = U(\varphi \times \Phi) = \sum (\varphi^{a_i}, \varphi) \varphi^{a_i} \times \Phi^{Z_i}$  after the premeasurement.  $U$  is the unitary operator which provides the premeasurement of the observable  $A$ . For simplicity we have assumed here an *ideal* unitary premeasurement. If the compound system is in the pure state  $W' = P[\Psi']$  the subsystems  $S$  and  $M$  will be in the states  $W_S' = \sum p(\varphi, a_i) P[\varphi^{a_i}]$  and  $W_M' = \sum p(\varphi, a_i) P[\Phi^{Z_i}]$  with  $p(\varphi, a_i) = |(\varphi, \varphi^{a_i})|^2$ , which are in general mixed states.<sup>(1)</sup>

If the system  $S$  is separated from the apparatus  $M$  without thereby changing the state  $W'$  of the compound system, then the system  $S$  will be in the mixed state  $W_S'$  and  $M$  in the mixed state  $W_M'$ . The question whether mixed states which are prepared in this way by separation, admit an "ignorance interpretation" - in the strong or weak sense - is answered by the following theorems.<sup>(2)</sup>

*Theorem III* : Let  $S$  be a subsystem of the compound system  $S + M$  which is prepared in a pure state  $W' = P[\Psi']$  and let  $A$  be a discrete observable of  $S$  with values  $a_i$  and eigenstates  $\varphi^{a_i}$  such that  $W'$  is not an eigenstate of the extension  $\hat{A} = A \times \mathbf{1}_M$  of  $A$  with respect to the compound system. Then  $S$  is prepared in a well defined mixed state  $W_S'$  which does not allow the assumption that  $S$  is actually in a state  $\varphi^{a_i}$  which is merely unknown to the observer. Hence the system  $S$  in state  $W_S'$  does not generally admit a "strong ignorance interpretation".

*Theorem III* equally applies to the apparatus  $M$  in the mixed state  $W_M'$ . In this case it means that it is not possible to assume that after the premeasurement the apparatus  $M$  is *actually* in an eigenstate  $\Phi^{z_i}$  of the pointer observable  $Z$  which is merely unknown to the observer. *Theorem III* does not necessarily exclude the weak objectification for mixed states prepared by separation. However this hypothesis is excluded by<sup>(2)</sup>

*Theorem IV* : Let  $S$  be a system in the mixed state  $W_S'$  which is prepared by separation and assume that the compound system  $S + M$  is in a pure state  $W = P[\Psi']$ . Then it is not possible to attribute a value  $a_i$  of the observable  $A$  to  $S$  such that  $a_i$  pertains to  $S$  with probability  $p(W_S', a_i)$  - except when  $W$  is an eigenstate of the extension  $\hat{A} = A \times \mathbf{1}_M$  of the observable  $A$ . Hence a system  $S$  prepared in a mixed state does not generally admit a "weak ignorance interpretation"

#### 4. Results

*Theorems I and II* show that a proper quantum system  $S$  in a pure state  $W$  does generally not admit a strong or weak objectification of the observable  $A$ , except when  $W$  is an eigenstate of  $A$ . These statements are essential for any interpretation of quantum mechanics. However, even *after the premeasurement* of the observable  $A$  the objectification of  $A$  can not be achieved. The systems  $S$  and  $M$  will then be in the mixed states  $W_S'$  and  $W_M'$ , and *theorems III and IV* show that these mixed states do not admit an ignorance interpretation, neither in its strong nor in its weak version. Hence the system  $S$  and the measuring apparatus  $M$  will not be in eigenstates of  $A$  and  $Z$ , nor will these systems objectively possess values  $a_i$  and  $Z_i$  of  $A$  and of the pointer observable  $Z$ , respectively. On the other hand the "minimal interpretation" of quantum mechanics assumes that the pointer observable is weakly objectified and thus presupposes that the mixed state  $W_M'$  of the measuring apparatus after the premeasurement admits at least the weak ignorance interpretation. Consequently a unitary premeasurement cannot justify the "minimal interpretation" of quantum mechanics usually considered as one of the starting points of the theory.<sup>(1)</sup>

#### Notes and References

- (\*) This paper is based essentially on a more extensive investigation by P. Busch and the present author ( ref. (2) ).
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## Measurements in a Quantum Universe

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A measurement has to be considered as a physical process, that is, as an interaction between the system S to be measured and an apparatus A. The 'pointer position' must thereby change in dependence on the value of the measured quantity. In the quantum formalism this has been described by von Neumann's interaction

$$\psi_i^S \Psi_0^A \rightarrow \psi_i^S \Psi_i^A,$$

where, more generally, the state of the system may also change due to a back reaction.

However, the superposition  $\sum_i c_i \psi_i^S$  of the system S leads then to an entangled state for the whole system,

$$\left( \sum_i c_i \psi_i^S \right) \Psi_0^A \rightarrow \sum_i c_i \psi_i^S \Psi_i^A,$$

instead of the observed states with specific pointer positions. For measurements proper (that is, when the pointer position is read), this measurement problem is usually 'solved' by applying the collapse of the wave function which is defined to pick out one of the factorizing components. This leads to the open questions of what this second dynamics means, and what distinguishes a measurement from a 'normal' interaction.

Even when the apparatus is not read, there remains an effect of measurements on the system S. As is best known from the two-slit experiment with measurement of the particle's passage, one observes a loss of interference between the contributions from different values of the measured quantity. In order to describe this effect, no collapse is required. The resulting entanglement corresponds to the *local* loss of interference

described by the local density matrix  $\rho^S = \sum_i |c_i|^2 \psi_i^S \psi_i^{S*}$  which results from tracing out system A.

Such a local loss of coherence results also from 'measurement-like processes'. They are defined by the same type of von Neumann interaction, but under the more general assumption that the 'pointer position' is uncontrollable (*not macroscopic*), so that no reading would in general be possible.

The point to be stressed here is that this type of interaction is extremely important for all 'macroscopic' systems S in leading to their unavoidable 'continuous measurement' by their normal environment. This interaction leads to the local disappearance of *certain* phase relations ('decoherence') which is characteristic for classical phenomena. When I first tried to point this out in a preprint of 1968, it was only Professor Eugene Wigner who responded positively, encouraged me to further elaborate on the idea, and later helped me to get it published.<sup>1</sup> I am thus very grateful for this opportunity to express my thanks to him!

Some examples may illustrate the widespread occurrence of decoherence in quantum mechanics:

a) **Chiral states** are found for many large or medium sized molecules like sugar or alanine instead of the energy (parity) eigenstates observed in microscopic objects. 'Measurement' of chirality by means of the scattering of unavoidably present particles would decohere superpositions of different chirality. In contrast, an ammonia molecule is still 'microscopic' in not being continuously measured in this sense. A side effect for such discrete quantum states is Zeno's quantum paradox, that is, the suppression of transitions between different chirality states. In contrast to popular belief (also expressed at this conference), continuous measurement does *not* lead to the freezing of motion in general, that is, not for systems with sufficiently dense spectra.<sup>2</sup>

b) **Superselection rules** can be explained, for example, by unavoidable irreversible 'measurements' of the charge of a particle by its environment<sup>3</sup> (to be distinguished from its mere 'dressing' by a cloud of 'virtual' particles).

c) **Positions and orientations of macroscopic objects** like dust grains are continuously measured by scattered photons and other particles even in intergalactic space. Classical (here particle) properties *emerge* thereby from a Schrödinger wave function in configuration space.<sup>4</sup>

d) **Quantum jumps** occur apparently instead of a unitary evolution if the decay status is permanently 'measured'. They may be described by master equations which lead to an exact exponential decay law and to the exclusion of any 'revival' of the decaying state in this case.<sup>2</sup> This difference between closed systems and those under continuous measurement can be nicely studied in laser physics.

e) **Incoherent light** ('different photons') may be understood by the decoherence of the corresponding parts of the quantum state of the electromagnetic field caused by its source (as by the decay of different atoms leading to orthogonal final states of the source).

f) **Classical spacetime geometry** emerges from quantum gravity, since curvature is unavoidably 'measured' by the matter fields.<sup>5,6</sup>

The quantitative estimate of these effects can in all cases be based on the same assumptions: ordinary quantum mechanics of interacting systems together with a *realistic* model for their normal environment. (It is the second part where models often fail.)

These considerations also allow one to describe a realistic measurement situation. Not only is the system measured by the apparatus - the macroscopic pointer positions (like Schrödinger's cats) are further decohered by their environment.

All measurements of this kind (without reading) are based on an arrow of time:<sup>7</sup> they transform product states into entangled states. As usual this requires very special initial conditions (in this case initially non-entangled states). The origin of this quantum arrow of time is a major problem of quantum cosmology. It is intimately connected with the thermodynamical arrow, although far more delicate. For example, even two thermal equilibrium systems with the same temperature  $T$  still offer considerable capacity for correlations to arise according to

$$\exp\left\{-\frac{H_1}{kT}\right\} \otimes \exp\left\{-\frac{H_2}{kT}\right\} \rightarrow \exp\left\{-\frac{H}{kT}\right\},$$

because of the strong entanglement of the eigenstates of  $H \neq H_1 + H_2$  as a result of their dense spectra.

The above arguments are still unrealistic if the environment itself possesses another environment. For a realistic quantum mechanical treatment of all but the smallest systems,

one either has to apply **open systems quantum mechanics** or to accept **quantum cosmology**, that is, to consider the wave function of the whole universe.

The first approach is phenomenological. In an entangled universe, 'outside the system' means 'under the rug'. Major problems swept under the rug thereby are the arrow of time, the collapse of the wave function, and the role of the observer.

Quantum cosmology, on the other hand, is meant to be fundamental: For example, Gell-Mann and Hartle<sup>8</sup> claim that "Quantum mechanics is best and most fundamentally understood in the framework of quantum cosmology." ... "It is the initial condition of the universe that explains the origin of quasiclassical domains within quantum theory itself."

As the universe contains its observer (who is assumed to be local), the collapse can then not be attributed to the intervention of an *outside* observer. Hence, decoherence does not yet *solve* the problem of measurements proper. For this purpose one either has to introduce an explicit collapse dynamics,<sup>9</sup> or to accept some variant of the Everett interpretation. The former approach must lead to deviations from the unitary Schrödinger dynamics which would have to show up somewhere.

Since the universe as a whole depends essentially on gravity, quantum cosmology must necessarily contain **quantum gravity** and its entanglement with matter. Vice versa, quantum gravity was a motivation for studying quantum cosmology.<sup>10</sup> This leads to new and dramatic conceptual problems. Since quantum gravity quantizes the spacetime metric, time itself has to be quantized.

In its canonical form, quantum gravity leads to the Wheeler-DeWitt equation which is of the form of a stationary Schrödinger equation,  $H\Psi = 0$ . This is the way how Mach's principle has to be taken into account in the absence of classical *orbits* in configuration space which could otherwise be parametrized by a time parameter. However, if there is no time, how can the dynamics of measurements, and how can the arrow of time (with its special *initial* state) be described?

Conventional quantum theory tells us that dynamics (time dependence) has now to be replaced by the entanglement of *physical* clocks with all other variables.<sup>11</sup> In particular, there is a fundamental clock in general relativity: spatial geometry on space-like hypersurfaces.<sup>12</sup> Banks has therefore proposed to derive a semiclassical fundamental clock from a Born-Oppenheimer approximation with respect to the Planck mass.<sup>13</sup> One may then parametrize the orbits of geometrical optics in the configuration space of gravity by a variable called 't', and derive from the Wheeler-DeWitt equation a unitary dynamics for the matter part  $\phi_{\text{matter}}$  of the universal wave function *along each orbit*:<sup>14</sup>

$$i \frac{\partial}{\partial t} \phi_{\text{matter}} = H_{\text{matter}} \phi_{\text{matter}} \quad \text{if} \quad \Psi = e^{iS_{\text{geometry}}} \phi_{\text{matter}} .$$

In order to solve it, one also needs an initial condition for  $\phi_{\text{matter}}$  at each orbit. This must also be obtained from the total wave function  $\Psi$ . If this initial condition is appropriate, the solution may describe measurements (increasing entanglement and branching into dynamically independent components) in *one* direction of t.

The question remains of how to determine the universal  $\Psi$ . Only its structure can explain the arrow of time required for measurements in a general sense.

The Wheeler-DeWitt equation  $H\Psi = 0$  seems to define an eigenvalue problem with fixed eigenvalue. As conjectured by DeWitt, it may possess a unique solution in a realistic theory. However, the 'stationary' Wheeler-DeWitt equation turns out to be

hyperbolic on Friedmann type topologies, with the logarithm of the expansion parameter,  $\alpha = \ln a$ , as a 'time-like' variable. The Wheeler-DeWitt Hamiltonian can therefore be written in the form<sup>13</sup>

$$H = \frac{\partial^2}{\partial \alpha^2} - \sum_k \frac{\partial^2}{\partial x_k^2} + V(\alpha, \{x_k\}),$$

that is, as an *intrinsic* Klein-Gordon dynamics which defines an initial value problem in  $\alpha$ . The potential  $V$  has a typical structure of the form

$$V(\alpha, \{x_k\}) = f(\alpha) + \sum_n e^{p_n \alpha} v_n(\{x_k\}) \quad \text{with } p_n > 0 \\ \rightarrow f(\alpha) \quad \text{for } \alpha \rightarrow -\infty,$$

This asymptotic independence of  $V$  of all  $x_k$  allows one to impose a completely symmetric intrinsic initial condition (SIC!)<sup>15</sup>

$$\Psi(\alpha, \{x_k\}) \rightarrow \frac{1}{(-V)^{1/4}} \exp \left[ \int_{-\infty}^{\alpha} \sqrt{-V(\alpha', \{x_k\})} d\alpha' \right] \rightarrow \Psi_0(\alpha)$$

at the big bang *and* the big crunch (which form one common 'state' with respect to the intrinsic dynamics).

The time arrow of measurement is hence given by the expansion of the universe. For extensions of the universe beyond the Planck scale it leads to decoherence growing with increasing  $a$ . It thereby turns out that  $a$  is itself the 'most classical' quantity of the universe in being most efficiently 'measured' by the matter fields.<sup>6</sup> So we can at least be quite sure that we may never observe any interference between different sizes (i. e. ages) of the universe.

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## SOME REMARKS ON QUANTUM PROBABILITIES

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### 1. Introduction

In the quantum description of a physical system the states can be viewed as probability measures on the lattice  $P(H)$  of projectors in the Hilbert space  $H$  associated with the physical system. When  $\dim H \geq 3$ , Gleason theorem says that a probability measure  $\alpha$  on  $P(H)$  has to take the form  $\alpha(P) = \text{tr}(DP)$ ,  $P \in P(H)$ , for some density operator  $D$ . Given  $D$  the probability that the value of some observable  $A$  is in the Borel set  $E$  of  $R$  is accordingly given by  $\text{tr}(DP_A(E))$  where  $P_A(E)$  is the projector associated to the pair  $A, E$  by the spectral theorem. Thus the notion of quantum probability rests on the triple  $(H, P(H), \alpha)$  in which the elements of  $P(H)$  are viewed as the "events", while  $\alpha$  is a state.

In a similar way, the notion of classical probability rests on a triple  $(\Omega, \Sigma, \mu)$  where  $\Omega$  is a set interpreted as the "phase space" of the system,  $\Sigma$  a Boolean  $\sigma$ -algebra of subsets of  $\Omega$  and  $\mu$  a probability measure on  $\Sigma$ . The elements of  $\Sigma$  are viewed as the events, while  $\mu$  is a state.

The key difference between the classical and the quantum case is that  $\Sigma$  is distributive, while  $P(H)$  is only orthomodular. This causes a series of departures between the two cases. One might for instance think of the notion of conditional probability which in the classical situation fits with a rule (Bayes rule) which is untenable in the quantum situation.

In this paper we shall approach the problem from a different point of view: given a set of empirical probabilities how to decide whether they come from a classical or a quantum situation? Besides the mathematical interest of the problem, it seems indeed closer to the experience the idea of looking at the probabilities as primitive, and derive from them the structure of the events.

### 2. Some results

To present in a more compact way some results we come to a few definitions. If  $S$  is a non-empty set, interpreted as the set of states, we define an  $S$ -probability as an  $S$ -indexed probability, i.e., a function  $p : S \rightarrow [0,1]$ . Should  $S$  be a singleton, an  $S$ -probability would be an ordinary probability; should  $S$  contain  $n$  states, an  $S$ -probability would be a sequence  $(p^{(1)}, \dots, p^{(n)})$  of  $n$  probabilities.  $S$ -probabilities can be endowed with a partial ordering  $p \leq q \iff p^{(i)} \leq q^{(i)} \forall i \in S$ , and with an orthogonality relation  $p \perp q \iff p + q \leq 1$ . A triple  $p_1, p_2, p_3$  of pairwise orthogonal  $S$ -probabilities will be denoted  $\Delta(p_1, p_2, p_3)$ . By an  $S$ -probability measure  $\alpha$  on an orthomodular poset  $L$  we shall mean an  $S$ -indexed sequence of probability measures on  $L$ ; we shall say that  $\alpha$  is complete when  $a \leq b \iff \alpha(a) \leq \alpha(b)$ ,  $a, b \in L$ .

Now we can quote some recent results<sup>(1)</sup>



**Representability:** Let  $K$  be a set of S-probabilities. Then  $K$  is the range of a complete S-probability measure on some orthomodular event system  $L$  if and only if i)  $0 \in K$ , ii)  $p \in K \Rightarrow 1 - p \in K$ , iii)  $\Delta(p_1, p_2, p_3) \in K \Rightarrow p_1 + p_2 + p_3 \in K$ .

**Classical Representability:** Let  $K$  be the range of a complete S-probability measure on  $L$ . Then  $L$  is classical if and only if for any  $p_1, p_2 \in K$  there exists  $\Delta(q_1, q_2, q_3) \in K$  such that  $p_1 = q_1 + q_2, p_2 = q_2 + q_3$ .

**Non-classical Representability:** Let  $K$  be the range of a complete S-probability measure on  $L$ . Then  $L$  is non-classical if and only if there exist  $p_1, p_2 \in K$  such that whenever  $q_1 \leq q_2$  and  $p_1 \leq q_1 + q_2$  for some  $q_1, q_2 \in K$  then  $p_1 - p_2 \neq q_1 + q_2 - 1$ .

It is easy, with these criteria, to check the nature of given probabilities. Let us quote just two simple examples, both referring to the case  $S = \{\alpha_1, \alpha_2\}$ . The set of S-probabilities  $K = \{(0,0), (1,0), (0,1), (1,1)\}$  is classically representable (it might come, e.g., from observations of a coin with  $\alpha_1, \alpha_2$  referring to the two coin states). The set  $K = \{(0,0), (1,0), (0,1), (3/4, 1/4), (1/4, 3/4), (1,1)\}$  is instead non-classically representable (it might come, e.g., from observing transmission of a photon through properly positioned analyzers,  $\alpha_1, \alpha_2$  being two orthogonal states of linear polarization). In both examples (0,0) and (1,1) correspond to observing nothing or anything.

### 3. Other approaches

When the considered set of S-probabilities is sufficiently small (and somewhat more structured) other approaches become tractable. We refer to Accardi's "statistical invariants"<sup>(2)</sup> and to Pitowski's polytope approach<sup>(3)</sup>. By a "correlation sequence" of S-probabilities we mean a sequence like  $(p_1, \dots, p_n, \dots, p_{ij}, \dots)$ ,  $1 \leq i < j \leq n$ , and we say that it is classically representable if there exists a classical event system  $L$ , a sequence  $(a_1, \dots, a_n)$  in  $L$ , and an S-probability measure  $\alpha$  on  $L$  such that  $p_i = \alpha(a_i), p_{ij} = \alpha(a_i \wedge a_j)$ . We can now state some facts<sup>(1,3,4,5)</sup>.

*case  $n = 2$ :* the correlation sequence  $(p_1, p_2, p_{12})$  is classically representable if and only if  $0 \leq p_{12} \leq p_1 \leq 1$ ,  $0 \leq p_{12} \leq p_2 \leq 1$  and  $p_1 + p_2 - p_{12} \leq 1$ .

*case  $n = 3$ :* the correlation sequence  $(p_1, p_2, p_3, p_{23}, p_{13}, p_{12})$  is classically representable if and only if  $0 \leq p_{ij} \leq p_i \leq 1$ ,  $0 \leq p_{ij} \leq p_j \leq 1$ ,  $p_1 + p_j - p_{ij} \leq 1$ , and the Bell's inequalities ( $p_1 - p_{12} - p_{13} + p_{23} \geq 0$ ,  $p_2 - p_{23} - p_{12} + p_{13} \geq 0$ ,  $p_3 - p_{13} - p_{23} + p_{12} \geq 0$ ,  $p_1 + p_2 + p_3 - p_{12} - p_{23} - p_{13} \leq 1$ ) hold true.

We may notice that in the special case  $p_1 + p_2 = 1$  ( $p_1, p_2 \neq 0$ ), and  $p_{12} = 0$  one recovers a 2-slit like situation,  $p_1, p_2$  being the probabilities of going through one or the other hole, and  $p_3$  the probability of reaching some region on the screen. In this case the above inequalities lead to  $p_3 = p_{13} + p_{23}$  and  $p_1 = (p_3 - p_{23}/p_2)(p_{13}/p_1 - p_{23}/p_2)^{-1}$  from which (and by use of Bayes formula for conditional probabilities) one gets Accardi's statistical invariant<sup>(2)</sup> for the 2-slit experiment.

The case  $n = 4$  would lead<sup>(3)</sup> to the "Clauser-Horne" inequalities. But further increasing  $n$ , the number and complexity of inequalities grows so fast that this approach becomes soon untractable.

#### 4. On the quantum case

Let us say that the correlation sequence  $(p_1, \dots, p_n, \dots, p_{ij}, \dots)$  has a quantum representation if there exists an Hilbert space  $H$ , a sequence  $(P_1, \dots, P_n)$  in  $P(H)$ , and a probability measure  $\alpha$  on  $P(H)$  s.t.  $p_j = \alpha(P_j)$ ,  $p_{ij} = \alpha(P_i \wedge P_j)$ . Call  $Q_n$  the set of the correlation sequences that have a quantum representation; by contrast call  $C_n$  the polytope of those which are classically representable. Let  $\Lambda_n$  the closed convex polytope of the correlation sequences that satisfy the basic inequalities  $0 \leq p_{ij} \leq p_i \leq 1$ ,  $0 \leq p_{ij} \leq p_j \leq 1$  ( $0 \leq i < j \leq n$ ). Then we have<sup>(3)</sup>: (i)  $\Lambda_n \supset Q_n$ , (ii)  $\Lambda_n \setminus Q_n$  is the set of vertices not classically representable, (iii)  $Q_n \supset C_n$ .

Let us add a remark on a theorem by Accardi and Fedullo<sup>(4)</sup> which says that, given three probabilities  $p, q, r$  (i.e., numbers in  $[0,1]$ ) and denoting  $f = |p + q + r - 1| - 2\sqrt{pqr}$ , there exist three vectors  $\varphi, \psi, \chi$  in a two-dimensional Hilbert space  $H_2$  such that  $p = |(\varphi, \psi)|^2$ ,  $q = |(\varphi, \chi)|^2$ ,  $r = |(\psi, \chi)|^2$  if and only if  $f \leq 0$ , and moreover  $H_2$  is over  $C$  if  $f < 0$ , over  $R$  if  $f = 0$ . The proposed interpretation for  $p, q, r$  is in terms of conditional probabilities among three events  $a, b, c$ , say  $p = \alpha(a|b)$ ,  $q = \alpha(a|c)$ ,  $r = \alpha(b|c)$  (which requires, in view of the scalar-product forms above,  $a, b, c$  to be atomic events).

If, for instance, we deal with a spin- $\frac{1}{2}$  system and suppose  $a, b, c$  to be polarizations along corresponding axes, then the quoted theorem can be used to draw conclusions on the Hilbert space of the system: if  $a, b, c$  are not coplanar we get  $f < 0$  (with the above conditional probability interpretation) so that  $H_2$  is on  $C$ , if they are coplanar  $f = 0$  and  $H_2$  is on  $R$  (this fits with known properties of the 2-dimensional projective representations of rotations).

If, as another example, the three polarizations  $a, b, c$  refer to a spin-1 system then we can have  $f < 0$ ,  $f = 0$ , or  $f > 0$  according to different positionings of the polarization axes, but we know that a spin-1 system needs  $R^3$  as its Hilbert space. This outlines that the last mentioned theorem does not refer, in the general case, to the Hilbert space needed by the quantum mechanical description of the physical system: it might however suggest the possibility of generalizations for Hilbert space dimensions higher than two.

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## Conditional Probability and Strong Correlations in the Quantum Theory of Measurement

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**1. Introduction.** The last few years have witnessed an increase of interest in the study of the measurement process in quantum mechanics. A systematic theory of measurement has emerged which allows one to analyse and reanalyse various aspects of measurements. In particular, several probabilistic and information theoretical characterizations of the measurement process have been worked out, see, e.g. [1]. In this contribution we shall present a continuation of this work analysing the notion of conditional probability in the quantum theory of measurement.

The notion of conditional probability touches the quantum theory of measurement in several crucial points, in its probability reproducibility condition, in various correlation conditions, in ideality conditions, and also in the theory of sequential measurements. Here we shall try to explain a characterization of strong correlation measurements in terms of appropriate conditionings. The very idea that the final states of the object system and the measuring apparatus are their states after the measurement with the condition that one of the pointer values occurs is a typical question on conditional probability. The fact that in quantum mechanics the notion of conditional probability is not always additive with respect to a partitioning of the conditioning event calls for a special attention to that notion.

We shall formulate our results within the ordinary Hilbert space formulation of quantum mechanics. We follow the notations and terminology of Ref. [1]. Here we recall only that any pair  $(E, T)$  of an observable  $E: \mathcal{F} \rightarrow \mathcal{P}(\mathcal{H}_S)$  [a projection operator valued measure on a measurable space  $(\Omega, \mathcal{F})$ ] and a state  $T \in \mathcal{T}(\mathcal{H}_S)_+^*$  [positive trace one operators] defines a probability measure  $E_T: \mathcal{F} \rightarrow [0, 1]$  through the trace formula  $E_T(X) \doteq \text{tr}[TE(X)]$ . According to the minimal interpretation, the number  $E_T(X)$  is the probability that a measurement of the observable  $E$  performed on the system  $S$  in the state  $T$  leads to a result in the set  $X$ . If  $T = P[\varphi]$  is a vector state (with a generating unit vector  $\varphi$ ), then this probability is simply equal to  $\langle \varphi | E(X) \varphi \rangle$ .

The notion of conditional probability is crucial for this study. Hence we briefly recall this notion in the context of Hilbert space quantum mechanics. According to Gleason's theorem the probability measures in quantum mechanics are generated by states  $T$  through the formula:  $P \mapsto \text{tr}[TP]$ ,  $P \in \mathcal{P}(\mathcal{H}_S)$ . Consider a state  $T$ , and let  $R \in \mathcal{P}(\mathcal{H}_S)$  be a projection operator such that  $\text{tr}[TR] \neq 0$ . The conditional probability with respect to  $R$  is the probability measure induced by the state  $T^R \doteq \frac{RTR}{\text{tr}[TR]}$ . Contrary to the classical probability theory, the conditional probability  $\text{tr}[T^R \cdot]: \mathcal{P}(\mathcal{H}_S) \rightarrow [0, 1]$  is not, in general, additive with respect to a partitioning of the conditioning event. If  $R = \sum_i R_i$  is a decomposition of the condition  $R$  into mutually orthogonal conditions  $R_i \in \mathcal{P}(\mathcal{H}_S)$ , the

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conditional probability with respect to  $R$  which is additive over  $R = \sum_i R_i$  is the probability measure defined by the state  $T^{\Sigma R_i} \doteq \sum_i \frac{R_i \text{tr} R_i}{\text{tr} [TR]} = \sum_i \frac{\text{tr} [TR_i]}{\text{tr} [TR]} T^{R_i}$ . It is the probability measure defined by the property  $\text{tr} [T^{\Sigma R_i} P] = \text{tr} [TRP]$  for each projection operator  $P$  in any segment  $[O, R_i]$  [2]. Clearly,  $\text{tr} [T^{\Sigma R_i} R] = 1$  and  $\text{tr} [T^{\Sigma R_i} R_i] = \text{tr} [TR_i] / \text{tr} [TR]$  for each  $i$ .

In its usual formulation the measurement theory of an observable  $E$  of the object system  $\mathcal{S}$  starts with fixing a measuring apparatus  $\mathcal{A}$  (with a Hilbert space  $\mathcal{H}_{\mathcal{A}}$ ), its initial (vector) state  $\Phi \in \mathcal{H}_{\mathcal{A}}$ , a pointer observable  $P_{\mathcal{A}} : \mathcal{F} \rightarrow \mathcal{L}(\mathcal{H}_{\mathcal{A}})^+$ , and a measurement coupling  $U : \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{A}} \rightarrow \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{A}}$  (a unitary operator). The interpretation of the resulting quadruple  $\langle \mathcal{H}_{\mathcal{A}}, P_{\mathcal{A}}, \Phi, U \rangle$  as a measurement of  $E$  starts with the assumption that if  $\varphi \in \mathcal{H}_{\mathcal{S}}$  is an initial (vector) state of the object system  $\mathcal{S}$  then  $U(\varphi \otimes \Phi)$  is the final (vector) state of the compound object-apparatus system  $\mathcal{S} + \mathcal{A}$ . The reduced states  $\mathcal{R}_{\mathcal{S}}[P[U(\varphi \otimes \Phi)]]$  and  $\mathcal{R}_{\mathcal{A}}[P[U(\varphi \otimes \Phi)]]$  are then the final states of  $\mathcal{S}$  and  $\mathcal{A}$ , respectively. As a rule, they are mixed states. (Here  $\mathcal{R}_{\mathcal{S}}, \mathcal{R}_{\mathcal{A}}$  denote the partial traces over the apparatus and object system Hilbert spaces  $\mathcal{H}_{\mathcal{A}}$  and  $\mathcal{H}_{\mathcal{S}}$ , respectively.) The basic requirement for  $\langle \mathcal{H}_{\mathcal{A}}, P_{\mathcal{A}}, \Phi, U \rangle$  to constitute a measurement of  $E$  is the probability reproducibility condition:  $\langle \varphi | E(X) \varphi \rangle = \text{tr} [\mathcal{R}_{\mathcal{A}}[P[U(\varphi \otimes \Phi)]] P_{\mathcal{A}}(X)]$  for all  $X \in \mathcal{F}$ , and for any vector state  $\varphi \in \mathcal{H}_{\mathcal{S}}$ . Quadruples  $\mathcal{M}_U \doteq \langle \mathcal{H}_{\mathcal{A}}, P_{\mathcal{A}}, \Phi, U \rangle$  which fulfil this condition are known as normal unitary premeasurements of  $E$  [1].

We close this preliminary section with recalling the notion of a reading scale. A reading scale is a countable partition of the value space  $\Omega$ , that is  $\Omega = \cup X_i$ ,  $X_i \in \mathcal{F}$ ,  $X_i \cap X_j = \emptyset$  for  $i \neq j$ . Such a reading scale will be denoted  $\mathcal{R}$ . A reading scale  $\mathcal{R}$  determines a discrete, coarse-grained version of the pointer observable  $P_{\mathcal{A}}$ ,  $P_{\mathcal{A}}^{\mathcal{R}} : i \mapsto P_{\mathcal{A},i} \doteq P_{\mathcal{A}}(X_i)$ . The  $P_{\mathcal{A}}^{\mathcal{R}}$ -value  $i$  refers to the pointer value  $X_i$  which, in turn, may refer to the value  $X_i$  of the measured observable  $E$ . It is with respect to such a reading scale that measurement results are to be recoded.

**2. Conditional final states.** Consider a premeasurement  $\mathcal{M}_U$  of an observable  $E$ . If  $\varphi$  is the initial state of  $\mathcal{S}$ , then  $U(\varphi \otimes \Phi)$  is the final state of  $\mathcal{S} + \mathcal{A}$ . Together with a condition  $I \otimes P_{\mathcal{A}}(X)$ , this state defines a (nonnormalized) conditional final state of  $\mathcal{S} + \mathcal{A}$ :

$$P[U(\varphi \otimes \Phi)] \mapsto I \otimes P_{\mathcal{A}}(X) P[U(\varphi \otimes \Phi)] I \otimes P_{\mathcal{A}}(X)$$

Up to a normalization, this is the final state of  $\mathcal{S} + \mathcal{A}$  with the condition  $I \otimes P_{\mathcal{A}}(X)$ , that is, with the condition that the pointer observable  $P_{\mathcal{A}}$  has the value  $X$ . The corresponding final (nonnormalized) states of  $\mathcal{S}$  and  $\mathcal{A}$  are

$$\mathcal{R}_{\mathcal{S}}[I \otimes P_{\mathcal{A}}(X) P[U(\varphi \otimes \Phi)] I \otimes P_{\mathcal{A}}(X)] = \mathcal{R}_{\mathcal{S}}[P[U(\varphi \otimes \Phi)] I \otimes P_{\mathcal{A}}(X)]$$

and

$$\mathcal{R}_{\mathcal{A}}[I \otimes P_{\mathcal{A}}(X) P[U(\varphi \otimes \Phi)] I \otimes P_{\mathcal{A}}(X)] = P_{\mathcal{A}}(X) \mathcal{R}_{\mathcal{A}}[P[U(\varphi \otimes \Phi)] P_{\mathcal{A}}(X)]$$

respectively. Let  $N_X^2 \doteq \langle \varphi | E(X) \varphi \rangle$ , and denote

$$T_{\mathcal{S}}(X, \varphi) \doteq N_X^{-2} \mathcal{R}_{\mathcal{S}}[P[U(\varphi \otimes \Phi)] I \otimes P_{\mathcal{A}}(X)]$$

$$T_{\mathcal{A}}(X, \varphi) \doteq N_X^{-2} P_{\mathcal{A}}(X) \mathcal{R}_{\mathcal{A}}[P[U(\varphi \otimes \Phi)] P_{\mathcal{A}}(X)]$$

whenever  $N_X^2 \neq 0$ . We also define  $T_S(X, \varphi) = O = T_A(X, \varphi)$  whenever  $\langle \varphi | E(X) \varphi \rangle = 0$ . We call  $T_S(X, \varphi)$  and  $T_A(X, \varphi)$  the final ( $X$ -)component states of  $\mathcal{S}$  and  $\mathcal{A}$ , respectively. These states are *conditional states*, that is, they give rise to conditional probabilities. Indeed,  $T_A(X, \varphi)$  is the final state of  $\mathcal{A}$  with the condition  $P_A(X)$ . In this state the pointer observable has the value  $X$  in the sense that  $\text{tr}[T_A(X, \varphi)P_A(X)] = 1$ . Similarly,  $T_S(X, \varphi)$  is the final state of  $\mathcal{S}$  with the condition  $P_A(X)$ . Clearly,  $\text{tr}[T_S(X, \varphi)E(X)] = 1$  need not hold now; the probability for  $E(X)$  in the final  $X$ -component state  $T_S(X, \varphi)$  of  $\mathcal{S}$  need not be equal to one. (For further details, see [3]).

Consider still a premeasurement  $\mathcal{M}_U$  of  $E$ , and fix a reading scale  $\mathcal{R}$ . For any  $X_i \in \mathcal{R}$  we denote  $T_S(i, \varphi) \doteq T_S(X_i, \varphi)$ ,  $T_A(i, \varphi) \doteq T_A(X_i, \varphi)$ , and  $N_i^2 \doteq N_{X_i}^2$ . These states are conditional states. They are the final ( $i$ -)component states of  $\mathcal{S}$  and  $\mathcal{A}$ , with the condition that the pointer observable  $P_A$  has the value  $X_i$ . The reading scale  $\mathcal{R}$  defines a partitioning of the trivial condition  $P_A(\Omega)$  into (mutually orthogonal) conditions  $P_{A,i} \doteq P_A(X_i)$ . The final states of  $\mathcal{S}$  and  $\mathcal{A}$  which give rise to conditional probabilities which are additive over the decomposition  $P_A(\Omega) = \sum_i P_{A,i}$  induced by  $\mathcal{R}$  can be determined. The conditional state of  $\mathcal{A}$ , defined by the final apparatus state  $T_A(\Omega, \varphi) = \mathcal{R}_A[P[U(\varphi \otimes \Phi)]]$  and the condition (reading scale)  $\mathcal{R}$  is

$$T_A(\mathcal{R}, \varphi) \doteq \sum_i P_{A,i} T_A(\Omega, \varphi) P_{A,i} = \sum_i N_i^2 T_A(i, \varphi)$$

In general,  $T_A(\mathcal{R}, \varphi) \neq T_A(\Omega, \varphi)$ , since  $T_A(\Omega, \varphi)$  need not commute with each  $P_{A,i}$ . Due to the special structure of the states  $T_S(X, \varphi)$  we, however, have

$$T_S(\Omega, \varphi) = \sum_i N_i^2 T_S(i, \varphi) \doteq T_S(\mathcal{R}, \varphi)$$

for any reading scale  $\mathcal{R}$  (and for any initial state  $\varphi$  of  $\mathcal{S}$ ).

**3. Strong correlation measurements.** Consider a premeasurement  $\mathcal{M}_U$  of an observable  $E$ , and fix a reading scale  $\mathcal{R}$ . With respect to this reading scale, if  $\varphi$  is the initial state of  $\mathcal{S}$ , then  $T_S(i, \varphi)$  and  $T_A(i, \varphi)$  are the final  $i$ -component states of  $\mathcal{S}$  and  $\mathcal{A}$ . The final apparatus state is  $T_A(\Omega, \varphi)$  and it can be expressed as  $T_A(\Omega, \varphi) = \sum_{i,j} P_{A,i} T_A(\Omega, \varphi) P_{A,j}$ . The final state of  $\mathcal{A}$  with the (additive) condition  $\mathcal{R}$  is, however,  $T_A(\mathcal{R}, \varphi) = \sum_i P_{A,i} T_A(\Omega, \varphi) P_{A,i}$ . The question which we should like to address here is the following: when do the above two states coincide? In other words, under which conditions the final apparatus state equals to the final apparatus state with the condition that one of the mutually exclusive pointer readings occurs. It turns out that for a given reading scale  $\mathcal{R}$  and for a given initial state  $\varphi$  of  $\mathcal{S}$  this is the case exactly when the final component states  $T_S(i, \varphi)$  and  $T_A(i, \varphi)$  are strongly correlated [3]. To formulate this result properly we remind ourselves that the correlation  $\rho(T_S(i, \varphi), T_A(i, \varphi), U(\varphi \otimes \Phi))$  of the component states  $T_S(i, \varphi)$  and  $T_A(i, \varphi)$  in the final state  $U(\varphi \otimes \Phi)$  of  $\mathcal{S} + \mathcal{A}$  is strong exactly when the (marginal) probability measures defined by the pairs  $(T_S(i, \varphi), T_S(\Omega, \varphi))$  and  $(T_A(i, \varphi), T_A(\Omega, \varphi))$  are linearly dependent [4].

**Theorem 1.** Let  $\langle \mathcal{H}_A, P_A, \Phi, U \rangle$  be a normal unitary premeasurement of an observable  $E$ . For any reading scale  $\mathcal{R}$  and for any initial vector state  $\varphi$  of the object system, the following two conditions are equivalent:

- (a)  $T_A(\Omega, \varphi) = T_A(\mathcal{R}, \varphi)$ ;
- (b)  $\rho(T_S(i, \varphi), T_A(i, \varphi), U(\varphi \otimes \Phi)) = 1$  for each  $i = 1, 2, \dots$  for which  $N_i^2 \neq 0$ .

We go on with studying a premeasurement  $\mathcal{M}_U$  of an observable  $E$ . If  $\varphi$  is the initial state of  $S$ , then  $T_S(X, \varphi)$ ,  $X \in \mathcal{F}$ , is the final  $X$ -component state of  $S$  after the measurement with the pointer condition  $P_A(X)$ . Moreover, with respect to any reading scale  $\mathcal{R}$ ,  $T_S(\Omega, \varphi) = \sum_i N_i^2 T_S(i, \varphi) = T_S(\mathcal{R}, \varphi)$ . As it was already pointed out, the conditional interpretation of  $T_S(i, \varphi)$  does not imply that  $\text{tr}[T_S(i, \varphi)E_i] = 1$  (with  $E_i = E(X_i)$ ). Thus the final  $X_i$ -component state  $T_S(i, \varphi)$  of  $S$  need not be the final state of  $S$  with the condition  $E_i$ . Though  $T_S(\Omega, \varphi) = T_S(\mathcal{R}, \varphi)$ , the equality  $T_S(\Omega, \varphi) = \sum_i E_i T_S(\Omega, \varphi) E_i$  need not hold. We shall now address to the question under which conditions the final state  $T_S(\Omega, \varphi)$  of  $S$  admits a conditional interpretation with respect to the (additive) condition  $I = \sum_i E_i$ .

There are, at least, two ways to tackle this question. The first, and, perhaps, the most natural approach is to ask when does the equality  $T_S(\Omega, \varphi) = \sum_i E_i T_S(\Omega, \varphi) E_i$  hold for a given reading scale  $\mathcal{R}$  and initial state  $\varphi$ . It appears that this is the case exactly when the values of  $E$  and  $P_A$  associated with  $\mathcal{R}$  are strongly correlated in the final object-apparatus state  $U(\varphi \otimes \Phi)$ . The other, more restrictive approach is to ask when does the equality  $T_S(\Omega, \varphi) = \sum_i E_i P[\varphi] E_i$  hold. It turns out that this is the case if and only if the measurement is ideal with respect to  $\mathcal{R}$ . Here we shall concentrate only on the first approach. To do that we shall recall that the correlation  $\rho(E_i, P_i, U(\varphi \otimes \Phi))$  of the values  $E_i$  and  $P_i$  of  $E$  and  $P_A$  (with respect to  $\mathcal{R}$ ) in the final state  $U(\varphi \otimes \Phi)$  can directly be computed, and the strong value correlation  $\rho(E_i, P_i, U(\varphi \otimes \Phi)) = 1$  is obtained exactly when the equality  $\text{tr}[T_S(\Omega, \varphi)E_i] = \text{tr}[T_A(\Omega, \varphi)P_i]$  holds true [4]. Then we obtain [3]:

**Theorem 2.** Let  $\langle \mathcal{H}_A, P_A, \Phi, U \rangle$  be a normal unitary premeasurement of an observable  $E$ . For any reading scale  $\mathcal{R}$  and for any initial vector state  $\varphi$  of the object system, the following two conditions are equivalent:

- (a)  $T_S(\Omega, \varphi) = \sum_i E_i T_S(\Omega, \varphi) E_i$ ;
- (b)  $\rho(E_i, P_i, U(\varphi \otimes \Phi)) = 1$  for each  $i = 1, 2, \dots$  for which  $N_i^2 \neq 0$ .

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A DISCUSSION  
ON SOME THEORIES OF QUANTUM MEASUREMENT\*

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1. In the ordinary formulation of quantum mechanics the principles concerning measurement — the probability law (PL) and the reduction postulate (RP) — deal with the measured system  $S$  alone. The measuring apparatus  $A$  is supposed to be there but it remains outside the description. The theory of measurement consists in applying the quantum principles — first of all the Schrödinger equation (SE) — to the compound system  $S + A$  and in deducing from such a description the usual postulates dealing with  $S$  alone (or at least in showing the consistency of the two descriptions). This program is not universally accepted: according to the Copenhagen interpretation the description of the apparatus is necessarily classical so that a quantum treatment of  $S + A$  is impossible.

In the ideal situation the dynamics of the system  $S + A$  shall be such that

$$|\psi_m\rangle|A_0\rangle \xrightarrow{\text{SE}} |\psi_m\rangle|A_{m,s_m}\rangle,$$

where  $|\psi_m\rangle$  are the eigenvectors of the measured variable and  $|A_{m,s}\rangle$  those of the pointer variable. Then because of linearity of SE one gets

$$(1) \quad \sum_m a_m |\psi_m\rangle \xrightarrow{\text{SE}} \sum_m a_m |\psi_m\rangle |A_{m,s_m}\rangle,$$

while according to PL+RP (and to common sense) one should find

$$(2) \quad \sum_m a_m |\psi_m\rangle \xrightarrow{\text{PL+RP}} |\psi_l\rangle |A_{l,s_l}\rangle,$$

each  $l$  happening with probability  $|a_l|^2$ . In its essence this is the problem of the theory of quantum measurement.

In the *orthodox* theories of measurement, the unrestricted validity of the Schrödinger equation and the completeness of the description provided by the state vector are not questioned. Examples, which will be discussed here, are the orthodox effective incoherence (OEI) theories and the approaches based on the concept of history. Counterexamples are models describing reduction as a real physical process and hidden variable theories.

2. OEI theories go back to the work of Daneri, Loinger and Prosperi.<sup>1</sup> They constitute the most numerous family of theories of measurement. The few references we give here<sup>1-3</sup> do not exhaust even the most relevant ones. The logical structure of these theories is schematically

$$\text{SE} + \text{PL} + \text{AM} \quad \text{equivalent to} \quad \text{SAME} + \text{RP}.$$

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where AM indicates a suitable assumption on measurability of the quantities of the system  $S + A$  (or  $S + A + E$ ,  $E$  being the environment). A preliminary statement is that quantum mechanics deals only with ensembles. Then it is shown that, for any actually measurable quantity of  $S + A$  (or  $S + A + E$ ), probabilities can be calculated omitting the cross contributions between the terms of the r.h.s. of eq. (1). It is concluded that the superposition appearing in (1) is effectively incoherent, i.e. equivalent for all purposes to the mixture given by (2). The various theories differ from one another essentially in the type of limitation which is invoked. The above account is obviously incomplete and oversimplified but it gives the essence of the argument.

The question immediately arises about OEI theories whether we can reconcile the fact that in a single measurement we actually get a definite result with a description of the system devoid of any counterpart of such a definite result. Remember that the state vector is always given by the Schrödinger equation and that the only link between the state vector and the results of any (allowed) measurement is the probability law. Let us consider two successive measurements of the same quantity of  $S$ . The state vector undergoes the evolution

$$|\psi\rangle|A_0^1\rangle|A_0^2\rangle \longrightarrow \sum_m a_m |\psi_m\rangle|A_{m,s_m}^1\rangle|A_0^2\rangle \longrightarrow \sum_m a_m |\psi_m\rangle|A_{m,s_m}^1\rangle|A_{m,s_m}^2\rangle.$$

If we apply the probability law to a third measurement consisting in looking at the pointers of the first and second apparatus, we find that the probability of equal results is 1. If, instead, we apply the probability law after the first to the second measurement of the subensemble, say, for which the result  $l$  has been obtained, we find  $\text{Pr}(l) = |a_l|^2 < 1$ . We meet an inconsistency, which comes from allowing a selection of the members of the ensemble. A similar inconsistency is met<sup>4</sup> if the description is applied to finite ensembles. We conclude that OEI theories are consistent only if applied to indivisible infinite ensembles. One can easily show that this kind of inconsistency disappears in those theories of measurement in which the description of the system includes a counterpart of the obtained result. However, effective incoherence is a fact, of vital importance for any theory of measurement.

3. The history approach was initiated recently by Griffiths<sup>5</sup> and continued by Omnès<sup>6</sup> and Gell-Mann and Hartle.<sup>7</sup> A critical study, to which the present discussion is largely inspired, has been given by d'Espagnat.<sup>8</sup> We shall refer here to the version of Ref. 7. It is a probabilistic theory of the universe (which implies to interpret probabilities as propensities). The key concept is that of *decoherent set of alternative histories*. One starts considering projection operators in the Heisenberg picture:  $P(t) = e^{iHt/\hbar} P(0) e^{-iHt/\hbar}$ . Then a set of alternatives is a set  $P_{\{\alpha\}}(t)$  such that  $P_{\alpha}(t) P_{\beta}(t) = \delta_{\alpha\beta} P_{\alpha}(t)$ ,  $\sum_{\alpha} P_{\alpha}(t) = 1$ . A particular history  $[P_{\alpha}]$  is a particular time sequence  $[P_{\alpha_1}^1(t_1), P_{\alpha_2}^2(t_2), \dots, P_{\alpha_n}^n(t_n)]$  of alternatives and a set of alternative histories (SAH)  $[P_{\{\alpha\}}]$  is a time sequence of sets of alternatives. A history is a coarse(fine)-graining of another if the set  $[P_{\alpha}]$  of the first (second) history consists of sums of the  $[P_{\alpha}]$  of the second (first) history. One proceeds further defining for a SAH the decoherence functional

$$(3) \quad D([P_{\alpha'}], [P_{\alpha}]) = \text{Tr} [P_{\alpha'_n}^n(t_n) \dots P_{\alpha'_1}^1(t_1) |\psi\rangle \langle \psi| P_{\alpha_1}^1(t_1) \dots P_{\alpha_n}^n(t_n)]$$



where  $|\psi\rangle$  is the initial state vector of the universe. A SAH is decoherent (DSAH) when

$$D([P_{\alpha'}], [P_{\alpha}]) \approx 0, \quad \text{for any } \alpha_k' \neq \alpha_k.$$

It is a property of a SAH which depends on  $|\psi\rangle$  and on the dynamics embodied in the time evolution operator. SAH's specified at only one time are all decoherent. It is easily seen that coarse-graining preserves decoherence and fine-graining can destroy it. Therefore some DSAH's are maximal with respect to fine-graining. Let  $[P_{\{\alpha\}}]$  be a DSAH, and  $[\bar{P}_{\{\beta\}}]$  a coarse-graining of it, which is also a DSAH. Then

$$D([\bar{P}_{\{\beta\}}], [\bar{P}_{\{\beta\}}]) \approx \sum_{\text{all } P_{\alpha} \text{ not fixed by } [\bar{P}_{\{\beta\}}]} D([P_{\alpha}], [P_{\alpha}])$$

This additivity property allows one to interpret the (diagonal) elements of  $D$  as the probabilities of the various histories within the DSAH to which  $D$  refers. One writes therefore the *fundamental formula*<sup>7</sup>

$$D([P_{\alpha'}], [P_{\alpha}]) \approx \delta_{\alpha_1' \alpha_1} \dots \delta_{\alpha_n' \alpha_n} \text{Pr}([P_{\alpha}]).$$

The origins of decoherence have been extensively discussed by the authors of the history approach. In the situations of interest for the theory of measurement, decoherence is related to the same limitations on measurability on which are based OEI theories.

In the history theories, the element of our description of physical systems (except the universe) is no more the state vector, it is the (decoherent) history. This characteristic fact has important consequences. The first one is that the kind of inconsistency we find in OEI theories is overcome. In fact, in such theories the probability law is to be applied to the state vector any time we perform a measurement; here, instead, probabilities refer to whole histories. In the situation considered above, one simply finds that, in the relevant DSAH, histories with different pointer positions have probability zero because of dynamics.

Given the initial state of the universe and its dynamics there are many DSAH's and even many maximal DSAH's. The following question is legitimate: what does decide, if anything does, between two histories belonging to different DSAH's? It is not chance, because probabilities can be assigned only within one DSAH. One can answer that, given  $|\psi\rangle$ , it is the dynamics of the universe that decides what is the relevant DSAH. Alternatively, one can state that we have free will and that it is our decision about what to measure that chooses the DSAH. The first answer is probably more adherent to the spirit of a cosmological theory like that of Gell-Mann and Hartle, even though, in my opinion, denying free will goes beyond a reasonable application of present day science. Anyhow, the first answer implies that one needs something besides the fundamental formula, which, on the other hand, seems to incorporate completely the initial condition and the dynamics through eq. (3). The second type of answer appears to me less metaphysical. It has some odd consequences as it will be discussed below. I think that the structure of the space of DSAH's should be studied and the problem of the choice of the relevant DSAH discussed thoroughly.

At any rate, it has been pointed out by d'Espagnat<sup>8</sup> that, as a rule, there are different equally relevant DSAH's the choice among which is completely arbitrary.

Suppose that at time  $t_0$  the component  $S_z$  and at time  $t_2$  the component  $S_x$  of the spin of a certain particle are measured. We know from the above discussion that  $[P_{\{m_0\}}^{S_z}(t_0), P_{\{m_2\}}^{S_x}(t_2)]$  is a DSAH. But then also  $[P_{\{m_0\}}^{S_z}(t_0), P_{\{m_1\}}^{S_z}(t_1), P_{\{m_2\}}^{S_x}(t_2)]$  and  $[P_{\{m_0\}}^{S_z}(t_0), P_{\{m_1\}}^{S_x}(t_1), P_{\{m_2\}}^{S_x}(t_2)]$  are DSAH's. Since the two specifications at time  $t_1$  are incompatible they cannot be both true. According to Omnès they are *reliable*. But it remains the fact that histories cannot be thought simply as unfoldings of true assertions about the system.

Another problematic feature of history theories is the future-past relation.<sup>9</sup> In order to make predictions about a tomorrow's measurement I must foresee what type of measurement I will perform, consider the corresponding SAH, ascertain its decoherence (no problem), calculate the probabilities. Assuming free will, what are the relevant DSAH's depends on what I will decide tomorrow. Since the set of relevant DSAH's specifies what quantities today have reliable values, we must conclude that in some sense the future can influence the past.

4. History theories appear to be an important improvement of the Copenhagen interpretation, essentially because they provide a quantum description of the system  $S + A$ . They are also an improvement with respect to OEI theories, which we found to be not fully consistent. However, the fact that only a part of the assertions making up a history can be said to be true (in practice, those more directly related to readings of instruments) is a severe limitation of our faculty of predicating something about the system we are considering. If, on the other hand, we tend to give some value of truth to reliability, then we must either deny free will, or accept that the future can influence the past.

My conclusion, not a new one, is that the history interpretation of quantum mechanics provides a consistent set of precise rules for making previsions, but that it does not allow to ascribe individual properties to physical systems beyond those directly related to measurements.

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## Local Position Measurements and State Preparation

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### 1. Quantum Measurements.

Measurement and state preparation procedures are often discussed in terms of the projection postulate, i.e. sudden and non-unitary processes are employed to accomplish a measurement or a state preparation. Our object here is to try to show that the mathematical structure of orthodox quantum mechanics is rich enough to allow the formulation of "continuous" unitary processes for measurements and for state preparation.

Consider the measurement of an observable  $A$  having eigenvalues  $a_j$ . A typical quantum measurement process involves the following two steps.

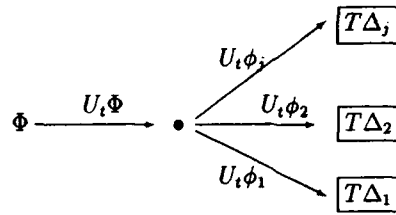
Step one is referred to as a spectral separation step. This involves the splitting up of the initial wave function into spatially separated components; each component corresponding to an eigenvalue  $a_j$  (M. Namiki, *Found. Phys.* **18** (1988) 28). These components are then guided towards detectors at different locations. Step two involves the direct detection of the particle by detectors at various locations; we call this the local position measurement step.

A spectral separation is a time evolution process. The question is whether there is an interaction described by an observable  $H$  which can produce the desired spectral separation of the wave function. Mathematically we seek a unitary evolution group  $U$ , where  $U_t = e^{-itH/\hbar}$  ( $t \in \mathbf{R}$ ), which will cause the initial wave function to undergo a spectral separation in time.

Suppose  $A$  has a discrete and nondegenerate spectrum with eigenvectors  $\phi_j$  corresponding to eigenvalues  $a_j$ . Then  $A = \sum a_j P_j$  where  $P_j = |\phi_j\rangle\langle\phi_j|$ . Let  $\Phi$  be a given initial state. Then we have  $\langle\Phi|A\Phi\rangle = \sum |c_j|^2 a_j$ , where  $\Phi = \sum c_j \phi_j$ , and  $c_j = \langle\Phi|P_j\Phi\rangle = \langle\phi_j|\Phi\rangle$ . The quantity  $|c_j|^2$  represents the probability of obtaining the value  $a_j$ . Under an evolution group  $U$  we have  $\Phi_t = \sum c_j U_t \phi_j$ .

To achieve a spectral separation we want the  $U_t \phi_j$  to evolve into different spatial regions at large times. Let  $\Delta_j$  be a collection of disjoint regions in space  $\mathbf{R}^3$ , and let  $t\Delta_j$  denote the set  $\{tx|x \in \Delta_j\}$ . Then the regions  $t\Delta_j$  in  $\mathbf{R}^3$  are mutually disjoint for each  $t > 0$ . We want each state  $U_t \phi_j$  to evolve asymptotically into the region  $t\Delta_j$ .

The following diagram illustrates the idea.



Let  $\chi_{t\Delta}$ , be the characteristic function of the region  $t\Delta_j$ . We want

$$\lim_{t \rightarrow \infty} \|\chi_{t\Delta} U_t \phi_j\| = 1, \quad \text{or equivalently} \quad \lim_{t \rightarrow \infty} \langle U_t \phi_j | \chi_{t\Delta} U_t \phi_j \rangle = 1.$$

The next step is to detect the arrival of the particles at detectors located in various regions. The count or count rate detected in region  $t\Delta_j$  should give, with arbitrary accuracy, the probabilities  $|c_j|^2$  needed to obtain  $\langle \Phi | A \Phi \rangle$ .

Let us now carry out a more detailed analysis. Given a finite number  $N$  of detectors we can place them in  $N$  disjoint regions, e.g.  $\{T\Delta_j | 1 \leq j \leq N\}$ , at some large time  $T$ . This implies we can only positively detect the components  $\{U_T \phi_j | 1 \leq j \leq N\}$  and therefore can only measure the observable  $A_N = \sum_{j=1}^N a_j P_j$ . This is good enough because for every  $\psi$  in the domain of  $A$  we have  $A\psi = \lim_{N \rightarrow \infty} A_N \psi$ , and the operator  $A_N$  has the same eigenvectors  $\phi_j$  and eigenvalues  $a_j$  as  $A$  for  $j = 1, \dots, N$  plus an additional infinitely degenerate eigenvalue 0. Writing  $\Phi = \Phi_N + \Phi_N^\perp$  where  $\Phi_N = \sum_{j=1}^N c_j \phi_j$  and  $\Phi_N^\perp$  is orthogonal to  $\Phi_N$ , we obtain  $\langle \Phi | A_N \Phi \rangle = \sum_{j=1}^N |c_j|^2 a_j$ .

The question now is whether there is an evolution group  $U$  which can effect a spectral separation of the initial state  $\Phi$  into its components  $U_t \phi_1, \dots, U_t \phi_N, U_t \Phi_N^\perp$ . The answer is given by the following result.

**Theorem 1.** (K.K. Wan and R.G. McLean, J. Phys. **24A** (1991) L425) *Let  $\{\psi_j | 1 \leq j \leq K\}$  be a finite orthonormal set in  $L^2(\mathbb{R}^3)$  and let  $\{\Delta_j | 1 \leq j \leq K\}$  be a set of disjoint regions in  $\mathbb{R}^3$ . Then there is an evolution group  $U$  on  $L^2(\mathbb{R}^3)$  such that*

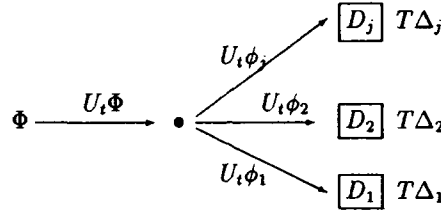
1.  $\lim_{t \rightarrow \infty} \langle U_t \psi_j | \chi_{t\Delta_j} U_t \psi_j \rangle = 1$ , i.e.  $U_t \psi_j$  evolves into  $t\Delta_j$  asymptotically;
2. every element of  $L^2(\mathbb{R}^3)$  is a scattering state of  $U$ ;
3. the wave operator  $\Omega^+ = s\text{-}\lim_{t \rightarrow \infty} U_t^* U_t^0$  exists and is unitary, where  $U^0$  is the free evolution group.

With  $K = N + 1$ ,  $\psi_j = \phi_j$ ,  $\psi_{N+1} = \Phi_N^\perp$  the theorem ensures the desired spectral separation, i.e. the existence of an evolution group  $U$  such that

$$\lim_{t \rightarrow \infty} \langle U_t \phi_j | \chi_{t\Delta_j} U_t \phi_j \rangle = 1 \quad (1 \leq j \leq N)$$

for some disjoint regions  $\Delta_1, \dots, \Delta_N$ .

We can now obtain each  $|c_j|^2$  by a local position measurement at some large time  $T$  using a detector sited in the region  $T\Delta_j$ ; these detectors  $D_j$  measure the local position observables  $\chi_{T\Delta_j}$ . Intuitively we have:



Mathematically we have the following consequence of Theorem 1.

**Corollary.** 
$$\lim_{t \rightarrow \infty} \langle U_t \Phi | \chi_{t\Delta_j} U_t \Phi \rangle = |c_j|^2.$$

It follows that the probabilities  $|c_j|^2$  ( $1 \leq j \leq N$ ) are obtainable by measuring the local position observable  $\chi_{T\Delta_j}$  at time  $T$ . Hence we can obtain

$$\langle \Phi | A_N \Phi \rangle = \sum_{j=1}^N |c_j|^2 a_j.$$

**Conclusion.** *We have indeed a unitary evolution process which reduces a general measurement to local position measurements.*

## 2. State Preparation.

We want to prepare an arbitrarily chosen state  $\phi$  by a unitary evolution process without a sudden wavepacket reduction. This can be achieved by the following.

**Theorem 2.** (K.K. Wan and R.G. McLean, preprint, 1991) *Given any unit vector  $\phi$  in  $L^2(\mathbb{R}^3)$  there is a selfadjoint operator  $H$  with the following properties:*

1.  $H\phi = 0$  i.e.  $\phi$  is an eigenvector of  $A$  corresponding to the eigenvalue 0;
2. every vector orthogonal to  $\phi$  is a scattering state of  $H$ ;
3. the wave operator  $\Omega_+ = s\text{-}\lim_{t \rightarrow \infty} U_t^* U_t^0$  exists, where  $U$  is the evolution group generated by  $H$ , i.e.  $U_t = e^{-itH/\hbar}$  ( $t \in \mathbb{R}$ ).

First consider the case of an ideal particle source, i.e. suppose that we are given a source which produces particles in a definite but unknown pure state  $\Phi$  at the rate of one particle per unit time. We want to "process" these particles to produce some particles in the desired state  $\phi$ . Let  $\Phi = a\phi + b\phi^\perp$  where  $\phi^\perp$  is a unit vector orthogonal to  $\phi$ . Theorem 2 then implies the existence of an evolution group  $U$  under which  $\phi$  is a bound state and  $U_t\phi = \phi$  for all  $t$ , and  $\phi^\perp$  is a scattering state (W.O. Amrein, *Non-Relativistic Quantum Dynamics*, Reidel, Dordrecht, 1981).

Now for any bounded operator  $B$  on  $L^2(\mathbb{R}^3)$ ,  $\lim_{t \rightarrow \infty} \langle \phi | BU_t \phi^\perp \rangle = 0$  (K.K. Wan and F.E. Harrison, preprint, 1991). So at large times the scattering state  $\phi_t^\perp$  is not correlated to the bound state  $\phi$  by any observables and the linear combination

$$U_t \Phi = a\phi + bU_t \phi^\perp$$

becomes *indistinguishable* from the mixture represented by the density operator

$$\rho_t = a^* a P_\phi + b^* b P_{U_t \phi^\perp}.$$

That is

$$\lim_{t \rightarrow \infty} \langle U_t \Phi | BU_t \Phi \rangle = \lim_{t \rightarrow \infty} \text{Tr}(B\rho_t)$$

for all bounded operators  $B$ . In other words at large times  $T$  we have a mixture of the desired state  $\phi$  and the state  $U_T \phi^\perp$  which is far away.

We can make everything precise by introducing the concepts of states at infinity (K.K. Wan and R.G. McLean, preprint, 1991) and of asymptotic superselection rules (K.K. Wan and F.E. Harrison, preprint, 1991).

Now consider the case of a random particle source, i.e. suppose that the particle source produces particles in various states in some random manner, e.g. these various states form a dense set in  $L^2(\mathbb{R}^3)$  and they appear with certain probabilities. Then the same evolution group  $U$  again produces particles in the desired state  $\phi$ . This is a useful result since in practice a particle source is likely to be a random source.

Let us now consider spin. The traditional Stern-Gerlach set-up does not lead to spin measurement with arbitrary accuracy because the set-up cannot eliminate the overlap of the spin-up and spin-down beams even asymptotically (P. Busch and F.E. Schroeck Jr., *Found. Phys.* **19** (1989) 807).

*Question:* Can spin be accurately measured at all, or equivalently, can the spin-up and spin-down components be asymptotically separated by some interaction?

*Answer:* Our present analysis can be extended to prove the existence of an evolution group  $U^s$  acting on the Hilbert space  $\mathcal{H}^s = L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$  which can produce the desired spectral separation, i.e. to separate the spin-up and the spin-down components asymptotically. This should allow the measurements of spin with arbitrary accuracy. Our present analysis can also be extended to allow the preparation of definite spin states (K.K. Wan and R.G. McLean, preprint, 1991).

### 3. Final Remarks.

In practice we often use the *waiting technique* to prepare the ground state of a system, i.e. we simply wait for the system to settle down into the ground state. This is a method of preparing a state asymptotically in time. L.E. Ballentine (*Quantum Mechanics*, Prentice-Hall, 1990) discusses this method and the type of potential required to prepare certain states. Our analysis is a general mathematical formulation of an asymptotic method for measurements and for state preparation.

## Wigner-Araki-Yanase Theorem for Continuous Observables

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### 1. Introduction

In [5], Wigner posed a claim that observables which do not commute with the additive conserved quantities cannot be measured precisely. Araki and Yanase [1] gave a proof of this claim, later called the Wigner-Araki-Yanase theorem, for measurements of discrete observables and bounded conserved quantities, and showed that, if the conserved quantity  $L_1$  is of finite rank, an observable  $A$  not commuting with  $L_1$  can be measured approximately. A little later, Yanase [8] obtained, by macroscopic approximation, an upper limit for the accuracy of the measurement of the spin component in terms of the size of the apparatus. Since then, there has been continuing researches on this subject, and the following statements are currently supported in the literature: No observable which does not commute with the additive conserved quantities (such as linear or angular momentum or electric charge) can be measured precisely, and in order to increase the accuracy of the measurement one has to use a very large measuring apparatus [7, p. 14]. However, no rigorous approach to this subject has been advanced much, and the following problems still remain unanswered; 1) to prove the Wigner-Araki-Yanase theorem for continuous observables; 2) to determine if the theorem holds for unbounded conserved quantities; 3) to obtain the bound for the accuracy of the spin measurement without any macroscopic approximation. In order to solve these problems, in this paper we shall introduce a new rigorous approach based on the analysis of commutation relations obeyed by the noise operators, and give rigorous results for the above problems.

### 2. Noise commutation relations

Let  $\mathcal{H}_1$  be the Hilbert space of an object and  $\mathcal{H}_2$  the Hilbert space of the measuring apparatus. Let  $A$  be the observable (self-adjoint operator) of the object, which is to be measured by the apparatus, and  $B$  the observable in the apparatus actually measured by the observer. Let  $U$  be the unitary operator on  $\mathcal{H}_1 \otimes \mathcal{H}_2$  representing the time evolution of the object-plus-apparatus during the measurement. Suppose

that there is a pair of observables  $L_1$  in  $\mathcal{H}_1$  and  $L_2$  in  $\mathcal{H}_2$  representing an additive conserved quantity, i.e.,

$$[U, L_1 \otimes 1 + 1 \otimes L_2] = 0.$$

In order to generalize the Wigner-Araki-Yanase theorem to measurements of continuous observables, it is convenient to discuss the problem in the Heisenberg picture. The noise operators  $N_1$  and  $N_2$  are then defined by

$$N_1 = U^\dagger(1 \otimes B)U - A \otimes 1,$$

$$N_2 = U^\dagger(A \otimes 1)U - A \otimes 1.$$

The following theorem gives fundamental commutation relations obeyed by the noise operators.

**Theorem 1.** *We have the following relations:*

$$[N_1, U^\dagger(L_1 \otimes 1)U] + [N_2, U^\dagger(1 \otimes L_2)U] = [L_1, A] \otimes 1, \quad (1)$$

$$[N_1, L_1 \otimes 1 + 1 \otimes L_2] = [L_1, A] \otimes 1 - U^\dagger(1 \otimes [L_2, B])U, \quad (2)$$

$$[N_2, L_1 \otimes 1 + 1 \otimes L_2] = [L_1, A] \otimes 1 - U^\dagger([L_1, A] \otimes 1)U, \quad (3)$$

$$\begin{aligned} [N_1, U^\dagger(1 \otimes L_2)U] + [N_2, U^\dagger(L_1 \otimes 1)U] \\ = [L_1, A] \otimes 1 - U^\dagger([L_1, A] \otimes 1 + 1 \otimes [L_2, B])U. \end{aligned} \quad (4)$$

### 3. A generalized Wigner-Araki-Yanase theorem

We say that the unitary  $U$  gives an *exact* measurement of  $A$  with meter  $B$ , if for any  $\varepsilon > 0$  there is some  $\xi \in \mathcal{H}_1$  with  $\|\xi\| = 1$  such that for any  $\psi \in \mathcal{H}_2$  with  $\|\psi\| = 1$  satisfying

$$\langle \psi \otimes \xi | N_1 | \psi \otimes \xi \rangle = 0, \quad (5)$$

$$\langle \psi \otimes \xi | N_2 | \psi \otimes \xi \rangle = 0, \quad (6)$$

$$\|N_1(\psi \otimes \xi)\| < \varepsilon, \quad (7)$$

$$\|N_2(\psi \otimes \xi)\| < \varepsilon. \quad (8)$$

It is easily seen that the above conditions are rigorous requirements for that, when the apparatus is prepared in the absolute precision before the measurement, the measurement represented by  $U$  satisfies the statistical formula for the outcomes of the measurement [4, pp. 200–201], and the repeatability hypothesis [4, p. 335].

The following theorem generalizes the Wigner-Araki-Yanase theorem to the case where the observable has a continuous spectrum.



**Theorem 2.** *If  $[A, L_1] \neq 0$ , and if  $L_1$  and  $L_2$  are bounded, then  $A$  cannot be measured exactly.*

*Proof.* Suppose that  $A$  can be measured exactly. From Eq. (1) and the Schwarz inequality,

$$\begin{aligned} |\langle \psi | [A, L_1] | \psi \rangle| &\leq 2(\|N_1(\psi \otimes \xi)\| \|L_1\| + \|N_2(\psi \otimes \xi)\| \|L_2\|) \\ &< 2\varepsilon(\|L_1\| + \|L_2\|). \end{aligned}$$

Since  $\varepsilon$  and  $\psi$  are arbitrary, we have  $[A, L_1] = 0$ .  $\square$

Contrary to the standard understanding of the Wigner-Araki-Yanase theorem, the following theorem shows in particular that the momentum conservation law does not prevent exact position measurements.

**Theorem 3.** *If  $[A, L_1]$  is a c-number, then  $A$  can be measured exactly with  $B$  such that  $[B, L_2] = 0$ .*

*Proof.* By uniqueness of representations of the CCR, it suffices to consider the case where the object is a one-dimensional mass with position  $x$  and momentum  $p_x$  and  $A = \hat{x}$ ,  $L_1 = \hat{p}_x$ . For this case, a counter example is obtained in [2].  $\square$

#### 4. Bound for the accuracy of spin measurements

Now we shall consider the case where  $[B, L_2] = 0$ . In this case the measurement of  $B$  is not limited by the additive conservation law of  $L_2$ .

**Theorem 4.** *If  $[B, L_2] = 0$ , we have, for any  $\psi \in \mathcal{H}_1$  and  $\xi \in \mathcal{H}_2$  with  $\|\xi\| = 1$ ,*

$$|\langle \psi | [L_1, A] | \psi \rangle| \leq 2\|N_1(\psi \otimes \xi)\| \|\Delta L_1 \psi \otimes \xi + \psi \otimes \Delta L_2 \xi\|.$$

*Proof.* From Eq. (2) and the Schwarz inequality,

$$\begin{aligned} |\langle \psi | [L_1, A] | \psi \rangle| &= |(\psi \otimes \xi | [N_1, L_1 \otimes 1 + 1 \otimes L_2] | \psi \otimes \xi)| \\ &\leq 2\|N_1(\psi \otimes \xi)\| \|\Delta L_1 \psi \otimes \xi + \psi \otimes \Delta L_2 \xi\|. \end{aligned}$$

$\square$

For  $w = x, y, z$ , let  $\hat{s}_w$  be the spin observable in the  $w$ -direction of an object with spin  $1/2$ ,  $\alpha_w$  the positive spin state,  $\beta_w$  the negative spin state, and  $\hat{L}_w$  the angular momentum of the apparatus in the  $w$ -direction. Yanase [8] considered the following measuring interaction  $U$  of a measurement of the  $x$ -component of the spin,

$$\begin{aligned} U(\alpha_x \otimes \xi) &= \alpha_x \otimes X + \beta_x \otimes \eta, \\ U(\beta_x \otimes \xi) &= \beta_x \otimes X' + \alpha_x \otimes \eta'. \end{aligned}$$

where  $\xi$  is the initial state of the apparatus, and  $X$ , and  $X'$  are eigenvectors of  $B$  for different eigenvalues. In this case,  $A = s_x$ , and we shall consider the additive conservation law of the  $z$ -component of the angular momentum, i.e.,

$$[U, \hat{s}_z + \hat{L}_z] = 0.$$

We suppose that  $B$  commutes with  $\hat{L}_z$ , and define the total error probability of this measuring apparatus by

$$\epsilon = \|\eta\|^2 + \|\eta'\|^2.$$

Then the following relation is obtained by applying Theorem 4 to  $\psi = \alpha_y$  (cf. [3]):

$$\epsilon \geq \frac{\hbar^2}{2\hbar^2 + 8(\Delta L_z)^2}.$$

This relation coincides with the bound obtained by Yanase [8]

$$\epsilon > \frac{\hbar^2}{8\langle L_z^2 \rangle} + O(\langle L_z^2 \rangle^{-3/2}),$$

in the macroscopic case where  $\hbar^2 \ll \Delta L_z^2 \approx \langle L_z^2 \rangle$ , and coincides with the bound obtained by Wigner [6]

$$\epsilon > \frac{\hbar^2}{2\langle L_z^2 \rangle},$$

in case where  $\hbar^2 \ll \Delta L_z^2 \approx \langle L_z^2 \rangle/4$ , the latter relation of which follows for instance from that  $L_z$  is uniformly distributed in the positive part of the spectrum.

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## Variations on a Theme by E.P.Wigner

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The problem of quantizing a simple harmonic oscillator is examined on the basis of some general formulae. All known, as well as new, methods of quantization are incorporated into our framework. Remarks are also made on some related problems.

### 1. Theme

Professor Wigner's original theme on which our following discussions will be based is the one presented in his 1950 paper<sup>1)</sup>, titled "Do the Equations of Motion Determine the Quantum Mechanical Commutation Relations?" For the case of a simple harmonic oscillator in which we shall hereafter be interested his question can be stated more explicitly as: whether the canonical commutation relation between the coordinate  $Q$  and its conjugate momentum  $P \equiv \dot{Q}$ , that is,

$$[Q, P] = i \quad (1)$$

is the unique possibility of quantization, when both the classical equation of motion

$$\ddot{Q} + Q = 0, \quad (2)$$

and the classical expression for the Hamiltonian

$$H = \frac{1}{2}(P^2 + Q^2) \quad (3)$$

are to be preserved after quantization, where our units are such that  $m = \omega = \hbar = 1$ . In addition to these the Heisenberg equation of motion, of course, is taken for granted

$$i\dot{A} = [A, H] \quad (4)$$

for any operator  $A$ .

Now, the answer given by Wigner himself to the above question is that there exists an infinite family of possibilities, parametrized by  $E_0 \geq 0$ , the minimum eigenvalue of  $H$ , and that the canonical quantization corresponds to the simplest nontrivial case  $E_0 = 1/2$ .

In the present paper we shall try, so to speak, a number of variations on this theme by Wigner, and supplement them with some remarks on related problems.

### 2. Formulation of the problem

Instead of  $Q$  and  $P$  we employ three operators  $a, a^\dagger$  and  $N \equiv H$  such that

$$a \equiv \frac{1}{\sqrt{2}}(Q + iP), \quad a^\dagger \equiv \frac{1}{\sqrt{2}}(Q - iP), \quad N = N^\dagger. \quad (5)$$

Let us then assume that

$$\begin{aligned} (I) \quad & [a, N] = a, \quad [a^\dagger, N] = -a^\dagger; \\ (II) \quad & N = F([a^\dagger, a]_\alpha) \quad \text{or} \quad [a^\dagger, a]_\alpha = F^{-1}(N) = G(N); \end{aligned} \quad (6) \quad (7)$$

where  $[A, B]_\alpha \equiv AB + \alpha BA$  with  $\alpha \neq 0$ , and  $G(x)$  is a real function of  $x$ , and (III) that there exists at least one representation in which the spectrum of  $N$  is bounded below 2).

Since  $H = N$  as noted above, (4), (5) and (6) lead us to (2). On the other hand, (I) and (III) imply that the  $n$ -th eigenvalue  $N_n$  of  $N$  is of the form:  $N_n = N_0 + n$  with  $n = 0, 1, 2, \dots$  (hence  $E_0 = N_0$ ). Thus,  $a, a^\dagger$  and  $(N - N_0)$  play the roles of annihilation, creation and occupation number of energy-quanta, respectively. Further, the use of (7) enables us to see that the eigenstate  $|\tilde{N}_n\rangle \equiv |n\rangle$  of  $N$  with eigenvalue  $N_n$  is non-degenerate, provided  $|0\rangle$  is taken to be so. All matrix elements of  $a$  and  $a^\dagger$  are then fixed to be

$$\langle n|a|n+1\rangle = \langle n+1|a^\dagger|n\rangle = \sqrt{I(n)}, \quad (8)$$

$$I(n) \equiv \|a^\dagger|n\rangle\|^2 = \sum_{m=0}^n (-1)^m \alpha^{-(m+1)} G(n-m) \geq 0, \quad (9)$$

where  $G(n) \equiv G(N_n)$ .

Quantization in our formulation is thus completely specified by  $\alpha, G$  and  $N_0$ . The condition (9) determines whether  $n$  has no maximal value (Bose-like) or has the maximal value  $n_{max} < \infty$  (Fermi-like).

### 3. Variations

By suitably choosing  $\alpha, G$  and  $N_0$  we can accommodate, in our framework, various methods of quantization so far proposed, and moreover invent new ones.

*Theme: Wigner quantization (Bose-like,  $N_0 > 0$ ).* Wigner's results<sup>1)</sup> are reproduced by putting  $\alpha = 1, G(N) = 2N$ , whence  $I(n) = n + 2N_0 (= n + 1)$  for  $n = \text{even(odd)}$ . Bose (or canonical) quantization ( $N_0 = 1/2$ ) and para-Bose quantization<sup>3)</sup> ( $N_0 = p/2, p = 1, 2, 3, \dots$ ) are special cases thereof.

*Var. 1: para-Fermi quantization<sup>3)</sup> (Fermi-like,  $N_0 = -p/2 = -n_{max}$ ).*  $\alpha = -1, G(N) = 2N$ , whence  $I(n) = -(n + 2N_0)(n + 1)$ . Note that no other values than  $N_0 = -p/2$  are permissible here. Fermi quantization ( $p = 1$ ) is a special case.

*Var. 2:  $q$ -deformed quantization (Bose-like,  $q > 0, N_0$  real).* The choice<sup>4)</sup>  $\alpha = -q^{-1}, G(N) = -q^{-(N+1)}$  leads to  $I(n) = [n+1]_q q^{-N_0}$ , where  $[x]_q \equiv (q^x - q^{-x})/(q - q^{-1})$ . On the other hand, the choice<sup>5)</sup>  $\alpha = -1, G(N) = [N]_q - [N+1]_q$  leads to  $I(n) = [n+1 + N_0]_q - [N_0]_q$ . Note that the above two choices are equivalent only for  $N_0 = 0$  (contrary

to the statement often made in the literature).

*Var.3: q-deformed Wigner quantization (Bose-like,  $q > 0, N_0 > 0$ ).* To generalize the original theme we put  $\alpha = 1, G(N) = [2N]_q$ . Then  $I(n) = [n + 2N_0]_q \{n + 1\}_q (= \{n + 2N_0\}_q [n + 1]_q)$  for  $n = \text{even(odd)}$ , where  $\{x\}_q \equiv (q^x + q^{-x}) / (q + q^{-1})$ .

*Var.4: q-deformed para-Fermi quantization (Fermi-like,  $q > 0, N_0 = -p/2$ ).* To generalize Var. 1 we put  $\alpha = -1, G(N) = [2N]_q$ . Then  $I(n) = [-(n + 2N_0)]_q [n + 1]_q$ . Variations similar to Var. 3 and Var. 4 were also discussed by other authors<sup>6</sup>.

*Var.5: O'Raifeartaigh-Ryan-Gruber quantization<sup>7</sup> (Bose-like,  $N_0 > 0$ ).* In this case  $\alpha > 0, G(N) = N$ . The expression for  $I(n)$  is easily found from (9).

*Var.6: T-D cut-off deformation (Bose-like,  $q > 1, N_0$  real).* The choice  $\alpha = -q, G(N) = -q^{-N}$  leads to  $I(n) = (n + 1)q^{-(N_0 + n + 1)}$ . Since  $I(n) \rightarrow 0$  as  $n \rightarrow \infty$ , the mechanism of the so-called Tamm-Dancoff cut-off is built in here.

*Var.7: Greenberg's q-mutator<sup>8</sup> (Bose-like,  $-1 < q < 1, N_0 = 0$ ).* Having  $\alpha = G(N) = -q^{-1}$ , this is an exceptional case in that (7) no longer defines  $N$ , but serves instead as a commutation relation. By using this relation, however, we can determine  $N$  in such a way that (6) holds true, the result being

$$N = \sum_{n=0}^{\infty} \frac{(1-q)^n}{(1-q^n)} (a^\dagger)^n a^n. \quad (10)$$

#### 4. Coda: Supplementary remarks

(a) It is possible to express general operators  $a$  and  $a^\dagger$  in terms of the ordinary Bose operators  $b, b^\dagger$  and  $N_b \equiv b^\dagger b$  together with  $N_0$ :

$$\begin{aligned} a &\equiv U b, & a^\dagger &\equiv b^\dagger U^\dagger; \\ U &\equiv \sum_{n=0}^{n_{\max}} \frac{1}{n!} \sqrt{\frac{I(n)}{n+1}} (b^\dagger)^n \frac{\sin(\pi N_b)}{\pi N_b} b^n. \end{aligned} \quad (11)$$

(b) It is not difficult either to construct a single coherent state for the operator  $a$ . In the Bose-like case, for example, it is given as

$$|\xi\rangle = [\text{Exp}(\xi^* \xi)]^{-1/2} \text{Exp}(\xi a^\dagger) |0\rangle, \quad (12)$$

where  $\text{Exp}(x) \equiv \sum_{n=0}^{\infty} x^n / \langle n \rangle!$ ,  $\langle n \rangle! \equiv \langle 1 \rangle \langle 2 \rangle \cdots \langle n \rangle$ ,  $\langle n \rangle \equiv I(n-1)$  and  $\xi$  is taken to be a complex number. When a set of coherent states are concerned, corresponding to different eigenvalues  $\xi, \xi', \dots$ , a question arises, however, as to what the most convenient way is to express  $\xi$ 's. It may be necessary to introduce new types of numbers more general than e.g. para-Grassmann numbers,  $g$ -numbers<sup>9</sup>,

(c) In order to deal with (4) for the case of a general oscillator in self-interaction we need the (anti-) commutator  $[a, a^\dagger]_{\mp} \equiv J^{(\mp)}$ , where  $\langle n | J^{(\mp)} | n \rangle \equiv I(n) \mp I(n-1)$ . Owing to the appearance of  $J^{(\mp)}$ , the Heisenberg equation for  $a$  takes, in general, a different and more nonlinear form than the corresponding classical (Hamilton) equation of motion.

(d) We would like to make a corresponding generalization of the commutation relations for the operators  $a_k, a_k^\dagger$  and  $N_k (k = 1, 2, \dots, f)$  of many oscillator system. Certainly, the naive choice  $[a_k, a_l]_{\mp} = 0$ , etc. ( $k \neq l$ ) is mathematically possible, but physically of very limited applicability. We need, therefore, a more sophisticated, possibly (quantum) group theoretical way of generalization, such as found useful in the case of parastatistics<sup>9)</sup>. If we want to apply the resulting commutation relations to field theory, however, a further condition has to be imposed on them: that is, those relations be invariant under unitary transformations of the operators  $a_k \rightarrow a'_k = \sum_l c_{kl} a_l$ . If such is not the case, the statistics to be obeyed by field quanta, which is a consequence of field quantization, will no longer be a general, but merely state-dependent, property.

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## The Action Uncertainty Principle in Continuous Quantum Measurements

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The path-integral approach to quantum theory of continuous measurements developed in preceding works of the author is applied to derive a simple estimation for variance of the measurement outputs. This estimation is given in the form of an inequality containing the action functional  $S$  and called the action uncertainty principle (AUP). The most general form of AUP is  $\delta S \gtrsim \hbar$ . It can also be written (for ideal measurements performed in the quantum regime) symbolically as  $\Delta(\text{Equation})\Delta(\text{Path}) \simeq \hbar$ . This means that deviation of the observed (measured) motion from the classical motion is inversely proportional to the uncertainty in a path (resulting from the measurement error). The consequence of AUP is that improving the measurement precision beyond the threshold of the quantum regime leads to decreasing information resulting from the measurement.

### 1 Introduction

The question of measurements was one of the central conceptual problems of quantum mechanics from the very beginning and became recently even more burning (see for example [1] and references therein). An important example of a complicated regime of measurement is continuous (prolonged in time) measurement which attracted much attention in connection with the so-called quantum Zeno effect (paradox) [2]. This effect consists in that the state of a quantum system "is frozen" (its change is prevented) if it is observed (measured) continuously.

As a result of the quantum Zeno effect a continuous measurement should give the trivial output (a constant function). However this is valid only for precise measurement of an observable having a discrete spectrum. If an observable having a continuous spectrum is measured approximately (an unsharp measurement), then the continuous measurement leads to a nontrivial output (expressed by a function of time). It is this type of continuous measurement that will be investigated in the present paper in the framework of the path-integral approach.

The formalism of Feynman path integrals was applied to continuous quantum measurements in [3, 4] and proved to be natural and efficient. A probability amplitude  $A_\alpha$  for each output  $\alpha$  of a continuous measurement can be presented in the form of a path integral. The variance  $\delta\alpha$  of probable outputs around the most probable one  $\alpha_{\text{class}}$  is one of the most important conclusions of the theory. The objective of the present paper is expression of this variance in a simple form called the action uncertainty principle (AUP).

The simplest form of AUP is  $\delta S \gtrsim \hbar$  and more complicated form valid for ideal measurements in the quantum regime is  $|\int_{q'}^{q''} \frac{\delta S[q]}{\delta q(t)} \Delta q(t) dt| \simeq \hbar$ .

## 2 Paths and Continuous Measurements

The amplitude  $A(q'', q')$  for a quantum system to go over from the point  $q'$  (of configuration space) to the point  $q''$  was expressed by Feynman [5] in the form of an integral of the amplitudes  $A[q] = \exp(\frac{i}{\hbar} S[q])$  corresponding to all possible paths  $[q]$  connecting  $q'$  and  $q''$ . This is however valid only if there is no possibility to find out what path is realized as a channel for the transition.

Let however suppose that a *continuous measurement* is performed simultaneously with this transition. Then this measurement gives some information about the path. This information can be expressed by some set of paths  $\alpha$ . If the measurement gives the result (output)  $\alpha$  then the transition is realized through one of the paths  $[q]$  belonging to the set  $\alpha$ . Then the amplitude for transition from  $q'$  to  $q''$  can be expressed [3] as an integral over paths belonging to  $\alpha$ :

$$A_\alpha(q'', q') = \int_\alpha e^{i S[q]/\hbar} d[q]. \quad (1)$$

If  $q', q''$  are fixed, the amplitude  $A_\alpha$  can be thought of as the probability amplitude for the continuous measurement to give the result  $\alpha$ . Taking a square modulus  $|A_\alpha|^2$  of the amplitude one can obtain the probability density for different results of the continuous measurement.

## 3 Classical Regime of Measurement

The natural task arises to obtain restrictions of the type of the uncertainty principle but for continuous measurements. This can be achieved by analyzing Eq. (1).

The value  $A_\alpha$  can be interpreted as an amplitude for the measurement to give the result  $\alpha$ . This means that only those measurement results are probable for which  $A_\alpha$  is large in absolute value. We shall try to estimate this value analyzing behavior of the action functional  $S[q]$ . In what follows we shall suggest that all sets  $\alpha$  have the same "width" differing only by their forms. Then the value of  $A_\alpha$  depends only on how rapidly  $S[q]$  varies along  $\alpha$ .

It is known that only paths close in a sense to the classical trajectory  $[q_{\text{class}}]$  contribute the unrestricted Feynman integral. This is because the action functional  $S[q]$  changes slowly in the vicinity of  $[q_{\text{class}}]$  and all partial amplitudes  $A[q] = \exp(\frac{i}{\hbar} S[q])$  have close phases in this vicinity giving "constructive interference".

Constructive interference maintains in the limits of the set

$$I_{\text{class}} = \{[q] \mid |S[q] - S[q_{\text{class}}]| \lesssim \hbar\}$$

and changes into "destructive interference" outside this set when oscillation becomes rapid. Therefore, main contribution into the Feynman path integral is given by the set of paths  $I_{\text{class}}$ .

It is evident (by analogous argument) that the measurement amplitude (1) has maximum value for  $\alpha = \alpha_{\text{class}}$  where the set  $\alpha_{\text{class}}$  contains  $[q_{\text{class}}]$  as its middle path.



Now classical regime of measurement can be defined by the requirement that  $\alpha_{\text{class}} \supset I_{\text{class}}$ . In this case the integral  $A_{\alpha_{\text{class}}}$  is practically equal to complete Feynman integral. It is evident that  $A_{\alpha}$  remains the same for  $\alpha \supset I_{\text{class}}$ . However  $A_{\alpha}$  decays when an intersection  $\alpha \cap I_{\text{class}}$  grows less tending to zero for  $\alpha \cap I_{\text{class}} = \emptyset$ .

From the argument of the preceding paragraph one can see that the probable outputs of the measurement performed in classical regime may be characterized by the condition  $[q_{\text{class}}] \in \alpha$  in agreement with classical theory.

## 4 Quantum Regime and AUP

Consider now quantum regime of measurement when no set  $\alpha$  includes  $I_{\text{class}}$ . The most probable in this case is the output corresponding to the set  $\alpha_{\text{class}}$  lying in the middle of  $I_{\text{class}}$ . However other  $\alpha$  are probable too provided that variation of the action in the limits of this set is small enough,  $\Delta S \lesssim \hbar$ . For some of probable outputs  $\alpha$  therefore variation of  $S$  is  $\Delta S \simeq \hbar$ . Taking all probable (emerging with high probability) sets  $\alpha$  and estimating variation  $\delta S$  of  $S$  for all of them we obtain for this variation  $\delta S \gtrsim \hbar$ .

This conclusion is made for the quantum regime of measurement. However in the classical regime  $\Delta S \gtrsim \hbar$ . We conclude thus that for the measurement outputs arising with comparatively large probability the variance  $\delta S$  of  $S$  satisfies the following inequality:

$$\delta S \gtrsim \hbar. \quad (2)$$

This is the simplest form of the *action uncertainty principle*, AUP.

In the case of linear system different components of some linear decomposition of the system motion can be analyzed separately. For a given measurement some of these components may turn out to be measured in classical regime while other components are measured (by the same measurement) in quantum regime. The analysis based upon such a decomposition gives the following form of AUP:

$$\left| \int_{\nu'}^{\nu''} \frac{\delta S[q]}{\delta q(t)} \Delta q(t) dt \right| \lesssim \hbar. \quad (3)$$

Here  $[q]$  is the middle path of the set of paths  $\alpha$ , and  $[\Delta q]$  is a deviation of  $[q]$  which does not drive it from  $\alpha$  so that  $[q + \Delta q] \in \alpha$  too. If the inequality (3) is fulfilled for a given set  $\alpha$  and for any deflection  $[\Delta q]$  not driving out from  $\alpha$ , then the corresponding measurement output  $\alpha$  emerges with high probability.

Since Eq. (3) is a condition for an output  $\alpha$  to be probable, there are some probable outputs  $\alpha$  for which the l.h.s. of Eq. (3) is of the order of  $\hbar$ . Thus

$$\left| \int_{\nu'}^{\nu''} \frac{\delta S[q]}{\delta q(t)} \Delta q(t) dt \right| \simeq \hbar \quad (4)$$

is valid for typical probable outputs of measurement performed in the quantum regime. The inequality (3) and the equality (4) present another form of the action uncertainty principle, AUP.

The entity  $\delta S[q]/\delta q(t)$  is nothing else than the left-hand side of the classical equation of motion,  $\delta S[q]/\delta q(t) = 0$ . This is why Eq. (4) can be symbolically rewritten as follows:

$$\Delta(\text{Equation})\Delta(\text{Path}) \simeq \hbar.$$

This means that deviation from classical picture has to be small for rough measurements (when  $\Delta(\text{Path})$  is large) but it can be large for fine measurements (with small  $\Delta(\text{Path})$ ).

The qualitative difference of AUP from the Heisenberg uncertainty principle is that the first one deals with variance of outputs of a single measurement, not of two measurements (of complementary variables) as in the second one. This of course is a direct consequence of this single measurement being continuous, thus containing in itself information about both complementary observables.

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## Quantum theory and causality

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" . . . quantum mechanics and general relativity theory are widely separated both in applications and in concepts and this is a very significant weakness of present day physics."

(E. P. Wigner)

The strange spacetime behavior of quantum mechanical systems in an EPR type experiment (or that of a single neutron in a Rauch interferometer[5]) leads to investigation: What is the correct relation of the spacetime causality to the basic ideas of quantum theory? According to the standard interpretation of the violation of Bell inequalities we have to make a dramatic revision of our conception of space and time[7]. I suggested such a new theory of the structure of spacetime, intrinsically based on quantum theory[16]. The primary object of the current formulation of general relativity is an "underlying set" on which the (causal, topological, geometrical) structure of spacetime is defined. The elements of this set are called "events". The meaning of an event is rather vague, especially if we want to abstain from tautology: defining an event with reference to the spacetime structure itself.

In quantum theory an event means a possible result of a possible measurement or observation performed on a physical object. I consider the set of all physical events belonging to the whole universe, rather than to a separate physical system.

This set of events has an immanent "logical" structure, a non-Boolean orthomodular lattice structure on which the quantum-mechanical probability theory can be defined. *The (quantum) spacetime is a structure defined over the quantum lattice of physical events.*

One can make a longer justification of the assumption that the elements of the quantum lattice play the same role as the subsets of the (classical) spacetime. In general, a physical event corresponds to a subset of spacetime[16].

A *quantum causal structure* is given by the causal and chronological future and past defined as maps

$$J^{\pm} : S \rightarrow S$$

$$I^{\pm} : S \rightarrow S$$

with the following properties:

- Q1.  $A < J^\pm(A)$
- Q2.  $I^\pm(A) < J^\pm(A)$
- Q3.  $J^\pm(J^\pm(A)) = J^\pm(A)$
- Q4.  $J^\pm(A \vee B) = J^\pm(A) \vee J^\pm(B)$
- Q5.  $I^\pm(A \vee B) = I^\pm(A) \vee I^\pm(B)$
- Q6.  $J^\pm(I^\pm(A)) = I^\pm(J^\pm(A)) = I^\pm(A)$
- Q7.  $I^\pm(\mathcal{M}) \neq \emptyset$
- Q8.  $x \notin I^\pm(x)$
- Q9.  $x < J^+(y) \iff y < J^-(x)$
- Q10.  $x < I^+(y) \iff y < I^-(x)$
- Q11.  $J^\pm(x) = J^\pm(y) \implies x = y$

where  $S$  is the dual of the quantum lattice,  $x$  and  $y$  are atoms of  $S$ ,  $A, B \in S$ ,  $\mathcal{M}$  and  $\emptyset$  denote the maximal element and the minimal element of  $S$ .

The causality and chronology relations can be defined as

$$A \prec B \iff B < J^\pm(A) \text{ or } A < J^\pm(B)$$

$$A \ll B \iff B < I^\pm(A) \text{ or } A < I^\pm(B)$$

$A$  and  $B$  are *spatially separated* iff neither  $A \prec B$  nor  $B \prec A$  hold.

These axioms are straightforward non-Boolean generalization of the Kronheimer-Penrose axioms describing an abstract causal space[12]. If  $S$  is a Boolean lattice, it can be represented by a suitable subset lattice and the quantum causality leads to the usual causality of Kronheimer and Penrose on an "underlying set."

The causality is the most important part of the structure of spacetime. On the ground of the causal structure one can introduce further structures. In absence of an underlying set one cannot introduce a topology that could make the quantum spacetime a topological space. Fortunately, the so called T-structure can generalize the notion of topology for a non-Boolean case[17]. As is well known the causal structure on an underlying set singles out a topology called Alexandrov topology that is the coarsest topology in which each chronological future and past is open. One can introduce the *Alexandrov T-structure* in a similar way: as the coarsest T-structure in which each  $I^\pm(A)$  is open.

On the ground of the Alexandrov T-structure one can introduce the covering dimension of quantum spacetime. *Dimension* of a quantum causal structure at a given atom  $x$  is the smallest integer  $d(x)$  such that for each open covering there is a refined open covering with overlap  $d(x) + 1$  at  $x$ .

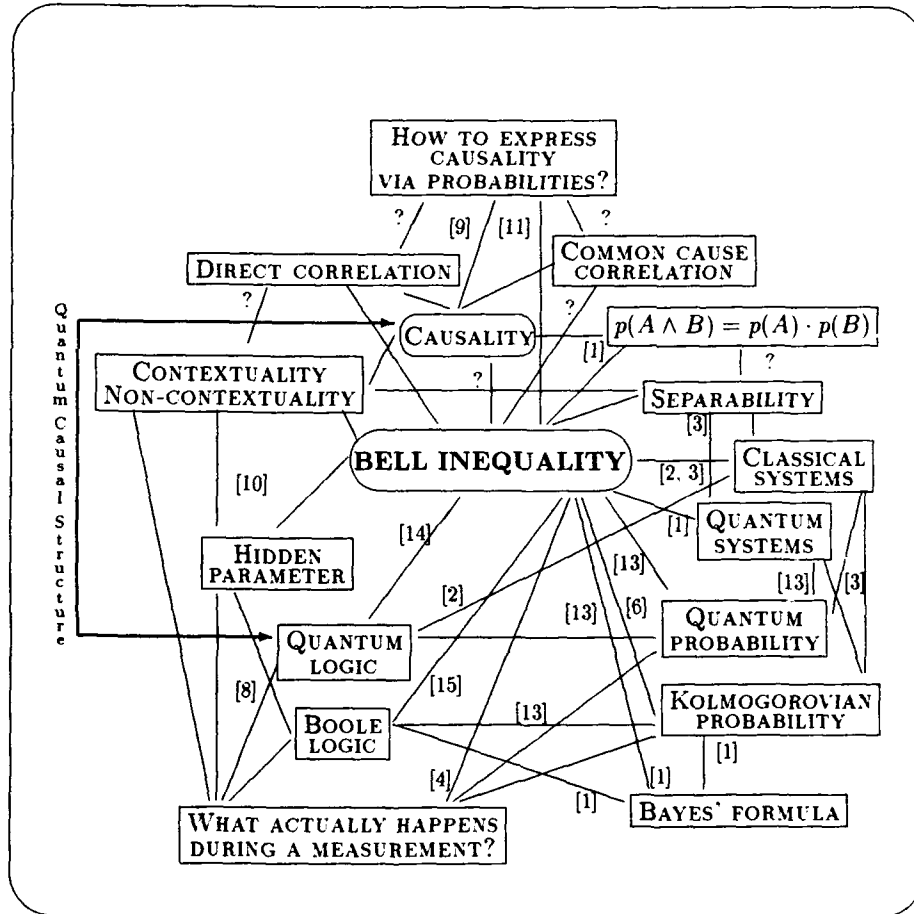


Fig. 1. This scheme shows how many questions are related to the violation of Bell inequalities. It is obvious from this picture that the heuristic stimulating force of the violation of Bell inequalities is much more important than its strict logical consequence. The theory of quantum causal structure connects the proposition calculus of quantum logic to causality. The numbers denote items from the "References" belonging to the corresponding topic.

I also investigated the spin correlation experiments in the light of the quantum causal structures[18]. Exploring all possible quantum causal structures on a model quantum lattice I found the following surprising result. None of them makes the crucial EPR events spatially separated, therefore the causal structure of events does not exclude the possibility of a direct correlation. If the similar result were valid on the whole infinite quantum lattice (this question is open yet) it would mean an interesting resolution of the EPR "paradox".

While some of the most common explanations of the violation of Bell inequalities provide a good background for such speculations, to avoid the wishful thinking I would like to emphasize that the violation of Bell inequalities has nothing to do with the problem of causality. As it was shown[1, 13] it only indicates the fact that quantum theory is a non-Kolmogorovian probability theory defined over a non-Boolean lattice of events. Figure 1. shows a scheme on which I summed up the most important aspects arising with respect to the violation of Bell inequalities. Some of these aspects have been hotly discussed in the literature. The heuristic stimulating force of the Bell theorems seems much more important than the real logical consequences of them.

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## DOES THE COVERING LAW FOLLOW FROM LORENTZ COVARIANCE?

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**ABSTRACT:** *We present arguments that Piron's covering law of quantum logic can be substantiated by lorentz covariance and the assumption that correlations in a physical state cannot be used to send signals between apparatuses testing commuting propositions, in particular across space-like intervals.*

One sideline of the debates about correlated quantum events taking place at space-like separations has been concerning the existence or not of superluminal influences or signals that would in principle mitigate some of the puzzling "non-realist" or "non-local" aspects of these correlations. By now it has been reasonably well established that very general features of ordinary hilbert-space quantum mechanics are sufficient to prevent humanly controlled superluminal communication using such long-range quantum correlations.

Here we ask a reciprocal question. Assuming the "no-signal hypothesis" (NS) of the impossibility of using long-range correlations as a superluminal signaling device, what can one deduce about the structure of physical theories? Assuming special relativity, NS is a very natural hypothesis, for the existence of the proscribed signals would give rise to well known and paradoxical problems of causality and free will.

Our first result in this direction 1 shows that certain deviations from hilbert space are precluded by NS. We consider an EPR apparatus in which a physical system decomposes into two parts which then separate in opposite directions. Focusing on the internal degrees of freedom of each part which we assume to be described by finite-dimensional hilbert spaces of the same dimension which is at least three, and assuming for observables stationed on one arm of the apparatus and for the state vector of the composite system, the usual projection postulate with the usual calculus of probabilities, we prove, essentially by Gleason's theorem, that NS implies that at the other arm of the apparatus: 1) the expected value in a pure quantum vector state  $\Phi$  of any observable must be given by the usual expectation value formula  $(\Phi, A\Phi)$  for some self-adjoint operator  $A$ . 2) any state-transformer (such as time-evolution) must act linearly on density matrices. 3) any state-transformer that transforms pure state into pure states and whose range contains more than one state must be given by either a linear or anti-linear operator on hilbert space.

What this shows is that once a large enough portion of physical phenomena obey the usual quantum mechanical rules, such rules must be universal in order to preclude superluminal communication. Long-range quantum correlations thus have a globalizing effect on the quantum formalism making it structurally unstable.

Now this result is "local" in that it shows that no deviations of certain types from ordinary hilbert-space quantum mechanics, no matter how small, can be allowed if NS is to be maintained. But one is then immediately led to ask the interesting "global" question as to whether NS can be used as an axiom leading to hilbert space itself. This would provide a truly physical basis for hilbert-space quantum theory motivated by



the space-time condition of lorentz covariance. We present some preliminary results in quantum logics that show that such a thesis is plausible. We know from Piron 2 that a generalized hilbert-space model can be constructed if the quantum logic obeys the so-called covering law. It is this law that we shall try to substantiate by space-time considerations.

Although the detection process, which underlines the notion of signals, is not the usual fare for quantum logic, it's fortunate that the literature already contains significant progress in this direction, especially in the papers of the late polish quantum logician Wawrzyniec Guz 3 - 5. Guz presents various axiomatic schemes in which the covering law follows from other physically clearer assumptions. One of these schemes concerns the existence of *generalized filters*, that is, state transformers that generalize the hilbert-space projection postulate to more general quantum logics.

Due to space limitations we can only present here a sketch of the axiom scheme. We use the notation of Reference 5 to which the reader must refer for more details.

A physical system is represented by a pair of sets  $(L, S)$  where  $L$  is an orthomodular poset whose elements are called *propositions* and  $S$  is a  $\sigma$ -convex set of probability measures on  $L$  which are called *states*. A further axiom (A3) asserts the existence of a subset  $P \subseteq S$  of extreme points whose elements correspond to *pure states* and from whose properties one deduces that  $L$  is atomic and atomistic and that if  $A(L)$  is the set of atoms of  $L$ , then there is a bijection  $s : P \rightarrow A(L)$  satisfying  $p(s(p)) = 1$  and  $p(a) = 1 \Rightarrow s(p) \leq a$ .

The atom  $s(p)$  is interpreted as the proposition that identifies the state.

For two pure states  $p, q \in P$  the number  $(p : q) = \inf\{p(a) | a \in L, q(a) = 1\}$  is called the *transition probability* of  $p$  to  $q$ , with the usual physical interpretation, and can be shown to be equal to  $p(s(q))$ .

Given  $a \in L$ , a partial mapping  $E_a$  from  $P$  into itself,  $E_a : D(E_a) \rightarrow P$  is called a *generalized filter* if  $D(E_a) = \{p \in P | p(a) > 0\}$  and, setting  $p_a = E_a p$ , if  $p(a) = (p : p_a)$  and  $p_a(a) = 1$ .

Generalized filters provide for state-collapse, that is, subject to an observation of  $a \in L$  a state  $p$  undergoes a "collapse" to the state  $p_a$  with transition probability  $(p : p_a)$ . The covering law follows from the existence of generalized filters provided some additional properties hold. One of these is:  $(p : p_a) = (p : q_a) \neq 0 \Rightarrow p_a = q_a$ . It is this property that we shall try to substantiate by *lorentz covariance* and the *no-signal hypothesis*.

We shall assume that given a set of pair-wise orthogonal propositions  $\{b_1, \dots, b_n\}$  there is a physical apparatus that tests them simultaneously with mutually exclusive outcomes. If the state is  $p \in P$  then the outcome that renders proposition  $b_j$  true and all the others false occurs with frequency  $p(b_j) = (p : p_{b_j})$ . Note that in this case the "detection rate" of the whole apparatus is  $\sum_{j=1}^n (p : p_{b_j}) = \sum_{j=1}^n p(b_j) = p(b_1 \vee \dots \vee b_n)$ . If  $\bigvee_{j=1}^n b_j = 1$ , then after the state passes through the apparatus we suppose it becomes the *mixed state*  $\sum_{j=1}^n (p : p_{b_j}) p_{b_j}$ .

Let us now contemplate again an EPR-type space-time situation in which one has a state  $p$  and at one site  $A$ , an apparatus corresponding to a proposition  $a \in L$ , and at a distant site  $B$  an apparatus corresponding to the pair-wise orthogonal set of propositions  $\{b_1, \dots, b_n\}$  with  $\bigvee_{j=1}^n b_j = 1$ . The arrangement is to operate in such a way that the events corresponding to registries in the apparatuses are space-like separated. We thus

assume that the propositions  $a$  along with the  $b_j$  form a commuting set. For there to be no signals from site  $B$  to site  $A$  due to correlations present in state  $p$ , the detection rate at  $A$  must be independent of the apparatus used at site  $B$ .

Now because of space-like separation, the temporal order of the registries of the apparatuses depend on the inertial frame, and in fact in some frame they are simultaneous. From this one can argue, if lorentz covariance is to be maintained, that the following *space-like equivalence hypothesis (SE)* should hold: *The situation described above can be equivalently viewed as either an observation of  $\{b_1, \dots, b_n\}$  followed by  $a$  or as a single observation of  $\{a \wedge b_1, \dots, a \wedge b_n\}$ .*

Now, viewed as successive observations, the rate of detection at  $A$  is given by  $\sum_{j=1}^n (p : p_{b_j}) p_{b_j}(a)$  and when viewed as a single observation the rate is  $p(\bigvee_{j=1}^n (a \wedge b_j)) = p(a)$ . This last number is independent of the apparatus at  $B$  and we see that the no-signal hypothesis follows from SE. Lorentz covariance thus motivates SE which in turn implies the no-signal hypothesis. Now it doesn't seem possible to deduce the covering law from SE, yet a slightly stronger hypothesis implies both. Let us assume the no-signal hypothesis not only for space-like separated observations but for any situation formally as above where only commutativity is assumed. We thus introduce the *Commutative no-signal hypothesis (CNS)*: *Let  $a, b_1, \dots, b_n \in L$  be a commutative set and suppose the  $b_j$  pair-wise orthogonal with  $\bigvee_{j=1}^n b_j = 1$ . then  $\sum_{j=1}^n (p : p_{b_j}) p_{b_j}(a)$  is independent of the set  $\{b_1, \dots, b_n\}$ .*

Picking the singleton  $\{1\}$  for the set of  $b_j$  one concludes that the number whose independence is pos ted has to be  $p(a)$ . From this SE for detection rates follows immediately.

But CNS likewise implies the covering law. Let  $a$  and  $b$  be two commuting propositions, then from the CNS we deduce:  $(p : p_b) p_b(a) + (p : p_{b'}) p_{b'}(a) = p(a)$  and  $(p : p_{b \wedge a'}) p_{b \wedge a'}(a) + (p : p_{b \wedge a}) p_{b \wedge a}(a) + (p : p_{b'}) p_{b'}(a) = p(a)$  from which using the fact that  $p_{b \wedge a'}(a) = 0$  and  $p_{b \wedge a}(a) = 1$  we deduce:  $(p : p_b) p_b(a) = (p : p_{b \wedge a})$  which is the same as  $(p : p_b) (p_b : (p_b)_a) = (p : p_{a \wedge b})$  which says that the detection rate with the successive observations of  $b$  followed by  $a$  is equal to the detection rate of the observation of  $a \wedge b$ . By complete symmetry we also have  $(p : p_a) (p_a : (p_a)_b) = (p : p_{a \wedge b})$ .

Consider now  $b = s(q_a)$  where  $q$  is any other state. As  $b \leq a$ , the two commute and we can apply the above results. Now, since  $b$  is an atom, for any state  $r$  one has  $r_b = s-1(b) = q_a$  provided  $r(b) \neq 0$ . Assume  $p(b) \neq 0$ , then by the last formula of the previous paragraph one deduces  $(p : p_a) (p_a : q_a) = (p : q_a)$ . If now  $(p : p_a) = (p : q_a)$  one concludes that  $(p_a : q_a) = 1$  which implies that  $p_a = q_a$  and this is precisely the property that leads to the covering law in the present axiomatic context.

Some final considerations are in order concerning these results. We have been gathering evidence to substantiate the following strong form of our thesis: *The covering law of quantum theory follows from the requirement of lorentz covariance and the hypothesis that correlations within physical states cannot be used to transmit superluminal signals.*

Now we haven't quite proven this. We have a weaker result in that a stronger hypothesis (CNS), leads to the covering law in at least one axiomatic framework. Now CNS can itself be motivated by lorentz covariance as it leads to SE which guarantees lorentz covariance for detection rates. Within the axiomatic scheme lorentz covariance and NS do not seem to play independent roles, as the thesis suggests they should. This is likely due to another apparent weakness in the scheme, the fact that it cannot distinguish

space-like separated propositions from other commutative pairs. This last weakness is at the present stage unavoidable for the only relation that one normally posits between space-time and quantum logic is that space-like separation leads to commutativity. Only through a joint axiomatization of both lorentzian space-time and hilbert-space quantum mechanics can one hope for anything better. Now commutativity is generally interpreted as "commensurability" and this means the ability to make simultaneous measurements which in turn suggests that space-like separations always play some role in commutativity, yet this has never been thoroughly examined. Furthermore if one believes in general unitary symmetry in hilbert space, it's not hard to find examples in which a commutative pair of observables pertaining to a single localized system (a bound pair of particles say) is unitarily equivalent to a pair of observables at the opposite arms of an EPR apparatus. Such a symmetry would thus extend the requirements on space-like commutativity to commutativity in general. That arguments pertaining to space-like commutativity can be carried over to general commutativity has already been pointed out by Home and Sengupta 6 in relation to Bell's inequalities. It may thus well be that the weaker thesis with the CNS assumption is not too distant from the strong thesis especially if one imposes strong symmetry requirements. In any case, it is clear that one cannot hope to reach a full understanding of hilbert-space quantum mechanics without linking it to space-time structure.

After this text was written I discovered through e-mail conversations with Nicolas Gisin that he too has postulated a connection between the covering law and the no-signaling constraint with results similar to the ones presented here. Details should appear in a forthcoming publication.

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**Greenberger-Horne-Zeilinger correlations in quadrature phase  
measurements**

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*ABSTRACT: E. P. Wigner [1] was one of the first to realize the importance of Bell theorem for our understanding of quantum mechanics.*

*Recently Greenberger, Horne and Zeilinger (GHZ) have shown, that this theorem becomes much stronger for the case of more than two particles.*

*It will be shown, that the state of light generated in the process of parametric down conversion produces the following phenomenon. Under special conditions the correlated counts at four balanced homodyne detectors show the highly non-classical, non-local, behavior of the type predicted by GHZ.*

The work of Greenberger Horne and Zeilinger (GHZ) [2] has shed new light on the problem of hidden variable theories. By considering three and four particle correlation (*gedanken*) experiments, they were able to show that the very premises which led Einstein, Podolsky and Rosen to their criterion for an element of reality are ruled out for the case of more than two particles. In contradistinction with the (two particle) Bell's theorem [3] a contradiction arises already for the case of perfect correlations.

It will be shown here, that under certain conditions the two photon emissions of the parametric down conversion process can lead to the four particle GHZ correlations. If four balanced homodyne detectors are used, the (weak) coherent local oscillator fields will deliver the two other photons. The coincident counts reveal the traits of the four particle GHZ correlations.

The possibility of using measurements of quadrature phases as tests of other nonclassical classical phenomena (Bell's inequalities) has been suggested earlier in refs 4-7.

The experimental configuration proposed here is shown in figs 1 and 2. The first represents the typical situation for a non-degenerate spontaneous down conversion process, the second one the actual measurements to be performed.

Fig 1: a pump beam  $L$  of frequency  $\omega$  falls on a nonlinear crystal, and in the interaction between the light and the medium the process of parametric down conversion occurs, which results in fission of some pump photons into pairs of two lower frequency photons known as the signal ( $s$ ) of frequency  $\omega(s)$  and the idler ( $i$ ) of frequency  $\omega(i)$ . If the phase matching condition is satisfied, then within the medium the three wave vectors and frequencies are related by

$$\mathbf{k} = \mathbf{k}_i + \mathbf{k}_s, \quad (1)$$

$$\omega = \omega(i) + \omega(s). \quad (2)$$

Under suitable conditions, outside the medium the down converted signal and idler beams form cones with their axes centered on the pump beam. The most essential feature of (1) is the following one. Since the wave vector of the pump field,  $\mathbf{k}$ , is well defined, one can select, by placing a diaphragm with suitable apertures, pairs of photons of well correlated directions, and frequencies. The apertures should be followed by very

narrow filters centered at  $\omega(I) \approx \omega(i)$  and  $\omega(S) \approx \omega(s)$ . The arrangement of the apertures of the diaphragm D of Fig. 1 is that tested by Rarity and Tapster [8]. Four beams 1, 2, 3 and 4, emerge from four linearly placed pinholes. The phase matching condition, and the filters enable us to assume that one can expect here only two cases. Either the pair of photons emerges via beams 1 (signal) and 4 (idler), or 2 (signal) and 3 (idler). Of course one has here the case of quantum superposition of those two possibilities. With the use of suitably placed mirrors M, the four beams (but two photons) are directed towards balanced homodyne detectors.

The field emerging from the four pinholes can be described by

$$|\Phi\rangle = (1 - |\beta|^2)^{1/2} |0\rangle + 2^{-1/2} \beta [b(1)b(4) + b(2)b(3)] |0\rangle, \quad (3)$$

where, for the simplicity of further calculations, the discrete mode creation operators  $b$  were used. They satisfy  $[b(k), b(l)] = \delta(kl)$ . Each mode has well defined frequency  $\omega(I)$  or  $\omega(S)$ . They play no role in the following calculations.

Fig. 2: each of the four beams of fig. 1 is directed to a balanced homodyne detector. The following intensity correlation function is measured:

$$E(\phi_1, \phi_2, \phi_3, \phi_4) = \langle \prod_{j=1}^4 (I(+,j) - I(-,j)) \rangle \langle \prod_{j=1}^4 (I(+,j') + I(-,j')) \rangle^{-1}, \quad (4)$$

where  $I(\pm, j)$  is the intensity at the detector  $D(\pm, j)$ . The balanced homodyne detector  $B(j)$  constituting of  $D(+, j)$  and  $D(-, j)$  and a perfect 50%-50% beam splitter BS is fed by the light coming from the parametric down conversion source in mode  $b(j)$ , and by coherent local oscillator field (at a suitable frequency) in mode denoted as  $a(j)$ . If one denotes the mode reaching  $D(-, j)$  by  $c(j)$ , and  $D(+, j)$  by  $d(j)$ , the quantum mechanical formula for (10) is given by

$$E_{QM}(\phi_1, \phi_2, \phi_3, \phi_4) = \langle : \prod_{j=1}^4 (d(j)d(j) - c(j)c(j)) : \rangle \langle : \prod_{j=1}^4 (d(j')d(j') + c(j')c(j')) : \rangle^{-1} \quad (5)$$

where  $::$  denotes the normal ordering of the creation and annihilation operators. The average is over the full state of the whole system  $|\Psi\rangle$  which includes the state of the local oscillator fields:

$$|\Psi\rangle = |\phi\rangle |\alpha(1), \alpha(2), \alpha(3), \alpha(4)\rangle, \quad (6)$$

where  $\alpha(j)$  denotes the amplitude of the coherent field feeding the balanced homodyne detector  $B(j)$ . It will be assumed that

$$\alpha(j) = \alpha \exp(i\phi_j), \quad (7)$$

i.e. the fields have the same average intensity. It should be remembered that the local oscillator fields are of frequency  $\omega(I)$  ( $j=3,4$ ) or  $\omega(S)$  ( $j=1,2$ ).

In order to compute the quantum mechanical predictions, one has to recall the well known relations between the input and output modes of a perfect 50%-50% beam splitter. For the modes considered here they read:

$$c(j) = 2^{-1/2} [a(j) + ib(j)] = -2^{-1/2} i [ia(j) - b(j)],$$

$$d(j) = 2^{-1/2} [ia(j) + b(j)]. \quad (8)$$

Let us assume that the local oscillator fields are very weak. Therefore, one should compute the lowest order, in  $\alpha$ , contribution to (5).

Thus one gets

$$\begin{aligned} \langle \prod_{j=1}^4 (d(j)d(j)) \rangle &\approx \beta^2 \alpha^4 / 8 [1 + \cos(\phi_1 + \phi_4 - \phi_2 - \phi_3)], \\ \langle \prod_{\substack{j=1 \\ j \neq k}}^4 (d(j)d(j)) c(k)c(k) \rangle &\approx \beta^2 \alpha^4 / 8 [1 - \cos(\phi_1 + \phi_4 - \phi_2 - \phi_3)], \end{aligned} \quad (9)$$

etc. The final quantum mechanical prediction, in the limit of small  $\alpha$ , reads

$$E_{\text{QM}}(\phi_1, \phi_2, \phi_3, \phi_4) = \cos(\phi_1 + \phi_4 - \phi_2 - \phi_3). \quad (10)$$

We have the four particle correlations of the type considered by GHZ. As it has been shown in the paper of GHZ [2] only quite bizarre, non-local hidden variable theories could reproduce the above quantum mechanical prediction, even for the case of perfect correlations, i.e. for those phases for which  $|E_{\text{QM}}(\phi_1, \phi_2, \phi_3, \phi_4)| = 1$ . Neither, the classical description of light can lead to this result. For lack of space, the reader is asked to consult the work of GHZ, or ref [9], for a confrontation of the formula (10) with the basic assumptions of local hidden variable theories.

The proposed experimental configuration is very difficult to realize in practice. Nevertheless, one may be tempted to say that it is a feasible one. The main problem is in locking of the coherent local oscillator fields of two different frequencies to the signal and idler frequencies  $\omega(S)$  and  $\omega(I)$ .

The derivation presented here uses the simplest possible approach (discrete mode description, monochromaticity). The time dependence is absent within it. Therefore, as we are here interested in coincident counts, the full description should use the formalism for photons of finite coherence length (compare e.g. [10]). Results of such a calculation suggest that the four coincident photons should emerge via a common generation process. I.e., the sources of the coherent local oscillator fields must be also involved in the generation of the pump field for parametric down conversion process. This technical point will be discussed elsewhere.

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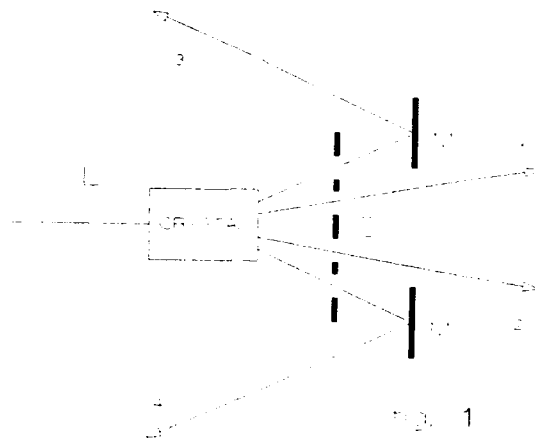


fig. 1

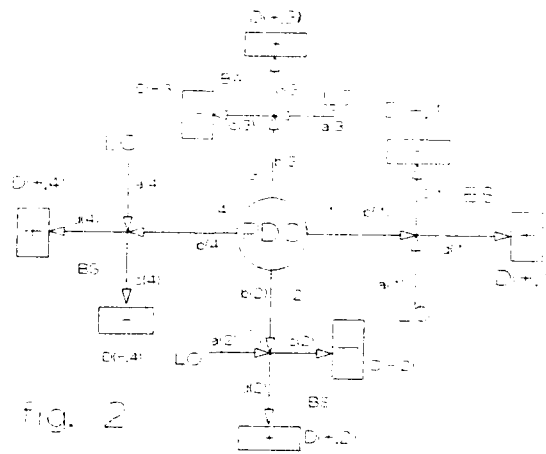


fig. 2

## Testing Bell's Inequalities With a Mach-Zehnder Interferometer

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A Mach-Zehnder interferometer with a single photon source is arranged so that one output mode is dark. An atom is then placed in each arm of the interferometer. By considering only those events for which a photon is detected in the previously dark output mode we find that the atoms are in an entangled state. Certain measurements on this entangled state will violate Bell's inequalities.

In a recent paper by Tan, Walls and Collett<sup>1</sup> it was shown that single photons can be used to demonstrate nonlocality. In this paper we consider a very different experimental set up where, again, the nonlocality of a single photon is demonstrated. We will see that, unlike the proposal of Tan, Walls and Collett, this demonstration can be generalised to all single particles, fermions as well as bosons.

The experiment is motivated by a recent paper by Elitzur and Vaidman<sup>2</sup> illustrating the possibility of interaction free measurements using a Mach-Zehnder interferometer with a single photon source. Their idea is simple. The interferometer is arranged so that, when both routes through it are open, no photons will arrive at the detector  $D_2$  (see fig. 1) due to destructive interference. On the other hand, if an obstacle is placed so that it blocks light through one route, then it is possible for the photon to arrive at  $D_2$  because there is no longer any destructive interference. However, if the photon has arrived at  $D_2$  then it must have taken the other route through the interferometer otherwise it would have been blocked by the obstacle. Therefore it did not 'touch' the obstacle. This means that we can deduce the presence of an obstacle without 'touching' it. The obstacle could, for example, be an atom which, when it is in the state  $|A\rangle$ , absorbs the photon with probability equal to one. If the atom is in any other state it will not absorb the photon. Thus a detection at  $D_2$  lets us know that the atom is in the state  $|A\rangle$  without interacting with it.

Now consider what happens if we place an atom in each route of the interferometer (fig.1). Suppose that when the atoms are in the states  $|A\rangle$  for atom  $A$  and  $|B\rangle$  for atom  $B$  then they absorb the incident photons with probability equal to one and when they are in the states  $|A'\rangle$  for atom  $A$  and  $|B'\rangle$  for atom  $B$  then they are transparent to incident photons. If a photon arrives at detector  $D_2$  then one route must be open and the other must be closed. Therefore the state of the atoms is given by

$$u|A\rangle|B'\rangle + v|A'\rangle|B\rangle, \quad (1)$$

In fact, it is not necessary that atoms in the state  $|A\rangle$  or  $|B\rangle$  absorb the photons with a probability of one to get the entangled state (1). To see this, first let the atoms  $A$  and  $B$  be prepared in in the states

$$|\psi_A\rangle = c|A\rangle + c'|A'\rangle, \quad (2)$$



$$|\psi_B\rangle = d|B\rangle + d'|B'\rangle. \quad (3)$$

The state of the photon moving to the right in the interferometer is represented by  $|1\rangle$  and the state of the photon moving up is represented by  $|2\rangle$ . The operations of the beam splitters on the state of the photon are

$$|1\rangle \longrightarrow \frac{1}{\sqrt{2}}(|1\rangle + i|2\rangle). \quad (4)$$

$$|2\rangle \longrightarrow \frac{1}{\sqrt{2}}(|2\rangle + i|1\rangle). \quad (5)$$

The operations of the fully-silvered mirrors are

$$|1\rangle \longrightarrow i|2\rangle. \quad (6)$$

$$|2\rangle \longrightarrow i|1\rangle. \quad (7)$$

Now, let us relax the requirement that atoms in the state  $|A\rangle$  or  $|B\rangle$  absorb photons with a probability one. Thus, the operations of the atoms are

$$|1\rangle|A\rangle \longrightarrow \alpha|1\rangle|A\rangle + \alpha'|0\rangle|A''\rangle, \quad (8)$$

$$|1\rangle|B\rangle \longrightarrow \beta|1\rangle|B\rangle + \beta'|0\rangle|B''\rangle, \quad (9)$$

$$|1\rangle|A'\rangle \longrightarrow |1\rangle|A'\rangle, \quad (10)$$

$$|1\rangle|B'\rangle \longrightarrow |1\rangle|B'\rangle, \quad (11)$$

where  $|0\rangle$  indicates that the photon has been absorbed and  $|A''\rangle$  and  $|B''\rangle$  are the states of atoms  $A$  and  $B$  respectively when they have absorbed the photon. The evolution of the state of the photon and two atoms is described by

$$\begin{aligned} |1\rangle|\psi_A\rangle|\psi_B\rangle &\longrightarrow \frac{1}{\sqrt{2}}(|1\rangle + i|2\rangle)|\psi_A\rangle|\psi_B\rangle, \\ &\longrightarrow \frac{1}{\sqrt{2}}(\alpha c|1\rangle|A\rangle + \alpha'c|0\rangle|A''\rangle + c'|1\rangle|A'\rangle)|\psi_B\rangle + \frac{i}{\sqrt{2}}|2\rangle|\psi_A\rangle|\psi_B\rangle, \\ &\longrightarrow \frac{1}{\sqrt{2}}(i\alpha c|2\rangle|A\rangle + \alpha'c|0\rangle|A''\rangle + ic'|2\rangle|A'\rangle)|\psi_B\rangle - \frac{1}{\sqrt{2}}|1\rangle|\psi_A\rangle|\psi_B\rangle, \\ &\longrightarrow \frac{1}{\sqrt{2}}(i\alpha c|2\rangle|A\rangle + \alpha'c|0\rangle|A''\rangle + ic'|2\rangle|A'\rangle)|\psi_B\rangle \\ &\quad - \frac{1}{\sqrt{2}}(\beta d|1\rangle|B\rangle + \beta'd|0\rangle|B''\rangle + d'|1\rangle|B'\rangle)|\psi_A\rangle, \\ &\longrightarrow \frac{1}{\sqrt{2}}\left(\frac{i\alpha c}{\sqrt{2}}(i|1\rangle + |2\rangle)|A\rangle + \alpha'c|0\rangle|A''\rangle + \frac{ic'}{\sqrt{2}}(i|1\rangle + |2\rangle)|A'\rangle\right)|\psi_B\rangle \\ &\quad - \frac{1}{\sqrt{2}}\left(\frac{\beta d}{\sqrt{2}}(|1\rangle + i|2\rangle)|B\rangle + \beta'd|0\rangle|B''\rangle + \frac{d'}{\sqrt{2}}(|1\rangle + i|2\rangle)|B'\rangle\right)|\psi_A\rangle. \end{aligned}$$

Simplifying, we get for the final state

$$\begin{aligned}
 |\Psi\rangle = & -\frac{1}{2}|1\rangle \left( cd(\alpha + \beta)|A\rangle|B\rangle + 2c'd'|A'\rangle|B'\rangle \right. \\
 & \left. + cd'(1 + \alpha)|A\rangle|B'\rangle + c'd(1 + \beta)|A'\rangle|B\rangle \right) \\
 & + \frac{i}{2}|2\rangle \left( cd(\alpha - \beta)|A\rangle|B\rangle - cd'(1 - \alpha)|A\rangle|B'\rangle + c'd(1 - \beta)|A'\rangle|B\rangle \right) \\
 & + \frac{1}{\sqrt{2}}|0\rangle \left( \alpha'c|A''\rangle|\psi_B\rangle + \beta'd|B''\rangle|\psi_A\rangle \right). \tag{12}
 \end{aligned}$$

We consider only those events in which the photon is detected at  $D_2$ . For these events the state vector collapses onto the second term on the RHS of (12). If the atoms  $A$  and  $B$  are identical so that  $\alpha = \beta$  then the unnormalised state of the atoms becomes

$$|\varphi\rangle = c'd|A'\rangle|B\rangle - cd'|A\rangle|B'\rangle. \tag{13}$$

To demonstrate nonlocality take identical atoms  $A$  and  $B$ . By preparing them so that they have spin  $+\frac{1}{2}$  along the  $x$ -axis, their states can be written

$$|\psi_A\rangle = \frac{1}{\sqrt{2}}(|+\rangle_A + |-\rangle_A), \tag{14}$$

$$|\psi_B\rangle = \frac{1}{\sqrt{2}}(|+\rangle_B + |-\rangle_B), \tag{15}$$

where the states  $|\pm\rangle_A$  and  $|\pm\rangle_B$  have spin  $\pm\frac{1}{2}$  along the  $z$ -axis. Now, if a magnetic field is applied along the  $z$ -axis then Zeeman splitting will occur. We choose the frequency of our single photon so that it corresponds to some excitation energy of the  $|-\rangle_A$  and  $|-\rangle_B$  states but not to any excitation energy of the  $|+\rangle_A$  and  $|+\rangle_B$  states. Thus, the  $|-\rangle_A$  and  $|-\rangle_B$  states will absorb some of the photons and, therefore, can be identified with the  $|A\rangle$  and  $|B\rangle$  states respectively. On the other hand, the  $|A'\rangle$  and  $|B'\rangle$  states will be transparent to the photons and, therefore, can be identified with the  $|A'\rangle$  and  $|B'\rangle$  states respectively. The initial state of the atoms can be written as a product:

$$|\varphi_i\rangle = \frac{1}{2}(|+\rangle_A + |-\rangle_A)(|+\rangle_B + |-\rangle_B) \tag{16}$$

However, after interaction with the photon, the state of the atoms becomes entangled:

$$|\varphi\rangle = \frac{1}{\sqrt{2}}(|+\rangle_A|-\rangle_B - |-\rangle_A|+\rangle_A) \tag{17}$$

This is a singlet state. It is well known that measurements of spin along appropriate directions on a singlet state will violate Bell's inequalities.<sup>3</sup>

The experiment proposed by Tan, Walls and Collett depends on the ability to produce coherent states, and so can only be generalized to single boson sources. However,

it is clear that the experiment discussed above can be generalised to all single particle sources, fermions as well as bosons. This is, then, the first demonstration of the nonlocality of a single particle that holds for all particles.

The author thanks Professor E. J. Squires for helpful discussions on this topic.

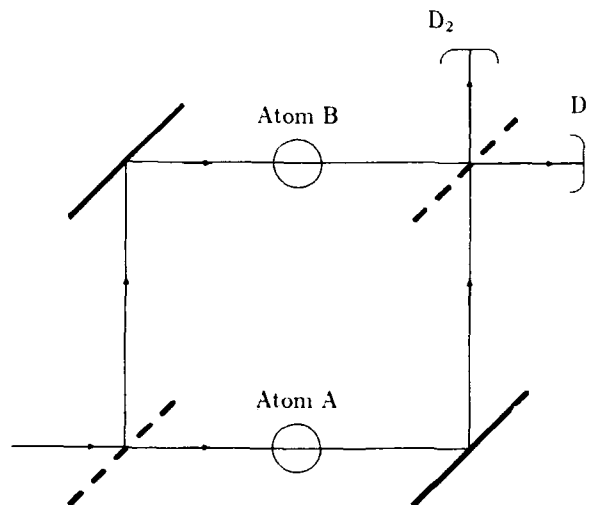
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**Fig. 1** Mach-Zender interferometer with single photon source arranged so that there are no objects in the path of the routes through it no photons arrive at detector  $D_2$ .



PREDICTIONS FOR BELL'S TYPE EXPERIMENTS  
IN  
STOCHASTIC OPTICS

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**Abstract**

T.W. Marshall and E. Santos have proposed recently a theory, called Stochastic Optics, that explains in a local way some experiments thought to be non classical.

We generalize their results in order to calculate the difference between quantum mechanics and the stochastic optics model used by them for the optical test of Bell's inequalities.

The quantum-optical prediction for the coincidence rate in a typical atomic cascade experiment or the decay of the metastable state of atomic hydrogen is, for ideal experiments [0]:

$$R_{12}(\theta) = \frac{1}{4} R_0 (1 + 2 \cos 2\theta), \quad (1)$$

where  $R_0$  is the rate at which cascades occur.

Most of the experiments designed to test the Bell inequalities using single-channel polarizers have measured the quantity:

$$r_{12}(\theta) = \frac{R(a, a + \theta)}{R(\infty, \infty)} = \frac{1}{4} (1 + \cos 2\theta), \quad (2)$$

where  $R(\infty, \infty)$  is the corresponding prediction for the rate with the polarizers removed.

In order to reproduce this results Marshall and Santos defined the function:

$$r(\theta) = \langle Q(\cos 2\varphi) Q(\cos 2\Phi) \rangle \equiv \frac{1}{2\pi} \int_0^{2\pi} d\psi \int_0^{\pi/2} \sin 2\varphi Q \cos(2\varphi) Q \cos(2\Phi) d\varphi \quad (3)$$

where  $\varphi$  and  $\psi$  play the role of hidden variables, with domains  $0 \leq \varphi \leq \pi/2$  and  $0 \leq \psi \leq 2\pi$ , and:

$$Q(\cos 2\Phi) = \frac{1}{\beta^2 - \gamma} \int_0^{\pi/2} (\beta^2 \cos^2(\Phi) + \sin^2 \varphi_0 - \gamma)_+ \sin 2\varphi_0 d\varphi_0 \quad (4)$$

where:

$$\cos 2\Psi = \cos 2\varphi \cos 2\theta + \sin 2\varphi \sin 2\theta \cos \psi, \quad (5)$$

with  $0 \leq \Psi \leq \pi/2$ , and  $\beta$  and  $\gamma$  are parameters of the theory. The notation  $(\dots)_+$  means putting zero if the argument is negative. Marshall and Santos wrote  $Q(\lambda)$  in terms of Legendre polynomials:

$$Q(\lambda) = \sum_n a_n P_n(\lambda) \quad (6)$$

and they showed that the first order approximation  $Q(\lambda) \approx a_0 + a_1 \lambda$  gave:

$$r(\theta) = a_0^2 + \frac{1}{3} a_1^2 \cos 2\theta \quad (7)$$

so, choosing the values  $a_0 = 1/2$  and  $a_1 = \sqrt{3}/2$  they got the quantum prediction (2). They have also shown that:

$$\langle P_2(\cos 2\varphi) P_0(\cos 2\Psi) \rangle = \langle P_2(\cos 2\varphi) P_1(\cos 2\Psi) \rangle$$

and

$$\langle P_2(\cos 2\varphi) P_2(\cos 2\varphi) \rangle = \frac{1}{5} P_2(\cos 2\theta) \quad (8)$$

making possible to consider a second order approximation  $Q(\lambda) \approx a_0 + a_1 \lambda + a_2 P_2(\lambda)$  which leads to the result:

$$r(\theta) \approx a_0^2 + \frac{1}{3} a_1^2 \cos 2\theta + \frac{1}{5} a_2^2 P_2(\cos 2\theta) \quad (9)$$

We have found that considering the complete Legendre expansion (6) the corresponding expression for  $r(\theta)$  is:

$$r(\theta) = \sum_n \frac{a_n^2}{2n+1} P_n(\cos 2\theta) \quad (10)$$

In order to prove (10) we need to demonstrate first that:

$$\langle P_n(\cos 2\varphi) P_m(\cos 2\Psi) \rangle = \frac{\delta_{mn}}{2n+1} P_n(\cos 2\theta) \quad (11)$$

To prove (11) we use definition (3):

$$\langle P_n(\cos 2\varphi) P_m(\cos 2\Psi) \rangle = \frac{1}{2\pi} \int_0^{\pi/2} \sin 2\varphi P_n(\cos 2\varphi) d\varphi \int_0^{2\pi} P_m(\cos 2\Psi) d\Psi \quad (12)$$

Applying now the addition theorem for associated Legendre functions:

$$P_n(\cos 2\Psi) = P_n(\cos 2\varphi) P_n(\cos 2\theta) + \sum_{k=1}^n \frac{(n-k)!}{(n+k)!} P_n^k(\cos 2\varphi) P_n^k(\cos 2\theta) \cos k\psi \quad (13)$$

to the integral  $\int_0^{2\pi} P_m(\cos 2\Psi) d\Psi$  in (12), we get:

$$\int_0^{2\pi} P_m(\cos 2\Psi) d\Psi = 2\pi P_m(\cos 2\varphi) P_m(\cos 2\theta) \quad (14)$$

Putting (14) in (12):

$$\langle P_n(\cos 2\varphi)P_m(\cos 2\Psi) \rangle = P_m(\cos 2\theta) \int_0^{\pi/2} \sin 2\varphi P_n(\cos 2\varphi)P_m(\cos 2\varphi)d\varphi \quad (15)$$

and using the orthogonality condition for Legendre functions:

$$\int_0^{\pi/2} \sin 2\varphi P_n(\cos 2\varphi)P_m(\cos 2\varphi)d\varphi = \frac{\delta_{nm}}{2n+1} \quad (16)$$

we get, finally (11).

To obtain the Legendre expansion of the function  $r(\theta)$  we use the full expansion (6) of  $Q(\chi)$  in the definition (3):

$$r(\theta) = \langle Q(\cos 2\varphi)Q(\cos 2\Psi) \rangle = \sum_n \sum_m a_n a_m \langle P_n(\cos 2\varphi)P_m(\cos 2\Psi) \rangle \quad (17)$$

and using (11) we get (10).

We will apply now the previous results to the optical test of the Bell's inequalities. We will do it assuming, as Marshall and Santos did, a perfect correlation between the hidden variables of the two signals. In this case we can define the angular relations:

$$\begin{aligned} \chi_a &= \cos(2\varphi) \cos(2a) + \sin(2\varphi) \sin(2a) \cos(\Psi) \\ \chi_b &= \cos(2\varphi) \cos(2b) + \sin(2\varphi) \sin(2b) \cos(\Psi) \end{aligned} \quad (18)$$

The probability of a coincidence count, as proposed by Bell, can be written:

$$\langle P_n(\chi_a)P_m(\chi_b) \rangle$$

We will now prove:

$$\langle P_n(\chi_a)P_m(\chi_b) \rangle = \frac{\delta_{mn}}{2n+1} P_n(\cos 2\theta) \quad (19)$$

where  $\theta = a - b$ .

Using the definition (3):

$$\langle P_n(\chi_a)P_m(\chi_b) \rangle = \frac{1}{2\pi} \int_0^{\pi/2} \sin 2\varphi d\varphi \int_0^{2\pi} P_n(\chi_a)P_m(\chi_b)d\Psi \quad (20)$$

and applying the addition theorem for associated Legendre functions to  $\int_0^{2\pi} P_n(\chi_a)P_m(\chi_b)d\Psi$ , we get:

$$\begin{aligned} \int_0^{2\pi} P_n(\chi_a)P_m(\chi_b)d\Psi &= 2\pi P_n(\cos 2a)P_n(\cos 2\varphi)P_m(\cos 2b)P_m(\cos 2\varphi) + \\ &+ 4\pi \sum_{k=1}^n \sum_{l=1}^m \frac{(n-k)!(m-l)!}{(n+k)!(m+l)!} P_n^k(\cos 2a)P_n^k(\cos 2\varphi)P_m^l(\cos 2b)P_m^l(\cos 2\varphi)\delta_{kl} \end{aligned} \quad (21)$$

Putting back (21) in (20), this last expression can be written:

$$\langle P_n(\chi_a)P_m(\chi_b) \rangle = I_1 + I_2 \quad (22)$$

with an obvious identification.

After a straightforward calculation the integrals  $I_1$  and  $I_2$  give:

$$I_1 = \frac{1}{2n+1} P_n(\cos 2a) P_n(\cos 2b) \delta_{nm}$$

$$I_2 = \frac{2}{2n+1} \sum_{k=1}^n \frac{(n-k)!}{(n+k)!} P_n^k(\cos 2a) P_n^k(\cos 2b) \delta_{nm} \quad (23)$$

Putting (23) in (22):

$$\langle P_n(\lambda_a) P_m(\lambda_b) \rangle = \frac{\delta_{nm}}{2n+1} \left[ P_n(\cos 2a) P_n(\cos 2b) + 2 \sum_{k=1}^n \frac{(n-k)!}{(n+k)!} P_n^k(\cos 2a) P_n^k(\cos 2b) \right] \quad (24)$$

and applying again the addition theorem of associated Legendre functions to (23), we get (19).

Using this last result it is very easy to show that:

$$r_{12}(a, b) = \langle Q(\lambda_a) Q(\lambda_b) \rangle = \sum_n \frac{a_n^2}{2n+1} P_n(\cos 2\theta) \quad (25)$$

We think that with this results it should be possible the generalize the Marshall and Santos calculations [0] and get the difference between the predictions made by quantum mechanics and the stochastic model proposed by them for atomic cascade experiments. This problem will be the subject of a future work.

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## MAGNETIC TOP: ON THE ORIGIN OF QUANTUM SPIN AND MAGNETIC MOMENT

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### Abstract

The quantum spin can be modelled by a top in which the magnetic field couples to the angular velocity  $\vec{\omega}$ . A complete symplectic theory can be given and the system is integrable. Although the top performs a complicated motion the magnetic moment has the exact Pauli precession. The system is quantized, and a point charge model is developed to calculate the gyromagnetic ratio  $g$ .

### I. Introduction

The spin of the electron was introduced into the quantum theory directly by the Pauli or Dirac matrices without quantizing any preexisting classical spin model. This fits quite well into the statistical interpretation of quantum mechanics, but the understanding of the physical nature and origin of spin and spin magnetic moment is lacking.<sup>1,2</sup> Neither of the proposed theoretical explanations (see for example Refs.3-6) was generally accepted. Spin models have not been very conclusive in the past because the phase space of classical models seems to be larger than that of quantum spin.<sup>7</sup>

In our recent work<sup>8</sup> we removed these shortcomings of previous classical models by a proper coupling of the magnetic field  $\vec{B}$  to the angular velocity  $\vec{\omega}$  of the top. The top with such coupling we call the magnetic top. The magnetic moment  $\vec{\mu}$  of the magnetic top performs a simple precession motion leading immediately to the Pauli equation, whereas the motion of the top itself is more complicated.

In order to show how essential the form of the coupling is, we compare the magnetic top with the case when the top couples to a fixed vector on the top (as in the case of gravitational coupling). There is a big difference.

Because this model is successful in providing the Pauli equation, one might ask, if



the possible underlying structure of the top could be revealed in future experiments, or in understanding the change of phase of the spin under  $2\pi$  rotations. We elaborate further here on the origin and value of the magnetic moment and study a simple model where a point charge  $\epsilon$  produces the magnetic moment due to the motion of the top and the relative motion of charge on the top.

## II. Lagrangian and Hamiltonian of the symmetric magnetic top

The Lagrangian and Hamiltonian for the symmetric magnetic top in a magnetic field are<sup>8</sup>:

$$L = T - U = \frac{1}{2}I\vec{\omega}^2 + gI\vec{\omega} \cdot \vec{B} \quad (1)$$

$$H = \frac{1}{2}I\vec{\omega}^2 = \frac{\vec{\Sigma}^2}{2I} = \frac{1}{2I}(\vec{s} - gI\vec{B}) \quad (2)$$

where  $I$  is the moment of inertia,  $\vec{\Sigma} = I\vec{\omega}$  the **kinetic** angular momentum and

$$\vec{s} = \vec{\Sigma} + gI\vec{B} \quad (3)$$

is the **canonical** angular momentum (in analogy with the kinetic linear momentum  $\vec{\pi} = m\dot{\vec{r}}$  and **canonical** momentum  $\vec{p} = m\dot{\vec{r}} + \epsilon\vec{A}$  for the point particle Hamiltonian  $H = \frac{1}{2m}(\vec{p} - \epsilon\vec{A})^2$ ). The phase space of the top consists of  $(\theta, \varphi, \chi, p_\theta, p_\varphi, p_\chi)$ , the three Euler angles and their conjugate momenta. We have here a gauge theory with  $\vec{B}$  playing the role of the gauge field.

Using the Lagrange and Hamilton equations we showed that in magnetic field  $\vec{B}(t)$  **kinetic** and **canonical** angular momenta as well as **kinetic** ( $\vec{M} = g\vec{\Sigma}$ ) and **canonical** ( $\vec{\mu} = g\vec{s}$ ) magnetic moments satisfy the torque equation.

$$\frac{d\vec{\mu}}{dt} = g\vec{\mu} \times \vec{B}(t) \quad (4)$$

Therefore, in the time-independent field  $\vec{B}_s$ , the vectors  $\vec{\Sigma}, \vec{s}, \vec{\mu}, \vec{M}$  simply precess around the field.

In contrast to this simple one-dimensional motion of  $\vec{\mu}$  of the magnetic top, a magnetic moment fixed to the body, e. g.  $\vec{m} = m\vec{e}_z$  satisfies a more complicated equation of motion and consequently performs a complicated precession with nutation.

## III. Quantum magnetic top

The usual quantization via Poisson brackets gives the usual spin algebra

$$[\hat{s}_i, \hat{s}_j] = i\hbar\epsilon_{ijk}\hat{s}_k \quad (5)$$

but does not provide us immediately wave functions on the configuration space  $(\theta, \varphi, \chi)$  of the top. In the Schrödinger quantization the canonical angular momenta operators  $\hat{s}_i$  are represented by differential operators on the space  $(\theta, \varphi, \chi)$  which gives the Hamiltonian, without the ambiguity of operator ordering, as

$$\hat{H} = \frac{1}{2I} \hat{s}^2 + g\hbar B \left( i \frac{\partial}{\partial \varphi} \right) + \frac{1}{2} g^2 I B^2 \quad (6)$$

$$\hat{s}^2 = -\hbar^2 \left[ \frac{\partial^2}{\partial \theta^2} + ctg\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \left( \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial \chi^2} \right) - 2 \frac{ctg\theta}{\sin \theta} \frac{\partial^2}{\partial \varphi \partial \chi} \right] \quad (7)$$

In particular, for  $s = 1/2$  we obtain the wave functions  $u_{1/2}(\theta, \varphi, \chi)$  and  $u_{-1/2}(\theta, \varphi, \chi)$  which span the subspaces  $D^{1/2}$  of the two-valued representations of the rotation group<sup>3</sup>

$$\begin{aligned} u_{1/2}(\theta, \varphi, \chi) &= \frac{i}{2\pi\sqrt{2}} e^{i[(\varphi/2)+(\chi/2)]} \cos \frac{\theta}{2} \\ u_{-1/2}(\theta, \varphi, \chi) &= \frac{1}{2\pi\sqrt{2}} e^{i[-(\varphi/2)+(\chi/2)]} \sin \frac{\theta}{2} \end{aligned} \quad (8)$$

#### IV. Magnetic moment and g - factor

We address ourselves now to the question of the origin of the magnetic moment  $g$  which so far was introduced via the coupling  $U = -gI\vec{\omega} \cdot \vec{B}$  in the Lagrangian. We now show that a point charge  $e$  which moves on the top and which is coupled to the electromagnetic potential  $\vec{A}$  by the term  $\vec{j} \cdot \vec{A} = e\vec{R} \cdot \dot{\vec{A}}$  can lead to a magnetic moment coupling  $gI\vec{\omega} \cdot \vec{B}$ . Then we will compute  $g$ .

For a homogeneous magnetic field, we can set  $\vec{A}(\vec{R}) = \frac{1}{2}\vec{B} \times \vec{R}$ . Hence

$$U = e\vec{R} \cdot \vec{A}(\vec{R}) = e\vec{R} \cdot \frac{1}{2}\vec{B} \times \vec{R} = \frac{e}{2}\vec{R} \times \vec{R} \cdot \vec{B} = -\vec{M} \cdot \vec{B} \quad (9)$$

with

$$\vec{M} = -\frac{e}{2}\vec{R} \times \dot{\vec{R}} \quad (10)$$

Using the relation

$$\dot{\vec{R}}_{space} = \dot{\vec{R}}_{body} + \vec{\omega} \times \vec{R}, \quad (11)$$

we find:

$$\vec{M} = -\frac{e}{2}[\vec{R} \times (\vec{\omega} \times \vec{R}) + \vec{R} \times \dot{\vec{R}}_b] = -\frac{e}{2}[\vec{\omega}R^2 - \vec{R}(\vec{\omega} \cdot \vec{R}) + \vec{R} \times \dot{\vec{R}}_b] \quad (12)$$

We now require the relation

$$\vec{M} = -\frac{e}{2}[\vec{\omega}R^2 - \vec{R}(\vec{\omega} \cdot \vec{R}) + \vec{R} \times \dot{\vec{R}}_b] = -gI\vec{\omega} \quad (13)$$

One simple solution of (13) is obtained by setting:

$$\vec{\omega} \cdot \vec{R} = 0; \quad \vec{R} \times \dot{\vec{R}} = \Lambda \vec{\omega} \quad (14)$$

where  $\Lambda$  is a constant parameter which is different from zero. The charge moves with respect to the body in such a way that  $\vec{R}$  is always perpendicular to the time dependent vector  $\vec{\omega}$ . Then the  $g$  factor is equal to:

$$g = \frac{e}{2I}(\Lambda + R^2) \quad (15)$$

Taking into account that moment of inertia,  $I = \int_V \rho d\vec{r}^2$ , of the extended particle, is different from the moment of inertia  $I = mR^2$ , of the point particle, we conclude that the  $g$  factor takes values different from  $g = e/2m$  value of the point charged particle.

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TOPOLOGICAL ARROW OF TIME AND QUANTUM-MECHANICAL  
EVOLUTION

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The quantum measurement problem remains still unsolved<sup>1</sup>. Actually, it just expresses the essentially dual character of evolution in quantum mechanics. On the one hand, the state vector evolves according to deterministic, time-symmetric, local and unitary laws, such as the Schroedinger equation; during this kind of evolution the state as a whole remains unobservable. On the other hand, the essential probabilistic nature of the quantum-mechanical predictions requires that the wave function should evolve following nonunitary, time-asymmetric and non-local laws during the measurement process which, moreover, is associated with a loss of quantum coherence. Thus, if we want to render quantum mechanics a predictive theory, then we will need a collapse of the wave function, and this implies an *arrow of time* for the measurement process.

Clearly, quantum mechanics describes the behaviour of ordinary matter in flat space. At least at the microscopic level, this represents an extremely good approximation whenever spacetime is considered as a classical construct which is independent of the observer. However, if quantum fluctuations of the spacetime topology are allowed to occur, one could still expect the appearance of gravitationally-induced important effects during evolution of microscopic matter. I will show that such effects are precisely those required to induce an arrow of time for quantum measurement.

Topological fluctuations of spacetime are best described as wormholes which are microscopic connections between two otherwise disconnected flat regions of spacetime<sup>2</sup>. A wormhole represents a topology change in that it induces an initial state which is just flat space to evolve into a final state which is flat space plus a given number of baby universes. Baby universes are very little closed universes described by just the Lorentzian version of the corresponding wormhole spacetime which, in turn, can only crop up in Euclidean gravity.

From a topological point of view, two types of wormholes which insert into the large flat region at only one contact point can be considered. Simply-connected wormholes are those wormholes describable by a pure quantum state given as a wave function<sup>3</sup>

$$\Psi[h_{ij}, \phi_0] = \int_{C_\psi} d[g_{\mu\nu}] d[\phi] e^{-i|g_{\mu\nu}, \phi|}, \quad (1)$$

where  $\tilde{I}$  is the Euclidean action and  $C_\psi$  describes the class of paths in the path integral (1) associated with Euclidean asymptotically flat four-geometries and asymptotically vanishing scalar fields  $\phi$  which match the configuration  $[h_{ij}, \phi_0]$  given by the three-metric  $h_{ij}$  and the scalar field value  $\phi_0$  on an inner three-boundary  $S$ , which divides the four-dimensional wormhole manifold into two disconnected

parts.

Nonsimply-connected wormholes possess a quantum state given by a density matrix<sup>4</sup>,  $\rho[h_{ij}, \phi_0; h'_{ij}, \phi'_0]$ , that describes a mixed state. It is also given by a path integral as in (1), but for a class of paths,  $C_\rho$ , having as inner three-boundary the sum of the disjoint three-surfaces  $S$  and their copy  $S'$ . This expresses the distinguishing feature of this kind of wormholes that  $S$  does not divide the whole manifold into two disconnected parts, and leads to a crucial difference between the effects that the two kinds of wormhole topology have on ordinary matter in the asymptotic region. In general, the effects of wormholes on conformally invariant scalar particles at low energy are evaluated by using the propagator<sup>3</sup>

$$\langle 0 | \phi(x_1) \dots \phi(x_r) | O_w | \phi(y_1) \dots \phi(y_r) | 0 \rangle, \quad (2)$$

where the  $x$ 's and the  $y$ 's are points on the two asymptotic regions, and  $O_w$  is a filter operator which is given by  $|\Psi\rangle\langle\Psi|$  for simply-connected wormholes<sup>3</sup> or by  $\rho$  for nonsimply-connected wormholes<sup>4</sup>. The Green function (2) can be worked out to give finally an effective Hamiltonian density in the asymptotic region

$$H = H_0(\phi) + \sum_i H_i(\phi) A_i. \quad (3)$$

Here,  $H_0$  is the usual local Hamiltonian function for scalar particles. The new scalar particle-wormhole interaction terms  $H_i A_i$  (in which the discrete index  $i$  denotes the type of wormhole) contain both scalar field operators  $H_i$  and baby universe operators  $A_i$ . If the wormhole quantum state is given by a wave function, then<sup>3</sup>  $A_i = a_i^\dagger + a_i$  which is associated with an uncorrelated production or annihilation of baby universes characterized by Fock operators  $a_i$ . If we consider mixed quantum states given a density matrix, then it turns out that  $A_i$  will correspond to the time-correlated production or annihilation of a number of baby universes, all at a time. For the case of a doubly-connected wormhole e.g., we would have  $A_i \simeq a_i^{\dagger 2} + a_i^2$ .

Now, since uncorrelated baby universes are all closed, such baby universes will be disconnected from the asymptotic region, and cannot therefore be observed from it. However, owing to their mutual correlations, all baby universes which are branched off from nonsimply-connected wormholes should be connected to the asymptotic region, making thereby observable the quantum state  $\rho$ .

Single-time correlations among baby universes are destroyed when one chooses different time orientations on the disjoint three-surfaces from which baby universes are branched off. An observer in the asymptotic region does not know anything about the number of disjoint three-surfaces on the inner boundary. It follows that if, for the sake of making the Euclidean action positive definite (so that the path integral becomes convergent), one uses complex metrics, then the contributions from complex metrics and their complex conjugate metrics should be the same in the case of the wave function, but these contributions ought to be necessarily

different for a density matrix. This is best seen by using the Laplace transform of the density matrix  $\rho$

$$P[\tilde{h}_{ij}, \phi_0, K; \tilde{h}'_{ij}, \phi'_0, K'] \\ = \int_0^\infty d[h^{\frac{1}{2}}] d[h'^{\frac{1}{2}}] e^{-\frac{1}{i2\pi} (\int K h^{\frac{1}{2}} d^3x + \int K' h'^{\frac{1}{2}} d^3x')} \rho[h_{ij}, \phi_0; h'_{ij}, \phi'_0], \quad (4)$$

where  $\tilde{h}_{ij}$  is the three-metric up to a conformal factor and  $K$ , the trace of the Euclidean second fundamental form, is taken to play the role of a cosmological time concept<sup>5</sup>. The Laplace transform  $P$  is holomorphic if the real part of  $K$  is positive;  $K$  can then be analytically extended so that  $K \rightarrow iK_L$ , with  $K_L$  being the Lorentzian version of  $K$ . It can be readily seen that complex metrics lead to a positive real part in  $K_L$ , and their conjugate counterpart metrics lead to a negative real part of  $K_L$ . One has then,

$$\Psi[\tilde{h}_{ij}, K_L, \phi_0] = \Psi^*[\tilde{h}_{ij}, -K_L, \phi_0] \quad (5)$$

$$P[\tilde{h}_{ij}, K_L, \phi_0] \neq P^*[\tilde{h}_{ij}, -K_L, \phi_0], \quad (6)$$

where we have omitted, for the sake of simplicity in the formulas, all primed arguments for  $P$ . It follows that, whereas (5) is a statement of  $T$  invariance of the quantum state of simply-connected wormholes, (6) is a statement of  $T$  noninvariance for the quantum state of nonsimply-connected wormholes. Thus, if we first replace the real scalar field by a complex one  $\Phi$ , and then introduce a triad of covectors  $e_k^a$  on the three-surfaces and a fermion field  $\psi$ , statements of  $CT$  and  $CTP$  invariance for the wave function and the corresponding statements for  $CT$  and  $CTP$  noninvariance for the density matrix are obtained, i.e. for  $P$  one has

$$P[\tilde{h}_{ij}, K_L, \Phi_0] \neq P^*[\tilde{h}_{ij}, -K_L, \Phi_0^*] \quad (7),$$

$$P[\tilde{e}_k^a, K_L, \psi_0] \neq P^*[-\tilde{e}_k^a, -K_L, \psi_0^*], \quad (8)$$

(and the corresponding equalities for  $\Psi$ ) where  $\psi_0^c = C\psi_0^*$  and  $\tilde{e}_k^a$  is the triad on the  $S$ 's up to a factor.

This result ultimately implies that causal locality as expressed in terms of the baby universe operators  $A_i$  is violated for nonsimply-connected wormholes, i.e.  $[A_i(x), A_j(y)] \neq 0$ , at least for some particular spacelike separations. The commutator will be always zero nevertheless for simply-connected wormholes.

Requiring then causal locality for operator  $H$  in the asymptotic region<sup>6</sup>, it follows that the scalar particle interaction operators  $H_i$  in (3) should always commute for spacelike separations for wormhole wave functions, but do not for wormhole density matrices, i.e.

$$[H_i(x), H_j(y)] \neq 0, \quad (9)$$

at least for some  $(x - y)^2 < 0$ . Assuming then  $CP$  invariance for quantum fields at low energy, it may be concluded that, whereas simply-connected wormholes

must induce an evolution for these quantum fields which is unitary, deterministic, time-symmetric and local, nonsimply-connected wormholes make that evolution nonunitary, time-asymmetric and nonlocal, inducing thus a loss of quantum coherence for the quantum fields. Now, bearing in mind the observability properties of  $\Psi$  and  $\rho$ , we see that the evolution of the state vector of any quantum-mechanical system in flat space is dual in exactly the same way as that which is induced by wormholes. Hence, we can see why quantum fluctuations of spacetime topology can make a suitable candidate for the missing physical origin of the arrow of time required by quantum measurement. It is worth noticing that, since the probability of occurrence of wormholes is proportional to  $e^{-S_m}$ , where  $S_m$  denotes the action corresponding to the wormhole cross section with minimal radius, the nonlocality induced by multiply-connected wormholes can even hold over the macroscopic distances implied by e.g. any *EPR* experiment.

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### III. General Group Theoretical and Quantization Methods



### Tensor Product of Quaternion Hilbert Modules

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*Abstract:* A solution to the problem of the construction of a tensor product of quaternion modules is given. Annihilation-creation operators are constructed, with Bose-Einstein or Fermi-Dirac statistics. The commutation relations are distorted.

Quaternion Hilbert modules are of interest because they form admissible representations of quantum theory in its abstract form[1]. The analog of a ray in complex quantum mechanics carries a full quaternionic phase, and one might expect such a structure to be useful as a model for theories with non-Abelian gauge symmetry. The one particle theory has undergone significant development[2][3]. Furthermore, a Fock space was constructed, with annihilation-creation operators, which is linear on a complex subalgebra [3], and a somewhat restricted form of quaternion field theory has been developed using path integral methods [4], but a tensor product with full quaternion symmetry has only recently been worked out [5]. We wish to review this result and emphasize some points not explicitly stated previously.

We first define a set of elements  $f \in V_H$  such that  $fa + gb \in V_H$ ;  $a, b \in H$ , the quaternion (star) algebra generated over the reals by  $1, \epsilon_1, \epsilon_2, \epsilon_3$ , for which  $\epsilon_i^2 = -1, \epsilon_i^* = -\epsilon_i, \epsilon_i^2 = 1, 2, 3$ , and  $\epsilon_1 \epsilon_2 = \epsilon_3$ . A scalar product is defined by  $(f, g) = (g, f^*)$ ;  $(f, g + h) = (f, g) + (f, h)$ ;  $(f, ga) = (f, g)a$ , and the norm is given by  $(f, f) \equiv \|f\|^2 \geq 0$ ;  $\|f\| = 0$  if and only if  $f = 0$ . A right module with these properties, closed under the topology defined by this norm, will be called a Hilbert  $H$ -module,  $\mathcal{H}_H$ . Such a space may be thought of as describing a one-particle quantum theory. To construct a tensor product space, we make use of a theorem of Horwitz and Biedenharn[3][6].

Let us choose an algebra  $\mathcal{L}$  of mappings  $\mathcal{H}_H \rightarrow \mathcal{H}_H$  which is  $\star$ -isomorphic to the quaternion algebra  $H$ , generated by  $1, E_i (i = 1, 2, 3)$  over the reals (with  $E_1 E_2 = E_3, E_i^2 = -1, E_i^* \equiv -E_i$ ), and require the operator bound  $\|E_i\| = 1$  for each  $i$ . It then follows that the adjoint  $E_i^\dagger = E_i^* = -E_i$ . With the help of this lemma, the theorem [3] states that there is a decomposition

$$f = \sum_{i=0}^3 f_i \epsilon_i, \quad (1)$$

where

$$f_i = \sum_{j=0}^3 E_j (f \epsilon_i^*) \epsilon_j^* \quad (2)$$

satisfies

$$E_k f_i^* = f_i \epsilon_k. \quad (3)$$

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and the scalar products  $(f_i, f_j)$  are real. We call such  $f_i$  "formally real". In fact, with the choice  $E_i = \int dx |x > \epsilon_i < x|$ , where  $|x >$  is the Dirac ket for the spectral representation of the self-adjoint position operator,  $\langle x | E_j f_i \rangle = \epsilon_j \langle x | f_i \rangle = \langle x | f_i \rangle \epsilon_j$ , and hence one has  $\langle x | f \rangle = \sum_{i=0}^3 \langle x | f_i \rangle \epsilon_i$ , with  $\langle x | f_i \rangle$  real. We identify the choice of  $\mathcal{L}$  in each copy of  $\mathcal{H}_H$  in the tensor product  $\mathcal{H}_H \otimes \mathcal{H}_H \otimes \dots$ , and write  $\epsilon_k$  for  $E_k$  on the left as well as the right (the  $f_i$  then appear as real valued). We may then represent the tensor product as

$$f_1 \otimes f_2 \otimes \dots \otimes f_N = \sum_{i_1, \dots, i_N} (f_1^{i_1} \otimes f_2^{i_2} \otimes \dots \otimes f_N^{i_N}) \cdot (\epsilon_{i_1} \otimes \epsilon_{i_2} \otimes \dots \otimes \epsilon_{i_N}); \quad (4)$$

the problem of constructing a tensor product for quaternion Hilbert modules therefore reduces to that of constructing the tensor product for quaternion algebras. The linear span  $D$  of the elements  $\epsilon_{i_1} \otimes \epsilon_{i_2} \otimes \dots \otimes \epsilon_{i_N}$  forms a large algebra (multiplication is defined by products in the corresponding place). To embed elements of the form (4) (a right  $D$  module) into quaternionic Hilbert space, it suffices to define a scalar product with values in  $H$ . Fermi-Dirac and Bose-Einstein statistics are reflected in the symmetrization of the formally real elements; the scalar product of elements in  $D$  must be totally symmetric, multilinear and non-negative.

Let us define a map of  $d \in D$  into the complex  $d \mapsto (d)_\epsilon \in R[\epsilon]$  by extracting only the complex part  $(1, \epsilon)$  for some  $\epsilon \in \text{Im} H$  of each factor, and carrying out ordinary (complex) multiplication. This operation puts into correspondence the element  $\epsilon$  in any position in the tensor product. We define the left ideal  $A_\epsilon$  generated by the elements  $\{[\epsilon]_i - [\epsilon]_i\}$ , where  $[\epsilon]_i = 1 \otimes 1 \dots \otimes \epsilon \otimes 1 \otimes \dots \otimes 1$ , with  $\epsilon$  in the  $i^{\text{th}}$  position ( $A_\epsilon = \{0\}$  if  $N = 1$ ). The quotient  $D/A_\epsilon = D_\epsilon$  is a left module. For  $d \in D$ ,  $(d)_\epsilon = 0$  if  $d \in A_\epsilon$ ,  $(d^* d)_\epsilon \geq 0$ , and is zero if and only if  $d \in A_\epsilon$ . With these properties, we can define an  $R[\epsilon]$  valued scalar product in  $D_\epsilon$ . Define  $1_{D_\epsilon} = 1_D + A_\epsilon$ ; every  $d \in D_\epsilon$  has the form  $d + A_\epsilon = d \cdot 1_{D_\epsilon}$ ; then,

$$(d_1 \cdot 1_{D_\epsilon} \cdot d_2 \cdot 1_{D_\epsilon})_{R[\epsilon]} = (d_1, d_2)_\epsilon = (d_1^* d_2)_\epsilon. \quad (5)$$

With (4), this definition defines the embedding of the tensor product of quaternion Hilbert modules into a complex Hilbert space ((5) is complex linear). This construction is equivalent to the complex tensor product proposed by Horwitz and Biedenharn [3] [7]. The generalization to the full quaternion case is made by defining  $A = \bigcap_\epsilon A_\epsilon$ , as the appropriate ideal, and  $D/A = D(H)$ . The natural projection of  $d$  into the reals (see (2) for  $i = 0$ ),  $\sum \epsilon_{j_1}^* \otimes \dots \otimes \epsilon_{j_N}^* d \epsilon_{j_1} \otimes \dots \otimes \epsilon_{j_N}$  contains a part in  $A$ ; in fact, a mapping into the reals which satisfies all of our requirements is given by

$$(d)_{\mathcal{R}} \cdot 1_{D(H)} = \frac{1}{(N+1)! 2^N} \sum_{\sigma \in S_N} \sum_{j_1, \dots, j_N} \epsilon_{j_1}^* \otimes \dots \otimes \epsilon_{j_N}^* d \epsilon_{\sigma(j_1)} \otimes \dots \otimes \epsilon_{\sigma(j_N)} \epsilon_A. \quad (6)$$

from which one can prove the iterative formula [5]  $((q)_{\mathcal{R}} = Rcq)$

$$(q_1 \otimes \dots \otimes q_N)_{\mathcal{R}} = \frac{1}{N+1} [(q_N q_1 \otimes q_2 \otimes \dots \otimes q_{N-1})_{\mathcal{R}} + \dots + (q_1 \otimes \dots \otimes q_N q_{N-1})_{\mathcal{R}} + 2Rcq_N(q_1 \otimes \dots \otimes q_{N-1})_{\mathcal{R}}]. \quad (7)$$

The scalar product in  $D(H)$ , defined by

$$(d_1 \cdot 1_{D(H)}, d_2 \cdot 1_{D(H)})_H = \sum_i (d_1^* (1 \otimes \dots \otimes e_i^*) d_2) \kappa e_i \quad (8)$$

provides an embedding of the space (4) into a quaternion Hilbert module (vectors in the Fock space contain  $d \otimes 1$  in the algebraic factor to preserve symmetry under (8)). One can then construct annihilation-creation operators for BE and FD statistics which, however, satisfy the algebraic relations

$$\begin{aligned} & [a_{N+1}(f) a_N^\dagger(g) \mp \frac{N+2}{N+3} a_N^\dagger(g) a_N(f)] \Psi_H(g_1, \dots, g_N) = \\ & = \Psi_H(g_1, \dots, g_N) \bullet \frac{1}{N+3} \sum_{j=1}^N (\pm)^{N-j} \Psi_H(g_1, \dots, g_{j-1}, g_{j+1}, \dots, g_N, g(f, g_j)). \quad (9) \end{aligned}$$

where the heavy dot indicates (normalized) distributive multiplication, indicating a deformation of the usual annihilation-creation algebra (the same method applied to the complex structure (5) does not contain the extra factor). This type of deformation occurs, for example, in the well-known quantum group  $SU_q(2)$ . We also note that the scalar product for  $N=1$  does not coincide with  $(f, g)$ :

$$(\Psi(f), \Psi(g)) = \frac{1}{3} [(f, g) + 2\text{Re}(f, g)]. \quad (10)$$

In fact, multiplying  $\Psi(g)$  by the annihilation operator  $a(f)$ , one obtains

$$a(f) \Psi(g) = \Psi_0 \cdot \frac{1}{3} [(f, g) + 2\text{Re}(f, g)]. \quad (11)$$

The one-particle state is the non-trivial tensor product of one particle with the vacuum; the one particle sector therefore does not coincide with the one-particle theory which is not embedded in a Fock space.

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## A RELATIONSHIP BETWEEN LOCAL AND IRREDUCIBLE REALIZATIONS OF LIE GROUPS

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**Abstract.** We study the form of an operator which connect local and irreducible realizations (which are not necessarily representations) of Lie groups which have a general semidirect product structure. We apply this development to a simple but non trivial case: the unidimensional Galilei group

### 1. INTRODUCTION

Since the work of Hoogland [1] it is well known the important role that local realizations (l.r.l.) of a group have in describing the kinematical symmetries of a physical system in Quantum Mechanics. Along this paper we shall use the term "realization" instead of "representation up to a factor", as sometimes appears in the literature and will be defined precisely bellow. The mathematical development of this theory can be found in [2], where it was introduced the idea of the representation group in order to classify and compute l.r.l. by means of local representations (l.r.p.) in the easiest possible way. In general, a l.r.l. is not irreducible and we will study here a particular case of the problem of decomposing a l.r.l. of a Lie group in terms of its unitary irreducible (sub)realizations (u.i.r.l.). Each irreducible subrealization can be selected by making use of invariant equations and may be associated to an elementary system. An investigation of this question in the language of fibre bundles for a certain kind of groups can be found, for instance in [3]. Here we shall work within a space of wave functions and we shall restrict ourselves to groups which have a general semidirect product structure so that our results comes into well known formulas when some of its components are abelian.

### 2. LOCAL VERSUS IRREDUCIBLE GROUP REALIZATIONS

Let  $G = X \oplus H$  a connected Lie group of symmetries whose structure is that of a semidirect product, where  $X$  is an invariant subgroup. The group  $X$  can be thought of as a homogeneous space under the action of  $G$  and from now on it will be identified physically with a submanifold of the space-time. We shall designe by  $\mathcal{H}$  the vector space of wavefunctions of  $n$  components,  $\psi : X \rightarrow \mathbb{C}^n$ . Then a local realization [1,2] of  $G$  in  $\mathcal{H}$  is a set of local operators of  $\mathcal{H}$ ,  $\mathcal{U}(g), \forall g \in G$ , which are defined by

$$(\mathcal{U}(g)\psi)(gx) = A(g,x)\psi(x), \forall x \in X, g \in G, \quad (2.1)$$

and such that

$$\mathcal{U}(g)\mathcal{U}(g') = \omega(g,g')\mathcal{U}(gg'), \forall g, g' \in G, \quad (2.2)$$

where  $A : G \times X \rightarrow GL(\mathbb{C}^n)$  is a non singular Borel matrix valued function and  $\omega : G \times G \rightarrow U(1) \equiv T$  is a Borel factor system of  $G$ , i.e.,  $\omega$  belongs to a class of

$H^2(G, U(1))$ . When the factor system  $\omega$  in (2.2) is always equal to one it is said that  $\mathcal{U}$  is a local representation of  $G$ . In physics one is usually interested in realizations rather than in representations, and this means one more complication. However, local realizations can be studied in terms of local representations by means of a representation group. It can be shown [2] that, given  $G$ , there exists a (non-unique) group  $\overline{G}$ , minimal in some sense, called representation group of  $G$ , which consists in a central extension of  $G$  by an abelian group  $A$ , i.e., the sequence  $1 \rightarrow A \rightarrow \overline{G} \rightarrow G \rightarrow 1$  is exact, and which has the property that any l.r. of  $G$  can be lifted to a local representation (l.r.p.) of  $\overline{G}$ . More precisely, if  $\mathcal{U}$  is a l.r. of  $G$ , there exists a l.r.p.  $\overline{\mathcal{U}}$  of  $\overline{G}$  (where the action of  $\overline{G}$  on  $X$  is via the canonical epimorphism  $p: \overline{G} \rightarrow G$ , by  $\overline{g}x \equiv p(\overline{g})x, \overline{g} \in \overline{G}, x \in X$ ), and a Borel section  $r: G \rightarrow \overline{G}$ , such that

- a)  $\overline{\mathcal{U}}(A) \subset T$
- b)  $\mathcal{U}(g) = \overline{\mathcal{U}}(r(g)), \forall g \in G$ .

Condition a) is referred to as saying that  $\overline{\mathcal{U}}$  is an  $A$ -split l.r.p. of  $\overline{G}$ . When  $G = X \odot H$ , as it is our present case, it can be checked that  $\overline{G} = (AX) \odot H$ , where the invariant subgroup  $(AX)$  is a central extension of  $X$  by  $A$ . It can also be proved that all the l.r.p. of  $\overline{G}$  can be found by induction in the following way. Let  $D$  an  $A$ -split matrix representation of  $A \otimes H$  and  $\sigma: X \rightarrow \overline{G}$  a normalized Borel section, then a l.r.p.  $\mathcal{U}_D$  of  $\overline{G}$  is obtained by induction from  $D$  and has the form:

$$(\mathcal{U}_D(\overline{g})\psi)(\overline{g}x) = D(\sigma(\overline{g}x)^{-1}\overline{g}\sigma(x))\psi(x). \quad (2.3)$$

Now we turn to characterize the unitary irreducible representations (u.i.r.p.) of  $G$ , which can be used to study the u.i.r. of  $G$  in the same way as it happened with local realizations. The u.i.r.p. of  $\overline{G}$  can be computed by means of a well known theorem of Mackey [4]. Roughly speaking the method can be applied to the present situation as follows. Let  $\chi$  be a u.i.r.p. of  $(AX)$ , and  $h \in H$ , then  $h[\chi]$  is a new representation defined by  $h[\chi](\overline{a}) = \chi(h^{-1}\overline{a}h), \forall \overline{a} \in (AX)$ . If  $(AX)^*$  designates the set of classes of u.i.r.p. of  $(AX)$ , the foregoing action of  $H$  on  $(AX)^*$  divides that set into orbits. We construct the closed subgroup  $\overline{G}_\chi = (AX) \odot H_\chi$ , made up of the elements  $\overline{g} \in \overline{G}$  such that  $\overline{g}[\chi] \simeq \chi$ , and  $W(h), h \in H_\chi$ , will denote the operator that realizes the equivalence

$$h[\chi](\overline{a}) = W(h)^{-1}\chi(\overline{a})W(h), \forall \overline{a} \in (AX). \quad (2.4)$$

Fixed a u.i.r.p.  $L$  of  $H$ , a unitary realization  $(\chi, L)$  of  $\overline{G}_\chi$  has the form:

$$(\chi, L)(\overline{a}, h) = \chi(\overline{a})W(h) \otimes L(h), \forall (\overline{a}, h) \in (AX) \odot H_\chi.$$

When the second cohomology group is trivial, i.e.,  $H^2(H, U(1)) = 0$ , and if there exists a Borel set that intersects in a single point each orbit of  $(AX)^*$ , Mackey's theorem states that any u.i.r.p. of  $G$  can be obtained, up to equivalence, by induction from the representations  $(\chi, L)$  of  $\overline{G}_\chi$ . Explicitly, if  $s: \rho \subset \overline{G}/\overline{G}_\chi \rightarrow \overline{G}$ , being  $\rho$  an orbit under  $G$ , is a normalized Borel section, a u.i.r.p.,  $\mathcal{U}_{(\chi, L)}$ , have the form

$$(\mathcal{U}_{(\chi, L)}(\overline{g})\tilde{\psi})(\overline{g}c) = (\chi, L)(s(\overline{g}c)^{-1}\overline{g}s(c))\tilde{\psi}(c), \overline{g} \in \overline{G}, c \in \rho. \quad (2.5)$$

### 3. CHARACTERIZATION OF AN INTERTWINING OPERATOR

We shall express the operator which realizes the equivalence between a subrepresentation of  $\mathcal{U}_D$  and the u.i.r.p.  $\mathcal{U}_{(\chi, L)}$  by means of a functional in the form

$$\psi(x) = \int_{\rho} K(x, c) \tilde{\psi}(c) d\mu(c), \quad (3.1)$$

where  $\rho$  is an orbit of  $(AX)^*$  under  $G$ ,  $\mu(c)$  the invariant measure (in general it can be quasiinvariant) of the homogeneous space  $\overline{G}/\overline{G}_{\chi}$  under  $G$ , and  $K(x, c)$  an operator which acts on the space that supports the representation  $(\chi, L)$  of  $G_{\chi}$ . In these circumstances, after a straightforward computation we find

$$K(x, c) = D(s(c)) K(\hat{c}) s(c) [\chi](\sigma(x))^{-1}, \quad (3.2)$$

where  $\hat{c}$  is the class of the identity in  $\overline{G}/\overline{G}_{\chi}$ , and the operator  $K(\hat{c})$  verifies

$$D(\gamma) K(\hat{c}) = K(\hat{c}) W(\gamma) \otimes L(\gamma), \forall \gamma \in A \otimes H_{\chi}. \quad (3.3)$$

So,  $K(\hat{c})$  is an intertwining operator between the representations  $D$  and  $W \otimes L$  when they are restricted to  $A \otimes H_{\chi}$ , in particular they must coincide in the abelian subgroup  $A$ . The form of the operator (3.1) given by (3.2) and (3.3) allow us to classify the kind of equations that a wavefunction of a l.r.l. must obey if it is to be associated to an elementary system. First, due to the fact that the integral (3.1) is extended to an orbit, it must satisfy an equation that we call of orbital type. When  $D(\gamma)$  and  $L(\gamma)$  in (3.3) are matrices, condition (3.3) gives rise to an equation of a matricial character and even if they are unidimensional we get an equation which we call of equivalence. If  $(AX)$  is a trivial extension of  $X$  by  $A$  and if  $X$  is abelian then the nucleus (3.2) becomes that of a Fourier transform and this happens to be the case for the three-dimensional (in space) Poincaré and Galilei groups. However, in general it may be not so simple as can be seen in the following example.

### 4. THE UNIDIMENSIONAL GALILEI GROUP

The elements of  $\overline{G}$  have the form [5,6]  $(\eta, \theta, b, a, v) \equiv (\eta, \theta, g) \equiv \overline{g}$ , with  $(\eta, \theta) \in A$ , and  $a, b, v$  the parameters of the space-time translations and inertial transformations, respectively. The composition law is

$$(\eta, \theta, b, a, v)(\eta', \theta', b', a', v') = (\eta + \eta' + \omega_1(g, g'), \theta + \theta' + \omega_2(g, g'), b + b', a + a' + b'v, v + v'),$$

with  $\omega_1(g, g') = \frac{1}{2}vb'^2 + ab'$ ,  $\omega_2(g, g') = \frac{1}{2}v^2b' + va'$ . In this case,  $(AX) \equiv \langle (\eta, \theta, b, a) \rangle$  and  $H \equiv \langle (v) \rangle$ . Physically, the extension corresponding to the parameter  $\theta$ ,  $[K, P] = I_{\theta}$ , is associated to the mass of the system, while that corresponding to  $\eta$ ,  $[P, H] = I_{\eta}$ , has to do with the possibility of a non null background constant field which is consistent with the Galilei (and Poincaré) group only in one space dimension. The u.i.r.p. of  $(AX)$  take the form

$$\left( \chi(\eta, \theta, b, a) \tilde{\psi} \right) (p) = \exp\{i(m\theta + f\eta + pa)\} \tilde{\psi}(p - fb), m, f \in \mathbf{R}^* \quad (4.1)$$

and the representation  $W \otimes L$  of  $H_\chi = H = \langle (v) \rangle$  is given by

$$W \otimes L(v) = \exp\{\Omega v + ikv\}, k \in \mathbf{R}^*,$$

where  $\Omega = ip^2/(2f) + m\partial_p$ . As for the local representations of  $\overline{G}$ , they are induced by the matrix representation  $D$  of  $A \otimes H \equiv \langle (\eta, \theta, v) \rangle$  which is of the form  $D(\eta, \theta, v) = e^{i(f\eta + m\theta)}$ . In this special case, since  $H_\chi$  coincides with  $H$ , the orbit  $\rho$  consists just in one single point  $\hat{c}$ , so the only components of the nucleus (3.2) to be found are  $\chi(\sigma(x))^{-1}$  and  $K(\hat{c})$ . Let  $x = (t, \mathbf{x})$  and  $\sigma(x) = (0, 0, t, \mathbf{x}, 0)$ , then

$$\chi(\sigma(x))^{-1} = \exp\{ft\partial_p - ip\mathbf{x} + \frac{1}{2}if\mathbf{x}t\}, \quad (4.2)$$

while it is easy to check that  $K(\hat{c})$  is, up to a constant factor, a projection operator of the form

$$K(\hat{c}) = \int F(p)^* \tilde{\psi}(p) dp,$$

$F(p)$  verifying  $(\Omega + ik)F(p) = 0$ ; explicitly  $F(p) = \exp\{-ip^3/(6mf) - ikp/m\}$ . Finally we have for the operator (3.1):

$$\psi(t, \mathbf{x}) = \int F(p)^* \exp\{ft\partial_p - ip\mathbf{x} + \frac{1}{2}if\mathbf{x}t\} \tilde{\psi}(p) dp. \quad (4.3)$$

It can be checked that this operator is unitary with respect to the inner products defined in the spaces supporting these representations. We must remark here that this operator is useful in order to clarify the physical meaning of the parameters  $m$ ,  $f$  and  $k$  which label each u.i.r.p. of  $\overline{G}$  and to establish the invariant equations [6].

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## Skein relations for any pair of irreducible representations of any Lie group

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A general approach to monodromy skein relations for any pair of irreducible representations of any semi-simple Lie group in the framework of topological Yang-Mills theory is proposed. The explicit form of the monodromy relations as well as the explicit solutions of some reduction formulas are given.

Skein relations are an important tool in studying low-dimensional topology of knots and links, giving a possibility of computing polynomial invariants in a recursive way. The recently discovered *physical* approach to skein relations utilizes some models of topological Yang-Mills theory, namely Chern-Simons theory [1] and the, so-called, *BF* theory [2]. It seems that any pair of irreducible representations of any semi-simple Lie group gives rise to a skein relation. In the first paper on this subject [1] the three-dimensional Chern-Simons approach was effectively reduced to two dimensions, and next the apparatus of conformal field theory was employed to the case of the fundamental representation of  $SU(N)$  group. The case of fundamental representations of other Lie groups has been considered in [3]. A genuinely three-dimensional approach appeared in [4], and a non-perturbative version of it in the framework of Chern-Simons theory in [5], and also in [6] in the framework of *BF* theory (with the possibility of a generalization to some higher-dimensional links), but only for the fundamental representation of  $SU(N)$ . A four-dimensional Yang-Mills formulation of link invariants has been suggested in [7], whereas a generalization to other representations (in conjunction with the concept of quantum groups) is worked out in [8].

In this work we propose a general and elementary approach to skein relations for any pair of irreducible representations of any semi-simple Lie group  $G$  in the framework of topological Yang-Mills theory with the same gauge group  $G$  [9]. Strictly speaking, since the analytical and geometrical side of the approach has already been developed we will confine ourselves only to the algebraical aspects (which are independent of the specific geometrical description), and our present work should be considered as a group-theoretical complement of [5-7], where only the case of the fundamental representation of  $SU(N)$  group has been analysed. It appears that the key calculational object in these approaches is the monodromy matrix  $M$  of two Wilson lines (possibly, in the higher-dimensional case, generalized Wilson observables [6,7]), rather than the braid matrix. It means that the prominent role is played by the, so-called, coloured (or pure) braid group rather than by the (full) braid group.

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Let  $t_\sigma^a$  be generators of the irreducible representations  $R_\sigma(G)$  of the semi-simple  $d_G$ -dimensional Lie group  $G$ ,  $a = 1, 2, \dots, d_G$ ,  $\sigma = 1, 2$ . Physically, the group  $G$  is the gauge group of some topological Yang-Mills theory, whereas the representations  $R_\sigma(G)$  appear in the (generalized) Wilson observables  $W_{R_\sigma}$ . From topological field theory follows the monodromy matrix [5-7]

$$M = e^{2\lambda T}, \quad T = \sum_{a=1}^{d_G} t_1^a \otimes t_2^a, \quad (1.2)$$

where the coupling constant  $\lambda = \frac{2\pi}{ik}$ , and the level  $k \in \mathbf{Z}^\pm$ . The expression (2) can be rewritten in the form

$$T = \frac{1}{2} \sum_{a=1}^{d_G} (t_1^a \otimes \mathbf{1}_2 + \mathbf{1}_1 \otimes t_2^a)^2 - \frac{1}{2} \sum_{a=1}^{d_G} (t_1^a t_1^a \otimes \mathbf{1}_2 + \mathbf{1}_1 \otimes t_2^a t_2^a). \quad (3)$$

The first term in (3) contains the generator of  $G$  in the (in general, reducible) product representation  $R(G) = R_1(G) \otimes R_2(G)$

$$t^a = t_1^a \otimes \mathbf{1}_2 + \mathbf{1}_1 \otimes t_2^a. \quad (4)$$

For irreducible representations the second term considerably simplifies, namely

$$\sum_{a=1}^{d_G} t_\sigma^a t_\sigma^a = C_\sigma, \quad (5)$$

where  $C_\sigma$  is the second order Casimir operator. Hence

$$T = \frac{1}{2} \sum_{a=1}^{d_G} t^a t^a - \frac{1}{2} C, \quad C = C_1 + C_2. \quad (6)$$

The Clebsch-Gordan expansion

$$R_1(G) \otimes R_2(G) = \bigoplus_{\alpha=1}^D R_\alpha(G) \quad (7)$$

gives rise to the following generalization of (5):

$$\sum_{a=1}^{d_G} t^a t^a = \sum_{\alpha=1}^D C_\alpha \mathcal{P}_\alpha. \quad (8)$$

where  $\mathcal{P}_\alpha$  are orthogonal projectors onto irreducible components  $R_\alpha(G)$  of  $R(G)$ .  $\sum_{\alpha=1}^D \mathcal{P}_\alpha = \mathbf{1}$ ,  $\mathcal{P}_\alpha \mathcal{P}_\beta = \delta_{\alpha\beta} \mathcal{P}_\alpha$ . According to (6) and (8) the monodromy matrix (which geometrically can correspond to the full-twist of two Wilson lines) is expressed by the spectral decomposition

$$M = e^{-\lambda C} \sum_{\alpha=1}^D e^{\lambda C_\alpha} \mathcal{P}_\alpha. \quad (9)$$

Analogously, the zero and the positive integer powers of  $M$  (which can correspond to the zero-twist and the  $n$ -twist respectively) are given by the following spectral formulas

$$M^0 = \mathbf{1}, \quad M^n = \epsilon^{-\lambda C n} \sum_{\alpha=1}^D \epsilon^{\lambda C_\alpha n} \mathcal{P}_\alpha. \quad (10)$$

A general monodromy skein relation is expressed by the polynomial in  $M$  of the degree  $N$  ( $N \geq 2$ )

$$\sum_{n=0}^N A_n M^n = 0. \quad (11)$$

Using (10) we can put (11) in the form

$$\sum_{\alpha=1}^D \mathcal{P}_\alpha \sum_{n=0}^N A_n \epsilon^{-\lambda C n} \epsilon^{\lambda C_\alpha n} = 0, \quad (12)$$

which is equivalent to the system of equations

$$\sum_{n=0}^N A_n q_\alpha^n = 0, \quad \alpha = 1, 2, \dots, D, \quad q_\alpha \equiv \exp \lambda(C_\alpha - C) \equiv q^{C_\alpha - C}. \quad (13, 14)$$

Due to possible multiplicities in the Clebsch-Gordan expansion (7), and possible accidental degenerations (in general) not all  $q_\alpha$  are different. Thus we can confine ourselves to the subset

$$\{q_\mu \mid q_\mu \neq q_\nu \text{ for } \mu \neq \nu\} \subseteq \{q_\alpha\}, \quad \mu, \nu = 1, 2, \dots, d \leq D, \quad (15)$$

to eliminate identical equations in the system (13). We should also assume  $N = d$  because for  $N = d - 1$  the solubility of the system (13) would be guaranteed by the vanishing Vandermonde determinant

$$V_d \equiv \det q_\mu^n = \prod_{1 \leq \mu < \nu \leq d} (q_\mu - q_\nu) = 0, \quad (16)$$

which is impossible by virtue of our earlier assumption (15). The explicit solution of the system (13) is of the form

$$A_n = \begin{cases} (-)^{d-n} \sum_{1 \leq \mu_1 < \mu_2 < \dots < \mu_{d-n} \leq d} \prod_{i=1}^{d-n} q_{\mu_i}, & \text{for } 0 \leq n < d, \\ 1 \text{ (normalization),} & \text{for } n = d. \end{cases} \quad (17)$$

A link  $L$  is defined as a finite collection of disjoint simple curves in the three-dimensional manifold  $\mathcal{M}^3$ . The links entering a skein relation can differ, in a suitable way, inside some ball  $\mathbf{B}^3$  containing only four-valent graphs ( $\mathbf{B}^3 \subset \mathcal{M}^3$ ). They are denoted by  $L_{2n}$ , where  $2n$  is the number of half-twists (in general, also an odd number of half-twists is allowed). For simplicity,  $L_n$  denotes not only a link itself but also (in algebraical relations) a Laurent polynomial associated to  $L_n$  or, in physical language,

the expectation value  $\langle W_{R_n}(L_n) \rangle$  of related Wilson observables. Since the monodromy matrix  $M$  corresponds to the operation of the twist one can obtain the (monodromy) skein relation

$$\sum_{n=0}^d A_n L_{2n} = 0. \quad (18)$$

saturating the indices of  $M$ 's in (11) with the terms that correspond to the part of the links  $L_{2n}$  outside the ball  $\mathbf{B}^3$ .

It is often possible and even desirable (intuitively, it is obvious that the less range of the indices of  $L_n$  the more calculational power of the skein relation) to find the corresponding braid relation

$$\sum_{n=0}^d B_n L_n = 0. \quad (19)$$

where the number of half-twists is no longer necessary even. The explicit form of the coefficients  $B_n$  follows from the consistency relations

$$\sum_{m=0}^d \beta_m \sum_{n=0}^d B_n L_{m+n} = \sum_{n=0}^d A_n L_{2n}, \quad (20)$$

which is geometrically equivalent to the reduction of the monodromy skein relation (18) by means of the repeated use of (19) with shifted  $n$ 's:  $\beta$ 's are some auxiliary coefficients. It should be emphasized that the existence itself of the relation (19) is by no means guaranteed, and depends on the case under consideration. For example, in the case of different representations  $R_1(G)$  and  $R_2(G)$  the reduction is excluded. The solution of the equation(s) (20) is of the form

$$B_n = \begin{cases} \sum_{1 \leq \mu_1 < \mu_2 < \dots < \mu_{d-n} \leq d} \prod_{i=1}^{d-n} Q_{\mu_i}, & \text{for } 0 \leq n < d, \\ 1 \text{ (normalization)}, & \text{for } n = d. \end{cases} \quad (21)$$

$$Q_{\mu_i} = \pm \sqrt{q_{\mu_i}}, \quad \beta_n = (-)^{d-n} B_n.$$

The braid skein relation (19) is, in a sense, a square root of the monodromy skein relation (18).

It is also possible to derive some other reduction formulas. For example, in the case of non-oriented links one can perform an additional reduction obtaining the relation

$$\sum_{n=0}^{d-1} C_n L_n + C_\infty L_\infty = 0. \quad (22)$$

Making use of (22) twice we get the following consistency relation

$$\sum_{n=0}^{d-1} C_n L_n + C_\infty L_\infty + \gamma \left( \sum_{n=0}^{d-1} C_n L_{n+1} + C_\infty L_{\infty 1} \right) = \sum_{n=0}^d B_n L_n. \quad (23)$$

where  $\gamma$  is an auxiliary coefficient. The links  $L_\infty$  and  $L_{\infty 1}$  are identical to the links  $L_n$  except of the inside of the ball  $\mathbf{B}^3$  where  $L_{\infty 1}$  possesses one half-twist in its lower part. Solving (23) we obtain  $d$  solutions of the relation (22) numbered by  $\mu$

$$C_n = \sum_{l=0}^n (-)^l \gamma^l B_{n-l}, \quad 0 \leq n < d, \quad \gamma = Q_\mu^{-1}, \quad \mu = 1, 2, \dots, d, \quad (24)$$

where  $B_n$  and  $Q$ 's are given in (21). More explicitly, the solutions have the form

$$C_n^{(\mu)} = \begin{cases} \sum_{1 \leq \mu_1 < \mu_2 < \dots < \mu_{d-n-1} \leq d}^{(\mu)} \prod_{i=1}^{d-n-1} Q_{\mu_i}, & \text{for } 0 \leq n < d-1, \\ 1 \text{ (normalization)}, & \text{for } n = d-1. \end{cases} \quad (25)$$

where the symbol  $\sum^{(\mu)}$  denotes that  $\mu_i \neq \mu$  for  $i = 1, 2, \dots, d-n-1$ . Unfortunately, the coefficient  $C_\infty$  cannot be determined by the consistency relation (23) (an additional, e. g. rotational symmetry of the graph has to be invoked), instead we obtain a relation

$$L_{\infty 1} = -\gamma^{-1} L_\infty. \quad (26)$$

Since, in general,  $\gamma \neq 1$  the third Reidemeister move is not preserved and the relation (22) describes a link invariant of regular isotopy rather than of ambient isotopy.

In the context of quantum groups, it is worth to note that the monodromy matrix (1.2.9) is proportional to the generalized  $R$ -matrix [S] of the quasi-triangular quasi-Hopf algebra. Thus topological field theory can be used to generate solutions of  $R$ -matrix for any pair of irreducible representations of any semi-simple Lie group.

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## GROUP EXTENSIONS AND UNIFICATION THEORIES

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ABSTRACT. The structure of a group extension  $1 \rightarrow N \hookrightarrow G \rightarrow D \rightarrow 1$  and its representations are analyzed from the point of view of unification theories. In particular, a realization of a  $G$  representation in a space of functions on the quotient group  $D \simeq G/N$  is given explicitly.

**1. Introduction**

An unification of two symmetry theories based on groups  $G_0$  and  $G_1$  consists usually in finding a larger group  $G$  that contains  $G_0$  and  $G_1$  as subgroups. The simplest possibility is a group  $G$  containing the direct product  $G_0 \times G_1$ . Such an embedding was utilized in Wigner's unification of spin and isospin [1], where:

$$G_0 = \text{SU}(2)_{\text{isospin}} \subset \text{SU}(4) \supset \text{SU}(2)_{\text{spin}} = G_1.$$

In this case, the link between states of the two unified theories is given by transformations that are not directly related to  $G_0$  or  $G_1$ .

The mathematical structure of a semidirect product  $G = G_0 \rtimes G_1$  or, generally, of a group extension automatically includes an unification link between the symmetries. A group  $G$  is an extension of  $G_0$  by means of  $G_1$  iff the following short sequence is exact:

$$1 \rightarrow G_0 \rightarrow G \rightarrow G_1 \rightarrow 1.$$

It implies, in particular, that  $G_0$  is isomorphic to a normal subgroup  $N \subset G$ , that  $G_1 \simeq D = G/N$ , and that there exists a mapping  $\psi: G_1 \rightarrow \text{Aut } G_0$  which provides a link between both groups. This link might be interpreted in the following way: the group  $G_1$  *knows* how to act on observables related to the symmetry  $G_0$  (although, in general, it *might not know* how to act on the corresponding states). If  $\psi$  is a homomorphism then  $G$  is called a central extension and if, moreover,  $G_1$  can be pulled back into  $G$  as a complementary subgroup to  $G_0$  then  $G$  is a semidirect product.

Our goal is to present and analyze some results supporting the idea of using group extensions for unification purposes. For instance, many physical relations implied by a symmetry based on a semisimple Lie algebra  $\mathfrak{su}(n)$  are, in fact, a consequence of an action of a Weyl group  $G_1 = W(R)$  on observables that is constructed by using a representation of an extension  $1 \rightarrow T \hookrightarrow G \rightarrow W(R) \rightarrow 1$ , where  $T$  is a maximal torus. A combination of this extension with an extension of the group  $W(R)$  to the group  $A(R)$  (of all automorphisms of the corresponding root system  $R$ ) leads to an unification capable of describing conjugation of general charges (e.g. particle-antiparticle conjugation), cf. Section 2.

In Section 3, we present a strictly mathematical result whose derivation was motivated by the fact that (e.g. from the point of view of physical relations discussed above) quotient groups are the most essential. Namely, we explicitly show how representations of an arbitrary group extension  $G$  can be realized in a subspace of the space of vector valued functions on the quotient group  $D = G/N$ .

## 2. Group extensions and physical relations

Let  $\mathfrak{g}$  be a complex semisimple Lie algebra whose compact form (e.g.,  $\mathfrak{su}(n)$ ) describes a given physical symmetry. The group  $\text{Int } \mathfrak{g}$  (of all inner automorphisms) is a Lie group corresponding to  $\mathfrak{g}$  and could be considered as (a complexification of) a symmetry group.

In a previous paper [2], it was shown that in order to explain the mathematical origin of physical relations such as mass or magnetic momentum formulas, we must choose first a Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{g}$  that describes the basic physical observables of the theory. One might say that  $\text{Int } \mathfrak{g}$  unifies a symmetry related to the freedom of choice of the Cartan subalgebra (choice of a reference system?) with a symmetry related to a fixed Cartan algebra. The latter symmetry is described by those elements in  $\text{Int } \mathfrak{g}$  which preserve the Cartan algebra  $\mathfrak{h}$ , i.e., elements from the group  $\text{Int}(\mathfrak{g}, \mathfrak{h})$ . This group is the following extension by means of the Weyl group  $W(R)$ :

$$1 \rightarrow e^{\text{ad } \mathfrak{h}} \hookrightarrow \text{Int}(\mathfrak{g}, \mathfrak{h}) \rightarrow W(R) \rightarrow 1.$$

On the other hand, in [3] and [4], it was shown that conjugation of general charges (for instance under particle-antiparticle symmetry) can be included in the group description of the theory by considering extensions of  $\text{Int}(\mathfrak{g}, \mathfrak{h})$  and  $W(R)$  by means of the group

$$\text{Out } \mathfrak{g} := \text{Aut } \mathfrak{g} / \text{Int } \mathfrak{g} \simeq A(R) / W(R) \simeq D(R),$$

where  $D(R)$  is the group of all automorphisms of the corresponding Dynkin diagram. In other words, we have to consider two short exact sequences:

$$1 \rightarrow \text{Int}(\mathfrak{g}, \mathfrak{h}) \hookrightarrow \text{Aut}(\mathfrak{g}, \mathfrak{h}) \rightarrow \text{Out } \mathfrak{g} \rightarrow 1 \quad \text{and} \quad 1 \rightarrow W(R) \hookrightarrow A(R) \rightarrow D(R) \rightarrow 1.$$

Both these extensions are semidirect products, see [5].

All the considered extensions can be combined into the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \rightarrow & e^{\text{ad } \mathfrak{h}} & \rightarrow & \text{Int}(\mathfrak{g}, \mathfrak{h}) & \rightarrow & W(R) \rightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \rightarrow & e^{\text{ad } \mathfrak{h}} & \rightarrow & \text{Aut}(\mathfrak{g}, \mathfrak{h}) & \rightarrow & A(R) \rightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \rightarrow & 1 & \rightarrow & \text{Out } \mathfrak{g} & \rightarrow & D(R) \rightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 1 & & 1 & & 1
 \end{array}$$

An 'equivalent' diagram can be created if we replace (the mathematically canonical group)  $\text{Int } \mathfrak{g}$  by any connected covering  $G$  of  $\text{Int } \mathfrak{g}$ . (This fact was checked for classical simple Lie algebras and we believe this to be true for all semisimple algebras.) More precisely, let  $G_{\mathfrak{h}} \subset G$  denote the corresponding covering of  $\text{Int}(\mathfrak{g}, \mathfrak{h})$  and let  $T := \exp_G(\mathfrak{h})$  be the maximal torus corresponding to the Cartan algebra  $\mathfrak{h}$ . Then we have:

$$\begin{array}{ccccccc}
& & 1 & & 1 & & 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & T & \rightarrow & G_{\mathfrak{h}} & \rightarrow & W(R) \rightarrow 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & T & \rightarrow & G_{\mathfrak{h}} \times D(R) & \rightarrow & A(R) \rightarrow 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & 1 & \rightarrow & D(R) & \rightarrow & D(R) \rightarrow 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 1 & & 1 & & 1
\end{array}$$

Now, let  $\mathcal{P}$  be a representation of  $G_{\mathfrak{h}} \times D(R)$  acting in a vector space  $U$ . (We usually obtain  $\mathcal{P}$  by restricting a representation of  $G \times D(R)$ .) The derivation of physical relations (that are characteristic for states from  $U$ ) is based on two properties of extensions.

Namely, in the case of the 'horizontal' extensions, the most essential is the fact that the image of  $T$  coincides with the preimage of  $1 \in W(R)$  (or  $A(R)$ ). It enables us to define a canonical representation(s) of  $W(R)$  (or  $A(R)$ ) in a subspace  $\mathcal{P}(\mathfrak{h})' \subseteq \text{End } U$  consisting of observables that commute with representants of basic observables from  $\mathfrak{h}$ .

In the case of the 'vertical' extensions, the structure of representations of group extensions becomes more essential, in particular, the fact that elements of the quotient group  $D(R)$  can act between distinct invariant subspaces of  $\mathcal{P}$ , cf. [3] and [4].

### 3. Realization of representations

We shall identify  $G_0$  with the normal subgroup  $N \subseteq G$ , and  $G_1$  with the quotient group  $D = G/N$ , i.e., we shall consider the following short exact sequence:

$$1 \rightarrow N \rightarrow G \rightarrow D \rightarrow 1.$$

The set  $G$  can be identified with the set  $N \cdot D$  in such way that the subgroup  $N$  coincides with  $\{(n, 1) | n \in N\}$ . For any  $d \in D$ , the mapping  $(n, 1) \rightarrow (1, d)(n, 1)(1, d)^{-1}$  defines an automorphism in  $\text{Aut } N$ . We shall denote this automorphism by  $c(d)$ , and its value on  $n \in N$  by  $[c(d)n]$ , cf. [6]. The group operation in the set  $N \cdot D$  is given by:

$$(m, c)(n, d) = (m[c(c)n] \chi(c, d), cd),$$

where the function  $\chi(\cdot, \cdot) \in N^{D \cdot D}$  satisfies the following conditions:

$$\chi(c, d)[c(cdm)] = [c(c)v(d)m] \chi(c, d), \quad [c(b)\chi(c, d)] \chi(b, cd) = \chi(b, c) \chi(bc, d).$$

To describe the realization of  $G$  representations in function spaces on  $D$ , we consider first a  $N$  representation  $\mathcal{R}$  acting in a (complex) vector space  $V$ . The corresponding inertia group of  $\mathcal{R}$  is given by:

$$G_{\mathcal{R}} := \{g \in G | \mathcal{R}_{g \cdot g^{-1}} \cong \mathcal{R}_{(1)}\} \subseteq N.$$

This definition implies that for the representation  $\mathcal{R}$  there exists a projective representation  $(Q, V)$  of the inertia group  $G_{\mathcal{R}}$  such that:

$$\mathcal{R}_{g \cdot g^{-1}} = Q_g \mathcal{R}_{(1)} Q_g^{-1}, \quad Q_{g_1} Q_{g_2} = \mu(g_1, g_2) Q_{g_1 g_2}.$$

where  $\mu \in Z^2(G_{\mathcal{R}}, \mathbb{C}^*)$  is a two cocycle satisfying

$$\mu(g_1, g_2)\mu(g_1g_2, g_3) = \mu(g_1, g_2g_3)\mu(g_2, g_3).$$

We can normalize  $Q$  as follows:  $Q|_N = \mathcal{R}$  and  $\mu((n, 1), g) = 1$ .

Let us consider additionally a projective representation  $T$  of the quotient group  $D_{\mathcal{R}} := G_{\mathcal{R}}/N$  acting in a vector space  $M$ . The representation  $T$  can be lifted to a projective representation of the group  $G_{\mathcal{R}}$  in the natural way:

$$T_g := T_{[g]}, [g] \in G_{\mathcal{R}}/N, \text{ i.e., } T|_N = id.$$

Due to our normalization of the two-cocycle  $\mu$ , we can assume that the lifted projective representation  $T$  also has  $\mu$  as its two-cocycle.

Now, a  $G$  representation can be realized in the space  $L^D$  of functions on  $D$  with values in  $L := \mathcal{L}(M, V)$  in the following way. For any  $g = (n, d) \in G = N \times D$ ,  $b \in D$ , and  $F(\cdot) \in L^D$ , the formula:

$$({}^gF)(b) := \mathcal{R}_{[n(b)n] \setminus (b,d)} F(bd)$$

defines a  $G$  representation. On the other hand, for any  $a \in D_{\mathcal{R}} \subset D$ , the formula:

$$({}^aF)(b) := \mathcal{R}_{\setminus (a,a^{-1}b)}^{-1} Q_{(1,a)} F(a^{-1}b) T_{(1,a)}^{-1}$$

defines a  $D_{\mathcal{R}}$  representation that commutes with  $G$  representation introduced above. Thus the subspace  $U \subset L^D$  of functions that are invariant with respect to this  $D_{\mathcal{R}}$  action carries a  $G$  representation  $\mathcal{P}$ . If we assume that  $D/D_{\mathcal{R}}$  is finite, the following refinement of the Clifford's theorem [7] can be proven:

**THEOREM.**

1. The representation  $(\mathcal{P}, U)$  coincides with the induced representation  $Ind_{G_{\mathcal{R}}}^{G_i} (Q; T^*)$ .
2. Every irreducible finite dimensional  $G$  representation is equivalent to a representation  $(\mathcal{P}, U)$ .

Here  $T^*$  denotes the representation contragredient to  $T$ . For proofs and some other results concerning the structure of representations for group extensions, see [8].

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## SEPARATION OF PURE GAUGE AND DYNAMIC DEGREES OF FREEDOM IN CHROMODYNAMICS

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### Abstract

The  $SU(3)$  gauge field theory or chromodynamics is the most popular non-Abelian gauge invariant theory. As a consequence of gauge invariance the superfluous degrees of freedom not describing the field dynamics appear in the theory. These pure gauge degrees of freedom are necessary while studying topological properties of solutions. Moreover, correct quantization must take into account any gauge transformations. It is the purpose of our work to apply the representation theory of the  $SU(3)$  group to the  $SU(3)$  gauge field theory.

Let us introduce the  $SU(3)$  group generators as the components of irreducible tensor of rank  $(1, 1)$ . They satisfy the commutation relations:

$$\left[ \hat{J}_{(y',j',m')}, \hat{J}_{(y'',j'',m'')} \right] = \left[ \begin{matrix} (1,1) & (1,1) & (1,1)_a \\ y',j',m' & y'',j'',m'' & y,j,m \end{matrix} \right] \hat{J}_{(y,j,m)}. \quad (1)$$

The first factor in the RHS is the Clebsch-Gordan coefficient of the  $SU(3)$  group and corresponds to the structure constant of the  $SU(3)$  algebra.

The  $SU(3)$  Yang-Mills field can be defined on the  $SU(3)$  algebra for the arbitrary irrep  $(\lambda, \mu)$ :

$$A_\nu(x) = A_\nu^{(\lambda,\mu)}(x) \left\langle \begin{matrix} (\lambda,\mu) \\ y,j,m \end{matrix} \left| \hat{J}_{(y,j,m)} \right| \begin{matrix} (\lambda,\mu) \\ y',j',m' \end{matrix} \right\rangle. \quad (2)$$

We have chosen the  $SU(3)$  group parametrization very close to Holland's used in [0]. The  $SU(3)$  group representation matrices are used in the form:

$$D_{y,j,m; y',j',m'}^{(\lambda,\mu)} = \left\langle \begin{matrix} (\lambda,\mu) \\ y,j,m \end{matrix} \left| T_2 T_3 T_2' \right| \begin{matrix} (\lambda,\mu) \\ y',j',m' \end{matrix} \right\rangle, \quad (3)$$

where

$$T_2 = e^{i\sqrt{3}\alpha \hat{J}_{(0,1,0)}} e^{-\frac{\sqrt{3}}{2}\beta (\hat{J}_{(0,1,1)} + \hat{J}_{(0,1,-1)})} e^{i\sqrt{3}\gamma \hat{J}_{(0,1,0)}} \quad (4)$$

$$T_3 = e^{i2\rho \hat{J}_{(0,0,0)}} e^{-\frac{\rho}{2\sqrt{3}}\varphi \left( \hat{J}_{(-1, \frac{1}{2}, \frac{1}{2})} + \hat{J}_{(1, \frac{1}{2}, -\frac{1}{2})} \right)}. \quad (5)$$

$T_2'$  is the Operator (4) with primed variables. The eight angles are defined as follows:

$$\begin{aligned} 0 \leq \alpha, \alpha' < 2\pi, \quad 0 \leq \beta, \beta' \leq \pi, \quad 0 \leq \gamma, \gamma' < 4\pi, \\ 0 \leq \rho \leq 3\pi, \quad 0 \leq \varphi \leq \pi \end{aligned} \quad (6)$$

Let us note that there remain some ambiguities of  $\alpha, \gamma$  and  $\alpha', \gamma'$ , when  $\beta, \beta'$  or  $\varphi = 0, \pi$ . The situation is similar as in the  $SU(2)$  case. The parametrization (3) allows the  $SU(3)$  group representation matrix element to be expressed through the  $SU(2)$  group Wigner  $D$ -functions:

$$D_{y,j,m; y',j',m'}^{(\lambda,\mu)}(\alpha, \beta, \gamma, \rho, \varphi, \alpha', \beta', \gamma') = \sum_{y'',u} D_{m; -\frac{1}{2}y-y''}^j(\alpha, \beta, \gamma) D_{-\frac{1}{2}y'-y; -\frac{1}{2}y''-y'}^u(0, \varphi, 0) \times \times D_{-\frac{1}{2}y'-y''; m'}^{j'}(\alpha', \beta', \gamma') e^{-iy\rho} [(2j+1)(2u+1)^2(2j'+1)]^{\frac{1}{2}} \times \times \left\{ \begin{array}{ccc} \frac{1}{2}\lambda & \frac{1}{2}(y_0-y) & j \\ \frac{1}{2}(y_0+y+y'') & \frac{1}{2}(y_0-y'') & u \end{array} \right\} \left\{ \begin{array}{ccc} \frac{1}{2}\lambda & \frac{1}{2}(y_0-y') & j' \\ \frac{1}{2}(y_0+y'+y'') & \frac{1}{2}(y_0-y'') & u \end{array} \right\}, \quad (7)$$

where  $y_0 = \frac{1}{3}(\lambda + 2\mu)$  and the two last factors in the RHS are  $SU(2)$  6j-symbols. All factors in the RHS are well defined and known as the  $SU(2)$  quantities. A very similar equation was derived by projection technique in [0]. It is known that parameters of the gauge transformations depend on the point of space. For brevity, let us not show the functional dependence parameters on  $\kappa$ . The  $SU(3)$  matrices (7) induced the gauge transformations of the Yang-Mills field:

$$A_{\nu}^{(y,j,m)} = D_{y,j,m; y',j',m'}^{(1,1)}(\alpha, \beta, \gamma, \rho, \varphi, \alpha', \beta', \gamma') A_{\nu}^{(y',j',m')} + \{ \alpha, \beta, \gamma, \rho, \varphi, \alpha', \beta', \gamma' \}_{\nu}^{(y,j,m)}, \quad (8)$$

where the last items of the RHS depend on functions  $\alpha(\kappa), \beta(\kappa), \gamma(\kappa), \dots$  and its derivatives. For example:

$$\{ \alpha, \beta, \gamma, \dots \}_{\nu}^{(0,0,0)} = -2i\partial_{\nu}\rho - \frac{3}{4}i(1 - \cos\varphi)(\partial_{\nu}\alpha' + \cos\beta'\partial_{\nu}\gamma'). \quad (9)$$

The additive component of the  $SU(2)$  gauge transformations of Yang-Mills field was obtained in [0]. The field strength tensor is defined as usual and its components are transformed under gauge transformations as components of the tensor of rank (1, 1):

$$F_{\nu\kappa}^{(y,j,m)} = D_{y,j,m; y',j',m'}^{(1,1)}(\alpha, \beta, \gamma, \dots) F_{\nu\kappa}^{(y',j',m')}. \quad (10)$$

The Lagrangian of pure Yang-Mills chromodynamics for the arbitrary irrep  $(\lambda, \mu)$  is described by

$$L_{YM} = \frac{[\lambda, \mu]^2}{4} F_{\nu\kappa}^{(y,j,m)} F_{(y,j,m)}^{\nu\kappa} \quad (11)$$

where

$$[\lambda, \mu]^2 = \frac{1}{2 \cdot 9 \cdot 16g^2} (\lambda + 1)(\mu + 1)(\lambda + \mu + 2)(\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu).$$

We can consider the Yang-Mills field as a Hamiltonian system. The the generalized coordinates are  $A_k^{(y,j,m)}$  and the generalized momenta are  $E_{k(y,j,m)} = F_{k0(y,j,m)}$ . After

renormalization with the factor  $[\lambda, \mu]$  the constraints of the Hamiltonian system may be written as follows:

$$\Gamma_{(y,j,m)} = \partial_k E_{k(y,j,m)} + [\lambda, \mu] \begin{bmatrix} (1,1) & (1,1) & (1,1)_a \\ y',j',m' & y,j,m & y'',j'',m'' \end{bmatrix} A_k^{(y',j',m')} E_{k(y'',j'',m'')}. \quad (12)$$

The Poisson brackets of the constraints are:

$$\left\{ \Gamma_{(y,j,m)}(\vec{x}), \Gamma_{(y',j',m')}(\vec{y}) \right\} = -[\lambda, \mu] \begin{bmatrix} (1,1) & (1,1) & (1,1)_a \\ y,j,m & y',j',m' & y'',j'',m'' \end{bmatrix} \Gamma_{(y'',j'',m'')}(x) \delta^3(\vec{x} - \vec{y}). \quad (13)$$

The gauge transformations transform them as the contravariant components of the irreducible tensor of rank (1,1).

$$\Gamma'_{(y,j,m)} = D_{(y,j,m)(y',j',m')}^{(1,1)}(-\gamma', -\beta', \dots) \Gamma_{(y',j',m')}. \quad (14)$$

Equations (8),(10),(14) present evident dependence of the quantities of the theory on the eight functions  $\alpha(\kappa), \beta(\kappa), \gamma(\kappa), \dots$  i.e. on gauge degrees of freedom. It is necessary for the topological analysis as well as for the definition of continual integration [0]. The correct formulation of the theory is possible only when there exists the global exact gauge. The problem of the exact global gauge is not solved for non-Abelian theories. The Coulomb and Lorenz gauges  $\partial_\nu A_\nu = 0$  are local and ambiguities (by Gribov) can be formulated as solutions  $\alpha(\kappa), \beta(\kappa), \gamma(\kappa), \dots$  of the system of differential equations.

$$\begin{aligned} & \begin{bmatrix} (1,1) & (1,1) & (1,1)_a \\ y',j',m' & y'',j'',m'' & y,j,m \end{bmatrix} \{-\gamma', -\beta', \dots\}_{\nu}^{(y'',j'',m'')} A_\nu^{(y',j',m')} + \\ & + D_{y,j,m; y'',j'',m''}^{(1,1)}(-\gamma', -\beta', \dots) \partial_\nu \{\alpha, \beta, \dots\}_{\nu}^{(y'',j'',m'')} = 0. \end{aligned} \quad (15)$$

System (15) is convenient to consider the symmetry of gauge.

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# Fate of an Invariant Operator in the Enveloping Algebra of $SO(d-1,1)$

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## Abstract

Applying Wigner's method of induced representations to the Lorentz group in higher even dimensions, the eigenvalue of a generalized 'helicity operator' is computed on physical states representing massless particles. An antisymmetric invariant in the enveloping algebra of the higher dimensional Lorentz group is shown to factorize into this generalized 'helicity operator' and an operator that yields the boost eigenvalue. A physical application of these results is the derivation of the higher dimensional analog of helicity suppression rules of four-dimensional S matrix theory.

## 1 Introduction

In the algebra of the four-dimensional Poincare group, there are two well-known invariant operators:  $P^\mu P_\mu$  and  $W^\mu W_\mu$  ( $\mu = 0 \dots 3$ ), where

$$W_\mu \equiv \frac{1}{2} \epsilon_{\mu\alpha\beta\gamma} J^{\alpha\beta} P^\gamma \quad (1)$$

is the Pauli-Lubanski pseudo-vector.  $J^{\alpha\beta}$  is a generator of the Lorentz group and  $P^\gamma$  generates translations. For a state in the Hilbert space corresponding to a particle with mass  $m$  and spin  $s$  the eigenvalue of  $P^2$  is  $m^2$  and the eigenvalue of  $W^2$  is proportional to  $m^2 s(s+1)$ . Note that  $W^\mu P_\mu = 0$  in general. In the massless limit, this relationship remains true, and in addition  $W^2 = P^2 = 0$  which implies that as operators  $W^\mu$  and  $P^\mu$  are proportional.<sup>1</sup> The constant of proportionality is the *helicity* of the state. To see this more explicitly, consider a massless particle moving in the 3 direction with a momentum  $k^\mu$ ,  $\mu = 0, 1, 2, 3$ . In its "rest-frame",  $k^\mu =$

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<sup>1</sup>Minkowski metric  $g_{\mu\nu} = \text{diag}(-1, 1, \dots, 1, 1)$ .

$(k, 0, 0, k)$ . Using the definition of  $W$ , we obtain  $W_0 = \frac{1}{2}(J^{12} - J^{21})P^3 = J^{12}P^3$  so  $\frac{W_0}{k} = J^{12}$ . Acting on a state in the Hilbert space  $J^{12}$  gives its helicity, as it is the sole diagonalizable (Cartan) generator of the (restricted)<sup>2</sup> little group (which, for massless states in four dimensions is  $SO(2) \sim U(1)$ ), and the little group classifies the helicity states of the particle. So  $J_{12}$  is the *helicity operator*.<sup>3</sup>

The outline of the paper is as follows: In section 2 we construct a vector operator in higher (even) dimensions analogous to the Pauli-Lubanski operator in four dimensions (we shall call it the generalized Pauli-Lubanski operator). For massless particle states this operator is shown to be proportional to the momentum operator, and since the constant of proportionality is reminiscent of the helicity, we call an analogous quotient the 'helicity operator'. It belongs to the enveloping algebra of the (restricted) little group and commutes with the raising and lowering operators: so it has the same eigenvalue on every vector of an irreducible representation obtained by acting with the lowering operators on a highest weight vector. It suffices thus to compute its eigenvalue on highest weight vectors. For sake of clarity we choose to demonstrate our general ideas by explicit examples, e.g. we compute the eigenvalue of the helicity operator in a non-trivial case (for six-dimensional massless particle states) using the Cartan-Weyl approach (see e.g. [0]). In section 3, we show explicitly in two non-trivial cases that some of the terms in the expansion of a similar operator in the enveloping algebra of the *full* Lorentz group can always be written in terms of Euclidean 'translations'. The eigenvalue of the remaining terms is known from the general results of section 2 from a similar computation in two lower dimensions. Finally we show that if the translations are required to vanish on physical states, the general antisymmetric invariant factorizes into the 'helicity operator' and a pure boost piece. Due to space limitations we refer the reader to [0] for detailed proofs and discussion of physical applications.

## 2 'Helicity operator' in higher dimensions

Analogous to the four dimensional Pauli-Lubanski pseudo-vector, we will construct in this section an antisymmetric operator in the enveloping algebra of the generalized Poincaré group of higher even dimensions. The Lorentz group commutation relations in  $d$  dimensions are

$$[J^{AB}, J^{CD}] = i(g^{AD}J^{BC} + g^{BC}J^{AD} - g^{AC}J^{BD} - g^{BD}J^{AC}) \quad (2)$$

with  $A, B = 0, \dots, (d-1)$ .

<sup>2</sup>The little group is actually the  $E(2)$  like group. But we set Euclidean translations to zero on the physical Hilbert space to ensure finite dimensional representations, reducing it to  $SO(2)$  [0] [0].

<sup>3</sup>A familiar example will make this more concrete; note that for spinors  $J^{12} \propto \gamma^1 \gamma^2$  and since the product of gamma-matrices is nothing but the Pauli matrix, we get the usual result that  $J^{12} = \sigma \frac{\mathbf{k}}{|\mathbf{k}|}$ . But this is the projection of the spin in the direction of the momentum, i.e. the helicity.

Consider a massless particle moving in the  $d-1$  direction with a  $d$ -momentum  $k^A$ . Choose as 'standard momentum frame' the frame in which the momentum  $d$ -vector of the particle is  $k^A = (k, 0, \dots, 0, k)$ . The group of transformations ('little' group) that leaves this  $d$ -momentum invariant is the Euclidean group  $E(d-2)$ , which consists of rotations and translations in hyperplanes not formed out of the 0 or  $d-1$  axes. The  $d-2$  "translation" generators<sup>4</sup> are given by

$$L_i \equiv J_{i0} + J_{i(d-1)}. \quad (3)$$

We can find the commutation relations of the generators  $J_{ij}$  and  $L_i$  using eq. 3 and eq. 2:

$$[L_i, L_j] = 0 \quad (4)$$

$$[J_{ij}, L_k] = i(\delta_{ik}L_j - \delta_{jk}L_i) \quad (5)$$

$$[J_{ij}, J_{kl}] = i(\delta_{ik}J_{jl} - \text{permutations}). \quad (6)$$

As usual [0], require the Euclidean translations, which form an Abelian subgroup, to vanish on physical Hilbert space states ( $\lambda$  is used as a generic index for the eigenvalues of all the commuting generators):

$$L_i|\lambda\rangle = 0 \quad (7)$$

in order to ensure finite dimensional representations. This may also be thought of as a 'gauge-condition' [0] and it forces one to use only gauge-independent Lorentz group representations. The little group thus reduces from  $E(d-2)$  to  $SO(d-2)$ .

Now consider the following operator (generalized 'Pauli-Lubanski' operator)

$$W_\mu \equiv \frac{1}{(-1)^{\frac{n(n-1)}{2}} 2^n n!} \times \epsilon_{\mu\alpha_1\beta_1\alpha_2\beta_2\dots\alpha_n\beta_n\nu} J^{\alpha_1\beta_1} J^{\alpha_2\beta_2}\dots J^{\alpha_n\beta_n} P^\nu \quad (8)$$

in the enveloping algebra of the generalized Poincaré group in  $d \equiv 2n+2$  dimensions. Since all space components of the  $d$ -momentum except the  $d-1$  component are zero for our particle, we obtain the 'helicity' operator<sup>5</sup>

$$\frac{W_0}{k} \equiv \frac{1}{(-1)^{\frac{n(n-1)}{2}} 2^n n!} \times \epsilon_{0\alpha_1\beta_1\alpha_2\beta_2\dots\alpha_n\beta_n(d-1)} J^{\alpha_1\beta_1} J^{\alpha_2\beta_2}\dots J^{\alpha_n\beta_n}. \quad (9)$$

Note that in the limit that the masses go to zero, not only are the operators  $W^\mu$  and  $P^\mu$  orthogonal, but also null:  $P^2 = m^2$  is indeed zero by definition when  $m = 0$ .

<sup>4</sup>Note each translation indexed by  $i$  is a combination of a boost in the  $i$  direction and a rotation in the  $i, (d-1)$  plane.

<sup>5</sup>A note on normalization: The factor of  $2^n$  appears because of degeneracy in the expansion from permutation of indices within generators. The minus sign due to antisymmetry of a generator is cancelled by a minus sign from the  $\epsilon$  symbol. Since the generators appearing in each term commute, one gets a factor of  $n!$ . There are  $C(n, 2) = \frac{n(n-1)}{2}$  ways of choosing indices, which accounts for the remaining factor.

Now  $W^\mu W_\mu$  has two  $\epsilon$ -tensors with indices contracted and a sequence of  $J$ 's and a pair of  $P$ 's. It can be shown by tedious but straightforward index manipulation that the only surviving terms have both  $P$ 's contracted, i.e.  $W^2 \propto P^2$ . The other terms cancel because of signs from the  $\epsilon$ -tensor. But since  $P^2 = m^2$ , in the massless limit,  $W^2 \rightarrow 0$  as required. So  $W$  is proportional to  $P$  and one is tempted to call, in analogy with four dimensions, the eigenvalue of eq. 9 on highest weight states the 'helicity' of that state.

We will now compute this eigenvalue explicitly on highest weight states.<sup>6</sup> Recall that highest weight states are given by specifying the weight vector in terms of the eigenvalues of the simultaneously diagonalizable (Cartan) generators. For  $d$  dimensional spacetime, the Lorentz group is  $SO(2n+1,1)$  with  $d \equiv 2n+2$ , so we specify the highest weight vector as  $\Lambda \equiv \Lambda_{(m_1, m_2, \dots, m_{n+1})}$  where  $m_i$  are the eigenvalues of the Cartan generators.

*Explicit evaluation of the eigenvalue of six-dimensional helicity operator or the  $SO(4)$  invariant:* Specify the little group  $SO(4) \sim D_2$  highest weight vectors by  $\Lambda \equiv \Lambda_{(m_1, m_2)}$  with  $m_1$  and  $m_2$  respectively the eigenvalues of the Cartan generators  $J_{12}$  and  $J_{34}$ . We want the eigenvalue of  $\epsilon^{\mu\nu\alpha\beta} J_{\mu\nu} J_{\alpha\beta}$  on  $\Lambda$ . Expansion of eq. 9 with  $n = 2$  gives

$$\epsilon^{\mu\nu\alpha\beta} J_{\mu\nu} J_{\alpha\beta} \propto J_{12}J_{34} - J_{13}J_{24} + J_{14}J_{23}. \quad (10)$$

The raising and lowering operators defined by  $[H_i, E_\alpha] = \alpha_i E_\alpha$  (where  $H_i$  are the Cartan generators) are

$$\begin{aligned} E_{+\alpha} &\equiv E_{(+1,+1)} = \frac{1}{2}[(J_{13} + iJ_{23}) - (J_{24} - iJ_{14})] \\ E_{-\alpha} &\equiv E_{(-1,-1)} = \frac{1}{2}[(J_{13} - iJ_{23}) - (J_{24} + iJ_{14})] \\ E_{+\beta} &\equiv E_{(-1,+1)} = \frac{1}{2}[(J_{13} - iJ_{23}) + (J_{24} + iJ_{14})] \\ E_{-\beta} &\equiv E_{(+1,-1)} = \frac{1}{2}[(J_{13} + iJ_{23}) + (J_{24} - iJ_{14})] \end{aligned} \quad (11)$$

where the subscripts denote the root vectors given by their coordinates in root space.  $E_{+\alpha} \equiv E_\alpha$  and  $E_{+\beta} \equiv E_\beta$  will be taken to be the positive simple roots (first nonzero entry from *right* is positive.). In terms of the raising and lowering operators, the generators can be re-expressed as

$$\begin{aligned} J_{13} &= \frac{1}{2}(+E_\alpha + E_\beta + E_{-\beta} + E_{-\alpha}) \\ J_{24} &= \frac{1}{2}(-E_\alpha + E_\beta + E_{-\beta} - E_{-\alpha}) \\ J_{14} &= \frac{1}{2}(-E_\alpha - E_\beta + E_{-\beta} + E_{-\alpha}) \\ J_{23} &= \frac{1}{2}(-E_\alpha + E_\beta - E_{-\beta} + E_{-\alpha}). \end{aligned} \quad (12)$$

Substitution in eq. 10 and straightforward algebra gives

$$J_{12}J_{34} - J_{13}J_{24} + J_{14}J_{23} = J_{12}J_{34} + 1/2[\{E_\alpha, E_{-\alpha}\} - \{E_\beta, E_{-\beta}\}] \quad (13)$$

<sup>6</sup>It may be proved [0] that the conditions of eq. 3 ensure that the states are indeed highest weight states.

where the braces indicate anticommutators. But by definition, on highest weight states,

$$E_\alpha \Lambda = E_\beta \Lambda = 0 \quad (14)$$

so we can replace the anticommutators by commutators, when operating on highest weight states:

$$\{E_\alpha, E_{-\alpha}\} \Lambda = [E_\alpha, E_{-\alpha}] \Lambda \quad (15)$$

and similarly for  $E_{\pm\beta}$ . The action of the commutator on a highest weight vector is simply the scalar product between the root vector and the weights, i.e.

$$[E_\alpha, E_{-\alpha}] \Lambda_{(m_1, m_2)} = (J_{12} + J_{34}) \Lambda_{(m_1, m_2)} = (m_1 + m_2) \Lambda_{(m_1, m_2)} \quad (16)$$

and

$$[E_\beta, E_{-\beta}] \Lambda_{(m_1, m_2)} = (-J_{12} + J_{34}) \Lambda_{(m_1, m_2)} = (-m_1 + m_2) \Lambda_{(m_1, m_2)}. \quad (17)$$

Also, by definition

$$J_{12} J_{34} \Lambda_{(m_1, m_2)} = m_1 m_2 \Lambda_{(m_1, m_2)} \quad (18)$$

so substituting the result of the last five equations into eq. 13, we get finally for the six dimensional helicity operator:

$$[J_{12} J_{34} - J_{13} J_{24} + J_{14} J_{23}] \Lambda_{(m_1, m_2)} = m_1(m_2 + 1) \Lambda_{(m_1, m_2)}. \quad (19)$$

*Eigenvalue of helicity operator in general even dimensions:* A similar computation for  $SO(2n)$  gives the following beautiful general result (see [0] or [0] for proof):

The eigenvalue of the helicity operator eq. 9 for  $d = 2n + 2$  on highest weight states is

$$m_1(m_2 + 1)(m_3 + 2) \dots (m_n + n - 1). \quad (20)$$

### 3 Factorization

We will now prove that if we require the generalized translations  $L_i$  in arbitrary dimensions to vanish on physical states, then the full invariant

$$C'_{SO(2n-1,1)} \equiv \frac{1}{(-1)^{\frac{n(n-1)}{2}} 2^n n!} \times \epsilon_{\alpha_1 \beta_1 \alpha_2 \beta_2 \dots \alpha_n \beta_n} J^{\alpha_1 \beta_1} J^{\alpha_2 \beta_2} \dots J^{\alpha_n \beta_n} \quad (21)$$

for  $SO(2n - 1, 1)$  factorizes into a product of the generalized 'helicity' operator and the 'boost' operator. The idea is to try and express the combination of non-Cartan generators in the expansion of eq. 21 in terms of the translations  $L_i$ , which themselves are required to vanish. Let us first demonstrate this strategy in four dimensions, for the  $SO(3, 1)$  invariant:

Assume  $L_1 = J_{10} + J_{13}$  and  $L_2 = J_{20} + J_{23}$  vanish on highest weight states given by  $\Lambda_{(m_1, m_2)}$ . Expanding r.h.s. of eq. 21

$$C'_{SO(3,1)} \equiv J_{23} J_{10} - J_{13} J_{20} + J_{12} J_{30} \quad (22)$$



where we have used the subscript 0 on the non-compact generators. The first two terms in eq. 22 can be rearranged in terms of the translations by using the commutation relations of eq. 2 :

$$\begin{aligned} & 1/2[-(J_{20} - J_{23})(J_{10} + J_{13}) + (J_{10} - J_{13})(J_{20} + J_{23})] \\ &= 1/2[-(J_{20} - J_{23})L_1 + (J_{10} - J_{13})L_2]. \end{aligned} \quad (23)$$

but since the translations are required to vanish on the physical states, the only remaining part of the  $SO(3,1)$  invariant (see eq. 22 ) gives

$$J_{12}J_{30} \Lambda_{(m_1, m_2)} = im_1m_2 \Lambda_{(m_1, m_2)}. \quad (24)$$

The origin of the  $i$  is the Weyl trick <sup>7</sup> to relate the compact and non-compact generators.

We obtain the advertised result that the helicity eigenvalue ( $m_1$ ) and boost eigenvalue ( $im_2$ ) factor. This is an example of the 'factorization' that we mentioned earlier. Here, it is almost too trivial. We shall explicitly do a more non-trivial example:

In six dimensions, the Lorentz group is  $SO(5,1)$  and the little group for massless states is  $SO(4)$ . We may work with the Euclideanized  $SO(6)$  and use the Weyl trick to relate  $J_{i0}$  and  $J_{i6}$ . Then the expansion of the invariant is (see eq. 21 )

$$\begin{aligned} C'_{SO(5.1)} \equiv & (J_{12}J_{34} - J_{13}J_{24} + J_{14}J_{23})J_{50} \\ & + (-J_{24}J_{35} + J_{23}J_{45} + J_{34}J_{25})J_{10} \\ & + (-J_{34}J_{15} + J_{13}J_{45} + J_{14}J_{35})J_{20} \\ & + (+J_{12}J_{45} - J_{14}J_{25} + J_{15}J_{24})J_{30} \\ & + (-J_{12}J_{35} + J_{13}J_{25} + J_{15}J_{23})J_{40} \end{aligned} \quad (25)$$

and we want to determine its eigenvalue on highest weight states specified by  $\Lambda_{(m_1, m_2, m_3)}$ . The first term in  $C'_{SO(5.1)}$  has the common "boost" generator  $J_{50}$  with the definition

$$J_{50}\Lambda_{(m_1, m_2, m_3)} = im_3\Lambda_{(m_1, m_2, m_3)} \quad (26)$$

and the part inside the brackets we know to give

$$(J_{12}J_{34} - J_{13}J_{24} + J_{14}J_{23})\Lambda_{(m_1, m_2, m_3)} = m_1(m_2 + 1)\Lambda_{(m_1, m_2, m_3)} \quad (27)$$

from the discussion for  $SO(4)$  in section 2. We still have to deal with the remaining terms. Since every  $D_n \sim SO(2n)$  has rank  $n$  and dimension  $n(2n-1)$ , for  $SO(6)$  there

<sup>7</sup>The Weyl trick achieves the following algebraic 'Wick rotation': Given two real forms, one compact, e.g.  $SO(4)$  and one non-compact e.g.  $SO(3,1)$ , with the same Universal Covering Group, if the algebra and the representations of the compact real form are known, then those of the non-compact real form can be found by replacing the compact generators with  $i$  times the corresponding non-compact generators.

are  $15 - 3 = 12$  raising and lowering operators, and 3 linearly independent raising operators (= rank). Also the little group (here  $SO(4)$ ) involves four translations

$$\begin{aligned} L_1 &\equiv J_{10} + J_{15} \\ L_2 &\equiv J_{20} + J_{25} \\ L_3 &\equiv J_{30} + J_{35} \\ L_4 &\equiv J_{40} + J_{45}, \end{aligned} \quad (28)$$

which will vanish on  $\Lambda$ . As in the four-dimensional case, we now wish to be able to simplify the invariant by expressing them in terms of the  $L_i, i = 1, \dots, 4$ . So we want to write

$$\begin{aligned} C'_{\text{non-Cartan}} &\equiv (-J_{24}J_{35} + J_{23}J_{45} + J_{34}J_{25})J_{10} \\ &+ (-J_{34}J_{15} + J_{13}J_{45} + J_{14}J_{35})J_{20} \\ &+ (+J_{12}J_{45} - J_{14}J_{25} + J_{15}J_{24})J_{30} \\ &+ (-J_{12}J_{35} + J_{13}J_{25} + J_{15}J_{23})J_{40} \\ &= aL_1 + bL_2 + cL_3 + dL_4, \end{aligned} \quad (29)$$

where  $a, b, c, d$  are undetermined quadratic forms of the generators and can be determined by a recursive procedure:

$$a = +J_{34}(J_{25} - J_{20}) - J_{24}(J_{35} - J_{30}) + J_{23}(J_{45} - J_{40}) \quad (31)$$

$$b = -J_{34}(J_{15} + J_{10}) + J_{14}(J_{35} - J_{30}) - J_{13}(J_{45} - J_{40}) \quad (32)$$

$$c = +J_{24}(J_{15} + J_{10}) - J_{14}(J_{25} + J_{20}) + J_{12}(J_{45} - J_{40}) \quad (33)$$

$$d = -J_{23}(J_{15} + J_{10}) + J_{13}(J_{25} + J_{20}) - J_{12}(J_{35} + J_{30}) \quad (34)$$

So that using eq. 7

$$C'_{\text{non-Cartan}} \Lambda_{(m_1, m_2, m_3)} = 0. \quad (35)$$

Hence

$$C'_{SO(5,1)} \Lambda_{(m_1, m_2, m_3)} \propto im_1(m_2 + 1)m_3 \Lambda_{(m_1, m_2, m_3)}. \quad (36)$$

Note that in the present calculation, we used our knowledge of the eigenvalue of the four dimensional operator to determine the eigenvalue of the six-dimensional operator. The terms not tied to  $J_{50}$  could be manipulated to give zero eigenvalue because the translations are required to vanish. This persists in each higher dimension. In  $d$  dimensions the invariant is an operator whose expansion is a sum of terms multilinear in the generators and degree equal to  $d/2$ . There are some terms in this expansion whose eigenvalue is known from our general theorem for the eigenvalue of the helicity operator in  $d - 2$  dimensions and the remaining terms can be put into a form with translation generators at the end. Since translations are required to vanish on highest weight states, these other terms have zero eigenvalue and we are left with only the part whose eigenvalue we know by referring to a similar computation in two lower

dimensions. In the six-dimensional example,  $im_3$  is the boost eigenvalue, and  $m_1(m_2 + 1)$  is the helicity eigenvalue (it comes from the part in the invariant composed of only the little group,  $SO(4)$  generators). But the eigenvalue of the full group invariant has been shown to be  $im_1(m_2 + 1)m_3$ , so *the boost and helicity eigenvalues factorize*. In general (see [0] for details), for the special class of representations satisfying eq. 3 the boost eigenvalue depends only on the last index of the representation weight vector: by examination of the invariant in eq. 21 we can see that in  $d$  dimensions the surviving term has a common  $J_{(d-1,0)}$  factored out of what is the helicity operator (which is formed out of  $SO(d-2)$  generators only).

As an application of the factorization property, it can be shown that to determine the behavior of matrix elements of massless particles coupled to currents in the limit of vanishing energy, the minimum set of information required is the representation of the fields and their tensor products, and in particular, the boost eigenvalues. Discussion of this issue and the special simplification occurring in four dimensions is left to other works [0] [0].

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## UNITARY STRUCTURES FOR SPANNORS

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### 1. Spannors

We recall the definition of the spannors representation,  $S_d$ , of  $SU(2, 2)^-$ , the universal covering group of  $SU(2, 2)$  [1].  $S_d$  is the tensor product of the scalar representation of  $SU(2, 2)^-$  of weight  $d$ ,  $V_d$ , with the spin representation of  $SU(2, 2)^-$ , which we denote by  $\Sigma$ :

$$S_d = V_d \otimes \Sigma. \quad (1.1)$$

For the physically important case,  $d = 2$ , the representation space may essentially be identified with  $\mathcal{L}^2(M_0) \otimes \mathbb{C}^2$ , where  $M_0$  is Minkowski space, and the representation  $V_2$  of  $SU(2, 2)^-$  is just that multiplier representation with *real* multiplier and which preserves the  $\mathcal{L}^2$  norm on  $M_0$ . We shall henceforth consider only this case.

### 2. K Finite Basis and Weight Diagram for Spannors

It suffices, in what follows, to consider just the front half spannors, which can be defined as the section space of all sections of the spannors bundle, whose lower four components vanish in the tensor product parallelization [1]. Convenient basis fields, which are defined in the standard curved parallelization [1], are the so-called K-finite basis fields [2]:

$$|e; khlmn \rangle \quad (2.1)$$

where all  $k, l, m$  are integers and  $e$  and  $h$  take the values  $+1$  and  $-1$ ; in addition,  $n - k - l = \lambda \pmod{2}$ ;  $k \geq 0, l \geq 0, -l - 1 \leq m \leq l$  if  $h = +1$ ; and  $k \geq 1, l \geq 0, -l - 1 \leq m \leq l$  if  $h = -1$ . ( $\lambda = d + \frac{1}{2}$ ). The  $|e; khlmn \rangle$  "diagonalize" the quantum numbers  $k, h, l, m$  and  $n$  (c.f. [2]).

Define the  $K$  types

$$b(n, k + l, h)^\pm = \{|e = \pm 1; khlmn \rangle \text{ with } n, h, k + l \text{ fixed; } + \text{ for } e = +1 \text{ and } - \text{ for } e = -1\}. \quad (2.2)$$

Also define the spaces

$$E_{\frac{1}{2}}^+ = \sum_{n, k+l, h} \oplus b(n, k + l, h)^+ \text{ and } E_{-\frac{1}{2}}^- = \sum_{n, k+l, h} \oplus b(n, k + l, h)^-. \quad (2.3)$$

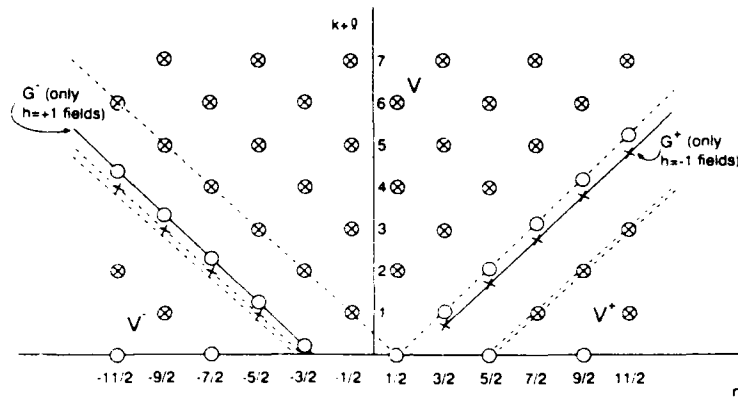


FIGURE 1. WEIGHT DIAGRAM FOR FRONT-HALF SPANNORS. Each circle or cross represents two K types, namely  $b(n, k + \ell, h)^+$  and  $b(n, k + \ell, h)^-$ . Circles denote  $h = +1$  fields and crosses denote  $h = -1$  fields. Points to the right of and along the dashed half-line in the first quadrant represent  $V^-[V^+]$ . Points above and along the dashed half-lines represent  $V^-[V^+]$ . Points to the left of and along the dashed double half-line in the second quadrant represent  $V^+[V^-]$ . Points along the solid half-lines represent  $G^-[G^+]$  and  $G^+[G^-]$ . They have been slightly displaced in order to make clear their disjointness from  $V^-[V^+]$  and  $V^+[V^-]$ .

The weight diagram for the front half spanners, which is shown in Figure 1, describes these spaces. (See section 3 for definitions of  $V^-$ ,  $V$  and  $V^+$  etc.)

### 3. Irreducible Composition Factors and their Unitarity

Now recall the notation for induced representations introduced in [1]:

$$Ind_{\hat{P}}^{\hat{G}}(\rho([n, m]; d)). \tag{3.1}$$

$\rho$  denotes an irreducible representation of  $\hat{P}$ , the universal cover of the extended Poincaré group, that acts trivially on the translations. ( $\hat{G}$  and  $\hat{P}$  denote the Lie algebras of  $SU(2, 2)$  and  $\mathcal{P}$ , respectively.)  $[n, m]$  labels the representation of  $SL(2, \mathbb{C})$ , and scaling acts with weight  $d$  [1]. Further we introduce the Dirac operator on  $\hat{M}$ , the double cover of the conformal compactification of  $M_0$  [2]:

$$D_c = \gamma_0 X_0 + \gamma_1 X_1 + \gamma_2 X_2 + \gamma_3 X_3 - \frac{3}{2} \gamma_4 \gamma_5, \tag{3.2}$$

where the  $\gamma$  matrices are defined in [2], and

$$X_0 = L_{-10}, X_1 = L_{14} - L_{23}, X_2 = L_{24} - L_{31}, X_3 = L_{34} - L_{12}.$$

Explicit expressions for the vector fields,  $L_{ab}$ , are given in [3].

**Theorem 1**

The 10 irreducible composition factors of the front half spanners are ( $s = \frac{1}{2}$ ):

$$V^+ = \sum_{h=-2s}^{2s} \sum_{k+l=s-\frac{h}{2}}^{\infty} \sum_{n=k+l+\frac{5}{2}}^{\infty} b(n, k+l, h)^+, \quad (3.3a)$$

$$V = \sum_{h=-2s}^{2s} \sum_{k+l=s-\frac{h}{2}}^{\infty} \sum_{n=-(k+l)+\frac{1}{2}}^{(k+l)-\frac{1}{2}+h} b(n, k+l, h)^+, \quad (3.3b)$$

$$V^- = \sum_{h=-2s}^{2s} \sum_{k+l=s-\frac{h}{2}}^{\infty} \sum_{n=(k+l)+\frac{5}{2}+h=0,-1,-2,\dots} b(n, k+l, h)^+, \quad (3.3c)$$

$$G^+ = \sum_{\substack{k+l=0 \\ n=k+l+\frac{1}{2}}}^{\infty} b(n, k+l, -1)^+, \quad (3.3d)$$

$$G^- = \sum_{\substack{k+l=0 \\ n=-(k+l)-\frac{1}{2}}}^{\infty} b(n, k+l, +1)^+, \quad (3.3e)$$

and  $V^{+'}, V', V^{-'}, G^{+'}$  and  $G^{-}'$ , which are  $V^+, V, V^-, G^+$  and  $G^-$  with  $b(n, k+l, h)^+$  replaced by  $b(n, k+l, h)^-$ .  $\bar{G}$  acts as  $Ind_{\mathcal{P}}^{\bar{G}}(\rho([2, 1]; \frac{5}{2}))$  on  $V^+, V$  and  $V^-$ ;  $\bar{G}$  acts via the quotient action of  $Ind_{\mathcal{P}}^{\bar{G}}(\rho([2, 1]; \frac{5}{2}))$  on  $G^+ = T^+/\{V + V^+\}$  and  $G^- = T^-/\{V^- + V\}$ , where  $T^+$  and  $T^-$  are the essential extensions of  $V + V^+$  and  $V^- + V$  given by eqns. (14.12) and (14.13) in Theorem 14.2 of [4].  $\bar{G}$  acts as  $Ind_{\mathcal{P}}^{\bar{G}}(\rho([1, 2]; \frac{3}{2}))$  on  $G^{+'}$  and  $G^{-}'$ ; and  $\bar{G}$  acts via the quotient action of  $Ind_{\mathcal{P}}^{\bar{G}}(\rho([1, 2]; \frac{3}{2}))$  on  $V^{+'} = X^+/G^{+'}, V' = X/\{G^{+'} + G^{-}'\}$  and  $V^{-}' = X^-/G^{-}'$ , where  $X^+, X$  and  $X^-$  are essential extensions of  $G^{+'}, G^{+'} + G^{-}'$  and  $G^{-}'$ , respectively.

The proof of this theorem follows from results in section 14.1 of [4] and from [1]. The spaces  $X^+, X$  and  $X^-$  have definitions, which are very similar to the definitions of the spaces given in eqns. (14.7), (14.8) and (14.9) of [4], but we do not have space here to write them down.

**Theorem 2**

Let the actions of  $\bar{G}$  on the irreducible composition factors of the front half spanners be as in Theorem 1.

Then,  $V^{+'}$  and  $V^{-}'$  are infinitesimally unitary with scalar products

$$(\psi, \phi)_{\pm}' = \pm \int_M \bar{\psi}^T \gamma_4 D_c \phi d^4 u \quad (3.4a)$$

for  $\psi, \phi \in V^{+'}$  or  $V^{-}'$  (+ for  $V^{+'}$ , - for  $V^{-}'$ );  $G^{+'}, G^{-}', G^+$  and  $G^-$  are infinitesimally unitary with scalar products

$$\langle\langle\psi, \phi\rangle\rangle = \int_{S^3} \bar{\psi}^T \phi d\Omega_{S^3}; \quad (3.4b)$$

$V^+$  and  $V^-$  are infinitesimally unitary with scalar products

$$(\mathcal{D}_c\psi, \mathcal{D}_c\phi)_\pm = (\psi, \phi)'_\pm \quad (3.4c)$$

for  $\mathcal{D}_c\psi, \mathcal{D}_c\phi \in V^+$  or  $V^-$  (+ for  $V^+$ , - for  $V^-$ ). Neither  $V$  nor  $V'$  are infinitesimally unitary. ( $\bar{\psi}^T$  means transposed complex conjugate of  $\psi$ , and  $d\Omega_{S^3}$  and  $d^4u$  are defined in [2].)

Most of the proof of this theorem follows from section 14.2 of [4] and also from results in [2]. A similar result for real spanners appears in [5]. We have also determined the Poincaré content of these representations, but we do not have space to report our results here.

#### Acknowledgements, references and footnotes

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## DISCRETE WEYL-WIGNER TRANSFORMATIONS

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1. Weyl realizations<sup>1</sup> of the Heisenberg group are built up in terms of two unitary operators  $U$  and  $V$  such that  $U^N = 1$  and  $V^N = 1$ , satisfying the basic relation

$$V U = e^{i(2\pi/N)} U V, \quad (1)$$

which entails

$$V^n U^m = e^{i(2\pi/N) m n} U^m V^n. \quad (2)$$

There is one realization for each integer  $N$ . The monomials in (2) constitute a complete basis for quantum operators<sup>2</sup>. A more convenient complete basis is the set  $\{S_{mn}\}$  of operators

$$S_{mn} = e^{i(\pi/N) m n} U^m V^n. \quad (3)$$

A general operator  $A$  can be then put into the form

$$A = \frac{1}{N} \sum_{m,n} A_{mn} S_{mn}. \quad (4)$$

2. The  $S_{mn}$ 's are unitary,  $S_{mn}^{-1} = S_{mn}^\dagger = S_{-m,-n}$ ;  $S_{00}$  is the identity; Also  $\text{tr}[S_{mn}] = \delta_{m0}\delta_{n0}N$ ,  $\text{tr}[S_{mn}^\dagger S_{rs}] = \delta_{mr}\delta_{ns}N$ ;  $\text{tr}A = A_{00}$ ;  $\text{tr}A^\dagger A = (1/N) \sum_{m,n} |A_{mn}|^2$  and  $A_{mn} = \text{tr}[S_{mn}^\dagger A]$ . The basis is orthogonal by the trace inner product,  $(A, B) = \text{tr}[B^\dagger A]$ .

3. The  $S_{mn}$ 's have some more remarkable properties:

- a) They reduce to the Pauli matrices for  $N = 2$ , and, for  $N \geq 2$ , they are generalizations of Pauli matrices providing the finest grading of the linear complex Lie algebra  $gl(N, \mathbb{C})^3$ .  $\{S_{mn}\}$  is so a preferred basis in what concerns additive quantum numbers.
- b) They lead to a classification of the quantum degrees of freedom in terms of prime decompositions of integer numbers. When  $N$  is a prime number, the pair  $(U, V)$  describes one  $N$ -valued degree of freedom. Otherwise  $N$  is a product of prime numbers and

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the basis  $\{S_{mn}\}$  factorizes correspondingly into a product of independent sub-basis, one for each degree of freedom.

c) The  $S_{mn}$ 's realize the Heisenberg group as a projective representation of  $Z_N \otimes Z_N$ . They satisfy

$$S_{rs} S_{mn} = e^{i\alpha_2(mn, rs)} S_{(m+r)(n+s)} \quad (5)$$

with  $\alpha_2(mn, rs) = \frac{\pi}{N} [ms - nr]$ . The basis commutators are

$$[S_{rs}, S_{mn}] = 2i \sin[\alpha_2(mn, rs)] S_{(m+r)(n+s)} \quad (6)$$

d)  $\{S_{mn}\}$  is a Fourier operator basis. Our objective here is precisely to point out this property and comment on some consequences. Besides making the  $S_{mn}$ 's specially convenient for the consideration of the semi-classical limit, this fact turns (4) automatically into a discrete version of the Weyl prescription.

4. All numbers  $m, n$ , etc, are defined  $\text{mod}(N)$  and all angles are  $\text{mod}(2\pi)$ . The lattice torus formed by the double-integer labels  $(m, n)$  constitutes the quantum phase space. The continuum limit comes from taking to infinite both the torus radii while  $N \rightarrow \infty$  and the area of each lattice plaquette (which is  $\frac{2\pi}{N}$ ) tends to zero. It is a phase space because the phase  $\alpha_2(mn, rs)$  above provides a symplectic structure<sup>4</sup>: it is a 2-cocycle, has properties analogous to the classical symplectic form  $\Omega$  and tends to  $\Omega$  in the classical limit. It is at work even in the discrete case, but the simplest example is the usual position-momentum  $(\mathbf{q}, \mathbf{p})$  case, which comes out if we put  $S_{m0} = U^m = e^{i\sqrt{2\pi/N} m \mathbf{q}}$ ,  $S_{0n} = V^n = e^{i\sqrt{2\pi/N} n \mathbf{p}}$ . In the continuum limit the values of  $\sqrt{2\pi/N} m$  and  $\sqrt{2\pi/N} n$  tend to constants  $a, b$ , so that the eigenvalues  $\sqrt{2\pi/N} k$  of  $\mathbf{q}$  and  $\mathbf{p}$  tend to numbers  $q, p$ . So,  $S_{m0} \rightarrow S_{a0} = e^{ia\mathbf{q}}$ ,  $S_{0n} \rightarrow S_{0b} = e^{ib\mathbf{p}}$  and

$$S_{mn} \rightarrow S_{ab} = e^{i(a\mathbf{q} + b\mathbf{p})} \quad (7)$$

As higher order terms vanish in the Weyl-Wigner transformation, only the Poisson bracket remain and  $\alpha_2(a0, 0b) = -(\hbar/2)\{a\mathbf{q}, b\mathbf{p}\}$ . In the more general case we obtain the Glauber formula  $S_{cd} S_{ab} = e^{(i/2)[ad - cb]} S_{(a+c)(b+d)}$ .

5. Again in the  $(\mathbf{q}, \mathbf{p})$  case, the Wigner function<sup>5</sup>  $A_W(\mathbf{q}, \mathbf{p})$  and the Weyl operator are the usual and the operator Fourier transform the Wigner density  $A(a, b)$ ,

$$A_W(\mathbf{q}, \mathbf{p}) = F[A] = \int \int da db e^{i(aq+ibp)} A(a, b); \quad (8)$$

$$\mathbf{A}(\mathbf{q}, \mathbf{p}) = \hat{F}[A] = \int \int da db S_{ab} A(a, b). \quad (9)$$

So,  $\mathbf{A} = \hat{F}[F^{-1}[A_W]]$  and  $\{S_{mn}\}$  appears as the natural basis for Weyl-Wigner transformations. Expansions like (4) are discrete versions of the Weyl<sup>6</sup> prescription (9) and the coefficients play the role of Wigner densities.

6. We are giving examples for the  $(\mathbf{q}, \mathbf{p})$  case but the formalism holds in principle for any degree of freedom. The product and the commutator of two operators are, from (5) and (6),

$$\mathbf{AB} = \frac{1}{N^2} \sum_{j,k} \left[ \sum_{m,n} A_{mn} B_{j-m,k-n} e^{i\alpha_2(jk,mn)} \right] S_{jk}; \quad (10)$$

$$[\mathbf{A}, \mathbf{B}] = \frac{1}{N^2} \sum_{p,q} \sum_{m,n} A_{mn} B_{p-m,q-n} 2i \sin[\alpha_2(pq,mn)] S_{pq}. \quad (11)$$

7. Basis  $\{S_{mn}\}$  is consequently a (discrete finite at first, or continuum infinite in the limit) operatorial Fourier basis. To illustrate how easy it is to find some results in this formalism, let us recall the twisted product<sup>7</sup> in the special case of phase space  $\mathbf{R}^4$ , with  $\mathbf{x} = (a,b)$  and  $\mathbf{y} = (c,d)$ . To the usual convolution  $(f \cdot g)(\mathbf{x}) = \int f(\mathbf{y})g(\mathbf{x}-\mathbf{y})d\mathbf{y}$  will correspond the expression

$$\frac{1}{N} \sum_{m,n} A_{mn} B_{p-m,q-n}. \quad (12)$$

The twisted convolution of index  $\eta$  is defined by

$$(f \star_{\eta} g)(\mathbf{x}) = \int e^{i(\eta/2)(\mathbf{x} \times \mathbf{y})} f(\mathbf{y})g(\mathbf{x}-\mathbf{y})d\mathbf{y}, \quad (13)$$

where  $(\mathbf{x} \times \mathbf{y}) = ad - cd$ , so that the coefficient in (10),

$$(\mathbf{AB})_{jk} = \frac{1}{N} \sum_{m,n} A_{mn} B_{j-m,k-n} e^{i\alpha_2(jk,mn)}, \quad (14)$$

is just a twisted convolution of index  $\eta = \frac{2\pi}{N}$ , the area of an elementary plaquette on the lattice torus. The usual product of functions is  $f \cdot g = F^{-1}[F[f] \cdot F[g]]$ . The twisted product is defined as

$$f \circ g = F^{-1}[F[f] \star_{\eta} F[g]]. \quad (15)$$

We immediately see from (11) that  $\mathbf{AB} = \hat{F}[F^{-1}[A_W] \star_{\eta} F^{-1}[B_W]]$ .

It follows that  $(AB)_W = \hat{F}^{-1}[AB] = [F^{-1}[A_W] \cdot_{\eta} F^{-1}[B_W]]$  and

$$(AB)_W = F[F^{-1}[A_W] \cdot_{\eta} F^{-1}[B_W]] = A_W \circ B_W, \quad (16)$$

a twisted product in which the phase  $\alpha_2$  gives the twisting. As  $\alpha_2 \neq 0$ , "classical" dynamical quantities in correspondence with quantum quantities multiply each other no more by the simple product but with the noncommutative twisted product. In the process of quantization, it is not the dynamical quantities which change but their algebra<sup>8</sup>. In the Weyl expression (11) for the commutator we recognize the Moyal bracket. It is to be confronted to the Wigner transform  $([A, B])_W$  of the commutator of two operators  $A$  and  $B$  in terms of their transforms  $A_W(q,p)$  and  $B_W(q,p)$  in the continuum limit,

$$([A, B])_W(q,p) = 2i \sin \left[ \frac{1}{2} (\partial^A_q \partial^B_p - \partial^A_p \partial^B_q) \right] A_W(q,p) B_W(q,p) \quad (17)$$

We see that  $\alpha_2$  is a shortwriting of all the intricate action of the "Poisson" operator in the sine argument.

8. The formalism has a great formal simplicity and holds for any degree of freedom, discrete or (in the limit) continuum. Degrees of freedom assuming a finite number of values fall necessarily in the discrete case. It has been easier to uncover the role of the fundamental cocycle, which in the continuum case replaces the action of the Poisson double-derivative operator. It has been applied to membranes and quantum groups<sup>9</sup>. Analysis of Yang-Baxter equations, both quantum and classical, are under way. Many other questions are susceptible of improvement with this formalism: semi-classical problems, aspects of classical mechanics of quantum origin, measurement problems, uncertainty principle for general variables, the study of other correspondences, 2nd quantization, quantum relativity.

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# Many Particle Dirac Identities

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## 1. Sets of generators for the center of $CS_n$

The representation theory of the symmetric group ( $S_n$ ) is considerably simplified by considering the corresponding group algebra ( $CS_n$ ). As is well known, the class-sums of  $S_n$  form a linear basis for the center of  $CS_n$ . Any member of the center can be expressed as a polynomial in an appropriate set of generators, which need not necessarily be members of the center. The only requirement we make is that the generators be mutually commutative. Several independent sets of operators were found to generate the center of  $CS_n$ , each set having certain specific advantages. Kramer [1] showed that the set of  $n$  single-cycle class-sums  $\{[(k)]_n; k = 1, 2, \dots, n\}$  generates the center. In particular, any other class-sum can be expressed as a polynomial in Kramer's set of generators, in a manner which he did not explicate. Another set of  $n$  generators was proposed by Farahat and Higman [2]. Their  $k$ 'th generator, which we denote by  $F_k$ , is the sum of all the classes consisting of  $k$  cycles. Thus  $F_n = [(1)^n]_n$ ,  $F_{n-1} = [(2)]_n$ ,  $F_{n-2} = [(3)]_n + [(2)^2]_n$ ,  $F_{n-3} = [(4)]_n + [(3)(2)]_n + [(2)^3]_n, \dots$ ,  $F_1 = [(n)]_n$ , where cycles of unit length are usually suppressed. Two sets of generators which do not themselves belong to the center of the algebra, although they do belong to the algebra, were proposed by Jucys [3] and by Chen [4], respectively. Jucys' set consists of  $\{\pi_k = \sum_{i=1}^{k-1} (i, k); k = 2, 3, \dots, n\}$ . His  $k$ 'th element is the sum of the transpositions between the index  $k$  and all the preceding indices. Explicitly,  $\pi_2 = (1, 2)$ ,  $\pi_3 = (1, 3) + (2, 3)$ , etc. This set has to be augmented by  $\pi_1 = 1$ . Chen's set of generators consists of the transposition class-sums in the group-subgroup chain  $S_1 \subset S_2 \subset \dots \subset S_n$ , i.e.,  $1, [(2)]_2 = \pi_2, [(2)]_3 = \pi_2 + \pi_3, \dots, [(2)]_n = \sum_{i=2}^n \pi_i$ . One immediate and important consequence of the form of Chen's set of generators is that the center of  $CS_n$  can be generated by adding the class of transpositions of  $S_n$ ,  $[(2)]_n$ , to the center of  $CS_{n-1}$ .

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This was recently extended to the statement that the center of  $CS_n$  can be generated by adding the set of class-sums  $[(2)]_n, [(3)]_n, \dots, [(k)]_n$  to the center of  $CS_{n-k}$  [5]. Note that this statement interpolates between Chen's result, with which it coincides for  $k = 1$ , and Kramer's result, with which it coincides for  $k = n$ . This result was recently found to be useful in the context of cluster physics [6].

These various sets of generators are mutually interconnected more or less straightforwardly. Thus, the Farahat-Higman operators are symmetric polynomials in Jucys' operators:  $F_{n-1} = \sum_{i=2}^n \pi_i$  ;  $F_{n-2} = \sum_{i < j} \pi_i \pi_j$  ;  $F_{n-3} = \sum_{i < j < k} \pi_i \pi_j \pi_k$  ;  $\dots$  ;  $F_1 = \pi_2 \pi_3 \dots \pi_n$ . The connection between Kramer's generators and those of Farahat and Higman can be obtained by systematic application of the expressions for the products of class-sums in  $CS_n$  [7-9]. One obtains  $[(2)]_n = F_{n-1}$  ;  $[(3)]_n = F_{n-1}^2 - 2F_{n-2} - n(n-1)/2$  ;  $[(4)]_n = F_{n-1}^3 + 3F_{n-3} - 3F_{n-2}F_{n-1} - (2n-3)F_{n-1}$  etc. The class-sum products also provide explicit expressions for arbitrary class-sums in terms of either set of generators.

## 2. The Dirac identity: some properties of spin-operators

For a pair of identical particles with an elementary spin equal to  $\frac{1}{2}$  the Dirac identity [10] expresses the effect of a transposition  $(i, j)$  of the particle indices on the two particle spin functions in terms of the spin operator  $t_{ij} = \vec{s}_i \cdot \vec{s}_j$ . Explicitly,  $(i, j) = 2t_{ij} + \frac{1}{2}$ . The expression for a transposition of two identical particles with an arbitrary elementary spin was presented by Schrödinger [11]. Since the transpositions generate the whole group algebra it is obvious that the Dirac-Schrödinger identities can be used to derive expressions for the operation of class-sums of the symmetric group on spin functions of an arbitrary number of identical particles. This program was outlined in [12], where further references are listed. This talk presents recent progress towards achieving this goal. To carry out the required computations, for particles with an elementary spin  $\sigma = \frac{1}{2}$ , we need the following quadratic identities:  $t_{ij}^2 = \frac{3}{16} - \frac{1}{2}t_{ij}$  ;  $t_{ij}(t_{ik} + t_{jk}) = \frac{1}{4}(t_{ik} + t_{jk})$  ;  $t_{ij}t_{ik} + t_{ik}t_{ij} = \frac{1}{2}t_{jk}$ . A somewhat interesting immediate application is the identity  $t_{ij}t_{kl}(t_{ik} + t_{il} + t_{jk} + t_{jl}) = \frac{1}{16}(t_{ik} + t_{il} + t_{jk} + t_{jl})$ . We emphasize that these identities are satisfied by spin  $\frac{1}{2}$  particles only.

Similarly, for spin 1 particles we find the cubic identities

$$t_{ij}^3 = -2t_{ij}^2 + t_{ij} + 2$$

$$t_{ij}^2(t_{ik} + t_{jk}) = t_{ij}(t_{ik} + t_{jk})t_{ij} = (t_{ik} + t_{jk})t_{ij}^2 = t_{ik} + t_{jk}$$

$$t_{ij}t_{ik}t_{il} + t_{il}t_{ik}t_{ij} = t_{ij}t_{kl} + t_{jk}t_{li}$$

$$t_{ij}(t_{ik}^2 + t_{jk}^2) + (t_{ik}^2 + t_{jk}^2)t_{ij} = (t_{ik} + t_{jk})^2 - (t_{ik}^2 + t_{jk}^2) + 4t_{ij} - t_{ij}(t_{ik} + t_{jk})$$

The systematic derivation of the corresponding identities for higher spin values will be considered elsewhere.

### 3. Symmetric polynomials in sets of spin-operators

The properties of the fundamental symmetric polynomials in a set of commuting variables are well known and of central importance in the representation theory of the symmetric group. Here, we consider the set of  $n(n-1)/2$  non-commuting variables  $\{t_{ij} ; i < j = 1, 2, \dots, n\}$ , for the case  $\sigma = \frac{1}{2}$ . We define the following set of fundamental symmetric polynomials in these variables  $U_1(n) = \sum_{i < j} t_{ij} ;$   
 $U_2(n) = \sum_{\{i < j, k < l\}} t_{ij} t_{kl} ; \dots ; U_p(n) = \sum_{\{i_1 < j_1 ; i_2 < j_2 ; \dots ; i_p < j_p\}} t_{i_1 j_1} t_{i_2 j_2} \dots t_{i_p j_p}$ .  
 The prime means that the various pairs of indices are ordered and are chosen in such a way that each one of the indices is distinct. Note that in the present context we only have  $\lfloor \frac{n}{2} \rfloor$  fundamental symmetric polynomials. For a set of  $n(n-1)/2$  commuting variables we would need that many fundamental symmetric polynomials. The advantage of the fundamental symmetric polynomials over the symmetric power-sums  $\{O_k = \sum_{i < j} t_{ij}^k, k = 1, 2, \dots\}$  introduced in ref. [12] is that the former are found to be very directly linked to the Higman-Farahat generators, as we show in the following section. Note that the equivalence between the symmetric power-sums and the fundamental symmetric polynomials was only established for commuting variables, although it holds in our  $\sigma = \frac{1}{2}$  case.

Using the identities presented in the previous section we find that

$$U_1^2 = \frac{3}{32}n(n-1) + \frac{n-3}{2}U_1 + 2U_2$$

$$U_1U_2 = \frac{5}{32}(n-2)(n-3)U_1 + 2\frac{n-5}{2}U_2 + 3U_3$$

$$U_1U_3 = \frac{7}{32}(n-4)(n-5)U_2 + 3\frac{n-7}{2}U_3 + 4U_4$$

⋮

$$U_1U_p = \frac{2p+1}{32}(n-2p+2)(n-2p+1)U_{p-1} + \frac{p}{2}(n-(2p+1))U_p + (p+1)U_{p+1}$$

Using these equations consecutively we can express  $U_p$  as a  $p$ 'th degree polynomial in  $U_1$ . Similar results, for  $\sigma \geq 1$ , will be presented elsewhere.

### 4. The center of $CS_n$ in terms of the symmetric polynomials in the spin operators

For a system of  $n$  identical particles with elementary spin  $\sigma = \frac{1}{2}$  we obtain the Farahat-Higman generators of the center of  $CS_n$  in the form

$$\sum_{i=2}^n \pi_i = \frac{n(n-1)}{4} + 2U_1(n)$$

$$\sum_{i < j} \pi_i \pi_j = \frac{n(n-1)(n-2)(3n-1)}{96} + \frac{(n-2)(n+1)}{2}U_1(n) + 4U_2(n)$$

$$\sum_{i < j < k}^n \pi_i \pi_j \pi_k = \frac{n^2(n-1)^2(n-2)(n-3)}{384} +$$

$$+ \frac{(n-2)(n-3)(n^2-5n+28)}{16} U_1(n) + (n^2-n-8)U_2(n) + 8U_3(n)$$

$$\cdots \pi_2 \pi_3 \cdots \pi_n = \frac{(n-1)!}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{2^{2i}}{(2i-1)!!} U_i(n)$$

Using the results of the previous section we can express each one of the Farahat-Higman operators as a polynomial in  $U_1(n)$ , which is essentially the resultant spin operator, i.e.,  $S^2 = \frac{3}{4}n + 2U_1(n)$ . In order to express the class-sums themselves in terms of  $U_1(n)$  we have to use the relations between the Kramer and the Farahat-Higman operators, discussed in section 1. This is simply a manifestation of the well known fact that for a system of identical particles with an elementary spin  $\sigma = \frac{1}{2}$  the class of transpositions is sufficient to determine the irreducible representations uniquely. When Young diagrams with at most  $k$  rows are possible, one needs the first  $k-1$  single-cycle class-sums to specify the irreducible representations uniquely [5]. Thus, for a system of particles with  $\sigma = 1$  one needs the two class-sums  $[(2)]_n$  and  $[(3)]_n$ . These class-sums can be expressed in terms of the symmetric power sums  $O_1$  and  $O_2$ , obtaining  $[(2)]_n = O_2 + O_1 - n(n-1)/2$  and  $[(3)]_n = \frac{1}{3(n+1)} \{4O_2^2 + 4O_1^2 - 4(n^2-n+4)O_2 + n(n-1)(n^2-n+6)\}$ . It was shown in [12] that  $[O_1, O_2] = 0$ .

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# QUANTIZATION AND CONTRACTION OF AN ELEMENTARY SYSTEM ON A CURVED SPACETIME

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## 1. Introduction

It is a well known fact that the Poincaré group,  $\mathcal{P}_+^\uparrow(3,1)$ , the kinematical group of Minkowski spacetime, can be obtained by means of a contraction from the anti-de Sitter (AdS) group,  $SO_0(3,2)$ , the kinematical group of anti-de Sitter spacetime. The contraction parameter is the constant positive curvature  $\kappa$  of the anti-de Sitter spacetime. This contraction procedure is thus nothing but a zero curvature limit. According to this fact, one would like to approximate  $\mathcal{P}_+^\uparrow(3,1)$ -invariant theories by  $SO_0(3,2)$ -invariant ones, hoping that such approximations give rise to regularized relativistic theories. Indeed, the nonzero curvature equips the AdS theories with a lengthlike parameter, which is actually the source of the sought regularizations.

Up to now, this very stimulating idea has not been fully exploited, though it has received a large amount of attention for its potential implications in the context of quantum field theories. The main drawback of the known approaches arises from the emphasis made on the spacetime or the momentum space realizations of those theories. Indeed, it is a known fact that such realizations, in both Poincaré and AdS cases, lack of a natural notion of localization. Moreover the modulus of the wave functions corresponding to the one particle quantum states of a Poincaré, as well as an AdS, free massive theory can not be interpreted, in those realizations, as a probability distribution. The regularizing role of  $\kappa$  is thus not effective for such realizations.

In this short contribution we propose the phase space realization as the regularizing alternative (for more details see [1], [2] and [3]). In fact, for the case of a free massive spinning particle in AdS spacetime, the phase space is a Kähler  $SO_0(3,2)$  homogeneous space, whose (geometric) quantization gives rise to a discrete series representation of  $SO_0(3,2)$ . The latter is known to be a square integrable representation, so its Hilbert space contains a particular family of quantum states: the coherent states. A natural notion of localization is attached to these states. They are *optimally localized* states in phase space. Moreover the modulus of the wave functions of the quantum states in this realization can be actually interpreted as a probability distribution.

Here we exhibit the explicit form of these coherent states and we show how their physical interpretation arises. We also stress the disappearance of this notion of localization in the flat space limit, confirming the effectiveness of the regularizing character of  $\kappa$ . We proceed as follows. In section 2 we describe the classical theory, in order to fix both the notations and the physical interpretations. In section 3, the quantum theory

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is obtained through the application of geometric quantization, then the explicit form and the zero curvature limit of the optimally localized states is given. For more details we refer to papers [1], [2] and [3].

## 2. The classical theory

The phase space description of the classical theory of a spin  $s$  and mass  $m \neq 0$  free particle in AdS spacetime finds its best formulation within the scheme developed by Souriau. The latter construction starts with the determination of an *evolution space*,  $(E_\kappa^{m,s}, \omega_E)$ , which is a presymplectic manifold ( $\omega_E$  is a closed but degenerate 2-form) with a projection on the AdS spacetime of constant curvature  $\kappa$ ,  $M_\kappa$ . The symmetries of  $M_\kappa$  are helpful guides in doing so. In fact,  $M_\kappa$  is just the one sheeted hyperboloid in  $(\mathbf{R}^5, \eta)$ , with  $\text{diag } \eta = \begin{pmatrix} 5 & 0 & 1 & 2 & 3 \\ - & - & + & + & + \end{pmatrix}$ ,

$$y \cdot y \equiv \eta_{\alpha\beta} y^\alpha y^\beta = -(y^5)^2 - (y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2 = -\kappa^{-2}, \quad (2.1)$$

$\alpha, \beta \in \{5, 0, 1, 2, 3\}$ . Clearly,  $O(3, 2)$  is the isometry group of (2.1), its connected component to the identity,  $SO_0(3, 2)$ , is the so-called AdS group.

We choose for  $E_\kappa^{m,s}$  the  $SO_0(3, 2)$ -principal homogeneous space,  $E_\kappa^{m,s} \cong SO_0(3, 2)$ , realized through the following  $SO_0(3, 2)$ -invariant constraints in  $\mathbf{R}^{25}$  (five copies of  $(\mathbf{R}^5, \eta)$ ),

$$y \cdot y = -\kappa^{-2}, \quad q \cdot q = -m^2, \quad u \cdot u = 1, \quad v \cdot v = 1 \quad \text{and} \quad t \cdot t = m^2 s^2. \quad (2.2a)$$

$$y \cdot q = 0 = \text{all the other scalar products} \quad (2.2b)$$

$$\epsilon_{\alpha\beta\gamma\rho\sigma} y^\alpha q^\beta u^\gamma v^\rho t^\sigma = \frac{m^2 s}{\kappa} \quad \text{and} \quad y^5 q^0 - y^0 q^5 > 0. \quad (2.2c)$$

The physical interpretation of the coordinates  $(y, q, u, v, t)$  is then as follows: in (2.2a)  $y$  is the position on the hyperboloid (2.1),  $q$  is its conjugate momentum,  $t$  is what we call the AdS-Pauli-Lubanski vector. The remaining five-vectors  $u$  and  $v$  are introduced in order to have a covariant description of  $E_\kappa^{m,s}$ , i.e.  $E_\kappa^{m,s} \cong SO_0(3, 2)$ . They shall represent the spin part in the quantum theory. The two last constraints (2.2b-c) are needed in order to fix an orientation.

The choice of  $\omega_E$  is constrained by the requirement that the projection on  $M_\kappa$  of each integral curve of the completely integrable distribution generated by  $\ker \omega_E$  in  $E_\kappa^{m,s}$ , results in a time-like geodesic of  $M_\kappa$ , i.e. the dynamic of the theory is obtained from  $\ker \omega_E$ . Such an  $\omega_E$  is provided by,

$$\omega_E = dy \wedge dq + s du \wedge dv. \quad (2.3)$$

This choice is not unique but it fulfils the above dynamic generating requirement. The phase space of the theory,  $(\Sigma_\kappa^{m,s}, \omega_E)$ , is obtained by symplectic reduction of  $(E_\kappa^{m,s}, \omega_E)$ . It appears, for  $\frac{m}{\kappa} \neq s$ , to be the  $SO_0(3, 2)$  symplectic homogeneous space  $SO_0(3, 2)/SO(2) \times SO(2)$ . For symmetry reasons, i.e. obvious action of  $SO_0(3, 2)$  on  $E_\kappa^{m,s}$ , we use  $(E_\kappa^{m,s}, \omega_E)$  as the arena for the forthcoming constructions.

In order to carry out the zero curvature limit in a meaningful way, we introduce a new set of coordinates on  $E_\kappa^{m,s}$ . This is the set of four-vectors  $(x, p, a, b, s)$ . Interpreted

in the same way as the five-vectors  $(y, q, u, v, t)$ , they are related to the letters through the following equations,

$$y^5 = Y \cos \kappa x^0, \quad y^0 = Y \sin \kappa x^0 \quad \text{and} \quad \vec{y} = \vec{x}, \quad (2.4a)$$

where  $-\pi \leq \kappa x^0 \leq \pi$ ,  $\vec{x} \in \mathbf{R}^3$  and  $Y = \sqrt{\kappa^{-2} + (\vec{x})^2}$ ; and

$$q \cdot dy = g_{\mu\nu} p^\mu dx^\nu, \quad u \cdot dy = g_{\mu\nu} a^\mu dx^\nu, \quad v \cdot dy = g_{\mu\nu} b^\mu dx^\nu \quad \text{and} \quad t \cdot dy = g_{\mu\nu} s^\mu dx^\nu. \quad (2.4b)$$

Here  $g_{\mu\nu}$  is the metric of  $M_\kappa$  for the global coordinates  $(x^0, \vec{x})$  and  $\mu, \nu \in \{0, 1, 2, 3\}$ . The zero curvature limit of  $g_{\mu\nu}$  is just the flat Minkowski metric. The constraints (2.2a-c) translated in terms of the new coordinates become,

$$g_{\mu\nu} p^\mu p^\nu = -m^2, \quad g_{\mu\nu} a^\mu a^\nu = 1, \quad g_{\mu\nu} b^\mu b^\nu = 1 \quad \text{and} \quad g_{\mu\nu} s^\mu s^\nu = m^2 s^2, \quad (2.5a)$$

$$g_{\mu\nu} p^\mu s^\nu = 0 = \text{all the other scalar products of the subset } (p, a, b, s), \quad (2.5b)$$

$$\epsilon_{\mu\nu\lambda\delta} p^\mu a^\nu b^\lambda s^\delta = m^2 s \quad \text{and} \quad p^0 > 0. \quad (2.5c)$$

The physical interpretation of the above constraints can now be confirmed by their zero curvature limits.

### 3. The quantum theory and the optimal localization

The methods of geometric quantization allow one to quantize the classical theory described above. In other words, using those methods one is able to construct the unitary irreducible representation of  $SO_0(3, 2)$  associated to the coadjoint orbit of  $SO_0(3, 2)$  for which the phase space  $\Sigma_\kappa^{m,s}$  is a covering. Exploiting the principal bundle structure  $E_\kappa^{m,s} \cong SO_0(3, 2) \rightarrow SO_0(3, 2)/SO(2) \times SO(2) \cong \Sigma_\kappa^{m,s}$ , the prequantum Hilbert space,  $\mathcal{H}$ , is realized as follows,

$$\mathcal{H} = \left\{ \psi : E_\kappa^{m,s} \rightarrow \mathbf{C} \mid \int_{E_\kappa^{m,s}} |\psi|^2 d\mu_\kappa^{m,s} < \infty, \quad Y_{50}\psi = i \frac{m}{\kappa} \psi \quad \text{and} \quad Y_{12}\psi = is\psi \right\}. \quad (3.1)$$

Here  $d\mu_\kappa^{m,s}$  is the invariant measure on  $E_\kappa^{m,s}$  and  $Y_{50}$  and  $Y_{12}$  are the left invariant vector fields generating  $\ker \omega_E$ . Since  $E_\kappa^{m,s} \cong SO_0(3, 2)$ , there exists a natural action of  $SO_0(3, 2)$  in  $L^2(E_\kappa^{m,s}, d\mu_\kappa^{m,s})$ . This yields the left regular representation of  $SO_0(3, 2)$ . The latter restricts to a unitary (reducible) representation in  $\mathcal{H}$ , i.e. the representation of  $SO_0(3, 2)$  induced by the character  $e^{i(\frac{m}{\kappa}\tau + s\tau')}$  of  $SO(2) \times SO(2)$ . Indeed, this holds provided  $\frac{m}{\kappa}$  and  $s$  are both integers.

There actually exists a positive invariant Kählerian polarization of  $\Sigma_\kappa^{m,s}$  allowing one to select in  $\mathcal{H}$  an invariant subspace  $\mathcal{H}_\kappa^{m,s}$ . The restriction of the previous unitary representation to the latter gives rise to a unitary irreducible representation of  $SO_0(3, 2)$ . Concretely,

$$\mathcal{H}_\kappa^{m,s} = \{ \psi \in \mathcal{H} \mid \bar{Z}_i \psi = 0, \quad i \in \{1, 2, 3\} \quad \text{et} \quad \bar{\Xi} \psi = 0 \}; \quad (3.2)$$

where  $Z_i = Y_{0i} + iY_{i5}$ ,  $i \in \{1, 2, 3\}$  and  $\Xi = Y_{23} + iY_{31}$ . The  $Y_{\alpha\beta}$ 's are the left invariant vector fields. The way one obtains the unitary irreducible representation carried by  $\mathcal{H}_\kappa^{m,s}$  is known in the mathematic litterature as the holomorphic induction, it yields

the discrete series representation of  $SO_0(3,2)$  with highest weight  $(\frac{m}{\kappa}, s)$ . (A necessary condition for the unitarity is  $\frac{m}{\kappa} > s$ .)

The quantum states of the theory are represented by well defined wave functions belonging to  $\mathcal{H}_\kappa^{m,s}$ . The physical interpretation of their modulus as probability distributions on  $\Sigma_\kappa^{m,s}$  is also well defined. The particular states belonging to the orbit,  $\mathcal{O}_{\varphi_0} \subset \mathcal{H}_\kappa^{m,s}$ , of the unitary representation of  $SO_0(3,2)$  passing through the highest weight state  $\varphi_0$  possess many interesting properties. These states, which are nothing but the generalized coherent states of  $SO_0(3,2)$ , are in a natural way optimally localized in phase space. In fact, by construction they are labeled by points  $w \in E_\kappa^{m,s}$ , specifying them through the equations,

$$\langle \varphi_w | \hat{L}_{\alpha\beta} | \varphi_w \rangle = L_{\alpha\beta}(w), \quad \forall \alpha, \beta \in \{5, 0, 1, 2, 3\}; \quad (3.3)$$

here the  $L_{\alpha\beta}$ 's are the classical observables and the  $\hat{L}_{\alpha\beta}$ 's are their quantum counterparts. The determination through (3.3) of the ten  $L_{\alpha\beta}(w)$  specifies in fact uniquely the leave of the distribution  $\ker \omega_E$  passing through  $w$ . Thus by symplectic reduction a unique point  $\tilde{w} \in \Sigma_\kappa^{m,s}$  is specified by (3.3). The state  $\varphi_w$  is then said to be localized in  $\tilde{w} \in \Sigma_\kappa^{m,s}$ . Moreover, since the coherent states minimize the uncertainty relations associated to the commutation relations of the  $\hat{L}_{\alpha\beta}$ 's, this notion of localization is said to be optimal.

The optimally localized states are given by the following formula.

$$\varphi_{z', \xi'}(z, \xi) = (-2)^{\frac{m}{\kappa}} (2)^{-s} (z' \cdot z)^{-\frac{m}{\kappa} - s} \left[ (z' \cdot z)(\xi' \cdot \xi) - (z' \cdot \xi)(\xi' \cdot z) \right]^s. \quad (3.4)$$

Here  $(z, \xi) \equiv w$  are the complex coordinates of  $E_\kappa^{m,s}$  associated to the Kählerian polarization, they are related to the coordinates given in (2.2) through the transformations  $z = \kappa y - im^{-1}q$  and  $\xi = u - iv$ .

The zero curvature limit of these states is as follows.

$$\lim_{\kappa \rightarrow 0} \left( \frac{m}{4\pi\kappa} \right)^{\frac{3}{2}} \varphi_{z', \xi'}(z, \xi) = m^2 p^0 \delta(\bar{p} - \bar{p}') e^{-ip_\mu(x'^\mu - x^\mu)} \left( \frac{\bar{\zeta}' \cdot \zeta}{2} \right)^s, \quad (3.5)$$

where  $\zeta_\mu = a_\mu - ib_\mu$ ,  $\mu \in \{0, 1, 2, 3\}$ . Clearly, these states are no longer optimally localized. They are completely delocalized in position ( $x$ ), perfectly localized in momentum ( $p$ ) and still optimally localized in spin ( $\zeta$ ). This zero curvature behaviour supports the regularization argument stressed in the introduction. In fact one can consider the AdS states in (3.4) as regularizations of the (generalized) Poincaré states in (3.5).

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## Contraction of the discrete series Fock-Bargmann representation of $SU(1,1)$

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*Abstract.* It is well-known from the Inönü-Wigner contraction procedure that the Poincaré group  $P_+^1(1,1)$  in one time and one space dimension can be viewed as the contraction of the group  $SU(1,1)$ . We present here a way to contract the discrete series Fock-Bargmann representation of  $SU(1,1)$  to the Wigner positive-energy representation of  $P_+^1(1,1)$ .

This is a report on joint work with J.P. Gazeau [1]. In this work we study some aspects of the contraction process  $SO_0(1,2) \rightarrow$  Poincaré  $P_+^1(1,1)$ . We begin with the choice of a suitable parametrisation for the desitterian phase-space  $SO_0(1,2)/SO(2) \cong SU(1,1)/U(1)$  realised as the open unit disk  $\mathcal{D}$ . Then we consider the Fock-Bargmann spaces of holomorphic functions in  $\mathcal{D}$  that carry the discrete series representations of  $SU(1,1)$  and show how this representation contracts at the global level onto the Wigner representation  $P(m)$  of  $P_+^1(1,1)$ . We insist here on the analytic aspect and give a "semiclassical" expansion of the general element of the Fock-Bargmann space in terms of the curvature parameter.

Let us recall that  $G = SU(1,1)$  is the group of  $2 \times 2$  complex matrices  $g$  of the form

$$g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad \alpha, \beta \in \mathbb{C} \quad (1)$$

with unit determinant. It admits the well-known Cartan decomposition [2]  $G = PH$ , where  $H = U(1)$  and  $P$  is the subset of hermitian matrices in  $G$ . Explicitly,  $g = p(z)h(\theta)$  with

$$p(z) = \begin{pmatrix} \delta & \delta z \\ \delta \bar{z} & \delta \end{pmatrix}, \quad z = \beta \bar{\alpha}^{-1}, \quad \delta = (1 - |z|^2)^{-1/2} \quad (2)$$

and

$$h(\theta) = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}, \quad \theta = 2 \operatorname{Arg} \alpha, \quad 0 \leq \theta < 4\pi. \quad (3)$$

The bundle section  $z \in \mathcal{D} \rightarrow p(z) \in P$  gives the open unit disk  $\mathcal{D} = \{z \in \mathbb{C}, |z| < 1\}$  a symmetric space realisation as the coset space  $G/H$ .  $SU(1,1)$  acts on  $\mathcal{D}$  by left action on the set of matrices  $p(z)$ , i.e.  $g : p(z) \rightarrow p(z')$  defined by  $gp(z) = p(z')h'$ . Explicitly, the action of  $g$  on  $z$  is homographic

$$z' \equiv g \cdot z = (\alpha z + \beta)(\bar{\beta} z + \bar{\alpha})^{-1}. \quad (4)$$

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As it is well-known [3, 4],  $\mathcal{D}$  is a Kählerian manifold and has the  $SU(1, 1)$ -invariant Kählerian 2-form

$$\omega = 2i(1 - |z|^2)^{-2} dz \wedge d\bar{z}. \quad (5)$$

$G$  also admits the so-called "space-time" factorisation:

$$\begin{aligned} g &= s(\theta, \psi)l(\phi) \\ &= \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \begin{pmatrix} \cosh \frac{\psi}{2} & \sinh \frac{\psi}{2} \\ \sinh \frac{\psi}{2} & \cosh \frac{\psi}{2} \end{pmatrix} \begin{pmatrix} \cosh \frac{\phi}{2} & i \sinh \frac{\phi}{2} \\ -i \sinh \frac{\phi}{2} & \cosh \frac{\phi}{2} \end{pmatrix} \varepsilon. \end{aligned} \quad (6)$$

$(\theta, \psi) \equiv s$  is actually a system of global coordinates for Anti De Sitter space-time [1].

Let us now come to the new parametrisation of  $\mathcal{D}$ . Any matrix representation (2) of a point  $z$  in  $\mathcal{D}$  admits a space-time factorisation (6) so that it is easy to deduce the expression of  $z$  as

$$z = z(\psi, \phi) = \frac{\sinh \psi + i \cosh \psi \sinh \phi}{1 + \cosh \phi \cosh \psi} \quad (7)$$

and to see that  $\theta$  satisfies the constraint

$$e^{i\theta} = \frac{\cosh \psi + \cosh \phi + i \sinh \psi \sinh \phi}{1 + \cosh \psi \cosh \phi}. \quad (8)$$

In order to give the above expressions a more familiar meaning let us adopt a parametrisation of a "Minkowski-Lorentz" type, namely

$$q^0 = \kappa^{-1} \theta, \quad q = \kappa^{-1} \psi, \quad \frac{p}{mc} = \sinh \phi, \quad \frac{p_0}{mc} = \cosh \phi, \quad (9)$$

where the "curvature"  $\kappa^{-1}$  is inverse lengthlike. The vector  $(p_0, p)$  belongs to the forward mass hyperbola

$$\mathcal{V}_m^+ = \{(p_0, p) \in \mathbb{R}^2 \mid p_0 > 0, p_0^2 - p^2 = m^2\}.$$

We deduce the coordinate transformation as the map  $z \in \mathcal{D} \longmapsto (q, p) \in \mathbb{R}^2$  defined by

$$z(q, p, \kappa) = \frac{mc \sinh \kappa q + ip \cosh \kappa q}{mc + p_0 \cosh \kappa q}. \quad (10)$$

The relation (8) imposes a specific value for  $q_0$  which satisfies

$$e^{i\kappa q_0} = \frac{p_0 + mc \cosh \kappa q + ip \sinh \kappa q}{mc + p_0 \cosh \kappa q}. \quad (11)$$

The Kählerian 2-form (5) is given in terms of  $(q, p)$  by

$$\omega = \kappa \cosh \kappa q dq \wedge \frac{dp}{p} = d(\sinh \kappa q) \wedge \frac{dp}{p_0}. \quad (12)$$

We now introduce Fock-Bargmann structures on  $\mathcal{D}$ . We denote  $\mathcal{F}^{E_0} = \{f(z) : z \in \mathcal{D}\}$  the Fock-Bargmann space [3] of functions holomorphic inside the unit circle, satisfying the square-integrability condition  $(f, f) < \infty$  with respect to the scalar product

$$(f_1, f_2) = \frac{E_0 - 1/2}{2\pi} \int_{\mathcal{D}} \bar{f}_1(z) f_2(z) (1 - |z|^2)^{2E_0} \omega \quad (13)$$

with  $\omega$  given by (5) and  $E_0 > 1/2$  is a fixed real number. The representation operator  $T^{E_0}(g)$  of  $SU(1, 1)$  is defined by

$$(T^{E_0}(g)f)(z) = (\beta z + \bar{\alpha})^{-2E_0} f(g^t \cdot z). \quad (14)$$

$T^{E_0}$  is unitary, irreducible and belongs to the discrete series for  $E_0 > 1/2$  [2].

Since our final aim is to understand better some aspects of the contraction process  $SU(1, 1) \rightarrow \mathcal{P}_+^1(1, 1)$ , as  $\kappa$  goes to zero and  $E_0$  goes to the infinity, it is apparent that the form (13) or equivalently the space  $\mathcal{F}^{E_0}$  is not well-adapted to such a limit procedure. Therefore we introduce a "weighted" Fock-Bargmann space

$$\mathcal{F}_W^{E_0} = \{F(z, \bar{z}) = (1 - |z|^2)^{E_0} f(z), f \in \mathcal{F}^{E_0}\} \quad (15)$$

which is the Hilbert space of square-integrable non analytic functions inside the unit disk with the scalar product

$$(F_1, F_2) = \frac{E_0 - 1/2}{2\pi} \int_{\mathcal{D}} \bar{F}_1(z, \bar{z}) F_2(z, \bar{z}) \omega. \quad (16)$$

The representation operator  $T_W^{E_0}(g)$  on  $\mathcal{F}_W^{E_0}$  is deduced from  $T_0^E(g)$  defined by (14):

$$(T_W^{E_0}(g)F)(z, \bar{z}) = \left( \frac{\beta \bar{z} + \alpha}{\beta z + \bar{\alpha}} \right)^{E_0} F(g^t \cdot z, g^t \cdot \bar{z}). \quad (17)$$

The last step before the contraction procedure is to translate this abstract machinery where no operational physical quantity appears into the familiar language where physical dimensions are present. Besides the (three) fundamental constants  $\kappa$ ,  $m$ , and  $c$  already injected into the formalism, the quantum context now introduces action-dimensional physical quantities at the order of  $\hbar$ . The pure number  $E_0$ , which is actually the minimal weight of the representation  $T_W^{E_0}$ , is taken as  $E_0 = E_0(\kappa) = \frac{mc}{\hbar \kappa}$ .

It is well-known [5] that the representation  $T^{E_0}(g)$  or  $T_W^{E_0}(g)$  of  $SU(1, 1) \simeq SO_0(2, 1) \times \mathbf{Z}_2$  must contract to the Wigner positive-energy representation  $P(m)$  of  $P_+^1(1, 1)$  as  $\kappa$  goes to zero and  $E_0$  goes to infinity while keeping the product  $\kappa E_0$  equal to  $\frac{mc}{\hbar}$ . Let us see how it works explicitly on the space  $\mathcal{F}_W^{E_0}$ .

In order to eliminate the singular terms in the expansion in  $\kappa$  of  $F(z, \bar{z})$ , we must impose some constraints on the form of  $F \equiv (15)$  and more particularly of the original analytic function  $f(z)$ . Indeed we must factorise  $f(z)$  as

$$f(z, \kappa) = N(\kappa)(1 + z^2)^{-E_0(\kappa)} h(z, \kappa), \quad (18)$$

where the function  $h$  is now analytic in both  $z \in \mathcal{D}$  and  $\kappa \geq 0$ .  $N(\kappa)$  is a normalisation factor possibly nonanalytic in  $\kappa$ . In the following normalisation will not be imposed in order to ignore this nonanalytic  $N(\kappa)$ . We accordingly restrict our considerations by working on the subspace of  $\mathcal{F}_W^{E_0}$  which consists of functions of the form

$$F(z, \bar{z}, \kappa) = \left( \frac{1 - |z|^2}{1 + z^2} \right)^{E_0(\kappa)} h(z). \quad (19)$$

where  $h(z) \equiv h(z, 0)$ . If we now introduce  $\Psi(q, p) \equiv \lim_{\kappa \rightarrow 0} F(z, \bar{z}, \kappa)$ , we have proved [1] by expanding (19) in  $\kappa$  that

$$\Psi(q, p) = e^{-\frac{i}{\hbar} \underline{q} \cdot \underline{p}} \Phi(p) \quad (20)$$

where

$$\Phi(p) = h\left(\frac{ip}{p_0 + mc}\right) \quad (21)$$

The 2-vectors  $\underline{p}$  and  $\underline{q}$ , are given by

$$\underline{p} = (p_0, \underline{p}), \quad \underline{q} = (q_0 = \frac{qp}{p_0 + mc}, \underline{q}) \quad (22)$$

and  $\underline{q} \cdot \underline{p} = q_0 p_0 - \underline{q} \cdot \underline{p}$ . We recognize in (22) the flat space-time limit of the section (8) which defines, in the space of parameters  $(\theta, \psi, \phi)$  for  $SU(1, 1)$ , the Cartan phase space for this group. The remarkable properties of the desitterian section (22) for Poincaré are listed elsewhere [5].

Let us state the crucial result about the limit of square integrability condition (see [1] for the proof):

**PROPOSITION.** Let  $h(z)$  be an holomorphic function in the unit disk  $\mathcal{D}$  such that the integral

$$\int_{\infty} d\Re z d\Im z \left(\frac{1 - |z|^2}{|1 + z|^2}\right)^{2E_0} |h(z)|^2 \quad (23)$$

is finite. Then the function on  $\mathbf{R}$  defined by (24) is square integrable on  $\mathbf{R}$  with respect to the Lorentz invariant  $\frac{dp}{p_0}$ .

The last thing to do is to show how the representation  $T_W^{E_0}(g)$  of  $SU(1, 1)$  on  $F \equiv (19)$  contracts. The first step is to write  $g \in SU(1, 1)$  on the form

$$g = h(\kappa a_0) b(\kappa a) l(\phi), \quad (24)$$

where  $\phi$  is here given by  $\phi = \tanh^{-1}\left(\frac{k}{k_0}\right)$ ,  $k \in \mathcal{V}_m^+$ . It is well-known from the Inönü-Wigner contraction procedure [6] that, when  $\kappa \rightarrow 0$ , the set  $(a_0, a, k)$  turns out to be the set of parameters characterising the Poincaré group  $P_+^1(1, 1) = ((a_0, a), \Lambda_k)$ . The second step is to express the dependence on  $\kappa$  of (17) where  $F$  has the form (19). This will be done by replacing  $z$  by (10) and  $\alpha, \beta$  by their expression in terms of  $a_0, a$  and  $\phi$  according to (24). The last step is to expand (18) in  $\kappa$  and to take the limit when  $\kappa \rightarrow 0$ .

Let us give the final result expressed in terms of the phase-space variables. We introduce the transformed variables  $(q', p')$  given by

$$q' = q - \left(a_0 \frac{p}{mc} - a \frac{p_0}{mc}\right), \quad p' = \frac{k_0}{mc} p - \frac{k}{mc} p_0, \quad p'_0 = \frac{k_0}{mc} p_0 - \frac{k}{mc} p. \quad (25)$$

We then get

$$\begin{aligned} \lim_{\kappa \rightarrow 0} \left(T_W^{E_0(\kappa)}(g(a_0, a, k; \kappa)) F\right) \left(z(q, p, \kappa), \bar{z}(q, p, \kappa)\right) \\ \equiv \left(U_m(a_0, a, k) \Psi\right)(q, p) = e^{\frac{i}{\hbar} (\zeta_0 + \zeta_1)} \Psi(q', p') \end{aligned} \quad (26)$$

where  $\Psi$  has been already defined in (20) and

$$\zeta_0 = \zeta_0(q, p; k) = \frac{-2m^2 c^2 q k}{((p_0 + mc)(k_0 + mc) - kp)} \quad (27a)$$

and

$$\zeta_1 = \zeta_1(p; a_0, a, k) = m^2 c^2 \frac{a_0(k_0 + p_0) - a(k + p)}{m^2 c^2 + k_0 p_0 - kp}. \quad (27b)$$

The representation (26) gives the action of  $P_+^I(1, 1)$  on the function  $\Psi(q, p)$ . We can easily deduce the corresponding action on  $\Phi(p)$  from its relation (20) to  $\Psi(q, p)$ . It turns out to be, as expected, the momentum version of the Wigner representation  $P(m)$ , i.e.

$$(U_m(a_0, a, k)\Phi)(p) = e^{\mathbf{\hat{x}} \cdot \mathbf{a}} P\Phi(p'). \quad (28)$$

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# CONTRACTIONS OF THE REPRESENTATIONS OF THE ORTHOGONAL ALGEBRAS

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### Abstract

About forty years ago Wigner and Inonu [0] have introduced the contraction of the group representations. This operation gives the representations of the contracted group from the representations of the initial group. Contraction was later used by many authors, in particular by Wigner and Kim [0] to describe the photon little group, by Smirnov and Asherova [0] in the symplectic collective model of the complex nucleus, by Celeghini, Tarlini and Vitiello [0] in the theory of the spontaneous breakdown of symmetry, by Doebner and Melsheimer [0] to describe the limiting process connecting the dynamical group of the interacting system with the dynamical group of the corresponding free particle.

There is another approach in group theory, namely Weyl unitary trick [0], which gives the representations of the pseudo-orthogonal group starting from the classical orthogonal group. Kuriyan, Mukunda and Sudarshan [0] have developed this trick to the method of master analytic representation.

The purpose of this report is to present the unified description of the orthogonal Cayley-Klein algebras and their representations starting from well known Gel'fand-Tsetlin representations [0] of the classical orthogonal algebras. The orthogonal Cayley-Klein algebras are obtained from the classical ones by Wigner-Inonu contractions and analytical continuations (or Weyl unitary trick).

The unified description [0, 0, 0] is achieved by introducing parameters, which may be equal to real unit or to imaginary unit or to Clifford dual units.

Under the map

$$\begin{aligned} \psi: \mathbb{R}_{n+1} &\rightarrow \mathbb{R}_{n+1}(j) \\ \psi x_0^* &= x_0, \quad \psi x_k^* = x_k \prod_{m=1}^k j_m, \quad k = 1, 2, \dots, n, \end{aligned} \quad (1)$$

where  $j = (j_1, j_2, \dots, j_n)$ , each of the parameters  $j_k = 1, i_k, i$ , and  $i_k$  is the Clifford dual unit with the algebraic properties  $i_k \neq 0$ ,  $i_k^2 = 0$ ,  $i_m i_m = i_m i_k \neq 0$ ,  $k \neq m$ , the euclidian space  $\mathbb{R}_{n+1}$  is transformed into the Cayley-Klein space ( $j$ ) with the quadratic form

$$(x, x) = x_0^2 + \sum_{k=1}^n x_k^2 \prod_{m=1}^k j_m^2. \quad (2)$$

The map  $\psi$  also induces the transformations of the generators  $\chi^*$  of the orthogonal algebra  $so(n+1)$  into the generators  $\chi$  of the orthogonal Cayley-Klein algebras  $so(n+1; j)$

$$\chi_{rs}(j) = \left( \prod_{m=r+1}^s j_m \right) \chi_{rs}^*(\rightarrow), \quad r < s, \quad (3)$$

where  $\chi^*(\rightarrow)$  denote the Wigner-Inonu [0] singular transformed generators and the products  $\prod j_m$  play the role of zero tending parameters, when some parameters  $j_m$  are equal to the dual units. When some parameters  $j_m$  are equal to imaginary unit, the Eqs.(3) give the transformations, corresponding to the Weyl unitary trick or the analytic continuations of the representations. Therefore, two different operations in group theory are naturally unified by Eqs.(3) and the same nature both of the operations, namely the continuation of the real group parameters into the complex (Weyl trick) or dual (Wigner-Inonu contraction) number field is well displayed in this manner.

The orthogonal Cayley-Klein algebra  $so(n+1; j)$  have the following commutation relations:

$$[\chi_{r_1 s_1}, \chi_{r_2 s_2}] = \begin{cases} \left( \prod_{l=r_1+1}^{s_1} j_l \right) \chi_{s_1 s_2}, & r_1 = r_2, s_1 < s_2 \\ \left( \prod_{l=r_2+1}^{s_2} j_l \right) \chi_{r_1 r_2}, & r_1 < r_2, s_1 = s_2 \\ -\chi_{r_1 s_2}, & r_1 < r_2 = s_1 < s_2 \end{cases} \quad (4)$$

The structure of the transformations (3) is the same for all representations of the orthogonal algebra. Only the Generators  $\chi^*(\rightarrow)$  are defined in a different way for a different representation of the orthogonal algebra. In the case of the Gel'fand-Tsetlin representation [0] the singular transformed generators  $\chi^*(\rightarrow)$  are specified by the transformations of the components of the patterns. The general nondegenerate representation of  $so(n+1; j)$  with the nonzero eigenvalues of all Casimir operators are obtained [0], when the components of the Gel'fand-Tsetline patterns are transformed as follows:

$$m_{p,r} = m_{p,r}^* \prod_{l=n+p-r}^{n+1-p} j_l, \quad p < r \quad (5)$$

In particular for the algebra  $so(4; j)$ ,  $j = (j_1, j_2, j_3)$  the transformation laws of the generators and of the components of the patterns are defined in the following manner:

$$\begin{aligned} \chi_{01} &= j_1 \chi_{01}^*(\rightarrow), & \chi_{02} &= j_1 j_2 \chi_{02}^*(\rightarrow), & \chi_{03} &= j_1 j_2 j_3 \chi_{03}^*(\rightarrow), \\ \chi_{12} &= j_2 \chi_{12}^*(\rightarrow), & \chi_{13} &= j_2 j_3 \chi_{13}^*(\rightarrow), & \chi_{23} &= j_3 \chi_{23}^*(\rightarrow), \end{aligned} \quad (6)$$

$$|m\rangle = \begin{Bmatrix} m_{13} & m_{23} \\ m_{12} \\ m_{11} \end{Bmatrix}, \quad \begin{aligned} m_{13} &= j_1 j_2 j_3 m_{13}^*, & m_{23} &= j_2 m_{23}^*, \\ m_{12} &= j_2 j_3 m_{12}^*, & m_{11} &= j_3 m_{11}^*, \end{aligned}$$

Then after some computations we obtain the explicit expressions of the generators of the all  $3^3 = 27$  orthogonal Cayley-Klein algebras  $so(4; j)$ . In particular for  $j_1 = i$ ,  $j_2 = j_3 = 1$  we have the Lorentz algebra  $so(1, 3)$ , for  $j_1 = i_1$ ,  $j_2 = j_3 = 1$  we have the Euclidian algebra  $e_3$ , for  $j_1 = i_1$ ,  $j_2 = i$ ,  $j_3 = 1$  we have the Poincare algebra  $p_{1,2}$  and for  $j_1 = i_1$ ,  $j_2 = i_2$ ,  $j_3 = 1$  we have the Galilean algebra  $g_3$ .

If different contractions or analytic continuations give isomorphic algebras, for example  $e_3 \equiv so(4; i_1, 1, 1) \cong so(4; 1, 1, i_3)$  in the case of inhomogenous algebras or  $so(1, 3) \equiv so(4; i, 1, 1) \cong so(4; 1, 1, i)$  in the case of the pseudoorthogonal algebras, then we obtain from the general explicit the representations of the contracted algebras in a different basisies, in particular in discrete and continuos ones.

The Gel'fand-Tsetlin representations of the unitary Cayley-Klein algebras  $[0, 0]$  and the Jordan-Schwinger representations of the orthogonal, unitary and symplectic Cayley-Klein algebras  $[0, 0]$  are described in a unified manner with help of the dual numbers. This approach is applicable also for quantum algebras  $[0, 0]$ . In the last case the unified description is split in two different methods: one for the contractions and another for the analytical continuations of the quantum algebras.

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**CLASSIFICATION OF ALL NONCANONICAL THREE-DIMENSIONAL QUANTUM  
OSCILLATORS GENERATING CLASSICAL LIE SUPERALGEBRAS**

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By a noncanonical (nonrelativistic) quantum system we understand a system for which the Heisenberg and the Hamiltonian equations are identical. This is always the case if the canonical commutation relations (CCR's) between the coordinates  $q_j$  and the momentum operators  $p_k$  hold [1],

$$[q_j, p_k] = i\hbar \delta_{jk}, \quad [q_j, q_k] = [p_j, p_k] = 0, \quad (1)$$

or, equivalently, if the operators

$$a_k^+ = (2\hbar)^{-1/2} (q_k - ip_k), \quad a_k^- = (2\hbar)^{-1/2} (q_k + ip_k) \quad (2)$$

are Bose creation and annihilation operators (CAO's):

$$[a_j^-, a_k^+] = \delta_{jk}, \quad [a_j^+, a_k^+] = [a_j^-, a_k^-] = 0. \quad (3)$$

In general however the class of operators for which the Heisenberg and the Hamiltonian equations are identical is much larger. This was demonstrated by Wigner in 1950 [2] on the example of a one-dimensional harmonic oscillator. Wigner has found an infinite set of solutions for  $q$  and  $p$ , all of them satisfying one and the same operator identities [3]. In terms of the operators (2) these identities read (throughout  $[x, y] = xy - yx$ ,  $\{x, y\} = xy + yx$ ):

$$[\{a^\xi, a^\eta\}, a^\epsilon] = (\epsilon - \xi)a^\eta + (\epsilon - \eta)a^\xi, \quad \xi, \eta, \epsilon = \pm 1. \quad (4)$$

In Ref. 3 it was shown that the operators  $\{a^\xi, a^\eta\}$  and  $a^\epsilon$ ,  $\xi, \eta, \epsilon = \pm$ , are the even and the odd generators of the classical Lie superalgebra (LS)  $osp(1/2)$  [4]. Thus Wigner was the first who constructed a class of representations of a Lie superalgebra, namely of  $osp(1/2)$ .

Coming back to the canonical operators (3) we observe that they satisfy relations which generalize eqs. (4), namely

$$[\{a_i^\xi, a_j^\eta\}, a_k^\epsilon] = \delta_{ik} (\epsilon - \xi)a_j^\eta + \delta_{jk} (\epsilon - \eta)a_i^\xi, \quad \xi, \eta, \epsilon = \pm 1. \quad (5)$$

The latter indicates [5] that any  $n$  pairs of Bose CAO's give a representation of the orthosymplectic Lie superalgebra  $osp(1/2n) \cong B(0, n)$  [4] with  $a_k^\pm$  constituting a basis in the odd subspace of it.

We conclude that the canonical position and the momentum operators in quantum mechanics and also the generalization found by Wigner lead to representations of classical orthosymplectic Lie superalgebras. In the present note we shall list all possible

noncanonical three-dimensional quantum oscillators with position and momentum operators generating classical Lie superalgebras. The precise meaning of "all" is given below.

Consider in the frame of a nonrelativistic quantum mechanics two point particles with masses  $m_1$  and  $m_2$  and a Hamiltonian

$$H_{tot} = \frac{1}{2m_1} p_1^2 + \frac{1}{2m_2} p_2^2 + \frac{m\omega^2}{2} (r_1 - r_2)^2. \quad (6)$$

Introduce the centre of mass (CM) and the internal variables

$$R = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}, \quad P = p_1 + p_2, \quad r = r_1 - r_2, \quad p = \frac{m_2 p_1 - m_1 p_2}{m_1 + m_2}. \quad (7)$$

Independently of the properties of the operators  $R, P, r, p$  one has

$$H_{tot} = H_{cm} + H, \quad H_{cm} = \frac{P^2}{2\mu}, \quad H = \frac{p^2}{2m} + \frac{m\omega^2}{2} r^2, \quad (8)$$

where  $\mu$  and  $m$  are the total and the reduced masses, respectively. For the equations of motion one has

Heisenberg equations:

$$\dot{P} = \frac{i}{\hbar} [H_{cm}, P], \quad \dot{R} = \frac{i}{\hbar} [H_{cm}, R],$$

$$\dot{p} = \frac{i}{\hbar} [H, p], \quad \dot{r} = \frac{i}{\hbar} [H, r]$$

Hamiltonian equations:

$$\dot{P} = 0, \quad \dot{R} = \frac{1}{\mu} P, \quad (9)$$

$$\dot{p} = -m\omega^2 r, \quad \dot{r} = \frac{1}{m} p. \quad (10)$$

If the CCR's hold the Heisenberg equations are identical with the Hamiltonian equations.

We wish to find all noncanonical solutions of the two-particle system which obey the following conditions.

**Postulate 0.1:** The CM operators  $R$  and  $P$  commute with the internal variables  $r, p$ .

**Postulate 0.2:** The CM operators  $R$  and  $P$  are canonical.

Thus it remains to consider the internal variables, which actually describe a three-dimensional oscillator, and to find interesting solutions with noncanonical  $r$  and  $p$ .

**Postulate 1.1:** The Heisenberg and the Hamiltonian equations (10) of the internal oscillator are identical.

**Postulate 1.2 (Rotation invariance):** There exists a vector operator  $M$ , the internal angular momentum, so that  $r$  and  $p$  are vector operators ( $j, k, l=1, 2, 3$ ):

$$[M_j, M_k] = i\epsilon_{jkl} M_l, \quad [M_j, r_k] = i\epsilon_{jkl} r_l, \quad [M_j, p_k] = i\epsilon_{jkl} p_l. \quad (11)$$

**Postulate 1.3:** There exists a representation space (state space)  $W$  of  $r, p, M, H$ , which is a Hilbert space, so that the physical observables ( $r, p, H, M, \dots$ ) are selfadjoint operators.

**Postulate 1.4:** The spectrum of  $H$  is bounded from below.

Taking into account the algebraical properties of the

canonical and the Wigner quantum systems we require in addition:

**Postulate 2:** The position and the momentum operators  $r$  and  $p$  of the internal oscillator constitute a basis in the odd subspace of a classical LS and generate it.

We proceed now to list all LS's, which admit solutions in the frame of the above postulates.

### 1. The LS osp(1/6). Para-Bose oscillators

This case was discussed in Ref.3. The operators

$$a_k^\pm = \left(\frac{m\omega}{2\hbar}\right)^{1/2} r_k \mp i(2m\omega\hbar)^{-1/2} p_k, \quad k=1,2,3 \quad (12)$$

satisfy the eqs.(5). They are odd generators and generate the LS osp(1/6) [6]. These operators are known in quantum field theory. They were introduced by G.äsen [7] as a possible generalization of the statistics of the integer-spin fields and are called para-Bose operators. A class of representations obeying postulates 1.1-1.4 is given with the representations corresponding to a certain order of the statistics (for more details see Ref.8 and the references therein). All such representations are infinite-dimensional. The Hamiltonian has infinitely many different eigenvalues.

### 2. The LS sl(1/3). A-oscillators

This case was studied in detail in Ref.3. The CAO's read

$$a_k^\pm = \left(\frac{m\omega}{2\hbar}\right)^{1/2} r_k \pm i(2m\omega\hbar)^{-1/2} p_k, \quad k=1,2,3. \quad (13)$$

They satisfy the relations ( $i, j, k=1,2,3$ ;  $\xi=\pm$ )

$$[[a_i^+, a_j^-], a_k^\pm] = \delta_{jk} a_i^+ - \delta_{ij} a_k^\pm, \quad [[a_i^+, a_j^-], a_k^\pm] = \delta_{jk} a_i^+ - \delta_{ij} a_k^\pm, \quad \{a_i^\xi, a_j^\xi\} = 0. \quad (14)$$

In terms of  $a_k^\pm$  one has

$$H = \frac{1}{2} \omega \hbar \sum_{k=1}^3 \{a_k^+, a_k^-\}, \quad r^2 = \frac{\hbar}{2m\omega} \sum_{k=1}^3 \{a_k^+, a_k^-\}, \quad p^2 = \frac{m\omega\hbar}{2} \sum_{k=1}^3 \{a_k^+, a_k^-\}, \quad (15)$$

$$M_k = i\hbar \sum_{k=1}^3 \epsilon_{k1m} \{a_1^-, a_m^+\}.$$

The representation space is finite-dimensional. The spectrum  $E_k$  of  $H$  and  $r^2$  has no more than 4 eigenvalues. For the representations considered in Ref.3, for instance, we have ( $p > 3$  is a fixed integer, labeling the representation)  $E_k = \omega\hbar(3p-2k)/2$ ,  $k=0,1,2,3$ . The maximal distance between the particles is  $r_{\max} = (3\hbar p/2m\omega)^{1/2}$ .

### 3. $osp(3/2)$ oscillators

Let  $e_{ij}$ ,  $i, j=1, \dots, 5$  be the Weyl generators of the LS  $gl(3/2)$  with supercommutation relations

$$[[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - (-1)^{(\langle i \rangle + \langle j \rangle)(\langle k \rangle + \langle l \rangle)} \delta_{il} e_{kj}, \quad (16)$$

where  $\langle 1 \rangle = \langle 2 \rangle = \langle 3 \rangle = 0$ ,  $\langle 4 \rangle = \langle 5 \rangle = 1$ ;  $\deg(e_{ij}) = (\langle i \rangle + \langle j \rangle) \bmod 2$ . The CAO's

$$\begin{aligned} a_1^+ &= -(3)^{-1/2}(e_{14} - e_{52} + e_{24} - e_{51}), & a_1^- &= -(3)^{-1/2}(e_{15} + e_{42} + e_{25} + e_{41}), \\ a_2^+ &= i(3)^{-1/2}(e_{14} - e_{52} - e_{24} + e_{51}), & a_2^- &= i(3)^{-1/2}(e_{15} + e_{42} - e_{25} - e_{41}), \\ a_3^+ &= (2/3)^{1/2}(e_{34} - e_{33}), & a_3^- &= (2/3)^{1/2}(e_{35} + e_{43}). \end{aligned} \quad (17)$$

constitute a basis in the odd subspace of the LS  $osp(3/2)$  and generate it. For  $H$  and  $M$  one has

$$H = \frac{1}{2} \omega \sum_{i=1}^3 [a_i^+, a_i^-] = \omega (e_{44} - e_{55}), \quad (18)$$

$$M_1 = \frac{1}{\sqrt{2}}(e_{13} - e_{32} + e_{31} - e_{23}), \quad M_2 = \frac{i}{\sqrt{2}}(e_{31} - e_{23} - e_{13} + e_{32}), \quad M_3 = e_{11} - e_{22}. \quad (19)$$

The requirements of the postulate 1.3 lead to infinite-dimensional representations. In the metaplectic representation of  $osp(3/2)$  [9] for instance the internal angular momentum is  $1/2$  and the energy is the same as for an one-dimensional canonical oscillator,

$$E_n = \omega(n + 1/2), \quad n \in \mathbb{Z}_+. \quad (20)$$

To conclude with we formulate a proposition.

**Proposition.** The Lie superalgebras  $osp(1/6)$ ,  $sl(1/3)$  and  $osp(3/2)$  are the only classical LS's to satisfy the postulates of a noncanonical 3-dimensional quantum oscillator.

The proof will be given elsewhere.

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## KAC-MOODY-MALCEV ALGEBRAS AND SUPERALGEBRAS

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We define the Kac-Moody-Malcev and super-Kac-Moody-Malcev algebras as infinite-dimensional Malcev and super-Malcev algebras, describe possible central extensions and prove that the Kac-Moody-Malcev algebras, associated with the simple non-Lie Malcev algebras, have trivial (i.e. zero) central extensions only and super-Kac-Moody-Malcev algebras, associated with the simple non-Lie Malcev algebras, have nontrivial central extensions for Grassmann generators only. We describe the Sugawara construction elsewhere.

Analogously the Kac-Moody-Malcev type algebras on generic (compact) Riemann surfaces, associated with simple non-Lie Malcev algebras, have trivial central extensions for the algebras and nontrivial for the Grassmann generators of the super-Kac-Moody-Malcev-Krichever-Novikov algebras. The detailed description for the case of generic Riemann surfaces will be given elsewhere. Here we consider the case of complex plane only.

Let  $M$  be a  $Z_2$ -graded algebra, that is  $M = M_0 \oplus M_1$ . We define the intrinsic  $Z_2$ -grading  $\partial x$  of the element  $x \in M$ ,  $\partial x = 0$  if  $x \in M_0$  and  $\partial x = 1$  if  $x \in M_1$ . The bilinear (super)bracket  $[\cdot, \cdot]$  is defined on  $M$ . We suppose that

$$[M_i, M_j] \subset M_{i+j}, \quad i, j = 0, 1 \quad \text{and} \quad M_2 \equiv M_0, \quad (1)$$

and

$$[x, y] = -(-1)^{\partial x \partial y} [y, x]$$

for elements with definite gradation. We define

$$J(x, y, z) = (-1)^{\partial x \partial z} [x, [y, z]] + (-1)^{\partial y \partial x} [y, [z, x]] + (-1)^{\partial z \partial y} [z, [x, y]]$$

for elements with definite gradation.

An algebra is called the super-Malcev algebra if the identities

$$[J(x, y, z), x] = J(x, y, [x, z]) \quad (2)$$

are satisfied for elements with definite gradation.

In particular, the Kac-Moody-Malcev algebra, associated with a Malcev algebra  $g$ , is defined by the set of generators  $(T_m^a, 1 \leq a \leq \dim g, m \in \mathbb{Z})$  and brackets

$$[T_m^a, T_n^b] = \sum_c f^{abc} T_{m+n}^c, \quad (3)$$

where  $f^{abc}$  are the structure constants of the algebra  $g$ .

The super-Kac-Moody-Malcev algebra, associated with a Malcev algebra  $g$  and (one) Grassmann generator, is given by the set of generators  $(T_m^a, \theta_m^b, 1 \leq a, b \leq \dim g, m, n \in \mathbb{Z})$  and brackets

$$[T_m^a, T_n^b] = \sum_c f^{abc} T_{m+n}^c, \quad [T_m^a, \theta_n^b] = \sum_c f^{abc} \theta_{m+n}^c, \quad (4)$$

$$[\theta_m^a, \theta_n^b] = 0.$$

Here  $\partial T_m^a = 0$  and  $\partial \theta_m^a = 1$  for all  $a$  and  $m$ .

It is easy to see that (3) and (4) are generated the Malcev algebra and super-Malcev algebra, correspondingly.

Let us consider now a more general algebra, which is a central extension of the algebra (4) and which is generated by the set of generators ( 1,  $T_m^a$ ,  $\theta_n^b$ ,  $1 \leq a, b \leq \dim g$ ,  $m, n \in \mathbb{Z}$ ) and commutation relations

$$\begin{aligned} [T_m^a, T_n^b] &= \sum_c f^{abc} T_{m+n}^c, \quad [T_m^a, \theta_n^b] = \sum_c f^{abc} \theta_{m+n}^c, \\ [\theta_m^a, \theta_n^b] &= \delta^{a,b} \varepsilon_{m+n}. \end{aligned} \quad (5)$$

Here  $\partial 1 = \partial T_m^a = 0$ ,  $\partial \theta_m^a = 1$  for all  $a, m$ .

We shall verify (2) for elements of algebra (5) with definite gradation.

Let now

1)  $x, y, z \in M_0$ . The foregoing implies that (2) is satisfied.

2) Two elements belong to  $M_0$  and the third elements belongs to  $M_1$ .

2a)  $x \in M_0$ . The fact that  $M_0$  is a Malcev algebra implies (2) for this case also.

2b)  $x \in M_1$ . It is sufficient to consider the case  $x = \sum_{a,m} x(a,m) \theta_m^a$ ,  $y = T_n^b$ ,  $z = T_p^c$ .

We have

$$\begin{aligned} [J(x, y, z), x] &= ([x, [y, z]] + [y, [z, x]] + [z, [x, y]]), x \\ &= \sum_{a, a', d, m, m'} (f^{bcd} f^{ada'} + f^{cad} f^{bda'} + f^{abd} f^{cda'}) x(a, m) x(a', m') \varepsilon_{m+m'+n+p}. \\ J(x, y, [x, z]) &= -[x, [y, [x, z]]] + [y, [[x, z], x]] + [[x, z], [x, y]] \\ &= \sum_{a, a', d, m, m'} (-f^{acd} f^{bda'} + f^{acd} f^{a'bd}) x(a, m) x(a', m') \varepsilon_{m+m'+n+p}. \end{aligned}$$

It is clear that

$$\sum_{a, a', d, m, m'} f^{bcd} f^{ada'} x(a, m) x(a', m') \varepsilon_{m+m'+n+p} = 0$$

and

$$[J(x, y, z), x] = J(x, y, [x, z]).$$

3) Two elements belong to  $M_1$  and the third element belongs to  $M_0$ .

3a)  $x \in M_0$ ,  $y, z \in M_1$ . It is sufficient to consider  $x = \sum_{a,m} x(a,m) T_m^a$ ,  $y = \theta_n^b$ ,  $z = \theta_p^c$ . Then

$$\begin{aligned} [J(x, y, z), x] &= ([x, [y, z]] + [y, [z, x]] - [z, [x, y]]), x = 0 \\ J(x, y, [x, z]) &= [y, [[x, z], x]] - [[x, z], [x, y]] \\ &= \sum_{a, a', d, m, m'} (f^{acd} f^{da'b} - f^{acd} f^{a'bd}) x(a, m) x(a', m') \varepsilon_{m+m'+n+p} = 0. \end{aligned}$$

i.e. (2) is fulfilled.

3b)  $y \in M_0$ ,  $x, z \in M_1$ . It is sufficient to consider  $x = \sum_{a,m} \theta_m^a$ ,  $y = T_n^b$ ,  $z = \theta_p^c$ . We have

$$J(x, y, z) = -[x, [y, z]] + [z, [x, y]] = \sum_{a,m} (-f^{bca} + f^{abc})x(a, m)\varepsilon_{m+n+p} = 0$$

and  $J(x, y, [x, z]) = 0$ , because  $[x, z] = \text{const}$ .

3c)  $z \in M_0$ ,  $x, y \in M_1$ . It is sufficient to consider  $x = \sum_{a,m} x(a, m)\theta_m^a$ ,  $y = \theta_n^b$ ,  $z = T_p^c$ . Analogously to the previous case

$$J(x, y, z) = [x, [y, z]] - [y, [z, x]] = \sum_{a,m} (f^{bca} - f^{cab})x(a, m)\varepsilon_{m+n+p} = 0$$

and  $J(x, y, [x, z]) = 0$ , so (2) is fulfilled.

4) Three elements belong to  $M_1$ . In this case the superbracket of any two elements belongs to  $\mathbb{R}$  and so  $J(x, y, z) = 0$ ,  $J(x, y, [x, z]) = 0$ , and again (2) is fulfilled.

Thus, (4) and (5) are super-Malcev algebras and the algebra with commutation relations (5) is a central extension of the algebra (4).

Let us consider now general central extensions (the central extensions of Kac-Moody and super-Kac-Moody algebras, cf., for instance, in [1]).

Commutation relations for central extensions of Kac-Moody-Malcev algebra (3) have form

$$[T_m^a, T_n^b] = \sum_c f^{abc} T_{m+n}^c + d_{m,n}^{a,b} k, \quad [T_m^a, k] = 0. \quad (6)$$

It is easy to see that a central extension (6) is nontrivial in the class of binary Lie algebras. In the class of Malcev algebras the central extensions of Kac-Moody-Malcev algebra, associated with simple non-Lie Malcev algebra, are trivial, that is,  $d_{m,n}^{a,b} = 0$ . Indeed, as in [2], we obtain  $d_{m,n}^{a,b} = \delta^{a,b} m \delta_{m,-n} \text{const}$ . Then putting in (7) in [2]  $l = 1$ ,  $m = 0$ ,  $n = 0$ ,  $p = -1$ ,  $a = 1$ ,  $b = 2$ ,  $c = 4$ ,  $d = 7$ , we obtain  $\text{const} = 0$ .

Let us consider now central extensions  $\tilde{M}$  of super-Kac-Moody-Malcev algebras  $M$  given by commutation relations (4). We suppose that  $M_i \subset \tilde{M}_i$ ,  $i = 0, 1$ . Commutation relations in the case of central extensions of super-Kac-Moody-Malcev algebras have the form

$$\begin{aligned} [T_m^a, T_n^b] &= \sum_c f^{abc} T_{m+n}^c + d_{m,n}^{a,b} k, \quad [T_m^a, \theta_n^b] = \sum_c f^{abc} \theta_{m+n}^c + \Delta_{m,n}^{a,b} \kappa, \\ [\theta_m^a, \theta_n^b] &= \varepsilon_{m,n}^{a,b} \delta \\ [T_m^a, k] &= [\theta_m^a, k] = [\kappa, k] = [\delta, k] = 0, \\ [T_m^a, \kappa] &= [\theta_m^a, \kappa] = [\delta, \kappa] = [T_m^a, \delta] = [\theta_m^a, \delta] = 0. \end{aligned} \quad (7)$$

The central extensions in the class of binary super-Lie algebras are given by any  $d_{m,n}^{a,b}$ ,  $\Delta_{m,n}^{a,b}$ ,  $\varepsilon_{m,n}^{a,b}$ . (Here  $d_{m,n}^{a,b}$ ,  $\Delta_{m,n}^{a,b}$  ( $\varepsilon_{m,n}^{a,b}$ ) must be antisymmetric (symmetric)).

The central extensions in the class of super-Malcev algebras, associated with simple non-Lie Malcev algebra, have the form (in the case of one Grassmann generator)  $d_{m,n}^{a,b} = \Delta_{m,n}^{a,b} = 0$ ,  $\varepsilon_{m,n}^{a,b} = \delta^{a,b} \varepsilon_{m+n}$ .

To prove this assertion we remark that (1) and (7) imply that  $\partial k = \partial \delta = 0, \partial \kappa = 1$  (and then, if, for instance,  $[T_m^a, k] = 0$  then  $[k, T_m^a] = 0$  also and so on). Further we consider elements with zero gradation. The same consideration as for (6) gives  $d_{m,n}^{a,b} = 0$ . Further let  $x = \sum x(a, m) T_m^a, y = \sum y(b, n) T_n^b, z = \sum z(c, p) \theta_p^c$ . Then coordinates of vectors  $x, y, z$  generate the identities, which coincide with identities (2) for Malcev algebra with commutation relation (10), where instead of  $d_{m,n}^{a,b}$  stand  $\Delta_{m,n}^{a,b}$ . But in this case analogously to the previous one  $\Delta_{m,n}^{a,b} = 0$ .

Now, we take  $x = T_m^a, y = \theta_n^b, z = \theta_p^c$ . Then we have  $[J(x, y, z), x] = 0$  due to the fact that  $J(x, y, z) = \text{const } \delta$ , and

$$J(x, y, [x, z]) = \sum_{d,e} (f^{acd} f^{dae} \varepsilon_{n,2m+p}^{b,e} - f^{acd} f^{ade} \varepsilon_{m+p,m+n}^{d,e}) \delta.$$

So it must be

$$\sum_{d,e} (f^{acd} f^{dae} \varepsilon_{n,2m+p}^{b,e} - f^{acd} f^{ade} \varepsilon_{m+p,m+n}^{d,e}) = 0 \quad (8)$$

Let  $\alpha, \beta, \gamma$  be such that  $f^{\alpha\beta\gamma} \neq 0$ . Take  $a = \alpha, b = \alpha, c = \beta$ . Then (8) and properties of structure constants give that  $\varepsilon_{n,2m+p}^{\alpha,\beta} = 0$ . This fact implies that all  $\varepsilon_{m,n}^{a,b} = 0$  for  $a \neq b$ .

Let now  $a = \alpha, b = \beta, c = \beta$  in (8). Then we obtain  $\varepsilon_{n,2m+p}^{\beta,\beta} = \varepsilon_{m+p,m+n}^{\gamma,\gamma}$ . Now we put  $p = -m$  and obtain  $\varepsilon_{n,m}^{\beta,\beta} = \varepsilon_{0,m+n}^{\gamma,\gamma}$ . Taking into account that  $\varepsilon_{m,n}^{a,b} = \varepsilon_{n,m}^{a,b}$  the foregoing gives that  $\varepsilon_{m,n}^{a,b} = \delta^{a,b} \varepsilon_{m+n}$  for all  $1 \leq a, b \leq \dim g = 7$  and  $m, n \in \mathbb{Z}$ .

Thus, the general central extensions of algebra (4) is given by commutation relations (7) with  $d_{m,n}^{a,b} = \Delta_{m,n}^{a,b} = 0$  and  $\varepsilon_{m,n}^{a,b} = \delta^{a,b} \varepsilon_{m+n}$ .

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## The local behaviour of supersymmetric quantum mechanics potentials

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**abstract.** - In supersymmetric quantum mechanics, the differential equations corresponding to exactly solvable potentials may be treated by algebraic methods. By use of a system of geodesic polar coordinates on a Riemannian manifold, and the subsequent transformation of the Laplace Beltrami eigenvalue problem to a Schrödinger equation with a potential in two ways, it is demonstrated that the local behaviour of the exactly solvable potentials corresponds to isotropic harmonic oscillator and Pöschl-Teller potential problems

Symmetry considerations have proven to be very effective in solving differential equations of mathematical physics. To these methods count the separation of variables, the similarity solutions, the dynamical symmetry, etc. In dynamical symmetry the differential equations are related to invariant differential operators of some Lie algebra. In this manner the calculation of physically relevant quantities such as energies, masses, transition probabilities, or scattering matrices can be achieved by algebraic manipulation<sup>1</sup>. The range of potentials that can be treated this way corresponds to the factorizable potentials<sup>2</sup> or the exactly solvable potentials, as considered in supersymmetric quantum mechanics<sup>3</sup>. This powerful technique has however the following disadvantage: in physics the system and not the symmetry is generally specified, and there is no general theory to determine which symmetry, if any, is of relevance. We provide here a result which is intended to remove this difficulty regarding the application of the algebraic technique to systems whose description can be reduced to the study of an ordinary differential equation. Specifically, we demonstrate that those physical systems which can be treated by algebraic techniques behave locally like an isotropic oscillator or a Pöschl-Teller potential problem.

In this paper the right choice of coordinates is relevant. We work with the generalization of the polar coordinates: the geodesic polar coordinates. To define them, let  $M$  be a Riemannian manifold and  $x$  be some point of  $M$ . Then exists a neighbourhood  $N$  of  $x$  in  $M$  on which every point can be parametrized by the length  $r$  of the shortest geodesic reaching the point, and by the direction of that geodesic, i.e. by the coordinates of the intersection between the tangent to the geodesic and the unit sphere on the tangential space  $M_x$ . The mapping  $x \mapsto (r, \theta_1, \dots, \theta_n)$  is called a system of geodesic polar coordinates.

For  $M$  of dimension two, the metric takes in these coordinates the following form,

$$ds^2 = dr^2 + h^2(r, \theta) d\theta^2. \quad (1)$$

For small values of  $r$  we have  $h(r, \theta) = r - \frac{1}{6}\kappa r^3 + R(r, \theta)$ , where  $\lim_{r \rightarrow 0} (R(r, \theta)/r^3) = 0$ , and  $\kappa$  is the Gaussian curvature of  $M$  at the point  $x$ . We are interested in the local behaviour of the systems, hence we restrict our study to scaling functions depending on  $r$  only. The Laplace Beltrami operator  $\Delta_M$  takes then the form

$$\Delta_M = \frac{1}{h(r)} \frac{\partial}{\partial r} h(r) \frac{\partial}{\partial r} + \frac{1}{h^2(r)} \frac{\partial}{\partial \theta^2}.$$

From this equation it follows that the eigenfunctions  $F_{\lambda, m}(r, \theta)$  of  $\Delta_M$  are given by

$$\Delta_M F_{\lambda, m}(r, \theta) = \lambda F_{\lambda, m}(r, \theta),$$

where  $F_{\lambda, m}(r, \theta) = \psi_\lambda^m(r) e^{im\theta}$ , with  $\psi_\lambda^m(r)$  obeying the following differential equation:

$$\left[ \frac{1}{h(r)} \frac{\partial}{\partial r} h(r) \frac{\partial}{\partial r} - \frac{m^2}{h^2(r)} \right] \psi_\lambda^m(r) = \lambda \psi_\lambda^m(r). \quad (2)$$

Equation (2) is of the Sturm-Liouville type and can be transformed into the form of a Schrödinger equation with a potential in two ways:

**Case (a):** After multiplying equation (2) by  $h^2(r)$  and using the new variable  $\xi = \int \frac{1}{h(r)} dr$  we obtain the equation

$$\left[ -\frac{d^2}{d\xi^2} + \lambda h^2(\xi) \right] \Phi_m^\lambda(\xi) = -m^2 \Phi_m^\lambda(\xi), \quad (3)$$

which corresponds to a one-dimensional Schrödinger equation in the variable  $\xi$  with eigenvalue  $-m^2$  and potential function  $\lambda h^2(\xi)$ .

**Case (b):** The normalization condition corresponding to the metric  $h(r)$ ,  $\int \Phi_\lambda^m(r)^2 h(r) dr = 1$ , suggests the introduction of the function  $\Psi_\lambda^m(r)$ , defined by

$\Psi_\lambda^m(r) = \Phi_\lambda^m(r) h^{1/2}(r)$ . Then equation (2) yields

$$\left[ -\frac{d^2}{dr^2} + \frac{h''(r)}{2h(r)} - \left( \frac{h'(r)}{2h(r)} \right)^2 + \frac{m^2}{h^2(r)} \right] \Psi_\lambda^m(r) = -\lambda \Psi_\lambda^m(r), \quad (4)$$

which corresponds to a one-dimensional Schrödinger equation in the variable  $r$  with eigenvalue  $-\lambda$  and a potential function depending on  $h(r)$ ,  $h'(r)$  and  $h''(r)$ .

After this more general treatment, in order to apply the powerful tools of group theory, we take for the manifold  $M$  a globally symmetric space  $X = G/K$  of dimension two: i.e. such that  $G$  is a connected semisimple Lie group and  $K$  is a maximal subgroup. Then, it is well known that the Casimir of the group can be related to a Schrödinger equation with some potential connected with the curvature of the space. Essentially two applications can be distinguished: that of a potential group<sup>4</sup> and that of a symmetry group<sup>5</sup>. In the first case the representation of the group determines the energy and the representation of the subgroup determines the potential strength. In the second case the roles are interchanged. Transformation (a) relates the group  $G$  with the symmetry group of a physical system. Transformation (b) with the potential group. These two

transformations therefore cover both applications. The following table presents some examples.

| Examples                                                                    |                                                                                                                          |
|-----------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------|
| *** SO(3)/SO(2) ***                                                         |                                                                                                                          |
| $h(r) = \sin(r), \quad 0 < r < \pi, \quad -\infty < \xi < \infty$           |                                                                                                                          |
| Trans. (a)                                                                  | $\left[ -\frac{d^2}{d\xi^2} + \frac{\lambda}{\cosh^2(\xi)} \right] \Phi_m^\lambda(\xi) = -m^2 \Phi_m^\lambda(\xi)$       |
| Trans. (b)                                                                  | $\left[ -\frac{d^2}{dr^2} + \frac{m^2 - 1/4}{\sin^2(r)} \right] \Psi_\lambda^m(r) = (-\lambda + 1/4) \Psi_\lambda^m(r)$  |
| *** SO(2,1)/SO(2) ***                                                       |                                                                                                                          |
| $h(r) = \sinh(r), \quad -\infty < r < \infty, \quad -\infty < \xi < \infty$ |                                                                                                                          |
| Trans. (a)                                                                  | $\left[ -\frac{d^2}{d\xi^2} + \frac{\lambda}{\sinh^2(\xi)} \right] \Phi_m^\lambda(\xi) = -m^2 \Phi_m^\lambda(\xi)$       |
| Trans. (b)                                                                  | $\left[ -\frac{d^2}{dr^2} + \frac{m^2 - 1/4}{\sinh^2(r)} \right] \Psi_\lambda^m(r) = -(\lambda + 1/4) \Psi_\lambda^m(r)$ |
| *** E(2)/SO(2) ***                                                          |                                                                                                                          |
| $h(r) = r, \quad 0 < r < \infty, \quad -\infty < \xi < \infty$              |                                                                                                                          |
| Trans. (a)                                                                  | $\left[ -\frac{d^2}{d\xi^2} + \lambda e^{2\xi} \right] \Phi_m^\lambda(\xi) = -m^2 \Phi_m^\lambda(\xi)$                   |
| Trans. (b)                                                                  | $\left[ \frac{d^2}{dr^2} + \frac{m^2 - 1/4}{r^2} \right] \Psi_\lambda^m(r) = -\lambda \Psi_\lambda^m(r)$                 |

Table

Systems possessing dynamical symmetry. Transformation (a) describes a symmetry group problem. Transformation (b) describes a potential group problem.

As the table shows, there is a variety of systems which can be studied by algebraic methods. However, in spite of this great range of systems, the local behaviour of all of them is characterized by only two potentials: the harmonic oscillator and the Pöschl-Teller potential. To prove this result, consider for sufficiently small values of  $r$  the scaling function  $h$  of any manifold of dimension two. This function has according to equation (1) the form

$$h(r) = r - \frac{1}{6} \kappa r^3.$$

**Case (a) :** Using the above defined scaling function, transformation (a) gives

$$\left[ -\frac{d^2}{d\xi^2} + \frac{\lambda e^{2\xi}/4}{\cosh^2(\xi - \mu)} + m^2 \right] \Phi_m^\lambda(\xi) = 0, \quad (5)$$

with  $(\kappa/6) = e^{-2\mu}$ . By making the replacements  $\xi - \mu \rightarrow \zeta$  and  $(\lambda/4)e^{2\mu} \rightarrow -\gamma(\gamma + 1)$ , this equation takes the form of the Schrödinger equation corresponding to a Pöschl-Teller potential, namely

$$\left[ -\frac{d^2}{d\zeta^2} - \frac{\gamma(\gamma + 1)}{\cosh^2(\zeta)} - E_n^{(\gamma)} \right] \Phi_n^{(\gamma)}(\zeta) = 0. \quad (6)$$

**Case (b) :** From equation (4), with  $h(r) = r - (\kappa/6)r^3$ , we have

$$\left[ \frac{d^2}{dr^2} + \frac{(m^2 - 1/4)}{r^2} + \frac{(m^2 - 1)\kappa^2}{12} r^2 - \frac{\kappa}{3}(m^2 - 1) + \lambda \right] \Psi_\lambda^m(r) = 0. \quad (7)$$

When  $m = 0$ , the coefficient of  $r^2$  is negative, and there are no bound states (cf. the Pöschl-Teller potential, where  $E_0 = 0$  if  $m = 0$ ). When  $m = \pm 1$ , the equation becomes

$$\left[ -\frac{d^2}{dr^2} + \frac{3/4}{r^2} \right] \Psi_{\lambda}^{(\pm 1)}(r) = 0.$$

We note that this equation also arises for general  $m$  in the special case  $\kappa = 0$  i.e. when dealing with flat spaces. When  $|m| = 1$ , the equation reduces to the Schrödinger equation for the isotropic oscillator :

$$\left[ -\frac{d^2}{dr^2} - \frac{m^2 - 1/4}{r^2} + \mathbf{F}(m)r^2 - E_n^m \right] \Psi_n^m(r) = 0.$$

where  $\mathbf{F}(m) \equiv [m^2 - 1]\kappa/12$  is the force constant and the term  $m^2 - 1/4$  corresponds to  $l(l+1)$  such that  $l = m - 1/2$ , where  $m$  can be positive or negative.

We conclude by noting that the Pöschl-Teller and the isotropic harmonic oscillator potentials appear to occupy a central position within the set of exactly solvable potentials. They have not only been exhaustively treated, but they also describe the local behaviour of all such potentials in so far as the corresponding physical system possesses a symmetry group or a potential group. Thus, this result can be used to determine the existence of a dynamical symmetry for a particular physical system, through a study of the local behaviour of the relevant potentials. They also provide a natural starting point for any perturbational expansion, if appropriate.

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## MOUFANG LOOPS AND CONSERVATION LAWS

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It is well known how the Noether theorem binds invariance (symmetry) principles and conservation laws in Lagrangian formalism. For a point  $g$  from a manifold  $G$ , consider a pair  $(s,t)$  of the field transformations  $u \rightarrow s_g u$ ,  $u \rightarrow t_g u$ . The (prolonged) action  $(S,T)$  of  $G$  on the Lagrangian  $L(u)$  is defined by

$$(S_g L)(u) := L(s_g^{-1} u) \quad , \quad (T_g L)(u) := L(t_g^{-1} u). \quad (1)$$

Lagrangian  $L(u)$  is said to be  $G$ -invariant if

$$L(s_g u) = L(t_g u) = L(u) \quad \text{for all } g \text{ of } G. \quad (2)$$

Let  $e$  be a fixed point in  $G$ , and let  $x$  denote a vector from the tangent space  $T_e(G)$  of  $G$  at  $e$ . Denote infinitesimal operators of  $(S,T)$  at  $e$  as  $S_x, T_x$  ( $x \in T_e(G)$ ). Infinitesimally,  $G$ -invariance conditions (2) read

$$S_x L(u) = T_x L(u) = 0 \quad \text{for all } x \text{ of } T_e(G). \quad (3)$$

Rearranging the terms according to the canonical prescription, these conditions can be rewritten as Noether's identities, where the Euler-Lagrange expressions and divergence terms for the Noether currents are explicitly separated. Conservation laws for the Noether charges follow from the Noether identities under the assumptions that field equations hold and  $G$ -currents vanish on the spatial integration boundary via the Gauss theorem.

Infinitesimal  $G$ -invariance conditions (3) can be considered as a constraining set of partial differential equations for  $G$ -invariant Lagrangian  $L(u)$ . To close (3), we must first extend it with the commutator equations

$$[S_x, S_y]L(u) = [T_x, T_y]L(u) = [S_x, T_y]L(u) = 0 \quad \text{for all } x, y \text{ of } T_e(G), \quad (4)$$

and then proceed with extension. In this way, we may obtain additional (new) conservation laws from the commutator Noether identities. Certainly, one will have much trouble to find the latter if no algebraic assumptions are not assumed for  $G$  and  $(s, t)$ , to say nothing of the topological complications. To clarify the algebraic aspect, suppose for a moment that  $g$  takes its values from a Lie group  $G$  with the identity element  $e$ , and  $(s, t)$  obeys the common group geocentric identities

$$s_g s_h = s_{gh}, \quad t_g t_h = t_{hg}, \quad s_g t_h = t_h s_g \quad \text{for all } g, h \text{ of } G. \quad (5)$$

We call such  $(s, t)$  associative. It is well known that infinitesimal associative  $G$ -transformations obey the following Lie algebra commutation relations (CR):

$$[S_x, S_y] - S_{[x, y]} = [T_x, T_y] + T_{[x, y]} = [S_x, T_y] = 0 \quad \text{for all } x, y \text{ of } T_e(G), \quad (6)$$

where  $[x, y] \in T_e(G)$  is the product of  $x$  and  $y$  in the tangent Lie algebra of  $G$ . As a result, we can say that at least for associative  $(s, t)$  the additional conservation laws never appear from the commutator Noether identities (4).

Now consider a more general case of the Moufang symmetry.

Suppose that  $g$  takes its values from the analytic Moufang loop  $G$  [1-3]. The latter is an analytic quasigroup with the identity element  $e \in G$  in which the Moufang identity

$$(ag)(ha) = a(gh)a \quad (7)$$

holds. The (anti-commutative) tangent algebra of  $G$  can be defined similarly [3,4] to the tangent (Lie) algebra of the Lie group. Geometrically, the tangent algebra of  $G$  coincides with the tangent space  $T_e(G)$  of  $G$  at  $e$ , but it need not be no more a Lie algebra, i.e. there may be a triple  $x, y, z$  in  $T_e(G)$  that the Jacobi identity fails:

$$J(x, y, z) := [x, [y, z]] + [y, [z, x]] + [z, [x, y]] \neq 0.$$

Instead, for all  $x, y, z$  of  $T_e(G)$ , we have a more general Mal'tsev identity [3]

$$[J(x, y, z), x] = J(x, y, [x, z]), \quad (8)$$

and anti-commutative algebras with this identity are called the Mal'tsev algebras [5].

Once more discarding the group geocentric prejudices, suppose that the following identities hold for all  $g, h$  of  $G$ :

$$s_g t_g s_h = s_{gh} t_g, \quad s_g t_g t_h = t_{hg} s_g. \quad (9)$$

Nevertheless,  $s_e = t_e = E$  (identity transformation) are assumed to survive. The pair  $(s, t)$  with such properties is called [6-9] the birepresentation of  $G$ .

It can be easily verified that  $(S, T)$  turns out to be a birepresentation of  $G$  as well. Infinitesimally, the non-associativity of  $(S, T)$  reveals itself as minimal but

an open extension (generalization) of the Lie algebra (6):

$$[S_x, S_y] - S_{[x,y]} = [T_x, T_y] + T_{[x,y]} = -2[S_x, T_y] \quad \text{for all } x, y \text{ of } T_e(G). \quad (10)$$

We call this algebra the Moufang-Mal'tsev algebra and outline a way of closing of the latter, which in fact means its embedding into a Lie algebra [10-13].

Start by rewriting the Moufang-Mal'tsev algebra as follows:

$$[S_x, S_y] = 2 F(x;y) + 1/3 S_{[x,y]} + 2/3 T_{[x,y]}, \quad (11)$$

$$[S_x, T_y] = -F(x;y) + 1/3 S_{[x,y]} - 1/3 T_{[x,y]}, \quad (12)$$

$$[T_x, T_y] = 2 F(x;y) - 2/3 S_{[x,y]} - 1/3 T_{[x,y]}. \quad (13)$$

Here, (11) or (12) or (13) can be assumed as the defining expression for the Yamaguti operator  $F(x;y)$ . These operators are not linearly independent, since

$$F(x;y) + F(y;x) = 0, \quad (14)$$

$$F([x,y];z) + F([y,z];x) + F([z,x];y) = 0 \quad (15)$$

for all  $x, y, z$  of  $T_e(G)$ . Constraints (14) trivially descend from the anti-symmetry of the commutator bracketing, but the proof of (15) is not so trivial. It turns out that

$$6[F(x;y), S_z] = S_{[x,y,z]}, \quad 6[F(x;y), T_z] = T_{[x,y,z]} \quad (16)$$

for all  $x, y, z$  of  $T_e(G)$ , where the Yamaguti triple product  $[x, y, z]$  in  $T_e(G)$  is defined as

$$[x, y, z] := [x, [y, z]] - [y, [x, z]] + [[x, y], z]. \quad (17)$$

CR (16) are natural to call the reductivity conditions. The Yamaguti operators obey the Lie algebra

$$6[F(x; \cdot), F(z; w)] = F([x, y, z]; w) + F(z; [x, y, w]). \quad (18)$$

Computations which in fact prove CR (10-18) were carried out in [10, 11]. Denoting  $r := \dim(G)$ , the dimension of the Lie algebra (11-18) does not exceed  $2r + r(r-1)/2$ , meanwhile the dimension of its subalgebra (18) does not exceed  $r(r-1)/2$ . Jacobi identities for the Lie algebra (11-18) are guaranteed [10] by the defining identities of the Lie [14] and general Lie [15, 16] triple systems associated with the tangent Mal'tsev algebra  $T_e(G)$  of  $G$ .

We can now summarize our discussion of the Moufang symmetry with the observation that equations (3) can be closed by

$$F(x;y) L(u) = 0, \quad x, y \text{ are in } T_e(G). \quad (19)$$

By rewriting the latter as Noether's identities, we may obtain desired additional conservation laws generated by the Moufang symmetries. Our method uses up the fact that infinitesimal Moufang transformations generate the Lie algebra (11-18). In this sense,

we can say that the collection of conservation laws obtainable from (3) and (19) is closed (complete) as well. Non-associativity hides itself in (19). Remind that one needs a closed (complete) collection of conserved charges to construct a basis in the linear space of physical states after the quantization. What concerns the topological aspect, then it must be stressed that every finite-dimensional real Mal'tsev algebra is proved [17-19] to be the tangent algebra of some analytic Moufang loop.

Finally, remark that the Lie algebra (11-18) is well acceptable from the point of view of alternative algebras and octonions [20-23]. Also, it is quite trivial to foresee the Noether charge density algebra generated by Moufang transformations.

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## PATH INTEGRAL REALIZATION OF DYNAMICAL SYMMETRIES

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**Abstract.** The path integral for certain systems is shown to be soluble if their dynamical group structure is known. In particular we consider the radial path integral for a harmonic oscillator having  $SU(1,1)$  as dynamical group.

### 1 Introduction

Sixty years ago in his book [1] Eugen Wigner demonstrated the power of group theoretical methods in quantum mechanics. Without solving the Schrödinger equation many important quantum mechanical results can be obtained purely from symmetry considerations. Even group theory may provide exact solutions of the Schrödinger equation for certain systems. Most of the exactly soluble problems have been classified by the factorization method of Schrödinger, Infeld and Hull [2]. The factorization method is indeed related with Lie theory [3]. There are only two elementary systems to which all others can be reduced by changing variables and transforming the wavefunction in the Schrödinger equation. The two systems are the radial harmonic oscillator having solutions of confluent hypergeometric type and the Pöschl-Teller oscillator possessing solutions of hypergeometric type. The underlying group structures of the exactly soluble problems are associated with symmetries of dynamical origin rather than geometrical ones.

The aim of the present report is to demonstrate that group theory is also very useful in the path integral approach to quantum mechanics [4]. The application of group theory to path integrals with geometrical symmetries has already been reported in the last Wigner Symposium [5]. Here we wish to discuss the path integral realization of dynamical symmetries. As an example we shall consider the radial path integral of a harmonic oscillator in  $\mathbb{R}^d$ . First, we shall explicitly demonstrate that this system has the  $SU(1,1)$  dynamical group. Then we shall present its path integral representation. For the realization of dynamical symmetries  $SU(2)$  and  $SU(1,1)$  of the Pöschl-Teller systems we refer to refs. [6-8]. In particular, for the local time rescaling technique, which is equivalent to the change of variable and transformation of wavefunction in the Schrödinger approach, see refs. [8,9].

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## 2 The dynamical group $SU(1, 1)$ of the harmonic oscillator in $\mathbb{R}^d$

The Lie algebra of  $SU(1, 1)$  may explicitly be given by the commutators

$$[J_1, J_2] = -iJ_3, \quad [J_2, J_3] = iJ_1, \quad [J_3, J_1] = iJ_2. \quad (1)$$

The quadratic Casimir operator is  $\mathbf{J}^2 = -J_1^2 - J_2^2 + J_3^2$ . In a given unitary irreducible representation (UIR) it is proportional to the unit operator, i.e.  $\mathbf{J}^2 = J(J+1)\mathbf{1}$ , where the "angular momentum" quantum number  $J$  may be used for labelling all UIR's. There are two continuous and two discrete series of UIR [10]. Spectra of the compact operator  $J_3$  for the continuous series are unbounded. Whereas, the spectra of  $J_3$  for the discrete series denoted by  $D_{\bar{J}}$  and  $D_{\bar{J}}^{\dagger}$  are bounded from above and below, respectively. It is the series  $D_{\bar{J}}^{\dagger}$  which is realized by the harmonic oscillator. On the standard discrete basis we have for  $D_{\bar{J}}^{\dagger}$  [10]:

$$\mathbf{J}^2|J, m\rangle = J(J+1)|J, m\rangle \quad \text{with} \quad -1 < J, \quad (2)$$

$$J_3|J, m\rangle = m|J, m\rangle \quad \text{with} \quad m = J+1, J+2, \dots \quad (3)$$

It is also possible to choose a continuous basis where a noncompact operator is diagonalized [11]. The one we are interested in is  $K := J_1 + J_3$  and has the positive real line as the spectrum for  $D_{\bar{J}}^{\dagger}$ :

$$K|J, \eta\rangle = \eta|J, \eta\rangle \quad \text{with} \quad \eta \in \mathbb{R}^+. \quad (4)$$

For the matrix element for a finite transformation in the continuous basis we find [11] in analogy to Wigner's  $d$ -function [1]

$$\langle J, \eta | e^{-2i\varphi J_3} | J, \eta' \rangle = \frac{1}{i \sin \varphi} \exp\{i(\eta + \eta') \cot \varphi\} I_{2J+1} \left( \frac{2\sqrt{\eta\eta'}}{i \sin \varphi} \right) \quad (5)$$

where  $I_\nu(z)$  is the modified Bessel function,  $0 < \varphi < \pi$  and  $\eta, \eta' \in \mathbb{R}^+$ .

In the following we consider a creation and an annihilation operator for each degree of freedom in  $\mathbb{R}^d$ , that is  $[a_i, a_i^\dagger] = \delta_{ij}$  ( $i, j = 1, \dots, d$ ). A realization of the algebra of  $SU(1, 1)$  is given by

$$J_1 := \frac{1}{4} \sum_{i=1}^d (a_i^\dagger{}^2 + a_i^2), \quad J_2 := -\frac{i}{4} \sum_{i=1}^d (a_i^\dagger{}^2 - a_i^2), \quad J_3 := \frac{1}{2} \sum_{i=1}^d \left( a_i^\dagger a_i + \frac{1}{2} \right). \quad (6)$$

The Hamiltonian for the harmonic oscillator in  $\mathbb{R}^d$  is  $H = 2\hbar\omega J_3$  which is bounded below (i.e.  $D_{\bar{J}}^{\dagger}$ ). We observe that the total angular momentum associated with  $SO(d)$  in  $\mathbb{R}^d$  is related to the Casimir operator (2) by  $\mathbf{L}^2 = 4\mathbf{J}^2 - d(d-4)/4$  and has eigenvalues  $l(l+d-2)$ ,  $l \in \mathbb{N}_0$ . Therefore, a fixed angular momentum subspace corresponds to the representation  $D_{\bar{J}}^{\dagger}$  with  $J := l/2 + d/4 - 1$ . The spectrum (3) of  $J_3$  leads to the energy eigenvalues  $E_n := \hbar\omega(2n + l + d/2)$ ,  $n \in \mathbb{N}_0$ . Finally, we note that the time evolution operator  $\exp\{-(i/\hbar)H\tau\} = \exp\{-2i\omega\tau J_3\}$  is a group element and that  $SU(1, 1)$  is indeed the dynamical group of the system.

### 3 Path integral realization of the dynamical group $SU(1, 1)$

According to Feynman [4] the propagator, i.e. the matrix element of the time evolution operator, may be given by a path integral,

$$\langle \mathbf{x}'' | e^{-(i/\hbar)H\tau} | \mathbf{x}' \rangle = \int_{\mathbf{x}'=\mathbf{x}(0)}^{\mathbf{x}''=\mathbf{x}(\tau)} \mathcal{D}[x(t)] \exp \left\{ \frac{i}{\hbar} \int_0^\tau \left( \frac{M}{2} \dot{\mathbf{x}}^2 - \frac{M}{2} \omega^2 \mathbf{x}^2 \right) dt \right\},$$

which reads on the sliced time basis  $[\varepsilon := \tau/N, \mathbf{x}_j := \mathbf{x}(j\varepsilon)]$

$$\langle \mathbf{x}'' | e^{-(i/\hbar)H\tau} | \mathbf{x}' \rangle = \lim_{N \rightarrow \infty} \left( \frac{M}{2\pi i \hbar \varepsilon} \right)^{Nd/2} \int \prod_{j=1}^N \exp \left\{ \frac{i}{\hbar} \left[ \frac{M}{2\varepsilon} (\mathbf{x}_j - \mathbf{x}_{j-1})^2 - \frac{M}{4} \omega^2 \varepsilon (\mathbf{x}_j^2 + \mathbf{x}_{j-1}^2) \right] \right\} \prod_{j=1}^{N-1} d\mathbf{x}_j. \quad (7)$$

Due to its spherical symmetry we immediately can perform the angular path integration [7]:

$$\langle \mathbf{x}'' | e^{-(i/\hbar)H\tau} | \mathbf{x}' \rangle = \sum_{l=0}^{\infty} K_l(r'', r'; \tau) \sum_{\mathcal{M}} \frac{\Gamma(d/2)}{2\pi^{d/2}} Y_{l\mathcal{M}}(\mathbf{x}''/r'') Y_{l\mathcal{M}}^*(\mathbf{x}'/r'). \quad (8)$$

where  $Y_{l\mathcal{M}}(\mathbf{e})$  are the hyperspherical harmonics in  $\mathbb{R}^d$  and  $\mathcal{M}$  stands for a  $(d-2)$ -tuple counting the degeneracy of the angular momentum  $l$ . The radial propagator is given by the remaining path integral  $[r_j := |\mathbf{x}_j|]$

$$K_l(r'', r'; \tau) = \left( \frac{1}{r'' r'} \right)^{(d-2)/2} \lim_{N \rightarrow \infty} \int \prod_{j=1}^N R(r_j, r_{j-1}) \prod_{j=1}^{N-1} r_j dr_j \quad (9)$$

where

$$R(r_j, r_{j-1}) := \frac{M}{i\hbar\varepsilon} \exp \left\{ \frac{i}{\hbar} (r_j^2 + r_{j-1}^2) \left( \frac{M}{2\varepsilon} - \frac{M}{4} \omega^2 \varepsilon \right) \right\} I_{l+(d-2)/2} \left( \frac{M r_j r_{j-1}}{i\hbar\varepsilon} \right). \quad (10)$$

It is the dynamical group  $SU(1, 1)$  which now enables us to complete the path integration. First, we set  $\eta_j := M\omega r_j^2/2\hbar$  which gives for the exponential in (10)

$$\exp\{\dots\} = \exp \{ i(\eta_j + \eta_{j-1})(1/\omega\varepsilon - \omega\varepsilon/2) \}. \quad (11)$$

Note also that  $r_j dr_j = (\hbar/M\omega) d\eta_j$ . Secondly, we define  $\sin \varphi := \omega\varepsilon$  which leads to

$$1/\omega\varepsilon - \omega\varepsilon/2 = \cot \varphi + \mathcal{O}(\varepsilon^3). \quad (12)$$

Inserting this in (11) [note that terms of  $\mathcal{O}(\varepsilon^3)$  may be ignored in path integration] we find

$$\frac{\hbar}{M\omega} R(r_j, r_{j-1}) = \frac{1}{i \sin \varphi} \exp \{ i(\eta_j + \eta_{j-1}) \cot \varphi \} I_{l+(d-2)/2} \left( \frac{2\sqrt{\eta_j \eta_{j-1}}}{i \sin \varphi} \right). \quad (13)$$

Now we observe that this is identical with the matrix element (5) if we set for the representation label  $J := l/2 + d/4 - 1$ . Therefore, the radial path integral reads

$$K_l(r'', r'; \tau) = \frac{M\omega}{\hbar} \left( \frac{1}{r''r'} \right)^{(d-2)/2} \lim_{N \rightarrow \infty} \int \prod_{j=1}^N \langle J, \eta_j | e^{-2i\varphi_j J_3} | J, \eta_{j-1} \rangle \prod_{j=1}^{N-1} d\eta_j \quad (14)$$

and is easily performed using the completeness relation  $\int_0^\infty d\eta_j |J, \eta_j\rangle \langle J, \eta_j| = \mathbf{1}$  for the continuous basis (4). Finally, the radial propagator is given by

$$K_l(r'', r'; \tau) = \frac{M\omega}{\hbar} \left( \frac{1}{r''r'} \right)^{(d-2)/2} \langle J, \eta'' | e^{-2i\Phi J_3} | J, \eta' \rangle \quad (15)$$

where  $\eta := M\omega r^2/2\hbar$  and

$$\Phi := \lim_{N \rightarrow \infty} [N\varphi] = \lim_{N \rightarrow \infty} [N \arcsin(\omega\tau/N)] = \omega\tau. \quad (16)$$

The present path integral treatment is an alternative to the earlier approach [12]. Besides its simplicity the present group theoretical approach has the advantage to realize the dynamical group  $SU(1, 1)$  explicitly [see eq. (15)]. It can also be applied to the radial path integral for the  $1/r$ -problem in  $\mathbb{R}^d$  and the generalized Morse potential in  $\mathbb{R}$ , both of which have the dynamical group  $SU(1, 1)$  [8].

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## Total Space Quantization of Kähler Manifolds\*

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In this contribution, a novel globalized generalization of the concepts of coherent states to arbitrary Kähler manifolds will be described [1]. The formalism to be developed generalizes previous work by Berezin, Rawnsley and others [1, 2, 3, 4] and, in particular, allows for a more convenient and unified discussion of the classical limit.

On the elementary phase space  $P = \mathbb{R}^{2n}$ , coherent states can be defined in the following way:

First, one regards  $\mathbb{R}^{2n}$  as a complex space  $\mathbb{C}^n$ , e.g. by taking  $z_k = \frac{1}{\sqrt{2}}(q_k + ip_k)$ . The Bargmann Hilbert space consists of antiholomorphic functions

$$s : \mathbb{C}^n \rightarrow \mathbb{C}$$

of finite norm w.r.t. the scalar product

$$\langle s, t \rangle = \frac{1}{\pi} \int d^{2n}z \overline{s(\bar{z})} \epsilon^{-|z|^2} t(z) \quad (1)$$

Coherent states  $e_\zeta$  are given by the functions

$$e_\zeta(\bar{z}) = e^{z\zeta} \quad (2)$$

and obey the defining relation

$$\langle e_\zeta, s \rangle = s(\zeta) \quad (3)$$

Operators  $A$  are definable as normal ordered products of the creation and annihilation operators

$$a_k^\dagger s = \bar{z}_k s \quad , \quad a_k s = \frac{\partial}{\partial z_k} s \quad (4)$$

The symbol

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\*Talk given by H. Römer

$$\sigma(A)(\zeta) = \frac{\langle \epsilon_\zeta, A\epsilon_\zeta \rangle}{\langle \epsilon_\zeta, \epsilon_\zeta \rangle} \quad (5)$$

of an operator  $A$  is the starting point for discussions of the classical limit.

These concepts have to be generalized to arbitrary Kähler manifolds  $(M, \omega, I)$ , where  $M$  is a manifold with symplectic form  $\omega$  and complex structure  $I$ .

Berezin [2] gave a first definition of coherent states on Kähler manifolds in terms of local complex coordinates.

For an intrinsic definition one needs the following structural data:

1. a Kähler manifold  $(M, \omega, I)$  of real dimension  $2n$
2. a holomorphic line bundle  $L$  over  $M$  with hermitean metric  $h$
3. The connection  $\nabla$  on  $L$  with the properties of
  - (a) compatibility with  $h$ :  $Xh(v_1, v_2) = h(\nabla_X v_1, v_2) + h(v_1, \nabla_X v_2)$
  - (b) holomorphy:  $\nabla_X v = 0$  if  $v$  is a holomorphic section of  $L$  and  $X$  an antiholomorphic direction
  - (c) quantization condition for the curvature  $F_\nabla$  of  $\nabla$ :

$$F_\nabla = i\omega .$$

Bergmann's Hilbert space is then given by the  $h$ -antiholomorphic sections  $s$  of  $L$ , which by definition fulfil

$$\nabla_X s = 0 \quad \text{for every holomorphic direction } X$$

and are requested to have finite norm

$$\langle s, s \rangle = \int \omega^n h(s, s) < \infty . \quad (6)$$

It is also useful to consider the  $m^{\text{th}}$  tensorial powers  $L^m$  of  $L$ . The quantization condition

$$F_\nabla = im\omega \quad (7)$$

shows that the classical limit  $\hbar \rightarrow 0$  corresponds to  $m \rightarrow \infty$ .

Rawnsley [3] gave a global definition of coherent states in terms of sections on  $L^m$ , which is most easily understood by using the identification of sections of  $L^m$  with equivariant functions on  $L_0 = L \setminus M$  ( $M$  is identified with the zero section of  $L$ ) in the following way:

There is a natural  $\mathbb{C}^*$  action  $\lambda \mapsto \lambda z_0$  on  $L$ .

A function  $\hat{\psi} : L_0 \rightarrow \mathbb{C}$  with

$$\hat{\psi}(\lambda z_0) = z_0^{-m} \hat{\psi}(\lambda) \quad (8)$$

gives rise to a section  $\psi$  of  $L^m$  via

$$\psi(\tau_0(\lambda)) = \pi_m(\lambda, \hat{\psi}(\lambda)) \quad , \quad (9)$$

where  $\tau_0$  is the projection  $L \rightarrow M$  and  $\pi_m$  the projection

$$L_0 \times \mathbb{C} \rightarrow L^m \quad .$$

We perform a further step in direction shown by Rawnsley by dropping the equivariance condition and directly quantizing the total space  $L_0$ .

First, we notice that  $L_0$  can be endowed with a Kähler structure  $\omega_L$  by using the natural complex structure of  $L_0$  and defining

$$\omega_L = i\partial\bar{\partial}\log h_L \quad , \quad (10)$$

where  $h_L : L_0 \rightarrow \mathbb{C}$  is the function

$$\lambda \mapsto \exp(h(\lambda, \lambda)) \quad . \quad (11)$$

For the quantization of  $L_0$  one considers the trivial line bundle  $\hat{L} = L_0 \times \mathbb{C}$  over  $L_0$ . Taking  $h_L$  as hermitean fibre metric on  $\hat{L}$  it is easy to find the compatible connection  $\hat{\nabla}$  with quantization condition  $F_{\hat{\nabla}} = i\omega_L$ . The Hilbert space  $\mathcal{H}_L$  consists of all  $h_L$ -antiholomorphic sections  $\psi_L$  with finite norm

$$\langle\langle \psi_L, \psi_L \rangle\rangle = \int \omega_L^{n+1} h_L(\psi_L, \psi_L) \quad . \quad (12)$$

It turns out that all the sections  $\psi_m$  of the bundles  $L^m$  can be identified with sections  $\psi_L^m$  fulfilling a certain equivariance condition and that there are embeddings

$$\mathcal{H}^m \rightarrow \mathcal{H}_L \quad , \quad \psi^m \mapsto \psi_L^m$$

such that

$$\langle\langle \psi_L^m, \psi_L^{m'} \rangle\rangle = 2\pi(m+n)! \langle \psi^m, \psi^{m'} \rangle_m \delta_{mm'} \quad (13)$$

Thus, all the bundles  $L^m$  are unified in a single global object. Moreover, it is possible to define coherent states on  $L_0$  as certain sections  $\epsilon_\lambda$  of  $\hat{L}$  which fulfil the defining property

$$\langle\langle \epsilon_\lambda, \psi_L \rangle\rangle = \psi_L(\lambda) \quad . \quad (14)$$

The symbol of an operator  $A$  on  $\mathcal{H}_L$  commuting with the  $U(1)$ -action on  $L_0$  has the form

$$\sigma(A)(\lambda) = \frac{\langle\langle \epsilon_\lambda, A\epsilon_\lambda \rangle\rangle}{\langle\langle \epsilon_\lambda, \epsilon_\lambda \rangle\rangle} = \frac{\sum_m \frac{\epsilon^{(m)}(\tau(\lambda))}{(n+m)!} |\lambda|^{-2m} \sigma(A_m)(\tau(\lambda))}{\sum_m \frac{\epsilon^{(m)}(\tau(\lambda))}{(n+m)!} |\lambda|^{-2m}} = \sum_m w_m \sigma(A_m) \quad . \quad (15)$$

where  $A = \sum_m A_m$  (each operator  $A_m$  acting in  $\mathcal{H}^m$ ),  $\sigma(A_m)$  is the Berezin-Rawnsley symbol of  $A_m$ ,  $|\lambda|^2 = \hbar(\lambda, \lambda)$  and Rawnsley's function  $\epsilon^{(m)}(\tau(\lambda))$  (equalling sometimes the dimension of  $\mathcal{H}^m$ ) is nonnegative. Thus, our symbol is just a weighted average of the Berezin-Rawnsley symbols  $\sigma(A_m)$  with weights  $0 \leq w_m \leq 1$  and  $\sum_m w_m = 1$ . If  $|\lambda|^2$  tends to zero the mean value

$$\langle m \rangle = \sum_m w_m m$$

tends to infinity. For example, taking  $L_0 = \mathbb{C}^{n+1} \setminus \{0\}$  over  $M = \mathbb{C}P^n$  (which corresponds to the harmonic oscillator) we obtain a Poisson distribution

$$w_m = \frac{|\lambda|^{-2m}}{m!} e^{-\frac{1}{|\lambda|^2}}$$

with mean value  $\langle m \rangle = \frac{1}{|\lambda|^2}$ . Hence, in our approach, Planck's constant  $\hbar$  does not appear as a parameter but rather as a coordinate along the fibre direction of  $L_0$ . It is a continuous parameter unlike  $\frac{1}{m}$  and it even acts on  $L_0$ . The classical limit can be discussed by taking

$$\lim_{\hbar \rightarrow 0} \sigma(A)(\lambda\hbar) = \sigma_{cl}(A) \quad (16)$$

which may considerably simplify and unify such considerations. Moreover, star products are definable in a unified way such that  $\hbar$  is a real coordinate on  $L_0$  rather than a formal parameter.

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## QUANTIZATION ON SEMISIMPLE SYMMETRIC SPACES

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### 1 Quantization according to Berezin

Let  $M$  be a symplectic manifold. It is required:

- (a) to construct a family of associative algebras  $\mathcal{A}_h \subset C^\infty(M)$  depending upon a parameter  $h$  (called the Planck constant), a product is denoted by the star  $*$ , these algebras must satisfy a correspondence principle (CP):

$$\lim_{h \rightarrow 0} f_1 * f_2 = f_1 f_2 \quad \text{the pointwise product}$$

$$\lim_{h \rightarrow 0} j h^{-1} (f_1 * f_2 - f_2 * f_1) = \{f_1, f_2\} \quad \text{the Poisson bracket}$$

$j = \sqrt{-1}$  or  $j = 1$ , and some less important conditions, we drop them here:

- (b) to construct representations  $f \mapsto \hat{f}$  of these algebras by operators in Hilbert spaces.

For  $M = \mathbb{C}^n$  this construction gives the well known Wick quantization.

F.A. Berezin himself has constructed the quantization for classical Hermitian symmetric spaces [2, 2, 2, 2].

**Example 1.**  $M$  is the unit disk  $z\bar{z} < 1$  in  $\mathbb{C}$  (the Lobachevsky plane). It is the Hermitian symmetric space  $SU(1,1)/U(1)$ . Operators mentioned in (b) act in a Fock space of analytic functions on  $M$ . A supercomplete system (see sect.2) consists of functions  $L_{\bar{v}}(z) = L(z\bar{v}) = (1 - z\bar{v})^{-1/h}$ . For Details, see [2].

We would like to construct the Berezin quantization on symplectic semisimple symmetric spaces. It is sufficient to consider indecomposable ones. Such spaces are of two classes (for details, see [2]): Hermitian (they are Riemannian,  $j = \sqrt{-1}$ ) and para-Hermitian (they are non-Riemannian,  $j = 1$ ). So we must study para-Hermitian symmetric spaces. Before going to the general case we consider in sect.3 a crucial example.

## 2 Covariant symbols of operators

This notion is a basis for the construction of algebras  $\mathcal{A}_h$ . Let  $H$  be a Hilbert space and  $\{x\}$  a set equipped with a measure  $dx$ . A system of vectors  $\epsilon_x$  in  $H$  is said to be supercomplete if

$$f = \int (f, \epsilon_x) \epsilon_x dx \quad (\forall f \in H).$$

For an operator  $\hat{A}$  in  $H$ , the following function  $A(x, y)$  is called the covariant symbol of  $\hat{A}$ :

$$A(x, y) = (A\epsilon_y, \epsilon_x) / (\epsilon_y, \epsilon_x) \quad (1)$$

Inversely, the operator  $\hat{A}$  is uniquely defined by its symbol. The unit operator has the symbol 1. If  $C = AB$  then

$$C(x, y) = \int A(x, z) B(z, y) (\epsilon_y, \epsilon_z) (\epsilon_z, \epsilon_x) (\epsilon_y, \epsilon_x)^{-1} dz$$

So we get an algebra of symbols:  $C = A * B$ .

## 3 Example 2: the hyperboloid of one sheet [6]

The hyperboloid  $M : -x_1^2 + x_2^2 + x_3^2 = 1$  in  $\mathbb{R}^3$  is the para-Hermitian symmetric space  $G/H$  where  $G = SO_0(1, 2)$ ,  $H = SO_0(1, 1)$ . It is of Cayley type, see sect.4. Let us take coordinates  $t, s$  on  $M$  as follows:

$$x_1 = \frac{t+s}{1-ts}, \quad x_2 = \frac{t-s}{1-ts}, \quad x_3 = \frac{1+t}{1-ts}.$$

Then an invariant measure and the Laplace-Beltrami operator are

$$dn(t, s) = 2(1-ts)^{-2} dt ds \quad \text{and} \quad \Delta = (1-ts)^2 \partial_t \partial_s$$

respectively. Operators  $\hat{A}$  act in a space of functions  $f(t)$ . A supercomplete system consists of functions

$$L_s(t) = L(t, s) = |1-ts|^{-1/h}.$$

It is the kernel of an intertwining operator for elementary representations of  $G$ . So there is a reproducing property:

$$f(t) = c(h) \int f(p) L(t, s) L(p, s)^{-1} dn(p, s) \quad (2)$$

where  $c(h) = \cot(\pi/2h)(1-h)/4\pi h$ . A covariant symbol of an operator  $\hat{A}$  is defined by

$$A(t, s) = (\hat{A}L_s)(t) / L_s(t) \quad (3)$$

It may be written like (1) using a scalar product for the complementary series. The action of  $\hat{A}$  is expressed by its symbol as follows:

$$(Af)(t) = c(h) \int A(t, s)L(t, s)L(p, s)^{-1}f(p)dn(p, s) \quad (4)$$

The product  $*$  is

$$(A * B)(t, s) = \int A(t, \hat{s})B(\hat{t}, s)I(t, s, \hat{t}, \hat{s})dn(\hat{t}, \hat{s}) \quad (5)$$

where

$$I(t, s, \hat{t}, \hat{s}) = c(h)L(t, \hat{s})L(\hat{t}, s)/L(t, s)L(\hat{t}, \hat{s}).$$

Now introduce other variables  $\alpha, \beta$ :

$$t = \cot\left(\frac{\pi}{4} - \frac{\alpha}{2}\right), \quad s = \tan\left(\frac{\pi}{4} - \frac{\beta}{2}\right).$$

Then  $M$  is embedded in a torus  $S \times S$  where  $S$  is the unit circle, and symbols become functions on this torus. Let  $W^+$  (resp.  $W^-$ ) be the subspace of  $C^\infty(S \times S)$  generated by  $\exp(in\alpha + im\beta)$ ,  $n, m \geq 0$  (resp.  $\leq 0$ ). These spaces  $W^+$  and  $W^-$  are algebras w.r.t.  $*$ ,  $CP$  is held on them. The latter statement follows from an asymptotic decomposition  $I = id + h\Delta + o(h)$ , where  $I$  is the operator with the kernel  $I(t, s, \hat{t}, \hat{s})$ .

The group  $G$  has 3 series of unitary irreducible representations: continuous and 2 discrete ones. Correspondingly  $L^2(M)$  is decomposed into the sum of 3 subspaces:  $L^2(M) = V_c + V^+ + V^-$ . Consider an operator  $Q$  (an integral over generating lines intersecting at a point  $x$ ):

$$(Qf)(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \{f(x + av(x)) - f(x + au(x))\}a^{-1}da$$

where

$$u(x) = (1, \sin \alpha, \cos \alpha), \quad v(x) = (1, \sin \beta, \cos \beta).$$

It turns out that  $Q$  separates the subspaces above:  $Q = 0, +1, -1$ , resp.

#### 4 On the quantization on para-Hermitian symmetric spaces

Let  $G$  be a simple Lie group (with the finite center, for simplicity),  $\sigma$  an involution of  $G$ , and  $H$  an open subgroup in the  $\sigma$ -fixed set. Then  $G/H$  is called a semisimple symmetric space. Let  $K$  be a maximal compact subgroup of  $G$  invariant w.r.t.  $\sigma$ . The Lie algebra  $\mathfrak{g}$  of  $G$  is decomposed into  $+1, -1$  subspaces w.r.t.  $\sigma$ :  $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ , so  $\mathfrak{h} = \text{Lie}H$ .

The space  $G/H$  is called a para-Hermitian symmetric space if there exists an operator  $U$  in  $\mathfrak{q}$  s.t.:  $U^2 = \text{id}$ ,  $U$  commutes with  $\text{Ad}H$ .

$$B(IX, Y) + B(x, IY) = 0, \quad X, Y \in \mathfrak{q},$$

here  $B$  is the killing form. Such space is symplectic:  $\omega(X, Y) = B(IX, Y)$ .

Let  $G/H$  be indecomposable. The  $\mathfrak{h}$  has an one-dimensional center  $\mathbb{R}Z_0$ . The tangent representation  $\mathfrak{h} \mapsto \text{Ad}h|_{\mathfrak{q}}$  of  $H$  is reducible:  $\mathfrak{q} = \mathfrak{q}^+ + \mathfrak{q}^-$ . These subspaces  $\mathfrak{q}^{\pm}$  are Abelian, and Lagrangian. We get two real polarizations on  $G/H$ .

The set  $P^+ = \text{Hex}p\mathfrak{q}^+$  is a maximal parabolic subgroup of  $G$ . The quotient space  $S = G/P^+$  is a compact symmetric space ( $\cong K/K \cap H$ ). We have an embedding  $G/H \hookrightarrow S \times S$ .

A space for operators  $\hat{A}$  and a supercomplete system consists of functions on  $S$ .

For  $\mu \in \mathbb{C}$  let  $U_{\mu}$  be an one-dimensional representation of  $p^+$  s.t.  $u_{\mu} = 1$  on  $\exp \mathfrak{q}^+$  and on the semisimple part of  $H$  and  $u_{\mu}(\exp aZ_0) = \exp \mu a$ . Put  $T_{\mu} = \text{Ind}_{p^+}^G u_{\mu}$ . There exists  $\mu_0 \in \mathbb{R}$  s.t. for any  $\mu \in \mathbb{C}$  there exists an operator  $B_{\mu}$  intertwining  $T_{\mu}$  and  $T_{\mu^*} \circ \tau$ ,  $\mu^* = -\mu_0 - \mu$  where  $\tau$  is the involution of  $G$  defining  $K$ . Let  $L(t, s) = L_{\mu}(t, s)$  be the kernel of  $B_{\mu^*}$ . These functions  $L$  on  $S \times S$  form a supercomplete system. The formulas (2)-(5) are preserved. The Planck constant is  $\hbar = -1/\mu$ .

**Example 3.**  $G = SL(n, \mathbb{R})$ ,  $H = GL(n-1, \mathbb{R})$ .

Here  $S = S^{n-1}/\pm$  where  $S^{n-1}$  is the unit sphere in  $\mathbb{R}^n$ ,  $L(t, s) = |t_1 s_1 + \dots + t_n s_n|^{\mu}$ ,  $t, s \in S^{n-1}$ .  $CP$  is held on the discrete series for  $G/H$ , see e.g. [2].

Among para-Hermitian symmetric spaces rather Cayley type ones are most of interest. They are characterized by  $K$  having nondiscrete center. Then  $S$  is the Shilov boundary of some complex manifold  $D$ . Let  $H^2(D \times D)$  be a Hardy space [2, 2, 2]. The  $CP$  is held on the subspace in  $L^2(G/H)$  consisting of boundary values of functions in  $H^2(D \times D)$ .

**Example 4.**  $G = SU(n, n)$ ,  $H = SL(n, \mathbb{C}) \times \mathbb{R}^*$ .

$G/H$  is of Cayley type.  $S = U(n)$ .  $D$  is the classical domain of type I.  $L(u, v) = |\det(u+v)|^{\mu}$ ,  $u, v \in U(n)$ .

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## DYNAMICAL GROUPS OF FREE MOTION

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**ABSTRACT.**

We exhibit features common to some simple quantum-mechanical problems like the  $N$ -dimensional oscillator, rotor or Kepler problem: they share i) maximal degeneracy ii) dynamical groups iii) spectrum generating groups iv) closed geodesics and v) separation in several systems of coordinates.

We point out the *free motion* nature in symmetric, rank-one spaces, as an explanation, the oscillator being even simpler. Generalizations to other spaces of same kind are possible, including free motion in complex projective space  $CP^N$ .

1.- Consider three standard problems in theoretical mechanics given by the hamiltonians

$$H = \frac{1}{2} \sum_{i=1}^N (p_i^2 + q_i^2) \quad N - \text{dim oscillator } (N \geq 1) \quad (1.1)$$

$$H = \frac{1}{2} \sum_{i,j=1}^N (l_{ij}^2) \quad N - \text{dim rotator (rotor)} (N \geq 2) \quad (1.2)$$

$$H = \frac{1}{2} \sum_{i=1}^N p_i^2 - \frac{k}{r} \quad N - \text{dim Kepler } (N \geq 1) \quad (1.3)$$

where  $l_{ij} = x_i p_j - x_j p_i$ . All the three cases are exactly soluble both in classical mechanics and in quantum mechanics; they share also an interesting list of common features, which we would like to understand and relate. They are:

1) The *classical* motion contains only closed orbits (we limit ourselves to  $E < 0$  in (1.3)):

oscillator: maximal circles  $z_i \rightarrow e^{it} z_i$ ;  $z_i = q_i + ip_i$ ,

rotator: maximal circles in  $S^{N-1}$

Kepler: ellipses

2) The *spectrum* depends on a single quantum number

$$\text{oscillator: } E_n = (n + N/2)\epsilon_0, \quad (n = 0, 1, \dots) \quad (1.4)$$

$$\text{rotator: } E_l = l(l + N - 2)/2, \quad (l = 0, 1, \dots) \quad (1.5)$$

$$\text{Kepler: } E_n = -\frac{1}{2n^2}\epsilon_0, \quad (n = 1, 2, \dots) \quad (1.6)$$

3) There is a *maximal* dynamical symmetry group

$$\text{oscillator: } U(N) = Sp(N) \cap O(2N)$$

rotor:  $O(N)$   
Kepler:  $O(N+1)$

4) The ireps selected by the symmetry group are always irreducible fully symmetric representations, i.e. they behave like  $[k]$  as Young tableaux (or its traceless part).

5) Therefore the *degeneracy* of the quantum levels is group-theoretical, i.e. it can be calculated by representation theory alone:

$$\text{oscillator: } \deg(n) = \binom{N+n-1}{n} \quad (1.7)$$

$$\text{rotator: } \deg(l) = \frac{(2l+N-2)(l+N-3)!}{l!(N-2)!} \quad (1.8)$$

Kepler: same as rotator ( $N+1$ ), see later

6) The hamiltonian separates in *more than one* system of coordinates: e.g.

oscillator: in cartesian and polar

Kepler: in polar and parabolic

7) The three problems admit compact or non-compact *spectrum generating groups* (Barut 1964)[1]; for example, the oscillator admits  $SU(N, 1)$ , the rotor admits  $SO_o(N, 1)$ .

This means that a *single* irep of the non-compact group gives *all* the ireps of the physical problem.

8) They all have supersymmetric extensions (partners), and in fact supersymmetry leads to the complete algebraic solution for the spectrum.

**2.-** The following questions arise in connection with these peculiar features:

1) Are these problems related?

2) Are these features themselves related?

3) What is the (deep?) mathematical reason for all that?

4) Are there more problems like them, i.e. can we find other similar systems?

Before entering into details let us already present the key characteristic:

- *we have essentially a problem of free motion (classical or quantized)*

*in a rank-one symmetric space.*

Let us first dispose of the Kepler case. As V. Bargmann showed back in 1935 for  $N=3$  [2], Alliluev (1957) [3] for the general quantum and Moser (1970) [4] for the classical case, see also Boya [5]:

- The  $N$ -dim Kepler problem is equivalent to the motion of a free particle in the surface of a  $N$ -sphere (bounded orbits only).

The equivalence is not completely trivial, however, and requires three steps: regularization of the  $r^{-1}$ -singularity, stereographic projection and Fourier transformation to momentum space.

Now free motion in  $S^N$  has obvious  $O(N+1)$  invariance, and it is exactly the rotator problem in  $N+1$ :

$$N - \text{Kepler problem} \approx (N+1)\text{rotator} \quad (2.1)$$

and we shall not have to worry about Kepler anymore: even the *unbounded* orbits, corresponding to the continuous spectrum, do correspond to free motion in hyperbolic space [4]; we shall not consider this case.

**3.-** The *oscillator* case is different from the rotator and somewhat simpler:

- The harmonic isotropic oscillator problem is equivalent to a particular rotation (imaginary dilatation) in  $\mathbb{C}^N$ :

If  $z = \{z_1, \dots, z_N\} \in \mathbb{C}^N$ , take

$$\text{Hamiltonian } H = \frac{1}{2}(z^* z), \quad (3.1)$$

$$\text{Vector field } X = \text{Im } z_i \frac{\partial}{\partial z_i} = p_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial p_i} \quad (3.2)$$

with dynamical flow  $\tau_t : z \rightarrow e^{it} z$ , i.e. motion along *some* great circles, so the flow is the  $U(1)$  generator central in  $U(N)$  acting naturally in  $\mathbb{C}^N$ ; hence there is *maximal reduction* in the Marsden-Weinstein sense [6], and the closure of the orbit is a 1-Torus. Therefore the quantum spectrum depends also only on a single quantum number. The space of orbits is again purely *geometric* namely  $\mathbb{C}^{N-1}$ ; in fact

$$\mathbb{C}^N = \text{phase space} = \mathbb{R}^{2N} \quad (3.3)$$

$$S^{2N-1} \subset \mathbb{C}^N, \text{ energy surface} \quad (3.4)$$

$$S^{2N-1}/U(1) = \text{orbit space} = \mathbb{C}P^{N-1} \quad (3.5)$$

4.- The *rotator* problem is just (free) motion in a sphere; now for free motion in symmetric spaces we have simple results: in particular

- The *classical* motion are geodesics running at constant speed: in particular at zero speed, i.e. the particle lies at rest at any point.

- The *quantum* problem is given by the spectrum of the laplacian.

- The duality geodesics  $\leftrightarrow$  laplacian, both particular cases of harmonic maps, is seen here as a duality between classical and quantum mechanics!

In group-theoretical terms we have that the quantum rotor problem with hamiltonian

$$H = \frac{1}{2} \vec{J} \cdot \vec{J} \quad (4.1)$$

is given by the spectrum of the *quadratic* Casimir operator for  $O(N+1)$ , dual to the Cartan-Killing form; it acts in a suitable space  $S^N = O(N+1)/O(N)$ .

In contrast, the oscillator hamiltonian  $H = z_i \frac{\partial}{\partial z_i}$  is given by the spectrum of the *linear* Casimir operator for  $U(N)$ ; the rotator case corresponds to simple groups, with no linear Casimir.

Now we can answer some of the questions posed before, namely

Q.: - why the classical orbits close, or equivalently why the spectrum depends only on one quantum number?

A: - because the classical orbits of the free motion are geodesics, and geodesics close precisely in *compact rank-one symmetric spaces* like

$$S^N = O(N+1)/O(N), \quad \mathbb{C}P^N = U(N+1)/U(1) \times U(N) \quad (4.2)$$

Q.: - For which other cases do we expect similar behaviour, like Energy =  $E(n)$ ?

A: - Try *free* motion in arbitrary *symmetric spaces*  $M = G/H$  of rank one. The following Table gives all the compact ones

- 1)  $S^N = SO(N+1)/SO(N)$ , spheres
- 2)  $\mathbb{R}P^N = SO(N+1)/O(N)$ , real projective spaces
- 3)  $\mathbb{C}P^N = SU(N+1)/U(N)$ , complex projective spaces
- 4)  $\mathbb{H}P^N = Sp(N+1)/Sp(N) \times Sp(1)$ , quaternionic projective space
- 5)  $\mathbb{O}P^2 = F_4/Spin(9)$ , Cayley plane.

Table I. Compact Rank-one symmetric spaces

We remind the reader the definition of *rank* for a Riemannian manifold: it is the dimension of the maximal totally geodesic flat submanifold (Helgason 1978 [7]). For a Lie group, this coincides with 's Cartan definition of rank (dim of maximal abelian subgroup), hence only  $U(1) = S^1$  and  $Sp(1) = SU(2) = S^3$  are rank-one groups. As a counterexample, free motion in the torus  $T^2 = S^1 \times S^1$  will have "dense" geodesics (with bidimensional closure); in fact, the spectrum depends on two quantum numbers.

5.- As another example of free motion, let us take case 3) of Table I: free motion in  $\mathbb{C}P^N$ ; the results are

- The geodesics close, with length  $2\pi$
- The energy spectrum is  $E_n = n(n+N)$  for  $\mathbb{C}P^N$

To single out the representations, notice that  $SU(N+1)$  is *not* effective on  $\mathbb{C}P^N$ , because of its center. The effective group is

$$PU(N+1) = U(N+1)/U(1) = SU(N+1)/Z_{N+1} \quad (5.1)$$

and therefore the relevant representations are the *irreducible* pieces of the *symmetric* product of the *adjoint* representation of  $SU(N+1)$ .

It might be interesting to look at the remaining cases of symmetric rank-one spaces, like  $\mathbb{H}P^N$  or the Cayley plane.

6.- Other type of symmetric spaces are important in quantum mechanics, namely the split-rank homogeneous spaces:

$$M = G/H \text{ is split rank if } \text{rank } M = \text{rank } G - \text{rank } H. \quad (6.1)$$

As examples we have 1) Lie groups themselves,  $G = (G_L \times G_R)/G_D$ . 2) all *odd* spheres. 3)  $SU(2n)/Sp(n)$  and 4)  $E_6/F_4$ .

These spaces are important because, as shown by Dowker and Camporesi [8],

- The propagator  $G(x, x')$  for split-rank spaces can be computed *exactly* from the heat kernel expansion.

7.- To complete our work we make some incomplete remarks on the *spectrum generating group* aspect of the problem. We hope to address the question of the *supersymmetry* connection in a future publication.

Both for  $O(N)$  and  $U(N)$ , the relation

$$\text{deg}(N, n) = \text{deg}(N-1, n) + \text{deg}(N, n-1) \quad (7.1)$$

leads at once to

$$\deg(N, n) = \sum_{n' \leq n} \deg(N - 1, n'). \quad (7.2)$$

This goes a long way to understand the embedding of dynamical groups on compact groups, as shown first by Sudarshan *et al.* (1965)[9]; now why also non-compact enveloping groups? A possible clue is the duality between  $G(N + 1)$ , compact, vs.  $G(N, 1)$ , non-compact.

But much more should be learnt before we claim we understand the use of non-compact groups as spectrum-generating groups; an interesting question, for example, is: do they have any implication for the set of *classical* solutions?

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### A new probabilistic quantization method and its semiclassical limit

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Our approach relies on a 1931-32 idea of E. Schrödinger [1], inspired by the following remark of A.S. Eddington on quantum dynamics: "The whole interpretation is very obscure but it seems to depend on whether you are considering the probability after you know what has happened or the probability for the purpose of prediction. The  $\psi\bar{\psi}$  is obtained by introducing 2 systems of  $\psi$  waves travelling in opposite directions in time: one of these must presumably correspond to probable inference from what is known (or is stated) to have been the condition at a later time." Such was Schrödinger's motivation to state a problem in the classical theory of Brownian motion, regarded by him as a revealing analogy with wave mechanics. This problem has been ignored by theoretical physicists until 1985-86, when it was solved, then interpreted as a new Euclidean quantization procedure (i.e. dealing with the "imaginary time" Schrödinger equation, or heat equation) [2]. The solution of Schrödinger's problem, however, has little to do with M.Kac's familiar reinterpretation of Feynman's path integral method [3,7]. Actually, it should be regarded as an alternative Euclidean interpretation of this technique [4].

Let us consider the Schrödinger equation for a single particle in space

$$(1) \quad i\hbar \frac{\partial \psi}{\partial \tau} = -\frac{\hbar^2}{2} \Delta \psi + V\psi = H\psi$$

$$\psi(x, s) = \chi(x)$$

For a (positive) Hamiltonian  $H$ , the initial state  $\chi$  determines quantum dynamics, for  $\tau$  in  $[s, \infty[$ . Via the change of variable  $\psi(x, \tau) = \rho^{\frac{1}{2}} e^{i\hbar S(x, \tau)}$ , Schrödinger asks: is it possible to describe quantum dynamics from the data of a pair  $\{\rho(x, s), \rho(x, u)\}$ , for an arbitrary interval  $[s, u]$ , instead of  $\chi$ ? In regular probability theory, a  $\mathbb{R}^3$ -valued stochastic process  $z_t = z(t)$  is Markov if the knowledge of its past information up to the time  $s$ ,  $\mathcal{P}_s$ , can be forgotten:  $P[z_t \text{ is in a set } B | \mathcal{P}_s] = P[z_t \text{ is in } B | z_s]$ ,  $t \geq s$ , where  $P[\cdot]$  is the probability of the event  $\{\cdot\}$ ,  $[\cdot | \cdot]$  means "knowing  $\cdot$ ", and the  $t$  has defines the usual (forward) transition probability. Kolmogorov has shown that the data of an initial probability,

$$P(z_s \text{ is in } dx) = \rho(x, s) dx$$

determines uniquely the process  $z_t$ ,  $t > s$  (in technical terms, determines uniquely its probability measure). But the Markov property, should also, according to Schrödinger, be formulated w.r.t the future information from time  $u > t$ ,  $\mathcal{F}_u : P^*[z_t \text{ is in } B | \mathcal{F}_u] = P^*[z_t \text{ is in } B | z_u]$ ,  $t \geq u$ , where the star  $*$  denotes this backward description. Then, the data of a final probability

$$P(z_u \text{ is in } dx) = \rho(x, u) dx$$

is necessary to determine  $z_t$ ,  $t \leq u$ . Clearly, in order that the forward and backward descriptions coincide, some unusual compatibility conditions are required. Schrödinger's Euclidean interpretation starts from the fact that the integral kernel of  $e^{-(t-s)H/\hbar}$ , denoted  $h(x, t-s, y)$  and supposed positive (this is an hypothesis on the potential  $\bar{V}$ ); is the propagator of two heat equations

$$(1) \quad -\hbar \frac{\partial \eta^*}{\partial t} = H \eta^*$$

$$(2) \quad \hbar \frac{\partial \eta}{\partial t} = H \eta$$

since a regular state  $\varphi$  is propagated forward in time to give  $\eta_\varphi^*(y, t)$  and backward in time to give  $\eta_\varphi(y, t)$ . The role of the quantum probability  $\psi\bar{\psi}$  is played by a product of positive solutions of (2), denoted  $\chi^*, \chi$ , respectively, are chosen so that they are consistent with the data of a pair of (never zero) initial and final probability densities  $P_s(y), P_u(y)$  [1,2]. The way to do it is due to Schrödinger [1]. An infinite collection of (Markov, "Bernstein") diffusions  $z(t)$ ,  $s \leq t \leq u$  is, in this way, associated with the given Hamiltonian  $H$  [2]. The two moments of such diffusions are computable via their forward and backward transition probabilities. For example, the limits of conditional expectations

$$(3) \quad \lim_{\Delta t \rightarrow 0} E \left[ \frac{z(t + \Delta t) - z(t)}{\Delta t} \middle| z(t) \right]$$

$$\lim_{\Delta t \rightarrow 0} E \left[ \frac{z(t) - z(t - \Delta t)}{\Delta t} \middle| z(t) \right]$$

$$(4) \quad \lim_{\Delta t \rightarrow 0} E \left[ \frac{(z(t + \Delta t) - z(t))^2}{\Delta t} \middle| z(t) \right] = \hbar Id$$

make sense. Clearly, the two first ones are regularized time derivatives (the "drifts"). The third one ( $Id$  = 3x3 identity matrix) is a constant ("diffusion") matrix. This should be compared with Feynman's powerful but heuristic analogy [3] between Brownian motion and quantum "paths"  $\tau \rightarrow \omega(\tau)$ . Using the "expectation"  $\langle \cdot \rangle_s$  w.r.t the "weight"  $e^{\frac{i}{\hbar} S[\omega(\cdot)]}$ , where  $S$  is the action functional  $S[\omega(\cdot)] = \int_0^t \left\{ \frac{1}{2} |\dot{\omega}(\tau)|^2 - V(\omega(\tau)) \right\} d\tau$ , he obtains, for instance

$$(4^*) \quad \left\langle \frac{(\omega(\tau + \Delta\tau) - \omega(\tau))^2}{\Delta\tau} \right\rangle_s = i\hbar Id.$$

Because of the factor  $i$  on the r.h.s, however, Feynman's approach has no probabilistic content (in spite of the unquestionable validity of Born statistical interpretation of the wave function  $\psi$ ). So, where do quantum probabilities come from, then? Eq.(4\*) is interpreted by Feynman as the nondifferentiability of quantum paths, a typically Brownian property [3]. Also notice that the continuum limit  $\Delta\tau \rightarrow 0$  is meaningless in real time.



In Feynman's framework, properties like (4\*) follow from a general integration by parts formula [3], for "arbitrary" functionals  $\mathcal{F}$  of the quantum paths  $\omega$  :

$$(5) \quad \langle \delta \mathcal{F}[\delta \omega](\delta \omega) \rangle_s = -\frac{i}{\hbar} \langle \mathcal{F} \delta S[\omega](\delta \omega) \rangle_s$$

where  $\delta$  denotes the directional derivative, and the same is true in our Euclidean context [5]. Our conclusions (although "only" Euclidean) hold in a much stronger sense, however, because of their probabilistic content. For instance, taking advantage of the existence of the continuum limit, one shows that the Bernstein diffusions solve regularized (i.e. quantized) versions of the (Euclidean) Euler-Lagrange equations, whose expectation corresponds with quantum mechanical predictions. Any Bernstein diffusion depends on  $\hbar$  via its diffusion matrix and its drifts. So, any  $z(\tau) = z^{\hbar}(\tau)$  can be expanded as

$$(6) \quad z^{\hbar}(\tau) = z^0(\tau) + \sqrt{\hbar} z^{sc}(\tau) + \hbar z^{(2)}(\tau) + \dots, \tau \in [t, u]$$

where  $z^0(\cdot)$  solves the classical (Euclidean) equations of motion. To stop after the first quantum correction  $z^{sc}$  is the "Semiclassical approximation" [6]. Then, the resulting diffusion is Gaussian, since  $z^{sc}$  (the "Bernstein Jacobi" process) is. Such Gaussian can be represented by

$$(7) \quad z^{sc}(\tau) = E_{y,t} z^{sc}(\tau) + \sum_{n=1}^{\infty} \xi_n \phi_n(\tau)$$

where  $E_{y,t}$  denotes the conditional expectation  $E[\dots | z^{sc}(t) = y]$ .  $\phi_n$  is an eigenfunction of the correlation operator  $C_t$  of  $z^{sc}$

$$(8) \quad (\text{whose integral kernel is } K_t(s, \tau) = E_{y,t} [z^{sc}(s) - E_{y,t} z^{sc}(s)] [z^{sc}(\tau) - E_{y,t} z^{sc}(\tau)])$$

and  $\xi_n$  are independant Gaussian random variables of zero mean and variance  $\lambda_n$ , the eigenvalues of  $C_t$ . As expected,  $z^{sc}$  solves the variational equation of the quantized Euler-Lagrange equation, here the one associated with the (Euclidean) semiclassical Lagrangian

$$(9) \quad \bar{L}^{sc}(\dot{q}, q, \tau) = \frac{1}{2} |\dot{q}|^2 + \frac{1}{2} \nabla^2 V(z^0(\tau)) |q|^2.$$

Actually,  $z^{sc}$  is entirely characterized by the data of a solution  $F$  of the classical Jacobi equation

$$(10) \quad \ddot{q}(\tau) = \nabla^2 V(z^0(\tau)) q(\tau), t \leq \tau \leq u$$

For instance, the correlation of this diffusion is

$$K_t(r, \tau) = \begin{cases} F(\tau) F_t^*(r) & , t < r < \tau < u \\ F(r) F_t^*(\tau) & , t < \tau < r < u \end{cases}$$

where  $F_t^*(\tau) = F(\tau) \int_t^\tau F^{-2}(r) dr$  is another solution of the Eq.(10). Such description holds if it is known that, in a past time  $t$ , the particle was precisely at  $y$  ("Past conditioning" through a narrow slit, for instance). In consequence, its dispersion (variance) is zero at time  $\tau = t$ . The eigenvalue equation for  $C_t$  shows that  $C_t^{-1} = J_t$ , where the Jacobi operator  $J_t$  is defined by

$$(11) \quad J_t \phi_n = \left( -\frac{d^2}{d\tau^2} + \nabla^2 V(z^0(\tau)) \right) \phi_n(\tau) = \lambda_n^{-1} \phi_n(\tau) \quad t < \tau < u$$

with  $\phi_n(t) = 0$  and  $\dot{F}(u)\phi_n(u) - F(u)\dot{\phi}_n(u) = 0$ . It follows, in particular, that  $F$  and  $F_t^*$  are linearly independent solutions of the Jacobi equation (10), with unit Wronskian. If it is known that the particle is in a sharp position at a future time  $t$ , the correlation becomes

$$K_s^t(v, \tau) = \begin{cases} F^t(v)F_*(\tau), & s < \tau < v < t \\ F^t(\tau)F_*(v), & s < v < \tau < t \end{cases}$$

with a nonzero solution  $F_*$  of Eq.(10) and  $F^t$  another, linearly independent one, built in terms of  $F_*$ . If nothing sharp is known about the particle on  $[s, u]$ , the Gaussian Bernstein Jacobi is characterized by the classical pair  $(F, F_*)$ . Its mean value  $m(\tau)$  and covariance  $c(\tau)$ , in one dimension, are [6]

$$(12) \quad \begin{cases} m(\tau) = w^{-1}(F(\tau)\delta_* + F_*(\tau)\delta) \\ c(\tau) = w^{-1}F(\tau)F_*(\tau) \end{cases}$$

where  $\delta, \delta_*$  are constant and  $w$  is the Wronskian of  $F$  and  $F_*$ . Notice that the effect of conditioning, in this framework, gives some weight to Eddington's remark. One shows that  $F$  and  $F_*$  are time reversed of each other, so that Eq(12) displays the time symmetry of the method [6]. The expansion (7) is a rigorous version of Feynman's "Fourier decomposition of quantum paths". Actually, he uses only a smooth (formal) cutoff approximation [3]

$$\mathcal{Z}_N^y(\tau) = E_{y, \tau} z^c(\tau) + \sum_{n=1}^N \xi_n \phi_n(\tau)$$

The  $\lim_{N \rightarrow \infty}$  is highly nontrivial in the traditional approaches [7]. Bernstein diffusions are associated in a one-to-one way with regular solutions  $v$  of the Schrödinger equation (1), so that the correspondence (after analytical continuation  $\tau \rightarrow -it$  of the time parameter) between time ordered  $n$ -points functions

$$\langle \psi | Q(\tau_1) Q(\tau_2) \dots Q(\tau_n) \psi \rangle_2$$

where  $Q(\tau)$  is the position observable at time  $\tau$ , in the quantum Hilbert space, and the Schwinger functions

$$E[z(t_1)z(t_2)\dots z(t_n)], \quad t_1 < t_2 < \dots < t_n$$

holds for any regular state  $\psi$ , not only for the ground state. All the measures "coming from quantum mechanics" by other routes than ours are included in the present family. Therefore, the research program founded on Schrödinger's idea provides, indeed, an alternative Euclidean interpretation of Feynman's strategy, promising both from the technical point of view and the interpretative one.

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**QUASICLASSICAL APPROXIMATION  
IN BARGMANN REPRESENTATION OF LIE GROUPS**

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Recently Voros [1] and Kurchan and al. [2] have used the Bargmann representation for a quasiclassical quantization. I would like to review my works on this subject [3-7], which did mostly in the seventies concerning the backbending phenomenon and the phase transition in nuclei.

**1. Bargmann representation of the  $SU(2)$  group**

The complex variable  $\zeta$  plays an important role in the Bargmann representation of the Lie groups [8]. It is determined by the stereographic projection of a sphere point with polar angles  $\vartheta$  and  $\varphi$  on the plane passing through the south pole:

$$\zeta = x + iy = e^{i\varphi} \cot \frac{\vartheta}{2}. \quad (1)$$

Rotations in the three-dimensional space correspond to a transformation

$$\mathbf{u}\zeta = \frac{\alpha\zeta + \beta}{-\beta^*\zeta + \alpha^*}, \quad (2)$$

where the unimodular matrix

$$\mathbf{u} = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \quad (3)$$

is determined by the Euler angles.

The Bargmann representation of the  $SU(2)$  group uses as a basis homogeneous polynomials  $f$  of the  $2j$  degree in two complex variables  $\zeta_1$  and  $\zeta_2$ . The linear transformation  $T(\mathbf{u})$  on the space of these polynomials has the form

$$T(\mathbf{u})f(\zeta_1, \zeta_2) = f(\alpha^*\zeta_1 - \beta\zeta_2, \beta^*\zeta_1 + \alpha\zeta_2). \quad (4)$$

**2. One-dimensional realization of the  $SU(2)$  group**

Let us make the change of variables  $\zeta = \zeta_1/\zeta_2$ ,  $\tau = \zeta_2$ . Then the polynomials  $f$  will be unambiguously determined by the functions

$$\varphi(\zeta) = \sum_{m=-j}^j a_m \zeta^{j+m}. \quad (5)$$

So, it is possible to construct the  $SU(2)$  group representation in the space of the polynomials (5) from one complex variable [9]. The result of operating with  $T(\mathbf{u})$  on  $\varphi(\zeta)$  is

$$T(\mathbf{u})\varphi(\zeta) = (\beta^*\zeta + \alpha)^{2j} \varphi\left(\frac{\alpha^*\zeta - \beta}{\beta^*\zeta + \alpha}\right) \quad (6)$$

Using the last equation and the parameterization of the  $u$  matrix by the Euler angles, we can find the angular momentum operators

$$j_x = j\zeta + \frac{1}{2}(1 - \zeta^2)\frac{d}{d\zeta}, \quad j_y = -i j\zeta + \frac{i}{2}(1 + \zeta^2)\frac{d}{d\zeta}, \quad j_z = -j + \zeta\frac{d}{d\zeta}. \quad (7)$$

The expression for the scalar product in space of the functions (5) is

$$(\varphi_1, \varphi_2) = \int \varphi_1^*(\zeta)\varphi_2(\zeta) \frac{(2j+1)d^2\zeta}{\pi(1+|\zeta|^2)^{2j+2}}. \quad (8)$$

The normalized eigenfunctions of the  $j_z$  operator with the eigenvalue  $m$  has the form

$$\psi_{jm} = [(2j)!/(j+m)!(j-m)!]^{1/2} \zeta^{j+m}. \quad (9)$$

The probability of a given orientation of the vector  $\mathbf{j}$  in the state (9) can be obtained from eqs. (8) and (9)

$$dw = \frac{(2j+1)!}{2\pi(j+m)!(j-m)!} \left(\sin \frac{\vartheta}{2}\right)^{2j-2m+1} \left(\cos \frac{\vartheta}{2}\right)^{2j+2m+1} d\vartheta d\phi. \quad (10)$$

Thus, the state (9) is described by a wave packet with maximum probability at  $\vartheta_0 = \arccos[m/(j+1/2)]$  and width  $\Delta\vartheta = (2j+1)^{-1/2}$ . In the classical limit it corresponds to a precession of the  $\mathbf{j}$  vector around the  $z$  axis.

The one-dimensional realization of the  $SU(2)$  group gives us another representation of the angular momentum operators. In this representation the orientation of the  $\mathbf{j}$  vector is determined by the angles  $\vartheta$  and  $\phi$ . On the contrary, the polar angles  $\theta$  and  $\varphi$  define the orientation of a rotor in the well known representation with spherical harmonics [10]. Relationship between these representations for  $j$  integer is realized by the angular momentum coherent state function satisfying the equation

$$(\mathbf{n}\mathbf{j})\Phi_{j\mathbf{n}}(\theta, \varphi) = j\Phi_{j\mathbf{n}}(\theta, \varphi). \quad (11)$$

where the orientation of the unit vector  $\mathbf{n}$  is determined by the angles  $\theta$  and  $\varphi$ . One can obtain

$$Y_{jm}(\theta, \varphi) = \frac{2j+1}{\pi} \int \frac{\Phi_{j\mathbf{n}}(\theta, \varphi)}{(1+|\zeta|^2)^{j+2}} \psi_{jm}(\zeta) d^2\zeta. \quad (12)$$

### 3. Quasiclassical approximation

The Bargmann representation of the  $SU(2)$  group has made it possible to separate the variables and reduce the dynamical problem to the Schrödinger equation in one complex variable. There are many problems in nuclear, atomic and molecular physics, which are described by the  $SU(2)$  invariant Hamiltonians. As a rule these Hamiltonians have a large parameter equal to the angular momentum quantum number  $j$ . So, we may use the quasiclassical approximation, which differs yet from the conventional WKB method on the real

axis. But this difference is not of principle because the Zwaan procedure [12] is used in both cases. Below we will discuss some of physical problems.

1. *One-particle motion in rotating nuclei* [3]. In heavy nuclei, particles in high- $j$  orbits play a crucial role in states with large spin. These states are described as usually in the cranking model [13] in which effects of the nuclear rotation on single-particle motion are considered by the mean Coriolis field. The Coriolis force is proportional to the single-particle angular momentum  $j$ . So, the nucleons in the high- $j$  orbits near the Fermi surface interact the most strongly with the nuclear rotation. The nuclear spin-orbit interaction leads to a unique-parity high- $j$  orbit in each major shell. Therefore  $j$  is a approximately good quantum number for such orbits, since the admixture of states with other values of  $j$  is small.

For the isolated  $j$ -level, the Schrödinger equation has the following form

$$[\epsilon_j + q(3j_x^2 - j) - \omega j_x] \psi = E\psi, \quad (13)$$

where  $\epsilon_j$  is the  $j$ -level energy in the spherical potential.  $q$  determines the splitting of the subshell levels due to the quadrupole deformation and  $\omega$  is the rotational frequency. To solve this equation, we use the angular momentum operators (7), introduce the new function  $\varphi(\zeta) = \zeta^j \psi(\zeta)$ , and make the change of the variable  $\zeta = \exp(2iz)$ . The last operation represents the conformal mapping of the  $\zeta$  plane onto the strip  $0 \leq \text{Re}z \leq \pi$ . Eq. (13) reduces to the Ince equation [14]

$$\varphi''(z) - \xi \sin(2z)\varphi'(z) + (\epsilon + 2j\xi \cos 2z)\varphi(z) = 0, \quad (14)$$

where  $\xi = 2\omega/3q$ ,  $\epsilon = 4[E - \epsilon_j + qj(j+1)]/3q$ . The substitution  $\varphi(z) = u(z) \exp[-\xi \cos(2z)/4]$  transforms the last equation to the standard form

$$u''(z) + g(z)u(z) = 0, \quad g(z) = \epsilon + (2j+1)\xi \cos 2z - \xi^2 \sin^2(2z)/4, \quad (15)$$

which is used to obtain quasiclassical solution for  $j \gg 1$ .

Two linearly independent solutions of eq. (15) in the quasiclassical approximation are

$$u_{1,2}(P) = [g(z)]^{-1/4} \exp[\pm i\phi_P(z)], \quad \phi_P(z) = i \int_{z_P}^z \sqrt{g(t)} dt, \quad (16)$$

where the coordinates of the turning points  $P$  are obtained from the equation  $g(z_P) = 0$ . The symmetry of the wave function allows us to restrict ourselves to determining the solution in the strip  $0 \leq \text{Re}z \leq \pi/2$ . In this strip there exist no more than four turning points, which lie either on the real axis or on the line  $z = \pi/2$ . The analytical continuation of the quasiclassical solutions (16) by the well known rules (see for example the book by Heading [12]) allows us to find the quantization conditions.

2. *Quasi-particle excitations in rotating nuclei* [4]. The conception of an isolated subshell can use in the more realistic model with pairing nucleon-nucleon interaction. This model was considered by the author for

a description of the backbending phenomenon. In the Hartree-Fock-Bogolyubov approximation for a high- $j$  subshell, the quasiparticle states are described by the equations for the amplitudes  $u$  and  $v$

$$\begin{aligned} [\varepsilon_j - \varepsilon_F + q(3j_x^2 - j^2) - \omega j_x]u - \Delta v &= Eu, \\ [\varepsilon_j - \varepsilon_F + q(3j_x^2 - j^2) + \omega j_x]u + \Delta v &= -Eu. \end{aligned} \quad (17)$$

where  $\Delta$  is the pair field,  $\varepsilon_F$  is the Fermi energy and  $E$  is the quasiparticle energy. The one-dimensional realization of the  $SU(2)$  group makes it possible to reduce eqs. (17) to a system of ordinary differential equations in the complex variable  $z = x + iy$ . For  $j \gg 1$ , the quantization conditions for  $E$  can be obtained from the quasiclassical solution of the eqs. (17) on the real axis if we use their symmetry. The system (17) on the real axis can be treated by the famous Stuekelberg method [15].

3. *Lipkin-Meshkov-Glick model* [7]. This model represents a system of  $N$  interacting fermions occupying the two  $N$ -fold degenerate levels separated by the energy  $\varepsilon$  [16]. Its many-body Hamiltonian can be rewritten in terms of the angular momentum operators

$$H = \varepsilon j_z + V(j_x^2 - j_y^2), \quad (18)$$

where  $V$  is the strength of a particle interaction. We can use the quasiclassical approximation as far as  $j = N/2 \gg 1$ . The problem is simplified if only the eigenvalue spectrum of the system is to be determined. Then it is sufficient to find the quasiclassical solution on the real axis. The energy levels are determined from the quantization conditions, which can be obtained by continuing even and odd quasiclassical solutions along the positive part of the real axis. Parity conservation of the solution under continuation is equivalent to a boundary condition. Therefore, the problem is reduced to the standard WKB method.

4. *Quantization of an asymmetric top* [5]. As an example of a more complicated Lie group, we consider the quantum problem of an asymmetric top. The corresponding  $SU(2) \otimes SU(2)$  group contains six operators of angular momentum projections on the axes of moving and laboratory frames and operators  $D_{mn}^j$ , which connect physical variables in two frames. These operators have homogeneous polynomials in four complex variables as a basis in the Bargmann representation. It can be shown [5] that variables in the Schrödinger equation are separated and the problem is reduced to the Heine equation [17]

$$\frac{d^2 \psi}{dt^2} + \left[ \frac{1}{2t} - \frac{2j-1}{2(t-1)} - \frac{2j-1}{2(t-a)} \right] \frac{d\psi}{dt} + \frac{2s + j(2j-1)t}{2t(t-1)(t-a)} \psi = 0, \quad (19)$$

where  $\zeta^2 = (\sqrt{1+\xi^2} - \xi)^2 t$ ,  $a = (\sqrt{1+\xi^2} + \xi)^4$ ,  $s = [2\varepsilon - (2\xi^2 + 1)j^2]\sqrt{a}/2$ ,  $\xi^2 = (A_3 - A_2)/(A_2 - A_1)$ ,  $A_1 < A_2 < A_3$  are the rotational constants, and the eigenvalue  $\varepsilon$  is connected with the system energy  $E$  by the equation

$$E = \frac{1}{4}(A_1 + A_2)j(j+1) + \frac{1}{2}(A_2 - A_1)\varepsilon. \quad (20)$$

The Schrödinger equation for a top can be written in a more symmetrical form if we use a conformal mapping of the plane  $\zeta$  onto the strip  $0 < \operatorname{Re} z \leq 2\pi$ . We obtain

$$(\xi^2 + \sin^2 z) \frac{d^2 \varphi}{dz^2} - (j - \frac{1}{2}) \sin(2z) \frac{d\varphi}{dz} + [\varepsilon + \frac{1}{2}j(j-1) \cos 2z] \varphi = 0. \quad (21)$$

On the real axis, eq. (21) coincides with the Schrödinger equation of a top in action-angle variables [18, 19].

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**GROUP-THEORETICAL CLASSIFICATION  
OF POLYNOMIAL FUNCTIONS OF THE RIEMANN TENSOR**

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**1. Introduction.** To motivate the kind of problem we are treating, consider the *heat kernel expansion* well known to researchers in general relativity, gauge theories, and differential geometry. Let  $\Delta = g^{pq}(x)\nabla_p\nabla_q$  be the Laplace-Beltrami operator of a Riemannian manifold (with Riemann curvature tensor  $R^a{}_{bcd}(x)$ ): let  $K(t, x, y)$  be the integral kernel of the operator  $e^{t\Delta}$  (which solves the initial value problem for  $\partial_t/\partial t = \Delta\psi$ ). Then the diagonal (coincidence) value of  $K$  has an asymptotic expansion at small  $t$ ,

$$K(t, x, x) \sim (4\pi t)^{-d/2} \sum_{n=0}^{\infty} a_n(x)t^n$$

( $d = \text{dimension}$ ). The first two coefficients are well known:

$$a_1 = \frac{1}{6} R, \quad a_2 = \frac{1}{30} \Delta R + \frac{1}{72} R^2 - \frac{1}{180} R^{pq} R_{pq} + \frac{1}{180} R^{pqrs} R_{pqrs}.$$

(Recall that the Ricci tensor is  $R_{ab} \equiv R^p{}_{apb}$  and the curvature scalar is  $R \equiv R^q{}_{q}$ .) The third term is also known [e.g., 6]: it is a linear combination of 17 terms, of which  $R^3$ ,  $R^{pq} R_{pq}$ ,  $R^{pqrs} R_{pqrs}$ , and  $\Delta^2 R$  are typical. It is clear that working to higher orders is giving rise to a combinatorial explosion.

Mathematicians and physicists have proposed a variety of algorithms for calculating  $a_n$  [e.g., 1, 2, 4, 6-7, 12, 13, 15, 17]. Advances in computer hardware and software are making high-order calculations increasingly practical. (*MathTensor*, the *Mathematica* tensor analysis program by Parker and Christensen [14], has been largely motivated by precisely this problem.) However, all methods eventually run into the same difficulty: combining a large number of similar terms into some comprehensible normal form. The symmetries of the Riemann tensor make this problem nontrivial. For example,  $R^{pqrs} R_{pqrs}$  is not linearly independent of  $R^{pq} R_{pq}$ , but this fact is not immediately

obvious from consideration of the index symmetries of each factor separately. A different kind of example is provided by  $R^p_q R^q_r R^r_s R^s_p$  — the trace of the fourth power of the Ricci tensor, regarded as a matrix. By a well known theorem of matrix theory, it is expressible as a polynomial in the lower-degree traces if  $d < 4$ .

Recognizing all such relationships, general and dimension-dependent, is a problem in group representation theory. The groups involved are  $S_n$  (the permutations of a tensor's indices),  $GL(d)$ , and  $O(d)$ . The methods required are known to physicists using group theory in atomic and nuclear physics [e.g., 18]. The lore is that associated with Young diagrams: indeed, the Young diagram representing the symmetries of  $R_{abcd}$  is the one with 4 blocks arranged in a square.

**2. The basis problem for Riemann polynomials.** Let us make the problem more precise with some formal definitions:

A *Riemann monomial* is an expression formed by tensor products and contractions from the Riemann tensor  $R$  and its covariant derivatives. A *Riemann polynomial* is a linear combination of these. (Actually, because of the rule relating commutation of covariant derivatives to  $R$ , we should work with *cosets* modulo terms of lower order and higher degree.)

Let  $\mathcal{R}_{s,q}^r$  be the vector space of Riemann polynomials of *rank*  $r$  (number of free tensor indices), *degree*  $q$  (number of factors  $R$ ), and *order*  $s$  (number of derivatives of  $g$  = number of covariant derivatives plus twice  $q$ ). Note that the heat kernel coefficient  $a_n$  belongs to  $\bigoplus_{q=1}^n \mathcal{R}_{2n,q}^0$ . We can further subdivide according to how the covariant derivatives are distributed among the factors: for example,  $\mathcal{R}_{6,2}^0 = \mathcal{R}_{\{2,0\}}^0 \oplus \mathcal{R}_{\{1,1\}}^0$ , where  $R^{pqrs} R_{pr,qs}$  belongs to the first of these sets and  $R^{pqr} R_{pr,q}$  to the second.

We can now state three increasingly ambitious versions of our problem: For  $\mathcal{R}_{s,q}^r$ .

- (1) Find its dimension — the number of elements in a basis.
- (2) Construct such a basis — list its elements. We want to choose the *best* basis — it should be “natural” or “simplest” or ...
- (3) Provide a *normal form algorithm* — i.e., tell how to express an arbitrary element in terms of the basis.

In view of the nonuniqueness of the basis, one might add a fourth objective:

- (4) Provide formulas or computer programs to convert from one basis to another.

**3. Tools.** The concepts employed include *irreducible representation*, *outer product*, *plethysm*, *branching rules*, *modification rules* [3, 8, 9, 11, 16, 18]. (Since there is no space here for a course in group representation theory, we can only cite the jargon.) A major tool is the computer program SCHUR written by Wybourne and his students [19].

**4. Results so far** [5]. On objective (1): SCHUR easily provides us with the number of scalars through order 12. For example, in order 6 one gets the table

| <i>class</i>              | 2 | 3 | 4 | 5 | 6 | <i>total</i> |
|---------------------------|---|---|---|---|---|--------------|
| $\mathcal{R}_{6,1}^0$     | 1 |   |   |   |   | 1            |
| $\mathcal{R}_{\{2,0\}}^0$ | 1 | 2 | 1 |   |   | 4            |
| $\mathcal{R}_{\{1,1\}}^0$ | 1 | 2 | 1 |   |   | 4            |
| $\mathcal{R}_{6,3}^0$     | 1 | 2 | 3 | 1 | 1 | 8            |
| Total                     | 4 | 6 | 5 | 1 | 1 | 17           |

where the column heading is the minimal dimension in which the object is independent of simpler ones. We find 92 scalars in order 8 (cf. [1]), 668 in order 10, and 6721 in order 12. (Since order is related to dimension in applications of  $a_n$ , these last are potentially relevant to Kaluza-Klein and string theories.)

On objective (2): We have lists of all the scalars through order 8 and all the higher rank tensors through order 6. For example, the table for  $\mathcal{R}_{5,2}^5$  reads

| <i>tensor</i>           | <i>representation</i>                      | <i>dimension</i> |
|-------------------------|--------------------------------------------|------------------|
| $R_{ab}R_{cd,e}$        | $[5]+2[4,1]+2[3,2]+[3,1^2]+[2^2,1]$        | 30               |
| $R_{,a}R_{bcde}$        | $[3,2]+[2^2,1]$                            | 10               |
| $RR_{abcd,e}$           | $[3,2]$                                    | 5                |
| $R^p_{a;b}R_{pcde}$     | $[4,1]+2[3,2]+2[3,1^2]+2[2^2,1]+[2,1^3]$   | 40               |
| $R_{ab}{}^p R_{pcde}$   | $[4,1]+[3,2]+[3,1^2]+[2^2,1]$              | 20               |
| $R^p_a R_{pbcd,e}$      | $[4,1]+2[3,2]+[3,1^2]+[2^2,1]$             | 25               |
| $R^p q_{ab} R_{pqcd,e}$ | $[4,1]+[3,2]+[3,1^2]+[2^2,1]$              | 20               |
| $R^p q_a R_{pcqd,e}$    | $[5]+[4,1]+2[3,2]+[3,1^2]+[2^2,1]+[2,1^3]$ | 30               |

The dimension stated is the number of independent index permutations, and the decomposition of the corresponding  $S_5$  and  $O(d)$  representation into irreducibles is given.

Objectives (3) and (4) are implicit in the foregoing results, but not yet realized in practice. Their proper embodiment is in computer software, not a published document.

The methods shown here can be applied to problems involving other tensors in addition to  $R$ .

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## IV. Coherent States

## Coherent States : from Group Representations to Relativistic Quantum Frames

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### 1. Introduction : What is a quantum frame ?

Frames are familiar objects, from the early perception of the world by children to space-time reference frames used in learning classical physics. In the latter case, a frame is simply a basis  $\{\epsilon_i, i = 1, \dots, 4\}$  of  $\mathbb{R}^4$ , orthogonal or not. The definition of the frame includes the law of transformation of the vectors  $\epsilon_i$  under the appropriate relativity group, such as the Galilei or the Poincaré group. In quantum physics, the same role is played by an orthonormal basis  $\{\psi_i\}$  in a Hilbert space  $\mathcal{H}$ , such as the familiar  $L^2(\mathbb{R}^3)$  over nonrelativistic configuration space or the  $L^2$  space over mass shell corresponding to the Wigner representation of the Poincaré group. Every such basis  $\{\psi_i\}$  may be associated to some physical observable, represented by a self-adjoint operator  $A$  with discrete spectrum  $\{a_i\}$ :

$$A = \sum_i a_i |\psi_i\rangle\langle\psi_i| = \sum_i a_i P_{a_i}. \quad (1)$$

However, whereas every vector in  $\mathcal{H}$  may be expanded in the basis  $\{\psi_i\}$ , not every observable may have a *diagonal* representation

$$B = \sum_i b_i P_{a_i}. \quad (2)$$

only those that commute (strongly) with  $A$ . This limitation may be overcome if one uses the generalized notion of frame introduced in a series of papers with S.T.Ali [1-3]. Given a measure space  $\{X, \nu\}$  of physical significance, a *frame* is defined as a triplet  $\{\mathcal{H}, F, A\}_n$ , where  $\mathcal{H}$  is a Hilbert space and  $F : X \rightarrow \mathcal{B}(\mathcal{H})_n^+$  a measurable function with values in the positive rank  $n$  operators on  $\mathcal{H}$ , such that the integral

$$\int_X F(x) d\nu(x) = A \quad (3)$$

converges weakly to a bounded positive operator  $A$  on  $\mathcal{H}$ , with bounded inverse. In such a frame, the state  $\psi \in \mathcal{H}$  may be 'resolved' through the relation:

$$A\psi = \int_X F(x)\psi d\nu(x), \quad (4)$$

and an observable  $B$  reads

$$B = \int_X b(x)F(x)d\nu(x), \quad (5)$$

where the function  $b(\cdot)$  is the symbol of  $B$ . Conversely, the function  $b$  is determined from experiment (necessarily in a sampled fashion) and the operator  $B$  is reconstructed through Eq.(5).

The usual process of 'quantization' is in fact a restricted version of this correspondence : the space  $\{X, \nu\}$  has a classical, macroscopic meaning (for instance, a continuous phase space) and Eq.(5) associates to the function  $b$  the operator  $B$  quantized according to the chosen frame. Thus a frame is a quantization over  $X$ , and our definition generalizes the usual quantization algorithm.

An interesting situation arises when the system has a symmetry group  $G$  (relativity group : Galilei, Poincaré, ...) and the phase space  $X$  is taken as a *coadjoint orbit* of  $G$ . According to Kirillov's theory, the latter corresponds to a unitary representation  $U$  of  $G$  in a Hilbert space  $\mathcal{H}$  [4]. If  $U$  is square integrable over  $X$  (see below), the theory developed in [1-3] yields an overcomplete family of *coherent states* (CS) based on  $X$ , which constitutes a frame as described above (possibly in a generalized sense, if  $A^{-1}$  is unbounded). Then the representation space  $\mathcal{H}$  is mapped unitarily onto a reproducing kernel Hilbert space, which for  $n = 1$  is simply a closed subspace of  $L^2(X, \nu)$ . This is the essence of the method of CS quantization [5]; for more details we refer to the contribution of Ali in these proceedings or to [6].

In the sequel we will review the theory of CS over homogeneous spaces developed in [1-3] and construct several classes of concrete quantum frames for the most relevant relativity groups, the Galilei and the Poincaré groups, both taken for simplicity in one space and one time dimensions.

## 2. The standard construction of coherent states

Let  $G$  be a locally compact group, with left Haar measure  $dg$ , and  $U$  a strongly continuous, irreducible, unitary representation of  $G$  into a Hilbert space  $\mathcal{H}$ . Assume  $U$  is *square integrable*, i.e. there exists a vector  $\eta \in \mathcal{H}$  such that :

$$c(\eta, \phi) = \int_G |\langle U(g)\eta | \phi \rangle|^2 dg < \infty, \forall \phi \in \mathcal{H} \quad (6)$$

(equivalently,  $U$  belongs to the discrete series). Choose a fixed vector  $\eta$  that satisfies the admissibility condition (6) and normalize it by  $c(\eta, \eta) = 1$ . Then the orbit of  $\eta$  under  $U$ , namely  $\mathfrak{S} = \{\eta_g = U(g)\eta, g \in G\}$ , is an overcomplete family of vectors, called *coherent states* associated to the representation  $U$ . In addition, the family  $\mathfrak{S}$  determines a *resolution of the identity*

$$\int_G |\eta_g\rangle \langle \eta_g| dg = I. \quad (7)$$

More precisely, the map  $W_\eta : \phi \mapsto \langle \eta_g | \phi \rangle$  is an isometry from  $\mathcal{H}$  onto a *closed* subspace of  $L^2(G, dg)$ , which is characterized by the *reproducing kernel*  $K(g, g') = \langle \eta_g | \eta_{g'} \rangle$ .

Let  $H_\eta$  denote the subgroup of  $G$  that leaves  $\eta$  invariant up to a phase :  $U(h)\eta = \exp i\alpha(h)\eta$ . Then the integrand in (6) does not depend on  $g$ , but only on the coset  $gH_\eta$ . Let  $\nu$  be an invariant measure on the coset space

$X = G/H_\eta$ . Then Perelomov [7] has shown that the whole construction of CS may be done under the weaker admissibility condition

$$\int_X |\langle U(g)\eta | \phi \rangle|^2 d\nu(x) < \infty, \quad \forall \phi \in \mathcal{H} \quad (x \equiv gH_\eta), \quad (8)$$

the difference being that the CS (rays)  $\eta_x$  are now indexed by points  $x \in X$ . In particular, the resolution of the identity (7) reads now

$$\int_X |\eta_x\rangle\langle\eta_x| d\nu(x) = I. \quad (9)$$

This formalism applies to a large number of interesting cases: Weyl-Heisenberg group (canonical CS), 'ax + b' group (wavelets), compact groups, discrete series representations of noncompact groups.

However the method fails for the Galilei or the Poincaré group, and other groups of similar structure. Consider for simplicity the 1 + 1 dimensional Poincaré group  $\mathcal{P}_+^\uparrow(1,1)$ . First, the familiar Wigner representation  $U_W$  is *not* square integrable ( $\mathcal{P}_+^\uparrow(1,1)$  has no discrete series!). Next, the quotient  $\Gamma = \mathcal{P}_+^\uparrow(1,1)/T$ , where  $T$  is the subgroup of time translations, may be interpreted as phase space (it is the coadjoint orbit corresponding to  $U_W$ ), with coordinates  $(q, p)$  and the unique invariant measure  $dq dp$ . Since  $T$  is not the stability subgroup of any vector, Perelomov's method indeed does not apply. However it is possible to find a *section*  $\sigma : \Gamma \rightarrow \mathcal{P}_+^\uparrow(1,1)$  and a vector  $\eta$  such that:

$$\int_\Gamma |\langle U_W(\sigma(q,p))\eta | \phi \rangle|^2 dq dp < \infty, \quad \forall \phi \in \mathcal{H}. \quad (10)$$

For such an admissible vector  $\eta$ , the family  $\mathfrak{S}_\sigma = \{\eta_{q,p} = U_W(\sigma(q,p))\eta, (q,p) \in \Gamma\}$  has all the properties of a CS system. However, instead of the resolution of the identity (9), one gets:

$$\int_\Gamma |\eta_{q,p}\rangle\langle\eta_{q,p}| dq dp = A, \quad (11)$$

where  $A$  is a positive operator, bounded with bounded inverse, and  $A \neq \lambda I$  in general. In other words, the set  $\mathfrak{S}_\sigma = \{\eta_{q,p}\}$  is a frame. The physical interpretation of this construction is clear. The state  $\eta$  is a quantum 'probe', the quantum equivalent of the analyzing wavelet in signal analysis. The states  $\{\eta_{q,p} = U_W(\sigma(q,p))\eta\}$  are obtained by displacing the probe  $\eta$  covariantly along the phase space  $\Gamma$ , and their collection  $\mathfrak{S}_\sigma$  is the frame in terms of which we analyze the system, as discussed in Section 1. The interesting fact is that this pattern extends to the general case.

### 3. A general theory of coherent states

Let  $G, U, \mathcal{H}$  be as before,  $H$  a closed subgroup of  $G$ ,  $X = G/H$  with an invariant measure  $\nu$  and  $\sigma : X = G/H \rightarrow G$  a Borel section. Then we say that  $U$  is *square integrable mod*  $(H, \sigma)$  for the vector  $\eta \in \mathcal{H}$  if the integral

$$\int_X U(\sigma(x))|\eta\rangle\langle\eta|U(\sigma(x))^* d\nu(x) \quad (12)$$



converges weakly to a bounded positive invertible operator  $A_\sigma$  on  $\mathcal{H}$ .

In that case, the general theory developed in [1-3] yields the following elements :

- an overcomplete family of CS based on  $X$  :  $\mathfrak{S}_\sigma = \{\eta_{\sigma(x)} = U(\sigma(x))\eta, x \in X\}$ ;

- a resolution

$$\int_X |\eta_{\sigma(x)}\rangle\langle\eta_{\sigma(x)}| d\nu(x) = A_\sigma; \quad (13)$$

- a linear map  $W_K : \mathcal{H} \rightarrow L^2(X, d\nu)$ , defined by  $(W_K\phi)(x) = \langle\eta_{\sigma(x)}|\phi\rangle$ , such that the range  $\mathcal{H}_K$  of  $W_K$  is complete with respect to the scalar product  $\langle\Phi|W_K A_\sigma^{-1} W_K^{-1}\Psi\rangle$  and  $W_K$  is unitary from  $\mathcal{H}$  onto  $\mathcal{H}_K$ ; in addition,  $\mathcal{H}_K$  has a reproducing kernel, given by  $K_\sigma(x, y) = \langle\eta_{\sigma(x)}|A_\sigma^{-1}\eta_{\sigma(y)}\rangle$  if

$$\eta_{\sigma(y)} \in \mathcal{D}(A_\sigma^{-1}).$$

Furthermore, the relation (13) may be transformed into a genuine resolution of the identity if one introduces the vectors  $\tilde{\eta}_x = T(x)\eta_{\sigma(x)}$ , called *quasi-coherent* states, where  $T(x)$  are suitable bounded operators (essentially  $A_\sigma^{-1/2}$  acting 'fiberwise').

Suppose now this setup describes a quantum system, with symmetry group  $G$ . Then, if  $A_\sigma$  has a bounded inverse, the set  $\mathfrak{S}_\sigma$  defines a (continuous) quantum frame and we have the same physical interpretation as in the Poincaré case above. Of course, the whole construction depends on the choice of a particular section  $\sigma$ , but this dependence is only apparent. Indeed, if two sections  $\sigma, \sigma'$  are admissible in the sense of (12), then the corresponding CS systems  $\mathfrak{S}_\sigma, \mathfrak{S}_{\sigma'}$  are in one to one correspondence: all those CS systems are physically equivalent.

This general theory covers all the (Perelomov) cases described above, but also that of the relativity groups (Euclidean, Galilei, Poincaré), that we shall describe in Sections 5-6 below. It may also be generalized in several directions:

- The projector  $|\eta\rangle\langle\eta|$  in (12) may be replaced by an operator of rank  $n$ .
- The measure  $\nu$  may be only quasi-invariant under the action of  $G$  (this allows to consider some infinite-dimensional groups) [6].
- It may be necessary to replace the section  $\sigma$  in (12) by the combined map  $\sigma \circ f$ , where  $f : X \rightarrow X$  is a homeomorphism (reparametrization); this happens, for instance [8], for massless representations of  $\mathcal{P}_+^\uparrow(1, 1)$ .

#### 4. Reproducing triples, frames

Actually the construction of CS just outlined is a particular case of a much more general setup, where no reference is made to group theory. Let  $(X, \nu)$  be a measure space,  $\mathcal{H}$  a Hilbert space,  $A$  a bounded positive invertible operator on  $\mathcal{H}$ ,  $F : X \rightarrow \mathcal{B}(\mathcal{H})^+$  a measurable positive operator valued function. Then  $\{\mathcal{H}, F, A\}$  is called a *reproducing triple* if the following relation holds, in the sense of a weak integral:

$$\int_X F(x) d\nu(x) = A. \quad (14)$$

Under these conditions, a full theory of CS based on  $X$  may be derived, along the same lines as above (see [1-3]).

The most interesting case arises when  $F(x)$  has constant finite rank  $n$ , and  $A^{-1}$  is also bounded: then  $\{\mathcal{H}, F, A\}$  is called a *frame*, as already mentioned in Section 1. Indeed the integrability condition (14) leads to the following inequalities:

$$m(A) \|\phi\|^2 \leq \sum_{i=1}^n \int_X |(\eta_x^i | \phi)|^2 d\nu(x) \leq M(A) \|\phi\|^2, \quad \forall \phi \in \mathcal{H}, \quad (15)$$

where  $\{\eta_x^i\}$  are the eigenvectors of  $F(x)$ , and  $m(A), M(A)$  the infimum and the supremum of the spectrum of  $A$ , respectively. If  $X$  is a discrete space, with  $\nu$  a counting measure, (15) means that  $\{\eta_x^i, i = 1, \dots, n, x \in X\}$  is a frame in the usual sense of nonorthogonal expansions [9]. Hence we call the general triple  $\{\mathcal{H}, F, A\}$  above a continuous frame. Its *width* is the positive number  $w(A) = [M(A) - m(A)][M(A) + m(A)]^{-1}$ . If  $A = \lambda I$ , i.e.  $w(A) = 0$ , the frame is said to be *tight*.

In the rest of this paper we shall construct such frames for the two main relativity groups in 1 + 1 dimensions: these are the *relativistic quantum frames* announced in Section 1.

#### 5. Galilean quantum frames

We apply the general method of Section 3 to the (extended) Galilei group:

- $\tilde{G} = \{g = (\omega, b, a, v)\}$ , where the parameters denote, respectively: phase, time translation, space translation, Galilean boost;
- unitary irreducible representation  $U_m$  on  $\mathcal{H}^G = L^2(\mathbb{R}, dk)$ :

$$(U_m(g)\psi)(k) = \exp i(\omega + \frac{k^2}{2m}b - ka) \psi(k - mv); \quad (16)$$

- phase space:  $\Gamma = \tilde{G}/\Omega \times T$  with coordinates  $(q, p)$  and invariant measure  $dq dp$  ( $\Omega$  = phase transformations,  $T$  = time translations).

Defining the Borel section  $\sigma_o : \Gamma \rightarrow \tilde{G}$  as:  $\sigma_o(q, p) = (0, 0, q, \frac{p}{m})$ , and the Galilean CS  $\eta_{q,p} = U_m(\sigma_o(q, p))\eta$ , a straightforward calculation shows that:

$$\int_{\Gamma} |\eta_{q,p}\rangle \langle \eta_{q,p}| dq dp = 2\pi \|\eta\|^2. \quad (17)$$

i.e. we get a tight frame for any  $\eta \in \mathcal{H}^G$ .

Given any other section  $\sigma : \Gamma \rightarrow \tilde{G}$  of the form  $\sigma(q, p) = \sigma_o(q, p) (0, q, \theta(p), 0, 0)$ , define the CS  $\eta_{q,p}^\sigma = U_m(\sigma(q, p))\eta$  and the corresponding resolution generator

$$A_\sigma^\eta = \int_{\Gamma} |\eta_{q,p}^\sigma\rangle \langle \eta_{q,p}^\sigma| dq dp. \quad (18)$$

Then the admissibility condition (12) must be strengthened by a support condition on  $\eta$ : the Galilean probe  $\eta \in \mathcal{H}^G$  is said to be admissible mod  $(\Omega \times T, \sigma)$  if it satisfies the following two conditions:

- (i)  $\forall k \in \mathbb{R}, p, \theta(k-p) \geq m \Rightarrow \eta(p) = 0$ ;
- (ii) the integral (18) converges weakly to a bounded positive operator, with bounded inverse.

Given an admissible  $\eta$ , the resolution operator  $A_\sigma^\eta$  turns out to be a multiplication operator:  $(A_\sigma^\eta \psi)(k) = \mathcal{A}_\sigma^\eta(k) \psi(k)$ , where

$$\mathcal{A}_\sigma^\eta(k) = 2\pi \int_{p, \theta(k-p) < m} (1 - \frac{1}{m} p \cdot \theta(k-p))^{-1} |\eta(p)|^2 dp. \quad (19)$$

## 6. Einsteinian quantum frames

We turn now to the 1+1 dimensional Poincaré group  $\mathcal{P}_+^1(1,1) = SO_o(1,1) \wedge \mathbb{R}^2$  and follow the same steps as before (this case has been treated at length in [1-3]).

- $\mathcal{P}_+^1(1,1) = \{g = (a, \Lambda_p)\}$ , with  $a \in \mathbb{R}^2$  and  $\Lambda_p \in SO_o(1,1)$ , a Lorentz boost indexed by  $p = (p_o, p)$ , where  $p_o = \sqrt{p^2 + m^2}$ ;
- Wigner representation  $U_m$  on  $\mathcal{H}^E = L^2(\mathcal{V}_m^+, \frac{dk}{k_o})$  where  $\mathcal{V}_m^+ = \{k = (k_o, k), k^2 = k_o^2 - k^2 = m^2, k_o > 0\}$ ;

$$(U_m(g)\psi)(k) = \exp ik \cdot a \psi(\Lambda_p^{-1}k); \quad (20)$$

In this representation, the energy and momentum operators read, respectively:

$$P_o \psi(k) = k_o \psi(k), \quad P \psi(k) = k \psi(k) \quad (21)$$

- phase space:  $\Gamma = \mathcal{P}_+^1(1,1)/T$  with coordinates  $(q, p)$  and invariant measure  $dq dp$ ;
- Galilean section  $\sigma_o : \Gamma \rightarrow \mathcal{P}_+^1(1,1) : \sigma_o(q, p) = ((0, q), \Lambda_p)$ .

We consider now an arbitrary section of the form (*affine section*) :

$$\sigma(q, p) = \sigma_o(q, p)((q, \theta(p), 0), I) \equiv ((\hat{q}_o, \hat{q}), \Lambda_p). \quad (22)$$

If we parametrize the section  $\sigma$  in terms of a momentum-dependent speed  $\beta$ :  $\hat{q}_o = \beta(p)\hat{q}, p|p|^{-1}$ , we see that the vector  $\hat{q}$  is time-like, resp. light-like, space-like, if  $|\beta| > 1$ , resp.  $|\beta| = 1, |\beta| < 1$ . It is useful to introduce also the dual speed  $\beta^*(p) = [|p| - p_o\beta(p)] \cdot [p_o - |p|\beta(p)]^{-1}$ . These two speeds obey several interesting identities, for instance:  $(p_o - |p|\beta)(p_o - |p|\beta^*) = m^2$ .

We define now the Poincaré CS  $\eta_{q,p} = U_m(\sigma(q, p))\eta$  and the corresponding resolution generator

$$A_o^\eta = \int_{\Gamma} |\eta_{q,p}\rangle \langle \eta_{q,p}| dq dp. \quad (23)$$

Then a detailed analysis [1-3] yields the following results:

- (i) Any time-like section, i.e.  $|\beta| > 1$ , implies support conditions on the probe  $\eta$ .
- (ii) Any space-like or light-like section, i.e.  $|\beta| \leq 1$ , produces a frame iff the probe is of finite energy:  $\eta \in \mathcal{D}(P_o^{1/2})$ .
- (iii) The width of such a frame is bounded:  $w(A_o^\eta) \leq \langle |P| \rangle_\eta \cdot \langle P_o \rangle_\eta^{-1}$
- (iv) More precisely, for  $|\beta| \leq 1$  and  $\eta \in \mathcal{D}(P_o^{1/2})$ ,  $A_o^\eta$  is a positive bounded multiplication operator with bounded inverse:  $(A_o^\eta \psi)(k) = \mathcal{A}_o^\eta(k) \psi(k)$ , where

$$\mathcal{A}_o^\eta(k) = \int_{v_m^+} \frac{2\pi m}{p_o + p \operatorname{sign}(k-p) \beta^*(\Lambda_p^{-1}k)} |\eta(p)|^2 \frac{dp}{p_o}. \quad (24)$$

The spectrum of the operator  $A_o^\eta$  obeys the following universal bounds:

$$\inf_{|\beta| \leq 1} \inf \operatorname{Spec}(A_o^\eta) = 2\pi \left\langle \frac{P_o - |P|}{m} \right\rangle_\eta, \quad \sup_{|\beta| \leq 1} \sup \operatorname{Spec}(A_o^\eta) = 2\pi \left\langle \frac{P_o + |P|}{m} \right\rangle_\eta.$$

Among the class of admissible sections, the following particular cases are remarkable:

- (1) Lorentz section,  $\beta = |p| p_o^{-1} \Leftrightarrow \beta^* = 0$ :  $A_o^\eta = 2\pi m \langle P_o^{-1} \rangle_\eta I$ , thus the frame is tight for any admissible probe  $\eta$ .
- (2) Galilean section,  $\beta = 0 \Leftrightarrow \beta^* = |p| p_o^{-1}$ :  $A_{o_o}^\eta = \frac{2\pi}{m} [\langle P_o \rangle_\eta I - \langle P \rangle_\eta \frac{P}{P_o}]$ , and  $w(A_{o_o}^\eta) = \langle \langle P \rangle_\eta \rangle \cdot \langle P_o \rangle_\eta^{-1}$ , thus the frame can be made tight with a 'polarized' probe, i.e.  $\langle P \rangle_\eta = 0$ .
- (3) Self-dual section,  $\beta = \beta^* = |p| (p_o + m)^{-1}$ : the frame is never tight, for any  $\eta$ ; for instance,  $w(A_o^\eta) = [\langle P_o \rangle_\eta - m \|\eta\|^2] \cdot [\langle P_o \rangle_\eta + m \|\eta\|^2]^{-1}$ , if  $\langle P \rangle_\eta = 0$ .
- (4) Light-like sections,  $\beta = -\beta^* = \pm 1$ : here too the frame is never tight.

## 7. Final remarks

A similar analysis can be made for the anti-deSitter group  $SO_o(1,2)$ , with phase space:  $SO_o(1,2)/SO(2) \simeq \mathcal{D}$ , the unit open disk in  $\mathbb{C}$ , and a discrete series representation [10]. However, the resolution generator  $A_\eta^0$  so obtained is complicated and it seems difficult to derive an explicit admissibility condition for a general probe  $\eta$ .

Yet this case, already treated in [2], is interesting, both physically and mathematically, because the three relevant groups are related by a process of contraction:

$$SO_o(1,2) \xrightarrow{\kappa \rightarrow 0} \mathcal{P}_+^1(1,1) \xrightarrow{c \rightarrow \infty} \tilde{G}. \quad (25)$$

The same is true for the respective representations that yield CS, and also for a particular type of 'quantum probe', namely Gaussian states  $\eta_G, \eta_E, \eta_S$ :

$$\eta_S \xrightarrow{\kappa \rightarrow 0} \eta_E \xrightarrow{c \rightarrow \infty} \eta_G, \quad (26)$$

$$\eta_G(k) \propto \epsilon^{-\frac{k^2}{2m\tilde{c}}}, \quad \eta_E(k) \propto \epsilon^{-\frac{k^2}{2\tilde{c}}}, \quad \eta_S(z) \propto \left(\frac{1-|z|^2}{1+z^2}\right)^{m/\kappa} \epsilon^{-\frac{2m/c}{1+z^2}} \quad (z \in \mathcal{D}). \quad (27)$$

The privileged position of those Gaussian CS is obviously related to the central role of the harmonic oscillator in quantum physics (see [10] for a detailed study).

An important question would be to clarify the relationship between the method of quantization described here, which implies the choice of a particular section, and the form of the commutation relations between the operators of position and momentum. In the Poincaré case, preliminary results [10] indicate that those CR take their canonical form only for the Galilean section  $\sigma_g$ , with the position operator in the Newton-Wigner form. Another interesting point is the extension of the analysis to non-affine sections.

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## VECTOR COHERENT STATE REPRESENTATIONS AS INDUCED REPRESENTATIONS

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This brief review, based on a recent article [1], presents the language of coherent states [2] as a natural setting for the various constructions of induced group representations [3]. The coherent state language is intuitively natural and highly practical and, as shown in numerous applications, give explicit constructions of the matrices of induced representations with a wide variety of potential applications.

### 1. Vector coherent state representations

Let  $T$  be a representation of a group  $G$  on a Hilbert space  $V$  with a hermitian inner product  $\langle \Psi | \Psi' \rangle$ . Let  $\{\Phi_\nu\}$  be an orthonormal basis for a subspace  $U \subset V$ . Then a state  $\Psi \in V$  can be represented as a vector-valued function,  $\psi : G \rightarrow U$  with

$$\psi(g) = \sum_\nu \Phi_\nu \langle \Phi_\nu | T(g)\Psi \rangle. \quad (1)$$

The set of such vector-valued functions carries a VCS representation  $\Gamma$  of  $G$  in which

$$[\Gamma(g)\psi](g') = \psi'(g') = \psi(g'g). \quad (2)$$

A VCS representation is isomorphic to a subrepresentation of  $T$ . It can also be viewed as a representation of  $G$  induced from a representation  $\rho$  of a subgroup  $H \subset G$ , where  $\rho$  is defined by

$$\rho(h)\Phi = T(h)\Phi, \quad \forall h \in H, \Phi \in U, \quad (3)$$

and  $H$  is the stability subgroup,

$$H = \{h \in G | T(h)\Phi \in U, \forall \Phi \in U\}, \quad (4)$$

of elements of  $G$  that leave the subspace  $U \subset V$  invariant. Since the projection operator

$$\pi : V \rightarrow U; \Psi \mapsto \sum_\nu \Phi_\nu \langle \Phi_\nu | \Psi \rangle \quad (5)$$

is the identity operator on the subspace  $U \subset V$ , the VCS wave functions satisfy

$$\psi(hg) = \sum_\nu \Phi_\nu \langle \Phi_\nu | T(hg)\Psi \rangle = \rho(h)\psi(g), \quad h \in H, \quad (6)$$

as required for an induced representation.

### 2. A more general VCS inducing construction

Let  $T$  be a representation of a group  $G$  on a complex linear space  $V$ . Let  $H \subset G$  be compact and have a representation  $\rho$  on a complex linear space  $U$ . We no longer require  $U$  to be a subspace of  $V$  or  $V$  to be a Hilbert space. However, we do require the restriction of  $T$  to  $H$  to contain a subrepresentation equivalent to  $\rho$  in the sense that there exists an  $H$ -intertwining operator  $\pi : V \rightarrow U$  such that

$$\pi \circ T(h) = \rho(h) \circ \pi, \quad \forall h \in H. \quad (7)$$

Then, the set of  $G \rightarrow U$  functions,

$$\{\psi | \psi(g) = \pi \circ T(g)\Psi, \Psi \in V\}, \quad (8)$$

is a module for a VCS representation  $\Gamma$  of  $G$  with

$$[\Gamma(g)\psi](g') = \psi(g'g). \quad (9)$$

The relationship  $\psi(hg) = \rho(h)\psi(g)$ , for  $h \in H$ , suggests an algorithm for constructing VCS representations. We first seek a factorization of  $G$

$$g = hk, \quad g \in G, \quad h \in H^+, \quad k \in K, \quad (10)$$

where  $H^+$  is a subgroup of  $G^c$  that contains  $H$ , and  $K$  is a subset of  $G^c$ . We also require that the representation  $\rho$  of  $H$  should extend naturally to a representation of  $H^+$  and the representation  $T$  should extend to  $K \subset G^c$ . We then choose a suitable set of  $K \rightarrow U$  functions  $\{\psi\}$  and extend them to  $G$  by means of the equation

$$\psi(hk) = \rho(h)\psi(k). \quad (11)$$

A suitable set is one which extends to a module for a representation  $\Gamma$  with the group action given by eq. (9). One way to choose a suitable set of functions, is by consideration of a representation  $T$  of  $G$  (e.g., the regular representation) and an  $H$ -intertwining operator. Finally, we seek appropriate invariant subsets of wave functions.

### 3. Holomorphic representations

Suppose  $G$  is a connected reductive Lie group and that  $H \subset G$  is compact and contains a Cartan subgroup  $K$  for  $G$ . One can think of the sequence  $G \supset H \supset K$  as a chain of matrix groups of the form

$$\left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \right\} \supset \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \supset \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\}. \quad (12)$$

The elements of  $G$  have the factorization

$$g = p(g)z(g), \quad p(g) \in P, \quad z(g) \in N_+, \quad (13)$$

where  $P \supset H^c$  and  $N_+$  are, respectively, the parabolic and nilpotent subgroups of  $G^c$

$$P = \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \right\}, \quad N_+ = \left\{ \begin{pmatrix} 1 & 0 & z_1 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \right\}. \quad (14)$$

We assume the existence of a representation  $T$  of  $G$  on a Hilbert space  $V$  and take  $U$  to be an  $H$ -invariant subspace of vectors in  $V$  that are annihilated by the raising operators of (the Lie algebra of)  $N_+$ . Then

$$\begin{aligned} T(z)\Phi &= \Phi, \quad \forall z \in N_+, \\ T(h)\Phi &= \rho(h)\Phi, \quad \forall h \in H, \end{aligned} \quad (15)$$

where  $\rho$  is a unitary representation of  $H$ . Let  $\pi$  be the orthogonal  $V \rightarrow U$  projection. Then each vector  $\Psi \in V$  is represented by the  $U$ -valued VCS wave function

$$\psi : G \rightarrow U : g \mapsto \psi(g) = \pi \circ T(g)\Psi = \rho(p(g))\chi(z(g)), \quad (16)$$

where  $\chi : N_+ \rightarrow U$  is the holomorphic function of the matrix elements of  $z \in N_+$  with

$$\chi(z) = \pi \circ T(z)\Psi \quad (17)$$

and  $\rho(p)$  is defined by

$$\rho \begin{pmatrix} a & 0 \\ c & b \end{pmatrix} = \rho \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}; \quad (18)$$

note that  $a, b$  and  $c$  may be interpreted as matrices.

One finds that the holomorphic functions defined in this way carry a VCS representation  $\Gamma$  and that they transform according to the equation

$$[\Gamma(\alpha)\chi](z) = \rho_\alpha(z)\chi(z \cdot \alpha), \quad \alpha \in G, \quad (19)$$

where  $z(g) \cdot \alpha = z(g\alpha)$  and

$$\rho_\alpha(z)\chi = \pi \circ T(z)T(\alpha)\chi, \quad \text{for } \chi \in U. \quad (20)$$

The subset of holomorphic vector-valued functions that span an irreducible VCS representation can be constructed starting from the  $z$ -independent functions

$$\varphi_\nu(z) = \pi \circ T(z)\Phi_\nu = \Phi_\nu, \quad \Phi_\nu \in U. \quad (21)$$

These are highest weight functions in the sense that they are annihilated by all the raising operators of  $N_+$ ; i.e.,

$$\Gamma(X_i^+)\varphi_\nu = 0, \quad \forall X_i^+ \in \mathfrak{n}_+, \quad (22)$$

where  $\mathfrak{n}_+$  is the Lie algebra of  $N_+$ . Thus, the carrier space for an irreducible holomorphic VCS representation is the space of vector-valued polynomials of the type

$$\chi_{\sigma\nu}(z) = \mathcal{P}_\sigma(\Gamma(X_i^-))\varphi_\nu, \quad (23)$$

where  $\{\mathcal{P}_\sigma\}$  is a set of polynomials in the lowering operators. The construction of such spaces is discussed in ref. [1] and in references quoted therein.

#### 4. Representations of $SL(2, \mathbb{R})$

We now start with the Iwasawa decomposition

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad (24)$$

of an  $SL(2, \mathbb{R})$  matrix. Thus, we can define VCS wave functions on  $SO(2)$  by

$$\psi(\theta) = \pi \circ T \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \Psi, \quad \Psi \in V. \quad (25)$$



One finds VCS representations, on subspaces of  $\mathcal{L}^2(SO(2))$ , of the  $\mathfrak{sl}(2, \mathbb{R})$  Lie algebra with

$$\Gamma \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i \frac{d}{d\theta}, \quad \Gamma \begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix} = e^{\mp 2i\theta} \left( 2s \pm i \frac{d}{d\theta} \right). \quad (26)$$

For different choices of the parameters  $\epsilon$  and  $2s$ , eq. (26) gives all the standard representations of  $SL(2, \mathbb{R})$ ; i.e., the unitary principle series, the discrete series, the supplementary series and the non-unitary tensor representations. A parallel construction gives the representations of  $SL(3, \mathbb{R})$ , and hence  $SU(3)$ , on subspaces of  $\mathcal{L}^2(SO(3))$ .

### 5. Semi-direct product groups

Suppose  $G = N \rtimes_t K$  is a semi-direct product group with product

$$(n, k)(n', k') = (nt_k(n'), kk'), \quad (27)$$

where  $t_k$  is an automorphism of  $N$ . We can induce a VCS representation of  $G$  from a representation  $\rho$  of  $N$ . However, such a representation is generally reducible even when  $\rho$  is irreducible. As Mackey has shown, it is preferable to induce from the larger subgroup  $H = N \rtimes_t C$  of  $G$ , where  $C$  is the subgroup of  $K$

$$C = \{h \in K | \rho(t_h(n)) = \rho(n), \forall n \in N\}. \quad (28)$$

Then, if  $\rho$  and  $\sigma$  are, respectively, irreps of  $N$  and  $C$ , the combination  $\rho \times \sigma$ , defined by

$$\rho \times \sigma(n, h) = \rho(n)\sigma(h), \quad n \in N, h \in C, \quad (29)$$

is an irrep of  $H$ . Application of the VCS construction now gives a representation of  $G$  that parallels Mackey's inducing construction [3].

### 6. Concluding remarks

There is no space to discuss inner products in this review. This topic is important for a discussion of unitarity and practical applications. It has been developed to a sophisticated state in terms of  $K$ -matrix theory and is treated in ref. [1] and references quoted therein. We mention only the important feature of  $K$ -matrix theory that it gives inner products for representations which may or may not be unitary and that it often works when the so-called "resolution of the identity" diverges.

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# GEOMETRIC QUANTIZATION AND COHERENT STATES METHODS

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**Abstract:**

On the basis of reproducing kernels and complex line bundles theories one unifies quantum and classical descriptions of physical systems.

This is a short version of the paper in reference [4]. It presents part of the results established there. The question we address here is how to recognize a physical microsystem when starting with elementary experimental dates. The situation encountered when dealing with experiments on microscopic events could be shortly described in the following way: (i) one wants to know the amplitude  $A$  of the transition that could be achieved by the considered system between any two of its states; (ii) states taken into account are parametrized by a finite-dimensional manifold  $M$  (the case  $\dim M = \infty$  is experimentally non-realizable). This manifold describes the geometry of classical measurement apparatus (classical frame system). In the case considered here it will be interpreted as a classical phase space of the microsystem. Let us assume for the transition amplitude  $A$  the following natural physical conditions:

$$a_{\alpha\alpha}(q, q) = 1 \quad \text{for } q \in \Omega_\alpha \quad (1.1)$$

$$\overline{a_{\alpha\beta}(q, p)} = a_{\beta\alpha}(p, q) \quad \text{for } q \in \Omega_\alpha \text{ and } p \in \Omega_\beta \quad (1.2)$$

$$\sum_{i,j=1}^N \bar{v}_i v_j a_{\alpha_i \alpha_j}(q_i, q_j) \geq 0 \quad \forall N \in \mathbf{N}, \forall q_i \in \Omega_{\alpha_i} \text{ and } \forall v_i \in \mathbf{C} \quad (1.3)$$

$$a_{\alpha\beta}(q, p) = g_{\beta\gamma}(p) a_{\alpha\gamma}(q), \quad g_{\beta\gamma} : \Omega_\beta \cap \Omega_\gamma \rightarrow U(1) \quad \text{s.t.} \quad g_{\beta\gamma} g_{\gamma\delta} = g_{\beta\delta}. \quad (1.4)$$

Open sets  $\{\Omega_\alpha\}_{\alpha \in I}$  are domains of charts of some fixed atlas of the manifold  $M$ .

In a geometrical language the category of physical microsystems is exactly the category  $\mathcal{A}$  of complex line bundles  $\mathbf{L} \rightarrow M$  with a distinguished positive hermitian kernel:

$$A = a_{\alpha\beta}(q, p) pr_1^* \bar{S}_\alpha^*(q) \otimes pr_2^* S_\beta^*(p) \in \Gamma^\infty(M \times M, pr_1^* \bar{\mathbf{L}}^* \otimes pr_2^* \mathbf{L}^*).$$

Conditions (1.1-4) are independant on the choice of frame  $S_\alpha : \Omega_\alpha \rightarrow \mathbf{C}^*$  and the bundle  $\mathbf{L} \rightarrow M$  is defined by the cocycle  $g_{\alpha\beta}$ . The set of morphisms between two objects  $(\mathbf{L}_1 \rightarrow M_1, A_1)$  and  $(\mathbf{L}_2 \rightarrow M_2, A_2) \in \mathcal{Ob}(\mathcal{A})$  consists of maps  $f : M_1 \rightarrow M_2$  such that  $f^* \mathbf{L}_2 = \mathbf{L}_1$  and  $f^* A_2 = A_1$ .

Let us also define the category  $\mathcal{C}$  whose objects are triples  $(M, \mathcal{M}, \mathcal{K} : M \rightarrow \mathbf{CP}(\mathcal{M}))$ , where  $\mathcal{M}$  is a complex Hilbert space and  $\mathcal{K}$  is a map from  $M$  in the complex

projectif Hilbert space  $\mathbf{CP}(\mathcal{M})$ . Morphisms of this category are given by commutative diagrams,

$$\begin{array}{ccc} M_1 & \xrightarrow{\kappa_1} & \mathbf{CP}(\mathcal{M}_1) \\ f \downarrow & & \downarrow [\varphi] \\ M_2 & \xrightarrow{\kappa_2} & \mathbf{CP}(\mathcal{M}_2) \end{array} \quad (1.5)$$

where  $\varphi$  is a complex Hilbert spaces monomorphism, such that  $\varphi^* \mathbf{E}_1 = \mathbf{E}_2$ .  $\mathbf{E}$  is the universal line bundle over  $\mathbf{CP}(\mathcal{M})$ .

Finally let  $\mathcal{L}$  be the category of line bundles with distinguished Hermitian metric  $H \in \Gamma^\infty(M, \bar{\mathbf{L}}^* \otimes \mathbf{L})$  and a metrical connection  $\nabla : \Gamma^\infty(\Omega, \mathbf{L}) \rightarrow \Gamma^\infty(\Omega, \mathbf{L} \otimes T^*M)$ ,  $\Omega \subset M$ . The following theorem states important relations between the above three categories.

**Theorem:**

(i) There exist natural functors given in the explicit way:

$$\begin{array}{ccc} \mathcal{A} & \cong & \mathcal{C} \\ & \searrow & \swarrow \\ & \mathcal{L} & \end{array} \quad (1.6)$$

(ii) Each object of the category  $\mathcal{L}$  is obtained as  $\mathcal{F}_{Ic}(C)$  and  $\mathcal{F}_{Ia}(A)$  for some  $A \in \mathcal{O}b(\mathcal{A})$  and  $C \in \mathcal{O}b(\mathcal{C})$

(iii) Categories  $\tilde{\mathcal{A}}$  and  $\tilde{\mathcal{C}}$  of classes of isomorphic objects are isomorphic.

(iv) The three statements above are also valid for symplectic subcategories  $\mathcal{A}_{sp}, \mathcal{C}_{sp}$  and  $\mathcal{L}_{sp}$ .

*Proof* (See [4]).

The symplectic character of categories  $\mathcal{A}_{sp}, \mathcal{C}_{sp}$  and  $\mathcal{L}_{sp}$  means the non-degeneracy of the 2-forms  $\Delta^* d_1 d_2 \log a_{\alpha_1 \alpha_2}, \mathcal{K}^* \omega_{FS}$  and  $i \text{curv} \nabla$ , respectively, where  $\Delta : M \rightarrow M \times M$  is diagonal map,  $\omega_{FS}$  is Fubini-study form on  $\mathbf{CP}(\mathcal{M})$  and  $d_i$  denotes differential with respect to the  $i$ -th argument of  $\log a_{\alpha_1 \alpha_2}$ .

In this way we could define a mechanical microsystem as an object,  $(M, \mathcal{M}, \mathcal{K} : M \rightarrow \mathbf{CP}(\mathcal{M}))$  of the symplectic subcategory  $\mathcal{C}_{sp}$ . A natural question now arises, namely how one identifies the system in the standard mechanical framework, when  $(\mathbf{L} \rightarrow M, A)$  or equivalently  $(M, \mathcal{M}, \mathcal{K} : M \rightarrow \mathbf{CP}(\mathcal{M}))$  are fixed by experimental or mathematical dates. In other words, how one constructs a Lagrangian for a given triple  $(M, \mathcal{M}, \mathcal{K} : M \rightarrow \mathbf{CP}(\mathcal{M}))$ . Before answering this question, let us introduce the following physical terminology: (a)  $(M, \mathcal{K}^* \omega_{FS})$  is the classical phase space of the system; (b)  $(\mathbf{CP}(\mathcal{M}), \omega_{FS})$  is the quantum phase space of the system; (c)  $\mathcal{K}(M)$  is the set of coherent states of the system; (d) the transition amplitude between two coherent states  $\mathcal{K}(q)$  and  $\mathcal{K}(p)$  is defined by

$$a_{\alpha\beta}(q, p) = \frac{\langle K_\alpha(q) | K_\alpha(p) \rangle}{\sqrt{\langle K_\alpha(q) | K_\alpha(q) \rangle} \sqrt{\langle K_\alpha(p) | K_\alpha(p) \rangle}}, \quad (1.7)$$

where  $K_\alpha : \Omega_\alpha \rightarrow \mathcal{M}$  is such that  $[K_\alpha(q)] = \mathcal{K}(q)$  and  $p \in \Omega_\beta$ ;

(e) for  $\rho \in L^1(M, d\mu_L)$ ,  $P(q) = \frac{|K_\alpha(q)\langle K_\alpha(q) |}{\langle K_\alpha(q) | K_\alpha(q) \rangle}$  and  $d\mu_L = \wedge^n \mathcal{K}^* \omega_{FS}$ , where  $2n = \dim M$ , we define

$$P(\rho) = \int_M p(q) \rho(q) d\mu_L(q) \quad (1.8)$$

as a mixed state of the system.

Let us fix  $H : M \rightarrow \mathbf{R}_+ \cup \{0\}$ ,  $\Delta_k \subset \mathbf{R}^k$  and a family of diffeomorphisms  $\varphi_{\vec{\beta}} : \mathbf{R}_+ \cup \{0\} \rightarrow \mathbf{R}_+ \cup \{0\}$  where  $\vec{\beta} \in \Delta_k$ . By definition  $\rho = \varphi_{\vec{\beta}} \circ H$  is an equilibrium state of the system iff

- (i)  $\exists \vec{\beta} \in \Delta_k$  s.t.  $\text{Tr}(P(\varphi_{\vec{\beta}} \circ H)P(q)) = (\varphi_{\vec{\beta}} \circ H)(q)$ ,
- (ii)  $P(\varphi_{\vec{\beta}_0} \circ H) = 1$  for some  $\vec{\beta}_0 \in \bar{\Delta}_k$ .

The condition  $P(\rho_0) = 1$  means that the system amplitudes satisfy the composition rule

$$a_{\vec{\alpha}\vec{\beta}}(q, p) = \int_M \sum_{\gamma} h_{\gamma}(r) a_{\vec{\alpha}\gamma}(q, r) a_{\gamma\vec{\beta}}(r, p) \rho_0(r) d\mu_L(r) \quad (1.9)$$

for some natural phase space measure  $\rho_0 d\mu_L$ , ( $\sum_{\gamma} h_{\gamma}(r) = 1$  and  $\text{supp} h_{\gamma} \subset \Omega_{\gamma}$ ). Iterating formula (1.9) and taking the number of iterations  $N$  to infinity we obtain the formal path integral expression

$$a_{\vec{\alpha}\vec{\beta}}(q, p) = \int \prod_{\tau \in [\tau_i, \tau_f]} d\mathcal{K}\gamma(\tau) \exp i \int_{\tau_i}^{\tau_f} \text{Im} \frac{\langle K | dK \rangle}{\langle K | K \rangle} \Big| \frac{d\gamma}{d\tau} d\tau \quad (1.10)$$

for the transition amplitude, where

$$\int \prod_{\tau \in [\tau_i, \tau_f]} d\mathcal{K}\gamma(\tau) = \lim_{N \rightarrow \infty} \prod_{j=2}^{N-1} \int \rho_0(\gamma(\tau_j)) d\mu_L(\gamma(\tau_j)) \quad (1.11a)$$

$$\exp i \int_{\tau_i}^{\tau_f} \text{Im} \frac{\langle K | dK \rangle}{\langle K | K \rangle} \Big| \frac{d\gamma}{d\tau} d\tau = \lim_{N \rightarrow \infty} \prod_{j=2}^{N-1} a_{\vec{\alpha}_j \alpha_{j+1}}(q_j, q_{j+1}); \quad (1.11b)$$

$\gamma$  is a piecewise curve which extrapolate a discrete process  $\gamma(\tau_i) = q, \gamma(\tau_2) = q_2, \dots, \gamma(\tau_f) = p$ , and  $\frac{\langle K | dK \rangle}{\langle K | K \rangle} \Big|_{\Omega_{\alpha}} = \frac{\langle K_{\alpha} | dK_{\alpha} \rangle}{\langle K_{\alpha} | K_{\alpha} \rangle}$  is a connection 1-form for  $\mathcal{K}^* \nabla^{\text{FS}}$ . So, the transition amplitude along  $\gamma$  is identified as a parallel transport for  $\mathcal{K}^* \nabla^{\text{FS}}$ .

In (1.10) one integrates over all possible processes joining the states  $q$  and  $p$ . However, if one restricts this integratives to those processes satisfying  $H(\gamma(\tau)) = E = \text{const}$  i.e. processes along which the equilibrium state of the system is preserved, we come by formal path integration to the following expression for the systems path integration (See [4] for details),

$$a_{\vec{\alpha}\vec{\beta}}(q, p, H = E, \lambda_0) = e^{-iE(t_f - t_i)} \int \prod_{t \in [t_f, t_i]} d\mathcal{K}\gamma(\tau) \exp i \int_{t_i}^{t_f} \left[ \text{Im} \frac{\langle K | dK \rangle}{\langle K | K \rangle} \Big| \frac{d\gamma}{d\tau} - H(\gamma(t)) \right] dt, \quad (1.12)$$

where  $t = \int_{\tau_i}^t \lambda_0(\tau) d\tau$  is the time parameter given by the fixing of some classical clock, i.e. the parametrization  $\lambda_0$ . Thus according to Feynman interpretation of path integral we identify the action of the system

$$S_{\mathcal{K}, H}[\gamma] = \int_{t_i}^{t_f} \left[ \text{Im} \frac{\langle K | dK \rangle}{\langle K | K \rangle} \Big| \frac{d\gamma}{d\tau} - H(\gamma(t)) \right] dt. \quad (1.13)$$

In real experiments one always deals with discrete processes, so it is enough to work with processes consisting of a finite number  $N$  of phase space events, restricted by the resolution of the measuring apparatus. Thus instead of the path integral formula (1.12) we evaluate well defined integrals on an  $(N - 1)$ -th Cartesian product of  $M$ . Therefore, the evaluation of the path integral appears in this context as a purely formal question. On the other hand, if one restricts oneself to the mathematical context, the problem of path integration is in our formulation equivalent to the finding of the map  $\mathcal{K} : M \rightarrow \mathbf{CP}(\mathcal{M})$  for a fixed Hamiltonian  $H$  and a connection  $\nabla = \mathcal{K}^* \nabla^{\text{FS}}$ . Suggesting that without additional assumptions about the transition amplitude, the above equivalence is impossible, since the connection one-forms  $\frac{\langle K_\alpha | dK_\alpha \rangle}{\langle K_\alpha | K_\alpha \rangle}$  depend only on the restriction of  $d_1 \log a_{\alpha\beta}$  to the diagonal  $\Delta \subset M \times M$ . The limiting case,  $N \rightarrow \infty$ , is however crucial for the correspondence of the quantum description to a classical one. In fact, it allows one to identify the system in the framework of classical mechanics.

As it follows from the above theorem objects of mechanical microsystems category  $\mathcal{A}_{sp}$  generate in a canonical way objects of the prequantum bundles category  $\mathcal{L}_{sp}$ . Hence, geometric quantization appears naturally in the theory of mechanical microsystems. In paper [4], we show on the basis of our model how Kostant-Souriau and Berezin quantizations procedures, as well as Ehrenfest theorem are related (see [1] [2]). Moreover, we test this model on different fundamental microsystems such as relativistic massive particles, Hermitian oscillators and the Kepler problem in papers [3], [4] and [5].

Finally let us note that the proposed model shows that such theories as complex geometry, complex analysis, representation theory and reproducing kernel theory contribute in a natural way in the description of physical microphenomena.

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# Massless Poincaré Coherent States and Wavelets

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## 1. Coherent states: a generalization

Various notions of square integrability of a group representation have been introduced in the literature, leading to the construction of sets of *coherent states* (see [1] and references therein). Here we want to build up a set of *massless coherent states* for the two dimensional Poincaré group  $\mathcal{P}_+^1(1,1)$ . This construction is plagued by additional difficulties with respect to the massive case [1], essentially because of the bad *infrared singularities*. For this reason we are forced to generalize further the previous definitions of square integrability of a representation. Let  $G$  be a locally compact topological group,  $H$  a closed subgroup of  $G$ ,  $U$  a unitary irreducible representation of  $G$  in a Hilbert space  $\mathcal{H}$ . We call *quasi-section* of the principal bundle  $(G, \pi, G/H, H)$  a map  $\sigma_f$  which satisfies the following condition:  $\sigma_f : G/H \rightarrow G$ , and  $\pi \cdot \sigma_f = f$  where  $f : G/H \rightarrow G/H$  is a homeomorphism. It is clear that the quasi-section  $\sigma_f$  is the composition  $\sigma \cdot f$  of a genuine section and a homeomorphism of the base manifold. We say that the representation  $U$  is *square integrable mod*  $(H, \sigma_f)$  if for some  $\zeta \in \mathcal{H}$  the following integral converges for all  $\phi \in \mathcal{D} \subset \mathcal{H}$ ,  $\mathcal{D}$  dense:

$$I_{\sigma_f}(\zeta, \phi) = \int_{G/H} |(U(\sigma_f(x)\zeta), \phi)_{\mathcal{H}}|^2 d\nu(x). \quad (1)$$

Notice that this definition allows also the *resolution of unbounded operators*. In certain favourable cases the coset space  $G/H$  corresponds to a coadjoint orbit of the group  $G$ , which has the interpretation of a *phase space*. Thus generalized coherent states may be indexed by points of a phase space (which is a nice way of recovering their classical character).

## 2. Massless representations and coadjoint orbits of $\mathcal{P}_+^1(1,1)$

As it is well known [2], there does not exist any Poincaré invariant positive-definite two-point distribution  $\mathcal{W}$  solving the equation  $\square\mathcal{W} = 0$  and such that  $\text{supp}\widehat{\mathcal{W}}(k) = C_+ = \{k \in \mathbb{R}^2 : k^\mu k_\mu = 0, k^0 \geq 0\}$ . However, if we relax the positivity condition, we may find a Poincaré invariant distribution having the desired support properties, namely  $W(\xi) = -\frac{1}{4\pi} \log(-\xi^2 + i\epsilon\xi_0)$ . This distribution is *not positive-definite* and therefore the Wightman reconstruction theorem gives only a nondegenerate sesquilinear form on  $\mathcal{S}$ :  $\langle f, g \rangle = \int \widehat{f}(x)W(x-y)g(y)d^2x d^2y$  and consequently a representation of  $\mathcal{P}_+^1(1,1)$  on a linear space. To obtain a *representation on a Hilbert space*, we have to complete  $\mathcal{S}$  using a Hilbert majorant topology [3,4]. The explicit

construction runs as follows [4,5]. Let  $\chi \in \mathcal{S}(\mathbf{R}^2)$  be such that  $\tilde{\chi}(0) = 1, \langle \chi, \chi \rangle = 0$ . Given  $f \in \mathcal{S}(\mathbf{R}^2)$ , define  $f_0(x) = f(x) - \tilde{f}(0)\chi$ . Then the desired inner product may be written as

$$(f, g) = \langle f_0, g_0 \rangle + \langle f, \chi \rangle \langle \chi, g \rangle + \tilde{f}(0)\tilde{g}(0). \quad (2)$$

Completing  $\mathcal{S}$  in this topology one obtains a Krein space [3] :

$$\mathcal{H} = L^2(C_+, \frac{dk}{|k|}) \oplus V \oplus X, \quad (3)$$

where  $X$  and  $V$  are one dimensional subspaces. Now the representation given by Wightman's theorem extends to a representation  $U$  of  $\mathcal{P}_+^1(1,1)$  defined on a dense subset of  $\mathcal{H}$ . This representation is  $\eta$ -unitary, i.e.  $\langle U(a, \Lambda)f, U(a, \Lambda)g \rangle = \langle f, g \rangle$  but  $\langle U(a, \Lambda)f, U(a, \Lambda)g \rangle \neq \langle f, g \rangle$ . Second, the representation  $U$  is neither irreducible nor completely reducible, but *indecomposable*. Finally we notice that the Hilbert subspace  $L^2$  may be decomposed into the following direct sum:  $L^2(C_+, \frac{dk}{|k|}) = \mathcal{H}_l \oplus \mathcal{H}_r$ ,  $\mathcal{H}_l = L^2(\mathbf{R}_-, \frac{dk}{|k|})$ ,  $\mathcal{H}_r = L^2(\mathbf{R}_+, \frac{dk}{k})$  (left and right Hilbert spaces). We may obtain two unitary irreducible representations of  $\mathcal{P}_+^1(1,1)$   $U_{l(r)}$  defined on  $\mathcal{H}_{l(r)}$  by *quotienting the representation  $U$* , that is by considering the matrix elements  $\langle \psi_1, U(a, \Lambda)\psi_2 \rangle_{\mathcal{H}_{l(r)}}$ ,  $\psi_1, \psi_2 \in \mathcal{H}_{l(r)}$ , and associating to the sesquilinear forms so defined the operators  $U_{l(r)}(a, \Lambda)$ . The final result is

$$(U(a, \Lambda)_{l(r)}\psi)(k) = e^{ika}\psi(\Lambda^{-1}k), \quad \psi \in \mathcal{H}_{l(r)}. \quad (4)$$

The coadjoint orbits of  $\mathcal{P}_+^1(1,1)$  are given by the following formula:

$$Ad^{\#}(g)(\xi, \lambda) = (\Lambda_p \xi, \lambda + \langle \Lambda_p \xi, \sigma_2 v \rangle_{\mathbf{M}^2}) \quad (5)$$

where  $g = (v, \Lambda_p)$  and  $\sigma_2 = \text{antidiag}(1,1)$ , with  $\Lambda_p \in \mathcal{L}_+^1, \xi, v \in \mathbf{M}^2$  and  $\langle \cdot, \cdot \rangle_{\mathbf{M}^2}$  denotes the Minkowski inner product. More explicitly, we can identify the coadjoint orbits of  $\mathcal{P}_+^1(1,1)$  with the families of hyperbolic cylinders ( $m > 0$ )  $\{\xi^0 = \pm\sqrt{\xi^1{}^2 + m^2}\}$ ,  $\{\xi^1 = \pm\sqrt{\xi^0{}^2 + m^2}\}$ , the four half-planes  $\{\xi^0 = \pm\xi^1, \xi^0 > 0\}$ ,  $\{\xi^0 = \pm\xi^1, \xi^0 < 0\}$ , and the degenerate orbits consisting of a single point  $\{\lambda = \text{const}, \xi = 0\}$ . The nondegenerate coadjoint orbits may be interpreted as *classical phase spaces* corresponding to elementary systems having  $\mathcal{P}_+^1(1,1)$  as relativity group. As we will see, they are particularly suited for the construction of systems of coherent states and each family of coherent states will be indexed by the points of a certain coadjoint orbit.

### 3. Massless Coherent States

Let us now pass to the construction of massless coherent states. The coadjoint orbits corresponding to a massless relativistic particle. are the half planes  $\Gamma_l = \{\lambda \in \mathbf{R}, \xi \in \mathbf{M}^2 : \xi^0 = -\xi^1, \xi^1 < 0\}$  (left orbit),  $\Gamma_r = \{\lambda \in \mathbf{R}, \xi \in \mathbf{M}^2 : \xi^0 = \xi^1, \xi^1 > 0\}$

(right orbit). Let us concentrate our attention on  $\Gamma_r \simeq \mathcal{P}_+^1(1,1)/L_r$ . A global parametrization of this orbit is given by  $(\tau, p)$  with  $\tau = \lambda$ ,  $p = \xi^1$ . The coadjoint action reads  $(\tau, p) \rightarrow (\tau', p') = (a, \Lambda_k)(\tau, p)$  with  $\tau' = \tau + \langle \Lambda_k p, \sigma_2 a \rangle_{\mathbb{M}^2}$ ,  $p' = (k^0 + k)p$ ; the invariant measure is  $d\mu(\tau, p) = d\tau dp/p$ . Correspondingly, we consider the representation  $U_r$  on the Hilbert space  $\mathcal{H}_r$  that we have displayed in Eq.(4). We must choose a quasi-section  $\sigma : \Gamma_r \rightarrow \mathcal{P}_+^1$  and try to construct coherent states out of it. Notice that the natural quasi-section  $\sigma_n(\tau, p) = ((0, \tau), \Lambda_p)$  cannot do the work. The reason is that  $(U_r(\sigma_n(\tau, p))\zeta)(k) = e^{i\tau k} \zeta((p^0 - p)k)$ . Since  $p > 0$ , it follows that  $0 < (p^0 - p) < 1$  and therefore the argument of the function  $\zeta$  cannot be arbitrarily dilated. A well-chosen quasi-section should have the following form:  $\sigma_\rho(\tau, p) = ((0, \tau), \Lambda_{\rho(p)})$ , where  $\rho : \mathbf{R}^+ \rightarrow \mathbf{R}$  is an auxiliary bijective map. An interesting explicit form for the function  $\rho$  is the following one:  $\rho(p) = \frac{1}{2p} - \frac{p}{2}$ . The nice features of this function are due to the following relation:  $\sqrt{\rho^2(p) + 1} = \frac{1}{2p} + \frac{p}{2}$ . Consequently we obtain that

$$(U_r(\sigma_\rho(\tau, p))\zeta)(k) = e^{i\tau k} \zeta(pk), \quad \zeta \in \mathcal{H}_r. \quad (6)$$

$$\begin{aligned} I_{\sigma_\rho}(\zeta, \phi) &= \int_{\Gamma_r} |\langle U_r(\sigma_\rho(\tau, p))\zeta, \phi \rangle_{\mathcal{H}_r}|^2 d\mu(\tau, p) = \int_0^\infty \frac{dp}{p} \int_0^\infty \frac{dk}{k^2} |\zeta(pk)|^2 |\phi(k)|^2 = \\ &= \int_0^\infty \frac{du}{u} |\zeta(u)|^2 \int_0^\infty \frac{dk}{k^2} |\phi(k)|^2 \end{aligned} \quad (7)$$

Define the following operator on  $\mathcal{H}_r$ :  $\mathcal{D}(H_r) = \{\phi \in \mathcal{H}_r ; \int_0^\infty dk k |\phi(k)|^2 < \infty\}$ ,

$$(H_r \phi)(k) = k \phi(k), \quad \phi \in \mathcal{D}(H_r). \quad (8)$$

$H_r$  is an unbounded self-adjoint operator on  $\mathcal{D}(H_r) \subset \mathcal{H}_r$  and  $I_{\sigma_\rho}(\zeta, \phi)$  exists if and only if  $\phi \in \mathcal{D}(H_r^{-1/2})$ . In this case it follows that

$$\int_{\Gamma_r} |U_r(\sigma_\rho(\tau, p))\zeta \langle U_r(\sigma_\rho(\tau, p))\zeta | d\mu(\tau, p) = C_\zeta H_r^{-1}, \quad (9)$$

in the sense of quadratic forms. Thus, the vectors given in Eq.(6) constitute a set of *massless coherent states* for every  $\zeta \in \mathcal{H}_r$ , but the operator that is "resolved" in Eq.(9) is *not bounded* and, moreover, *its inverse is also unbounded*. Because of these facts, this set of coherent states is more general than those considered in [1]. Let us exploit our freedom in the choice of quasi-sections to get a more appealing set of massless coherent states and define  $\sigma_r(\tau, p) = ((0, \frac{\tau}{p}), \Lambda_{-\rho(p)})$ . With the help of this quasi-section we get the following set of states:

$$(U_r(\sigma_r(\tau, p))\zeta)(k) = e^{i\frac{\tau}{p}k} \zeta\left(\frac{k}{p}\right), \quad \zeta \in \mathcal{H}_r. \quad (10)$$

In this case the integral (1) becomes:

$$I_{\sigma_r}(\zeta, \phi) = \int_0^\infty dp \int_0^\infty \frac{dk}{k^2} |\zeta\left(\frac{k}{p}\right)|^2 |\phi(k)|^2 = \int_0^\infty \frac{du}{u^2} |\zeta(u)|^2 \int_0^\infty \frac{dk}{k} |\phi(k)|^2 \quad (11)$$



Therefore, vectors  $\zeta \in \mathcal{H}_r$  are admissible for the quasi-section  $\sigma_r(\tau, p)$  if and only if they satisfy the condition  $\zeta \in \mathcal{D}(H_r^{-1/2})$ . If this condition is satisfied we obtain that

$$\frac{1}{c_{\sigma_r}(\zeta)} \int_{\Gamma_r} |U_r(\sigma_r(\tau, p))\zeta\rangle \langle U_r(\sigma_r(\tau, p))\zeta| d\mu(\tau, p) = I, \quad (12)$$

i.e. we get a genuine resolution of the identity! We call the states defined in equation (10) *right coherent states*. In a perfectly identical way we may construct a corresponding set of *left coherent states*. We finally obtain a resolution of the identity in the Krein space (3):

$$I = |v\rangle\langle v| + |\chi\rangle\langle\chi| + \frac{1}{c_{\sigma_l}(\phi)} \int_{\Gamma_l} |U_l(\sigma_l(\tau, p))\phi\rangle \langle U_l(\sigma_l(\tau, p))\phi| d\mu_l(\tau, p) \\ + \frac{1}{c_{\sigma_r}(\psi)} \int_{\Gamma_r} |U_r(\sigma_r(\tau, p))\psi\rangle \langle U_r(\sigma_r(\tau, p))\psi| d\mu_r(\tau, p), \quad (13)$$

with  $\phi \in \mathcal{H}_l$ ,  $\psi \in \mathcal{H}_r$ . An interesting feature of the sets of coherent states (10) is that they are exactly identical to wavelets, i.e. the coherent states of the affine group. Indeed, we may convince ourselves easily that the previous coherent states coincide with the wavelets given for instance in [6] making the following identifications:  $\frac{1}{p} = a$ ,  $\frac{\tau}{p} = -b$ ,  $\frac{\zeta(\omega)}{\sqrt{\omega}} = \hat{\varphi}(\omega)$ . The admissibility condition  $\zeta \in \mathcal{D}(H_r^{-1/2})$  becomes

$$\int_0^\infty \frac{dk}{k^2} |\zeta(k)|^2 = \int_0^\infty \frac{d\omega}{\omega} |\hat{\varphi}(\omega)|^2 < \infty \quad (14)$$

The invariant measure reads now  $d\mu(\tau, p) = \frac{d\tau dp}{p} = \frac{da db}{a^2}$ . Define the following operator:  $\mathcal{U} : \mathcal{H}_r = L^2(\mathbf{R}_+, \frac{dk}{k}) \rightarrow L^2(\mathbf{R}_+, dk)$ ,  $(\mathcal{U}\psi)(k) = \hat{\psi}(k) = \frac{\psi(k)}{\sqrt{k}}$ . Then we obtain that

$$\frac{1}{\sqrt{c_{\sigma_r}(\zeta)}} \langle U_r(\sigma_r(\tau, p))\zeta, \psi \rangle_{\mathcal{H}_r} = \int_0^\infty e^{-i\frac{\tau}{p}k} \bar{\zeta}\left(\frac{k}{p}\right) \psi(k) \frac{dk}{k} = \sqrt{\frac{a}{C_\varphi}} \int_0^\infty e^{ib\omega} \hat{\varphi}(a\omega) \hat{\psi}(\omega) d\omega, \quad (15)$$

which is exactly the wavelet transform for a progressive wavelet  $\varphi$  (cfr. [6] Sect. 3.1.). This correspondence opens a new range of applications for wavelet-analysis, namely *Quantum Field Theory*.

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## Phase space description of electrons in the one-band approximation

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Charge transport in semiconductors is usually discussed in terms of the the Wigner-Weyl (WW) formalism [1] which is adapted to wave mechanical problems with natural boundary conditions [2]. Recently a similar scheme has been proposed [3] to include - within an approximation - the hitherto neglected periodic potential of the ions and the resulting transformation properties of the wave functions. In this contribution the mathematical structure of the new scheme is presented in a more transparent way and an alternative phase space form of the approximation, formally corresponding to the Husimi-Coherent-State (HCS) formalism [4], is outlined.

We consider a fixed band of a one-dimensional periodic potential. If periodic boundary conditions are used the number of Bloch states  $|p\rangle$  belonging to this band is finite. They span the state space of the (simplified) quantum mechanical problem that is considered in the following. By a finite Fourier transform one may pass from Bloch to Wannier states  $|q\rangle$  which also form an orthonormal basis.

$$|q\rangle = \sum_p F_{q,p} |p\rangle, \quad F_{q,p} = \frac{1}{\sqrt{N}} \omega^{qp} \quad (1)$$

$$\omega = \exp(2\pi i/N) \quad (2)$$

Here  $N = 2M + 1$  is an odd integer,  $p, q \in \{-M, \dots, M\}$ , and expressions like  $pq$ ,  $2pq$ ,  $pq/2$ ,  $p_1 + p_2$ , etc. are considered as integers modulo  $N$ .

Every operator  $A$  acting in this state space may be defined by its action on the basis  $\{|q\rangle\}$ . Of special interest are the operators  $V$  (momentum translation),  $W$  (position translation), and the Weyl operators  $U(p, q)$ .

$$V |q\rangle = \omega^q |q\rangle, \quad W |q\rangle = |q+1\rangle \quad (3)$$

$$U(p, q) = \omega^{-pq/2} V^p W^q \quad (4)$$

If we combine the pair  $p, q$  into a vector  $\eta$  and introduce the (symplectic) product

$$p_1 q_2 - q_1 p_2 = \eta_1 \times \eta_2 = -\eta_2 \times \eta_1 \quad (5)$$

the multiplication law of the Weyl operators reads as follows.

$$U(\eta_1) U(\eta_2) = \omega^{\eta_1 \times \eta_2 / 2} U(\eta_1 + \eta_2) \quad (6)$$

It follows from (2)-(6) that  $U(-\eta) = U(\eta)^{-1} = U(\eta)^\dagger$ , and  $U(\eta) U(\eta') U(-\eta) = \omega^{\eta' \times \eta} U(\eta')$ , the latter relation showing that the  $U$ 's are tensor operators of rank 1.

Equation (6) shows that the operators  $\omega^r U(p, q)$  form a representation of a group of order  $N^3$ , or a projective representation of the Abelian group  $C_N \times C_N$ . This representation is irreducible as has been pointed out by Schwinger [5] who first introduced the operators (3)-(4); as a consequence every operator  $A$  can be represented as a linear combination of the  $U$ 's (see [5], [6], and, for the one-band approximation, also [7]). The idea of a linear space of operators, a complex vector space of dimension  $N^2$  in the present case, lies behind all phase space formulations of quantum mechanical problems. In such a formalism an operator  $A$  is characterized by a set of numbers, either its expansion coefficients with respect to a complete set of operators ('basis'), or by its 'projection' onto sufficiently many linearly independent operators. Here we use the second approach: starting from a fixed self-adjoint operator  $Z = Z^\dagger$  we form the  $N^2$  operators

$$Z(\eta) = U(\eta) Z U(-\eta). \quad (7)$$

Their labels  $\eta = (p, q)$  form a two-dimensional grid of  $N^2$  points, symmetrically arranged around the origin  $p = q = 0$ , and called 'phase space' in the following. To each operator  $A$  a function  $a$  is assigned by the definition

$$a(\eta) = \text{Trace } Z(\eta)^\dagger A = \langle\langle Z(\eta), A \rangle\rangle; \quad (8)$$

the last expression refers to the definition of a scalar product that makes the linear space of operators a unitary space. Relation (8) is therefore nothing but a projection where  $\alpha A + \beta B$  is mapped onto  $\alpha a + \beta b$  and  $A^\dagger$  onto  $a^*$ . If the unit operator  $E$  is to be represented by the constant function 1 the operator  $Z$  has to satisfy

$$\text{Trace } Z = 1. \quad (9)$$

The phase space function  $a$  contains the same information as the operator  $A$  if, and only if,  $a(\eta) = 0$  for all  $\eta$  implies  $A = 0$ . As the Weyl operators form a complete set of operators and the image of  $U(\eta')$  is proportional to the 'plane wave'  $\omega^{\eta' \times \eta}$  one-to-one correspondence of operators and phase space functions is guaranteed if none of the proportionality factors

$$\sigma(\eta') = \text{Trace } U(\eta') Z \quad (10)$$

vanishes. For WW

$$Z = 1 \quad (\text{inversion operator: } |q\rangle = |-q\rangle) \quad (11)$$

and  $\sigma(\eta) = 1$  for all  $\eta$ . For HCS

$$Z = |\tilde{0}\rangle\langle\tilde{0}|, \quad Z(\eta) = |\eta\rangle\langle\eta| \quad (\text{projection operators}) \quad (12)$$

where the state vectors  $|\eta\rangle = U(\eta)|\hat{0}\rangle$  are the 'coherent states' generated from  $|\hat{0}\rangle$ , and  $|\hat{0}\rangle$  is the normalized eigenvector of

$$H = (2E - W - W^{-1}) + (2E - V - V^{-1}) \quad (13)$$

that belongs to the lowest eigenvalue. Also in this case  $\sigma(\eta) \neq 0$  for all  $\eta$  so that the mapping of operators onto functions is bijective both for WW and HCS.

In order to be able to perform all the steps of a conventional quantum mechanical calculation also in the phase space formalism under consideration its mathematical structure has to be elaborated further. Here it is essential that the operators do not only form a linear space but a star-algebra, where products - and hence commutators - are defined. Completeness of the operators  $Z(\eta)$  implies the existence of a kernel  $M(\eta', \eta'')$  such that

$$c(\eta) = \sum_{\eta', \eta''} M(\eta', \eta'') a(\eta + \eta') b(\eta + \eta'') \quad (14)$$

if  $c$  is the function that corresponds to the operator  $C = AB$ . Explicit calculation gives

$$M(\eta', \eta'') = \frac{1}{N^4} \sum_{\eta_1, \eta_2} \omega^{-\eta' \times \eta_1 - \eta'' \times \eta_2 + \eta_1 \times \eta_2 / 2} \frac{\sigma(\eta_1 + \eta_2)}{\sigma(\eta_1)\sigma(\eta_2)} \quad (15)$$

Not only are these kernels very similar to those found in the continuous WW and HCS schemes but it is also possible to find 'local' forms of the multiplication law (14) where the first term of  $c(\eta)$  is  $a(\eta)b(\eta)$  and the succeeding terms contain more and more information on the form of the functions  $a$  and  $b$  in the neighborhood of  $\eta$ . The counterpart of Groenewold's formula [8.2] for the WW scheme reads

$$c(\eta) = \sum_{r,s=0}^{N-1} \gamma_{r,s} [\mathcal{D}_{r,s} a(\eta)] [\mathcal{D}_{s,r} b(\eta)] : \quad (16)$$

here the  $\gamma$ 's are numbers and the  $\mathcal{D}$ 's are difference operators that can be determined recursively. The equivalent of Ruschin's product formula [9.4] in the HCS scheme is

$$c(\eta) = \sum_{n=0}^{N-1} [\mathcal{D}_n a(\eta)] [\mathcal{D}_n^* b(\eta)] \quad (17)$$

where the  $\mathcal{D}$ 's are 'integral' operators with kernels  $d_n(\eta, \eta')$  that look like discrete versions of smoothed distributions  $\delta^{(n)}(\eta - \eta')$ . In principle, the  $n$ -th eigenfunction of (13) has to be known to obtain  $d_n$  but simple approximations exist for  $n \ll N$ .

With the formulas obtained up to now all algebraic calculations (linear combinations, products, transition to the adjoint) can be performed entirely within

the chosen phase space formalism. What is left is a mapping into the complex numbers that corresponds to the trace operation in the conventional formalism. However, it is easily derived from the irreducibility of the Weyl operators and the properties of the operators (11) and (12) that both for WW and HCS

$$\text{Trace } A = \frac{1}{N} \sum_{\eta} a(\eta). \quad (18)$$

Although the formulas given above are valid for arbitrary odd integers  $N \geq 3$  large values of  $N$  are more appropriate for the physical problem under consideration. The limit  $N \rightarrow \infty$  is best discussed by scaling the original phase space variables.

$$\eta = (p, q) \quad \text{---} \quad \xi = (k, l) = (2\pi p/Na, 2\pi q/Nb) \quad (19)$$

The multiplication law of the operators  $U(\eta) = \tilde{U}(\xi)$  is then identical in form with that of the usual Weyl operators except that  $\xi$  appears instead of the continuous vector  $(p, q)$  and the parameter

$$\alpha = 2\pi/Nab \quad (20)$$

in the place of  $\hbar$ . It is therefore not surprising that in the limit  $N \rightarrow \infty$ ,  $\alpha > 0$  formal coincidence with the usual WW and HCS schemes is obtained. If  $\alpha$  approaches zero (after  $N \rightarrow \infty$ ) the product is commutative up to terms of order  $\alpha$  and the quantum-Liouville equation approaches the semi-classical transport equation which is the starting point of [3].

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## V. Berry Phase

## INVARIANCE OF THE GEOMETRICAL PHASE UNDER TIME DEPENDENT UNITARY TRANSFORMATIONS\*

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### I. Introduction

The effects of time dependent unitary transformations on the geometric (Berry) phase have been discussed by several authors in different contexts. In the adiabatic treatments,<sup>1,2</sup> the Berry phase was said to become "dynamical" under these transformations. By becoming "dynamical" it loses geometrical meaning and could in this way be "removed".<sup>1</sup> It was then shown that even though the Berry phase could become "dynamical", it retained its geometrical meaning and was therefore not "removed".<sup>2</sup> Non-adiabatic treatments have been given for specific physical systems<sup>3</sup> and for periodic transformations.<sup>4</sup> We will consider arbitrary time dependent unitary transformations and work in the general non-adiabatic setting. By carefully considering the time evolution of both paths in an interference experiment, we will show that the geometrical and dynamical phases are invariant, and that the geometrical phase remains entirely geometrical.

### II. The Interference Experiment

The measurement of a relative phase is accomplished by performing an interference experiment.<sup>5,6</sup> This experiment measures the relative phase between two state vectors which have undergone different time evolutions, but represent the same initial and final physical state. In order to derive the measured relative phase, we must consider the time evolution equations for both paths. We will denote path 1 as the path which goes through the apparatus and path 2 as the path which goes around the apparatus. We will assume that the transit times have been set equal so that we can ignore the free space Hamiltonian. The time evolution equations for both paths are

$$i\dot{|c_1(t)\rangle} = h(t)|c_1(t)\rangle \quad (1)$$

$$i\dot{|c_2(t)\rangle} = 0 \quad (2)$$

where  $|c_1(t)\rangle$  is the state vector representing the state of path 1,  $|c_2(t)\rangle$  is the state vector representing the state of path 2 and  $h(t)$  is the Hamiltonian representing the apparatus. The Hamiltonian  $h(t)$  is chosen so as to produce cyclic evolution for path 1. By cyclic evolution we mean that the curve in the space of physical states (projective

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Hilbert space) is closed. We will denote the physical states of path 1 and 2 by the projection operators

$$\pi_1(t) \equiv |\psi_1(t)\rangle\langle\psi_1(t)|, \quad \pi_2(t) \equiv |\psi_2(t)\rangle\langle\psi_2(t)|. \quad (3)$$

We can express the condition for cyclic evolution over a time period  $T$  as

$$\pi_1(T) = \pi_1(0). \quad (4)$$

At the times  $t = 0$  and  $t = T$  the states of the two paths are equal

$$\pi_1(0) = \pi_2(0), \quad \pi_1(T) = \pi_2(T). \quad (5)$$

The measured relative phase between the two state vectors (which represent the same physical state at time  $T$ ) is given by

$$|\psi_1(T)\rangle = e^{i\alpha} |\psi_2(T)\rangle \quad (6)$$

where  $\alpha = \alpha_1 - \alpha_2 \pmod{2\pi}$ , and the phases  $\alpha_1$  and  $\alpha_2$  are the phases accumulated by the state vectors of path 1 and path 2. From Eq. (2) we see that  $\alpha_2 = 0$  which implies that  $\alpha = \alpha_1$ . In the usual way we can express the total relative phase  $\alpha$  in terms of a geometrical and dynamical part<sup>6</sup> (the connection defines this splitting<sup>7,8</sup>)

$$\alpha = \alpha_1 = i \int_0^T \langle \phi(t) | \dot{\phi}(t) \rangle dt - \int_0^T \langle \psi_1(t) | h(t) | \psi_1(t) \rangle dt \quad (7)$$

where the vectors  $|\phi(t)\rangle$  are related to the vectors  $|\psi_1(t)\rangle$  by a phase such that  $|\phi(T)\rangle = |\phi(0)\rangle$ . The geometric phase is

$$\beta = i \int_0^T \langle \phi(t) | \dot{\phi}(t) \rangle dt = \oint_c \mathcal{A} \quad (8)$$

where we have introduced the coordinates  $x$  of projective Hilbert space ( $c$  represents a closed path in this space) and defined the connection form  $\mathcal{A}$  as

$$\mathcal{A} \equiv i \langle \phi(x) | d | \phi(x) \rangle. \quad (9)$$

The operator  $d$  is the exterior derivative with respect to the coordinates  $x$ . The dynamical phase is

$$\delta = - \int_0^T \langle \psi_1(t) | h(t) | \psi_1(t) \rangle dt. \quad (10)$$

### III. Time Dependent Unitary Transformations

We want to consider time dependent unitary transformations acting on Hilbert space. All the vectors transform according to

$$|\tilde{\psi}(t)\rangle = U(t) |\psi(t)\rangle \quad (11)$$



where  $U(t) \cdot U^\dagger(t) = 1$  for all  $t$ . The transformation  $U(t)$  is acting on all the vectors in Hilbert space. It therefore effects the time evolution equations of *both* path 1 and path 2 of our interference experiment. Equations (1) and (2) become

$$i|\dot{\tilde{v}}_1(t)\rangle = \tilde{h}_1(t)|\tilde{v}_1(t)\rangle \quad (12a)$$

$$i|\dot{\tilde{v}}_2(t)\rangle = \tilde{h}_2(t)|\tilde{v}_2(t)\rangle \quad (12b)$$

where

$$\tilde{h}_1(t) = U(t)h(t)U^\dagger(t) + i\dot{U}(t)U^\dagger(t) \quad (13a)$$

$$\tilde{h}_2(t) = 0 + i\dot{U}(t)U^\dagger(t). \quad (13b)$$

Under the transformation  $U(t)$  the curves in projective Hilbert space represented by the projection operators  $\pi_1(t)$  and  $\pi_2(t)$  change according to

$$\tilde{\pi}_1(t) = U(t)\pi_1(t)U^\dagger(t), \quad \tilde{\pi}_2(t) = U(t)\pi_2(t)U^\dagger(t). \quad (14)$$

Thus under the transformation  $U(t)$  the time evolution of path 1 and path 2 generates two new curves in projective Hilbert space. The two curves start at the same point at  $t = 0$  and later meet at some other point at  $t = T$ . This can be seen explicitly from Eq. (14) upon setting  $t = 0$  and  $t = T$  and using Eq. (5)

$$\tilde{\pi}_1(0) = \tilde{\pi}_2(0), \quad \tilde{\pi}_1(T) = \tilde{\pi}_2(T). \quad (15)$$

Taken together these two curves form a new closed curve in projective Hilbert space which will be denoted by  $\tilde{c}$ .

It is straightforward to show that the measured relative phase is invariant

$$|\tilde{v}_1(T)\rangle = \epsilon^{i\alpha} |\tilde{v}_2(T)\rangle. \quad (16)$$

We will now evaluate the measured relative phase using the transformed vectors. We can express  $|\tilde{v}_1(t)\rangle$  and  $|\tilde{v}_2(t)\rangle$  in terms of vectors denoted by  $|\phi_1(t)\rangle$  and  $|\phi_2(t)\rangle$  where  $|\phi_1(t)\rangle$  and  $|\phi_2(t)\rangle$  are chosen so that they form a closed curve in Hilbert space (analogous to  $|\phi(t)\rangle$ ). By performing a few simple steps we obtain the final expression for  $\alpha$  under the transformation  $U(t)$

$$\alpha = \oint_{\tilde{c}} A - i \oint_{\tilde{c}} \langle \phi(x) | dU(x)U^\dagger(x) | \phi(x) \rangle - \int_0^T \langle \tilde{v}_1(t) | U^\dagger(t)h(t)U(t) | \tilde{v}_1(t) \rangle dt. \quad (17)$$

The transformed geometrical and dynamical phases are given by

$$\tilde{\beta} = \oint_{\tilde{c}} A - i \oint_{\tilde{c}} \langle \phi(x) | dU(x)U^\dagger(x) | \phi(x) \rangle \quad (18a)$$

$$\tilde{\delta} = - \int_0^T \langle \tilde{v}_1(t) | U^\dagger(t)h(t)U(t) | \tilde{v}_1(t) \rangle dt. \quad (18b)$$

By comparing Eq. (18b) with Eq. (10) we see that the transformed dynamical phase is equal to the original dynamical phase. Since we know that the relative phase  $\alpha$  is

invariant, this obviously implies that the geometrical phase is also invariant

$$\beta = \tilde{\beta}. \quad (19)$$

Substituting the expressions for  $\beta$  and  $\tilde{\beta}$  into Eq. (19) we find

$$\oint_c A = \oint_{\tilde{c}} A - i \oint_{\tilde{c}} \langle \phi(x) | dU(x) U^\dagger(x) | \phi(x) \rangle. \quad (20)$$

Equation (20) shows that even though the curve in projective Hilbert space has changed ( $c \rightarrow \tilde{c}$ ), the  $i \langle \phi | dU U^\dagger | \phi \rangle$  term compensates for this change keeping the geometric phase invariant. We can interpret  $i \langle \phi | dU U^\dagger | \phi \rangle$  as a connection form which is induced by the transformation to a "rotating" frame of reference. This is not a gauge transformation, the curvature two-form  $F = dA$  also changes. From Eqs. (12) and (13) we see that the operator  $U h U^\dagger$  is present only in the time evolution equation for path 1 just like the operator  $h$  in Eq. (1). This operator represents the apparatus and gives rise to the dynamical phase. Note that the  $i \langle \phi | dU U^\dagger | \phi \rangle$  term is common to both time evolution equations which allows it to be given a geometrical interpretation. From a mathematical point of view the choice of a connection is in general arbitrary. It is the physical quantities which must be invariant. The connection is chosen to divide the total phase  $\alpha$  into the physically observed geometrical and dynamical parts. A different choice for the connection must be made in order to calculate the observed geometrical phase under a time dependent unitary transformation.

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# Quantum Geometrical Phase for a Threefold Degenerate State

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## Abstract

Different expressions for the calculation of the quantum geometrical phase are reviewed and their limitations are discussed. The comparison between the Berry Phase [1] and the Aharonov-Anandan phase [2] is made. The expressions are illustrated for a circuit surrounding a triple degeneracy.

## 1 Introduction

In this paper we endeavour to show different ways of calculating the quantum geometrical phase. We give three expressions for the calculation of the Berry Phase of an adiabatic process. We discuss advantages and limitations of different expressions. Then we define the Aharonov-Anandan phase and describe the geometry of the fibre bundle based on the projective Hilbert space. Quantum states evolve along horizontal lifts in this bundle. We show that the phase, acquired during this evolution, in fact coincides with the phase introduced by Berry. The formulas are then applied to the calculation of the geometrical phase corresponding to the evolution of the eigenvector of a general threefold degenerate state [3].

## 2 The Geometrical Phase

Suppose that the Hamiltonian  $H(X)$  of a quantum system depends on an arbitrary parameter  $X \in \Omega$ . Then, in the adiabatic approximation the time evolution of a given state of this system  $|\psi(X)\rangle \in \mathcal{H}$  along a smooth curve  $X(t) \in \Omega$ ,  $t \in [0, T]$  is given by

$$|\psi(t)\rangle \approx |n(X(t))\rangle \cdot e^{i\mathcal{D}(t)}, \quad (1)$$

where  $|n(X(t))\rangle$  is a nondegenerate eigenstate of the stationary Schrödinger equation  $H|n\rangle = E_n|n\rangle$ , at  $X(t)$ , corresponding to the eigenvalue  $E_n(X(t))$  and  $\mathcal{D}(t) = -\hbar^{-1} \int_0^t E_n(X(t')) dt'$  is the usual dynamic factor

It turns out that only those states  $|n(X)\rangle$  which obey the parallel transport law

$$\text{Im}\langle n(X) | dn(X) \rangle = 0, \quad X \in \Omega \quad (2)$$

are in accordance with the Schrödinger equation [4]. Suppose that the system encircles a closed loop  $C$  in the space  $\Omega$ , i.e.  $X(T) = X(0)$ . Then eqs (1,2) imply that

$$\langle \psi_{final} | \psi_{initial} \rangle = \exp(-i(\mathcal{D}(T) + \beta_n(C))) \quad (3)$$

the final state differs from the initial one by the phase being the sum of dynamic  $\mathcal{D}(T)$  and geometrical  $\beta_n(C)$  factors. The latter is the Berry phase, which according to eqs (1,3) is equal to

$$\beta_n(C) = -\text{Im} \ln \langle n_{final} | n_{initial} \rangle. \quad (4)$$

It is worth noticing that the phase depends only on the path  $C$  in  $\Omega$ , no matter what were dynamic details of the evolution. The direct calculation of the phase  $\beta_n(C)$  from the definition (4) is in practice difficult because vectors  $|n(X)\rangle$  must be found in a gauge in which they obey the parallel transport law (2).

It is sometimes easier to bring the state  $|n\rangle$  to a gauge  $|n^\nu(X)\rangle = |n(X)\rangle e^{-i\nu(X)}$  in which it is continuous along  $C$ , including the point closing the loop  $X(0) = X(T)$ . Then the phase

$$\beta_n(C) = -\text{Im} \oint_C \langle n^\nu | dn^\nu \rangle \quad (5)$$

is given as a line integral, along  $C$ , of the 1-form  $-\text{Im}\langle n^\nu | dn^\nu \rangle$ .

In the expressions (4) and (5) specific gauge transformations were seen to be required what might make it appear that the phase itself is gauge dependent. This is however not the case. Using the Stokes' theorem one can convert (5) into a gauge independent form

$$\beta_n(C) = -\text{Im} \iint_S \langle dn | \wedge | dn \rangle, \quad (6)$$

where  $S$  must be an orientable surface bounded by the circuit  $C$ . This formula is less general than (4) also because the 2-form  $\langle dn | \wedge | dn \rangle$  does not always exist.

A more general way of defining the geometrical phase is due to Aharonov and Anandan (A-A). They considered cyclic evolutions, i.e. such that the initial and final states coincide, in the projective Hilbert space  $P(\mathcal{H})$ . In Berry's approach it is the adiabatic approximation (1) that allows a given physical system to remain, during evolution, at the same state  $|n\rangle$  up to a phase. Here it is assured by the definition of the cyclic evolution.

The A-A phase is defined as [2]

$$\alpha_\psi(C) = \phi + i \oint_C \langle \psi | d\psi \rangle,$$

where  $\phi$  is the total phase  $\langle \psi_{final} | \psi_{initial} \rangle = \exp(-i\phi)$  acquired along a closed loop  $C \subset P(\mathcal{H})$  by the state vector  $|\psi\rangle$ . We would like to stress that if in Berry's approach one omits the adiabatic approximation then the Berry phase becomes equal to the A-A phase. In fact, the second term in RHS of eq (7) is equal to minus the dynamic phase  $i \oint_C \langle \psi | d\psi \rangle = \hbar^{-1} \int_0^T \langle \psi(t') | H | \psi(t') \rangle dt = -\mathcal{D}(T)$  and hence, comparing (7) with (3) we have

$$\alpha_\psi(C) = \beta_n(C). \quad (8)$$

### 3 Geometrical Description

Formulation of the geometric phase in terms of the fibre bundle formalism provides us with a better insight into the whole concept [5].

The projective Hilbert space  $P(\mathcal{H})$  is the base space of a fibre bundle  $E = \mathcal{H} - \{0\}$  with the structure group  $U(1)$ . The variation of a given state  $|\psi\rangle$  in this bundle has its horizontal and vertical parts

$$d|\psi\rangle = d_H|\psi\rangle + d_V|\psi\rangle. \quad (9)$$

The horizontal variation  $d_H|\psi\rangle$  adds to the path in the projective space and is orthogonal to the fibre

$$\langle \psi | d_H | \psi \rangle = 0. \quad (10)$$

The vertical variation  $d_V|\psi\rangle$  composes the phase acquired by the vector during evolution. It is directed along the fibre

$$d_V|\psi\rangle = \omega^1|n\rangle. \quad (11)$$

where  $|n\rangle$  is a smooth field of normalized representatives of fibres, i.e.  $|\psi\rangle = e^{i\gamma}|n\rangle$  and  $\omega^1$  is a differential 1-form - the connection form on  $P(\mathcal{H})$ . Combining (9), (10) and (11) one gets its explicit form

$$\omega^1 = e^{i\gamma} \langle n | dn \rangle + d e^{i\gamma}. \quad (12)$$

The horizontal lift of a closed curve  $C$  in  $P(\mathcal{H})$  is the trajectory along which  $\omega^1 = 0$  i.e. evolving only in the horizontal direction. Such a trajectory is, in general, not closed in  $E$ . The resulting fibre element  $\gamma_n(C)$  that relates the initial and final states of this trajectory is the holonomy given from (12) by

$$\gamma_n(C) = -Im \oint_C \langle n | dn \rangle. \quad (13)$$

This holonomy is a purely geometrical object depending only on the path in the space of physical states and not on the Hamiltonian. Comparing (13) with (5) and (8) we recognize that the Berry phase as well as the A-A phase are both equal to the holonomy of the bundle  $E$  over the projective Hilbert space  $P(\mathcal{H})$  corresponding to the circuit  $C$ .

## 4 The Geometrical Phase for a $SU(3)$ state

The general normalized quantum state having  $SU(3)$  symmetry is given by

$$|\psi\rangle = e^{i\gamma}|n\rangle = e^{i\gamma}(\sin\theta\cos\phi e^{i(\alpha-\gamma)}|\xi\rangle + \sin\theta\sin\phi e^{i(\beta-\gamma)}|\eta\rangle + \cos\theta|\zeta\rangle) \quad (14)$$

where  $|\xi\rangle, |\eta\rangle$  and  $|\zeta\rangle$  form an arbitrary orthonormal basis,  $\alpha, \beta, \theta$  and  $\phi$  parameterizes the general element  $|n\rangle$  of the  $CP^2$  projective Hilbert space and  $e^{i\gamma}$  represents the element of the structure group. The phase parameters  $(\alpha, \beta)$  belong to the torus  $S_1 \times S_1 \simeq [0, 2\pi]^2$  whereas  $(\theta, \phi)$  are restricted to the positive octant  $S_2/8 \simeq [0, \pi/2]^2$  of the sphere [3].

The connection (12) in this parameterization of  $CP^2$  is  $\omega^1 = ie^{i\gamma}[(\cos^2\theta)d\gamma + \sin^2\theta(\sin^2\phi d\beta + \cos^2\phi d\alpha)]$ , which yields for the geometrical phase

$$\gamma_n(C) = - \oint_C \tan^2\theta(\sin^2\phi d\beta + \cos^2\phi d\alpha). \quad (15)$$

This result was already obtained [6] for a different parameterization of the  $CP^2$  space. In the particular case of  $\theta = \pi/2$  the phase (15) coincides with the  $SU(2)$  case considered by Berry [1].

**Summary** In this paper we have shown different ways of calculating the geometrical phase. We investigated the Berry phase obtained in the adiabatic approximation. We have given three expressions for that phase discussing differences among them. Then we have introduced the A-A phase in the framework of the geometry of the fibre bundle and shown that the adiabatic approximation is *not necessary* in this context. Finally we have calculated the geometrical phase for the parameter space  $CP^2$  of a threefold degenerate state.

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## On a Certain Class of Connections in the U(n)-bundle over Density Matrices

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### 1 Introduction

Recently a lot of work has been done for nonabelian generalizations of Berry's phase [1]. In particular, Uhlmann proposed a connection governing parallel transport along density matrices [2]. In the generic case of nondegenerate not normalized density matrices it leads to an  $u(n)$ -valued connection form  $\omega$  on the principal bundle

$$GL(n, C) \longrightarrow GL(n, C)/U(n) \quad (1)$$

defined implicitly by the equation

$$g^*g\omega + \omega g^*g = g^*dg - dg^*g; \quad g \in GL(n, C). \quad (2)$$

By the polar decomposition  $g = \rho^{\frac{1}{2}}u$ , the space  $GL(n, C)/U(n)$  can be identified with the space of positive matrices. The horizontal subspaces of this connection are just the orthogonal complements of the vertical subspaces with respect to the induced Riemannian metric  $h$ ,  $h(X, Y) = \text{Re Tr } XY^*$ , on  $GL(n, C) \subset gl(n, C)$ . It is rather difficult to write down explicit formulae for  $\omega$  and its curvature  $\Omega$  in terms of natural matrix operations. However, one finds

$$\omega = \omega_0 + (\text{Ad}p + \mathbf{1})^{-1} \circ (\text{Ad}p - \mathbf{1}) \left( \frac{\theta + \theta^*}{2} \right), \quad (3)$$

where

$$p := g^*g, \quad \theta = g^{-1}dg, \quad \omega_0 = \frac{1}{2}(\theta - \theta^*), \quad (4)$$

( $\theta$  - the canonical left invariant 1-form on  $GL(n, C)$ ;  $\omega_0$  - the canonical connection form on the above bundle [3]).

Here we consider a whole class of connections including the above one and calculate the curvature. For  $n=2$  we give explicit formulae for the above connection and its curvature. The proofs and technical details can be found in [5].

## 2 General Results

Note, that the operator

$$z := \text{Ad} p : \text{Gl}(n, \mathbb{C}) \longrightarrow \text{Gl}(n, \mathbb{C}) \quad (5)$$

has the positive spectrum  $\sigma(z) = \left\{ \frac{\lambda}{\mu} \mid \lambda, \mu \text{ are eigenvalues of } p \right\}$ . Therefore,  $f(z)$  is well defined for a function  $f$  on  $\mathbb{R}_+$ . Generalizing solution (3) of equation (2) we obtain the following

**Theorem 1:** Let  $f : \mathbb{R}_+ \longrightarrow \mathbb{C}$  be a  $C^1$ -function satisfying  $f(x) = -\overline{f(x^{-1})}$ . Then the 1-form  $\omega$  defined by

$$\omega := \frac{\theta - \theta^*}{2} + f(z) \left( \frac{\theta + \theta^*}{2} \right), \quad (6)$$

is a connection form on the principal bundle (1) invariant under natural left  $U(n)$ -action.  $\square$

The main difficulty in calculating the curvature  $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$  is to determine the operator valued 1-form  $df(z)$ . For this purpose we introduce a "commutator" of certain operator functions. Obviously, for  $X \in \mathfrak{gl}(n, \mathbb{C})$  the term " $\frac{1}{z-1}X$ " does not make sense, because 1 belongs to the spectrum of  $z$ . However, we may define a generalized commutator of the operators  $f(z)$  and ' $\text{ad} \frac{1}{z-1}X$ ' as follows:

**Definition:** If  $zX = \lambda X$ ,  $X \in \mathcal{G}$ ,  $\lambda \in \sigma(z)$ , let

$$\left[ f(z), \text{ad} \frac{1}{z-1}X \right] := \begin{cases} \frac{1}{\lambda-1} [f(z), \text{ad} X] & \text{if } \lambda \neq 1 \\ z \circ f'(z) \circ \text{ad} X & \text{if } \lambda = 1 \end{cases} \quad (7)$$

For general  $X$  we define the commutator by linearity.

The second case in the definition is just the formal limit of the first one if  $\lambda$  tends to 1. Moreover, the so defined commutator is compatible with the usual commutator and the Riesz-Dunford-calculus [4] (for analytic  $f$  on  $U$ ;  $\mathbb{R}_+ \subset U \subset \mathbb{C}$ ) in the following sense:

$$[f(z), \text{ad} X] = \left[ f(z), \text{ad} \frac{1}{z-1}((z-1)X) \right] \quad (8)$$

and

$$\left[ f(z), \text{ad} \frac{1}{z-1}X \right] = \frac{1}{2\pi i} \oint f(\zeta) \left[ \frac{1}{\zeta-1}, \text{ad} \frac{1}{z-1}X \right] d\zeta. \quad (9)$$

If  $\alpha$  is a  $\mathfrak{gl}(n, \mathbb{C})$ -valued differential form on  $\text{Gl}(n, \mathbb{C})$ , then  $[f(z), \text{ad} \frac{1}{z-1}\alpha]$  is in an obvious way defined as an operator-valued differential form by application of the above Definition to the values of  $\alpha$ . The above Definition is justified by

**Theorem 2:** The differential of the operator valued function  $f(z)$  is given by

$$d(f(z)) = \left[ f(z), \text{ad} \frac{1}{z-1}(dp p^{-1}) \right] \quad \text{with} \quad dp p^{-1} = z\theta + \theta^*. \quad (10)$$



To prove this Theorem one uses the following technical

**Lemma 3:** Let  $X \in \mathfrak{gl}(n, \mathbb{C})$  and  $\zeta \notin \sigma(X)$ . Then

$$\frac{1}{\zeta \mathbf{1} - X} \circ \text{ad } X \circ \frac{1}{\zeta \mathbf{1} - X} = \left[ \frac{1}{\zeta \mathbf{1} - X}, \text{ad} \frac{1}{X - \mathbf{1}} X \right] \circ X^{-1}. \quad (11)$$

□

Using Theorem 2 the curvature  $\Omega$  of the connection form  $\omega$  defined in Theorem 1 is now obtained by a straightforward calculation. The result is the following

**Theorem 3:**

$$\Omega = \Omega_0 + \left[ f(z), \text{ad} \frac{z + \mathbf{1}}{z - \mathbf{1}} \beta \right] \beta + \frac{1}{2} [f(z) \beta, f(z) \beta], \quad (12)$$

where  $\beta = \frac{1}{2}(\theta + \theta^*)$  and  $\Omega_0 = -\frac{1}{8}[\theta + \theta^*, \theta + \theta^*]$  (curvature of the canonical connection form  $\omega_0$ ).

### 3 Example

We consider the case

$$f(\zeta) = \frac{\zeta - \mathbf{1}}{\zeta + \mathbf{1}}. \quad (13)$$

Then

$$\omega = \frac{1}{2}(\theta - \theta^*) + \frac{1}{2} \frac{z - \mathbf{1}}{z + \mathbf{1}}(\theta + \theta^*) \quad (14)$$

coincides by (3) with Uhlmann's connection given by (2). From Theorem 3 we obtain for the curvature of this connection

$$\Omega = \Omega_0 + \left[ \frac{z - \mathbf{1}}{z + \mathbf{1}}, \text{ad} \frac{z + \mathbf{1}}{z - \mathbf{1}} \beta \right] \beta + \frac{1}{2} \left[ \frac{z - \mathbf{1}}{z + \mathbf{1}} \beta, \frac{z - \mathbf{1}}{z + \mathbf{1}} \beta \right]. \quad (15)$$

Thus, an explicit calculation of  $\omega$  and  $\Omega$  essentially amounts to explicit knowledge of  $(z + \mathbf{1})^{-1} \equiv (\text{Ad } p + \mathbf{1})^{-1}$ . We consider the case  $n=2$ . Using the characteristic equation of  $p$  we obtain

$$\frac{1}{z + \mathbf{1}} \equiv (\text{Ad } p + \mathbf{1})^{-1} = \frac{1}{2} \left( \mathbf{1} - \frac{1}{\text{Tr } p} \text{ad } p \right). \quad (16)$$

Denoting

$$q := \frac{1}{\text{Tr } p} p, \quad (17)$$

we get

$$\frac{z - \mathbf{1}}{z + \mathbf{1}} = \text{ad } q. \quad (18)$$

In this notation the connection form is given by the simple formula

$$\omega = \omega_0 + [q, \beta]. \quad (19)$$

The above Lemma, (15), (18) and the the Jacobi identity yield

$$\Omega = \frac{1}{2} \left( [\beta, \beta] - [q, [\beta, [q, \beta]]] - [\beta, [q, [q, \beta]]] \right) . \quad (20)$$

The term on the right hand side can be reduced using the following identity for traceless  $2 \times 2$ -matrices and their consequences for  $\mathfrak{sl}(2, \mathbb{C})$ -valued 1-forms:

$$[X, [Y, Z]] = -2\text{Tr}(XZ) Y + 2\text{Tr}(XY) Z; \quad X, Y, Z \in \mathfrak{sl}(2, \mathbb{C}) . \quad (21)$$

We obtain

$$\Omega = (1 - \text{Tr}(q^2)) [\beta, \beta] = \frac{2 \det p}{(\text{Tr } p)^2} [\beta, \beta] . \quad (22)$$

Thus, the final result for  $n = 2$  is:

$$\omega = \frac{1}{2} (\theta - \theta^*) + \frac{1}{2 \text{Tr } p} [p, \theta + \theta^*] . \quad \Omega = \frac{\det p}{2 (\text{Tr } p)^2} [\theta + \theta^*, \theta + \theta^*] . \quad (23)$$

We observe that this curvature coincides - up to a factor depending on  $p$  - with the curvature  $\Omega_0$  of the canonical connection.

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## The Bures metric and Uhlmann's transition probability: explicit results

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**Abstract:** We give explicit results for the Bures metric on the space of density matrices in the 2- and  $n$ -dimensional cases. At the same time, this applies to Uhlmann's transition probability for mixed states on a  $*$ -algebra. The Bures metric, Uhlmann's connection, the Fubini-Study metric and the Pancharatnam parallelity are special cases of a general construction.

### 1. Generalities

Let us consider two states  $\omega_1, \omega_2$  on a  $*$ -algebra  $M$  and all  $*$ -representations  $\pi : M \rightarrow B(\mathcal{H})$  on Hilbert spaces  $\mathcal{H}$ , for which there exist vectors  $|1\rangle, |2\rangle \in \mathcal{H}$  which induce the states  $\omega_1, \omega_2$ :

$$\omega_i(a) = \langle i | \pi(a) | i \rangle \quad \forall a \in M.$$

Uhlmann introduced in 1976 the generalized transition probability  $P(\omega_1, \omega_2)$  as a supremum

$$P(\omega_1, \omega_2) := \sup | \langle 1 | 2 \rangle |^2 \quad (1)$$

over all admissible representations  $\pi$  and all vector representatives  $|1\rangle, |2\rangle$  of the states [4]. This is intimately connected with the Bures metric  $\tilde{d}$  on the space of states:

$$\tilde{d}(\omega_1, \omega_2) := \inf \| |1\rangle - |2\rangle \| \quad (2)$$

Bures [2] introduced his metric in order to study infinite tensor products of von Neumann algebras, thereby generalizing classical results of Kakutani [1] to the noncommutative case. Besides many interesting facts, he showed the triangle inequality for  $\tilde{d}$ , thus proving the nontrivial proposition that (2) defines a metric after all. From (1) and (2) we get immediately [5]

$$\tilde{d}(\omega_1, \omega_2)^2 = 2(1 - \sqrt{P(\omega_1, \omega_2)}). \quad (3)$$

Thus all results for the Bures metric apply to Uhlmann's transition probability and vice versa. Araki and Raggio proved in [6] that Uhlmann's concept contains Cantoni's generalized transition probability [3] as a special case. The Bures distance was further clarified in deep papers by Araki [7] and Araki and Yamagami [8]. Alberti and Uhlmann proved intuitively appealing results on the behaviour of Uhlmann's transition probability with respect to stochastic maps on state spaces [9].

Results on the explicit form of the Bures metric were lacking for a long time. A. Uhlmann posed this to me as a problem and we collect our findings in the next section. More details can be found in [11,12,13].

### 2. Results

On a type I factor, states are represented by density matrices

$$\omega_i(a) = \text{tr } \rho_i(a)$$

on the representation Hilbert space  $\mathcal{H}$ . We identify  $\omega_i$  and  $\rho_i$ . Using a result of Uhlmann, we can establish

$$\bar{d}(\rho_1, \rho_2)^2 = \inf \operatorname{tr} (W_1 - W_2)(W_1 - W_2)^*, \quad (4)$$

where the infimum runs over all Hilbert-Schmidt matrices fulfilling  $W_i W_i^* = \rho_i$ . It is not hard to show

$$\bar{d}(\rho_1, \rho_2)^2 = \operatorname{tr}(\rho_1 + \rho_2) - 2 \operatorname{tr} \sqrt{\sqrt{\rho_1} \rho_2 \sqrt{\rho_1}}.$$

Consider the simplest nontrivial case, a 2-dimensional Hilbert space. Parametrizing the density matrices by Pauli matrices, we get after computation

$$P(\rho_1, \rho_2) = (\operatorname{tr} \sqrt{\sqrt{\rho_1} \rho_2 \sqrt{\rho_1}})^2 = \operatorname{tr} \rho_1 \rho_2 + 2 \sqrt{\det \rho_1 \rho_2} \quad (5)$$

and the Bures distance follows from (3).

The infinitesimal form of the Bures distance is more interesting. It turns out that the squared distance of two neighbouring density matrices is of second order in the infinitesimal translations. So the metric can be described by a metric tensor. This can be calculated and the result is in invariant matrix operations (i.e. without referring to a certain ONB of  $\mathcal{H}$ ):

$$\bar{d}(\rho, \rho + d\rho)^2 = \frac{1}{2} \operatorname{tr} (d\rho)^2 + (d \sqrt{\det \rho})^2. \quad (6)$$

The space of 2-dimensional normalized density matrices is topologically equivalent to the 3-dimensional solid ball. But (6) is not a metric tensor for a flat metric. Examining the last formula, we get the

**Theorem:** *The set of 2-dimensional normalized density matrices with the Bures metric is isometric to one closed half of the Euclidean 3-sphere with radius  $\frac{1}{2}$ .*

In  $n$  dimensions there is apparently no way to express the Bures metric in invariant matrix operations, avoiding the disturbing square root operation in the definition. But we can write down an explicit formula for the metric tensor of the Bures metric at a certain nonsingular density matrix  $\rho$ , if we have knowledge about the eigenvalues  $\{\lambda_i\}$  and eigenvectors  $\{|i\rangle\}$  of  $\rho$ . Defining the Hermitian matrix  $A(t)$  by the relation

$$A(t)^2 = \sqrt{\rho}(\rho + t d\rho)\sqrt{\rho}$$

( $t$  being a small real parameter), differentiating two times after  $t$  and setting  $t$  equal to zero gives

$$\bar{d}(\rho, \rho + d\rho)^2 = \sum_{i,j} \frac{|\langle i|d\rho|j\rangle|^2}{2(\lambda_i + \lambda_j)}. \quad (7)$$

With the aid of this formula we can obtain useful estimations of the Bures metric from below and above, see the references [12,13].

### 3. A unifying concept

Consider a bundle  $p: E \rightarrow B$  with structure group  $G$  acting transitively on the isomorphic fibres  $p^{-1}(b), b \in B$ . Suppose that the total space  $E$  is given a metric  $d$  and

assume the action of  $G$  on  $E$  to be isometric and continuous with respect to the metric topology on  $E$

$$d(g\epsilon, g\epsilon') = d(\epsilon, \epsilon').$$

Now define the function  $\tilde{d}$  on  $B \times B$  as

$$\tilde{d}(b, b') := \inf_{p(\epsilon)=b, p(\epsilon')=b'} d(\epsilon, \epsilon'). \quad (8)$$

Of course,  $\tilde{d}(b, b') \geq 0$  and  $\tilde{d}(b, b) = 0 \forall b, b' \in B$ . Symmetry of  $\tilde{d}$  in its arguments is evident. For compact fibres  $\tilde{d}$  takes on its infimum on the compact space  $p^{-1}(b) \times p^{-1}(b')$ , thus

$$\tilde{d}(b, b') > 0 \quad \text{for } b \neq b'.$$

Furthermore it is not hard to prove the triangle inequality for  $\tilde{d}$  - it is here, where we need the transitivity of the action on the fibres. We established that  $\tilde{d}$  is a *metric* on the base space  $B$  [12].

Now for even more structure on the bundle. Fix a certain  $\epsilon \in E$ . If the infimum in (8) occurs for a *unique*  $\epsilon' \in p^{-1}(b')$ , then  $\epsilon'$  is defined to be *parallel* to  $\epsilon$ . Every point in the total space  $E$  is parallel to itself. Applying transitivity of the action of  $G$  on the fibres one more time we see:

1. Parallelity is a symmetric relation on  $E$ .

2. If  $\epsilon, \epsilon'$  are parallel, then  $g\epsilon, g\epsilon'$  are parallel for all  $g \in G$ . So the parallelity relation is equivariant with respect to the group action.

This is a rather useful construction. Notice that the failure of this parallelity relation to be *transitive* corresponds to the curvature of the parallel transport.

Example 1: Take a Hilbert space and the Hopf bundle of its unit vectors over the corresponding projective Hilbert space. Then the metric  $\tilde{d}$  turns out to be the Fubini-Study metric on the complex projective space. Our definition of parallelity coincides with Pancharatnam's notion of parallelity: two unit vectors are parallel if their scalar product is positive [14]. (This breaks down for orthogonal subspaces.) Considering the parallel transport infinitesimally, we get the canonical connection in the Hopf bundle, obtained in fundamental work by Aharonov and Anandan [14]. If we pull this bundle back to a parameter space and consider Hamiltonian evolution with adiabatically changing parameters, we get the Berry phase.

Example 2: Take the injective Hilbert-Schmidt operators with the natural metric and project onto the injective positive trace class operators. Then the metric  $\tilde{d}$  is the Bures metric and the parallelity and connection is the one due to Uhlmann [10]. For many more details, see [12,13].

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## **VI. Phase Space and Wigner Distributions**

### The Great Orthogonality Theorem and Informational Completeness

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*Abstract.* From the orthogonality theorem for square integrable representations on homogeneous spaces of a locally compact group, a series of lemmas are derived showing the informational completeness for the generalized Wigner distributions (matrix elements of the group) as well as the informational completeness of the natural covariant localization operators. The results are applied to special (phase space) representations of the Heisenberg, Galilei, and Poincaré groups.

A set of quantum observables is informationally complete if a collection of values assigned to that set of observables uniquely determines a quantum state for which these values are the quantum expectation values of the observables in that state. To perform any determinative measurement, an experimenter must have a (practical) way of measuring such a set. An obvious penalty for measuring only an informationally incomplete set of observables is that the ambiguity of the results permits incorrect conclusions. On the other hand, measuring more than a complete set (oversampling) leads to various efficient methods of error/noise reduction.

Analysis shows that no commuting set of observables is informationally complete. Examples are the family of spectral projectors for position, or through Fourier transform, the family of spectral projectors for momentum. Even the union of these two spectral families is informationally incomplete although not commuting. The set of all bounded self-adjoint operators in the representing Hilbert space is complete, but it is impractical to attempt to measure everything in this set. We show here that the set of joint momentum and position localization observables (as well as spin for non-spin zero systems) has the property of informational completeness for non-relativistic quantum mechanics. This set is termed "the set of (generalized) phase space localization operators." Our mathematical approach also leads to a definition of phase space localization operators for Poincaré invariant quantum mechanics, for both massive and massless particles. In all cases, the localization operators are covariant with respect to the entire symmetry group, and not just with respect to the Euclidean subgroup.

The method used is the following in outline. We begin with a symmetry group  $G$  for the system ( $G =$  Weyl-Heisenberg or Galilean or Poincaré or other kinematical group.) For  $H$  a closed subgroup of  $G$ , consider the homogeneous space  $G/H$  (or  $H\backslash G$ ). Various choices for  $H$  will be considered, but we must be able to carry out the following steps:

Step 1. Find a left  $G$ -invariant measure  $\mu$  on  $G/H$ .

Step 2. In order to have a phase space interpretation of  $G/H$ , we must exhibit a symplectic form through which one can construct  $\mu$  by taking wedge products.

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† This is a report on joint work with D. Healy and J. A. Brooke.



Step 3. Form  $L^2(G/H, \mu)$ . Let  $U^{(i)}$  denote the  $i$ th irreducible representation of  $G$  on some Hilbert space  $\mathcal{H}^{(i)}$ . (These may be obtained by Mackey's method of induced representations.) One finds an isometry,  $W$ , of  $\mathcal{H}^{(i)}$  to a closed subspace  $K^{(i)}$  of  $L^2(G/H, \mu)$  and thereby intertwines  $U^{(i)}$  with a representation,  $V^{(i)}$ , of  $G$  on  $K^{(i)} \subset L^2(G/H, \mu)$ . For example, one might intertwine with the left (quasi-) regular representation,  $V^L$ , on  $L^2(G/H, \mu)$ .

Step 4. For  $f \in L^2(G/H, \mu)$ , and  $\Delta$  a Borel subset of  $G/H$ , define the localization operator  $A(\Delta)$  by  $A(\Delta)[f] = \chi_\Delta f$ , where  $\chi_\Delta$  is the characteristic function of  $\Delta$ .  $\{A(\Delta) | \Delta \text{ is Borel}\}$  is a projection valued measure. It is covariant with respect to  $V^L$ , but is informationally incomplete.

Step 5. With  $P^{(i)}: L^2(G/H, \mu) \rightarrow K^{(i)}$  being the orthogonal projection, define

$$A^{(i)}(\Delta) = W^* P^{(i)} A(\Delta) W.$$

Then  $\{A^{(i)}(\Delta) | \Delta \in \text{Borel}(G/H)\}$  is a positive operator valued measure (localization operator) on the irreducible representation space  $\mathcal{H}^{(i)}$ . If  $W$  intertwines  $U^{(i)}$  with  $V^L$ , then  $A^{(i)}$  is covariant with respect to the entire group  $G$ , not just with respect to a subgroup (such as the Euclidean group.)

Step 6. The localization operator  $A^{(i)}$  must form an informationally complete set in  $\mathcal{H}^{(i)}$ . This is a constraint on the pair  $\{\mathcal{H}^{(i)}, G/H\}$ .

Step 7. An explicit formula for reconstructing a state from the set of expected values of the  $A^{(i)}(\Delta)$  must be given. The formula should have a discretized approximation and must yield a numerically stable algorithm.

The smaller  $H$  is, the more likely the condition in step 6 can be met. On the other hand, the smaller  $H$  is, the more difficult step 3 becomes. Hence, there is an optimal range of choices if not a unique choice for  $H$ . The present analysis, although related to that of Perelomov (1971), differs from his approach in which  $H$  is determined as the stabilizer subgroup of a given vector in the representation space rather than on general principles.

In particular, for  $\eta, f \in \mathcal{H}^{(i)}$  the map  $W = W_\eta$  takes on the form

$$[W_\eta f](x) = \langle U^{(i)}(\sigma(x))\eta, f \rangle$$

where  $\sigma$  is a Borel cross section from  $G/H$  to  $G$ . The condition that  $W_\eta$  be an isometry is then closely related to the condition that  $\eta$  must be square integrable with respect to  $\sigma(G/H)$ . We wish to derive the orthogonality relations as a consequence of square integrability. These relations read

$$\int_{G/H} \langle \phi_1, U^{(i)}(\sigma(x))\eta_1 \rangle \langle U^{(i)}(\sigma(x))\eta_2, \phi_2 \rangle d\mu(x) = \langle C\eta_2, C\eta_1 \rangle \langle \phi_1, \phi_2 \rangle$$

for some self-adjoint operator  $C$  in  $\mathcal{H}^{(i)}$ ,  $\phi_j, \eta_j \in \mathcal{H}^{(i)}$ ,  $\eta_j$  admissible. If  $H$  is in the center of  $G$ , these results are known (Grossmann et al, 1985). From these orthogonality relations the informational completeness of the  $A^{(i)}$  follows (see below).

The details for steps 1 through 5 for the Heisenberg and Galilei groups are known (Guillemin and Sternberg, 1984). For the Poincaré group, they were recently investigated by J. Brooke and the author for obtaining fully covariant phase space localization operators for both massive and massless relativistic particles, such as the photon (Brooke and Schroeck, 1989 and in preparation). In the non-relativistic cases and the massive Poincaré case, the intertwining of the  $U^{(i)}$  is with the left regular representation  $V^L$ . In the non-relativistic cases the only choice for  $G/H$  such that all steps are satisfied seems to fix  $G/H$  to be configuration space  $\times$  momentum space  $\times$  spin space.

To satisfy steps (6) and (7) consider the following:

**Definition:** A nonzero  $\eta \in \mathcal{H}^{(i)}$  is admissible with respect to  $\sigma(G/H)$  iff

$$\int_{G/H} |\langle U^{(i)}(\sigma(x))\eta, \eta \rangle|^2 d\mu(x) < \infty.$$

If  $U^{(i)}$  is irreducible, we say  $U^{(i)}$  is square integrable over  $\sigma(G/H)$ .

Condition (a). There exists  $\alpha: H \rightarrow \mathbf{C}$  such that  $U(h)\eta = \alpha(h)\eta \forall h \in H$ . If this holds and  $\eta$  is admissible, we say  $\eta$  is  $\alpha$ -admissible.

Condition (b). There exists  $B: G/H \times G/H \rightarrow \mathbf{C}$  and a dense set of 1-admissible vectors  $\eta$  such that

$$B(x, y)\eta = U(\sigma(x)^{-1}\sigma(y)^{-1}\sigma(x)\sigma(y))\eta \quad \forall x, y \in G/H.$$

D. Healy and the author (in preparation) show that (a) & (b) imply that our square integrability condition implies that of Ali *et al* (1991). (a) implies that the results are section independent. (a) implies that the orthogonality theorem holds on the subspace of  $\mathcal{H}_\alpha$  generated by the  $\alpha$ -admissible vectors; (a) & (b) imply that the operator  $C$  in the orthogonality condition is a multiple of the identity.

Now let  $\psi \in \mathcal{H}_\alpha$ ,  $\|\psi\| = 1$ , and let  $\{e_j\}$  be an orthonormal basis for  $\mathcal{H}_\alpha$ . Then from the orthogonality theorem we obtain

$$\int_{G/H} \langle e_j, U^{(i)}(\sigma(x))\eta \rangle \langle U^{(i)}(\sigma(x))\psi, \psi \rangle d\mu(x) = \langle C\psi, C\eta \rangle \langle e_j, \psi \rangle.$$

Thus the set  $\{\langle U^{(i)}(\sigma(x))\psi, \psi \rangle \mid x \in G/H\}$  of "generalized Wigner coefficients" determines  $\psi$  up to a phase; conditions (a), (b) imply the informational completeness of the Wigner coefficients in  $\mathcal{H}_\alpha$ , modulo the fact that the  $U^{(i)}(\sigma(x))$  are not observables.

The localization operators  $A^{(i)}$  are shown to have a projection valued density  $T^\eta$ :  $A^{(i)}(\Delta) = \int_\Delta T^\eta(x) d\mu(x)$ . Healy and the author then show

$$\int_{G/H} B(x, y) \langle \psi, T^\eta(x)\psi \rangle d\mu(x) = \langle C\eta, CU(\sigma(y))\eta \rangle \langle U(\sigma(y))\psi, \psi \rangle,$$

for  $\eta \in \mathcal{H}_\alpha$ ,  $\psi \in \mathcal{H}$ ,  $y \in G/H$ . By the result on Wigner coefficients, this shows that the set  $\{T^\eta(x) \mid x \in G/H\}$ , which is equivalent to  $\{A^{(i)}(\Delta) \mid \Delta \in \text{Borel}(G/H)\}$ , is

informationally complete on  $\mathcal{H}_\alpha$ . The algorithm this yields is not as efficient as the following:

Since the  $T^n$  are informationally complete in  $\mathcal{H}_\alpha$ , any density operator  $\rho$  in  $\mathcal{H}_\alpha$  may be written in the form  $\rho = \int_{G/H} f(y) T^n(y) d\mu(y)$ . Then one shows

$$\int_{G/H} B(y, k) \text{Tr}(\rho T^n(y)) d\mu(y) = \langle C\eta, CU(\sigma(k))\eta \rangle \langle U(\sigma(k))\eta, \eta \rangle \\ \times \int_{G/H} f(x) \overline{B(x, k)} d\mu(x).$$

Thus, if

$$(c) \quad \langle U(\sigma(k))\eta, \eta \rangle \neq 0 \quad \text{a.e. } k \in G/H$$

holds, then one may reconstruct  $\rho$  via an integral formula if one can reconstruct  $f$  from its  $B$ -transform.

Healy and the author show that (a), (b), (c) may be satisfied in the non-relativistic cases and that the  $B$ -transform is the Fourier transform. Discrete approximations to this integral provide the numerically stable procedures desired in steps 6 and 7.

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## PHASE-SPACE- AND OTHER REPRESENTATIONS OF QUANTUM MECHANICS: A UNIFYING APPROACH

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*Abstract:* Bi-orthogonal systems in Liouville space, which are parametrized by phase space, provide a convenient way of formal insight into the common structure of all linear phase-space representations of quantum mechanics. The farthest reaching generalization of the early Weyl-Wigner representation is achieved by replacing phase space by a general parameter space  $\Gamma$  equipped with a measure and possibly with a group structure. Positivity and invariance properties of the representations with respect to transformation groups on  $\Gamma$  are studied in relation to coherent states.

### 1 Introduction: Motivation and goal

The choice of a representation (rep) of Hilbert-space operators is necessary for most concrete calculations in quantum mechanics. In view of many recent publications on Weyl-Wigner-Moyal theory, positive phase-space reps, coherent states, squeezed states, wavelets with applications in quasiclassical physics, quantum chaos, (quantum) optics, signal processing etc., we are concerned with a unifying approach which reveals the common (mathematical) structure of these kinds of reps. Combining and generalizing ideas of various authors [1-7], we provide an overall view for orientation and classification of special results and aim at a basis for better intuition. In doing so we stay at a formal level not caring about questions of mathematical rigor.

### 2 The general formalism: $\Gamma$ -representations

Consider the Liouville space  $\mathcal{L}(\mathcal{H})$  consisting of operators acting on a given Hilbert space  $\mathcal{H}$ . Expectation values are then given by the Liouville-space scalar product  $\text{Tr}[WA]$  of state operators  $W = W^+ \geq 0$ ,  $\text{Tr } W = 1$  with observable operators  $A = A^+$ . A set  $\Gamma$  equipped with a measure  $\mu$  serves as a parameter space. To represent elements of  $\mathcal{L}(\mathcal{H})$  by functions on  $\Gamma$ , it is useful to introduce bi-orthogonal operator systems  $\{\Delta(\alpha), \bar{\Delta}(\beta) : \alpha, \beta \in \Gamma\}$ , obeying the *bi-orthogonality relation*

$$\text{Tr} [\bar{\Delta}^+(\alpha) \Delta(\beta)] = \delta(\alpha, \beta) \quad (\alpha, \beta \in \Gamma) \quad (1)$$

and the *completeness relation* in  $\mathcal{L}(\mathcal{H})$

$$\int_{\Gamma} d\mu(\gamma) \bar{\Delta}^+(\gamma) A \Delta(\gamma) = \mathbf{1} \text{Tr } A \quad (A \in \mathcal{L}(\mathcal{H})). \quad (2)$$

$\bar{\Delta}$  is called the dual basis for  $\Delta$ ,  $\delta$  stands for a Kronecker- or Dirac-delta depending on whether  $\Gamma$  is discrete or continuous.

As is easily shown by considering  $A = |n\rangle\langle n'|$ , where  $\{|n\rangle\}$  denotes an orthonormal basis in  $\mathcal{H}$ , eq. (2) is equivalent to the operator expansion

$$A = \int_{\Gamma} d\mu(\gamma) \operatorname{Tr} [\bar{\Delta}^+(\gamma) A] \Delta(\gamma). \quad (3)$$

An operator  $A$  is thus represented by its expansion coefficients with respect to the  $\mathcal{L}(\mathcal{H})$ -basis  $\Delta$ . This complex-valued function on  $\Gamma$ ,

$$a_{\bar{\Delta}}(\gamma) := \operatorname{Tr} [\bar{\Delta}^+(\gamma) A], \quad (4)$$

is called, extending a notion of [1], the *contravariant symbol* of  $A$ . Replacing  $\bar{\Delta}$  by  $\Delta$ , one obtains the *covariant symbol*  $a_{\Delta}(\gamma)$ .

Given  $\Delta$  and  $\bar{\Delta}$ , the scalar product in  $\mathcal{L}(\mathcal{H})$  is representable as

$$\operatorname{Tr} [A^+ B] = \int_{\Gamma} d\mu(\gamma) a_{\bar{\Delta}}^*(\gamma) b_{\Delta}(\gamma). \quad (5)$$

Remarks:

- The symbols are linear in the represented operators. For nonlinear reps see [8].
- The contravariant symbol of  $\Delta(\gamma')$  is  $\delta(\gamma, \gamma')$ , see (1) and (4).
- A self-dual basis  $\Delta = \bar{\Delta}$  is orthonormal, see (1).
- Change of rep by use of a generalized matrix

$$a_{\Delta'}(\gamma') = \int_{\Gamma} d\mu(\gamma) t^*(\gamma', \gamma) a_{\Delta}(\gamma). \quad (6)$$

Here the transformation kernel  $t(\gamma', \gamma) := \operatorname{Tr} [\bar{\Delta}^+(\gamma) \Delta'(\gamma')]$  represents the new basis  $\Delta'(\gamma')$ ,  $\gamma' \in \Gamma'$  with respect to the old one  $\Delta(\gamma)$ ,  $\gamma \in \Gamma$ .

- Phase-space reps ( $\mathcal{H} \cong L^2(\mathbf{R})$ ,  $\Gamma = \mathbf{R}^2 = \{(p, q)\}$ ) are treated, within this approach, in [2]; for special cases cf. also [3,9].

Two important examples:

1. Ordinary Hilbert-space reps with respect to an orthonormal basis  $\{|n\rangle\}$  in  $\mathcal{H}$  fit into the  $\mathcal{L}(\mathcal{H})$ -approach [4]:  
 $\Gamma = \mathbf{N}^2$ ,  $\int d\mu(\gamma) = \sum_{n, n'}$ ,  $\Delta(\gamma) = |n\rangle\langle n'| \in \mathcal{L}(\mathcal{H})$ ,  $a_{\Delta}(\gamma) = \langle n|A|n'\rangle$ .
2. Let  $\Gamma$  be a group with a unitary irreducible rep  $U$  as an orthonormal basis in  $\mathcal{L}(\mathcal{H})$  and  $\mu$  a Haar-measure on  $\Gamma$ . (2) is obvious for  $\Delta = U$  by Schur's lemma. It can be shown that  $A \geq 0$  iff its  $U$ -symbol  $a_U(\gamma) := \operatorname{Tr} [U^+(\gamma) A]$  is  $\Gamma$ -positive definite, i.e. the  $n \times n$ -matrix  $(a_U(\gamma_i \circ \gamma_j^{-1}))_{i, j=1, \dots, n}$  is positive for all  $\{\gamma_1, \dots, \gamma_n\} \subset \Gamma$  and all  $n \in \mathbf{N}$  (cf.[10]).

For the Heisenberg-Weyl group  $\Gamma$ ,  $\Gamma$ -positive definiteness reduces to  $\hbar$ -positive definiteness [10,11] except for a phase factor arising because the Weyl operators  $U(\gamma) = D(p, q) := e^{i(pQ - qP)/\hbar}$  ( $QP - PQ = i\hbar$ ) constitute only a ray rep of  $\Gamma$ .

### 3 $G$ -covariant $\Gamma$ -representations and $G$ -coherent states

Now let  $G$  be a transformation group acting on  $\Gamma$ ,  $d\mu(\gamma) = d\mu(g\gamma)$  for all  $g \in G$  and  $U$  a unitary irreducible rep of  $G$  in  $\mathcal{L}(\mathcal{H})$ . The requirement that  $a_\Delta(g\gamma)$  represents  $U^+(g)AU(g)$ , when  $a_\Delta(\gamma)$  represents  $A$ , yields the  $G$ -covariance condition

$$\Delta(g\gamma) = U(g)\Delta(\gamma)U^+(g) \quad \text{for all } g \in G, \gamma \in \Gamma. \quad (7)$$

For  $G$  acting transitively on  $\Gamma$  (i.e. for all  $\gamma_1, \gamma_2$  there is  $g: \gamma_2 = g\gamma_1$ ), (7) reduces to

$$\Delta(\gamma) = U(g)\Delta(\gamma_0)U^+(g), \quad \gamma = g\gamma_0. \quad (8)$$

$\Delta$  is uniquely determined by  $\Delta(\gamma_0)$ , which has to be invariant under the isotropy group of  $\gamma_0$  (i.e. for  $g$  with  $g\gamma_0 = \gamma_0$ ).

$\Gamma$ -reps (with basis  $\Delta$  in  $\mathcal{L}(\mathcal{H})$ ) are called *real*, *positive* and *normalized* iff  $\Delta(\gamma) = \Delta^+(\gamma)$ ,  $\Delta(\gamma) \geq 0$  and  $\text{Tr}\Delta(\gamma) = 1$  for all  $\gamma \in \Gamma$ , respectively. Real reps associate  $a_\Delta = a_\Delta^*$  with  $A = A^+$ , positive reps associate  $a_\Delta \geq 0$  with  $A \geq 0$ , and normalized reps yield  $a_\Delta = 1$  for  $A = \mathbf{1}$ , according to (4).

For  $G$ -covariant positive normalized  $\Gamma$ -reps, (8) takes the form

$$W(g) = U(g)W_0U^+(g). \quad (9)$$

$W(g)$  are the  $G$ -coherent (mixed!) states [12] with fiducial state  $\Delta(\gamma_0) = W_0$ .

Examples and remarks:

1.  $G =$  translations on  $\Gamma = \mathbb{R}^2 = \{(p, q)\}$ ,  $W_0 = |\psi_0\rangle\langle\psi_0|$ . The rep of  $A$  by its expectation values  $\langle\psi_0|D^+(p, q)AD(p, q)|\psi_0\rangle$  in the well-known canonical coherent states is obviously positive for each fiducial vector  $|\psi_0\rangle$ . By choosing  $|\psi_0\rangle$  as a squeezed harmonic-oscillator ground state, one obtains the rep by squeezed states [13]. For the analogous introduction of coherent states in case of a discrete finite  $\Gamma$ , relevant for quantum lattice systems, see [14].
2.  $G =$  affine group:  $q$ -translations and scale transformations,  $W_0 = |\psi_0\rangle\langle\psi_0|$ ,

$$U(q, \lambda) = \exp\left\{-\frac{i}{\hbar}qP\right\} \exp\left\{-\frac{i}{\hbar}\frac{\ln \lambda}{2}(PQ + QP)\right\}, \quad \lambda > 0. \quad (10)$$

$A$  is represented as  $a(q, \lambda) = \langle\psi_0|U^+(q, \lambda)AU(q, \lambda)|\psi_0\rangle$ , especially  $|\psi\rangle\langle\psi|$  as  $\left|\int dx \frac{1}{\sqrt{\lambda}} \psi_0^*\left(\frac{x-q}{\lambda}\right) \psi(x)\right|^2$ , the square modulus of the *wavelet-transform* of  $\psi$  with analyzing wavelet  $\psi_0$  [15].

#### 4 Quantum mechanics in phase space

All translation-covariant phase-space reps are, according to (8), characterized by  $\Delta(0,0)$ , or alternatively, by the " $\Omega$ -function"  $\Omega(p,q) := \text{Tr} [D^+(p,-q)\Delta(0,0)]$ . Conversely,  $\Delta$  is given by  $\Omega$  as [2]

$$\Delta(p,q) = \int \frac{dp'dq'}{2\pi\hbar} \Omega(p',q') \exp \left\{ \frac{i}{\hbar} [p'(Q-q) + q'(P-p)] \right\}. \quad (11)$$

Analogously,  $\bar{\Delta}$  is given by  $1/\Omega^*$ . For special cases and detailed considerations in this  $\Omega$ -approach see [2,3,5] and cf. also [6]. For  $\Omega = 1$  one gets the famous (non-positive) Weyl-Wigner rep [7], corresponding to a  $\mathcal{L}(\mathcal{H})$ -basis of shifted parity operators [16], which is uniquely determined by the requirement of metaplectic covariance [17].

*Conclusion:* Bi-orthogonal systems in Liouville space are a natural general approach to  $\Gamma$ -representations ( $\Gamma$ -reps), leading to ordinary Hilbert-space reps, phase-space reps, coherent-state reps and wavelet transformations.

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## The Dual Nature of Phase-Space Representations

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In this communication we discuss the well-known class of phase-space representations defined by state-independent kernels,  $f(\theta, \tau)$ . We show that for a given  $f(\theta, \tau)$ , we must also consider the dual representation defined by the kernel  $g(\theta, \tau) = f(-\theta, -\tau)^{-1}$  and thus involve two different phase-space Hamiltonians. This is of particular importance for the construction of dynamical equations.

We tie our discussion to the one-dimensional motion of a physical system with Cartesian coordinate  $q$  and conjugate momentum  $p$ . The corresponding quantum-mechanical operators are denoted  $\hat{Q}$  and  $\hat{P}$ ; they satisfy the commutation relation  $[\hat{Q}, \hat{P}] = -i\hbar$ . The restriction to a single dimension is made for simplicity, and the extension to several dimensions is straightforward.

Let  $|\psi\rangle$  be a normalized state vector in the Hilbert space associated with our system, and let

$$\psi(q) = \langle q|\psi\rangle, \quad \phi(p) = \langle p|\psi\rangle \quad (1)$$

be the corresponding position and momentum wave functions, in the notation of Dirac. They are normalized to unity,

$$\langle \psi|\psi\rangle = \int dq \psi^*(q) \psi(q) = \int dp \phi^*(p) \phi(p) = 1 \quad (2)$$

and are connected by a Fourier transformation:

$$\psi(q) = \sqrt{\frac{1}{2\pi\hbar}} \int dp \phi(p) e^{ipq/\hbar}, \quad \phi(p) = \sqrt{\frac{1}{2\pi\hbar}} \int dq \psi(q) e^{-ipq/\hbar} \quad (3)$$

with all integrations here and elsewhere going from  $-\infty$  to  $\infty$ . Then the quantity  $\psi^*(q)\psi(q)$  measures the probability density in position space and  $\phi^*(p)\phi(p)$  the probability density in momentum space. We may also consider the more general quantities  $\psi_i^*(q)\psi_j(q)$  and  $\phi_i^*(p)\phi_j(p)$  which are probability densities when  $i$  and  $j$  refer to the same state ( $i = j$ ), and transition densities when  $i$  and  $j$  refer to different states ( $i \neq j$ ).

Now, let  $f(\theta, \tau)$  be any well behaved function for which

$$f(0, \tau) = f(\theta, 0) = 1 \quad (4)$$



Then each such function defines a phase-space representation [1, 2], according to the following prescriptions.

For any pair of states,  $|\psi_i\rangle$  and  $|\psi_j\rangle$ , we define the *phase-space distribution function*

$$\mathcal{D}_{ij}^f(q, p) = \frac{1}{4\pi^2} \int \int \int dud\theta d\tau \psi_i(u - \frac{1}{2}\hbar\tau)^* \psi_j(u + \frac{1}{2}\hbar\tau) f(\theta, \tau) e^{-i\theta q} e^{-i\tau p} e^{i\theta u} \quad (5)$$

Its marginal densities are, independently of the form of  $f(\theta, \tau)$ :

$$\int dp \mathcal{D}_{ij}^f(q, p) = \psi_i(q)^* \psi_j(q), \quad \int dq \mathcal{D}_{ij}^f(q, p) = \phi_i(p)^* \phi_j(p) \quad (6)$$

This is readily verified by using Equation 4 and noting that

$$\int dy e^{ixy} = 2\pi \delta(y) \quad (7)$$

Next, we construct a correspondence between operators on Hilbert space and *dynamical functions on phase space*. Thus, we take the dynamical function corresponding to the operator  $\hat{A}$  to be [3]:

$$a_f(q, p) = \frac{\hbar}{2\pi} \int \int \int dud\theta d\tau \langle u + \frac{1}{2}\hbar\tau | \hat{A} | u - \frac{1}{2}\hbar\tau \rangle \frac{1}{f(-\theta, -\tau)} e^{-i\theta q} e^{-i\tau p} e^{i\theta u} \quad (8)$$

We shall refer to  $a_f(q, p)$  as the *f-transform* of  $\hat{A}$ . The inverse of relation 8 is:

$$\hat{A} = \int \int d\theta d\tau f(\theta, \tau) \alpha_f(\theta, \tau) e^{i(\theta \hat{Q} + \tau \hat{P})} \quad (9)$$

where  $\alpha_f(\theta, \tau)$  is the Fourier transform of  $a_f(q, p)$ , obtained by writing:

$$a_f(q, p) = \int \int d\theta d\tau \alpha_f(\theta, \tau) e^{i(\theta q + \tau p)} \quad (10)$$

With the above definitions, we can show that

$$\langle \psi_i | \hat{A} | \psi_j \rangle = \int \int dq dp a_f(q, p) \mathcal{D}_{ij}^f(q, p) \quad (11)$$

This is a central relation in the phase-space formulation of quantum mechanics.

Let us now assume that  $a_f(q, p)$  and  $b_f(q, p)$  are the phase-space functions corresponding to the operators  $\hat{A}$  and  $\hat{B}$ , respectively. It is then a fundamental problem to determine the phase-space function corresponding to the operator product  $\hat{A}\hat{B}$ . We denote this function by  $c_f(q, p)$  and write:

$$\hat{C} = \hat{A}\hat{B}, \quad c_f(q, p) = a_f(q, p) * f * b_f(q, p) \quad (12)$$

As indicated by the symbol  $*f*$ , the actual form of this *twisted product* depends on the form of the function  $f(\theta, \tau)$ . It is relatively simple in some specific cases [1,4], much more complicated in others, but we shall not need explicit expressions in the

present context. Assuming the expressions to be known, we may also evaluate the *f*-bracket:

$$\{a_f(q, p), b_f(q, p)\}_f = \frac{1}{i\hbar} (a_f(q, p) * f * b_f(q, p) - b_f(q, p) * f * a_f(q, p)) \quad (13)$$

which corresponds to  $1/i\hbar$  times the commutator

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \quad (14)$$

Knowing the form of the *f*-bracket 13, we may go on and determine the phase-space equivalent of the dynamical equation:

$$i\hbar \frac{d\hat{A}}{dt} = [\hat{A}, \hat{H}] + i\hbar \frac{\partial \hat{A}}{\partial t} \quad (15)$$

where  $\hat{H}$  is the Hamiltonian and  $t$  the time. This is the familiar *Heisenberg equation*. It refers to operators expressed in the *Heisenberg picture*. The phase-space equivalent is:

$$\frac{da_f(q, p, t)}{dt} = \{a_f(q, p), h_f(q, p)\}_f + \frac{\partial a_f(q, p, t)}{\partial t} \quad (16)$$

where  $h_f(q, p)$  is the *f*-transform of the Hamiltonian  $\hat{H}$ .

Another important dynamical equation is the *quantum Liouville equation*, also known as *von Neumann's equation*. It refers to the *Schrödinger picture* and has the form:

$$i\hbar \frac{\partial}{\partial t} \hat{\rho}_{ij} = [\hat{H}, \hat{\rho}_{ij}] \quad (17)$$

where

$$\hat{\rho}_{ij} = \frac{1}{2\pi\hbar} |\psi_j\rangle\langle\psi_i| \quad (18)$$

The factor  $1/2\pi\hbar$  is optional. It is included here for convenience.

To determine the phase-space equivalent of the quantum Liouville equation we begin by noting that the relation 1 allows us to write the expression 5 for  $\mathcal{D}_{ij}^f(q, p)$  in the form:

$$\mathcal{D}_{ij}^f(q, p) = \frac{1}{4\pi^2} \int \int \int dud\theta d\tau \langle u + \frac{1}{2}\hbar\tau | \psi_j \rangle \langle \psi_i | u - \frac{1}{2}\hbar\tau \rangle f(\theta, \tau) \epsilon^{-i\theta q} \epsilon^{-i\tau p} \epsilon^{i\theta u} \quad (19)$$

or, by introducing  $\hat{\rho}_{ij}$  from Equation 18:

$$\mathcal{D}_{ij}^f(q, p) = \frac{\hbar}{2\pi} \int \int \int dud\theta d\tau \langle u + \frac{1}{2}\hbar\tau | \hat{\rho}_{ij} | u - \frac{1}{2}\hbar\tau \rangle f(\theta, \tau) \epsilon^{-i\theta q} \epsilon^{-i\tau p} \epsilon^{i\theta u} \quad (20)$$

This equation may also be written:

$$\mathcal{D}_{ij}^f(q, p) = \frac{\hbar}{2\pi} \int \int \int dud\theta d\tau \langle u + \frac{1}{2}\hbar\tau | \hat{\rho}_{ij} | u - \frac{1}{2}\hbar\tau \rangle \frac{1}{g(-\theta, -\tau)} \epsilon^{-i\theta q} \epsilon^{-i\tau p} \epsilon^{i\theta u} \quad (21)$$

where

$$g(\theta, \tau) = \frac{1}{f(-\theta, -\tau)} \quad (22)$$

The function  $g(\theta, \tau)$  will also define a phase-space representation, and by comparing with Equation 8 we see, in fact, that  $\mathcal{D}_{ij}^f(q, p)$  may be identified with the  $g$ -transform of  $\hat{\rho}_{ij}$ . To turn Equation 17 into an equation of motion for  $\mathcal{D}_{ij}^f(q, p)$  we must therefore take the  $g$ -transform on both sides, not the  $f$ -transform. This gives:

$$\frac{\partial}{\partial t} \mathcal{D}_{ij}^f(q, p, t) = \{h_g(q, p), \mathcal{D}_{ij}^f(q, p, t)\}_g \quad (23)$$

where  $h_g(q, p)$  is the  $g$ -transform of the Hamiltonian. It is generically different from  $h_f(q, p)$ , although it may be the same in particular cases.

The phase-space equivalents of the two fundamental equations of motion, i.e., the Heisenberg equation and the quantum Liouville equation, must accordingly be constructed by different procedures. This was already noted, albeit not explained, by Mehta [5].

The phase-space representations induced by  $f(\theta, \tau)$  and  $g(\theta, \tau)$  may be considered the duals of each other, and we see that both must be applied together to cover the phase-space description of a quantum system. A great simplification appears when  $f(\theta, \tau) = 1$ , for then  $f(\theta, \tau)$  and  $g(\theta, \tau)$  become the same, i.e., the phase-space representation becomes self-dual. This representation is the Weyl-Wigner representation.

As an example, we mention that the so-called standard and antistandard representations are the duals of each other. This special example has been discussed by the author in a different context [4].

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# New Constructions of Positive Probability Distributions in Phase Space

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## Abstract

The existence of phase space distributions which are *true* probabilities depends critically on Wigner's theorem, which is however only applicable to phase space functions which are *linear* functionals of the density matrix. We present therefore a complete analysis of distributions which are *quadratic* functionals of the density matrix. There exists a *unique* positive representation with the correct quantum-mechanical marginals. This representation corresponds to a new concept of joint probability distribution. We introduce a scheme which holds the possibility to describe a *single event* as well as the *ensemble*.

## 1 Positive joint probability distributions

Perhaps the most convenient way towards a realistic underpinning of non-relativistic quantum phenomena is the representation of quantum mechanics by joint distributions in phase space [1]. The existence of *true* (positive) joint probability densities (jpd) in phase space is severely restricted by Wigner's theorem [2], which states that the following requirements are incompatible: *i*) the distribution function is a linear functional of the density matrix, *ii*) it is a true (positive) probability function, *iii*) the marginal distributions coincide with the proper quantum-mechanical probabilities (in the stronger version of Kruszynski and de Muynck [3] only one marginal distribution has to coincide).

In contradistinction to a widespread belief, different ways are left open for the construction of jpd in different interpretations [4]. Jpd of the *first* kind (i.e. where the jpd is interpreted as the probability that the variables  $q$  and  $p$  have certain values, the variable being considered as a property of the object system) and jpd of the *second* kind (i.e. where the jpd is considered as the probability that on simultaneous measurements of the observables  $Q$  and  $P$  the values  $q$  and  $p$  are found) which are linear functionals of the density matrix, exist only with relation to a restricted class of functions (e.g. Wigner distributions corresponding to pure

states are necessarily Gaussians in phase space [5] [6]). Jpd which are nonlinear functionals of the state function are not restricted by Wigner's theorem. In this talk we present a complete analysis of jpd which are *bilinear* functionals of the density matrix; however, there exist also examples of jpd which are multilinear functionals of the density matrix [7]. Jpd of the *third* kind are not functionals of the density operator only, but also depend on the measurement arrangement of two incompatible observables  $Q$  and  $P$ . The measurements mutually influence each other in such a way that the singly measured quantum probability function cannot be reproduced from the measurement results. So it is no longer desirable that the marginal probability distributions equal the single measured ones, and hence Wigner's theorem does not restrict this class of jpd either. The jpd of the *third* kind can be written as a convolution of two Wigner functions, the first corresponding to the object system and the second representing the measurement apparatus.

## 2 Bilinear functionals of the density matrix

Using Galilei-invariance, jpd with proper marginals which are bilinear functionals of the density matrix  $\rho$  corresponding to a pure state, can be represented by

$$f(q, p) = \frac{1}{4\pi^2\hbar} \int \Omega(\theta, \theta', \tau, \tau') \exp\left\{i\left[\left(\frac{x+y}{2} - q - \frac{\tau}{2}\right)\theta - \left(\frac{x'+y'}{2} - q - \frac{\tau'}{2}\right)\theta' + \frac{p}{\hbar}(\tau - \tau')\right]\right\} \langle x | \rho | y \rangle \langle y' | \rho^\dagger | x' \rangle d\theta d\theta' dx dy dx' dy' \quad (1)$$

in two-dimensional phase space. For a general density matrix  $\rho = AA^\dagger$  we introduce a straightforward generalisation by replacing  $\rho$  by  $A$  and  $\rho^\dagger$  by  $A^\dagger$  in eq. (1). The requirements to obtain the proper marginals and positivity yield:

$$\int f(q, p) dp = \langle q | \rho | q \rangle \implies \Omega(\theta, \theta', \tau, \tau) = 1. \quad (2)$$

$$\int f(q, p) dq = \langle p | \rho | p \rangle \implies \Omega(\theta, \theta, \tau, \tau') = 1. \quad (3)$$

$$f(q, p) \geq 0 \implies \Omega(\theta, \theta', \tau, \tau') = \sum_i \lambda_i \Omega_i(\theta, \tau) \Omega_i^*(\theta', \tau'). \quad (4)$$

with  $\lambda_i \geq 0$ . On account of the bilinearity these requirements are *compatible*. Moreover, we can show that they have the *unique* solution  $\Omega(\theta, \theta', \tau, \tau') = 1$ , leading in general to the jpd

$$f(\vec{q}, \vec{p}) = |\langle \vec{q} | A | \vec{p} \rangle|^2. \quad (5)$$

This jpd corresponds to a quantummechanical state if  $A^\dagger A = AA^\dagger$  (for a pure state  $A = \rho$ ). E.g. if  $\langle q | A | p \rangle$ , the complex phase space amplitude, is Gaussian, then  $A^\dagger A = AA^\dagger$  yields Heisenberg's uncertainty relation in Schrödinger form:  $\sigma_q \sigma_p - \sigma_{q,p}^2 \geq \hbar^2/4$ .

The representation (5) corresponds to a *new concept* of jpd. Let us consider the wave function  $\psi(\vec{q}; E, L, L_z)$  in coordinate representation with eigenvalues  $E, L, L_z$

as the amplitude of a jpd of position, energy  $E$  and angular momentum  $L, L_z$  or  $\langle \vec{q} | A | E, L, L_z \rangle$ . By eliminating the eigenvalues  $E, L, L_z$  in favour of the momentum  $\vec{p}$  we obtain  $\Sigma \langle \vec{q} | A | E, L, L_z \rangle \langle E, L, L_z | \vec{p} \rangle = \langle \vec{q} | A | \vec{p} \rangle$  leading also to eq. (5).

### 3 Evolution in phase space

Because the phase space amplitude  $\psi(\vec{q}, \vec{p}) = \langle \vec{q} | A | \vec{p} \rangle$  has the same transformation properties as a density matrix, the evolution equation in phase space is

$$-\frac{\hbar}{i} \frac{\partial \psi(\vec{q}, \vec{p})}{\partial t} = \left[ H\left(\vec{q}, \frac{\hbar}{i} \frac{\partial}{\partial \vec{q}}\right) - H\left(\frac{\hbar}{i} \frac{\partial}{\partial \vec{p}}, \vec{p}\right) \right] \psi(\vec{q}, \vec{p}) \quad (6)$$

By setting  $\psi = f \exp(iS/\hbar)$  we obtain two coupled equations for the real functions  $f(\vec{q}, \vec{p})$  and  $S(\vec{q}, \vec{p})$ . For quadratic Hamiltonians we obtain a *linear* quantum Liouville equation and a generalised Hamilton-Jacobi equation where the last term, depending on the Lagrange function  $L$ , is the analogue of Bohm's quantum potential, but now in *phase space*:

$$\frac{\partial f(\vec{q}, \vec{p})}{\partial t} + \frac{\partial}{\partial \vec{q}} \left[ \frac{\partial H}{\partial \vec{p}} \left( \frac{\partial S}{\partial \vec{p}}, \vec{p} \right) f \right] - \frac{\partial}{\partial \vec{p}} \left[ \frac{\partial H}{\partial \vec{q}} \left( \vec{q}, \frac{\partial S}{\partial \vec{q}} \right) f \right] = 0 \quad (7)$$

$$\frac{\partial S}{\partial t} + H \left( \vec{q}, \frac{\partial S}{\partial \vec{q}} \right) - H \left( \frac{\partial S}{\partial \vec{p}}, \vec{p} \right) = \frac{\hbar^2}{\sqrt{f}} L \left( \frac{\partial}{\partial \vec{p}}, \frac{\partial}{\partial \vec{q}} \right) \sqrt{f} \quad (8)$$

In the classical limit  $\hbar \rightarrow 0$  these equations reduce to the classical Liouville equation and to  $S = \vec{q}\vec{p} - Et$ , generating the identity transformation. In general the coupled evolution equations are nonlinear in the derivatives, and hence a *causal* interpretation by means of *trajectories in phase space* is therefore restricted to quadratic Hamiltonians. The evolution equations are however well suited for the calculation of corrections to the classical limit.

### 4 Representation of single events and ensembles

Solutions  $f(\vec{q}, \vec{p})$  of the evolution eqs. (7) and (8) corresponding to a density matrix describe an ensemble, but there exist also solutions not corresponding to a density matrix ( $A^\dagger A \neq AA^\dagger$ ), which may be much more localised than is allowed by Heisenberg's uncertainty relation. These solutions,  $f_{se}(\vec{q}, \vec{p})$  say, can be associated with a representation of a *single event* (e.g. the observation during measurement of an individual electron), in contradistinction to an *ensemble*, which describes the expected behaviour of a single entity in *repeated* experiments. The "hidden parameters" of the single event are nothing more than position, velocity and shape parameters of the jpd. The superposition principle for  $\psi$  expresses the *rules for combining probabilities during the preparation*, but for  $\psi_{se}$  the superposition of two solutions represents two

individual particles. In this conception, the notion of the collapse of the wavefunction evaporates, and the j.d.f.  $f$  for the ensemble emerges after averaging over the "hidden parameters" of  $\psi_{se}$  such that  $A^\dagger A = AA^\dagger$  is fulfilled.

## 5 An exact solution for an individual particle

Take in eq. (8)  $\partial S/\partial t = -E = -mc^2 \gg H$  (which is allowed by relativistic considerations), then eq. (8) splits into two equations. The first equation leads to  $S = \vec{q}\vec{p} - Et$  and to the classical Liouville equation, while the second equation reduces, for the simple example of the free particle, to

$$\Delta \sqrt{f_{se}} + \frac{2m^2 c^2}{\hbar^2} \sqrt{f_{se}} = 0, \quad (9)$$

which determines the shape of the localised solution. The simplest spherical symmetric solution with maximum localisation in  $\vec{p}$  is

$$f_{se}(\vec{q}, \vec{p}) \sim \frac{\sin^2(\vec{q}'^2)}{\vec{q}'^2} \delta(\vec{p}') \quad (10)$$

where  $\vec{q}' = \vec{q} - \vec{q}_0 - (\vec{p}_0/m)t$ ,  $\vec{p}' = \vec{p} - \vec{p}_0$  are the classical trajectories. Eq. (10) describes a localised solution with a long oscillatory tail, the center following the classical path in *phase space*. The solution shows an important overlap of classical and quantum behaviour, and resembles the solution for a single particle derived by Barut [8] in configuration space. The limit  $m \rightarrow 0$  yields the classical point particle, while averaging over the "hidden parameters" leads to an ensemble description which is equivalent to the usual quantummechanical formalism.

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**Solution of a Variational Problem Confirming Wigner's Result  
on the Joint Probability Distribution**

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Wigner (1932) observed and later (Wigner, 1971) proved that the joint probability distribution of canonically conjugate observables does not exist in the sense that "it may take negative values, but of course this must not hinder the use of it in calculations as an auxiliary function which obeys many relations we would expect from such a probability." An ample study of such a quasi-probability was done by Mückenheim (1986).

The objective of this paper is to construct the joint (product) measure induced by the correlation between two observables and show that such a joint measure is a probability measure only if the correlation is very small or, in the cases of interest, even equal to zero. For strongly correlated observables the joint measure is not necessarily nonnegative, confirming, in a constructive way, Wigner's result.

Let  $P$  and  $Q$  be two observables and  $f(p)$  and  $g(q)$  their marginal probability densities. Let  $\mu_1$ ,  $\mu_2$  and  $\sigma_1$ ,  $\sigma_2$  be the mean values and the standard deviations of  $P$  and  $Q$ , respectively. Let  $\rho$  be the correlation coefficient measuring the (linear) dependence between  $P$  and  $Q$ . If the joint distribution has the density  $\phi(p,q)$ , then

$$\rho = (\sigma_1 \sigma_2)^{-1} \iint_D (p - \mu_1)(q - \mu_2) \phi(p,q) dp dq. \quad (1)$$

If  $\phi$  is known and the integral (1) does exist, then  $\rho$  is calculable in a unique way. The converse assertion, however, is not true. If  $\rho$ ,  $f(p)$ , and  $g(q)$  are given, then there are infinitely many densities  $\phi$  compatible with them. We construct the closest joint density  $\phi^*$  to the independent (direct) product  $fg$  of the two marginals subject to the given correlation coefficient  $\rho$ , ( $-1 \leq \rho \leq 1$ ), where closeness is measured by the Pearson's  $\langle \chi^2 \rangle$ -indicator,

$$\langle \chi^2 \rangle(\phi:fg) = \iint_D \{[\phi(p,q) - f(p)g(q)]^2 / [f(p)g(q)]\} dp dq \quad (2)$$

where the integrals (1) and (2) are taken on the Cartesian product of the supports of  $f$  and  $g$ ,  $D = \{(p,q); f(p)g(q) \neq 0\}$ . Mathematically, we have



to solve the least weighted squares variational problem with constraints:  
 $\min_{\phi} \langle \chi^2 \rangle (\phi:fg)$  subject to the given value of  $\rho$ . Using the standard  
 Lagrange multipliers method, the solution is

$$\phi^*(p,q) = f(p)g(q) \left[ 1 + \rho \frac{p - \mu_1}{\sigma_1} \frac{q - \mu_2}{\sigma_2} \right] \quad (3)$$

Remarks: (a)  $\phi^*$  is called the joint density induced by the correlation coefficient  $\rho$  between P and Q.

(b) If  $\rho = 0$  (i.e. P and Q are not correlated), then  $\phi^*(p,q) = f(p)g(q)$  and  $\langle \chi^2 \rangle (\phi^*:fg) = 0$ .

(c)  $\phi^*$  is normed, i.e.

$$\iint_D \phi^*(p,q) dp dq = 1.$$

(d)  $\phi^*$  is compatible with the given marginals, i.e.

$$\int_{D_1} \phi^*(p,q) dp = g(q), \quad \int_{D_2} \phi^*(p,q) dq = f(p),$$

where  $D_1 = \{p; f(p) \neq 0\}$ ,  $D_2 = \{q; g(q) \neq 0\}$ .

(e) However,  $\phi^*$  is not necessarily a joint probability density. Taking all  $(p,q) \in D$ , let  $m = \min \{(p - \mu_1)(q - \mu_2)\}$  and  $M = \max \{(p - \mu_1)(q - \mu_2)\}$ . Obviously,  $m < 0$  and  $M > 0$ , unless P and/or Q are degenerate, being then identically equal to  $\mu_1$  and/or  $\mu_2$ , respectively. Then  $\phi^*$  would be non-negative, and therefore a joint probability density, only if

$$-\sigma_1 \sigma_2 / M \leq \rho \leq -\sigma_1 \sigma_2 / m. \quad (4)$$

On the other hand,  $\phi^*$  can take on negative values if  $\rho < -\sigma_1 \sigma_2 / M$  and  $\sigma_1 \sigma_2 < M$ , or if  $\rho > -\sigma_1 \sigma_2 / m$  and  $-\sigma_1 \sigma_2 > m$ , which in fact does happen in the cases of practical interest, as predicted by Wigner (1932).

The deviation from independence due to the correlation coefficient  $\rho$  between P and Q is

$$\psi(p,q) = \phi^*(p,q) - f(p)g(q) = f(p) \frac{p - \mu_1}{\sigma_1} \rho \frac{q - \mu_2}{\sigma_2} g(q).$$

The set

$$W = \left\{ (p,q); 1 + \rho \frac{p - \mu_1}{\sigma_1} \frac{q - \mu_2}{\sigma_2} < 0 \right\}$$

is called the Wigner region.

Special case: Let Q be the coordinate (position) and P the momentum. Suppose that they are both normally distributed  $N(0,1)$ , in which case  $\mu_1 = \mu_2 = 0$ ,  $\sigma_1 = \sigma_2 = 1$ ,  $m = -\infty$ ,  $M = +\infty$ , and D is the whole two-dimensional

real space  $\mathbb{R}^2$ . If  $\rho$  is the correlation coefficient between P and Q, the joint density (3) becomes

$$\phi^*(p,q) = (2\pi)^{-1} e^{-(p^2+q^2)/2} (1 + \rho pq). \quad (5)$$

If P and Q were independent,  $\rho$  would be equal to zero, the joint density  $\phi^*$  would be a probability density (Fig.1), the contours would be like in Fig.2, and the corresponding Wigner region W would be the empty set. But, as P and Q are dependent, for any correlation coefficient  $\rho \neq 0$  the Wigner region W is not empty and  $\phi^*$  given by (3) is not a probability density. Fig.3 gives the graph of (5) when  $\rho = 0.999$ , while Fig.4 shows the corresponding Wigner region (the shaded sets), with an accuracy of three decimals in computation, provided by SAS/GRAPH computer package. Although the Heisenberg's uncertainty relation prevents at least the simultaneous knowledge of the coordinate and momentum of a particle, the joint density  $\phi^*$  shows the degree of implausibility (on the Wigner region), and the degree of plausibility (on the complementary set to the Wigner region) in locating the potential values of P and Q

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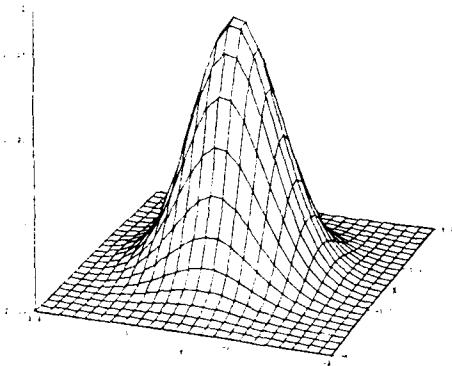


Fig. 1

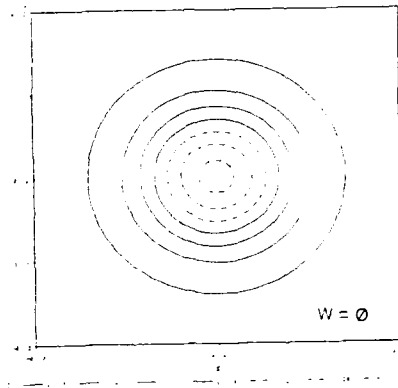


Fig. 2

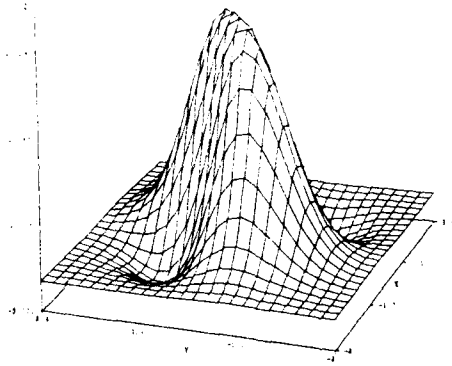


Fig. 3

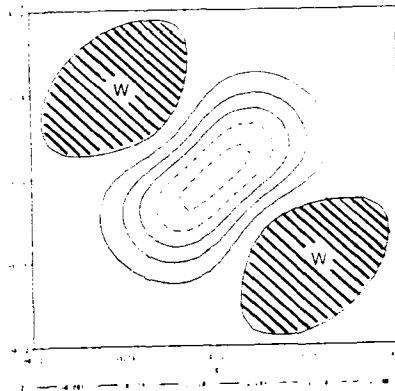


Fig. 4

## Lorentz Squeezed Harmonic Oscillators

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As has been noted in previous papers,[1, 2], the Wigner phase-space picture of quantum mechanics[3, 4] is the natural language for squeezed states of light. Since most of the squeezed states observed in the laboratory are two-mode states,[5] it is natural to study the canonical transformations of two-mode states within the framework of the phase-space picture of quantum mechanics. The basic symmetry, in this case, is that of Sp(4).[5]

The groups SU(2) and SU(1,1), which are locally isomorphic to the O(3) and O(1,2) subgroup of the (3+2)-dimensional Lorentz group,[6] are also important in quantum optics.[7] The (3+2)-dimensional Lorentz group, together with the (4+1)-dimensional Lorentz group, is often called the de Sitter group O(3,2). It is the local isomorphism between Sp(4) and O(3,2)[5, 8] that allows the study of spacetime symmetries of the relativistic world in terms of canonical transformations of the Wigner function for two-mode squeezed states. The mathematical language of squeezed states is based on the Lorentz group and the harmonic oscillator.[8]

It is possible to begin this study of squeezed states in terms of special relativity and vice versa by exploiting the correspondence between SU(1,1) and O(2,1).[1, 7, 9] There are many O(2,1)-like subgroups of O(3,2). One of the subgroups of O(3,2) is the "ordinary" (3+1)-dimensional Lorentz group which has one three dimensional rotation group and three (2+1)-dimensional Lorentz groups. It would thus be easy to be misled into believing that SU(2) [or O(3)] and SU(1,1) [or O(2,1) and O(1,2)] are those from the ordinary Lorentz group. However, there are only three rotation generators in O(3,1), hence the rotation generator of O(2,1) must be one of the three generators of the rotation group. To see that this is not true, let us look at the interferometers of Yurke, McCall and Klauder.[7] Their SU(2) interferometer is based on  $\hat{J}_1$ ,  $\hat{J}_2$  and  $\hat{J}_3$ , satisfying the commutation relations of the usual rotation generators:

$$[\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk}\hat{J}_k. \quad (1)$$

Their SU(1,1) interferometer is based on  $\hat{J}_0$ ,  $\hat{K}_3$  and  $\hat{Q}_3$ [5, 8] satisfying the commutation relations:

$$[\hat{K}_3, \hat{Q}_3] = i\hat{J}_0, \quad [\hat{J}_0, \hat{K}_3] = -i\hat{Q}_3, \quad [\hat{J}_0, \hat{Q}_3] = -i\hat{K}_3. \quad (2)$$

It is important to note that  $\hat{J}_0$ , while being a generator of rotation, is not one of the generators of the SU(2) group. SU(2) is locally isomorphic to the three-dimensional rotation group, while SU(1,1) is locally isomorphic to the (2+1)-dimensional Lorentz group. The rotation generator of SU(1,1) ( $J_0$ ) is not one of the three generators of SU(2). This confirms that there are four generators of rotation and indicates that

the SU(2) and SU(1,1) groups cannot be isomorphic to subgroups of the familiar (3+1)-dimensional Lorentz group which has only three rotation generators. Rather, these groups are isomorphic to O(3) and O(2,1) of the (3+2)-dimensional Lorentz group. Thus the SU(1,1) group given here is one of the three O(1,2) subgroups of the O(3,2) de Sitter group and not a subgroup of the (3+1)-dimensional Lorentz group.

In the system of interferometers of Yurke, McCall and Klauder, there are three generators for SU(2) and three for SU(1,1) making a total of six generators. They do not, however, form a closed system of commutation relations, but require four more generators. If we add them together, the result is the set of ten generators of Dirac's two-oscillator representation of the O(3,2) group.[6] One of the characteristics of O(3,2) is that  $\hat{K}_3$  in the SU(1,1) interferometer is capable of forming another SU(1,1) group. It would be very helpful if we could design experiments to test the set of O(2,1) commutation relations involving  $\hat{K}_3$  and the generators not contained in the interferometer of Yurke, McCall and Klauder.  $\hat{K}_3$  can form the following sets of closed commutation relations:

$$[\hat{J}_1, \hat{K}_3] = -i\hat{K}_2, \quad [\hat{J}_1, \hat{K}_2] = i\hat{K}_3, \quad [\hat{K}_2, \hat{K}_3] = -i\hat{J}_1, \quad (3)$$

or

$$[\hat{J}_2, \hat{K}_3] = i\hat{K}_1, \quad [\hat{J}_2, \hat{K}_1] = -i\hat{K}_3, \quad [\hat{K}_1, \hat{K}_3] = i\hat{J}_2. \quad (4)$$

The same reasoning is applicable to the expressions where the  $\hat{K}_i$  are replaced by the  $\hat{Q}_i$ .

An experiment based on one or more of the above four sets of commutation relations would prove the existence of both the O(2,1) and O(1,2) symmetries. This experiment would also reinforce the evidence of the O(3,2) symmetry in the two-mode system and would allow us to study space-time symmetries of squeezed states in terms of the canonical transformations of the Wigner function for these states. The nine (2+1)-dimensional subgroups in the O(3+2)-like symmetry group of two-mode squeezed states, are either separable or can be transformed into separable representations by a rotation, for example,  $\hat{K}_3$  and  $\hat{Q}_3$  can be transformed into  $\hat{K}_1$  and  $\hat{Q}_1$  by  $\hat{J}_2$ .

Although we cannot design an experiment which incorporates all ten generators of Dirac's two-oscillator representation, it is still possible to detect the O(3,2) characteristics with its subgroups. Let us begin by studying how we can extract measurable numbers from the Wigner function.

In quantum mechanics, we calculate those numbers from the overlap of the distribution functions and the expectation value of the operators for two given wave functions  $\psi(x)$  and  $\phi(x)$ , and their corresponding Wigner functions  $W_\psi(x,p)$  and  $W_\phi(x,p)$ . Then the transition probability takes the form

$$|(\phi(x), \psi(x))|^2 = 2\pi \int W_\psi(x,p)W_\phi(x,p)dxdp. \quad (5)$$

This expression is useful when we calculate the probability of a certain state being in a particular eigenstate. For two pairs of canonical variables, the overlap integral

may be written as

$$\begin{aligned} & |(\phi(x_1, x_2), \psi(x_1, x_2))|^2 \\ &= (2\pi)^2 \int W_\psi(x_1, x_2; p_1, p_2) W_\phi(x_1, x_2; p_1, p_2) dx_1 dx_2 dp_1 dp_2. \end{aligned} \quad (6)$$

We consider next the expectation value of an operator applicable to  $\psi(x)$  or the momentum operator  $f(p)$ . If the operator  $R$  is a function only of  $x$  or  $p$ , then the expectation value is

$$\langle R \rangle = (\psi(x), R(x)\psi(x)) = \int R(x)W(x, p)dxdp, \quad (7)$$

with a similar expression for  $R(p)$ . If  $R$  is a function of both  $x$  and  $p$ , we are not aware of any simple expression. It is possible to prove that

$$\begin{aligned} & \int W(x, p)(x^n p^m) dxdp \\ &= (-i)^m \left(\frac{1}{2}\right)^n \sum_{r=0}^n \binom{n}{r} \left(\psi(x), x^{n-r} \left(\frac{\partial}{\partial x}\right)^m x^r \psi(x)\right). \end{aligned} \quad (8)$$

The photon number and the (photon number)<sup>2</sup> operators are the two most important quantum mechanical operators in quantum optics. In the Schrödinger representation, the number operator takes the form

$$\hat{N} = \frac{1}{2} \left\{ x^2 - \left(\frac{\partial}{\partial x}\right)^2 - 1 \right\}. \quad (9)$$

Thus, in the Wigner phase-space picture, the expectation value of this operator is

$$\langle N \rangle = \frac{1}{2} \int (x^2 + p^2 - 1) W(x, p) dxdp. \quad (10)$$

The expression for  $N^2$  is more complicated because there is a term proportional to  $x^2 p^2$ . We obtain:

$$\langle N^2 \rangle = \frac{1}{4} \int \left\{ (x^2 + p^2 - 1)^2 - 1 \right\} W(x, p) dxdp. \quad (11)$$

In the single mode squeezed state, it was observed that every squeezed state can be represented by the Wigner function of Gaussian form localized within an elliptic region in phase space. If the region of localization is a circle, the Wigner function corresponds to an unsqueezed coherent state. The circle centered on the origin is the vacuum state. Without loss of generality, we can obtain every squeezed state by squeezing the vacuum followed by translation. Every squeezed state is a translated

squeezed vacuum. If  $\hat{J}_0$  measures the total number of photons,[8] then  $\hat{J}_3$  measures the difference between the photon numbers of the first and second kinds. In terms of the expectation values of  $N_1$  and  $N_2$  this is

$$\begin{aligned}\langle N_1 \rangle &= \left\langle \left( J_0 + J_3 + \frac{1}{2} \right) \right\rangle = \frac{1}{2} \int (x_1^2 + p_1^2 - 1) W(x_1, x_2, p_1, p_2) dx_1 dx_2 dp_1 dp_2, \\ \langle N_2 \rangle &= \left\langle \left( J_0 - J_3 + \frac{1}{2} \right) \right\rangle = \frac{1}{2} \int (x_2^2 + p_2^2 - 1) W(x_1, x_2, p_1, p_2) dx_1 dx_2 dp_1 dp_2\end{aligned}\quad (12)$$

Likewise,

$$\begin{aligned}\langle N_1^2 \rangle &= \frac{1}{4} \int \left[ (x_1^2 + p_1^2 - 1)^2 - 1 \right] W(x_1, x_2, p_1, p_2) dx_1 dx_2 dp_1 dp_2, \\ \langle N_2^2 \rangle &= \frac{1}{4} \int \left[ (x_2^2 + p_2^2 - 1)^2 - 1 \right] W(x_1, x_2, p_1, p_2) dx_1 dx_2 dp_1 dp_2, \\ \langle N_1 N_2 \rangle &= \frac{1}{4} \int (x_2^2 + p_2^2 - 1)(x_1^2 + p_1^2 - 1) W(x_1, x_2, p_1, p_2) dx_1 dx_2 dp_1 dp_2.\end{aligned}\quad (13)$$

These quantities are needed in calculating the photon number variations:

$$\langle (\Delta N)^2 \rangle, \quad \langle (\Delta N_1)^2 \rangle, \quad \langle (\Delta N_2)^2 \rangle, \quad (14)$$

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## Wigner Formalism for Quantum Statistics of Nonrelativistic and Relativistic Anharmonic Oscillator Systems

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In recent years, there has been growing interest in the foundations, extensions, and applications of the Wigner phase space formalism of quantum mechanics [1]. An important advantage of this formalism is the possibility to calculate exactly partition functions of quantum statistics by means of a phase space integration without the necessity to determine the energy eigenvalues of the system considered or to sum up the series of corresponding Boltzmann factors. In practice, a suitable tool for the evaluation of exact quantum corrections to the classical partition function and to thermodynamic functions is given by the Wigner-Kirkwood series in powers of  $\hbar^2$ .

The purpose of this contribution is threefold: Firstly, we generalize the Wigner-Kirkwood series expansion to include relativistic systems. Secondly, we give a survey over various anharmonic oscillator systems the partition functions of which can be calculated in terms of well-known special functions ( e.g. Bessel functions, Kelvin functions, gamma functions and parabolic cylinder functions ). Thirdly, we thus indicate the applicability and utility of Wigner formalism.

The relativistic quantum theory in the Wigner formalism has been treated in a number of papers for equilibrium as well as for nonequilibrium systems. The nonequilibrium processes considered range from multiparticle production and kinetic theory [2] to cosmology [3]. There exist only a few papers on relativistic phase space quantum mechanics [4-8] and equilibrium quantum statistical mechanics [9], cf [10,11]. In this paper we develop a frame-dependent Wigner formalism using the so-called synchronous gauge [12], i.e. a comoving frame.

Let us consider a system with one degree of freedom and the Hamiltonian operator  $\hat{H}$  and the Boltzmann operator  $\hat{\Omega} = \exp(-\beta\hat{H})$ ,  $\beta = \frac{1}{kT}$ . The Wigner equivalent of  $\hat{\Omega}$  given in the coordinate representation is defined as

$$\Omega_w(q, p; \beta) = \int dz \, e^{ipz/\hbar} \langle q - \frac{z}{2} | \hat{\Omega} | q + \frac{z}{2} \rangle \quad (1)$$

and the partition function is represented by

$$Z = \iint dq dp \, \Omega_w(q, p; \beta) \quad (2)$$



The Bloch equation for the evaluation of  $\Omega_w$  reads

$$\frac{\partial \Omega_w(q, p; \beta)}{\partial \beta} = -H_w(q, p) \cos\left(\frac{\hbar}{2}\Lambda\right) \Omega_w(q, p; \beta) \quad (3)$$

where  $H_w$  is the Wigner equivalent of  $\hat{H}$  and  $\Lambda$  denotes the symplectic differential operator ( Poisson bracket operator )

$$\Lambda = \frac{\partial}{\partial p} \frac{\partial}{\partial q} - \frac{\partial}{\partial q} \frac{\partial}{\partial p} \quad (4)$$

Let  $\hat{H} = \hat{H}_0(p) + \hat{V}(q)$  with

$$\hat{H}_0(p) = \sqrt{(mc^2)^2 + p^2 c^2} - mc^2 \quad (5)$$

then  $H_w(q, p) = H_0(p) + V(q)$ . The Wigner-Kirkwood series is

$$Z = \frac{1}{2\pi\hbar} \iint dq dp \{ \exp[-\beta(H_0(p) + V(q))] \sum_{n=0}^{+\infty} \hbar^{2n} \Phi_n(q, p; \beta) \} \quad (6)$$

with  $\Phi_0 = 1$ . Solving (3) we get

$$\Phi_1(q, p, \beta) = -\frac{1}{8} \beta^2 H_0'' V'' + \frac{1}{24} \beta^3 (H_0'' V'^2 + H_0'^2 V'') \quad (7)$$

with  $H_0' = \partial H_0 / \partial p$ ,  $V' = \partial V / \partial q$  etc. By use of the substitution  $p/mc = \sinh u$  we first perform the p-integration and get after some algebra, cf. [10],  $Z(q) = Z_0 + Z_1(q)$  with

$$Z_0 = \frac{1}{\pi\hbar} \epsilon^{\beta mc^2} mc K_1(\beta mc^2) \quad (8)$$

$$Z_1(q) = \hbar^2 \frac{1}{\pi\hbar} \epsilon^{\beta mc^2} \left[ -\frac{1}{8} \beta^2 V'' + \frac{1}{24} \beta^3 V'^2 \right] A \quad (9)$$

where  $A$  represents a linear combination of Kelvin functions ( modified Hankel functions [13]

$$K_\nu(\beta mc^2) = \int_0^\infty \epsilon^{-\beta mc^2 \cosh u} \cosh \nu u \, du \quad , \nu = 1, 3, 5, \dots \quad (10)$$

The full partition function up to the first quantum correction, proportional to  $\hbar^2$ , and relativistic corrections with second and fourth order in  $p/mc$  can now be calculated by the q-integration of (8), (9) using the potential function  $V(q)$  of the system to be considered. Eq. (9) is a main result of our paper: for a more detailed exposition cf. [10,11]. The description of nonrelativistic quantum systems can be obtained by performing the suitable limit or directly by use of  $\hat{H}_0 = p^2/2m$  instead of (5).

As an example we take the symmetric relativistic quantum Toda oscillator with

$$V(q) = V_0 (\cosh aq - 1) \quad (11)$$

One gets the partition function  $Z = Z_0 + Z_1$  with the classical part

$$Z_0 = \frac{1}{\pi \hbar} \epsilon^{\beta mc^2} mc K_1(\beta mc^2) \epsilon^{\beta V_0} \frac{2}{a} K_0(\beta V_0) \quad (12)$$

and the quantum part

$$Z_1 = \hbar^2 \frac{1}{\pi \hbar} c \epsilon^{\beta mc^2} K_1(\beta V_0) \beta^2 V_0 \epsilon^{\beta V_0} B \quad (13)$$

where B is given with  $K_\nu = K_\nu(\beta mc^2)$  .  $\nu = 1, 3, 5$  . by

$$B = -\frac{1}{6} \left[ \frac{206}{128} K_1 - \frac{93}{128} K_3 + \frac{15}{128} K_5 \right] + \frac{mc^2 \beta}{12} \left[ -\frac{6}{16} K_1 + \frac{7}{16} K_3 - \frac{1}{16} K_5 \right] \quad (14)$$

For the nonrelativistic Toda oscillator see [14]. Singular and so-called isotonic oscillators with the prototype potential

$$V(q) = V_0 \left( \frac{q}{a} - \frac{a}{q} \right)^2 \quad (15)$$

are treated in [15] and yield partition functions containing modified Hankel functions  $K_{n/2}(2\beta V_0)$  with half-integer index. For systems of restricted rotators or particles in multiwell potentials the partition function Z contains modified Bessel functions [16]. The partition functions of the full relativistic quantum Toda chain contains gamma functions, and those of polynomial oscillators and double-well systems parabolic cylinder functions (Weber functions or modified Hermite polynomials). From the partition functions the thermodynamic functions can be obtained in a well-known manner. More detailed papers are in preparation.

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## Wigner distribution function on a graded phase space and its application to optical systems

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Sixty years ago, Wigner [1] has shown that the quantum expectation value of an observable can be expressed in a form similar to the statistical average of the corresponding classical quantity with respect to a distribution function on phase space. Since this discovery, a lot of works have been done on the phase-space representation in the literature [2]. However, inclusion of fermions or spins is still nontrivial.

Recently, we have developed the phase-space representation of a fermion by using the Grassmann variables ( $G$ -variables) [3]. We have defined the Wigner distribution function (WDF) of a fermion as

$$W_f(\beta, \beta^*) = \text{Tr}_g[\hat{\rho}_f \hat{\Delta}_f(\beta, \beta^*)]. \quad (1)$$

Here  $\hat{\rho}_f$  is the density operator. The fermionic Wigner operator  $\hat{\Delta}_f$  is given by

$$\hat{\Delta}_f(\beta, \beta^*) = \int d^2\zeta \exp[(\hat{b}^\dagger - \beta^*)\zeta - \zeta^*(\hat{b} - \beta)]. \quad (2)$$

$\hat{b}^\dagger$  and  $\hat{b}$  are the usual creation and annihilation operators of a fermion.  $\beta^*$  and  $\beta$  are the corresponding classical phase-space  $G$ -variables. The Berezin integrations over  $G$ -numbers [4] are normalized as follows:

$$\int d^2\zeta \zeta \zeta^* = 1, \quad \int d^2\zeta \zeta = \int d^2\zeta \zeta^* = 0, \quad \int d^2\zeta = 0, \quad (d^2\zeta \equiv d\zeta^* d\zeta). \quad (3)$$

The symbol  $\text{Tr}_g$  in Eq.(1) denotes the graded trace operation. In terms of the Fock basis  $\{|0\rangle, |1\rangle = \hat{b}^\dagger|0\rangle\}$ , it can be expressed as

$$\text{Tr}_g(\hat{A}) = \sum_{n=0,1} (-)^n \langle n|\hat{A}|n\rangle. \quad (4)$$

We note that, *not with the standard trace operation but with the graded trace*

operation, the Weyl correspondence can be established in the fermion theory:

$$A(\beta, \beta^*) = \text{Tr}_g[\hat{A}_w(\hat{b}, \hat{b}^\dagger) \hat{\Delta}_f(\beta, \beta^*)], \quad (5a)$$

$$\hat{A}_w(\hat{b}, \hat{b}^\dagger) = \int d^2\beta A(\beta, \beta^*) \hat{\Delta}_f(\beta, \beta^*), \quad (5b)$$

where  $\hat{A}_w$  and  $A$  are a Weyl-ordered operator and its corresponding classical quantity in phase space, respectively. The quantum expectation value is represented in the form of a statistical phase-space average:

$$\langle \hat{A}_w \rangle = \text{Tr}[\hat{\rho}_f \hat{A}_w] = \int d^2\beta W_f(\beta, \beta^*) \tilde{A}(\beta, \beta^*), \quad (6a)$$

$$\tilde{A}(\beta, \beta^*) \equiv \frac{1}{2} \int d^2\zeta \exp[2(\beta^*\zeta - \zeta^*\beta)] A(\zeta, \zeta^*). \quad (6b)$$

$\tilde{A}$  is not directly equal to  $A$  but its  $G$ -Fourier transform, since the quantum expectation value is defined through the standard trace operation. Thus we see one of the qualitative differences between boson and fermion theories.

In what follows, we extend the above formalism to a coupled boson-fermion system and examine its possible application to the kinetic theory of an optical model [5]. Let us discuss it by employing the Dicke model of a two-level atom interacting with a monochromatic radiation field. The system Hamiltonian is given by

$$\hat{H} = \Omega \hat{a}^\dagger \hat{a} + \frac{1}{2} \omega \hat{\sigma}_z + g(\hat{a}^\dagger + \hat{a}) \hat{\sigma}_x, \quad (7)$$

where the usual creation and annihilation operators,  $\hat{a}^\dagger$  and  $\hat{a}$ , and the Pauli matrices denote, respectively, the radiation field with frequency  $\Omega$  and the atom of level distance  $\omega$ .  $g$  is a coupling constant.

To apply our formalism to this system, we represent the atom variables in terms of the fermionic oscillator variables. A simple method is the substitution:  $\hat{\sigma}_+ \rightarrow c\hat{b}^\dagger$ ,  $\hat{\sigma}_- \rightarrow \hat{b}c$ ,  $\hat{\sigma}_z \rightarrow 2\hat{b}^\dagger\hat{b} - 1$ , [ $\hat{\sigma}_\pm \equiv \frac{1}{2}(\hat{\sigma}_x \pm i\hat{\sigma}_y)$ ], where  $c$  is a real Clifford  $c$ -number having the properties:  $c^2 = 1$ ,  $c\hat{b}^\dagger + \hat{b}^\dagger c = c\hat{b} + \hat{b}c = 0$ ,  $c\beta + \beta c = c\beta + \beta c = 0$ . Under this substitution, the Pauli-matrix algebra is kept unchanged, and, furthermore, each term in the Hamiltonian becomes statistically bosonic. Thus, instead of the form (7), we employ

$$\hat{H} = \Omega \hat{a}^\dagger \hat{a} + \omega(\hat{b}^\dagger \hat{b} - \frac{1}{2}) + g(\hat{a}^\dagger + \hat{a})(c\hat{b}^\dagger + \hat{b}c). \quad (8)$$

Now the total WDF of the system is given by

$$W(\alpha, \alpha^*, \beta, \beta^*) = \text{Tr}_g[\hat{\rho} \hat{\Delta}_b(\alpha, \alpha^*) \hat{\Delta}_f(\beta, \beta^*)], \quad (9)$$

where  $\hat{\Delta}_b$  is the ordinary bosonic Wigner operator and  $\alpha$ 's are the complex phase-space variables. The standard trace operation is understood for the boson part.

In the Schrödinger picture, the WDF evolves in time as

$$\frac{\partial W}{\partial t} = -i \text{Tr}_g\{[\hat{H}, \hat{\rho}] \hat{\Delta}_b \hat{\Delta}_f\} = -i \text{Tr}_g[\hat{\Delta}_b \hat{\Delta}_f \hat{H} \hat{\rho}] + i \text{Tr}_g[\hat{\rho} \hat{H} \hat{\Delta}_b \hat{\Delta}_f], \quad (10)$$

provided that the cyclicity property of the graded trace operation (4) has been used. By virtue of the operator correspondence relations

$$\hat{\Delta}_b \hat{a}^\dagger = (\alpha^* - \frac{1}{2} \overrightarrow{\frac{\partial}{\partial \alpha}}) \hat{\Delta}_b, \quad \hat{\Delta}_f \hat{b}^\dagger = (\beta^* + \frac{1}{2} \overrightarrow{\frac{\partial}{\partial \beta}}) \hat{\Delta}_f, \quad (11)$$

and so on, we see there always exists the operator  $\overrightarrow{L}$  satisfying

$$\hat{\Delta}_b \hat{\Delta}_f \hat{H} = -i \overrightarrow{L}(\alpha, \alpha^*, \frac{\overrightarrow{\partial}}{\partial \alpha}, \frac{\overrightarrow{\partial}}{\partial \alpha^*}, \beta, \beta^*, \frac{\overrightarrow{\partial}}{\partial \beta}, \frac{\overrightarrow{\partial}}{\partial \beta^*}) \hat{\Delta}_b \hat{\Delta}_f. \quad (12)$$

Therefore we obtain the following generalized Fokker-Planck equation:

$$\frac{\partial W}{\partial t} + \overrightarrow{L} W + W \overleftarrow{L} = 0. \quad (13)$$

This equation may be regarded as a certain superfield equation defined on the graded phase space  $(\alpha, \alpha^*, \beta, \beta^*)$ .

As can be seen, the WDF (9) depends on the fermionic variables only through the combinations,  $\beta c$  and  $c \beta^*$ . Therefore, the expansion of the WDF in terms of these nilpotent variables has the following form:

$$W = W_0(\alpha, \alpha^*) + W_1^*(\alpha, \alpha^*) \beta c + W_1(\alpha, \alpha^*) c \beta^* + W_2(\alpha, \alpha^*) \beta c \beta^*. \quad (14)$$

In Ref.[5], it has been shown that a set of these coefficient functions gives the phase-space representation of the effective theory for the radiation field.

Substitution of Eq.(14) into Eq.(13) leads to an identity, which gives rise to the following set of equations:

$$\frac{\partial W_0}{\partial t} + i\left\{\Omega\left(\alpha^* \frac{\partial}{\partial \alpha^*} - \alpha \frac{\partial}{\partial \alpha}\right)W_0 + \frac{g}{2}\left(\frac{\partial}{\partial \alpha^*} - \frac{\partial}{\partial \alpha}\right)(W_1 + W_1^*)\right\} = 0, \quad (15a)$$

$$\frac{\partial W_1^*}{\partial t} + i\left\{\left[\Omega\left(\alpha^* \frac{\partial}{\partial \alpha^*} - \alpha \frac{\partial}{\partial \alpha}\right) - \omega\right]W_1^* + g\left\{\left(\frac{\partial}{\partial \alpha^*} - \frac{\partial}{\partial \alpha}\right)W_0 - (\alpha + \alpha^*)W_2\right\}\right\} = 0, \quad (15b)$$

$$\frac{\partial W_1}{\partial t} + i\left\{\left[\Omega\left(\alpha^* \frac{\partial}{\partial \alpha^*} - \alpha \frac{\partial}{\partial \alpha}\right) + \omega\right]W_1 + g\left\{\left(\frac{\partial}{\partial \alpha^*} - \frac{\partial}{\partial \alpha}\right)W_0 + (\alpha + \alpha^*)W_2\right\}\right\} = 0, \quad (15c)$$

$$\frac{\partial W_2}{\partial t} + i\left\{\Omega\left(\alpha^* \frac{\partial}{\partial \alpha^*} - \alpha \frac{\partial}{\partial \alpha}\right)W_2 + 2g(\alpha + \alpha^*)(W_1 - W_1^*)\right\} = 0. \quad (15d)$$

This set of equations determines the effective dynamics of the radiation field.

In this note, we have constructed the WDF on a graded phase space and applied it to the kinetic theory of the Dicke model. This has also a meaning of the phase-space representation of SU(2) spin with the complete Weyl correspondence. Recently, the optical models with two-level atoms have been revisited in the context of supersymmetry [6]. In the present approach, the atom and radiation field variables are treated on a completely equal footing with each other. Therefore, we expect that this approach enables us to discuss such a symmetry geometrically on the phase space. In the case of the supersymmetric quantum mechanics of Witten's type, such a discussion has been actually developed in Ref.[7].

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## EXACT WIGNER FUNCTION OR SOME NONQUADRATIC QUANTUM SYSTEMS

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The explicit expressions for the Wigner functions [0] of quantum systems with quadratic Hamiltonians were given in refs. [0]-[0]. The case of the quantum particle motion in the potential field  $U(x) = -Fx$ ,  $x > 0$  under the presence of an ideal reflecting wall at  $x = 0$  was investigated in [0].

Here we obtain the explicit expression for the equilibrium Wigner functions of two nonquadratic quantum systems and analyze their various limit cases.

Let us consider first the one-dimensional motion of a quantum particle with mass  $m$  in the following potential

$$\begin{aligned} U(x) &= 0 \quad \text{if } 0 < x < a \\ U(x) &= \infty \quad \text{if } x \geq a, \quad x \leq 0 \end{aligned} \quad (1)$$

The exact Green function of the Schrödinger equation with this potential was found by Pauli [0].

The explicit expression of the equilibrium Wigner function in this case has the following form [0]:

$$\begin{aligned} W(p, q, \beta) &= \frac{(-\hbar)}{a^2 p} \int_0^a \sin\left[\frac{2py}{\hbar}\right] \theta_3'\left(\frac{y}{a}, e^{-\beta\gamma}\right) dy \\ &= \frac{2}{a} \int_0^a \cos\left[\frac{2py}{\hbar}\right] \theta_3\left(\frac{y}{a}, e^{-\beta\gamma}\right) dy - \frac{\hbar}{ap} \sin\left[\frac{2qp}{\hbar}\right] \theta_3\left(\frac{q}{a}, e^{-\beta\gamma}\right) \end{aligned} \quad (2)$$

where  $\gamma = \frac{\pi^2 \hbar^2}{2ma^2}$ ,  $\beta = (kT)^{-1}$  and  $\Theta_3^I(\nu, \tau)$  is the derivative of the theta-function  $\Theta_3(\nu, \tau)$  over the first argument  $\nu$ .

In particular, near the wall (when  $q \rightarrow 0$ ) we get the following expansion,

$$\begin{aligned} W(p, q \rightarrow 0, \beta) &= -\frac{2q^3}{3a^3} \left\{ \Theta_3^I(0, e^{-\beta\gamma}) \left[ 1 - \frac{2p^2 q^2}{5\hbar^2} + \dots \right] \right. \\ &\quad \left. + \frac{q^2}{10a^2} \Theta_3^{(IV)}(0, e^{-\beta\gamma}) + \dots \right\} \end{aligned} \quad (3)$$

Note, that this expression starts from cubic terms on coordinates. Moreover, this expression changes its sign: when  $pq/\hbar \ll 1$  it is positive, while for  $pq/\hbar \gg 1$  it is negative.

The Wigner function for the half space can be obtained by making the asymptotic transition  $a \rightarrow \infty$  in (2).

$$\begin{aligned} W_\infty(q, p, \beta) &= \exp[-\beta p^2/2m] \operatorname{Re} \operatorname{erf} \left( q \left( \frac{2m}{3\hbar^2} \right)^{1/2} + ip \left( \frac{\beta}{2m} \right)^{1/2} \right) \\ &\quad - \left( \frac{2m}{\pi \beta p^2} \right)^{1/2} \sin\left(\frac{2pq}{\hbar}\right) \exp\left(-\frac{2mq^2}{3\hbar^2}\right) \end{aligned} \quad (4)$$

where the error function  $\operatorname{erf}(z)$  is defined as follows,

$$\operatorname{erf}(z) = \frac{2}{\pi^{1/2}} \int_0^z \exp(-t^2) dt.$$



The right-hand side of (4) leads to the classical distribution function if  $mq^2/\beta\hbar^2 \gg 1$ , i.e. for the distances from a wall which are much greater than de Broglie's wave length.

The second system is a particle moving in a one-dimensional delta-funtion potential. The Hamiltonian is as follows.

$$H = \frac{p^2}{2} - a\delta(x)$$

The propagator for this problem was obtained in [9-11]. The equilibrium Wigner function has the form:

$$\begin{aligned} W(p, q, \beta) = & \exp(-p^2\beta/2) + \\ & + \frac{a}{p} \exp(-2qa + a^2\beta/2) \operatorname{erf} \left( \frac{2q}{(2\beta)^{1/2}} - a(\beta/2)^{1/2} \right) \sin(2pq) + \\ & + 2 \exp(-a^2\beta/2 - p^2\beta/2) \operatorname{Re} \left[ \frac{\operatorname{erf} \left( \frac{2q}{(2\beta)^{1/2}} + \frac{ip(2\beta)^{1/2}}{2} \right)}{a + ip} \right] + \\ & + 2 \operatorname{erf} \left( \frac{2q}{(2\beta)^{1/2}} - a(\beta/2)^{1/2} \right) \operatorname{Re} \left[ \frac{\exp(-2q(a + ip))}{a + ip} \right] \end{aligned} \quad (5)$$

The asymptotic expression in the strong field case ( $a \rightarrow \infty$ ) of the Wigner function (5) is

$$W_\infty(q, p, \beta) \simeq \exp[\beta p^2/2] - (\beta/2)^{-1/2} \frac{\sin(2pq)}{2p} \exp(-2q^2/\beta) \left[ 1 - \frac{4q}{a\beta} + \dots \right] \quad (6)$$

In the weak field case the following expansion ( $a \rightarrow 0$ ) can be derived from (5):

$$\begin{aligned} W_0(p, q, \beta) \simeq & \exp(-p^2\beta/2) + \frac{\sin(2pq)}{p} \left[ \left( 1 - \frac{2q}{(2\beta)^{1/2}} \right) a \right. \\ & - \left( 2q + \frac{4q^2}{(2\beta)^{1/2}} + (\beta/2)^{1/2} \right) a^2 \\ & \left. + \left( 2q^2 + \frac{4q^3}{(2\beta)^{1/2}} + 2q(\beta/2)^{1/2} \right) a^3 + \dots \right] \end{aligned} \quad (7)$$

Comparing expression (6) with the analogous expression for the Wigner function of a free particle moving in the half space [6] we see that they do not coincide because the delta-potential is a half-penetrable one.

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## ON THE POSITIVENESS OF THE SMOOTHED WIGNER FUNCTION

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Wigner function was for a first time introduced in 1932 for a treatment of quantum correction to thermodynamics [1]. It found many applications afterwards, and by its use the phase space formulation as one of the equivalent representations of quantum mechanics was established. Although defined as an ordinary function on phase space, which plays the role analogous to that of classical distribution function [2], Wigner function can not be directly interpreted as a probability distribution function because, in the general case, it is necessarily negative in some regions of phase space. For an indirect probabilistic interpretation, a non-negative phase space function, related to Wigner function, is necessary. One example of such a function is Husimi distribution function [3] which can be represented as a Gaussian smoothing of Wigner function in phase space.

From early sixties [4] it was generally assumed, as most clearly formulated in [5] and quoted again in relatively recent review article [6] without comment, that Wigner function "once smoothed out over any phase space region of dimensions larger than or equal to unity ( $= h$ ) is always positive and smaller than one". In the review article on Wigner representation of quantum mechanics [7] it was claimed that averaging of Wigner function on finite phase space cells gives always non-negative results. To prove this, instead of direct averaging over finite phase space cells, the author performs Gaussian smoothing, tacitly assuming that both procedures give close results necessarily of the same sign.

Wlodarz [8] in 1988 proved that, in the general case, it is not possible to obtain a non-negative function on phase space by averaging Wigner function over finite phase space cells. This conclusion was directly related with the discontinuity of the characteristic functions of the corresponding cells at their boundaries. Recently de Aguiar and de Almeida [9], in a detailed and thorough analysis of Wigner function of a particle in a box, have shown that averaging it on a rectangle in phase space of area greater than  $2\pi h$  one can obtain negative values. When representing the characteristic function of a rectangle as a sequence of analytic functions, they noticed that this result in their concrete example remains valid even if the smoothing function is continuous on the boundary of the non-zero region. In this way they found one example to which Wlodarz's proof, strictly speaking, does not apply.

In the present work we give a simple proof that whenever smoothing function is zero outside a finite phase space region, but arbitrary otherwise, the smoothed Wigner function, in the general case, can not be non-negative.

For simplicity we shall restrict our considerations to one dimensional case, and in what follows we shall put  $h = 1$ .

For pure state Wigner function is defined as [2]

$$W(q, p) = 1/\pi \int \psi^*(q+y)\psi(q-y)\exp(2ipy)dy. \quad (1)$$

From this function the probability distributions in position and momentum coordinates can be derived [2]:

$$|\psi(q)|^2 = \int W(q,p) dp \quad (2)$$

$$|\tilde{\psi}(p)|^2 = \int W(q,p) dq \quad (3)$$

where  $\tilde{\psi}(p) = 1/\pi \int \psi(q) \exp(ipq) dq$  is the momentum wave function.

The absolute square of the scalar product of two wave functions  $\psi_k$  and  $\psi_n$  may be expressed through their Wigner functions as

$$|\langle \psi_k | \psi_n \rangle|^2 = \int W_k(q,p) W_n(q,p) dq dp.$$

This expression is zero when the two wave functions are orthogonal to each other, what shows that Wigner functions, in the general case, can not be positive everywhere in phase space. Due to this fact, the problem of smoothing arises.

The conditions which the smoothing function  $f(q,p)$  must satisfy in order that for every Wigner function the relation

$$P = \int f(q,p) W(q,p) dq dp \geq 0 \quad (4)$$

holds, were found in [2]. There was shown that the smoothing function  $f(q,p)$  must be of the form

$$f(q,p) = \sum_k \lambda_k W_k(q,p), \quad \lambda_k > 0. \quad (5)$$

In other words,  $f(q,p)$  must be, up to a multiplication factor, a Wigner function.

Now we shall prove that such a function can never take non-zero values on the finite part of phase space only.

Supposing the opposite, it would be always possible to find such  $q_m$  and  $p_m$  that whenever at least one of the inequalities  $|p| > p_m$  or  $|q| > q_m$  is satisfied one could write

$$f(q,p) = 0 = \sum_k \lambda_k W_k(q,p) \quad (6)$$

Taking into account the relations (2) and (3), after integration of (6) over  $p$  and  $q$  one would have, respectively

$$0 = \sum_k \lambda_k |\psi_k(q)|^2, \quad |q| > q_m \quad (7)$$

$$0 = \sum_k \lambda_k |\tilde{\psi}_k(p)|^2, \quad |p| > p_m \quad (8)$$

(  $\lambda_k > 0$  )

However, (7) and (8) can not hold simultaneously. If, for example, (7) were satisfied we would have

$$\psi_k(q) = 0 \quad \text{for } |q| > q_m$$

and

$$\tilde{\psi}_k(p) = \int_{-q_m}^{+q_m} \psi_k(q) \exp(ipq) dq. \quad (9)$$

The last equation implies the existence of all the integrals

$$\int_{-q_m}^{+q_m} q^n \psi_k(q) \exp(ipq) dq$$

for arbitrary  $n$  and  $p$ . This means that  $\tilde{\psi}_k(p)$  is an entire function of  $p$  and as such can not be zero on a segment, what is required for the validity of (8). This terminates the proof.

Our result shows that smoothing of a Wigner function over an infinite instead over a finite region of phase space, in order to get a non-negative smoothed Wigner function, is not just a matter of convenience or a matter of choice, as it is often, explicitly or implicitly assumed [5,7], but is a necessary condition for getting a non-negative smoothed Wigner function. We think that this fact should be taken into account, in an appropriate way, in all endeavors to give a probabilistic interpretation to the non-negative smoothed Wigner functions.

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THE PHASE SPACE REPRESENTATION OF QUANTUM STATES  
AND THE GIBBS ENTROPY \*

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*1. Introduction.* The uncertainty principle of Quantum Mechanics (QM) is a fundamental concept of the theory since it represents the breakdown of the possibility, that exists in classical physics, of a simultaneous measurement of physical observables with arbitrary degree of accuracy. QM introduced a fundamental limit on the simultaneous knowledge of observables, formally represented by noncommutative operators, which cannot share the same set of eigenstates and for some state  $|\psi\rangle$ , the product of their variances has a lower bound. The concept of entropy permits to introduce an alternative approach to represent the amount of uncertainty that is contained in a state  $|\psi\rangle^1$ . The aim of this contribution is to discuss the use of the Squeezed-Coherent States (SCS) within the entropy formalism since this is an adequate representation for noncompatible observables with continuous spectrum (generators of the Weyl-Heisenberg group)<sup>2</sup>.

*2. The Squeezed-Coherent States and the Squeeze Operator*

The squeezed-coherent states are defined as,

$$|pq; \gamma\rangle := D(p, q)S(\gamma)|0\rangle \quad (1)$$

where  $|0\rangle$  is the vacuum state,

$$D(p, q) := \exp\left[-\frac{i}{\hbar}(p\hat{Q} - q\hat{P})\right] \quad (2)$$

is the phase space displacement operator and the squeeze operator is

$$S(\gamma) = \exp\left[-\frac{i\gamma}{2\hbar}(\hat{P}\hat{Q} + \hat{Q}\hat{P})\right], \quad (3)$$

where the real squeeze parameter  $\gamma$  controls the sharpness of the states. The squeeze operator transforms  $\hat{P}$  and  $\hat{Q}$  by simply introducing a scale factor,  $S(\gamma)\hat{Q}S^\dagger(\gamma) = e^{-\gamma}\hat{Q}$ ,  $S(\gamma)\hat{P}S^\dagger(\gamma) = e^{\gamma}\hat{P}$ . Therefore,  $S(\gamma)$  squeezes or stretches  $\hat{Q}$  (for  $\gamma$  positive or negative, respectively), but keeps the product  $\hat{P}\hat{Q}$  invariant.

*3. The Differential Equation of the "Projection" Operator  $|pq, \gamma\rangle\langle pq, \gamma|$*

Let us consider the "projection" operator

$$\hat{\mathcal{P}}(p, q, \gamma) = |pq, \gamma\rangle\langle pq, \gamma|. \quad (4)$$

For any a density matrix  $\hat{\rho}$ ,  $P(p, q; \gamma) = \text{Tr} \hat{\rho}\hat{\mathcal{P}}(p, q, \gamma) \geq 0$ , while for a pure state,  $\hat{\rho} = |\psi\rangle\langle\psi|$ ,

$$P_\psi(p, q; \gamma) = |\langle pq, \gamma|\psi\rangle|^2. \quad (5)$$

$\hat{\mathcal{P}}(p, q, \gamma)$  has the important property<sup>3</sup>

$$\hat{\mathcal{P}}(p,q,\gamma) \left\{ \begin{array}{l} \hat{Q} \\ \hat{P} \end{array} \right\} = \left\{ \begin{array}{l} q + \frac{\hbar}{2} \frac{\partial}{\partial p} + \frac{\hbar e^{2\gamma}}{2} \frac{\partial}{\partial q} \\ p - \frac{\hbar}{2} \frac{\partial}{\partial q} + \frac{\hbar e^{-2\gamma}}{2} \frac{\partial}{\partial p} \end{array} \right\} \hat{\mathcal{P}}(p,q,\gamma) \quad (6)$$

which permits to verify the partial differential equation

$$\frac{\partial}{\partial \gamma} \hat{\mathcal{P}}(p,q;\gamma) = \left[ \frac{\hbar e^{2\gamma}}{2} \frac{\partial^2}{\partial q^2} - \frac{\hbar e^{-2\gamma}}{2} \frac{\partial^2}{\partial p^2} \right] \hat{\mathcal{P}}(p,q;\gamma). \quad (7)$$

Therefore, for a square integrable wavefunction  $\psi(x)$ ,  $P_\psi(p,q;\gamma)$  is a solution of the partial differential equation

$$\frac{\partial P_\psi(p,q;\lambda)}{\partial \lambda} = \frac{1}{4} \left[ \frac{\partial^2}{\partial p^2} - \frac{1}{\lambda^2} \frac{\partial^2}{\partial q^2} \right] P_\psi(p,q;\lambda) \quad (8)$$

This equation is similar to a diffusion equation in two dimensions, where the parameter  $\lambda$  plays the role of *time*. However, the diffusion coefficients,  $1/4$  and  $-1/(4\lambda^2)$ , have opposite signs, so it describes a diffusive process in the  $q$ -variable and an "infusive" process in  $p$ -variable. For this reason it is called *pseudo-diffusion equation*<sup>(4)</sup>.  $p$  and  $q$  have been made dimensionless and  $\lambda = e^{-2\gamma}$ . The coefficient of  $\partial^2/\partial q^2$  depends on  $\lambda$ , in such a way that for  $\lambda < 1$  there is more "infusion" ( $1/\lambda^2 > 1$ ) than diffusion and the other way around for  $\lambda > 1$ . Considering  $\langle \psi | \psi \rangle = 1$ , the probability functions  $P_\psi(p,q;\lambda)$  have the properties, *i)*  $0 \leq P_\psi(p,q;\lambda) \leq 1$ , *ii)*  $P_\psi(q,p;\lambda^{-1})$ ,  $P_\psi(-p,q;\lambda)$  and  $P_\psi(p,-q;\lambda)$  are also solutions of the *pseudo-diffusion equation*.

#### 4. Phase space entropy functionals

In probability theory two correlated random variables have their probability distribution represented by a joint distribution function rather than by two independent ones; also in QM the SCS permit to associate to each state  $|\psi\rangle$  the probability density function  $P_\psi(p,q;\lambda)$ , whereas the *marginal probabilities*,  $Q_\psi(q;\lambda)$  and  $R_\psi(p;\lambda)$ , are obtained by integrating on only one variable at a time.

As Wehrl's entropy for CS<sup>5</sup>, which in the limit  $\hbar \rightarrow 0$  goes to the classical Gibbs entropy, we introduce a class of functionals for the SCS, where the *joint entropy* or *p-q entropy* is given by

$$S_{P_\psi}(\lambda) = - \int - \frac{dp dq}{2\pi\hbar} P_\psi(p,q;\lambda) \ln P_\psi(p,q;\lambda), \quad (9)$$

while the *marginal entropies*,  $S_Q(\lambda)$  ( $q$ -entropy) and  $S_R(\lambda)$  ( $p$ -entropy) are defined similarly although the functions in the integral are now the *marginal probabilities*. The *conditional entropies* are defined as

$$S_{P_{R|Q}}(\lambda) = S_{P_\psi}(\lambda) - S_R(\lambda) \quad \text{and} \quad S_{P_{Q|R}}(\lambda) = S_{P_\psi}(\lambda) - S_Q(\lambda). \quad (10)$$

Comparing these equations one can establish the balance equation

$$S_{P_{R|Q}}(\lambda) + S_R(\lambda) = S_{P_{Q|R}}(\lambda) + S_Q(\lambda) \quad (11)$$

which is the only conservation law that one encounters in the entropy formalism. Some inequalities among these functionals can be verified:

Any pair  $(x, y)$  of non-negative variables satisfy the Klein inequality  $y \ln x - y \ln y - x - y \leq 0$ , then if we consider  $y = P_\psi(p, q; \lambda)$  and  $x = R_\psi(p; \lambda) Q_\psi(q; \lambda)$  and integrate on both variables,  $p$  and  $q$ , one obtains

$$S_P(\lambda) \leq S_Q(\lambda) + S_R(\lambda), \quad (12)$$

which means that the sums of the marginal entropies contains less information (about  $|\psi\rangle$ ) than the joint entropy which becomes a lower bound. The equality holds only when the variables are uncorrelated which is to say that the function (5) factorizes as  $R(p)Q(q)$ . Now comparing eqs. (10) - (12) one arrives to the inequalities

$$S_{P/Q}(\lambda) \leq S_Q(\lambda) \quad , \quad S_{P/R}(\lambda) \leq S_R(\lambda). \quad (13)$$

Therefore the conditional entropies contain more information than the marginal entropies and they become equal only when the variables are uncorrelated.

It is useful to define the  $p$ - $q$  correlation functional<sup>6</sup>,

$$C_P(\lambda) = S_Q(\lambda) - S_{P/R}(\lambda) = S_R(\lambda) - S_{P/Q}(\lambda) = S_Q(\lambda) + S_R(\lambda) - S_P(\lambda) \quad (14)$$

which has the properties of symmetry and positiveness. The  $p$ - $q$  correlation is zero when the joint probability factorizes and its value grows as the correlations become more manifest.

When one variable determines the other exactly the conditional entropies become zero, implying that the joint and marginal entropies are equal, and the correlation becomes equal to the marginal entropies

##### 5. Illustrative Application and Discussions

We consider the 1-d harmonic oscillator wavefunctions, which, have associated the probability distribution functions<sup>7</sup>

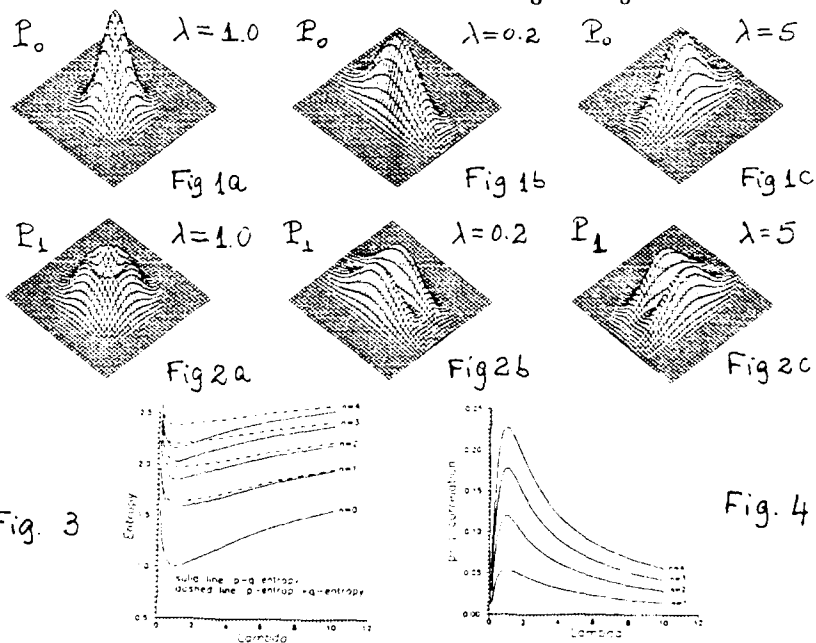
$$P_n(p, q; \lambda) = \frac{2\lambda^{1/2}}{(\lambda+1)} \left( \frac{\lambda-1}{\lambda+1} \right)^n \sum_{k=0}^n (-1)^k L_k^{-1/2} \left( \frac{2\lambda^2 q^2}{\lambda^2 - 1} \right) \\ \times L_{n-k}^{-1/2} \left( \frac{-2p^2}{\lambda^2 - 1} \right) \exp \left[ - \frac{\lambda q^2 + p^2}{\lambda + 1} \right] \quad (15)$$

where the  $L_k^{-1/2}$  are the associated Laguerre polynomials. For  $\lambda = 1$ , the  $P_n$ 's goes to the familiar Poisson distribution, so they are called *Generalized Poisson Distributions*. See Figs 1,2 for  $P_0$  and  $P_1$ ; for  $\lambda \neq 1$  one verifies that the symmetry present for  $\lambda = 1$  is broken.

In Fig 3 the entropies are plotted as function of  $\lambda$ , the joint entropy (solid line) and the sum of the  $p$ -entropy and  $q$ -entropy (dashed line), verify the inequality (12); for  $n = 0$  both curves coincide while for  $n \neq 0$ , the curves merge only in the limits



$\lambda = 0, \infty$ , implying the existence of correlations that are absent for the ground state wavefunction. For  $\lambda = 1$  (Glauber coherent states), all curves attain their lowest value because the  $p$ - $q$  uncertainty becomes a minimum. So one can appreciate the rôle of  $\lambda$ , as it squeezes or stretches the fundamental cell from its optimum value,  $\lambda = 1$ , the map blurs the information contained in the wavefunction; in the limits,  $\lambda = 0, \infty$ , the entropy becomes infinite and the phase space becomes useless for providing information on both variables, jointly. It is worth to remember that independently of how the fundamental cell is deformed its area  $h$  is always constant. In Fig. 4 the  $p$ - $q$  correlation attains a maximum value at  $\lambda = 1$ , then decreases in the extremes. Thus, one verifies that while the joint entropy attain a minimum value at  $\lambda = 1$ , for  $n \neq 0$ , the  $p$ - $q$  correlation attain a maximum; the only exception being  $n = 0$  which is zero because the joint distribution function factorizes as  $Q_0(q;\lambda)R_0(p;\lambda)$ .



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## NELSON'S STOCHASTIC MECHANICS AS THE PROBLEM OF RANDOM FLIGHTS

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In the framework of Nelson's stochastic mechanics<sup>(1)</sup> the notion of a particle path can be given a well defined meaning in the configuration space. We can view it as a sample trajectory followed by the mass point undergoing a Markovian diffusion process in  $R^3$  with dynamics constrained by the second Newton law in the conditional mean. Leaving aside the problem to what extent a particular solution of the respective stochastic differential equation with constraint is capable of simulating a genuine point particle path in  $R^3$ , we are left with a persuasive representation of each Schrödinger wave function by a collection of sample propagation scenarios.<sup>(1,2)</sup> To understand this classically motivated stochastic model of quantum phenomena on physically deeper grounds it appears necessary to reveal a random phase-space propagation, whose configuration space projection would imply stochastic mechanics.

The phase space route was originally abandoned by Nelson in his expository paper<sup>(3)</sup> although it was clearly stated there that "the situation appears to be related to diffusion processes in fluids with currents flowing in them". In fact this hydrodynamical aspect of the quantum mechanical formalism found a detailed analysis in the papers by Takabayasi.<sup>(4,5)</sup> The only serious attempt to derive stochastic mechanics from a random phase space model belongs to de la Peña and Cetto,<sup>(6)</sup> but it was finally reduced to the search for restrictions on the formalism which would give rise to the Wigner distribution. Apparently, the Wigner function is one only<sup>(7)</sup> (though best known to the physicists) example of the joint phase space distribution, unfortunately not everywhere positive and not a solution of the diffusion equation.

In the present note, we wish to report that<sup>(2,8)</sup> random phase space motions governed by white noise without friction, induce a diffusion equation, which if constrained by the appropriate equation of state (terminology borrowed from the kinetic theory of gases and fluids) implies:

1) Madelung fluid equations known to characterize wave functions of Schrödinger mechanics in the nonrelativistic case

2) Local conservation laws characterizing the positive energy solutions of the Klein-Gordon field in case of relativistic Markovian diffusions.

Let us recall that in the nonrelativistic situation the white noise Langevin problem without friction:

$$\frac{d\vec{x}}{dt} = \frac{\vec{p}}{m} \quad \frac{d\vec{p}}{dt} = \vec{F} + \vec{\beta}(t) \quad \langle \vec{\beta}(t) \rangle = 0 \quad (1)$$

$$\vec{F} = -\nabla V, \quad \langle \beta_i(t)\beta_j(t') \rangle = \zeta \delta_{ij}\delta(t-t')$$

implies the following evolution equation for the joint position - momentum probability distribution associated with the statistical ensemble of diffusing particles

$$\partial_t \Phi + \frac{\vec{p}}{m} \nabla_x \Phi + \vec{F} \nabla_p \Phi = \frac{\zeta}{2} \Delta_p \Phi \quad (2)$$

The white noise amplitude is not specified and  $\vec{x}, \vec{p}$  take values in  $R^3 \times R^3$ .

By introducing the marginal distribution  $\rho(\vec{x}, t) = \int d^3p \Phi(\vec{x}, \vec{p}, t)$  and local moments  $\langle p_i \rangle = \int d^3p p_i \Phi(\vec{x}, \vec{p}, t)$ ,  $\langle p_i p_j \rangle = \int d^3p p_i p_j \Phi(\vec{x}, \vec{p}, t)$  of  $\Phi$  we impose the following constraint (which is a direct analogue of the equation of state traditionally utilized in the kinetic theory of gases) on the admissible solutions of (2)

$$\langle p_i p_j \rangle = \frac{1}{\rho} \langle p_i \rangle \langle p_j \rangle + m^2 P_{ij} \quad (3)$$

$$P_{ij}(\vec{x}, t) = - \left( \frac{\hbar}{2m} \right)^2 \rho(\vec{x}, t) \frac{\partial^2}{\partial x_i \partial x_j} \ln \rho(\vec{x}, t)$$

Let  $\vec{v} = \langle \vec{p} \rangle / \rho m$  be an irrotational velocity field, then the constraint (3) relates first and second moments of  $\Phi(\vec{x}, \vec{p}, t)$  with  $\rho(\vec{x}, t)$ , and in the infinite hierarchy of local conservation equations (for moments) induced by  $\Phi$  it transforms the first two equations into a self-contained (closed) system. Its solution does not involve an explicit form of the joint distribution, and given the initial position and first moment distributions, we have uniquely determined  $\rho(\vec{x}, t)$  and  $\langle \vec{p} \rangle(\vec{x}, t)$  together with  $P_{ij}(\vec{x}, t)$  for all times. Under the assumption that (2) and (3) are compatible, we recover then Nelson's stochastic mechanics.<sup>(2)</sup>

The interesting situation is here obtained by comparing the Langevin problem (1) with the Nelson's diffusion

$$\begin{aligned} d\vec{X}(t) &= (\vec{u} + \vec{v})(\vec{X}(t), t)dt + \sqrt{\frac{\hbar}{m}} d\vec{W}(t) \\ \frac{\vec{F}}{m}(\vec{X}(t), t) &= \frac{1}{2}(D_+ D_- + D_- D_+) \vec{X}(t) = \\ &= (\partial_t + \vec{v} \nabla) \vec{v} - \left( \frac{\hbar}{2m} \Delta + \vec{u} \nabla \right) \vec{u} = \\ &= \partial_t \vec{v} + \nabla \left( \frac{v^2 - \vec{u}^2}{2} - \frac{\hbar}{2m} \operatorname{div} \vec{u} \right) \end{aligned} \quad (4)$$

where  $\vec{W}(t)$  is a normalized Wiener noise in  $R^3$ . Although (4) generates wildly random trajectories in  $R^3$  they cannot be identified with the individual particle paths in the sense of (1). The drift  $\vec{b} = \vec{u} + \vec{v}$  entering (4) indicates the local flow tendency of propagation and comes from configuration space conditioned moments of the phase-space probability distribution  $\Phi(\vec{x}, \vec{p}, t)$ . Hence is definitely not the individual particle velocity consistent with (1). In the above the joint distribution is not specified completely by the continuity and momentum balance equations, and quite strong additional restrictions would be necessary to replace it by the Wigner distribution in our formalism. At the moment no explicit phase space solutions of (1)-(3) are available, but the mathematical structure of previous arguments makes it plausible that the set of solutions is nonempty and sufficiently rich. One should here consult how solutions of the Boltzmann equation are looked for. Higher moment equations were not exploited at all in the above and they are non-trivial.

However, one should realize that the level of difficulty in obtaining explicit solutions of the diffusion equation (2) is similar to this encountered in case of the Boltzmann equation, or in search for general solutions of the Kramers equation, see e.g. <sup>(9)</sup> for an analysis of the innocent looking equation  $w_t = w_{xx} - xw_y$  arising in the analogous (random accelerations) context. In our investigation, the absence of friction locates the problem in the domain of the non-equilibrium statistical physics.

Let us pass to the relativistic generalization of the above observations. It is instructive to notice that the left-hand-side of (2) would precisely represent  $d\Phi/dt$  in case of no noise. By passing to the finite difference approximation with small time increments, we can evaluate the net change of  $\Phi$  due to noise in the time interval  $\Delta t$ , as follows:

$$\Phi\left(\vec{x} + \frac{\vec{p}}{m} \Delta t, \vec{p} + \vec{F} \Delta t, t + \Delta t\right) - \Phi(\vec{x}, \vec{p}, t) \cong \left(\frac{\zeta}{2} \Delta_p \Phi\right) \Delta t \quad (5)$$

In case of the relativistic invariant phase space distribution  $f(x, p)$  the role of  $\Delta t$  is taken by the proper time increment  $c\Delta t = \gamma\Delta\tau$  with  $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$ ,  $\vec{v} = \frac{\vec{p}}{m}$ . Then (4) should be replaced by:

$$f\left(x + \frac{p}{m} \Delta\tau, p + F\Delta\tau\right) - f(x, p) \cong (\text{noise term})\Delta\tau \quad (6)$$

where

$$\begin{aligned} x &= (ct, \vec{x}), & p &= (p_0, \vec{p}), & p_\mu p^\mu &= p_0^2 - \vec{p}^2 = m^2 c^2 \\ p \cdot m &= mcu = m \frac{dx}{d\tau} & p_0 &= mc\gamma & \vec{p} &= m\vec{v}\gamma \\ F &= \frac{dF}{d\tau} & F_\mu u^\mu &= 0 \end{aligned} \quad (7)$$

Apparently the noise term in (5) must be a relativistic invariant diffusion generator coming from the assumption that a Markovian random walk is taking place on the mass  $m$  hyperboloid.

The pertinent diffusion equation reads:

$$cu^\mu \partial_\mu f(x, u) + \frac{F^\mu}{mc} \frac{\partial}{\partial u^\mu} f(x, u) = \frac{\zeta}{2m^2 c^2} \Delta_{LB} f(x, u) \quad (8)$$

where the Laplace-Beltrami operator on  $u^\mu u_\mu = 1$  is given in the hyperbolic parametrization as follows:

$$\Delta_{LB} = \frac{1}{sh^2 r} \left( \frac{\partial}{\partial r} sh^2 r \frac{\partial}{\partial r} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) \quad (9)$$

With the notion of the configuration space conditioned local moment  $v_\nu(x) = \frac{1}{\rho} \int u_\nu f(x, u) d^3 u / u^0$  we arrive at the continuity and energy-momentum balance equations:

$$\partial_\mu (\rho v^\mu) = 0 \quad (10)$$

$$\partial_\nu T^{\mu\nu} = \rho F^\mu \quad \text{or} \quad \rho F^{\mu\nu} v_\nu \quad (11)$$

which upon:

$$T^{\mu\nu} = mc^2 \rho v^\mu v^\nu + \int f(x, u) mc^2 (u^\mu - v^\mu)(u^\nu - v^\nu) d^3 u / u_0 = mc^2 (\rho v^\mu v^\nu + P^{\mu\nu}) \quad (12)$$

and imposing the equation of state (osmotic) constraint:

$$P^{\mu\nu} = - \left( \frac{\hbar}{2mc} \right)^2 \rho \partial_\mu \partial_\nu \ln \rho \quad (13)$$

give rise to the equation:

$$\frac{1}{m} \left( \partial_\mu S - \frac{e}{c} A_\mu \right) \left( \partial^\mu S - \frac{e}{c} A^\mu \right) - \frac{\hbar^2}{2m} \{ (\partial^\mu \ln \rho) (\partial_\mu \ln \rho) + \square \ln \rho \} = M \quad (14)$$

In case of  $M = mc^2$  together with a continuity equation (14) forms a closed system providing us with the positive energy solutions of the Klein-Gordon equation:

$$\left\{ \left( \partial_\mu - i \frac{e}{c} A_\mu \right) \left( \partial^\mu - i \frac{e}{c} A^\mu \right) + \frac{m^2 c^2}{\hbar} \right\} \Psi(x) = 0 \quad (15)$$

under the identification  $\Psi = \exp \left( \ln \rho^{1/2} + \frac{i}{\hbar} S \right)$ , where the invariant scalar density is given by  $\rho(x) = \int \frac{d^3 u}{u^0} f(x, u)$ .

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**VII. Applications of  
Quantum Mechanics:  
From Channel Space  
to Wigner Crystal**

**CANONICAL TRANSFORMATIONS IN CHANNEL SPACE  
AND THEIR UNITARY REPRESENTATION:  
A STEP TOWARD UNDERSTANDING  
CHAOTIC SCATTERING IN QUANTUM MECHANICS**

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We define a classical scattering process in a four-dimensional phase space as a canonical map between the two-dimensional phase spaces of incoming and outgoing channels. Time reversal invariance is introduced and the iterated scattering map appears naturally. Using the theory of unitary representations of classical canonical transformations the transition to the quantum problem is achieved in the framework of a semi-classical approximation. Families of canonical scattering maps are defined by canonical transformations on channel space; the invariance properties of these families translate naturally into invariant ensembles of S-matrices that are unique according to Dyson. The transitivity of the action of a chaotic iterated scattering map indicates that its unitary representation is a characteristic member of the ensemble and thus by way of ergodicity the corresponding S-matrix has a distribution of eigenphases known as a COE.

The statistical behaviour of scattering processes has been of interest for many years in nuclear physics [1], but recently it has acquired more general importance in the context of quantum manifestations of chaos [2]. In this context one question asked, refers to the statistical distribution of the eigenphases of the S-matrix. The interplay between energy averages for specific models and ensemble averages for random matrix ensembles is usually the key point of such an analysis.

Smilansky has shown [3], that in the semi-classical limit, the two point function of the eigenphase distribution coincides with the one of the circular ensemble of unitary and symmetric matrices [4] (referred to in what follows as COE), if the corresponding iterated scattering map [5] is chaotic.

The purpose of this paper is first to formulate the classical problem in terms of classical canonical transformations that describe the scattering process and to discuss the symmetry properties related to chaoticity of the iterated scattering map. Next we use the theory of representation of classical canonical transformations in quantum mechanics [6,7]. The invariance properties derived for a family of classical transformations can be translated to the quantum case and, due to the well known group structure of  $U(n)$ , induce in the semi-classical limit the invariant ensemble known as COE. Invoking ergodicity we find the COE properties for the eigenphases of every generic member of the ensemble. Typical non-generic cases based on additional symmetries of the problem are discussed at the end. The entire problem will be restricted to the case of two degrees of freedom, which is the simplest case that displays general features of chaos.



First consider a free particle in four dimensional phase space with coordinates  $(p_1, q_1, p_2, q_2)$ . We can perform a canonical transformation

$$(p_1, q_1, p_2, q_2) \rightarrow (E, t, P_\phi, \phi) \quad (1)$$

to energy  $E$ , time  $t$ , angular momentum  $P_\phi$ , and angle  $\phi$ . In principle this transformation is two to one [8], but we shall be interested in particular subspaces. The purpose is to describe a scattering process by a time independent and time reversal invariant Hamiltonian that differs from the free particle Hamiltonian on a compact support only. Thus the energy is constant and the particles will be asymptotically free. Therefore we consider the free particle for large times  $t \rightarrow \infty$  and outgoing particles as final channel space and the same for times  $t \rightarrow -\infty$  and incoming particles as initial channel space. These are typical surfaces of section that on their own form a phase space each. The choice of incoming and outgoing particles respectively chooses between the two Riemann like surfaces we have in phase space [8]. We shall call the outgoing and incoming spaces respectively  $M^o$  and  $M^i$  and they are characterized by coordinates

$$(E = \text{const.}, t = \pm\infty, P_\phi, \phi) \rightarrow (P_\phi, \phi). \quad (2)$$

These spaces are thus effectively two dimensional as  $E$  and  $t$  are fixed.

Scattering by a Hamiltonian of the above mentioned type can be represented as a canonical transformation

$$C_S : M^i \rightarrow M^o, \quad (3)$$

which we call scattering map.

As we are interested in time reversal invariant scattering processes we have to consider the time reversal operation  $T$ . This operation is non-canonical in that it changes  $P_\phi$  to  $-P_\phi$  while simultaneously changing an outgoing into an incoming space or vice versa. Time reversal invariance implies that  $TC_S TC_S = \mathbf{1}$  is the identical transformation on  $M^i$ .

The iterated scattering map was introduced by Jungen [5] in order to be able to use for iterated maps the surface of section implicit in the definition of channel space. In our language we shall define the iterative map  $I_S = TC_F TC_S$ , where  $C_F$  is the map induced by the free particle dynamics. This map can readily be iterated as it maps  $M^i$  onto itself, and its iteration is indeed identical to the iterated scattering map [5].

We now consider bijective canonical transformations  $C_\alpha$  of  $M^i$  onto itself. The corresponding transformation on  $M^o$  is  $TC_\alpha T$ , and we can define a family of scattering maps -

$$C_{S^\alpha} = TC_\alpha^{-1} TC_S C_\alpha, \quad (4)$$

with a corresponding family of iterative scattering maps  $I_S^\alpha$ . These two families of scattering maps have by construction a remarkable invariance property. If we perform an arbitrary bijective canonical transformation on  $M^i$  and simultaneously the time reversed transformation on  $M^o$  the family of transformations  $C_{S^\alpha}$  remains invariant, and a similar argument holds for the iterative maps.

We now proceed to the quantum mechanical problem. To do this we follow ideas of Dirac, Moshinsky and others [6,7], that allow us to associate unitary operators to canonical transformations. In general these unitary operators do not represent the

quantum problem on hand "precisely" [9], as ordering problems will occur. On the other hand any unitary representation we may choose will, by construction, give the correct semi-classical limit [6]. We can therefore choose any representation in channel space and associate a unitary operator to a canonical transformation. We furthermore have to recall that the representation of conjugation by the time reversal operator is simply the anti-unitary operation of canonical conjugation.

If the S-matrix  $S$  is the unitary representation of  $C_S$  then  $TC_S TC_S = 1$  implies  $S^* S = 1$  or, due to the unitarity of  $S$ ,  $S = S^t$ . We are only interested in those parts of channel space, whose trajectories are affected by the non-free part of the Hamiltonian, which was chosen to have compact support. This part of channel space is in turn compact and for finite Planck constant will lead to a finite dimensional S-matrix. If we further denote the representation of  $C_{S^\alpha}$  by  $S^\alpha$ , we have a family of symmetric S-matrices that, at least in the semi-classical limit, represents the family of canonical scattering maps, which we constructed above.

The group of bijective canonical transformations is still very little understood and no invariant measure is known. By consequence we cannot construct an ensemble from the family of scattering maps, which we defined in eq. (4). On the other hand the corresponding S-matrices  $S^\alpha$  can readily and naturally be endowed with a measure. The invariance of the family of scattering maps discussed above, translates into an invariance of the family of S-matrices, such that if  $S^\alpha$  is a member of the family so is  $U^t S^\alpha U$  where  $U$  is any unitary transformation that in turn represents any bijective classical canonical transformation. Yet this invariance is well-known for random matrix ensembles and if we require it for the measure, that determines the ensemble it leads uniquely [10] to one of the classical circular ensembles, also known as classical domains [11], namely the orthogonally invariant ensemble of unitary symmetric  $n \times n$  matrices (COE).

Thus the family of scattering maps developed from classical notions leads naturally to the COE, but we now have to establish under what circumstances we can expect the individual S-matrix on hand to be typical in the sense that we can assume ergodicity. Ergodicity is here invoked not in the usual sense where time averages are replaced by ensemble averages. Rather we replace averages over eigenvalues by ensemble averages, and in this sense we proceed in a similar fashion as for the energy averages, commonly considered in random matrix theory. Ergodic behaviour has been proven for the Gaussian orthogonal ensemble [12], and can be extended to the closely related COE.

A first requisite is the transitivity of the scattering process on channel space. This is expressed in terms of chaoticity of the iterated scattering map on  $M'$ . This condition is necessary because otherwise we can select some subspace of  $M'$  such that no iteration of  $I$  will take us out of this space. Time reversal invariance insures the equivalence of incoming and outgoing spaces, and thus we could formulate a similar family of scattering maps on the reduced space. This in turn induces an invariant measure on the subspace and its complement, thus inducing a superposition of two COE's.

This leads us immediately to the next and well known restriction. In quantum mechanics spaces that belong to irreducible representations of a symmetry group will induce independent invariant measures and thus lead to superpositions of COE's. This happens in close analogy to the bound case, and obliges us to consider minimal invariant subspaces of each chaotic region of channel space.

Even if we separate symmetry classes or eliminate symmetries to take this problem into account, this is not sufficient to ensure COE behaviour as other types of symmetries may also occur. Consider the map  $C_{SS} = C_S I_S$  which is again a symmetric scattering map. The corresponding S-matrix  $S^2$  has eigenphases that are double the eigenphases of  $S$  according to the spectral theorem. If the eigenphases of  $S$  corresponded to a COE those of  $S^2$  correspond to a superposition of two COE's. This behaviour is not particular to the S-matrix; indeed the same is to be expected for time evolution operators connected with periodic hamiltonians, if we choose them over two periods rather than one, and this has been numerically confirmed [13] for the quasi energies of the Fermi oscillator [14].

Even now we can not be sure that we have identified all possible sources of non-generic behaviour, though we have eliminated the most obvious ones. We have thus to keep in mind that any property of the system that makes it possible to define separate Hilbert spaces corresponding to the same classically connected domain in phase space but not connected by the quantum Hamiltonian will cause non-generic behaviour.

We can thus summarize as follows: The chaoticity of the iterated scattering map is a necessary condition for a COE distribution of the eigenphases of the S-matrix and furthermore generically scattering processes with a chaotic iterative scattering map have COE distributions of eigenphases. We obtained this result using a new approach of the concept of invariance that in a sense indicates a dynamical symmetry, because we use the ergodic behaviour of the orbit to create an ensemble which by a second apparently independent ergodic hypothesis gives us results on energy averages. The dynamical aspect of this symmetry lies in the fact that the two ergodic behaviours must still be more closely related, which is not done in this paper. The main difficulty arises from the appearance of other symmetries which have to be globally understood to define the generic case properly.

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INTRINSIC IRREVERSIBILITY OF QUANTUM SYSTEMS  
RIGGED HILBERT SPACE FOR THE LEE-FRIEDRICHS MODEL.

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Resonances in quantum systems with continuous spectrum as in scattering or in interacting fields, give rise to divergencies in the conventional perturbation algorithms which should solve the Hamiltonian eigenvalue problem. These quantum resonances are of the same nature with the classical Poincaré resonances<sup>1</sup> of large systems<sup>2,3,4</sup> which prevent the analytic integration of Hamilton's canonical equations. However, a natural time ordering of the dynamical states leads to a rule for analytic continuation of the perturbation terms according to the processes they represent. The rule may be seen as a boundary condition, necessary to guarantee the uniqueness of the solution of the dynamical system with continuous spectrum, i.e. as a generalized Sommerfeld radiation condition. The construction developed<sup>5,6</sup> by the Brussels-Austin school, directed by I. Prigogine, not only integrates the system, but also leads to a representation of dynamics in an extended space, where the Hamiltonian has complex eigenvalues and the generated evolution is split into two semigroups<sup>7</sup>, one decaying in the future and the other in the past. Irreversibility emerges then, intrinsically as the selection of the semigroup compatible with our future observations. In the case of the Friedrichs model, a suitable mathematical framework for the time ordering construction is provided by the Rigged Hilbert space of Hardy class, constructed by A. Bohm and M. Gadella<sup>8</sup>.

### 1. COMPLEX EIGENVALUES FOR THE LEE-FRIEDRICHS MODEL.

The time ordering construction will be illustrated through the Lee-Friedrichs model<sup>8-12</sup>. This model is solvable in a non analytic way and served as a prototype for the decay problem in quantum mechanics. In the simplest version, a discrete state  $|1\rangle$ , is coupled with a continuum of states  $|\omega\rangle$ ,  $\omega \in [0, \infty)$  corresponding to field modes. The Hamiltonian operator is

$$H = H_0 + \lambda V = \omega_1 |1\rangle\langle 1| + \int_0^\infty d\omega \omega |\omega\rangle\langle \omega| + \lambda \int_0^\infty d\omega V_\omega (|\omega\rangle\langle 1| + |1\rangle\langle \omega|) \quad (1)$$

The Friedrichs solutions<sup>9</sup> are obtained through the wave operators<sup>10,11</sup>

$$|f_{\omega}^{\pm}\rangle = |\omega\rangle + \frac{\lambda V_\omega}{\eta_{\pm}(\omega)} \left( |1\rangle + \int_0^\infty d\omega' \frac{\lambda V_{\omega'}}{\omega - \omega' \pm i0} |\omega'\rangle \right) \quad (2)$$

Here  $\eta_{\pm}(\omega)$  are the boundary functions from above and below corresponding to the inverse partial resolvent

$$\eta(z) \equiv \langle 1 | \frac{1}{z - H} | 1 \rangle^{-1} = (z - \omega_1) + \int_0^\infty d\omega' \frac{\lambda^2 |V_{\omega'}|^2}{\omega' - z} \quad (3)$$

The interaction is assumed to be such that the branches  $\frac{1}{\eta_{\pm}(z)}$  of the partial resolvent have meromorphic extensions to the lower and upper half planes, with simple poles at  $z_1$  and  $z_1^*$  correspondingly. Conditions for the meromorphic structure of the partial resolvent are discussed for example by Exner<sup>12</sup>.

Friedrichs eigenvectors (2) present two difficulties: first they are not analytic in the coupling parameter<sup>4,6</sup> and second they provide a representation of the evolution as a shift in the continuum. Transitions and events do not appear in this representation. The unstable

state  $|1\rangle$  has disappeared, as there is no real point eigenvalue, and the complex poles  $z_1$  and  $z_1^*$  have instead emerged.

An eigenstate associated with the complex eigenvalue  $z_1$  was constructed by Nakanishi<sup>13</sup> and a complete set of right and left eigenvectors, including Nakanishi<sup>13</sup> solution, was constructed by Sudarshan, Chiu and Gorini<sup>14</sup>. These solutions were also obtained<sup>6</sup> with the time ordering rule and perturbation theory, as a first test of the method.

A more straightforward derivation<sup>7</sup> is based on the application of the analytic continuation rule to the Brillouin-Wigner iteration formulas. For the eigenstate  $f_1$  the Brillouin-Wigner formulas give:

$$z_1 = \omega_1 + \frac{\langle 1|\lambda V|f_1\rangle}{\langle \omega_1|f_1\rangle} \quad (4)$$

$$|f_1\rangle = \langle \omega_1|f_1\rangle|\omega\rangle + \frac{1}{z_1 - H_0} Q_1 \lambda V|f_1\rangle \quad (5)$$

$$Q_1 \equiv I - |1\rangle\langle 1| = \int_0^\infty d\omega |\omega\rangle\langle \omega|$$

The time ordering rule regularises the BW formulas (4), (5) according to the following natural convention. The terms representing transitions from the continuum modes  $|\omega\rangle$  to the eigenstate  $|1\rangle$ , i.e. excitations, are past oriented, therefore they are analytically continued from the lower to the upper half-plane. All other terms, i.e. de-excitations or mode-mode transitions, are future oriented and they are analytically continued from the upper to the lower half plane.

Formula (5), regularized according to the time ordering rule, gives:

$$|f_1\rangle = \langle 1|f_1\rangle \left( |1\rangle + \int_0^\infty d\omega \frac{\lambda V_\omega}{[z_1 - \omega]_+} |\omega\rangle \right) \quad (6)$$

Putting (6) into (4) we get the formula for  $z_1$

$$z_1 = \omega_1 + \int_0^\infty d\omega \frac{\lambda^2 |V_\omega|^2}{[\omega - z_1]_+} \quad (7)$$

Equation (7) means (3), that  $z_1$  is the simple pole of the meromorphic extension  $\frac{1}{\eta_{+1}(z)}$  of the partial resolvent to the lower half plane.

For the complex eigenvalue  $z_1$  we have also the left eigenvector  $\langle \tilde{f}_1|$  obtained in the same way

$$\langle \tilde{f}_1| = \langle \tilde{f}_1|1\rangle \left( \langle 1| + \int_0^\infty d\omega \frac{\lambda V_\omega}{[z_1 - \omega]_+} \langle \omega| \right) \quad (8)$$

Normalization fixes<sup>14,6</sup> the undetermined factors:

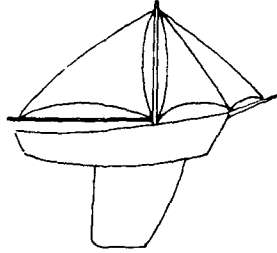
$$\langle 1|f_1\rangle = \langle f_1|1\rangle = [\eta'_+(z_1)]^{-\frac{1}{2}} \quad (9)$$

## 2. THE EIGENVECTORS ASSOCIATED TO THE COMPLEX EIGENVALUES LIVE IN THE RIGGED HILBERT SPACE OF HARDY CLASS.

The eigenvectors (6) and (8) cannot live in the Hilbert space as they correspond to complex eigenvalues of a self adjoint operator. However, they acquire meaning as generalized eigenvectors in a suitable Rigged Hilbert Space extension  $\Phi^\dagger$ .

Rigging the Hilbert space "boat" in order to extend the eigenvalue problem of the self-adjoint operator  $H$ , means that we choose a suitable "carine"  $\Phi$  of test vectors, which is

stable with respect to the operator  $H$  and construct the sails  $\Phi^\dagger$  as the continuous linear functionals on  $\Phi$



$\Phi^\dagger$  the topological dual of  $\Phi$

Hilbert space

$\Phi$  a test vectors subspace.

For the Rigged Hilbert Space extension of the eigenvalue problem see for example the monograph by Bohm and Gadella<sup>8</sup> and references therein. The possibility of complex generalized eigenvalues of self-adjoint operators was realized by Kuriyan, Mukunda and Sudarshan<sup>15</sup> and by Lindblad and Nagel<sup>16</sup> in the context of group representations.

From the defining formulas (6), (8) one can immediately guess suitable test spaces. The test vectors  $\phi$  for  $f_1$  should be chosen so that

$$\langle \phi | f_1 \rangle = \frac{1}{\sqrt{\eta_+(z_1)}} \left( \langle \phi | 1 \rangle + \int_0^\infty d\omega \frac{\lambda V_\omega \langle \phi | \omega \rangle}{[z_1 - \omega]_+} \right) \quad (10)$$

has meaning as a complex number.

It is enough therefore that  $\langle \phi | \omega \rangle$  has an analytic extension to the lower half-plane which includes the singularity  $z_1$ , so that the integral defines an analytic function evaluated at  $z_1$ . The simplest choice for  $\langle \phi | \omega \rangle$ , which does not depend upon the location of  $z_1$ , is that  $\langle \phi | \omega \rangle$  is a square integrable Hardy function from below. Therefore the spectral representation  $\phi(\omega) \equiv \langle \omega | \phi \rangle$  of the test vector  $\phi$  should be in the upper Hardy class  $\mathcal{H}_+^2$ . The test vectors  $\phi$  for the eigenvector  $f_1$ , may therefore be chosen to satisfy the condition that the spectral representations  $\langle \omega | \phi \rangle$  are restrictions to the positive semiaxis of functions in the upper Hardy class  $\mathcal{H}_+^2$ , which are also in the Schwartz class  $\mathcal{S}$  of infinitely differentiable, rapidly decreasing functions on the real line, i.e.:

$$\langle \omega | \phi \rangle \text{ are in } \theta[\mathcal{S} \cap \mathcal{H}_+^2]$$

$\theta$  is the Heaviside step function.

In the same way we can show that the test vectors  $\phi$  for the left eigenvector  $\tilde{f}_1$  may be chosen so that the spectral representations  $\langle \omega | \phi \rangle$  are in the lower Hardy class  $\mathcal{H}_-^2$ .

Let us denote by  $\Phi_+$  and  $\Phi_-$  the test spaces for the right and left eigenvectors  $f_1, \tilde{f}_1$  correspondingly

$$\Phi_\pm \equiv \{ \phi | \langle \omega | \phi \rangle \text{ are in } \theta[\mathcal{S} \cap \mathcal{H}_\pm^2] \} \quad (11)$$

The test spaces of Hardy class  $\theta[\mathcal{S} \cap \mathcal{H}_\pm^2]$  were introduced by Bohm and Gadella<sup>8</sup>, who also studied their properties.

The eigenvectors  $f_1$  and  $\tilde{f}_1$  live therefore in the duals  $\Phi_+^\dagger$  and  $\Phi_-^\dagger$ . The time ordering rule leads to an extension of the Friedrichs Hamiltonian (1) to the space  $\Phi_+^\dagger + \Phi_-^\dagger$ . Here we cannot use the direct sum, as the spaces  $\theta[\mathcal{S} \cap \mathcal{H}_+^2]$  and  $\theta[\mathcal{S} \cap \mathcal{H}_-^2]$  are not disjoint<sup>8</sup>. However the extension of the unitary group  $U_t = \exp(-iHt)$  generated by  $H$ , to the space  $\Phi_+^\dagger + \Phi_-^\dagger$  has not the group property. As a result of the time asymmetric evolution of the Hardy spaces,  $U_t$  can be extended<sup>7</sup> to  $\Phi_+^\dagger$  for positive times,  $t > 0$ , only and to  $\Phi_-^\dagger$  for negative times,  $t < 0$ .

only. The time asymmetry of Hardy spaces is the basis of Lax-Phillips scattering theory<sup>17</sup> and was used to show that relativistic fields are Kolmogorov systems<sup>18,19</sup>. The asymmetry in the time evolution associated with  $f_1$  and  $\tilde{f}_1$  was also noticed<sup>6</sup> as divergence in the formal expressions for the time evolution of  $f_1$  in the past and  $\tilde{f}_1$  in the future.

### 3. CONCLUDING REMARKS.

The time ordering rule leads to an extension of the eigenvalue problem to  $\Phi_+^\dagger + \Phi_-^\dagger$ , where the two roles of the Hamiltonian as the energy operator and as the generator of the evolution are separated. The analogy of the two semigroups of the Friedrichs model with the two semigroups of Kolmogorov systems<sup>3,18,19</sup> is striking. In the latter case, the two semigroups arise through the time asymmetric evolution of the stable and the unstable  $K$ -partitions. The above mentioned construction of the Friedrichs model shows the intrinsic irreversibility at the dynamical level, but does not deal with the probabilistic character of the macroscopic evolutions. This can be achieved for large Poincaré systems, at the level of the Liouville space of densities. The time ordering on the basis of correlations<sup>5</sup> leads to generalized eigendensities which live in appropriate Rigged Liouville spaces and provide an intrinsically irreversible and probabilistic representation of dynamics<sup>20</sup>.

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## CHIRAL SYMMETRY BREAKING IN ATOMS ON THE ROTATING EARTH

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Symmetry, as emphasised in the research and writings of Eugene Wigner, plays a vital role in both the kinematics and dynamics of interacting fields and particles [1]. Through the inextricable linkage of invariance principles and conservation laws, the existence of particular symmetries implies the nonexistence of particular processes. One long-known example of this is the strict disallowance of optical activity in systems of free atoms (to the extent that parity-breaking weak interactions between the nucleus and bound electrons can be neglected [2]). Optical activity is the generic term for a complex of optical phenomena (such as optical rotatory power and circular dichroism) that derive from the nonequivalent interaction of chiral structures with left and right circularly polarised light. Such structures lack a centre of symmetry and cannot be superposed on their mirror images.

Recent studies have shown, however, that unbound atoms (treated as purely electrodynamic systems) ought to manifest chirally asymmetric optical behaviour in a rotating reference frame [3]. All atoms on the spinning Earth, therefore, should exhibit a weak optical activity [4].

From the perspective of quantum mechanics, the coupling of the total internal angular momentum of the atom to the spin of the Earth lifts the degeneracy of magnetic substates [5] thereby leading to chirally asymmetric atomic polarisabilities or magnetisabilities, and ultimately to chirally asymmetric refractive indexes. From the perspective of classical mechanics [6], the Coriolis force of the Earth's rotation acts oppositely upon clockwise and counterclockwise circulating electron orbits to produce chirally asymmetric electric dipole moments that again give rise to an atomic circular birefringence.

For a sufficiently rarefied sample of  $N$  atoms per unit volume, the electric permittivity  $\epsilon$  and the magnetic permeability  $\mu$  are expressible in terms of the atomic polarisability  $\alpha$  and the magnetisability  $\beta$  as follows

$$\epsilon = 1 + 4\pi\alpha \quad (1a)$$

$$\mu = 1 + 4\pi\beta \quad (1b)$$

where  $\alpha$  and  $\beta$  relate the expectation values of the induced atomic electric and magnetic dipole moments to the electric and magnetic fields of the incident light. Thus, the index of refraction of the medium (for each of two orthogonal states of light polarisation) takes the form

$$n = \sqrt{\epsilon\mu} = 1 + 2\pi N(\alpha + \beta) \quad (1c)$$

To determine  $\alpha$  and  $\beta$  for an atom in a rotating reference frame, one must solve the Schrodinger equation where the Hamiltonian  $H$  (characterising a system at an

instantaneous angular location  $\theta$ ) is related to the corresponding Hamiltonian  $H^0$  in an inertial reference frame by the transformation [5]

$$H = UH^0U^{-1} + iU(dU^{-1}/dt) \quad (2a)$$

where

$$U = \exp(-iF_z\theta) \quad (2b)$$

is the rotation operator generated by the total angular momentum  $F$  of the system (where  $F_z$  is the projection onto the axis of rotation). For a nonrelativistic hydrogenic system the total Hamiltonian can be decomposed into a sum of two terms, one characterising the centre of mass (CM), the other characterising the internal dynamics of the atom. The effect of the Earth's rotation on the CM of a quantum system has already been demonstrated by the neutron Sagnac effect [7]. The internal Hamiltonian, which is the focus of interest here, takes the form

$$H_i = H_i^0 - \omega f_z \quad (2c)$$

where  $f$  is the total relative angular momentum of the electron and nucleus (with  $f_z$  the projection onto the rotation axis) and  $\omega$  is the angular velocity of the reference frame.

First-order perturbation theory applied to an atom interacting with the electric field of nonresonant left or right circularly polarised light waves leads to chiral polarisabilities (as a result of virtual transitions to levels of parity opposite that of the ground state) and hence, from Eq. (1c) with  $\beta = 0$ , to a rotationally induced circular birefringence. This circular birefringence, for a rotation rate  $\omega = 7.3 \times 10^{-5}$  rad/sec corresponding to that of the Earth, has been estimated for electromagnetic waves lying in the visible and ultraviolet to be on the order of  $10^{-18}$  [4]. Weak as this effect may be, it falls within the theoretical capability of large ring-laser interferometers now under development [8], [9].

Recent examination of the effect of rotation on the ground state hyperfine structure of atomic hydrogen [10] has shown that the splitting of the inertially degenerate states with total angular momentum quantum numbers  $f = 1$ ,  $m_f = \pm 1$  should be observable for rotational rates lower than 1 Hz. The extreme sensitivity to rotation derives in part from the long lifetimes, and consequently sharply defined energies, of the hydrogen 1S hyperfine states. This suggests the novel prospect of a ground state rotational optical activity deriving exclusively from the spin degrees of freedom and falling within the microwave region of the spectrum close to the hydrogen 1,420 MHz transition. The optical activity is manifested through the interaction of the hydrogen atoms with the magnetic field of the incident microwave radiation. Although strictly speaking the classical Coriolis force is zero (since the states have zero orbital angular momentum), one may heuristically attribute the chiral asymmetry to the precession of particle spins in the rotating frame:

$$dS/dt = -(2\pi i/\hbar)[S, H_i] = -\omega \times S \quad (3)$$

To first order in perturbation theory, the virtual magnetic dipole transitions induced by incident microwave radiation between the  $f = 0$  and  $f = 1$  states in the ground level of rotating atomic hydrogen leads to the magnetisabilities

$$\beta_{L,R} = (4\pi/\hbar) (eh/4\pi mc)^2 w_0[w_0^2 - (W \pm \omega)^2] \quad (4a)$$

and hence (from Eq. (1c) with  $\alpha = 0$ ) to the circular birefringence

$$\Delta n = n_L - n_R = (w_0^2 w_0 W \omega) / [w_0^2 - W^2] \quad (4b)$$

where  $\omega_0$  is the (hyperfine) Bohr transition frequency,  $W$  is the radiation frequency,  $\omega_p$  is the electronic plasma frequency of the medium

$$\omega_p = (4\pi N e^2 / m)^{1/2}, \quad (4c)$$

(where  $e/m$  is the charge to mass ratio of the electron), and  $\omega_C$  is the Compton frequency of the electron

$$\omega_C = 2\pi m c^2 / h. \quad (4d)$$

All things being equal, the circular birefringence deriving from magnetic dipole transitions between states of the same parity is of order  $\alpha^2$  smaller than that deriving from electric dipole transitions between states of opposite parity, where  $\alpha$  is the fine structure constant ( $\approx 1/137$ ). Under the particular circumstances, however, the sharply defined hyperfine energies and the relatively small Doppler width (proportional to the hyperfine level separation) compared with optical transitions give rise to a circular birefringence in the microwave region some *ten orders of magnitude* larger than that estimated previously for visible/UV radiation. As an example, consider atomic hydrogen gas at 1 Atm and 300 K, for which the atomic density  $N$  is about  $2.4 \times 10^{19}$  per cubic centimetre and the Doppler width is  $1.7 \times 10^4$  Hz. Assuming a hyperfine splitting of 1,420 MHz, an incident microwave frequency below resonance by ten times the Doppler width, and the rotational angular velocity of the Earth leads to a circular birefringence

$$\Delta n = 9.2 \times 10^{-9} \quad (5a)$$

and an associated optical rotary power (rotation of the plane of polarisation of incident linearly polarised radiation)  $\delta$  of

$$\delta = 10^{-7} \text{ degrees/cm}. \quad (5b)$$

Though small, this rotationally induced optical activity is now comparable to the optical activity produced in heavy metal vapours by the nuclear weak interactions. On the other hand, the technology that permits measurement of very weak chiral asymmetries in the optical domain (e.g. the use of photoelastic modulation combined with phase sensitive detection [11], [12] or of ring laser interferometry [8], [9]) does not yet exist for the microwave domain. Thus, current prospects for observing predicted effects of the Earth's rotation on the electromagnetic interactions of atoms centre upon optical methods.

Light waves of appropriate polarisations counterpropagating round a rotating ring laser interferometer containing a birefringent sample are frequency shifted in opposite directions and give rise to a beat frequency upon recombination. The beat frequency derives in part directly from the rotation of the interferometer and in part from the difference in phase velocities of the two differently polarised waves in the birefringent medium (which, in the present case, is also attributable to the rotation of the interferometer). Various procedures exist for measuring only that part of the beat frequency deriving from the rotationally induced circular birefringence. For example, the difference in beat frequencies between the configuration where right and left circularly polarised waves (emitted in opposite directions by the laser) propagate respectively clockwise and counterclockwise, and the configuration where the sense of propagation of the circularly polarised waves is reversed, is linearly proportional to the mean light frequency and the induced circular birefringence; that is,

$$(W^R - W_L^-) - (W_L^+ - W^R) \approx 2Wl \Delta n \quad (6)$$

where  $W_s^X$  is the frequency of light of polarisation  $X = R, L$  and propagation sense  $s = \pm$  (for  $s = -$  the wave propagates in the sense of interferometer rotation) and  $l$  is the filling factor (fraction of optical path occupied by the birefringent sample). The predicted

sensitivity (i.e. the measurable beat divided by the optical frequency) of realisable ring laser interferometers has been shown to be on the order of  $10^{-18}$ - $10^{-19}$ .

Success in observing the predicted atomic optical activity will depend critically on being able to eliminate spurious effects from two principal sources. One is the atomic optical activity from weak neutral currents (mediated by the exchange of a  $Z^0$  vector boson between the nucleus and the bound electrons). Of very short range (about  $10^{-16}$  cm), the weak interactions act on S states giving rise to parity violating effects (through mixing in of close-lying P states) that increase approximately with the cube of the atomic number and have the same symmetry characteristics as structural (i.e. chemical) optical activity. By contrast, the optical activity induced by the Earth's rotation is analogous to the Faraday effect. This suggests, therefore, that one may circumvent the problem of weak interactions by employing light atoms (e.g. hydrogen or helium), selecting a light frequency that avoids virtual transitions from S states, and taking advantage of signal enhancement by multiple passage through the sample. (It is worth noting that, in the case of microwave optical activity in the hydrogen ground level, there is no 1P state, and hence no significant contribution from the weak interactions.)

The second problem, however, is the true Faraday effect induced by stray magnetic fields, in particular the field of the Earth. As follows readily from Larmor's theorem (and the analogy between dynamics of field-free rotating systems and inertial systems in a static magnetic field [6], one must reduce the stray magnetic field along the rotation axis of the Earth to the extent that

$$B \leq 2m\omega/e \approx 8 \times 10^{-12} \text{ G} \quad (7)$$

Although quantum magnetometry currently permits the measurement of magnetic fields with a sensitivity on the order of femtotesla, the elimination or compensation of fields to this extent poses a significant technical challenge.

Difficult though such an experiment may be, its successful achievement would mark another step forward in the advance of high precision measurements and the concomitant exploration of unusual physical phenomena, for, in effect, one will have observed the coupling of the angular momentum of an atom to the spin of a planet.

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# A Method of Solving the Stern-Gerlach Problem

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## Abstract

We propose a method of decoupling the Pauli equation of a neutral spin  $\frac{1}{2}$  particle in a strong, inhomogeneous magnetic field (Stern-Gerlach problem).

For a special choice of the magnetic field the resulting pair of Schrödinger equations will be exactly solvable.

## 1 The Stern-Gerlach-Problem

The splitting of a beam of paramagnetic atoms in an inhomogeneous magnetic field (Stern-Gerlach-Experiment SGE [1]) is one of the fundamental experiments of quantum theory and has, moreover, often been used for illustrating or testing assumptions of theories concerning the quantum-mechanical measurement process. For example, E. Wigner [2] proposes the SGE as a model for the interaction between a quantum object ( $\hat{=}$  spin component) and measuring apparatus ( $\hat{=}$  orbital component of the wave function). Nevertheless, the SGE seems yet not to be completely understood. Many treatments of SGE use the replacement of the interaction term  $\vec{\sigma} \cdot \vec{B}$  by  $\sigma_z B_z$ , thereby decoupling the Pauli equation. The heuristical argument for this is the vanishing of the  $\sigma_x$ ,  $\sigma_y$ -components in the time-average due to the rapid spin precession about  $\vec{B}$  [3]. In this paper we will reformulate this heuristical argument in a more careful (though not completely rigorous) manner. Our method seems to support a recent analysis of the SGE as an "unsharp" spin measurement [4].

## 2 A Method of Decoupling the Pauli equation

After the usual simplifications we consider a dimensionless Pauli equation

$$i \frac{\partial}{\partial t} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = -\frac{1}{2} \Delta \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} - \frac{1}{2} \mu \begin{pmatrix} B_z & B_x \\ B_x & -B_z \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (1)$$

where

$$\mu := \frac{M \hbar^2}{m} B_0 \approx 3 \cdot 10^{12},$$

$M$ : mass of a silver atom,

$m$ : electron mass,

$d$ : width of the initial wave-packet  $\approx 10^{-4}$  m

$B_0$ : strength of the magnetic field  $\approx 1$  T.

Here length is measured in units of  $d$ , time in units of  $Md^2/\hbar \approx 17$  s, which is a typical time for the diffusion of the wave packet.  $B_x, B_z$  are assumed to depend only on  $(x, z)$  and  $B_y \equiv 0$ ; hence the free motion into  $y$ -direction can be separated and  $(\psi_1, \psi_2)$  in (1) only depend on  $(x, z)$ . The Stern-Gerlach-magnet occupies some region  $\mathcal{B} \subset \mathbb{R}^2$ ;  $B_x, B_z, \psi_1, \psi_2$  in (1) are defined on the complement  $\mathcal{B}^c = \mathbb{R}^2 \setminus \mathcal{B}$ , with Dirichlet boundary conditions on  $\partial\mathcal{B}$  for  $\psi_1$  and  $\psi_2$ .

After a transformation to "magnetic-field-coordinates"  $\rho, \beta$  defined by

$$B_x = \rho \sin \beta, \quad B_z = \rho \cos \beta \quad (2)$$

and a  $\beta$ -depending rotation in spin space

$$U_\beta := \begin{pmatrix} \cos \frac{\beta}{2} & \sin \frac{\beta}{2} \\ -\sin \frac{\beta}{2} & \cos \frac{\beta}{2} \end{pmatrix} \quad (3)$$

we obtain, using  $\operatorname{div} \vec{B} = 0, \operatorname{rot} \vec{B} = \vec{0}$ :

$$i \frac{\partial}{\partial t} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = -\frac{1}{2G} \Delta \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} + \frac{1}{2G\rho^2} \begin{pmatrix} \frac{1}{4}\phi_1 + \partial_\beta \phi_2 \\ \frac{1}{4}\phi_2 - \partial_\beta \phi_1 \end{pmatrix} - \frac{1}{2}\mu\rho \begin{pmatrix} \phi_1 \\ -\phi_2 \end{pmatrix}, \quad (4)$$

with

$$G := -\frac{\partial(B_x, B_z)}{\partial(x, z)}. \quad (5)$$

Writing  $\phi_{1,2} = r_{1,2} \exp(\frac{i}{2}(\delta \pm \varphi))$ ,  $\tau := \mu t$ ,  $\epsilon := 1/\mu$  we obtain equations of the form

$$\frac{\partial r_i}{\partial \tau} = \epsilon F_i(r, \delta, \varphi, \tau) + O(\epsilon^2) \quad i = 1, 2 \quad (6)$$

$$\frac{\partial \varphi}{\partial \tau} = \rho + \epsilon G(r, \delta, \varphi, \tau) + O(\epsilon^2) \quad (7)$$

$$\frac{\partial \delta}{\partial \tau} = \epsilon H(r, \delta, \varphi, \tau) + O(\epsilon^2). \quad (8)$$

$F_i, G$  and  $H$  are functionals depending on  $r_i, \delta, \varphi$  and its spatial derivatives of first and second order.

The crucial point is the following: the explicit calculation of  $F_i, G, H$  shows that neglecting the non diagonal terms  $\partial_\beta \phi_2, \partial_\beta \phi_1$ , in (4) will not produce any change in the averaged equations (6), (7), (8), where averaging is performed with respect to the rapidly oscillating phase  $\varphi$ . This is what we take as a justification for considering the decoupled form of (4). This justification is still heuristical in the sense that we did not prove that a

solution of (6), (7), (8) can be approximated by a solution of the corresponding averaged equations for times smaller than a typical time of flight of the silver atom. We do not know an averaging theorem for partial differential equations of the kind of (6), (7), (8) analogous to the averaging principle in the theory of ordinary differential equations, see e. g. [5]. Related theorems can be found in [6] and [7] but these, as they stand, are not applicable to our problem.

However, by this very justification it is clear, even without a detailed calculation, that the SGE cannot be an ideal measurement, because the direction of  $\vec{B}$  determining the up- and down-component varies from place to place.

### 3 A Solvable Example

The pair of Schrödinger equations obtained by neglecting  $\partial_3\phi_2$ ,  $\partial_3\phi_1$  in (4) will, nonetheless, in general not be explicitly solvable. However, the special choice

$$B_x = -2xz, \quad B_z = z^2 - x^2, \quad |x| \leq z/\sqrt{3} \quad (9)$$

(see figure 1) leads to a pair of Schrödinger equations equivalent to the 2-dimensional Coulomb problem.

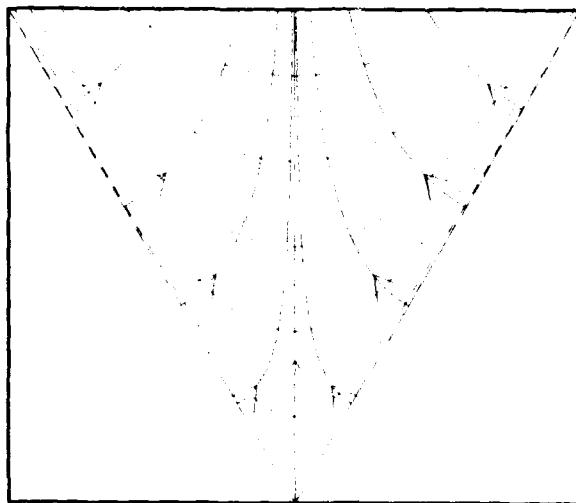


Figure 1: magnetic field (9) and its potential

The corresponding time-independent Schrödinger equations read

$$-2\rho(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_\beta^2)\phi_{1,2} + \frac{1}{2}\phi_{1,2} \mp \frac{1}{2}\mu\rho\phi_{1,2} = E\phi_{1,2} \quad (10)$$

with (unnormalized) solutions for the (-) sign

$$\phi_1^{(n,m)}(\rho, \beta) = \frac{\cos(\frac{3}{2}n\beta)}{\sin(\frac{3}{2}n\beta)} \exp(-\frac{1}{2}\sqrt{\mu}\rho) \rho^{3n/2} L_m^{(3n)}(\sqrt{\mu}\rho). \quad (11)$$

$$E_{n,m} = \frac{1}{2} + (3n + 2m + 1)\sqrt{\mu}, \quad (12)$$

where  $m \in \mathbb{N}$ ,  $n = 1, 3, 5, \dots$  for the cos case,  $n = 2, 4, 6, \dots$  for the sin case, and  $L_m^{(3n)}$  denoting the generalized Laguerre polynomials. These solutions which correspond to bound states of the Coulomb problem represent orbits of the silver atom trapped by the magnetic notch. The eigenfunctions for the (+)-sign (repulsive) case are

$$\phi_2^{(n,E)}(\rho, \beta) = \frac{\cos(\frac{3}{2}n\beta)}{\sin(\frac{3}{2}n\beta)} \exp(-\frac{i}{2}\sqrt{\mu}\rho) \rho^{3n/2} M(a, b, -i\sqrt{\mu}\rho), \quad (13)$$

where

$$a = \frac{1}{2} \left( (1 + 3n) + \frac{i}{\sqrt{\mu}} \left( E - \frac{1}{2} \right) \right) \quad (14)$$

$$b = 1 + 3n, \quad (15)$$

$n$  as above,  $E \in \mathbb{R}$  and  $M(a, b, z)$  denoting the confluent hypergeometric function.

A numerical evaluation of a special solution describing the splitting up of an unpolarized beam will be presented in a forthcoming paper.

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THE QUANTUM BEAMSPLITTER, THE 2-D OSCILLATOR AND THE 2-D  
HYDROGEN ATOM: GROUP THEORETICAL CONNECTIONS

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The purpose of this paper is to point out some striking analogies between the lossless optical beam splitter and the following two elementary quantum systems: (1) the two-dimensional isotropic oscillator and (2) the two-dimensional hydrogen atom. The analogies follow when one notes the underlying symmetries of these systems and exploits them by means of standard techniques from the quantum theory of angular momentum.[1]

Consider the lossless, two-port beamsplitter shown in Fig.1. Photons are

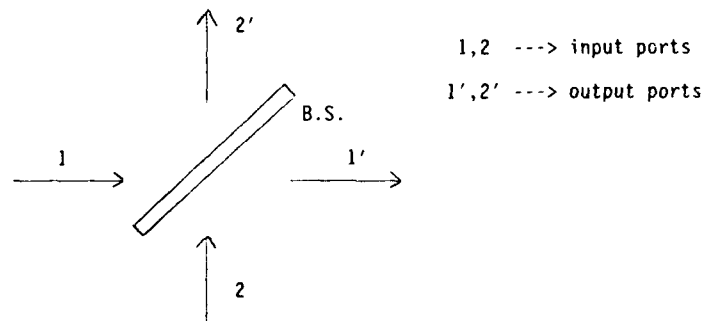


FIGURE 1. The lossless two-port beamsplitter.

incident at the input ports 1 and 2 and are partially transmitted to the output ports 1' and 2'. We will restrict the entire treatment to a single frequency mode so that all photons, both incident and outgoing ones, are at the same frequency  $\omega$ . Because the beamsplitter is assumed lossless, the average number of incident photons is equal to the average number of outgoing photons. The quantum mechanical theory of such a beamsplitter has been worked out by several authors[2-5] in recent years. We will use this theory to exhibit some analogies between the beamsplitter and the two quantum systems mentioned above.

We take as our starting point the quantum mechanical description of a lossless beamsplitter given by Prasad, Scully and Martienssen.[3] Denote by  $|\psi_{in}\rangle$

the joint input state of the photons at ports 1 and 2 and by  $|\Psi_{out}\rangle$  the joint output state at ports 1' and 2'. Then, as shown by Prasad et.al., [3] the action of the beamsplitter is to perform a unitary transformation on the input state  $|\Psi_{in}\rangle$  to yield the output state  $|\Psi_{out}\rangle$  :

$$|\Psi_{out}\rangle = B(\psi, \theta, \varphi) |\Psi_{in}\rangle \quad (1)$$

The beamsplitter operator  $B(\psi, \theta, \varphi)$  is given by

$$B(\psi, \theta, \varphi) = \exp(i\psi J_3) \exp(i\theta J_2) \exp(i\varphi J_3) \quad (2)$$

where  $J_1 = \frac{1}{2}(a_1^\dagger a_2 + a_1 a_2^\dagger)$ ,  $J_2 = -\frac{1}{2}i(a_1^\dagger a_2 - a_1 a_2^\dagger)$  and  $J_3 = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2)$ , with  $a_1$  and  $a_2$  being the annihilation operators for photons in the input ports 1 and 2. The angles  $\psi, \theta$  and  $\varphi$  occurring in (2) are related to the properties of the beamsplitter as follows:  $\psi$  and  $\varphi$  are phase shifts imparted by the beamsplitter while  $\cos^2(\theta/2)$  is the transmittance of the beamsplitter. In physical terms Eqn.(2) says that the beamsplitter performs a sequence of Euler rotations on the input state, with the generators of the rotations being the Schwinger operators  $J_1, J_2$  and  $J_3$  and the angles of rotation being determined by the characteristics of the beamsplitter.

A fundamental question one can ask about the beamsplitter is the following: if the joint Fock state  $|\Psi_{in}\rangle = |n_1\rangle|n_2\rangle$  (i.e.  $n_1$  photons in port 1 and  $n_2$  photons in port 2) is incident on the beamsplitter, what is the probability amplitude for observing  $n_1'$  photons in port 1' and  $n_2'$  photons in port 2' ? The answer, which we will denote  $P(n_1', n_2' | n_1, n_2)$ , is given by [5]

$$P(n_1', n_2' | n_1, n_2) = \langle n_1', n_2' | B(\psi, \theta, \varphi) | n_1, n_2 \rangle \quad (3a)$$

$$= R^{\frac{1}{2}(n_1+n_2), \frac{1}{2}(n_1-n_2), \frac{1}{2}(n_1'-n_2')}(\psi, \theta, \varphi) \quad (3b)$$

Equation (3b) is just a rotation matrix element [1] (or irreducible representation of the rotation group) with the angular momentum parameters  $j = \frac{1}{2}(n_1+n_2)$ ,  $m = \frac{1}{2}(n_1-n_2)$  and  $m' = \frac{1}{2}(n_1'-n_2')$  and the Euler angles  $\psi, \theta$  and  $\varphi$ .

### Analogy with the 2-d isotropic oscillator

Consider a two-dimensional isotropic harmonic oscillator. The steady state Schrodinger equation for this system can be separated in both cartesian and polar coordinates. Denote the normalized cartesian and polar eigenfunctions by  $\Psi_{n_1, n_2}(x, y)$  and  $\Phi_{n, \ell}^p(r, \varphi)$ , respectively, where  $n_1, n_2$  and  $n, \ell$  are the quantum numbers labelling the two sets of eigenfunctions; we will assume that an extra phase factor  $\exp[-\frac{1}{2}i(n-\ell)(\pi/2)]$  has been appended to each (otherwise real)  $\Phi_{n, \ell}^p(r, \varphi)$ , as this will prove convenient below. We ask now what the connection between the two sets of eigenfunctions is, i.e. how can one express a polar eigenfunction as a linear combination of the cartesian eigenfunctions and vice-versa ? The answer, which can be worked out with the aid of the SU(2) degeneracy group of the oscillator, is the following:

$$\Phi_{2j, 2m}^p(r, \varphi) = \sum_{m'=-j}^j R_{m, m'}^j(\frac{1}{2}\pi, \frac{1}{2}\pi, -\frac{1}{2}\pi) \Psi_{j+m', j-m'}(x, y) \quad (4)$$

Equation (4) says that any polar eigenfunction can be expressed as a linear combination of the degenerate cartesian eigenfunctions at the same energy, with the coefficients in the linear combination being given by the rotation matrix elements  $R_{m,m'}^j$  for the particular choice of Euler angles

$$\theta = \pi/2 ; \quad \psi = -\varphi = \pi/2 . \quad (5)$$

To exhibit the analogy between the beamsplitter and the oscillator, consider a beamsplitter possessing the Euler angles in (5) (such a beamsplitter has a transmittance of 50% and also imparts phase shifts of  $90^\circ$  to the outgoing beams). In this case the expression for  $P(n_1', n_2' | n_1, n_2)$ , Eqn.(3b), reduces exactly to that for  $R_{m,m'}^j(\frac{1}{2}\pi, \frac{1}{2}\pi, -\frac{1}{2}\pi)$  in (4), provided that the quantities  $(n_1, n_2, n_1', n_2')$  and  $(j, m, m')$  are related in the manner indicated below Eqn.(3b). In other words, the fundamental probability amplitudes for the beamsplitter (5) are identical to the coefficients relating the cartesian and polar eigenstates of the 2-d oscillator! As an amusing application of this observation, we show how the beamsplitter (5) can be used as an analog device to realize the eigenstates of the 2-d oscillator. Suppose that the joint Fock state  $|\psi_{in}\rangle = |n_1\rangle|n_2\rangle$  is incident on this beamsplitter. This input state can be thought of as a cartesian eigenstate of the oscillator with quantum numbers  $n_1$  and  $n_2$ . Then the output state produced by the beamsplitter is analogous to the polar eigenstate with quantum numbers  $n = n_1 + n_2$  and  $\ell = n_1 - n_2$ . It should be stressed that the output state does not consist of definite numbers of photons in the ports 1' and 2' but is rather a superposition of such states, with the coefficients in the superposition being given by the  $R_{m,m'}^j(\frac{1}{2}\pi, \frac{1}{2}\pi, -\frac{1}{2}\pi)$  of eqn.(4).

#### Analogy with the 2-d hydrogen atom

Consider a hydrogen atom in two spatial dimensions with a  $1/r$  potential between the electron and proton. The Schrodinger equation for this system can be separated both in polar  $(r, \varphi)$  and parabolic  $(u, v)$  coordinates (the latter coordinates[6] are related to the former by the equations  $u = (2r)^{\frac{1}{2}}\cos(\theta/2)$  and  $v = (2r)^{\frac{1}{2}}\sin(\theta/2)$ ; by restricting  $u$  and  $v$  to the ranges  $-x < u < x$  and  $0 \leq v < x$  one ensures that the correspondence between the two sets of coordinates is one-to-one). Denote by  $\phi_\ell^n(r, \varphi)$  and  $\psi_{n_1, n_2}(u, v)$  the normalized polar and parabolic eigenfunctions, respectively. All polar eigenfunctions with the same  $n$  (but different  $\ell$ ) are degenerate while all parabolic eigenfunctions with the same  $n_1 + n_2$  (which is restricted to be even) are degenerate. We will assume that a phase factor  $\exp[-\frac{1}{2}i\pi n_2]$  has been affixed to each (otherwise purely real)  $\psi_{n_1, n_2}(u, v)$ , as this will prove convenient below. As in the oscillator problem, we ask what the connection is between the two alternate sets of eigenfunctions (here polar and parabolic). The answer, obtained with the help of the  $SO(3)$  degeneracy group of this problem[7], is

$$\psi_{(-m, +m)}(u, v) = \sum_{m'=-\ell}^{\ell} R_{m, m'}^{\ell}(0, -\frac{1}{2}\pi, 0) \phi_{m'}^{\ell+1}(r, \varphi) \quad (6)$$

Equation (6) says that any parabolic eigenfunction can be expressed as a linear combination of the degenerate polar eigenfunctions at the same energy, with the coefficients in the linear combination being given by the rotation matrix elements  $R_{m, m'}^{\ell}$  for the particular choice of Euler angles  $\theta = -\frac{1}{2}\pi$  and  $\psi = \varphi = 0$ .

To exhibit the analogy between the beamsplitter and the hydrogen atom, consider a beamsplitter with the Euler angles  $\theta = \frac{1}{2}\pi$  and  $\psi = \varphi = 0$  (these imply a transmittance of 50% and no phase shifts imparted to the outgoing beams). In this case the matrix of probability amplitudes  ${}^L P(n_1', n_2' | n_1, n_2)$  becomes identical to the inverse of the rotation matrix  $R_{m, m'}^L(0, -\frac{1}{2}\pi, 0)$  appearing in (6). To put the matter more transparently, suppose that the joint Fock state  $|\psi_{in}\rangle = |n_1\rangle|n_2\rangle$  (with  $n_1 + n_2 = \text{even}$ ) is incident on a beamsplitter with  $\theta = \frac{1}{2}\pi$  and  $\psi = \varphi = 0$ . This input state can be thought of as a parabolic eigenstate with quantum numbers  $n_1$  and  $n_2$ . Then the output state produced by the beamsplitter is analogous to the polar eigenstate with quantum numbers  $n = \frac{1}{2}(n_1 + n_2) + 1$  and  $l = \frac{1}{2}(n_2 - n_1)$ . The output state does not consist of definite numbers of photons in the ports 1' and 2' but is rather a superposition of such states, with the coefficients in the superposition being determined by the quantities  $R_{m, m'}^L(0, -\frac{1}{2}\pi, 0)$  in (6).

### CONCLUSION

We have shown that for particular types of beamsplitters (namely, those with a transmittance of 50% and the appropriate phase shifting properties) the connection between the input and output photon statistics is identical to the connection between the cartesian and polar eigenstates of the 2-d oscillator or the parabolic and polar eigenstates of the 2-d hydrogen atom. The analogies between the optical beamsplitter and two fundamental quantum systems demonstrate once again the unity that can be brought to seemingly diverse phenomena by the application of symmetry methods.

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## HIGHER ORDER POLARIZATIONS AND THE RELATIVISTIC HARMONIC OSCILLATOR

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### ABSTRACT

A generalization of the symmetry group of the harmonic oscillator, which has the appropriate  $c \rightarrow \infty$  and  $\omega \rightarrow 0$  limits, is proposed. The quantization is achieved by using a higher-order polarization leading to a set of relativistic Hermite polynomials.

In a recent paper<sup>1</sup>, we have solved the quantization of a system defined by the following quantum operators commutation relations:

$$\begin{aligned} [\hat{x}, \hat{p}] &= i\hbar \left( 1 + \frac{1}{mc^2} \hat{E} \right) \\ [\hat{E}, \hat{x}] &= -i \frac{\hbar}{m} \hat{p} \\ [\hat{E}, \hat{p}] &= im\omega^2 \hbar \hat{x} \end{aligned} \quad (1)$$

where  $\hat{E}$ ,  $\hat{p}$  and  $\hat{x}$  stand for energy, momentum and "position" (this is a relativistic system) operator, respectively. Once the solution in configuration space was obtained, this quantum system was found completely adequate to describe a quantum relativistic harmonic oscillator (RHO) (even at the Lie algebra level, you can see how (1) leads to the non-relativistic harmonic oscillator algebra, under the contraction  $c \rightarrow \infty$ , and in the limit  $\omega \rightarrow 0$  to the algebra of the 1+1 dimensional Poincare group, when  $\hat{E}$  has the rest-mass energy subtracted, just as it should be for the RHO algebra). Also, (1) is an affine version of the algebra of the 1+1 dimensional antideSitter group. Thus, the quantum system defined by (1) can also be regarded as a free particle in antideSitter space.

Let us recall that the irreducible representations of (1) have been obtained by finding, among functions satisfying the differential polarization condition  $L_p \psi = 0$ , the vacuum defined by

$$\hat{E} \psi_0 = 0 \quad (2.a)$$

$$\hat{z} \psi_0 = \left( \frac{1}{\sqrt{2m\omega\hbar}} \hat{p} - i \sqrt{\frac{m\omega}{2\hbar}} \hat{x} \right) \psi_0 = 0 \quad (2.b)$$

and then applying repeatedly the creation operator  $\hat{z}^\dagger$  on the vacuum. As a result, we get a family of states  $\psi_n$  with energy levels equally spaced between them (just as in the case of the non-relativistic oscillator),

$$\psi_n = \frac{1}{(2\hbar)^{n/2}} e^{\frac{i}{\hbar} \hat{z}^\dagger} e^{-in\omega t} \Phi_n(\xi) \quad (3a)$$

$$\Phi_n(\xi) = \alpha^{-(N+n)} H_n^N(\xi) \quad (3b)$$

where  $\xi \equiv \sqrt{m\omega/\hbar} x$ ,  $N = mc^2/\hbar\omega$  and  $\alpha = \sqrt{1 + \omega^2 x^2/c^2}$ . The functions  $H_n^N$  are polynomials which generalize the non-relativistic ones, and therefore are called Relativistic Hermite Polynomials. They satisfy the following recursion relations

$$H_{n+1}^N(\xi) = \left( 2 \left( 1 + \frac{n}{N} \right) \xi - \left( 1 + \frac{\xi^2}{N} \right) \frac{d}{d\xi} \right) H_n^N(\xi) \quad (4a)$$

$$\frac{d}{d\xi} H_n^N(\xi) = \frac{n}{N} (2N + n - 1) H_{n-1}^N(\xi) \quad (4b)$$

and the differential equation

$$\left( 1 + \frac{\xi^2}{N} \right) \frac{d^2}{d\xi^2} H_n^N - \frac{2}{N} (N + n - 1) \xi \frac{d}{d\xi} H_n^N + \frac{n}{N} (2N + n - 1) H_n^N = 0 \quad (5)$$

They have as general expression

$$H_n^N(\xi) = \sum_{s=0}^{\lfloor n/2 \rfloor} a_{n, n-2s}^N (2\xi)^{n-2s} \quad (6a)$$

$$a_{n, n-2s}^N = \frac{(-)^s n!}{s! (n-2s)!} \frac{N^s (N - \frac{1}{2})!}{(N+s-\frac{1}{2})!} \frac{(2N+n-1)!}{(2N)^n (2N-1)!} \quad (6b)$$

The first polynomials are

$$\begin{aligned}
 H_0^N &= 1; \quad H_1^N = 2\xi; \quad H_2^N = 4 \left(1 + \frac{1}{2N}\right) \xi^2 \cdot 2; \\
 H_3^N &= 8 \left(1 + \frac{3}{2N} + \frac{1}{2N^2}\right) \xi^3 - 12 \left(1 + \frac{1}{N}\right) \xi.
 \end{aligned} \tag{7}$$

As expected, in the limit  $c \rightarrow \infty$  ( $N \rightarrow \infty$ ), all expressions above go to their non-relativistic counterparts. For example, we may explicitly see this for the vacuum.

$$\begin{aligned}
 \lim_{c \rightarrow \infty} \Phi_0 &= \lim_{c \rightarrow \infty} \left(1 + \frac{\omega^2 x^2}{c^2}\right)^{\frac{mc^2}{2\hbar\omega}} = \\
 &= \lim_{c \rightarrow \infty} \left[ \left(1 + \frac{\omega^2 x^2}{c^2}\right)^{c^2} \right]^{\frac{m}{2\hbar\omega}} = \\
 &= e^{\frac{m\omega x^2}{2\hbar}}
 \end{aligned} \tag{8}$$

(the function  $f$  in (3a) yields  $px$  in the limit  $c \rightarrow \infty$ ).

In this paper we want to use another method of solution that has been introduced recently<sup>2</sup> in order to reach the quantization of the RHO. Here will just sketch the method in order to apply it, since it has been discussed in detail in Ref. 2. First of all, let us note that the differential operators  $L_x, L_p, L_t$ , close on the same algebra (1) as the physical operators (but for a change of sign in each r.h.s. of (1)). Thus, the polarization condition  $L_p \psi = 0$  prevents the imposition of any other first order condition. Therefore, we used in Ref. 1 the method of determination of the vacuum and orbit through it in order to achieve a full reduction. However, we can find out another reduction condition if we decide to use elements of the enveloping algebra generated by  $L_x, L_p, L_t$ , thus leading to a HIGHER ORDER POLARIZATION<sup>2</sup> (HOP).

The condition which is to determine the higher order condition is that it closes a (HOP) subalgebra together with  $L_p$ . From the commutation relations it is easy to see that

$$\begin{aligned}
 [(L_t)^2, L_p] &= -\frac{1}{m} (L_t L_x + L_x L_t) \\
 [(L_x)^2, L_p] &= -\frac{1}{mc^2} (L_t L_x + L_x L_t) - \frac{2i}{\hbar} L_x,
 \end{aligned} \tag{9}$$

and thus we have

$$\left[ (L_t)^2 - c^2(L_x)^2 + \frac{2imc^2}{\hbar} L_t, L_p \right] = 0 \tag{10}$$

Therefore, we set  $L_p \psi = 0$  as in Ref. 1 leading to  $\psi(t,x,p) = \exp[i f(t,x,p)/\hbar] \phi(t,x)$ , and then we impose the higher order condition

$$\left( (L_t)^2 - c^2(L_x)^2 + \frac{2imc^2}{\hbar} L_t \right) \psi = 0 \tag{11}$$

In order to work out (11) we first calculate

$$\begin{aligned} L_t \psi &= e^{\frac{i}{\hbar} t} \left( \frac{i}{\hbar} m c^2 \varphi - \frac{i}{\hbar} \frac{P^0 c}{\alpha^2} \varphi + \frac{p}{m} \frac{\partial \varphi}{\partial x} + \frac{P^0}{m c \alpha^2} \frac{\partial \varphi}{\partial t} \right) \\ L_x \psi &= e^{\frac{i}{\hbar} t} \left( -\frac{i}{\hbar} \frac{p}{\alpha^2} \varphi + \frac{P^0}{m c} \frac{\partial \varphi}{\partial x} + \frac{P}{m c^2 \alpha^2} \frac{\partial \varphi}{\partial t} \right) . \end{aligned} \quad (12)$$

and then (11) turns into

$$\left[ \frac{1}{\alpha^2} \frac{\partial^2 \varphi}{\partial t^2} - \frac{2i m c^2}{\hbar \alpha^2} \frac{\partial \varphi}{\partial t} - 2\omega^2 x \frac{\partial \varphi}{\partial x} - c^2 \alpha^2 \frac{\partial^2 \varphi}{\partial x^2} - \frac{m^2 c^4}{\hbar^2 \alpha^2} \varphi + \frac{m^2 c^4}{\hbar^2} \varphi \right] = 0 . \quad (13)$$

As is well known, the change  $\varphi = \exp[i m c^2 t / \hbar] \phi$  restores the rest-mass energy (which was absent previously, see (1)) in the wavefunctions. Making this change in (13) you can check that  $\phi$  satisfies the wave equation

$$\frac{1}{c^2 \alpha^2} \frac{\partial^2 \phi}{\partial t^2} - \frac{2\omega^2 x}{c^2} \frac{\partial \phi}{\partial x} - \alpha^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{m^2 c^2}{\hbar^2} \phi = 0 . \quad (14)$$

or

$$\square \phi + \frac{m^2 c^2}{\hbar^2} \phi = 0 . \quad (15)$$

where  $\square$  is the D'Alembertian in antideSitter space. We want to note as an attractive feature of the HOP method that it leads directly to the wave equation in configuration space<sup>2</sup>. Moreover, the HOP fixes the indeterminated constant that appears in wave equations obtained by means of procedures of conventional quantization in curved space-time background<sup>3</sup>. On the other hand, we may compute the non-relativistic limit of the wave equation - it is clear that, for this purpose, it is (13) that should be employed. As a result, one gets immediately the Schrödinger equation with potential  $V = 1/2 m \omega^2 x^2$ .

<sup>1</sup>Aldaya V., Bisquet J. and Navarro-Salas J. "The quantum relativistic harmonic oscillator: generalized Hermite polynomials" *Phys. Lett. A* **156**, 381 (1991).

<sup>2</sup>Aldaya V., Bisquet J., Loll R. and Navarro-Salas J. "Symmetry and quantization: higher order polarizations and anomalies" preprint FTUV-90-28, IFIC-90-24, submitted to *J. Math. Phys.*

<sup>3</sup>Birrell N.D. and Davies P.C.W., *Quantum fields in curved space* (Cambridge University Press, Cambridge, 1982).



## PHASE MEASUREMENT IN QUANTUM MECHANICS

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### Abstract

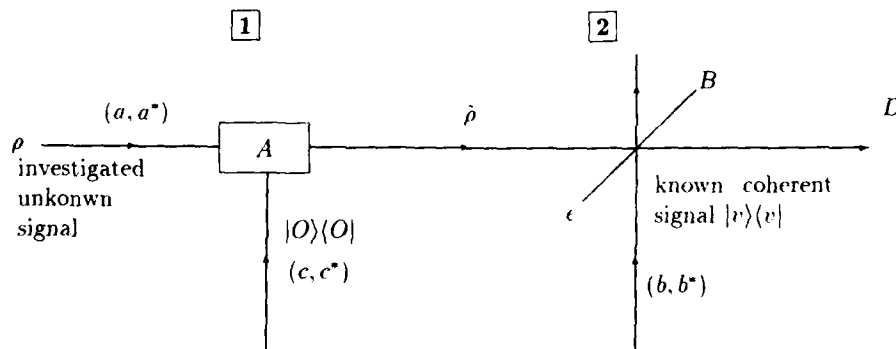
In quantum optics we can prepare the states which depend on the phase. The question of detection and measurement of phase arises.

### Difficulties

**1** In quantum optics there is essentially only one type of experiments — the photon counting. But it gives only an information about the number of photons and no knowledge about the phase. To obtain any information about the phase properties we should superpose the investigated signal with some known coherent impuls and from the interference pattern infer about the phase.

**2** Really operating in the quantum area, we have a very low intensity of the unknown signal. The interference picture is very vague. To overcome this difficulty we should amplify the investigated signal so that the information about the phase is not lost.

### Measurement scheme



**1** The amplified unknown signal  $\hat{\rho}$  is mixed in a partially transmitted beamsplitter with transmittance  $\epsilon$  with the known coherent signal  $|v\rangle\langle v|$  of the mode  $(b, b^*)$  and finally detected in the photodetector  $D$ . The beamsplitter transformation is the following unitary map  $U = U(\alpha) = \exp(\alpha a \otimes b^* - \alpha a^* \otimes b)$ , where  $\alpha = r \exp(i\varphi)$ ,  $\cos r = \sqrt{\epsilon}$ . The photodetector with unity quantum efficiency is as usual described by the spectral measure  $\{|n\rangle\langle n|\}$  connected with the number of photons  $N = a^*a = \sum_n n |n\rangle\langle n|$ . Treating  $B + D$  as a whole detector and eliminating the degrees of freedom of the mode  $(b, b^*)$  we obtain from the equation

$$\text{Tr } U(\hat{\rho} \otimes |v\rangle\langle v|)U^*(|n\rangle\langle n| \otimes I) = \text{Tr } \hat{\rho} E_n(v)$$

the positive operator-valued measure (POV-measure)  $E_n(v)$ , which describes our mixing and detecting device  $B + D$ . From calculations

$$\begin{aligned} E_n(v) &= D(xv)^* E_n D(xv), \\ \text{where } D(z) &= \exp(za^* - \bar{z}a), |x|^2 = (1 - \epsilon)/\epsilon \quad \text{and} \\ E_n &= \sum_{p=n}^{\infty} \binom{p}{n} \epsilon^n (1 - \epsilon)^{p-n} |p\rangle\langle p|. \end{aligned}$$

The coherent signal  $|v\rangle$  should be strong enough to ensure a good quality of the interference pattern. This can be achieved by taking  $|v| \rightarrow \infty$ . But then  $D(xv) \rightarrow 0$ . To obtain the reasonable result, we must use the arbitrariness of  $\epsilon$ . We take  $\epsilon \rightarrow 1$  and simultaneously  $|v| \rightarrow \infty$  but in such a way that  $|xv| \rightarrow |z|$ . The parameter  $z$  can change but it is finite. At the level of POV-measure the above limit induces the transition

$$E_n(v) \rightarrow D(z)^* |n\rangle\langle n| D(z).$$

the limiting process preserves the known phase of  $|v\rangle$ . The price we pay for obtaining the expression describing both the phase and number of photons is the appearance of the parameter  $|z|$ . This parameter makes our description of simultaneous measurement of phase and number of photons in the device  $(B + D + \text{limiting procedure})$  unsharp. From the expression  $D(z)^* |n\rangle\langle n| D(z)$  we can extract separate information about the phase or about the number of photons. It is done in the standard way using marginal observables. These observables (again POV-measures) we denote

$$F_n(\varphi) = \int_0^{\infty} d(|z|^2) D(z)^* |n\rangle\langle n| D(z) \quad , \quad P_n = \int_0^{2\pi} \frac{d\varphi}{2\pi} D(z)^* |n\rangle\langle n| D(z).$$

The phase observable is defined for the determined vector  $|n\rangle$ .

**2** We use the non-degenerate parametric amplifier. This amplifier couples a signal wave (mode  $a, A^*$ ) and an idler wave ( $c, c^*$ ) by the second and third order nonlinear processes produced by the pump wave. This pump wave is treated as a classical wave. the state of the mode  $(c, c^*)$  is the vacuum state  $|0\rangle$ . The interaction

between modes has the shape  $H = k a^* \otimes c^* + \bar{k} a \otimes c$ , where  $k$  is the parameter of interaction.  $\hat{U} = \exp(-iH)$  describes the interaction unitary operator. When we measure any observable (POV-measure)  $A$  in the part  $\boxed{1}$  of the device, the whole amplification process can be described as follows

$$\text{Tr } \hat{U}(\rho \otimes |0\rangle\langle 0|)\hat{U}(A \otimes I) = \text{Tr } \hat{\rho}A = \text{Tr } \rho\hat{A}$$

Tilde over  $A$  describes the amplification process in Heisenberg picture. We obtain the expressions for the observable of the number of quanta and the phase:

$$\hat{P}_n = \int_0^{2\pi} \sum_{p=0}^{\infty} \frac{|x|^{2p}}{p!} e^{-y(a^*a+1)} a^p D(z)^* |n\rangle\langle n| D(z) a^{*p} e^{-y(a^*a+1)} \frac{d\varphi}{2\pi}$$

$$\hat{F}_n(\varphi) = \int_0^{2\pi} \sum_{p=0}^{\infty} \frac{|x|^{2p}}{p!} e^{-y(a^*a+1)} a^p D(z)^* |n\rangle\langle n| D(z) a^{*p} e^{-y(a^*a+1)} d(|z|^2),$$

where  $y = \ln \text{ch } |k|$ ,  $x = e^{i\varphi} \text{th } |k|$ ,  $k = e^{i\varphi} |k|$ . In such a way we have the final expressions for the desired observables connected with the experimental set-up ( $A+B+D$ -limiting procedure) (see the picture). This kind of amplification increases the number of photons. It also changes  $F_n$  with the remarkable exception of  $n = 0$ . The noise introduced by non-degenerate parametric amplification does not influence the measurement of the lack of counts in the photodetector. A direct calculation shows

$$\hat{F}_0 = F_0.$$

The POV-measure we have obtained to describe the measurement of phase and lack of photons is  $|z\rangle\langle z|$ . In quantum optics  $|z\rangle\langle z|$  is assumed as the POV-measure describing the heterodyne detection. Hence our final result is a little bit in a spirit of the paper J. H. SHAPIRO and S. S. WAGNER *IEEE J. Quantum Electro.* **QE-20**, 803(1984).

**Quantum Stochastic Calculus,  
Measurements Continuous in Time,  
and Heterodyne Detection in Quantum Optics**

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Quantum stochastic calculus (QSC) [1] is a noncommutative analogue of Itô's stochastic calculus. Its usefulness in quantum optics is mainly due to the fact that it is based on the use of certain Bose fields which can be taken as an approximation of the electromagnetic field [2]. In particular, by QSC one can develop a quantum photodetection theory (direct, heterodyne and homodyne detection) [3-5]. This subject is also connected to the theory of measurements continuous in time in quantum mechanics [6,7]. In this note we want to show how this theory of heterodyne detection can be applied to the study of the fluorescence spectrum of a two level atom stimulated by a strong monochromatic laser (*dynamical Stark effect* [8,9]). Due to the shortage of space we shall refer to the paper [4] for what concerns theory, general results and references.

Let us introduce  $d$  Bose fields  $a_j(t)$ ,  $a_j^\dagger(t)$ , satisfying the canonical commutation rules  $[a_j(t), a_i(s)] = 0$ ,  $[a_j(t), a_i^\dagger(s)] = \delta_{ji} \delta(t-s)$ . We consider the Fock representation: this means that the Hilbert space on which the field operators act is the symmetric Fock space over the "one-particle space"  $\mathbb{C}^d \otimes L^2(\mathbb{R})$ . This space, which we denote by  $\mathcal{F}$ , is spanned by the exponential vectors  $\iota(f)$ , whose components in the  $0, 1, \dots, n, \dots$  particle spaces are  $\iota(f) = \left(1, f, \dots, (n!)^{-1/2} f \otimes \dots \otimes f, \dots\right)$ ,  $f \in \mathbb{C}^d \otimes L^2(\mathbb{R})$ . Apart from the normalization, the exponential vectors are the usual coherent vectors for the fields  $a_j(t)$ ; the vector  $\iota(0)$  is the Fock vacuum. Then we define the annihilation and creation processes by  $A_j(t) := \int_0^t a_j(s) ds$ ,  $A_j^\dagger(t) = \int_0^t a_j^\dagger(s) ds$ . The rigorous definition of these processes in terms of their action on the exponential vectors is given in [1], §2 and 4. In particular, the exponential vectors are eigenstates of  $A_j(t)$  with respect to the eigenvalues  $\int_0^t f_j(s) ds$ . A QSC of Itô type, based on the integrators  $dA_j(t)$ ,  $dA_j^\dagger(t)$  and  $dt$ , has been developed by Hudson and Parthasarathy [1].

The Bose fields  $a_j(t)$  can be considered as an approximation of the electromagnetic field: in this case the index  $j$  stays for polarization, direction of propagation (discretized), and so on. Then, let us consider a system (e.g. an atom; Hilbert space  $\mathcal{H}$ ) interacting with the electromagnetic field. In the so called broadband approximation [2], the evolution operator  $U_t$  of the composed system (in the interaction picture with

respect to the free dynamics of the fields) satisfies the quantum stochastic differential equation (QSDE) ([4], §1 and 2)

$$dU_t = \left\{ \sum_j \left[ -R_j^\dagger dA_j(t) + R_j dA_j^\dagger(t) - \frac{1}{2} R_j^\dagger R_j dt \right] - iH dt \right\} U_t, \quad U_0 = \mathbb{1}. \quad (1)$$

where  $R_j, H$  are bounded operators on  $\mathcal{H}$  and  $H = H^\dagger$ . The solution  $U_t$  of this equation exists and is unique and, for any  $t \geq 0$ ,  $U_t$  is a unitary operator on  $\mathcal{H} \otimes \mathcal{F}([1], §7)$ .

Concretely, we consider a two level atom with ground state  $|0\rangle$  and excited state  $|1\rangle$ ; its Hamiltonian is  $H = \omega|1\rangle\langle 1|$ , with  $\omega > 0$ . We take  $d = 2$ ; the index  $j = 2$  represents the modes of the field carrying the laser signal and  $j = 1$  represents all the other modes carrying fluorescence light (sideways scattering). The quantities  $R_j$  are atomic dipole operators  $R_j = \sqrt{\Gamma_j}|0\rangle\langle 1|$ , with  $\Gamma_j > 0$ . As initial state of the total system we take  $\varrho = \Psi(f)$ , where  $\varrho$  is the initial state of the atom (a statistical operator on  $\mathcal{H}$ ) and  $\Psi(f) := |\psi(f)\rangle\langle \psi(f)| / \|\psi(f)\|^2$ , with  $f(t) = (f_1(t), f_2(t))$ ,  $f_1(t) = 0$  and  $f_2(t) = i\lambda e^{-i\omega t}$ . The function  $f_2$  is not square integrable, but can be seen as a limit of  $L^2$ -functions [9]; it represents a coherent input signal (the laser field; in resonance, for simplicity).

The power spectrum of the emitted light can be obtained by measurements on the fields after the interaction with the atom. These *output fields* are given by ([4], §3)

$$A_j^{\text{out}}(t) := \lim_{T \rightarrow +\infty} U_T^\dagger A_j(t) U_T \quad (2)$$

and satisfy the QSDE's  $dA_j^{\text{out}}(t) = dA_j(t) + U_t^\dagger R_j U_t dt$ ,  $j = 1, 2$ .

Let us consider now *balanced heterodyne detection* ([4], §5). By means of a beam splitter the fluorescence light of the atom (output field 1) is made interfere with a strong laser signal of frequency  $\nu$  (local oscillator). Two identical photoelectron counters are used for detecting the photons coming out from the two output ports of the beam splitter. The two output currents are subtracted one from the other and resulting signal  $I(\nu, t)$  is analyzed. In [4] §5 it has been shown that the whole procedure described here corresponds to the measurement of the observables ( $t \geq 0$ )

$$I(\nu, t) = \int_0^t F(t-s) [f_0(s) dA_1^{\text{out}}(s)^\dagger + f_0^*(s) dA_1^{\text{out}}(s)]. \quad (3)$$

where  $F(t)$  is the detector response function and  $f_0(t) = i\lambda_0 e^{-i\nu t}$  represents the local laser field. Here the key point is that  $[I(\nu, t), I(\nu, s)] = 0$ , for any  $t, s$ . Therefore, we can use the standard prescriptions of quantum mechanics in order to obtain the joint probability distribution for  $I(\nu, t)$ ,  $t \geq 0$  (here we identify the stochastic output signal and the quantum observables (3)). We can say that our measuring procedure performs a measurement continuous in time of  $I(\nu, t)$ . Let us stress that the parameter  $\nu$  has a different role: it is the frequency of the local laser and to change it means to change the measuring apparatus and, indeed, for different  $\nu$  the output currents are not represented by commuting operators.

In [4] it is shown how the whole statistics of the process  $I(\nu, t)$ ,  $t \geq 0$ , can be obtained. By eliminating the degrees of freedom of the fields this statistics can be expressed in terms of atomic quantities only (see equations (1.17), (4.8) (4.10) and

(5.18) of [4]. This last level of description corresponds to the theory of continuous measurements developed in [6]. The power spectrum of the fluorescence light  $P(\nu)$  will be proportional to the power related to the current  $I(\nu, t)$ ; more precisely,

$$P(\nu) \propto \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \langle I(\nu, t)^2 \rangle dt. \quad (4)$$

The whole spectrum is explored by changing the frequency of the laser used as local oscillator in the heterodyne scheme. Because the whole statistics of  $I(\nu, t)$  can be obtained, in principle one could also compute the fluctuations of the detected spectrum.

A typical form for the detector response function in (3) is an exponential one; moreover, in order that (4) give the power spectrum we need that the detector integrates the input signal for a long time (bad time resolution, but good frequency resolution); precisely,

$$F(t) \propto \gamma e^{-\gamma t}, \quad 0 < \gamma \ll \Gamma := \Gamma_1 + \Gamma_2. \quad (5)$$

The first two moments  $\langle I(\nu, t) \rangle$  and  $\langle I(\nu, t)I(\nu, t') \rangle$  are given by equations (5.19) and (5.20) of [4]. In our concrete case, by a unitary transformation under trace and by changing the phases of the states, equation (5.20) of [4] gives

$$\begin{aligned} \langle I(\nu, t)^2 \rangle &\propto \frac{1}{2} |\lambda_0|^2 \gamma (1 - e^{-2\gamma t}) + \\ &+ 2 \int_0^t ds \int_0^s ds' \gamma^2 e^{-\gamma(2t-s-s')} \text{Tr}_R \left\{ \mathcal{J}(s) e^{\mathcal{L}(s-s')} \mathcal{J}(s') e^{\mathcal{L}s'} \varrho \right\}, \end{aligned} \quad (6)$$

$$\mathcal{J}(t)\varrho = i\lambda_0 \sqrt{\Gamma_1} e^{-i(\nu-\omega)t} \varrho |1\rangle\langle 0| - i\lambda_0^* \sqrt{\Gamma_1} e^{i(\nu-\omega)t} |0\rangle\langle 1| \varrho, \quad (7)$$

$$\mathcal{L}\varrho = -\frac{1}{2}\Omega [|0\rangle\langle 1| + |1\rangle\langle 0|, \varrho] + \frac{1}{2}\Gamma (2|0\rangle\langle 1|\varrho|1\rangle\langle 0| - \varrho|1\rangle\langle 1| - |1\rangle\langle 1|\varrho), \quad (8)$$

$$\Omega := 2|\lambda| \sqrt{\Gamma_2}. \quad (9)$$

$\Omega$  is called Rabi frequency.

Because of the limit in (4), many terms do not contribute and (4) reduces to

$$\begin{aligned} P(\nu) &\propto 1 + \lim_{T \rightarrow +\infty} \frac{4\gamma\Gamma_1}{T} \int_0^T dt \int_0^t ds \int_0^s ds' e^{-\gamma(2t-s-s')} \\ &\times \left\{ \langle 0| e^{\mathcal{L}(s-s')} [|0\rangle\langle 1|\varrho_\infty] |1\rangle e^{-i(\nu-\omega)(s-s')} + \text{c.c.} \right\} = \\ &= 1 + 2\Gamma_1 \int_0^{+\infty} dt \left\{ e^{-[\gamma+i(\nu-\omega)]t} \langle 0| e^{\mathcal{L}t} [|0\rangle\langle 1|\varrho_\infty] |1\rangle + \text{c.c.} \right\}, \end{aligned} \quad (10)$$

where  $\varrho_\infty$  is the equilibrium state satisfying the equation  $\mathcal{L}\varrho_\infty = 0$ , from which it can be easily computed. Note that the spectrum is determined by the equilibrium dipole-dipole correlation function.

By setting  $\varrho(t) := e^{Ct} [|0\rangle\langle 1| \varrho_\infty]$ , we obtain for its matrix elements

$$\begin{aligned} \frac{d}{dt} \varrho_{11}(t) &= \frac{1}{2} \Omega [\varrho_{10}(t) - \varrho_{01}(t)] - \Gamma \varrho_{11}(t) = -\frac{d}{dt} \varrho_{00}(t), \\ \frac{d}{dt} \varrho_{10}(t) + \frac{1}{2} \Gamma \varrho_{10}(t) &= \frac{1}{2} \Omega [\varrho_{11}(t) - \varrho_{00}(t)] = -\left[ \frac{d}{dt} \varrho_{01}(t) + \frac{1}{2} \Gamma \varrho_{01}(t) \right], \end{aligned} \quad (11)$$

with

$$\varrho_{11}(0) = \varrho_{10}(0) = 0, \quad \varrho_{00}(0) = -i \frac{\Omega \Gamma}{2\Omega^2 + \Gamma^2}, \quad \varrho_{01}(0) = \frac{\Omega^2}{2\Omega^2 + \Gamma^2}. \quad (12)$$

These equations can be solved by Laplace transform and we obtain, for  $\Omega \geq \Gamma/4$ ,

$$\begin{aligned} P(\nu) \propto 1 + \kappa_1 \left[ \kappa_2 \frac{\gamma}{\gamma^2 + (\nu - \omega)^2} + \frac{\gamma_0}{\gamma_0^2 + (\nu - \omega)^2} + \right. \\ \left. + \frac{\kappa_3 \gamma_1 - \kappa_4 (\nu - \omega - \tilde{\omega})}{\gamma_1^2 + (\nu - \omega - \tilde{\omega})^2} + \frac{\kappa_3 \gamma_1 + \kappa_4 (\nu - \omega + \tilde{\omega})}{\gamma_1^2 + (\nu - \omega + \tilde{\omega})^2} \right], \end{aligned} \quad (13)$$

$$\tilde{\omega} = \sqrt{\Omega^2 - \Gamma^2/16}, \quad \gamma_0 = \gamma + \Gamma/2, \quad \gamma_1 = \gamma + \frac{3}{4}\Gamma, \quad (14)$$

$$\kappa_1 = \frac{\Gamma_1 \Omega^2}{\Omega^2 + \Gamma^2/2}, \quad \kappa_2 = \frac{\Gamma^2}{\Omega^2 + \Gamma^2/2}, \quad \kappa_3 = \frac{\Omega^2 - \Gamma^2/2}{2\Omega^2 + \Gamma^2}, \quad \kappa_4 = \frac{\Gamma}{8\tilde{\omega}} \frac{10\Omega^2 - \Gamma^2}{2\Omega^2 + \Gamma^2}. \quad (15)$$

In the limit of a strongly stimulated atom,  $0 < \gamma \ll \Gamma \ll \Omega$ , we obtain

$$\tilde{\omega} \simeq \Omega, \quad \gamma_0 \simeq \Gamma/2, \quad \gamma_1 \simeq 3\Gamma/4, \quad (16)$$

$$\kappa_1 \simeq \Gamma_1, \quad \kappa_2 \simeq \Gamma^2/\Omega^2, \quad \kappa_3 \simeq 1/2, \quad \kappa_4 \simeq 5\Gamma/(8\Omega). \quad (17)$$

Equations (13), (16), (17) give the well known three-peaked spectrum typical of the dynamical Stark effect [8,9], apart from the first constant term, which is a white-noise contribution due to the continuous measurement [6].

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## Quantum phase fluctuations and correlations in nonlinear optical processes

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This contribution is aimed to give a short review of our results obtained by applying the Hermitian phase formalism of Pegg and Barnett [1]- [3] to study phase properties of the field generated in various nonlinear optical processes. The Pegg-Barnett formalism allows us to study a number of phase characteristics of the optical field that were inaccessible before, for example, the expectation values and variances of the Hermitian phase operators, the phase probability distribution functions, and the correlations between the phases of different modes.

We have systematically studied the phase properties of fields generated in a number of nonlinear processes, and results can be found in our original papers, which include the anharmonic oscillator model [4, 5], the elliptically polarized light propagating in a nonlinear Kerr medium [6]- [8], the two-mode squeezed vacuum states [10], the second harmonic generation [11]- [13], the parametric down-conversion with quantum pump [14], the Jaynes-Cummings model [15, 16], the fractional coherent states [17], and the displaced number states [18].

Our studies of the phase properties show that the phase distribution or the joint phase distribution (for the two-mode fields) that can be obtained according to the Pegg-Barnett formalism are new representations of the quantum state of the field alternate to, for example, the  $Q$  function or the Wigner function, and they carry quite a bit of information characterizing the field state. For instance, when the field is a superposition of well separated coherent states the phase distribution splits into separate peaks clearly indicating the components of the superposition [6], the phase distribution splits into separate peaks when the transition from the second harmonic generation to the down-conversion regime takes place [11]- [13], the multiplicity of the phase distribution in the multiphoton down-conversion indicates clearly the multiplicity of the process, etc. This gives the motivation for further studies.

Here, I would only like to explore the relationship between the Pegg-Barnett phase distribution and the phase distributions obtained by integrating the  $Q$  function and the Wigner function over the radial variable  $|\alpha|$ . It is interesting that all three phase distributions can be unified into one analytical formula which has the form

$$P_s(\theta) = \frac{1}{2\pi} \left\{ 1 + 2\text{Re} \sum_{n>k} \rho_{nk} \exp[-i(n-k)\theta] G_s(n, k) \right\}, \quad (1)$$



where the coefficients  $G_s(n, k)$  distinguish between the three distributions, and they are:

(i) for the Pegg-Barnett phase distribution

$$G_{PB}(n, k) = 1, \quad (2)$$

(ii) for the distribution  $P_Q(\theta) = \int_0^\infty Q(\alpha) |\alpha| d|\alpha|$

$$G_Q(n, k) = \frac{\Gamma[(n+k)/2 + 1]}{\sqrt{n!k!}}, \quad (3)$$

(iii) for the distribution  $P_W(\theta) = \int_0^\infty W(\alpha) |\alpha| d|\alpha|$

$$G_W(n, k) = \sum_{m=0}^p (-1)^{p-m} 2^{(n-k+2m)/2} \times \sqrt{\binom{p}{m} \binom{q}{p-m}} G_Q(m, |n-k| + m), \quad (4)$$

where  $p = \min(n, k)$ ,  $q = \max(n, k)$ . All the coefficients  $G_s(n, k)$  are symmetric,  $G_s(n, k) = G_s(k, n)$ , and  $G_s(n, n) = 1$ . Relation (1) is quite general and can be applied for any field described by the density operator with the number state basis matrix elements  $\rho_{nk}$ .

The coefficients  $G_Q(n, k)$  can be easily calculated with the following recurrence formula

$$G_Q(n+1, k) = \frac{G_Q(n, k)}{\sqrt{n+1}} \begin{cases} \left(\frac{n+k}{2} + 1\right) \sqrt{\pi} \prod_{s=1}^{s_m} \left(1 - \frac{1}{2s}\right) & \text{for } n+k \text{ even} \\ \left[\sqrt{\pi} \prod_{s=1}^{s_m} \left(1 - \frac{1}{2s}\right)\right]^{-1} & \text{for } n+k \text{ odd} \end{cases} \quad (5)$$

where

$$s_m = n + k + 1 - \left[ \frac{n+k+1}{2} \right]. \quad (6)$$

and  $[\times]$  in (6) denotes the greatest integer less than or equal to  $\times$ . The behaviour of the coefficients  $G_Q(n, k)$  is such that the farther away we go from the diagonal  $G_Q(n, n) = 1$ , the smaller are  $G_Q(n, k)$ , although the rate of decay decreases as the numbers  $n, k$  increase. Knowing the coefficients  $G_Q(n, k)$  we can directly calculate the phase distribution  $P_Q(\theta)$  according to (1), and inserting  $G_Q(n, k)$  into (4), calculating the coefficients  $G_W(n, k)$ , we can again use (1) to calculate the phase distribution  $P_W(\theta)$ .

The Pegg-Barnett phase distribution  $P_{PB}(\theta)$  and the distribution  $P_Q(\theta)$  are positive definite and normalized to unity, so they satisfy the requirements imposed for

the probability distributions. Since all the nondiagonal coefficients  $G_Q(n, k)$  are less than unity, the phase distribution  $P_Q(\theta)$  is obtained from the Pegg-Barnett phase distribution by weighting the nondiagonal density matrix elements  $\rho_{nk}$  with the "probabilities"  $G_Q(n, k)$ . This means a sort of averaging procedure that must be performed to get  $P_Q(\theta)$  from  $P_{PB}(\theta)$ . This procedure leads to the broadening of the Pegg-Barnett phase distribution [19] and, in some cases, to washing out some of the peaks present in the Pegg-Barnett phase distribution [18, 19]. This means that the Pegg-Barnett phase distribution  $P_{PB}(\theta)$  carries more phase information characterizing a given state than the distribution  $P_Q(\theta)$ .

Since the Wigner function  $W(\alpha)$  can take on negative values, for some states of the field, the phase probability  $P_W(\theta)$  can also be negative, and it cannot be considered as a true probability distribution, but rather as a quasiprobability distribution. However, even for the displaced number states for which  $W(\alpha)$  really takes the negative values, the distribution  $P_W(\theta)$  is positive and gives the phase structure very similar to the Pegg-Barnett distribution [18]. The positiveness of  $P_W(\theta)$  is, however, not guaranteed, and the anharmonic oscillator model is an example which leads to  $P_W(\theta)$  with negative values [20]. The coefficients  $G_W(n, k)$  can take the values that are both smaller and greater than unity (depending on  $n$  and  $k$ ), so their effect on the Pegg-Barnett phase distribution is not as simple as that of the coefficients  $G_Q(n, k)$ . Nevertheless, formula (1) allows to calculate all the three phase distributions and find the differences between them.

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**QUANTUM PARAMETRIC CHAIN IN WIGNER REPRESENTATION**

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A number of papers devoted to the problem of different kinds of quantum chains were published last time. The quantum unclosed nonparametric chain was discussed in Ref./1/. The closed nonparametric chain is considered in /2-4/. Different kinds of parametric chains are considered in /3,5,6/, and different kinds of damped parametric chains are considered in /7,8/.

In this paper let us consider a chain consisted of  $N$  harmonic coupled parametric oscillators. All oscillators vibrate with the frequency  $\Omega(t)$  which depends on time, and linearly interact with the neighbours. The interaction constant  $\Omega(t)$  depends on time too. When the distance between neighbours approaches zero, and number  $N$  tends to infinity, the chain turns into the parametric string. In order to consider the most simple case  $N$  must be an odd number  $N=2p+1$ . The Hamiltonian of this system is

$$\hat{H} = \frac{1}{2} \sum_{n=1}^N \left( \frac{\hat{p}_n^2}{m} + \Omega^2(t)m \left( \hat{q}_n - \hat{q}_{n+1} \right)^2 + \Omega_0^2 m \hat{q}_n^2 \right). \quad (1)$$

where  $\hat{q}_n$  is an operator of a shift from the equilibrium point of an  $n$ -th oscillator,  $\hat{p}_n$  is a momentum operator of the oscillator,  $m$  is a mass of oscillators, and the part  $\Omega^2(t)m \left( \hat{q}_n - \hat{q}_{n+1} \right)^2$  describes the interaction between the neighbours. The equations of motion corresponding to Hamiltonian (1) are /2/

$$\ddot{q}_n = \Omega^2(t)(q_{n+1} + q_{n-1} - 2q_n) - \Omega_0^2(t)q_n$$

Following the procedure suggested in /2,3/, let us introduce new variables:

$$x_s = \left( \frac{2}{N} \right)^{1/2} \sum_{m=1}^N q_m \cos \left( \frac{2\pi sm}{N} \right), \quad x_N = \frac{1}{N^{1/2}} \sum_{m=1}^N q_m,$$

$$y_s = \left( \frac{2}{N} \right)^{1/2} \sum_{m=1}^N q_m \sin \left( \frac{2\pi sm}{N} \right), \quad s=1,2,\dots,p.$$

This variables reduce the system of  $N$  coupled harmonic oscillators to a set of  $N$  free oscillators, vibrating independently according to the equations

$$\ddot{x}_s + \Omega_s^2(t)x_s = 0, \quad \ddot{x}_N + \Omega_N^2(t)x_N = 0,$$

$$\ddot{y}_s + \Omega_s^2(t)y_s = 0$$

with the time-dependent frequencies

$$\Omega_s^2(t) = 4\Omega^2(t) \sin^2(\pi s/N) + \Omega_0^2(t). \quad (2)$$

One can see that the Hamiltonian (1) turns to the sum of the Hamiltonians of free parametric oscillators, vibrating with frequencies (2). Following the usual procedure /9/ one can construct "annihilation" operators for variable-frequency chain.

$$\begin{aligned}\hat{A}_s(t) &= \frac{i}{N^{1/2}} \sum_{m=1}^N \left( \frac{l_s \epsilon_s}{\hat{p}_m} - \frac{\dot{\epsilon}_s}{\Omega_s(0) l_s} \hat{q}_m \right) \cos \frac{2\pi sm}{N}, \\ \hat{B}_s(t) &= \frac{i}{N^{1/2}} \sum_{m=1}^N \left( \frac{l_s \epsilon_s}{\hat{p}_m} - \frac{\dot{\epsilon}_s}{\Omega_s(0) l_s} \hat{q}_m \right) \sin \frac{2\pi sm}{N}, \\ \hat{A}_N(t) &= \frac{i}{(2N)^{1/2}} \sum_{m=1}^N \left( \frac{l_o \epsilon_o}{\hat{p}_m} - \frac{\dot{\epsilon}_o}{\Omega_o(0) l_o} \hat{q}_m \right), \quad (3) \\ l_o &= \left( \frac{\hbar}{m\Omega_o(0)} \right)^{1/2}, \quad l_s = \left( \frac{\hbar}{m\Omega_s(0)} \right)^{1/2}.\end{aligned}$$

where the functions  $\epsilon_o(t)$ ,  $\epsilon_s(t)$  are the solutions of the equations

$$\dot{\epsilon}_s + \Omega_s^2(t) \epsilon_s = 0, \quad \dot{\epsilon}_o + \Omega_o^2(t) \epsilon_o = 0$$

with additional conditions

$$\left[ \dot{\epsilon}_s \epsilon_s^* - \dot{\epsilon}_s^* \epsilon_s \right] = 2i\Omega_s(0), \quad \left[ \dot{\epsilon}_o \epsilon_o^* - \dot{\epsilon}_o^* \epsilon_o \right] = 2i\Omega_o(0).$$

One can check that operators (3) and their hermitian conjugate operators are integrals of motion for the quantum parametric chain. Operators (3) satisfy the commutation relations of the boson creation and annihilation operators. The ground state of the parametric chain can be constructed with the help of the integrals of motion, so the ground state is as follows,

$$\begin{aligned}\psi_o &= \pi^{-N/4} l_o^{-1/2} \epsilon_o^{-1/2} \prod_{s=1}^p \epsilon_s^{-1} l_s^{-1} \exp \left\{ \frac{i\dot{\epsilon}_o}{2\epsilon_o N \Omega_o(0) l_o} \left( \sum_{m=1}^N q_m \right)^2 + \right. \\ &\left. + \sum_{s=1}^p \frac{i\dot{\epsilon}_s}{\epsilon_s N \Omega_s(0) l_s} \left[ \left( \sum_{m=1}^N q_m \cos(2\pi sm/N) \right)^2 + \left( \sum_{m=1}^N q_m \sin(2\pi sm/N) \right)^2 \right] \right\}.\end{aligned}$$

The Wigner function of the quantum parametric chain is

$$\begin{aligned}
W(\mathbf{p}, \mathbf{q}, t) &= \hbar^{-N} (\det \sigma)^{-1/2} \otimes \\
&\exp \left[ - \frac{1}{N} \sum_{m, m'=1}^N \left\{ \frac{l_o^2 \omega_o^2 (|\epsilon_o \dot{\epsilon}_o|^2 / \omega_o^2 - 2)}{2 l_o^2 |\dot{\epsilon}_o|^2 (|\epsilon_o \dot{\epsilon}_o|^2 - 1)} p_m p_{m'} + \frac{|\epsilon_o \dot{\epsilon}_o|^2 / \omega_o^2 - 1}{2 l_o^2 |\epsilon_o|^2 |\epsilon_o \dot{\epsilon}_o|^2} q_m q_{m'} + \right. \right. \\
&\quad \left. \left. + \frac{(|\epsilon_o \dot{\epsilon}_o|)^{-2} \omega_o^2}{2 (|\epsilon_o \dot{\epsilon}_o|^2 / \omega_o^2 - 1)^{1/2}} (q_m p_{m'} + p_m q_{m'}) + \right. \right. \\
&\quad \left. \left. + \sum_{s=1}^p \cos(2\pi s(m-m')/N) \left[ \frac{l_s^2 \omega_s^2 (|\epsilon_s \dot{\epsilon}_s|^2 / \omega_s^2 - 2)}{2 l_s^2 |\dot{\epsilon}_s|^2 (|\epsilon_s \dot{\epsilon}_s|^2 / \omega_s^2 - 1)} p_m p_{m'} + \right. \right. \right. \\
&\quad \left. \left. \left. + \frac{(|\epsilon_s \dot{\epsilon}_s|^2 / \omega_s^2 - 1) \omega_s^2}{l_s^2 |\epsilon_s|^2 |\epsilon_s \dot{\epsilon}_s|^2} q_m q_{m'} + \frac{(|\epsilon_s \dot{\epsilon}_s|)^{-2} \omega_s^2}{(|\epsilon_s \dot{\epsilon}_s|^2 / \omega_s^2 - 1)^{1/2}} (q_m p_{m'} + p_m q_{m'}) \right] \right\} \right],
\end{aligned}$$

where  $\sigma$  is the matrix of dispersions. The quantum dispersions and correlations of coordinates and momenta in coherent states (4) are equal to

$$\begin{aligned}
\sigma_{q_k^2} &= N^{-1} \left( l_o^2 |\epsilon_o|^2 / 2 + \sum_{s=1}^p l_s^2 |\epsilon_s|^2 \right), \quad \sigma_{p_k^2} = N^{-1} \left( l_o^{-2} |\dot{\epsilon}_o|^2 / 2 \omega_o^2 + \right. \\
&\quad \left. + \sum_{s=1}^p l_s^{-2} |\dot{\epsilon}_s|^2 \omega_s^{-2} \right), \quad \sigma_{q_k p_k} = N^{-1} \hbar \left( \omega_o^{-2} |\epsilon_o \dot{\epsilon}_o|^2 (|\epsilon_o \dot{\epsilon}_o|^2 / \omega_o^2 - 1)^{-1/2} + \right. \\
&\quad \left. + \sum_{s=1}^p 2 \omega_s^{-2} |\epsilon_s \dot{\epsilon}_s|^2 (|\epsilon_s \dot{\epsilon}_s|^2 / \omega_s^2 - 1)^{1/2} \right) \quad (5)
\end{aligned}$$

The squeezing coefficients  $k = \left( \sigma_{q_k^2}(t) \sigma_{p_k^2}^{-1}(t) \sigma_{q_k^2}^{-1}(0) \sigma_{p_k^2}(0) \right)^{-1/2}$

and the correlating coefficients  $r = \sigma_{q_k p_k}(t) \sigma_{q_k^2}^{-1/2}(t) \sigma_{p_k^2}^{-1/2}(t)$

in coherent states (4) are equal to

$$\begin{aligned}
k &= \left( l_o^2 |\epsilon_o|^2 / 2 + \sum_{s=1}^p l_s^2 |\epsilon_s|^2 \right)^{1/2} \left( l_o^{-2} + 2 \sum_{s=1}^p l_s^{-2} \right)^{1/2} \left( l_o^2 + \sum_{s=1}^p 2 l_s^2 \right)^{-1/2} \\
&\otimes \left( l_o^{-2} |\dot{\epsilon}_o|^2 / 2 \omega_o^2 + \sum_{s=1}^p l_s^{-2} |\dot{\epsilon}_s|^2 \omega_s^{-2} \right)^{-1/2}
\end{aligned}$$

$$r = \frac{\omega_0^{-2} |\dot{\epsilon}_0 \dot{\epsilon}_0|^2 (|\epsilon_0 \dot{\epsilon}_0|^2 \omega_0^{-2} 1)^{\frac{1}{2}} + \sum_{s=1}^p 2\omega_s^{-2} |\dot{\epsilon}_s \dot{\epsilon}_s|^2 (|\epsilon_s \dot{\epsilon}_s|^2 \omega_s^{-2} 1)^{\frac{1}{2}}}{\left( \left[ 1_0^{-2} |\dot{\epsilon}_0|^2 \omega_0^{-2} + 2 \sum_{s=1}^p 1_s^{-2} |\dot{\epsilon}_s|^2 \omega_s^{-2} \right] \left[ 1_0^2 |\epsilon_0|^2 + 2 \sum_{s=1}^p 1_s^2 |\epsilon_s|^2 \right] \right)^{1/2}}$$

One can see from (5) that changings of the frequencies influence the squeezing coefficients and one can increase the dispersions of the coordinates due to decreasing of the dispersions of momenta and vice versa.

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ANALYTIC NUMBER THEORY AND NUCLEAR  
LEVEL DENSITIES

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Abstract:

We are presenting here an alternative approach to obtain a better description of the nuclear level density. We are applying methods from Analytic Number Theory. The obtained results are formally related to other areas of Mathematical Physics where the estimation of the coefficients of (formal) Fourier series of partitions functions plays an important role. The acid test of this approach and the primary reason to develop it is a comparison of an explicit theoretical calculation with experimental data.

This contribution was motivated by the original work of H.A. Bethe (1936/37), N. Bohr (1936), C. van Lier and G.E. Uhlenbeck (1937), S. Goudsmit (1937) and others physicists. These authors obtained a formula which showed with very simple analytical relations the most important behaviour of the nuclear level density, parametrized by means of quantities with a clear physical meaning. No parameter was introduced in an ad-hoc way to reproduce experimental results. The connection to number theoretical problems was also known. Nevertheless, their results are only a crude approximation which still today needs to be implemented.

By introducing the shell structure into the single particle spectra, we obtain here new results expressed with simple formulas using as guide methods from the modern Analytic Number Theory (see for example H. Rademacher (1973)). In this way it becomes easier to recognize the relevant mathematical quantities which must be related to the physical parameters.

We succeeded here to obtain a thorough analytical description, so that only a minimum of numerical computer calculations is needed to carry out the comparison with experimental results. During the last 30 years, there have been many extensive numerical works under the title "microscopic calculations". But of course they are done without the existence of a consistent fundamental nuclear theory. It is well known since many years that different spectra lead to the same average results for the level density. There is no unique way to fix the "correct" nuclear Hamiltonian from these considerations.

Here we adopt a different point of view. We consider classes of single particle spectra with common analytic properties. We stress the importance to recognize the most relevant parameters, which must be common to all single particle spectra, whenever they are to reproduce the experimental data. In this way, we do not need to limit ourselves to a special kind of Hamiltonian and it is also not necessary to diagonalize it. We attempt rather to introduce



nuclear structure properties in the form of well founded mathematical quantities with a corresponding physical interpretation. This approach should conduce to the study of invariants associated with heavy nuclei.

At the same time we introduce as many mathematical devices we feel reasonable for a serious study of nuclear spectra and to find a common basis for all analytic studies up to now, as well as for future developments. The most important criterium we observe is the achievement of a method for explicit numerical calculations ready to be compared with available experimental data. It is remarkable that many results are formally related with other branches of theoretical physics as the reader can recognize. But because of lack of space we will report many results separately.

As usual we follow the Darwin-Fowler method to obtain an analytic expression for the level density. The density of excited sates of a system of  $N$  neutrons and  $Z$  protons with total energy  $E$  and total angular momentum projection  $M$  is given by  $\rho(N,Z,M,E)$  in the expression for the grand partition function:

$$\begin{aligned} Z(\alpha, \beta) &= \prod (1 + \exp(\alpha_n + \alpha_3 m_{3\nu_n} - \beta \epsilon_{\nu_n})) \prod (1 + \exp(\alpha_p + \alpha_3 m_{3\nu_p} - \beta \epsilon_{\nu_p})) = \\ &= \sum_{Z,N,M,E} x_p^Z x_n^N x_3^M y^E \rho(N,Z,M,E), \end{aligned}$$

where  $x_k = \exp(\alpha_k)$  and  $y = \exp(-\beta)$ . The single particle spectrum is given by the numbers  $\{\epsilon_{\nu_{n(p)}}\}$  for neutrons (protons).

Using the saddle point method the level density is given approximately by:

$$\rho(N,Z,M,E) = \frac{\exp S(\alpha_0, \beta_0)}{(2\pi)^2 \sqrt{D}},$$

where  $S(\alpha, \beta) = \ln Z(\alpha, \beta) + \beta E - \alpha_n N - \alpha_p Z - \alpha_3 M$  is the entropy of the system and  $D$  is the determinant of the  $4 \times 4$  matrix formed with the second order partial derivatives of  $\ln Z(\alpha, \beta)$  evaluated at the saddle point given by the solution of:

$$\partial_{\alpha_n} S = 0, \quad \partial_{\alpha_p} S = 0, \quad \partial_{\alpha_3} S = 0, \quad \partial_{\beta} S = 0,$$

Our method consists now in the calculation of the entropy  $S(\alpha, \beta)$  using standard methods of Analytic Number Theory. Although the single particle energies  $\epsilon_{\nu}$  are real numbers, we follow the common practice of substituting them by integer numbers using a sufficiently small energy unit. Furthermore, we take  $\epsilon_n = 1, 2, \dots$ , and describe a general single particle spectrum by an adequated selection of degeneracy numbers  $a_n$ . Consider now the number of particles which can be allocated in the given spectrum up to level  $n$  starting on from the lowest level. For this number of particles we assume:

$$\sum_{j=1}^n a_j \approx \sum_m \frac{A_m}{d_m} n^{d_m}, \quad d_m > d_n \text{ for } m > n, \quad d_m > 0.$$

where we have introduced a finite set of parameters  $\{A_m, d_m\}$ , which for physical reasons must be real. With this assumption the most important objects are the following Dirichlet series: the partition function  $Z(x)$  and its Mellin transform  $D(z)$ :

$$Z(x) = \sum_{n>0} a_n e^{-xn}, \quad D(z) = \sum_{n>0} \frac{a_n}{n^z} = \frac{1}{\Gamma(z)} \int_0^{\infty} dx x^{z-1} Z(x),$$

We obtain after some effort the following expression:

$$\begin{aligned} \ln Z(\alpha, \beta) = \ln Z(-\alpha, -\beta) + \int_0^{\mu} \text{deg}(\varepsilon)(\alpha - \beta\varepsilon) + \sum_{j>0} \frac{(-)^{j-1} 2(2^{2j-1} - 1)\pi^{2j}}{(2j)!\beta^{2j-1}} B_{2j} g^{(2j-2)}(\mu) \\ - \sum_{j>0} \frac{g^{(j-1)}(\mu)}{\beta^j (j-1)!(2\pi i)^{j-1}} \partial_t^{j-1} t^{-1} D_j(\mu, \pi t), \end{aligned}$$

where  $\mu = \alpha/\beta$ ,  $t = 1/\beta$ ,  $D_j(\mu, \pi t) = \sigma_{j-1} C_j(\mu, \pi t) + i\sigma_j S_j(\mu, \pi t)$ ,  $\sigma_{2m} = 1$ ,  $\sigma_{2m+1} = 0$  and with the  $q$ -series :

$$C_{2m+1}(z, t) = \sum_{n>0} \frac{\cos(2\pi n z)}{n^{2m+1} \sinh(2\pi n t)}, \quad S_{2m}(z, t) = \sum_{n>0} \frac{\sin(2\pi n z)}{n^{2m} \sinh(2\pi n t)},$$

and the single particle level density defined by:

$$g(t) = \sum_m \frac{A_m}{d_m} \left[ t^{d_m} - (t-1)^{d_m} \right] + \sum_{k \geq 0} D(-k) \delta^{(k)}(t)/k!$$

Additionally, an equivalent expression for  $\ln Z(\alpha, \beta)$  in terms of  $\beta$  itself can be found. It corresponds to the modular transformation  $\beta \rightarrow 1/\beta$ , i.e. relates large and low temperatures. Both relations are very important for explicit numerical and analytical calculations. Additionally, the transformation formula establishes a link with other areas of Physics.

We have applied the preceding method to the most simple possible case. It is the periodic spectrum introduced by P. Kahn and N. Rosenzweig (1969)  $\varepsilon_{k,j} = (k + \nu(j)) \hbar\omega$ ,  $k \in \mathbb{N}$ ,  $j = 1, 2, \dots$ ,  $e$ ; where  $e$  gives the degeneracy of a shell. Our method yields explicitly the energy dependence of the shell effects not present in the considerations of Kahn and Rosenzweig.

In our calculation we computed the so called  $a$ -parameter appearing in the formula:

$$\rho(U) = \frac{\exp(2\sqrt{a(U-E_1)})}{12\sqrt{2} \sigma a^{1/4} (U-E_1)^{5/4}},$$

where  $U$  is the excitation energy. The constants  $a$  and  $E_1$  were adjusted by T. von Egidy, H. Schmidt and A. Behkami (1988) to reproduce experimental

data at the neutron binding energy.

For the periodic spectrum the shell width was set equal for all nuclei belonging to the same shell. Besides the shell widths we did not adjust any other parameter. In Figure 1, we show the results of the theoretical calculation in comparison with the experimental data compiled by von Egidy et al.. The agreement is encouraging. In a forthcoming publication we will show more details of this calculation as well as many other related results of mathematical and physical interest.

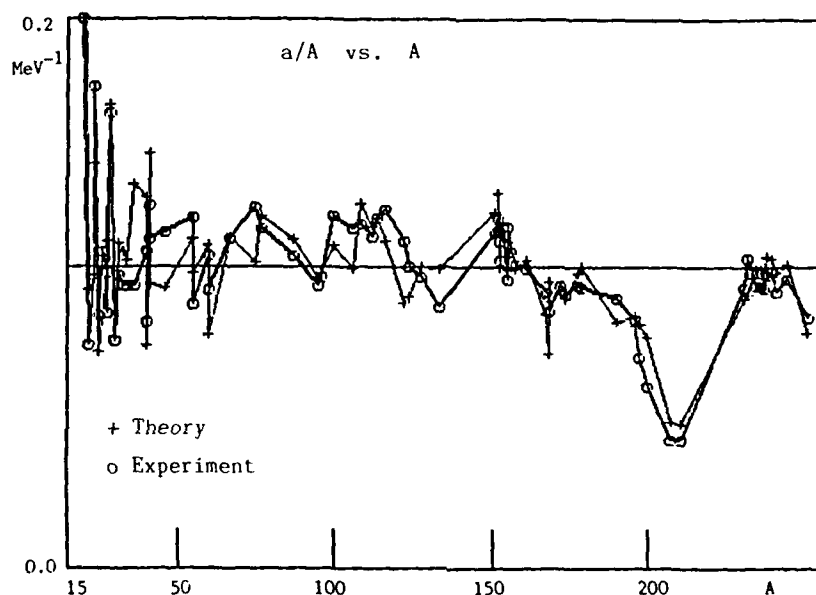


Figure 1 Comparison for a particular set of shell widths with the experimental data.

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## TYPES OF NUCLEAR REACTION MECHANISMS

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Nuclear reactions show a statistical structure as well as the wave structure of the nuclear potential size. But recently a transitional behaviour called precompound mechanism has been discovered. We treat the problem of properly distinguishing between these mechanisms and defining them. This purpose can be achieved by using properly defined average cross sections where we distinguish between statistical averages and averages by means of sum rules. This method may form a step towards a formulation of cross sections in terms of reaction mechanisms. It also might provide a hint to a suitable treatment of the non-weak interacting many-nucleon systems.

Information about microsystems as nuclei or particles can only be obtained by means of scattering or reaction experiments and their theoretical interpretation. As an example we may consider the inelastic scattering of neutrons on nuclei. For the case of  $^{56}\text{Fe}$  we obtained the following results:

**1. Direct Mechanism.**

Fig. 1 shows the secondary energy dependent inelastic scattering cross section of 14.6 MeV neutrons. The left and the right end of the curve are dominated by two different reaction mechanisms. The reaction mechanism at the right end has as its characteristic angular distribution of the inelastically scattered particle cross section the type shown in fig. 2. This is the typical angular distribution of a direct reaction mechanism. With its relative small rate of energy transfer (only 2 - 3 MeV of 14.6 MeV incident energy) it can be described by the DWBA-approach which yields this type of angular distribution. The wave structure is originated by the occurrence of the square of the spherical Besselfunction  $[j_\nu(qR)]^2$  where the argument  $qR$  is the product of the transferred momentum times the Radius  $R$  of the nuclear potential size. In the figure 2 there is  $\nu = 2$ .

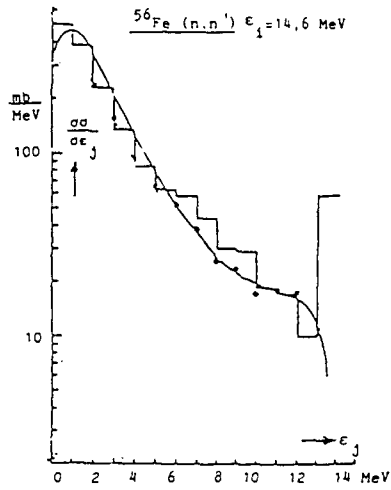


Fig. 1: Stepcurve and GDH-curve of the secondary energy dependent total inelastic scattering cross section of 14,6 MeV neutrons on  $^{56}\text{Fe}$ . Points: Dresden Measurements, /2/

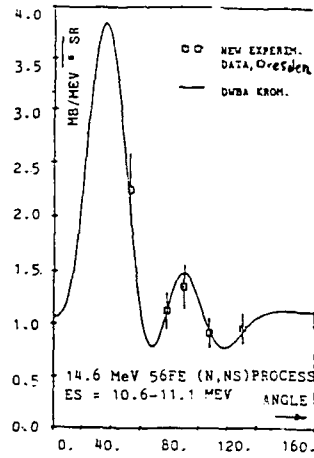


Fig. 2: Angular-distribution of the inelastic-scattering cross section of 14,6 MeV-neutrons on  $^{56}\text{Fe}$  averaged over the interval 10,6-11,1 MeV of the secondary neutron energies. Approx. DWBA-curve with  $R=6,5 \text{ fm}$ ,  $t=2,4 \text{ fm}$ ,  $a=1,5 \text{ fm}$  u.  $\gamma=0,7$

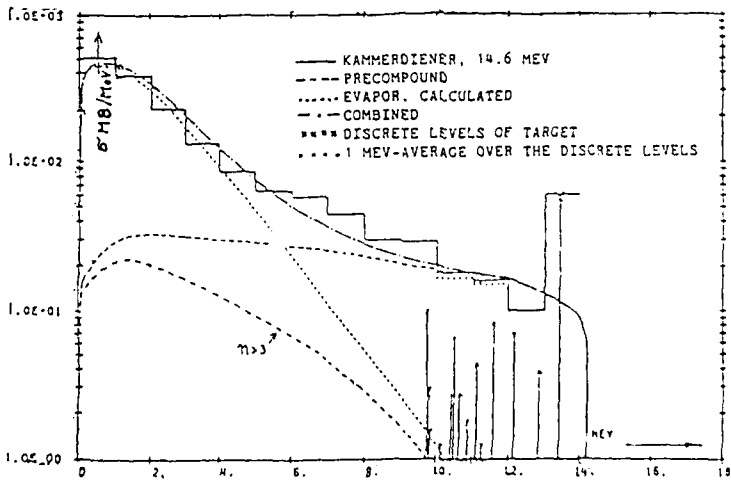


Fig. 3: Neutron emission cross-section of  $^{56}\text{Fe}$ ;  $\epsilon_i = 14,6 \text{ MeV}$

### 2. Compound mechanism.

In contrast to the just described reaction mechanism at the right end of fig.1 we have at the left end of fig.1 a reaction mechanism where a maximal amount of the incident energy is transferred. This is possible only if many nucleons are involved into the reaction process, where many impacts must occur to excite the nucleus to the considered energy before the secondary particle will be emitted. Because of the many impacts the correlations of the emitted particle to the incident particle ray have been lost. Thus the angular distribution of the emitted particle becomes nearly isotropic and the Hauser-Feshbach-Formula is valid with the emission cross-section proportional to the level-density of the target-nucleus remaining after the emission of the secondary particle. Anzaldo presents at this meeting for the first time a rigorous consistent and complete formalism for the calculation of the level densities of nuclei. His method is statistical corresponding to the statistical nature of the many impacts. But by means of the use of the methods of analytical number theory it takes a priori into account the shell structure of nuclei which is shown by considering the  $A$  - dependence of the level densities of the various nuclei.

### 3. Precompound mechanism.

Between these two reaction mechanisms at the right and at the left of fig.1 there is a transition region, which firstly has been considered by Griffin and in particular by Blann who developed the first three models by name pre-equilibrium or pre-compound to describe this transition region. Meanwhile there are in total 11 models:

Those are models by Cline and Blann and by Bunakov describing the formation of the statistical equilibrium at the left end of fig.1 by means of a master equation. Other models are based on reaction formalisms like those of Weidenmüller and Feshbach. All of these models have free parameters which have to be fitted like residual interactions or transition strength or amplitudes (see review of Hodgson/1/, in/2/ and/3/). But the experimental errors are too large to get a unique fit this way and no unique results can thus be obtained (see/4/). Only Blann's geometry dependent hybrid model (GDH) (see/3/,/5/) has no fit-parameters other than those of the usual optical model. Its results for our example are shown in fig.1.u.3., where fig.3. shows that the precompound excitation steps with particle-hole number  $n \geq 3$  mostly remain much smaller than 10% of the complete cross-section. Thus they can be neglected with regard to the errors of the measured results and merely the only geometry-dependent  $n = n_0 = 3$ - component has to be taken into account of which it can be proven to be equal to the averaged direct contributions. This proof has been carried out in /2/ with the result:

$$\int \left( \frac{d^2 \sigma_{ij}^{(n)}(c_i, c_j, 0_j)}{dc_j d\Omega_j} \right)_{DWBA} d\Omega_j = \left( \frac{d\sigma_j^{(n)}(c_i, c_j)}{dc_j} \right)_{n=n_0=3}^{GDH}$$

and the preceding equations in /2/. It should be pointed out that the averaging interval in (1) must have a minimum of length, which must include a minimum number of levels. Since this minimum number of levels needs to be only 3 to 5 with the corresponding minimum length of intervals between 1/2 MeV and 1 MeV the averaging in (1) cannot be considered as a statistical averaging. Alternatively we might call it a spectroscopical averaging since it can be related to the sum rules of Lane /6/. Thus as a result of the preceding considerations it can be concluded from the figures that in the region of nuclei around Fe and at incident neutron energies around 14 MeV and below the inelastic neutron cross-section is almost fully composed by the direct plus the compound reaction. A precondition of this conclusion is, that the averaged direct component has been fully exhausted throughout the scale of the secondary energy of the emitted particle. This we have achieved by using the  $n=3$ -component of the GDH as a substitute for the averaged direct component according to (1). As the essential result of our investigation we can conclude, that the direct and the compound component have to be fully exhausted before a precompound component must be taken into account. In our example it even could be neglected in agreement with the semimicroscopical results of /7/ and /8/. A proper tool for exhausting the direct component is the  $n=n_0=3$ -component of the GDH, which can be expressed by the sum rules of Lane /6/, and the exhausting of the compound component can be achieved by using the level density concept of Anzaldo /9/. Both of these tools have to be applied before a precompound component has to be considered.

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INFORMATION THERMODYNAMICAL APPROACH  
AND A COMMENT ON  
WIGNER'S STATISTICAL  
THEORY OF SPECTRA

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First we shall try to explain what the information thermodynamical approach is. *Information thermodynamics* is an empirical probability inference (assessment, fixing) by means of measurement of mean values of the related observables, and sometimes also their higher order statistical moments, and maximizing entropy (information) by these values as constraints. This method was proposed first in 1957 by E. T. Jaynes [4] for classical systems, in 1961 by R. S. Ingarden and K. Urbanik [4, 4] for quantum systems, and in 1963 it was generalized by R. S. Ingarden [4, 4] for higher order moments. The Lagrange indeterminate coefficients occurring by this variational procedure have been called by the present author *temperature coefficients* because of their physical meaning. In particular, one speaks about *higher order temperatures* as conjugated with higher order moments.

When higher order moments are taken into account, the method of information thermodynamics is more general than that of *central limit theorem* of probability theory. The latter works only in the limit of large number of components  $N$  and their *stochastic independence*, in particular, in the case of the so-called thermodynamic limit ( $N \rightarrow \infty$ , volume  $V \rightarrow \infty$ , but  $V/N = \text{const.}$ ).

Therefore, information thermodynamics (IT) can give essentially new results with respect to the usual thermodynamics (T) in the following cases:

- a) for small systems, e.g., for  $N$  in the region  $10^5$ - $10^6$ , where  $N^{-\frac{1}{2}}$  is of the order  $10^{-2}$ - $10^{-3}$ ,
- b) for strongly and intermediately interacting particles, as for nuclei, atoms, molecules, microclusters, etc.,
- c) for nonuniform systems with  $V/N \neq \text{const.}$ , where surface, size and shape effects are essential,
- d) for integrable systems (solitons) with infinite number of hidden symmetries and constants of motion which cause that the usual thermization is impossible and entropy is very low, theoretically zero, but practically small positive.

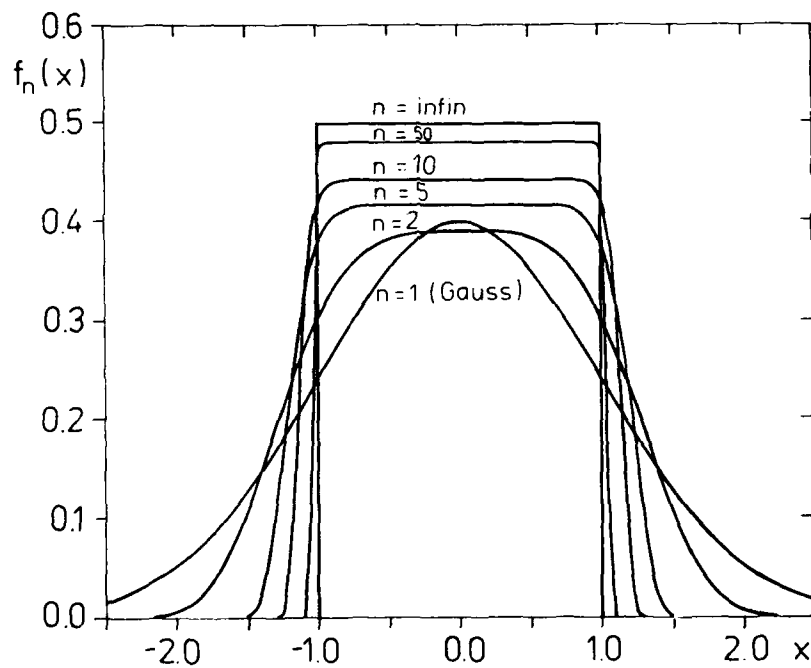
To show the meaning of higher order temperatures we shortly discuss the results concerning two important idealized cases.

**1. One real continuous observable,  $-\infty < x < +\infty$ , and one higher order moment**

$$\langle x^{2n} \rangle = \int_{-\infty}^{+\infty} x^{2n} f(x) dx = \sigma^{2n} = \frac{T_n^n}{(2n)^{\frac{1}{2}}}, \quad n = 1, 2, \dots$$

where  $\sigma > 0$  is a given number and  $T_n > 0$  is the conjugated temperature. The resulting distribution is the *generalized Gauss distribution*, cf. Fig. 1,

$$f(x) = \frac{1}{2(2n)^{\frac{1}{2n}} \sigma \Gamma(1 + \frac{1}{2n})} \exp(-\frac{x^{2n}}{2n \sigma^{2n}}) \geq 0, \quad \int_{-\infty}^{+\infty} f(x) dx = 1.$$



**Fig. 1.** Generalized Gauss distributions for  $n = 1, 2, 5, 10, 50, \infty$ ;  $\sigma = 1$

2. **3-dimensional ideal gas** with Hamiltonian  $H(p) = \frac{1}{2m} \sum_{i=1}^{3N} p_i^2$  and two moments  $\langle H \rangle = U = U_1$ ,  $\langle H^2 \rangle = U_2$ ,  $D = (U_2 - U_1^2)^{\frac{1}{2}}$ .

We obtain the partition function

$$\begin{aligned} Z(\beta_1, \beta_2) &= \frac{V^N}{N!} \int \exp(-\beta_1 H(p) - \beta_2 H^2(p)) dp \\ &= \frac{V^N (2m\pi)^{3N/2}}{N! (2\beta_2^2)^{3N/4}} \exp\left(\frac{\beta_1^2}{8\beta_2^2}\right) D_{-3N/2}\left(\frac{\beta_1}{\sqrt{2}\beta_2}\right), \end{aligned}$$

where  $D_p(x)$  is the Weber parabolic cylinder function. Now for large, but not infinite  $N(1 \ll N \ll \infty)$  we obtain using Darwin's asymptotic representation of  $D_p(x)$

$$\beta_1 = -\frac{1}{U} \left( \frac{U^2}{D^2} - 3N \right), \quad \beta_2 = \frac{1}{U^2} \left( \frac{U^2}{D^2} - \frac{3N}{2} \right) > 0, \quad \beta_i = \frac{1}{k_B T_i}$$

( $i = 1, 2$ ), with the condition of solubility

$$\frac{U^2}{D^2} \geq \frac{3N}{2} \quad \text{or} \quad \Delta = \frac{D}{U} \leq \sqrt{\frac{2}{3N}},$$

i.e., for  $N \rightarrow \infty : \Delta \rightarrow 0$  or  $\beta_2 \rightarrow 0$  ( $T_2 \rightarrow \infty$ ) which gives the correct correspondence with T. But IT can be applied when, e.g.,  $N = 10^5$  and  $\Delta$  is of the order of 1%, and therefore can be measured. Otherwise, we can go to the *2nd order thermodynamic limit*

$$N \rightarrow \infty, \quad V \rightarrow \infty, \quad \text{but} \quad \frac{V}{N^a} = \text{const.}, \quad a = 1 + \frac{9}{4} = 3.25,$$

which shows a vanishing mass density in infinity, but constant entropy density [4].

We see that IT can be applied in mesoscopic domain where surface effects are essential since higher order temperatures are not, in general, intensive and energy is not extensive. But how to measure thermodynamical parameters of such a small system? This can be done by means of spectroscopy, especially infrared spectroscopy. We have in mind polymers, biopolymers, microclusters, quasicrystals. To measure thermodynamical parameters from infrared absorption spectra, it is necessary to distinguish the constant and the stochastic part of the Hamiltonian:  $H = H_o + H_L$  ( $H_o$  = the Hamiltonian of the polymer,  $H_L$  = Langevin term = the Hamiltonian of the environment). Thus we come to the Wigner statistical theory of spectra [4, 4, 4, 4].

At present, three versions of this theory are mainly discussed: GOE (Gaussian orthogonal ensemble), GUE (Gaussian unitary ensemble) and GSE (Gaussian symplectic ensemble) with distributions of  $(n \times n)$ -matrices:

$$P_\beta(\{\lambda_i\}) = C_\beta \prod_{i < j} |\lambda_i - \lambda_j|^\beta \exp\left(-\frac{1}{4a^2} \sum_{i=1}^n \lambda_i^2\right),$$

where  $\beta = 1, 2, 4$  for GOE, GUE, and GSE, respectively. The multiplicative term is due to P. Pechukas [4] and should describe energy level repulsions, while the Gaussian factor stems from Wigner and corresponds to the central limit theorem approximation. Both factors can be considered as "ad hoc" and not general enough for our purpose. We have in mind an IT-generalization of the form

$$P(\{\lambda_i\}) = Z^{-1}(\beta_1, \beta_2, \dots) \exp(-\beta_1 \sum_{i=1}^n \lambda_i^2 - \beta_2^2 (\sum_{i=1}^n \lambda_i^2)^2 + \dots)$$

where  $\beta_1, \beta_2, \dots$  are physically measurable constants extracted from the absorption spectra. Such a project connected with an experimental work is in a preliminary stage in Toruń. We report on it already now because of the Wigner Symposium in Goslar.

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## Increase of Information and Decrease of Entropy due to Averaging by Doubly Stochastic Transformations

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**Abstract:** We show that the information increases by applying a doubly stochastic matrix (*DSM*) to the probability distribution vector, while the Boltzmann entropy decreases by applying a *DSM* on the spectral vector, provided that the expectation value of the energy is kept fixed. An algebraic proof of Schiffer's inequality follows as a corollary.

### 1 Introduction

Consider an experiment with  $N$  possible outcomes. These outcomes may have some quantitative additive values assigned to them, such as an energy  $\epsilon_i$  or a money reward, or they may not have such a value, if say, the outcomes refer to different colours of certain objects. There are two scalar functions which are commonly associated with such experiments:

(I) If the probability  $p_i$  of each outcome is known, then we can assign to the probability (distribution) vector  $\mathbf{p} := (p_1, \dots, p_N)^T$  the scalar quantity:

$$I(\mathbf{p}) := - \sum_{i=1}^N p_i \ln p_i, \quad (1)$$

which is known as (Shannon) information. However, in some mathematical books [3] the expression (1) is called entropy and is denoted by  $H(\mathbf{p})$ . Note that the information (1) is well defined, even if no values are assigned to the outcomes.

(II) In contrast, suppose the outcomes have assigned values,  $\mathbf{e} := (\epsilon_1, \epsilon_2, \dots, \epsilon_N)^T$  and suppose that we were given the average value  $\bar{e}$  of the outcomes, without any further details. Then there are many possibilities (for  $N \geq 3$ ) for choosing the probabilities, which would give the average  $E := \bar{e} = \sum_{i=1}^N p_i \epsilon_i =: \langle \mathbf{p} | \mathbf{e} \rangle$ . However, there is a well-defined probability density  $\mathbf{p}^B$  which is uniquely defined in terms of  $\mathbf{e}$  and  $E$ :

$$p_i^B(E, \mathbf{e}) = e^{-\beta \epsilon_i} / Z[\beta, \mathbf{e}], \quad \text{with} \quad Z[\beta, \mathbf{e}] := \sum_{i=1}^N e^{-\beta \epsilon_i}, \quad (2)$$

where  $Z$  is the "partition function" and  $\beta$  is a parameter which is adjusted to obtain the average value  $E$ . The vector  $\mathbf{e}$  can be thought of as the set of eigenvalues of a Hermitian matrix (an observable). Therefore,  $\mathbf{e}$  will be called the spectral vector. I shall call  $p_i^B(E, \mathbf{e})$  the Boltzmann (probability) density. This density gives the maximum information among all the probability densities that can reproduce the given average [8,9]:

$$S(E, \mathbf{e}) := I(\mathbf{p}^B(E, \mathbf{e})) \geq I(\mathbf{q}), \quad \forall \text{ prob. vector } \mathbf{q} \text{ with } E = \langle \mathbf{p}^B | \mathbf{e} \rangle = \langle \mathbf{q} | \mathbf{e} \rangle, \quad (3)$$

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where we use the symbol  $S(E, \mathbf{e})$  to denote the information of  $\mathbf{p}^B$  which will be called the (Boltzmann) entropy.

In the present talk, I shall explain what happens to information and entropy, when we apply a Doubly stochastic matrix (DSM) to the relevant vectors,  $\mathbf{p}$  and  $\mathbf{e}$ , respectively. I recall that a DSM is an  $N \times N$  square matrix with nonnegative elements,  $d_{ij} \geq 0$ , such that  $\sum_{i=1}^N d_{ij} = \sum_{j=1}^N d_{ij} = 1$ ,  $\forall i, j = 1, \dots, N$ . A special DSM will be called u-DSM, if there exists a unitary matrix  $U$ , such that  $d_{ij} = |U_{ij}|^2$ , for all components  $ij$ . When applied to a vector  $\mathbf{e}$ , a DSM  $D$  produces a kind of averaging of the components of  $\mathbf{e}$ , which is manifested as follows: Let  $\mathbf{h} := D\mathbf{e}$ , then [7]

- (1) The extreme components move closer together:  $\epsilon_{\max} \geq h_{\max} \geq h_{\min} \geq \epsilon_{\min}$ .  
 (2) The average "level splitting" becomes smaller:  $\sum_{ij} |h_i - h_j|^p \leq \sum_{ij} |\epsilon_i - \epsilon_j|^p$  for  $p = 1, 2$ .  
 The extreme DSM is the "equalizer"  $Q$ , which averages out all the components completely:

$$Q = (q_{ij}), \quad q_{ij} = 1/N, \quad Q\mathbf{e} = \bar{\epsilon}\mathbf{z}, \quad \text{where } \mathbf{z} := (1, 1, \dots, 1)^T. \quad (4)$$

## 2 Two Inequalities

**Theorem 1:** The following inequality holds for arbitrary DSM  $D$ :

$$I(D\mathbf{p}) \geq I(\mathbf{p}), \quad (\text{generalized von Neumann Inequality}). \quad (5)$$

The above inequality can be considered as a generalization of the usual von Neumann inequality on entropies [1,4,5], which can be shown to hold for u-DSM [6]. The proof of (5) is simple and follows essentially from the convexity of the function  $-x \ln x$  [6].

As an immediate corollary of (5), we obtain, by using  $D = Q$ , the upper bound on the information for a system of  $N$  levels:

$$\ln N = I(\mathbf{z}) = I(Q\mathbf{p}) \geq I(\mathbf{p}), \quad \text{where } \mathbf{z} := \frac{1}{N}(1, 1, \dots, 1)^T. \quad (6)$$

This inequality is usually proved by variation [8].

In contrast to information, the entropy decreases by the application of a DSM to the spectral vector  $\mathbf{e}$ :

**Theorem 2 (Decrease of Entropy by Averaging of the Spectrum):** Given two spectral vectors  $\mathbf{h}$  and  $\mathbf{e}$ , which are related by an arbitrary DSM  $D$ . Then their Boltzmann entropies satisfy the following inequality:

$$S(E, \mathbf{h}) \leq S(E, \mathbf{e}), \quad \text{where } \mathbf{h} = D\mathbf{e}, \quad \forall E \in B(D, \mathbf{e}), \quad (7)$$

where  $B(D, \mathbf{e})$  is the common  $E$ -domain to both systems, i.e.

$$B(D, \mathbf{e}) := \{E \mid \text{if } \beta \text{ and } \beta^* \text{ exist, such that } E = \sum_{i=1}^N \epsilon_i p_i^B[\beta, \mathbf{e}] = \sum_{i=1}^N h_i p_i^B[\beta^*, \mathbf{h}]\} \quad (8)$$

*Proof:* The equality in (8) for the average energy  $E$  can be written as scalar products between the spectral vectors and their corresponding Boltzmann probability distribution:

$$E = \langle \mathbf{e} | \mathbf{p}^B \rangle = \langle \mathbf{h} | \mathbf{p}^{B^*} \rangle = \langle D\mathbf{e} | \mathbf{p}^{B^*} \rangle = \langle \mathbf{e} | D^T \mathbf{p}^{B^*} \rangle = \langle \mathbf{e} | \mathbf{q} \rangle, \quad \text{where } \mathbf{p}^{B^*} := \mathbf{p}^B[\beta^*, \mathbf{h}], \quad (9)$$

Note that  $\mathbf{q} := D^T \mathbf{p}^{B^*}$  is a probability distribution, but not necessarily a Boltzmann distribution. Since  $D^T$  is also a *DSM*, (5) gives

$$S(\mathbf{e}, \mathbf{h}) = I(\mathbf{p}^B(E, \mathbf{h})) \leq I(D^T \mathbf{p}^B(E, \mathbf{h})) =: I(\mathbf{q}) \leq S(E, \mathbf{e}), \quad (10)$$

where the second inequality in (10) follows from (3). ♣

The upper and lower bounds of the  $E$ -domain  $B(D, \mathbf{e})$  were determined in [6].

The recent **inequality by Schiffer** [2] can be seen as a special case of (7) for *u-DSM*: Schiffer compares the entropies obtained by using the spectral vector  $\mathbf{e}$  of the Hamiltonian and the entropies obtained by using the diagonal matrix elements  $\mathbf{h} = (h_1, \dots, h_N)^T$ , where  $h_i := \langle \alpha_i | H | \alpha_i \rangle$  and  $\{|\alpha_i\rangle\}$  is any basis, such as the coherent states, which is not the energy (eigen)-basis. It is easy to show [6] that  $\mathbf{h} = D\mathbf{e}$ , where  $D_{ij} = |\langle \alpha_i | \epsilon_j \rangle|^2$ . By calling the entropy of  $\mathbf{h}$  information (since it is not the true physical entropy), Schiffer [2] concludes that the entropy is the upper bound of information:  $S(E, \mathbf{e}) \geq S(E, \mathbf{h})$ .

### 3 Zeeman Splitting

Theorem 2 is also useful if the spectra  $\mathbf{h}$  and  $\mathbf{e}$  of two Hamiltonians,  $H_0$  and  $H = H_0 + V$ , are related by a *DSM*. As a concrete example, we consider the Zeeman level splitting due to a weak uniform external magnetic field  $\mathbf{B}$ . The total Hamiltonian is

$$H = H_0 - \mu \hbar^{-1} L_z B, \quad \text{with } \mu := \frac{e\hbar}{2m_e c}, \quad (11)$$

where  $H_0$  is the Hamiltonian of an electron in a spherically symmetric potential and where  $e$  and  $m_e$  are the charge and mass of the electron. We shall neglect the spin of the electron, for simplicity [10]. Therefore, the eigenvalues  $h_{n,l}$  of  $H_0$  corresponding to a certain angular momentum  $l$ , are  $(2l+1)$ -degenerate, where  $n$  denotes all the other quantum numbers that characterize the quantum states.

This field  $\mathbf{B}$  breaks the degeneracy, and leads to the Zeeman splitting of these levels:

$$\epsilon_{n,l,m}(B) = h_{n,l} - m\mu B, \quad \text{for } -l \leq m \leq l. \quad (12)$$

We note that the above perturbation keeps the "center of energy" for every  $l$  invariant, i.e.

$$h_{n,l} = \frac{1}{2l+1} \sum_{m=-l}^l \epsilon_{n,l,m}. \quad (13)$$

Therefore, for each  $n$  and  $l$  the  $(2l+1)$  degenerate levels  $h_{n,l}$  are related to the corresponding split levels of the multiplet  $\epsilon_{n,l,m}(B)$ , by the  $(2l+1) \times (2l+1)$  equalizer matrix  $Q^{(2l+1)}$  which

is defined in (4). Therefore, the spectral vector  $\mathbf{h}$  of  $H_0$  is related to the spectral vector  $\mathbf{e}$  of  $H$  by a DSM  $D$ , which is a direct sum of equalizers, one for each  $n$  and  $l$ :

$$D = \bigoplus_{n,l} Q^{(2l+1)}. \quad (14)$$

Similarly, any breaking of the degenerate levels, which keeps, as in (13), the "center of energy" invariant, can be described by a DSM similar to (14). Eq. (7) tells us that the entropy of the system is increased (!) by the symmetry breaking, provided that the average energy  $E$  is kept the same:

$$S(E, H_0) = S(E, \mathbf{h} = D\mathbf{e}) \leq S(E, \mathbf{e}) = S(E, H). \quad (15)$$

Note that in many books [11],  $\langle H_0 \rangle_H = \text{Tr}(H_0 \rho^B(\beta))$  is called "internal energy", while our average energy  $E := \langle H \rangle_H = \text{Tr}(H \rho^B(\beta)) = \langle H_0 \rangle_H - M(B)B$  is called the "enthalpy", where  $\rho^B(\beta) := e^{-\beta H} / \text{Tr}(e^{-\beta H})$  is the density matrix of the total system  $H$ . The inequality (15) seems to surprise many people, since they expect the external field to align the spins and thus to produce more order and consequently less entropy. But the condition of comparing the entropies at the same average energy  $\text{Tr}(\rho_0^B(\beta^*)H_0) = \text{Tr}(\rho^B(\beta)H)$  plays a crucial role. The inequality (15) can also be proved by using thermodynamical arguments [6]. Here, we shall illustrate the inequality (15) by a numerical calculation for a simple 3-level system:

$$\mathbf{e} = (0, 2, 2)^T \quad \text{and} \quad \mathbf{h} = D\mathbf{e} = (1, 1, 2)^T, \quad \text{where} \quad D = Q^{(2)} \oplus Q^{(1)}. \quad (16)$$

For these spectral vectors, we readily obtain the partition functions:

$$Z(\beta) = 1 + 2e^{-2\beta}, \quad Z^*(\beta^*) = 2e^{-\beta^*} + e^{-2\beta^*}. \quad (17)$$

Using  $E(\beta) = -\frac{\partial \ln Z(\beta)}{\partial \beta}$  and  $S(\beta) = \beta E(\beta) + \ln Z(\beta)$ , we obtain the results in Table 1.

| $\beta^*$ | $\beta$             | $Z^*(\beta^*)$ | $Z(\beta)$ | $E$    | $S^*(\beta^*)$ | $S(\beta)$          |
|-----------|---------------------|----------------|------------|--------|----------------|---------------------|
| $\infty$  | $\frac{1}{2} \ln 2$ | 0              | 2          | 1      | $\ln 2$        | $\frac{3}{2} \ln 2$ |
| 25        | .3466               | 0.0000         | 2.0000     | 1.0000 | .6931          | 1.0397              |
| 1.5       | .2459               | 0.4960         | 2.2231     | 1.1004 | .9495          | 1.0695              |
| .6931     | .1438               | 1.2500         | 2.5000     | 1.2000 | 1.0549         | 1.0889              |
| 0         | 0                   | 3              | 3          | 4/3    | $\ln 3$        | $\ln 3$             |

Table 1 : Comparison of the partition functions  $Z$ , and entropies  $S$  of the two systems defined in (16), at the same values of the average energy  $E^*(\beta^*) = E(\beta)$ ; The quantities belonging to  $\mathbf{h}$  are denoted by \*.

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## QUANTUM THERMODYNAMICS IN CLASSICAL PHASE-SPACE

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The path-integral formulation of quantum statistical mechanics [1] constitutes an useful tool in order to reduce quantum statistical mechanics to classical-like calculations, based on an effective phase-space distribution which, in the classical limit, bears the usual form  $e^{-\beta H(p,q)}$ , where  $H(p,q)$  is the classical hamiltonian function. Indeed, the classical distribution is recovered by considering the only paths with minimal action, so it is expected that an expansion accounting for the neighboring paths gives the main modifications due to quanticity. In this way it turns out to be possible to define effective hamiltonians (more often, effective potentials) in terms of which the thermodynamic quantities of anharmonic systems can be expressed, and calculated by any classical technique, overcoming the strong difficulties of direct quantum mechanical approaches.

With this aim, Feynman [1] introduced a free-particle-like trial action  $S_0$  with variational parameters to be determined by the so-called Feynman-Jensen inequality:  $F \leq F_0 + \beta^{-1} \langle S - S_0 \rangle_{S_0}$ , where  $F$  and  $F_0$  are the free energies associated with the actions  $S$  and  $S_0$ , respectively. This method, however, is limited to almost classical systems, since it lacks the very desirable property of exactly accounting for the behavior of harmonic systems.

This shortcoming has been overcome a few years ago, when a variational approach using the same basic inequality and a quadratic trial action, was proposed and implemented in the calculation of the quantum specific heat of the sine-Gordon field [2]. The results for the case of one particle in an anharmonic  $\varphi^4$  potential were independently recovered also by Feynman and Kleinert [3]. Afterwards, a number of successful applications has been made to various model systems [4-7]. Among them, single particles in different potentials [4,5], nonlinear Klein-Gordon models in one space dimension [6], a unidimensional solid with Lennard-Jones (LJ) interaction [7]. In the last case it has been shown that the heavy quantum Monte Carlo calculations by McGurn et al. [8] on the same model could be reproduced within just a few seconds on a personal computer. It has also been shown that the method correctly reproduces the results of the self-consistent quantum harmonic approximation at lowest temperatures [6], and the results of the Wigner effective potential [9] in the high temperature limit up to order  $\beta^4 \hbar^6$  [4].

Further improvements regarded the calculation of quantum static correlation functions [10] and the extension of the formalism to the general hamiltonian case [11] (whereas previously only hamiltonians with a separated quadratic kinetic dependence on the conjugated momenta were considered), thus using the full phase-space formulation and opening the way to the treatment of complicated systems like spin chains [12].

Recently the method was used to obtain results for realistic nonlinear solid state systems [13,14], and because of the limited space in this paper we will shortly report

about this topic. We consider solid argon, a rare gas solid whose interaction potential is reasonably well understood [15], represented by a LJ-model. The full hamiltonian is

$$H = \sum_1 \frac{m}{2} \dot{\mathbf{x}}_1^2 + V(\mathbf{X}) = \sum_1 \frac{m}{2} \dot{\mathbf{x}}_1^2 + \frac{1}{2} \sum_1 \sum_n \sum_{\mathbf{d}_n} u(|\mathbf{x}_{1+\mathbf{d}_n} - \mathbf{x}_1|). \quad (1)$$

$$u(r) = 4\epsilon \left[ \left( \frac{\sigma}{r} \right)^{12} - \left( \frac{\sigma}{r} \right)^6 \right], \quad (2)$$

The  $N$  atoms are labeled by their equilibrium position  $\mathbf{l}$ ;  $\mathbf{x}_1$  is the position of the atom at lattice site  $\mathbf{l}$ ; and the sum over  $n = 1, 2, \dots$  accounts for the successive shells of neighbors of atom  $\mathbf{l}$  which, in equilibrium, are at relative positions  $\mathbf{d}_n$ .

It turns out as a result of the variational approach [7,14] that the quantum partition function is at best approximated by the configurational integral

$$Z = \left( \frac{m}{2\pi\hbar^2\beta} \right)^{\frac{3}{2}N} \int e^{-\beta V_{\text{eff}}(\mathbf{X})} d\mathbf{X}, \quad (3)$$

where

$$V_{\text{eff}}(\mathbf{X}) = \frac{1}{2} \sum_1 \sum_n \sum_{\mathbf{d}_n} u_n^{(\text{eff})}(|\mathbf{x}_{1+\mathbf{d}_n} - \mathbf{x}_1|) + \frac{1}{\beta} \sum_{\mathbf{k}\mu} \ln \frac{\sinh f_{\mathbf{k}\mu}}{f_{\mathbf{k}\mu}} \quad (4)$$

is the effective potential expressed in terms of an effective pairwise interaction

$$u_n^{(\text{eff})}(r) = u(r) + \frac{1}{2} [u''(r) - u''(d_n)] D_n^L + \frac{1}{2} \left[ \frac{u'(r)}{r} - \frac{u'(d_n)}{d_n} \right] D_n^T. \quad (5)$$

The logarithmic term in Eq. (4) contains the quantities  $f_{\mathbf{k}\mu} = \frac{1}{2} \beta \hbar \omega_{\mathbf{k}\mu}$ , and  $\omega_{\mathbf{k}\mu}$  are the phonon frequencies of the fcc lattice:

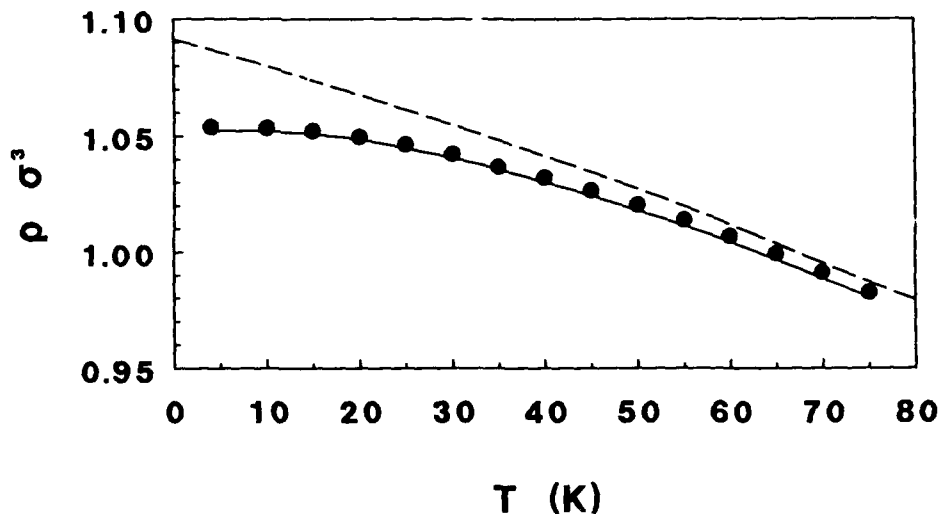
$$\begin{aligned} \omega_{\mathbf{k}\mu}^2 \delta_{\mu\nu} &= \sum_{ij} \epsilon_{\mu i}(\mathbf{k}) \omega_{\mathbf{k},ij}^2 \epsilon_{\nu j}(\mathbf{k}), \\ m \omega_{\mathbf{k},ij}^2 &= \sum_n \sum_{\mathbf{d}_n} u_{ij}(d_n) 2 \sin^2 \frac{\mathbf{k} \cdot \mathbf{d}_n}{2}, \end{aligned} \quad (6)$$

where the subscripts  $ij$  denote the derivatives of  $u(|\mathbf{x}|)$  and  $\mathbf{e}_\mu(\mathbf{k}) = \{\epsilon_{\mu i}(\mathbf{k})\}$  are the polarization vectors of the phonon modes. In Eq. (5)  $D_n^L$  and  $D_n^T$  are the longitudinal and transverse (with respect to  $\hat{\mathbf{d}}_n = \mathbf{d}_n/d_n$ ) "pure quantum" square fluctuations of the relative position  $\mathbf{x}_{1+\mathbf{d}_n} - \mathbf{x}_1$ :

$$\begin{aligned} D_n^L &= \sum_{\mathbf{k}\mu} (A_{\mathbf{k},1+\mathbf{d}_n} - A_{\mathbf{k}\mathbf{l}})^2 (\hat{\mathbf{d}} \cdot \mathbf{e}_\mu(\mathbf{k}))^2 \alpha_{\mathbf{k}\mu}, \\ D_n^T &= \sum_{\mathbf{k}\mu} (A_{\mathbf{k},1+\mathbf{d}_n} - A_{\mathbf{k}\mathbf{l}})^2 (1 - \hat{\mathbf{d}} \cdot \mathbf{e}_\mu(\mathbf{k}))^2 \alpha_{\mathbf{k}\mu}, \end{aligned} \quad (7)$$

where  $A_{\mathbf{k}\mathbf{l}}$  is the orthogonal matrix which realizes the Fourier transformation to  $\mathbf{k}$ -space and

$$\alpha_{\mathbf{k}\mu} = \frac{\hbar}{2m\omega_{\mathbf{k}\mu}} \left( \coth f_{\mathbf{k}\mu} - \frac{1}{f_{\mathbf{k}\mu}} \right). \quad (8)$$

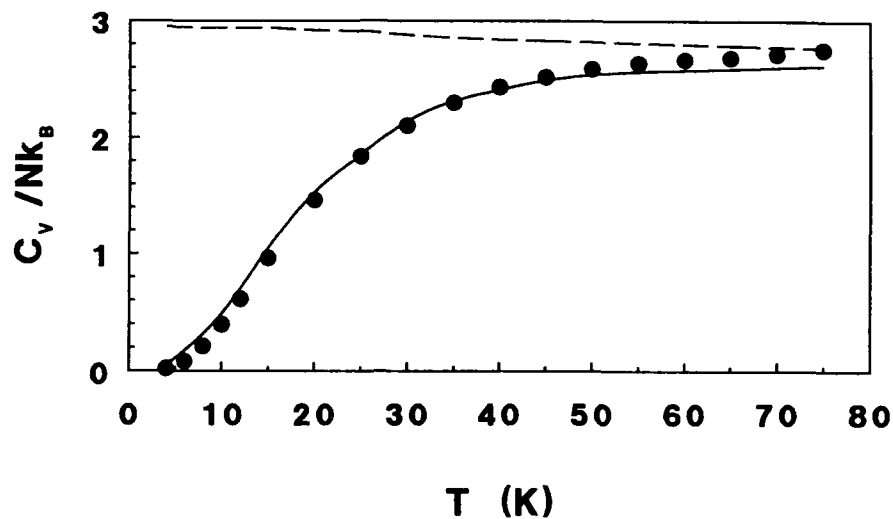


**Figure 1.** — Temperature dependence of the number density  $\rho$  for solid argon at zero pressure. Dashed line and solid line: classical MC and EPMC simulations with the LJ-model; circles: experimental data from Ref. [17].

By means of the above defined effective potential, the very long procedure needed to perform the path integral quantum Monte Carlo simulations (PIMC) can be shortened by a factor corresponding to the required Trotter number. Indeed the effective potential can be inserted in classical Monte Carlo simulations (EPMC) in order to calculate the partition function (3) and the related thermodynamic quantities. This has been done for a sample consisting of an fcc lattice of  $N = 108$  atoms enclosed in a cubical box to which periodic boundary conditions were applied. In these simulations the dynamical interaction beyond the nearest neighbor distance has been cut off and accounted for in a static approximation, appropriate for an infinite static fcc lattice. This cutoff correction allowed us to use the well-established LJ potential parameters for argon ( $\epsilon/k_B = 119.8$  K and  $\sigma = 3.405$  Å), which were determined from gas phase data [16] and give a reasonable representation of the true pair potential. With this simple procedure, excellent agreement with the experimental equation of state of solid argon is obtained, and the use of unphysical parameter values is avoided [13]. The results for the density at zero pressure and for the specific heat at constant volume are reported and compared with the corresponding experimental data [17] in the figures. In addition, the EPMC results are in excellent agreement with those obtained by PIMC [14]. However, taking the difficulties in obtaining "exact" PIMC results into account, the performance of the EPMC approach is impressive indeed [14].

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**Figure 2.** — Temperature dependence of the specific heat at constant volume  $C_V$  for solid argon at zero pressure. Dashed line and solid line: classical MC and EPMC simulations with the LJ-model; circles: experimental data from Ref. [17].

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## Magnetization of a Charged Particle Interacting with Reservoir

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The problem of investigation of the dissipation processes in quantum mechanical systems has a long history [1]-[5]. But in most of the papers the irreversible behaviour of the simplest one-dimensional system - an oscillator coupled with thermal bath consisting of a number of harmonic oscillators - has been studied. One purpose is to investigate the simple two-dimensional system - charged oscillator in a magnetic field in the presence of dissipation - in the framework of the Wigner function description of quantum systems [6]. According to [6] we assume that for the quadratic Hamiltonian  $H = \mathbf{qBq}$  all mean values are expressed through matrix  $\mathbf{R}$  which is a solution of the following equation [6]

$$\begin{aligned} \dot{\mathbf{R}} &= -\Sigma \mathbf{B} \mathbf{R}, \quad \Sigma_{\alpha\beta} = i\hbar^{-1} [q_\alpha, q_\beta] \quad \mathbf{R} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \\ \mathbf{R}(0) &= \mathbf{I} \end{aligned} \quad (1)$$

Besides that the bath to be presumed is at the equilibrium state at the initial moment.

We suppose the magnetic field to be directed perpendicular to the plane of motion of the oscillator. We assume that the oscillator is connected to the bath via the canonical momenta:

$$\begin{aligned} H &= \frac{\Omega^2(x^2 + y^2)}{2} + \frac{\pi_x^2 + \pi_y^2}{2} + \sum_k \frac{p_{kx}^2 + \omega_k^2 x^2}{2} + \sum_k \frac{p_{ky}^2 + \omega_k^2 y^2}{2} \\ &+ \sum_k \alpha_k (\pi_x p_{kx} + \pi_y p_{ky}) \end{aligned} \quad (2)$$

where  $\pi = \mathbf{p} - \frac{e\mathbf{A}}{c}$  is a canonical momentum,  $\mathbf{A}$  is the vector potential. The covariance matrix can be expressed in terms of matrix  $\mathbf{R}_{12}$  as follows

$$\begin{aligned} m_{qp} &= \sum_k f_k \left[ \omega_k^2 \left( \mathbf{R}_{12}^{qr} \mathbf{R}_{12}^{pk} + \mathbf{R}_{12}^{qN+k} \mathbf{R}_{12}^{oN+k} \right) \right. \\ &\quad \left. + \mathbf{R}_{12}^{q2N+k} \mathbf{R}_{12}^{p2N+k} + \mathbf{R}_{12}^{q3N+k} \mathbf{R}_{12}^{p3N+k} \right] \end{aligned}$$

where  $f_k = \frac{1}{2\omega_k} \coth \frac{\omega_k \hbar}{2T}$ ,  $k = 1, \dots, N$

The main problem is to calculate magnetic moment at the steady state ( $t \rightarrow \infty$ ):

$$M = \frac{e\hbar}{2c} (m_{23} - m_{14})$$

Solving equations (1) with the help of the Laplace transformation, we obtain at the end of rather complicated calculations the following expression

$$\begin{aligned} \mathbf{M}(t \rightarrow \infty) = & -\frac{e\hbar}{c} \sum_k \alpha_k^2 \omega_k^2 f_k \int_0^\infty \int_0^\infty d\tau d\tau' \cos \omega_k(\tau - \tau') \\ & \times \left[ \ddot{\bar{K}}_1(\tau) \left( \ddot{\bar{K}}_1 + \Omega^2 (\dot{\bar{K}}_1 - \dot{\bar{K}}_2) \right)_{\tau'} \right. \\ & \left. \left( \ddot{\bar{K}}_1(\omega^2 + \Omega^2) + (\bar{K}_1 - \bar{K}_2) \Omega^4 + \omega^2 \ddot{\bar{K}}_2 \right)_\tau \left( \ddot{\bar{K}}_1 - \ddot{\bar{K}}_2 \right)_{\tau'} \right] \omega^3 \end{aligned} \quad (3)$$

where the Laplace transforms  $\bar{K}(s) = \mathcal{L}[K(t)](s)$  are as follows:

$$\begin{aligned} \bar{K}_1^{-1}(s) &= -\omega^2 \left[ \Gamma(\Gamma - 1)(\Omega^4 + \omega^2 s^2) + s^4 \Gamma \right. \\ &\quad \left. + (s^4 + s^2(\omega^2 + 2\Omega^2) + \Omega^4)(1 - \Gamma) \right], \\ \bar{K}_2(s) &= \bar{K}_1(s) \Gamma(s), \\ \Gamma(s) &= \sum_k \frac{\omega_k^2 \alpha_k^2}{\omega_k^2 + s^2}, \quad \omega = \frac{e\mathcal{H}}{c}. \end{aligned}$$

$\mathcal{H}$  is a magnetic field. Let us consider the inverse Laplace transformation of the kernel  $\bar{K}(s)$

$$K(t) = \frac{1}{2\pi} \int_{-\infty - i\delta}^{\infty - i\delta} dy \bar{K}(s = iy) e^{iyt}, \quad \delta > 0, \quad (4)$$

where the integration contour lies in the lower half-plane. If the number of oscillators of the bath is finite then the integral (4) equals the sum of residues at the poles lying at the real axis. In the continual limit all poles merge and form a cut in the complex plane along the real axis, so that the integration contour turns into the path around the cut from  $-\infty$  to  $\infty$  along the down shore and then from  $\infty$  to  $-\infty$  along the up shore. For this reason in the continual limit the single function  $\Gamma(s)$  gives birth to two functions  $\Gamma_+$  and  $\Gamma_-$  differing in the meanings of the variable  $y$  on the down or up sides of the cut  $\Gamma_\pm = \sum_k \frac{\omega_k^2 \alpha_k^2}{\omega_k^2 - (y - i0)^2}$ . Accordingly we obtain two functions  $\bar{K}_\pm = \bar{K}(\Gamma_\pm)$ . Formula (4) takes the form

$$K_{1,2}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dy e^{iyt} [\bar{K}_{1,2-}(iy) - \bar{K}_{1,2+}(iy)]. \quad (5)$$

The character of the approach to the equilibrium state is determined completely by the function  $\Gamma_\pm$ . If we suppose that the spectrum of the bath oscillators is very dense and makes the continual limit by substitution  $\sum_k = \int d\omega \epsilon(\omega)$  with the density function  $\epsilon(\omega)$  satisfying the relation

$$\epsilon(\omega) \alpha^2(\omega) \pi = \frac{2\gamma}{\omega^2 + \Lambda^2}, \quad \gamma, \Lambda = \text{const},$$

then  $\Gamma_{\pm}(iy) = \frac{\gamma}{\Lambda \mp iy}$ . The positivity condition of the Hamiltonian (2) puts the following limitation on the constants  $\gamma$  and  $\Lambda$ :  $\gamma < \Lambda$ . Then only the  $\bar{K}_{-}(iy)$  function contains poles lying in the upper half plane. By evaluating integral (5) we obtain (in dimensionless form)

$$K_1(t) = \sum_{n=1}^b \frac{[\lambda - \varphi_n]^2 e^{-\varphi_n t}}{\prod_{j \neq n} (\varphi_n - \varphi_j)}; \quad K_2(t) = \sum_{n=1}^b \frac{\alpha [\lambda - \varphi_n] e^{-\varphi_n t}}{\prod_{j \neq n} (\varphi_n - \varphi_j)} \quad (6)$$

where  $\varphi = y/i\omega$ ,  $\alpha = \gamma/\omega$ ,  $\lambda = \Lambda/\omega$ ,  $b = \Omega/\omega$  and the values  $\varphi_n$ ,  $n = 1, \dots, b$  are solutions of the algebraic equation

$$\alpha^2 [b^4 + \varphi_n^2] - 2\alpha [\lambda - \varphi_n] [b^4 + \varphi_n (b^2 + 1)] + [\varphi_n^4 + \varphi_n^2 (2b^2 + 1) + b^4] [\lambda - \varphi_n]^2 = 0 \quad (7)$$

Substituting expressions (6) into the formula (3) we obtain

$$\mathbf{M} = -\frac{e\hbar}{2c} \alpha \sum_{n=1}^b \sum_{m=1}^b \frac{\Theta_{nm}}{\varphi_n + \varphi_m} \left[ -\frac{2\lambda + \varphi_n + \varphi_m}{2\lambda\beta(\varphi_n + \lambda)(\varphi_m + \lambda)} + \varphi_m \frac{\Psi\left(\frac{\beta}{\varphi_n}\right) - \Psi\left(\frac{\beta}{\varphi_m}\right)}{\pi(\varphi_m^2 - \lambda^2)} + \varphi_n \frac{\Psi\left(\frac{\beta}{\varphi_n}\right) - \Psi\left(\frac{\beta}{\varphi_m}\right)}{\pi(\varphi_n^2 - \lambda^2)} \right] \quad (8)$$

where  $\beta = \frac{\hbar\omega}{2T}$ ,  $\Psi(x) = \frac{d}{dx} \ln \Gamma(x)$ . Expression (8) is the exact formula for the magnetic moment of the charged oscillator in the magnetic field in the presence of an arbitrarily strong dissipation. In the weak coupling case  $\frac{\alpha\lambda}{\lambda^2+1} \ll 1$  it is possible to solve equation (7) in analytic form. In this limit we obtain the following corrections to Landau formula for the magnetic moment of a free charged particle ( $b \rightarrow 0$ ) in the magnetic field:

a) in the high temperature limit ( $\beta \ll 1$ )

$$\mathbf{M} \simeq -\frac{e\hbar}{2c} \beta \left( \frac{1}{3} - \frac{\alpha\beta}{\pi^3} \xi(3) \right) \quad (9)$$

where  $\xi(x) = \sum_{n=1}^{\infty} 1/n^x$ .

b) in the low temperature limit ( $\beta \gg 1$ )

$$\mathbf{M} = -\frac{e\hbar}{2c} \left[ 1 + \frac{2\alpha}{\lambda^2 + 1} \left( \frac{\lambda(2 - \lambda^2)}{2(1 + \lambda^2)} - \frac{1}{\pi} \left( 1 + \frac{1 - \lambda^2}{1 + \lambda^2} \ln \lambda \right) - \frac{1 + \lambda^2}{4\lambda^2\beta^2} \right) \right] \quad (10)$$

All presented calculations of the magnetic moment are directly applicable to the system obeying Boltzmann statistics. But it is possible to obtain a generalization to Fermi statistics case if we make use of the Rumer method [7, 8]. According to this method for the single particle density matrix exists integral representation:

$$\hat{\rho} = \frac{1}{\exp\{\xi(\hat{H} - \eta)\} + 1} = \frac{1}{2i\xi} \int_{\sigma-i\infty}^{\sigma+i\infty} dx \frac{e^{xy} e^{-x\hat{H}}}{\sin \frac{\pi x}{\xi}} \quad 0 < \sigma < \xi, \quad (11)$$

where  $\mu$  is a chemical potential,  $\hat{H}$  is the Hamiltonian of the system  $\xi = 1/kT$ . Using representation (11) and formula (9) in the case of strongly degenerated gas in a weak magnetic field  $\mu \gg T \gg \mu\mathcal{H}$ ,  $\gamma \ll \omega$ ,  $\mu = e\mathcal{H}/2mc$  we obtain (after rather complicated calculations) the following expression for the Fermi-gas magnetization  $\langle M \rangle_F = \langle M \rangle_F^s + \langle \tilde{M} \rangle_F$

$$\langle M^s \rangle_F = \frac{\mu^2 \mathcal{H}}{\mu} \left[ 1 + \frac{3\hbar\gamma\xi(3)}{8\pi^3\eta} \right]$$

$$\langle \tilde{M} \rangle_F = -\frac{\mu^2 \mathcal{H} 3\pi}{\xi \omega^{3/2}} \frac{1}{\sqrt{2\mu\hbar^3}} \sum_{n=1}^{\infty} \frac{(-1)^n \cos \frac{2\pi n \mu \mathcal{H}}{\hbar \omega} \sin \left( \frac{2\pi n \mu}{\hbar \omega} - \frac{\pi}{4} \right)}{\sinh^2 \left( \frac{2\pi^2 n}{\hbar \omega \xi} \right) \sqrt{n}} \left[ 1 + \frac{n^2 \hbar \gamma}{\mu} \right]$$

We see that in the case of weak interaction between Fermi gas and thermal bath the total paramagnetic moment (both its regular part and its amplitude of oscillations) increases. Perhaps, this effect can be explained by a suppression of the Landau diamagnetism (9). This suppression results in the increase of a total magnetic moment. Besides, we have assumed that coupling with thermal bath does not affect to the spin state of a gas.

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Wigner Crystallisation in One Component Coulomb Plasmas  
Some rigorous Estimates

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1. Introduction

In the regime of statistical mechanics of classical particles interacting via short range potentials, crystallisation in one and two dimensions is ruled out by the original Mermin argument. Matters are, however, far less transparent in the case of long range interactions (e.g. Coulomb), as most of the technical machinery is not directly applicable in this context. There exists in fact an old conjecture of Wigner, that electron systems may crystallize at sufficiently low temperatures.

Unfortunately, calculations for e.g. Coulomb systems turn out to be rather tedious and one has to develop a variety of relatively ingenious methods in order to arrive at really reliable results. In the following we want to treat, as a case in point, the so called one component Coulomb plasma (jellium), i.e. electrons being immersed in a positive neutralizing background.

In order to get quantitative results (and as the mere existence of the infinite volume limit is far from being trivial, cf. e.g. /1/), we start with the system being confined to a finite volume  $V$  and perform the limit  $V \rightarrow \infty$  in the end. Furthermore we impose periodic boundary conditions and an exterior field which fixes the position of the crystal (if it exists at all) and which is switched off in the end.

Remark: Note that this makes the Coulomb potential non-trivially dependent on the volume  $V$  (see below)!

2. The Model

With the kinetic energy integrated out the Hamiltonian reads:

$$H_V(r_1, \dots, r_N) = q^2 \sum_{i < j} \phi_V(r_i, r_j) - q\bar{\rho} \sum_i \int dx \phi_V(r_i, x) + \frac{1}{2\bar{\rho}^2} \int dx dy \phi_V(x, y) + \lambda \sum_i \phi_{ext.}(r_i) \quad (1)$$

We choose the canonical ensemble,  $-q$  denotes the charge of the electron,  $\bar{\rho}$  the neutralizing positive background charge density,  $\phi_V$  the adjusted (!) Coulomb potential.

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In the following we want to treat the whole class of possible scenarios, both with respect to space dimension  $d$  and 'type' of Coulomb potential, i.e:

| potential: | <u>ordinary plasma</u> | <u>surface plasma</u> |
|------------|------------------------|-----------------------|
| $d = 1$    | $\sim - x $            | $\sim +\ln r $        |
| 2          | $\sim +\ln r $         | $\sim 1/ r $          |
| 3          | $\sim 1/ r $           | etc.                  |

Remark: Some special cases have been studied e.g. in /2/ to /4/. On a broader scale and improving the various known estimates considerably (presumably optimizing them) the problem was attacked by ourselves in /5/.

### 3. General Ideas about Breaking of Translation Invariance in the Classical Regime

As to the general strategy applied by us cf. e.g. /6/ and further references given there. (in particular concerning the systematic use of Poisson brackets). We employ the following criterion of crystallinity: In the expansion of the one particle density,  $\rho(r) = \sum_K \exp(iKr) \cdot \hat{\rho}(K)$ ,  $K$  the vectors of the reciprocal lattice, occurs at least one  $K$  with  $\hat{\rho}(K) \neq 0$ .

The onset of crystallisation is signalled by a specific singularity structure of the Fourier transform of the pair correlation function  $\rho^{(2)}(r, r')$  resp. descendants thereof, e.g. the so called structure factor,  $S(K+k)$ , near some  $K$ , i.e. for  $k$ , taken from the first Brillouin zone, to zero.

In our case this will be employed with the help of the following (Mermin-type) inequality:

$$S(K+k) \geq [(K+k) \cdot e_t]^2 \cdot |\hat{\rho}(K)|^2 / [(k \cdot e_t)^2 + D_{tt}(k)] \quad (2)$$

with  $e_t$  an arbitrary unit vector, in the direction of which the overall momentum is taken,  $D_{tt}$  being related to a 'double commutator' between momentum  $P$  and Hamiltonian  $H$ . The main task consists in fact in estimating effectively this relatively complicated quantity.

#### 4. Some Calculational Intermediate Steps

In a first step one has to determine the 'effective' Coulomb potential of a given configuration of electrons and compensating background under periodic boundary conditions. The strategy is to solve the corresponding Poisson equation in the volume  $V$  under periodic b.c:

$$\Delta\phi(r) = -\text{const.} \cdot q \cdot [\sum_i \delta(r-r_i) - N/V] \quad (3)$$

by Fourier methods. This leads, in the end, to the potential energy of the microscopic configuration (ordinary jellium,  $m=2$ ; surface jellium,  $m=1$ ):

$$U = U_0 + 1/2 \cdot \text{const.} \cdot \sum_{i \neq j} \sum_{k \neq 0} q^2/V \cdot |k|^{-m} \cdot \exp(ik(r_i - r_j)) \quad (4)$$

with  $U_0$  a self-energy counter term;  $\phi_V(r) = \sum_{k \neq 0} q^2/V |k|^{-m} \cdot e^{ikr}$  may be viewed as the effective (Coulomb) pair potential.

In a next step we have to estimate in a subtle way (as most of the occurring expressions are at the border of being almost ill-defined due to the long range of the Coulomb interaction) the intricate behavior of  $D_{tt}(k)$  for  $k \rightarrow 0$  as  $V \rightarrow \infty$ . It turns out that

$$D_{tt}(k) \sim \int_V d^d r \bar{\rho}h(r) (1 - \cos kr) \cdot (e_t \cdot \nabla)^2 \phi_V(r) \quad (5)$$

with  $\bar{\rho}h(r) = N^{-1} \cdot \int dr' \rho^{(2)}(r', r'-r)$  a certain descendant of the pair correlation function.

As  $h(r)$  does not (!) decay at infinity one has to split it into an asymptotically oscillating and a decaying part which will then be estimated separately in Fourier resp. position space. In doing this we use the theory of harmonic functions, elliptic regularity, asymptotic Fourier analysis. In a last step we develop a (new) recursive optimisation procedure with a variant of the 'Mermin inequality' as input.

#### 5. Results and Conclusions

We briefly describe our results, the details of which can be found in /5/.

i) For ordinary Coulomb plasma crystallisation is incompatible with a decay of pair correlation

$$|\rho_T^{(2)}(r', r'-r)| \lesssim \begin{cases} (d=2): |r|^{-(1+\epsilon)}, \epsilon > 0 \\ (d=3): |r|^{-(3/2+\epsilon)}, \epsilon > 0 \end{cases} \quad (6)$$

ii) For surface Coulomb plasma the same holds with

$$|\rho_T^{(2)}(r', r'-r)| \lesssim \begin{cases} (d=2): |r|^{-(1/2+\epsilon)}, \epsilon > 0 \\ (d=1): |r|^0 \end{cases}$$

Our conclusion is that, in contrast to short range interactions, crystallisation is not (!) a priori ruled out for Coulomb systems in lower dimensions (apart from the surface plasma in  $d=1$ , as  $\rho_T^{(2)}$  has to decay in a pure phase on a priori grounds), but is accompanied by a very slow decay of the pair correlation function. This slow decay may then be critical for certain integral expressions which contain  $\rho_T^{(2)}$  as e.g. mean free energy density etc. Therefore a fine tuning will then be necessary in order that various quantities remain well defined in the thermodynamic limit.

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DYNAMIC DEPINNING OF 2D ELECTRON WIGNER CRYSTAL  
AT A SOLID STATE HETEROJUNCTION

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Wigner [1] predicted the phase transition from a liquid (disordered) state to a solid (crystalline) one in an electron Coulomb plasma. Such a transition was experimentally observed in a 2D case - in a low density nondegenerate 2D electron system at a liquid helium surface [2] and in a high density degenerate 2D electron system at a high-quality solid state GaAs/AlGaAs heterojunction in a strong magnetic field at low temperature [3,4]. In both cases a shear mode can propagate in 2D electron system, which is a defining property of a solid (crystalline) state. In the absence of magnetic field the shear mode in the system is dispersionless ( $\omega^{(0)}(k) = (\mu/mn_s)^{1/2}k \equiv c_t k$ ), but in a strong perpendicular magnetic field a gapless low-frequency magnetophonon branch has a characteristic dispersion

$$\omega(k) = \omega_t^{(0)}(k)\omega_p(k)/\omega_c = c(2\pi\mu)^{1/2}k^{3/2}/H, \quad \omega_c \gg \omega_p, \quad (1)$$

where  $\mu$ ,  $m$  and  $n_s$  are the shear modulus, effective mass and charge concentration of 2D electron solid,  $\omega_c = eH/mc$  and  $\omega_p(k) = [2\pi e^2 n_s k / \epsilon m]^{1/2}$  are the cyclotron and 2D plasmon frequencies.

The pinning of 2D electron Wigner crystal in a random potential of a solid state heterojunction breaks translational periodicity and creates a restoring force with respect to the host milieu [5]. In such a way the pinning has the effect of introducing a gap into the nominally gapless excitation branches. The recent experiments [6] reveal a sharp insulator-conductor threshold at the same boundary at which radio-frequency (rf) resonances signal the onset of transverse elasticity. These experiments established a quantitative relation between threshold conduction field and rf resonance frequency which is well accounted for by a model of pinned 2D electron Wigner crystal. Thus the effect of pinning on the magnetophonon frequencies was established in the experiments.

The dynamic equations of 2D Wigner crystal in external magnetic field have the following form, which in general case takes into account an effect of pinning on its dynamics:

$$m \dot{\mathbf{v}}_t = e\mathbf{E}_t + (e/c)[\mathbf{v}_t, \mathbf{H}] + \mathbf{f} \quad (2)$$

where  $\mathbf{v}_t$  and  $\mathbf{E}_t$  are the tangential components of electron velocity and electric field (the Greek indices  $\alpha, \beta = 1, 2$ ),

$$\mathbf{f}_\alpha = - (i/\omega) [C_{\alpha\beta\gamma\delta} v_{\gamma\delta, \beta} + A_{\alpha\beta} v_\beta], \quad v_{\alpha\beta} = 1/2 (v_{\alpha, \beta} + v_{\beta, \alpha})$$

are the elastic forces due to the lateral electron-electron interaction (the elastic tensor  $C_{\alpha\beta\gamma\delta}$ ) and due to the electron-substrate interaction (the tensor  $A_{\alpha\beta}$  of the pinning forces  $A'_{\alpha\beta}$  and of the electron-substrate friction  $A''_{\alpha\beta} = -i\omega G_{\alpha\beta}$ ,  $G_{\alpha\beta} = m\delta_{\alpha\beta}/\tau$ ,  $\tau$  is a characteristic momentum-relaxation time of electrons in 2D crystal). The explicit form of the dynamic matrix  $C_{\alpha\beta\gamma\delta}$  of 2D Wigner crystal was derived by Bonsall and Maradudin [7].

For the plane wave (propagating in the  $x$  direction) we can exclude the elastic and the relaxation terms from the dynamic equations (2) by introducing the anisotropic effective electron mass

$$m_{xx,yy} = m \{1 - (\omega_0^2 + c_{1,t}^2 k^2)/\omega^2 + i/(\omega\tau)\}, \quad (3)$$

where  $\omega_0^2 = A_{xx}/(n_s m) = A_{yy}/(n_s m)$  is the square of the bare (for  $H = 0$ ,  $k = 0$ ) gaps in both longitudinal and transverse branches of Wigner crystal oscillations due to the pinning in a solid state heterojunction. The anisotropic effective mass tensor (3) enters the tensor of 2D magnetoconductivity of the electron layer  $\sigma_{\alpha\beta} = \sigma_{\alpha\beta}(\omega, H)$ :  $\mathbf{j}_\alpha = en_s v_\alpha = \sigma_{\alpha\beta} \mathbf{E}_\beta$ .

If the 2D electron Wigner crystal is placed between two semiconductor layers ( $\epsilon_1, d_1$  and  $\epsilon_2, d_2$ ) with metallized external surfaces (a solid state heterojunction), then we have to satisfy the boundary conditions (BC) of zero tangential electric field at metal surfaces ( $\mathbf{E}_t = 0$  at  $z = -d_1$  and  $z = d_2$ ) and the BC on the jump of the tangential magnetic field at the 2D electron layer due to the surface current  $\mathbf{j}_t$  ( $[\mathbf{H}_1 - \mathbf{H}_2, \mathbf{n}] = (4\pi/c)\mathbf{j}_t$  at  $z = 0$ ).

In the collisionless regime  $\omega\tau \gg 1$  the dispersion equation (DE) of the coupled magneto-phonon modes may be written in the following form:

$$\omega^4(1 + (\Omega/cQ)) - \omega^2 [(1 + (\Omega/cQ))(cP\Omega + c_1^2 k^2 + \omega_0^2) + \omega_c^2 + c_t^2 k^2 + \omega_0^2] + (\omega_0^2 + c_t^2 k^2)(cP\Omega + c_1^2 k^2 + \omega_0^2) = 0, \quad (4)$$

where

$$\Omega = (4\pi n_s e^2)/(mc), \quad q_{1,2} = [k^2 - (\epsilon_{1,2} \omega^2/c^2)]^{1/2}$$

$$P = [(\epsilon_1/q_1) \text{cth} q_1 d_1 + (\epsilon_2/d_2) \text{cth} q_2 d_2]^{-1},$$

$$Q = q_1 \text{cth} q_1 d_1 + q_2 \text{cth} q_2 d_2.$$

In Eq.(4) the parameters  $cP$ ,  $cQ$  and  $\Omega$  are the characteristic frequencies of a bounded and a screened 2D electron system (with the finite thicknesses  $d_{1,2}$  of metallized semiconductor cladding). In the case of nonzero shear rigidity, when  $(c_t, \omega_0) > 0$ , the Eq (4) describes two branches of the coupled magnetophonon oscillations: a (quasi) longitudinal  $\omega_1(k)$  and a (quasi) transverse  $\omega_2(k)$  ones.

In the case of a weak collective Coulomb forces, when  $\Omega \ll \omega_c$ , the both branches have the gaps (for  $k = 0$ ) and their spectra also possess the dispersionless domains:

$$\omega_1 \approx \omega_c \gg \omega_2 \approx (\omega_0^2/\omega_c). \quad (5)$$

In the case of a strong collective Coulomb forces, when  $\Omega \gg \omega_c$ , the both branches are gapless and a transverse one possesses a characteristic dispersion (for  $kd \gg 1$ ):

$$\omega_t(k) \approx (\omega_0/\omega_c)(cP\Omega)^{1/2} \propto k^{1/2}/H. \quad (6)$$

Thus due to the collective Coulomb forces it happens the dynamic depinning of 2D electron Wigner crystal in a solid state heterojunction: the gaps in the coupled magneto-phonon branches disappear. It is worth mention that the frequencies  $\omega_1$  and  $\omega_2$  (5) are the characteristic frequencies of a charged oscillator in an external magnetic field. The screening of a 2D electron system (in the case  $kd \ll 1$ ) suppresses in general case the influence of the collective Coulomb forces on the coupled magnetophonon oscillations.

According to the available experimental data [6] in a GaAs/AlGaAs heterojunction one deals with a weak collective Coulomb forces in so far as  $\Omega \approx \omega_0 \approx (10^{10} + 10^{11}) \text{ s}^{-1} < \omega_c \approx 10^{12} \text{ s}^{-1}$  (for  $H = 5\text{T}$ ). In this case a coupled magneto-phonon spectrum has to present the finite gaps for  $k = 0$ .

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## VIII. Quantum Fields and Particles

## Relativistic generalization of the Breit-Wigner formula

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**Abstract.** We discuss the interaction of two Klein-Gordon or two Dirac particles, not as a problem of QCD, but of relativistic quantum mechanics. To keep the problem invariant under the Poincaré group we restrict the interaction to the space-time point of coincidence of the two particles, and assume that there a single compound particle is formed. From considerations of conservation of probability we derive the boundary condition which connects the two particle wave function, at the point where the two particles coincide, with the compound particle. In this way we derive cross-sections of scattering that can be associated with the one-level Breit-Wigner formula.

**1. Introduction**

The history behind this paper may be more interesting than the paper itself, particularly in a Conference that bears the name of Wigner. Thus I will take a few lines to tell it.

In 1946-1949 I was a graduate student in Princeton and my advisor was Eugene P. Wigner. In our weekly discussions Wigner used to touch on many subjects as his versatility is known to many of you.

One subject to which he returned many times was the one of interactions between particles, two of them to begin with, that was relativistically invariant. Wigner was of course familiar with the quantum electrodynamics of his time and I assume that later he also learned of at least the group theoretical background of quantum chromodynamics. Yet it was clear that he wanted a simpler representation of the interaction of particles.

When he mentioned his idea of interaction only at a space-time point of coincidence of two particles, it brought to me a responsive chord. As an undergraduate I

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got interested in vibrating systems of different dimensions, like the vibrating string (one dimension) with a point mass (0 dimension) or as the drum, where the air in the cylinder is three dimensional while the membrane at the end is two dimensional.

I thus thought that here I had to deal with the eight dimensional space time of two particles, while the interaction took place only in the four dimensional subspace-time where the particles coincide.

How to take into account this interaction? First of course pass to the center of mass and relative space-time coordinates. The former give rise to a plane wave and, in the center of mass frame, to just an irrelevant phase factor. The later then represent free relative motion so long as the relative coordinate does not vanish, but at the point of coincidence of the two particles one must impose a boundary condition.

I developed the appropriate relativistic formalism and the paper was accepted as a Ph.D. thesis. Unfortunately, the interaction I managed to introduce in all cases: Two Klein-Gordon, one Klein-Gordon and one Dirac, or two Dirac particles, did not actually include the resonance term that is so relevant for the Breit-Wigner formula, already known from the 1936 paper<sup>1)</sup>, and applied to non-relativistic interactions of neutrons with nuclei.

Due to this defect a paper on my Ph.D. thesis was never published. On my return to México I realized though how a resonance effect could be introduced in a non-relativistic version of my formalism, and developed what, in México, is known as the schematic theory of nuclear reactions.

At that time I was already engaged in other problems and did not have the interest to extend the ideas mentioned in the previous paragraph to the relativistic problem.

It was not until the second half of last year when giving a course on "Relativistic quantum mechanics from a different point view" that I reread my Ph.D. thesis and with a student, Guadalupe López Lurrabaquio, achieved what would actually could

have been my objective in 1949, a Relativistic Generalization of the Breit-Wigner formula.

This will be the subject of my paper and I will only give you the simplest examples, starting with the non-relativistic case and then the case of two Klein-Gordon particles. For the more interesting situation of two Dirac particles I will only present results for the  $R$  matrix and the scattering cross section.

## 2. A non relativistic point interaction model for deriving the one level Breit-Wigner formula

We want in this section to discuss a non-relativistic model in which two particles of masses  $m_1, m_2$  interact at the point of coincidence forming a compound particle of mass  $M$ . This model, introduced long ago,<sup>2)</sup> will be generalized in the next section to discuss the interaction of two relativistic Klein-Gordon particles at their point of coincidence, again forming a compound particle.

Our first step will be to consider, in the frame of reference where the center of mass is at rest, a state having two components

$$\Psi = \begin{bmatrix} \psi(\mathbf{r}, t) \\ \phi(t) \end{bmatrix}, \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \quad (2.1)$$

where the first one  $\psi(\mathbf{r}, t)$  represents the two particles with  $\mathbf{r}$  being the relative coordinate, while the second one  $\phi(t)$  will correspond to the compound particle. Note that in the frame of reference we chose, neither component depends on the center of mass coordinate.

The equation satisfied by  $\psi(\mathbf{r}, t)$  is

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2\mu} \nabla^2 \psi, \quad r \neq 0 \quad (2.2)$$

where we took units in which  $\hbar, c$  and some given mass, say of the electron, are given by 1. The  $\mu$  appearing in (2.2) is the reduced mass

$$\mu = m_1 m_2 (m_1 + m_2)^{-1} \quad (2.3)$$

How can we get the interaction between  $\psi(\mathbf{r}, t)$  at  $\mathbf{r} = 0$  and  $\phi(t)$ . We shall use for this purpose the argument of conservation of probability for our full state  $\Psi$  of (2.1). Considering two different states  $\bar{\Psi}, \Psi$  their scalar product can be defined by

$$(\bar{\Psi}, \Psi) = \int \bar{\psi}^*(\mathbf{r}, t)\psi(\mathbf{r}, t)d^3x + \bar{\phi}^*(t)\phi(t) \quad (2.4)$$

The derivative of  $(\bar{\Psi}, \Psi)$  with respect to time should be 0 and thus we get the

$$\frac{1}{i} \frac{d(\bar{\Psi}, \Psi)}{dt} =$$

$$\left[ \frac{2\pi}{\mu} \left( \frac{\partial r \bar{\psi}}{\partial r} \right) \right]_0^* (r\psi)_0 - (r\bar{\psi})_0^* \left[ \frac{2\pi}{\mu} \left( \frac{\partial r \psi}{\partial r} \right) \right]_0 + \bar{\phi}^* \left( \frac{1}{i} \frac{\partial \phi}{\partial t} \right) - \left( \frac{1}{i} \frac{\partial \phi}{\partial t} \right)^* \phi = 0 \quad (2.5)$$

where we restricted ourselves to s-waves, used the time dependent Schrödinger equation (2.2) and Gauss theorem on a sphere surrounding  $r = 0$  <sup>2)</sup>.

As we are interested in states of definite energy we can write

$$\psi = f(r) \exp(-iE_k t), \quad \bar{\psi} = \bar{f}(r) \exp(-i\bar{E}_k t) \quad (2.6a, b)$$

$$\phi = \varphi \exp(-iE_k t), \quad \bar{\phi} = \bar{\varphi} \exp(-i\bar{E}_k t) \quad (2.6c, d)$$

where  $E$  carries the index  $k$  to indicate that it is only the kinetic energy. The equation (2.5) becomes then

$$\bar{x}_1^* x_3 - \bar{x}_3^* x_1 + \bar{x}_2^* x_4 - \bar{x}_4^* x_2 = 0 \quad (2.7)$$

where

$$x_1 = \left[ \frac{2\pi}{\mu} \left( \frac{\partial r f}{\partial r} \right) \right]_0, x_2 = \varphi \quad (2.8a, b)$$

$$x_3 = [r f(r)]_0, x_4 = -E_k \varphi \quad (2.8c, d)$$

The bilinear form (2.7) can be satisfied by the linear equations<sup>2)</sup>

$$x_3 = c_{11}x_1 + c_{12}x_2, \quad x_4 = c_{21}x_1 + c_{22}x_2 \quad (2.9a, b)$$

where  $c \equiv \|c_{ij}\|$  is a constant  $2 \times 2$  hermitian matrix.

The stationary problem satisfies then the equation

$$-(2\mu)^{-1}\nabla^2 f = E_k f, \quad r \neq 0 \quad (2.10)$$

together with the boundary conditions at  $r = 0$  given by

$$(rf)_0 = c_{11}(2\pi/\mu)(\partial r f/\partial r)_0 + c_{12}\varphi \quad (2.11a)$$

$$-E_k\varphi = c_{21}(2\pi/\mu)(\partial r f/\partial r)_0 + c_{22}\varphi \quad (2.11b)$$

If there is only coupling through the formation of a compound particle  $c_{11} = 0$ . If besides also  $c_{12} = 0$ , then from (2.11b)  $c_{22}$  should be the negative of the energy associated with the mass of the compound particle at rest from which we subtract the masses of the two initial particles. Denoting this energy by  $E_0$  we have

$$c_{22} = -E_0 \equiv -[M - (m_1 + m_2)]. \quad (2.12)$$

Eliminating  $\varphi$  between equations (2.11a) and (2.11b) we get

$$(rf)_0 = R(E_k)(\partial r f/\partial r)_0, \quad R(E_k) \equiv (2\pi|c_{12}|^2/\mu)(E_0 - E_k)^{-1} \quad (2.13a, b)$$

that gives the explicit value of what is called the  $R$  matrix<sup>3)</sup> (in this case just  $1 \times 1$ ) of the problem.

As we are dealing only with an  $s$ -wave, the radial function  $f(r)$  multiplied by  $r$  is given by

$$rf(r) = e^{-ikr} - S(k)e^{ikr}, \quad (2.14)$$

where  $S(k)$  is the  $S$  matrix<sup>3)</sup> (also  $1 \times 1$  in this case) of the problem which from (2.13a) takes the value

$$S(k) = \frac{1 + ikR(E_k)}{1 - ikR(E_k)} \quad (2.15)$$

As the cross section  $\sigma$  is given for  $s$ -waves in terms of the  $S(k)$  by

$$\sigma = \frac{\pi}{k^2} |1 - S(k)|^2, \quad (2.16)$$

then we see that it takes the value

$$\sigma = \frac{4\pi\Gamma_0^2}{(E_0 - E_k)^2 + \Gamma_0^2 k^2} \quad (2.17)$$

where  $\Gamma_0$  is the reduced width defined by

$$\Gamma_0 = (2\pi|c_{12}|^2/\mu) \quad (2.18)$$

and the kinetic  $E_k$  is related with the wave number  $k$  by the usual non-relativistic expression

$$E_k = (k^2/2\mu) \quad (2.19)$$

We derived then the standard one level Breit-Wigner formula,<sup>1)</sup> by considering only interactions at the point of coincidence of the two initial particles, expressed by boundary conditions derived from arguments of conservation of probability. We shall now extend this type analysis to relativistic interactions.

### 3. Breit-Wigner formula for the scattering of two Klein-Gordon particles

Again we start with a wave function with two components, one for the two initial particles of masses  $m_1, m_2$  and one for the compound particle of mass  $M$ , but now the former depends explicitly on the space time coordinates of the two particles  $x_1^\mu, x_2^\mu, \mu = 0, 1, 2, 3$ , while the latter is a function of one space time coordinate  $X^\mu$ , which we will later identify with the center of mass coordinates of the two initial particles.

Our state can then be written as

$$\Psi = \begin{bmatrix} \psi(x_1^\mu, x_2^\mu) \\ \phi(X^\mu) \end{bmatrix}, \quad \mu = 0, 1, 2, 3. \quad (3.1)$$

So long as  $x_1^\mu \neq x_2^\mu$  the  $\psi(x_1^\mu, x_2^\mu)$  satisfies the equations

$$g^{\mu\nu} \frac{\partial^2 \psi}{\partial x_1^\mu \partial x_1^\nu} - m_1^2 \psi = 0, \quad g^{\mu\nu} \frac{\partial^2 \psi}{\partial x_2^\mu \partial x_2^\nu} - m_2^2 \psi = 0, \quad (3.2a, b)$$

On the other hand in the absence of interaction with the compound particle  $\phi$  satisfies

$$g^{\mu\nu} \frac{\partial^2 \phi}{\partial X^\mu \partial X^\nu} - M^2 \phi = 0, \quad (3.3)$$

where in this paper we use the metric

$$g^{\mu\nu} = 0 \quad \text{if} \quad \mu \neq \nu, \quad -g^{00} = g^{11} = g^{22} = g^{33} = 1 \quad (3.4)$$

#### a) Relative and center of mass space-time coordinates

We first note that the total four momentum

$$P_\mu = p_{\mu 1} + p_{\mu 2}. \quad (3.5)$$

is an integral of motion even in the presence of an interaction. This will hold also for the scalar product of  $P_\mu$  with itself i. e.

$$P^2 \equiv P_\mu P^\mu = -w^2 \quad (3.6)$$

which we denote as  $-w^2$  as from the metric (3.4)  $P^2$  is negative

Now following Crater and van Alstine<sup>4)</sup> we define first  $\varepsilon_1, \varepsilon_2$  by

$$\varepsilon_1 = \frac{w^2 + m_1^2 - m_2^2}{2w}, \quad \varepsilon_2 = \frac{w^2 + m_2^2 - m_1^2}{2w}, \quad \varepsilon_1 + \varepsilon_2 = w. \quad (3.7a, b, c)$$

so the relativistic relative and center of mass coordinates become

$$x^\mu = x_1^\mu - x_2^\mu, \quad X^\mu = w^{-1}(\varepsilon_1 x_1^\mu + \varepsilon_2 x_2^\mu). \quad (3.8a, b)$$



Because momenta are related to the derivative of the coordinates *i.e.*  $p_\mu = -i\partial/\partial x^\mu$  we see from (3.8) that

$$p_\mu = (\varepsilon_2/w)p_{\mu 1} - (\varepsilon_1/w)p_{2\mu}, \quad P_\mu = p_{\mu 1} + p_{\mu 2}. \quad (3.9a, b)$$

In terms of  $x^\mu, X^\mu$  the equations (3.2) for  $\psi$  will be

$$2i g^{\mu\nu} P_\mu \frac{\partial\psi}{\partial x^\nu} = 0 \quad (3.10a)$$

$$2g^{\mu\nu} \frac{\partial^2\psi}{\partial x^\mu \partial x^\nu} + \left[ \frac{w^2}{2} + \frac{(m_1^2 - m_2^2)}{2w^2} - (m_1^2 + m_2^2) \right] \psi = 0 \quad (3.10b)$$

where  $P_\mu$  can be interpreted as the operator  $(-i\partial/\partial X^\mu)$  or as a number as the dependence of  $\psi$  on  $X^\mu$  is  $\exp(i P_\mu X^\mu)$  even in presence of interactions.

If we pass now to the frame where the center of mass is at rest *i.e.*  $P_i = 0, i = 1, 2, 3$ , then  $w^2 = (P_0^2)$  and the two equations (3.10) becomes

$$-2iP_0(\partial\psi/\partial x^0) = 0 \quad (3.11a)$$

$$2\nabla^2\psi + \left[ (1/2)(P_0)^2 + (2P_0^2)^{-1}(m_1^2 - m_2^2)^2 - (m_1^2 + m_2^2) \right] \psi = 0 \quad (3.11b)$$

where in the second we already made use of the fact that  $(\partial\psi/\partial x^0) = 0$  that follows from the first.

As  $-\nabla^2$  is the square ordinary relative momentum, whose value we may denote by  $k^2$ , and  $P_0$  is the total energy which we may call  $E$ , we get from (3.11b) the following relation between the two

$$k^2 = (E^2/4) + (m_1^2 - m_2^2)^2(4E^2)^{-1} - (1/2)(m_1^2 + m_2^2) = \frac{(E^2 - m_+^2)(E^2 - m_-^2)}{4E^2} \quad (3.12)$$

where

$$m_\pm = m_1 \pm m_2 \quad (3.13)$$

If we invert (3.12) to get  $E$  in terms of  $k$ , we obtain for the positive value of  $E$  the expected relation

$$E = (\sqrt{k^2 + m_1^2} + \sqrt{k^2 + m_2^2}) \quad (3.14)$$

#### b) Currents associated with the one and two particle systems

As in the non-relativistic case<sup>2)</sup> we want to consider two states  $\Psi, \bar{\Psi}$ . The compound particle part of these states is represented by the second component i.e.  $\phi, \bar{\phi}$ , with which we can associate the current

$$\begin{aligned} j_\mu &= \frac{1}{2iM} \left( \bar{\phi}^* \frac{\partial \phi}{\partial X^\mu} - \frac{\partial \bar{\phi}^*}{\partial X^\mu} \phi \right) \\ &= \bar{\phi}^* \left[ \frac{1}{2iM} \left( \frac{\partial}{\partial X^\mu} - \overleftarrow{\frac{\partial}{\partial X^\mu}} \right) \right] \phi \end{aligned} \quad (3.15)$$

where on the right hand side we wrote it as an operator between the states  $\bar{\phi}^*, \phi$ , with arrows indicating when it acts on the right and when on the left.

For a single non interacting particle where  $\phi, \bar{\phi}$  satisfy the Klein-Gordon equation (3.3), the current (3.15) satisfies the continuity equation

$$\frac{\partial j^\mu}{\partial X^\mu} = 0 \quad (3.16)$$

The factor  $(2iM)^{-1}$  is introduced in (3.15) to have the same form for the spatial part  $j_i$  as in the non-relativistic case.

Clearly for two Klein-Gordon particles  $j^\mu$  must be replaced by a tensor  $T^{\mu\nu}$  with two indices given by

$$T_{\mu\nu} = \bar{\psi}^* \left[ \frac{1}{2im_1} \left( \frac{\partial}{\partial x_1^\mu} - \overleftarrow{\frac{\partial}{\partial x_1^\mu}} \right) \right] \left[ \frac{1}{2im_2} \left( \frac{\partial}{\partial x_2^\nu} - \overleftarrow{\frac{\partial}{\partial x_2^\nu}} \right) \right] \psi \quad (3.17)$$

which from (3.2) satisfies the corresponding continuity equations

$$\frac{\partial T^{\mu\nu}}{\partial x_1^\mu} = 0, \quad \frac{\partial T^{\mu\nu}}{\partial x_2^\nu} = 0. \quad (3.18a, b)$$

c) Scalar product

With the help of the vector and tensor currents discussed above we can define the scalar product of the two component states  $\Psi, \bar{\Psi}$

$$(\bar{\Psi}, \Psi) = \int \int T^{00} d^3 x_1 d^3 x_2 + \int j^0 d^3 X \quad (3.19)$$

which is a concept invariant under the Lorentz group as discussed in another publication<sup>5)</sup>.

Is possible to express the scalar product (3.19) purely in terms of the relative and center of mass coordinates as from (3.8) we see that

$$x_1^\mu = X^\mu + (\varepsilon_2 x^\mu / w), x_2^\mu = X^\mu - (\varepsilon_1 x^\mu / w) \quad (3.20a, b)$$

where we note that Jacobian of  $(x_1, x_2)$  with respect to  $(x, X)$  is 1 so that  $d^3 x_1 d^3 x_2 = d^3 x d^3 X$ .

For the derivatives we have from (3.9) the relations

$$\frac{\partial}{\partial x_1^\mu} = \frac{\partial}{\partial x^\mu} + \frac{\varepsilon_1}{w} \frac{\partial}{\partial X^\mu}, \quad \frac{\partial}{\partial x_2^\mu} = -\frac{\partial}{\partial x^\mu} + \frac{\varepsilon_2}{w} \frac{\partial}{\partial X^\mu} \quad (3.21a, b)$$

where we must be careful to note that they can be applied to  $\psi(x_1^\mu, x_2^\mu)$  as they appear in (3.21) but when applied to  $\bar{\psi}(x_1^\mu, x_2^\mu)$  the  $\varepsilon_1, \varepsilon_2$  must be replaced by  $\bar{\varepsilon}_1, \bar{\varepsilon}_2$  i.e.  $w$  by  $\bar{w}$  in which the first is associated with total energy  $E$  and the second with  $\bar{E}$ .

d) Basic argument

If we choose the frame of reference where the center of mass is at rest, the conservation of probability argument for the states can then be expressed as<sup>5)</sup>

$$\frac{\partial}{\partial X^0} \left[ \int \int T^{00} d^3 x d^3 X + \int j^0 d^3 X \right] = 0, \quad (3.22)$$

where now as  $P_i = \bar{P}_i = 0$  we can write the two body wave functions  $\psi, \bar{\psi}$  as

$$\psi = f(\mathbf{r}) \exp(-iEX^0), \bar{\psi} = \bar{f}(\mathbf{r}) \exp(-i\bar{E}X^0) \quad (3.23a, b)$$

while the compound particle  $\phi, \bar{\phi}$  take the form

$$\phi = \varphi \exp(-iEX^0), \bar{\phi} = \bar{\varphi} \exp(-i\bar{E}X^0) \quad (3.24a, b)$$

with  $\varphi, \bar{\varphi}$  depending only on the energies  $E, \bar{E}$ .

The equation (3.22) by an analysis entirely similar to the one leading to (2.5) in the non-relativistic case, can be written as

$$\frac{d}{dX^0}(\bar{\Psi}, \Psi) = \frac{-i(E + \bar{E})}{2M} \exp[-i(E - \bar{E})X^0] \int d^3X \left\{ \frac{1}{2\bar{\mu}} \int [\nabla \cdot (\bar{f}^* \nabla f - \bar{f} \nabla f)] d^3x + [\bar{\varphi}^*(-E\varphi) - \varphi(-\bar{E}\bar{\varphi})^*] \right\} = 0 \quad (3.25)$$

where we already replaced  $i\partial/\partial X^0$  by  $E$  as follows from (3.23), (3.24).

As only the curly bracket in (3.25) can vanish we get, again using Gauss theorem on a sphere surrounding  $r = 0$ , the equation

$$\begin{aligned} \frac{2\pi}{\bar{\mu}} \left( \frac{\partial r \bar{f}}{\partial r} \right)_0 (rf)_0 - (r\bar{f})_0^* \frac{2\pi}{\bar{\mu}} \left( \frac{\partial r f}{\partial r} \right)_0 \\ + \bar{\varphi}^*(-E\varphi) - (-\bar{E}\bar{\varphi})^* \varphi = 0 \end{aligned} \quad (3.26)$$

which differs from the non-relativistic case<sup>2,5)</sup> only in that the reduced mass  $\mu = m_1 m_2 (m_1 + m_2)^{-1}$  is replaced by  $\bar{\mu} = m_1 m_2 M^{-1}$ .

#### e) Linear relations

The linear relations which guarantee that the bilinear form (3.26) vanishes identically are then given by<sup>2,5)</sup>

$$(rf)_0 = c_{12} \varphi \quad (3.27a)$$

$$-E\varphi = c_{21} \frac{2\pi}{\bar{\mu}} \left( \frac{\partial r f}{\partial r} \right)_0 - M\varphi \quad (3.27b)$$

where  $c_{12} = c_{21}^*$  is the coupling constant relating the two particle system  $\psi(x_1^\mu, x_2^\mu)$  when  $x_1^\mu = x_2^\mu$  with  $\phi(X^\mu)$  both in the frame where the center of mass is at rest.

Eliminating  $\varphi$  between the equations (3.27a) and (3.27b) we get a relation between  $(rf)_0$  and its derivative  $(\partial rf/\partial r)_0$  given by

$$(rf)_0 = R(E)[\partial rf/\partial r]_0 \quad (3.28)$$

where the  $R$  matrix (in this case only  $1 \times 1$ ) is given by

$$R(E) = (2\pi|c_{12}|^2/\bar{\mu})(M - E)^{-1} \quad (3.29)$$

As only  $s$ -waves are involved in this analysis, because of the interaction takes place only at the point of coincidence of the two particles, the total scattering cross section is given by<sup>5)</sup>

$$\sigma = \frac{4\pi\Gamma_0^2}{(M - E)^2 + \Gamma_0^2 k^2} \quad (3.30)$$

with the reduced width  $\Gamma_0$  related to the coupling constant  $|c_{12}|^2$  by

$$\Gamma_0 = (2\pi|c_{12}|^2/\bar{\mu}), \quad (3.31)$$

The expression (3.30) for the cross section is then entirely equivalent to the non-relativistic one<sup>2)</sup> with the basic change that the relation between energy and momentum is not given by  $E_k = (k^2/2\mu)$  but by the relation (3.12) i.e.

$$k^2 = \frac{(E^2 - m_+^2)(E^2 - m_-^2)}{4E^2}, m_{\pm} = m_1 \pm m_2 \quad (3.32)$$

The boundary conditions (3.27), expressed only in terms of  $rf$  can be given a dynamical form if  $E$  is replaced by  $i\partial/\partial X^0$ . By using the transversal relative coordinates  $\hat{x}^\mu$  defined by

$$\hat{x}^\mu = x^\mu - (P_r P^r)^{-1} (x^\nu P_\nu) P^\mu \quad (3.33)$$

as well as the Poincaré invariant expression

$$\rho^2 = \hat{x}^\mu \hat{x}_\mu \quad (3.34)$$

we then get that the problem leading to the relativistic Breit-Wigner formula (3.30) is described by the two equations (3.2a,b) satisfied by  $\psi(x_1^\mu, x_2^\mu)$  when  $x_1^\mu \neq x_2^\mu$ , together with the boundary condition

$$\left[ M(\rho\psi) - i(-P_\nu P^\nu)^{-1/2} P^\mu (\partial\rho\psi/\partial X^\mu) \right]_{\rho=0} = \Gamma_0 (\partial\rho\psi/\partial\rho)_{\rho=0} \quad (3.35)$$

Thus the Breit-Wigner formula (3.30) is a consequence of a fully relativistic analysis for two Klein-Gordon particles.

#### 4. Breit-Wigner formula for the scattering of two Dirac particles

The analysis is entirely parallel to the one of the previous section, with the wave function given by (3.1) but now with  $\psi(x_1^\mu, x_2^\mu)$  satisfying two free particle Dirac equations when  $x_1^\mu \neq x_2^\mu$ . We assume that the compound particle is a scalar one satisfying, in the absence of interaction, the Klein-Gordon equation (3.3).

As shown in reference 5, a conservation of probability argument leads to a bilinear form equal to 0, though different in form from (3.26). The corresponding linear relations that satisfy it now give an  $R$  matrix of the form<sup>5)</sup>

$$R(E) = \frac{4M\Gamma E}{(M^2 - E^2)(E^2 - m_-^2)} \quad (4.1)$$

where the reduced width  $\Gamma$  is related with the coupling constant  $c_{12}$  by

$$\Gamma = 4\pi|c_{12}|^2 \quad (4.2)$$

The differential cross section  $d\sigma$  for scattering is now given by<sup>5)</sup>

$$d\sigma = (1/2) \sin^2(\delta_1 - \delta_2) \frac{16M^2\Gamma^2 E^2 d\Omega}{(M^2 - E^2)^2 (E^2 - m_-^2)^2 + 16M^2\Gamma^2 E^2 k^2} \quad (4.3)$$

where  $d\Omega$  is the differential of solid angle, and  $\delta_1, \delta_2$  are the angles that the spins of the two Dirac particles form with the direction of propagation.

Thus we have in (4.3) a generalized form of the Breit-Wigner scattering cross section for a single level for a system of two Dirac particles forming a scalar Klein-Gordon one at the point of coincidence.

If we assume, for example, that  $E$  is of the order of  $M$  and  $M \gg m_1$  or  $m_2$ , then

$$(E^2 - m_{\pm}^2) \simeq E^2, (E + M) \simeq 2M, k^2 \simeq E^2/4 \quad (4.4)$$

and the cross section takes the form

$$d\sigma \simeq \frac{1}{2} \sin^2(\delta_1 - \delta_2) \frac{4\Gamma^2}{(M - E)^2 + \Gamma^2 E^2} \quad (4.5)$$

close to the standard Breit-Wigner formula.

## 5. Conclusions

We have developed a relativistic formalism for the interaction, at their point of coincidence, of two Klein-Gordon or two Dirac particles. These interactions implied the formation of a single compound particle of the Klein-Gordon type.

In this way we obtained a single-level Breit-Wigner type of cross section for the two problems mentioned, in which the only parameters are the mass  $M$  of the compound particle and the reduced width  $\Gamma$ , where the latter is related to the strength of the coupling between the two particle component  $\psi(x_1^\mu, x_2^\mu)$  at  $x_1^\mu = x_2^\mu$ , and the single compound particle  $\phi(X^\mu)$ .

In the analysis presented here, as well as in reference 5, the relative orbital angular momentum of the two particles is taken as 0 implying that the compound particle has total angular momentum 0 *i.e.* is a scalar one.

It is possible though to generalize our reasoning to arbitrary relative orbital angular momentum of the two particles and thus, for example, consider the interaction of an electron with a neutrino with mass, in which they form a  $W^-$  compound particle which has total angular momentum 1 and parity -. In that case in our formula (4.3)  $M, \Gamma$  could be determined from the experimental measurement of the energy and width associated with the  $W^-$  particle which are respectively 80 and 2.5 GeV.

A more interesting example is the scattering of electrons by positron at very high energies where we can disregard the effects of the Coulomb interaction. In that case  $M, \Gamma$  will be determined by the energy and width of  $Z^0$  particle. We plan to discuss this problem in a future paper.

Finally we note that our analysis is restricted to scattering with a single resonant level. We can though introduce many levels by taking several functions  $\phi$  in (3.1) instead of one. To have many channels we take several functions  $\psi$  in (3.1). Both of these generalizations were already considered in reference 2 for the non-relativistic case and there is no essential change in the relativistic analysis presented in this paper, when we consider several  $\phi$  or  $\psi$ , or both.

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## ON THE SYMMETRY OF LATTICE FERMIONS

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The Dirac-Kähler equation [1] (DKE):  $(d - \delta + m)\Phi = 0$  is a generalization of the Dirac equation which allows a straightforward lattice approximation leading to a geometric interpretation of staggered lattice fermions  $\chi(x)$  [2]. We embed in  $R^4$  a cubical lattice with lattice points  $y$ , lattice vectors  $e_\mu$ , h-dim. lattice cubes  $[y, H]$  spanned at  $y$  by  $e_\mu, \mu \in H$ . DeRham mapping  $\phi(y, H) = \int_{[y, H]} \Phi = \chi(x), x = y + \frac{1}{2}e_H, e_H = \sum_{\mu \in H} e_\mu$  leads to the DKE on the lattice:  $(\check{\Delta} - \check{\nabla} + m)\Phi = 0$ .  $\check{\Delta}, \check{\nabla}$  are the boundary and coboundary operators acting on cochains  $\Phi = \sum_{y, H} \phi(y, H) d^{y, H}$ . By DeRham mapping the DK-action gets transformed into the action of staggered fermions:  $S_g = \int (\check{\Phi}, (d - \delta + m)\Phi)_0 \Rightarrow S_g(\check{\chi}, \chi)$ .

It is the aim of this talk to report on the group theoretical treatment of the symmetry problems posed by this scheme [3]. This includes: (1) the description of the symmetry group  $\mathcal{G}$  of the DKE; (2) the determination of the symmetry group  $\mathcal{G}_L$  of the lattice action which is determined as a sub-group of  $\mathcal{G}$  by the geometry of DeRham mapping; (3) the classification of the irreducible unitary representations ("irreps") of  $\mathcal{G}$  and  $\mathcal{G}_L$ ; (4) the decomposition of the irreps of  $\mathcal{G}$  into the irreps of  $\mathcal{G}_L$ . We mention some applications at the end.

1. For the understanding of the meaning of the DKE it is essential to know that the DKE is equivalent to four simultaneous Dirac equations. It follows that its symmetry group  $\mathcal{G}$  is generated by the transformations of the Dirac components under the 4-dim. Euclidean group  $\mathcal{SE}$  and by the  $SU(4)$  "flavour" transformations of the four degenerate Dirac fields. In addition there is a global  $U(1)$ -phase transformation. For mass  $m = 0$ , the  $SU(4) \times U(1)$  gets enlarged to  $U(4) \times U(4)$  by chiral transformations.

The extension of the calculus of differential forms by a Clifford product:  $dx^\mu \vee dx^\nu = g^{\mu\nu} + dx^\mu \wedge dx^\nu$ , (generally:  $dx^K \vee dx^L = \check{\rho}_{KL} dx^{K\Delta L}, \check{\rho}_{KL} = \pm 1$ ) allows the derivation of these facts. The transformations of the flavour symmetry group are generated by Clifford right multiplication with constant forms:  $\Phi \vee c(u)$ . The spinor rotations of the Dirac components  $(\phi_a^b(x)) = \sum_H \phi(x, H) (\gamma^H)_a^b$  can be expressed as operations on the forms:  $\delta_{\mu\nu} \Phi = (x_\mu \partial_\nu - x_\nu \partial_\mu) \Phi + \frac{1}{2} S_{\mu\nu} \vee \Phi$ .  $S_{\mu\nu} = dx_\mu \wedge dx_\nu$ . The Cartesian components transform as  $O(4)$ -tensors like:  $\delta_{\mu\nu}^G \Phi = (x_\mu \partial_\nu - x_\nu \partial_\mu) \Phi + \frac{1}{2} (S_{\mu\nu} \vee \Phi - \Phi \vee S_{\mu\nu})$ . These 'geometric' rotations differ from spinor rotations by flavour transformations. A general element of  $\mathcal{G}$ :  $(f, a, s)$  is composed by a flavour transformation  $(f)$ , a translation  $[a]$ , and a spinorial rotation  $(s)$ . Alternatively it may be composed by a geometric rotation  $R(s)$ , another flavour transformation  $[f]$  and a translation  $[a]$ :  $[f, a, R(s)]$ . With these notions and concepts we can describe the structure of the symmetry group of the DKE:

**PROPOSITION 1.** The symmetry group  $\mathcal{G} = \{(f, a, s)\} = \{[f, a, R(s)]\} \simeq \mathcal{SE} \times U(4)/Z_2$  is defined by  $(f, a, s) \circ (f', a', s') = (ff', R(s)a' + a, ss')$ , or

$$[f, a, R(s)] \circ [f', a', R(s')] = [fsf's^{-1}, R(s)a' + a, R(s)R(s')].$$

The relation between the two notations is given by  $g = (\bar{f}) \circ (s, a, s) = (\bar{f}s, a, s) = [\bar{f}, a, R(s)]$ . The second form of the group multiplication is essential for the understanding of the lattice symmetry group.

2. DeRham mapping defines the symmetry group  $\mathcal{G}_L$  of the lattice DKE as "lattice restriction" of the continuous symmetry group  $\mathcal{G}$  [4]. This is obvious for the translation group:  $\mathcal{T} \supset \mathcal{T}_L = \{a \mid a = b(n^i), n^i \in \mathcal{Z}\} = \{[a]\}$ . The geometric rotation group must be restricted to the symmetry group  $W_4$  of the 4-dim. cube. A similar geometric approach to the lattice restriction of the flavour transformations makes use of a definition of a  $\vee$ -product on the lattice:  $\epsilon d^K \Phi = \epsilon \Phi \vee (d^K)^{-1}$ ,  $\epsilon = \pm 1$ . The lattice flavour transformations  $\epsilon d^K \in \mathcal{FT}_L$  generate translations:  $(d^K)^2 = [-e_K]$ . The factor group  $\mathcal{F}_L = \mathcal{FT}_L/\mathcal{T}_L \simeq \mathcal{K}_4$  is isomorphic to the multiplicative group  $\mathcal{K}_4$  of the Dirac matrices  $\{\pm \gamma^K\}$ . However  $\mathcal{FT}_L$  is not a semi-direct product  $\mathcal{T}_L \rtimes \mathcal{K}_4$ , but a non-symmmorphous extension of the lattice translation group. We may summarize these facts on the group  $\mathcal{G}_L$  in the following

PROPOSITION 2. The lattice restriction of  $\mathcal{G}$  is  $\mathcal{G}_L = \{[\epsilon d^K, -\frac{1}{2}e_K + a, R] \mid a \in \mathcal{T}_L, R \in W_4\}$  with the composition law

$$[\epsilon d^K, -\frac{1}{2}e_K + a, R] \circ [\epsilon' d^L, -\frac{1}{2}e_L + a', R'] = [\rho d^{K \Delta R \circ L}, -\frac{1}{2}(e_K + Re_L) + Ra' + a, RR'].$$

$\rho = [\epsilon \epsilon' \rho(R, R \circ L) \check{\rho}_{K, R \circ L} = \pm 1$ . It is a symmetry group of the free DKE if it acts on staggered fermion fields according to:  $([a]\chi)(x) = \chi(x - a^\mu e_\mu)$ ,  $([R]\chi)(x) = \rho(R, H(x))\chi(R^{-1}x)$ ,  $R \in W_4$ ,  $(\epsilon d^K \chi)(x) = \epsilon \check{\rho}_{H(x), K} \chi(x + \frac{1}{2}e_K)$ . The sign  $\rho(R, H)$  is the same as in the transformation of the basis differentials of the continuum:  $Rdx^H = \rho(R, H)dx^{R^{-1} \circ H}$ .

3. The problem of the classification of the irreps of a group with a normal subgroup is greatly simplified by the induction procedure. This was illustrated for semi-direct products with abelian normal subgroup by E.P. Wigner [5] in a classical paper on the Poincaré group. The structure of the group  $\mathcal{G}_L$  is somewhat more complicated. Therefore the construction of the irreps of  $\mathcal{G}_L$  must be guided by a more general procedure which is due to G.W. Mackey [6].

MACKEY'S MAIN THEOREM states: All irreducible unitary representations of a group  $G$  with normal subgroup  $N$  are characterized by the  $G$ -orbits  $\Theta^i$  in  $\hat{N}$ , and the irreducible projective representations of the related little groups of second kind  $S_i^{(1)}/N = S_i^2 \ni s \rightarrow D(s)$  with multiplier of a certain equivalence class. The little group of first kind  $S_i^{(1)}$  is the stability group of  $\Theta^i$ .

The iterated application of the Wigner-Mackey procedure leads to a complete classification of the irreps of  $\mathcal{G}_L$  [7]. (See also [8], [9]). In a first step we consider the translation group  $\mathcal{T}_L$  as a normal subgroup of  $\mathcal{G}_L$ . The 1-dim. irreps of  $\mathcal{T}_L : [a] \rightarrow e^{ipa}$  are labelled by 'momenta'  $p$  varying in the Brillouin zone:  $-\frac{\pi}{b} < p_\mu \leq \frac{\pi}{b}$ . The star  $St_j$  is the orbit of the rotations  $R \in W_4$  applied to the momenta,  $p \rightarrow Rp$ , i.e.  $St_j = \{R\bar{p}_j \mid R \in W_4\}$ . There are 17 qualitatively different "momentum stars"  $St_j$  characterized by reference momenta  $\bar{p}_j$ . In the applications the irreps with a

"momentum at rest"  $\bar{p}_j = (0, 0, 0, p)$  are used mostly.  $S_j^{(1)}$  is generated by the translations, flavour transformations, and the rotations of  $S_j = \{R | R\bar{p}_j = \bar{p}_j\}$ . The little group of the second kind  $S_j^{(2)} \simeq S_j^{(1)}/\mathcal{T}_L$  is generated by  $S_j$  and the elements of  $\mathcal{K}_4$ . It is not a subgroup of  $\mathcal{G}_L$ . The  $St_j$  and the irreps of  $S_j^{(2)}$  determine the irreps of  $\mathcal{G}_L$ .

In order to determine the irreps of  $S_j^{(2)}$  we apply the Wigner-Mackey construction a second time. The group  $S_j^{(2)}$  contains  $\mathcal{K}_4$  as a normal subgroup. Therefore we consider first the irreps of  $\mathcal{K}_4$ . These are the 4-dim. representation by Dirac matrices (' $L = 0$ '), and the 16 one dimensional representations  $\epsilon d^K \rightarrow \epsilon \gamma^K$ ,  $\epsilon d^K \rightarrow e^{i\pi(\epsilon_L \cdot \epsilon_K)} \equiv \Gamma^L(\epsilon d^K)$ . The transformations of the irreps of  $\mathcal{K}_4$  under the rotations of  $S_j$  are:  $\Gamma^L(R^{-1}(\epsilon d^K)R) \simeq \Gamma^{R \circ L}$ . For  $L=0$ , this is an equivalence transformation for all  $R$ :  $R \circ (L=0) = (L=0)$ . The set of 1-dimensional representations decomposes under the rotations of  $S_j$  in 'flavour orbits'  $\Theta_{j,F}$ .  $S_{j,F}^{(1)}$  is a semidirect product of  $S_{j,F}$  with  $\mathcal{K}_4$  as normal subgroup:  $S_{j,F} = \{R | R\bar{L}_F = \bar{L}_F; R \in S_j\} \subset S_j$ ,  $\bar{L}_F$  reference point of  $\theta_{i,F}$ , and  $S_{j,F}^{(1)}/\mathcal{K}_4 \simeq S_{j,F}$ . We call  $S_{j,F}$  the "reduced spin group", the character of its irreps "reduced spin". The result is

**PROPOSITION 3.** The irreducible, unitary representations of the symmetry group  $\mathcal{G}_L$  of staggered fermions are determined by a 'momentum star', a 'flavour orbit', and the 'reduced spin'.

4. The decomposition of the irreps of the symmetry group  $\mathcal{G}$  of the DKE in the continuum into those of the lattice restriction  $\mathcal{G}_L$  plays an important role in the calculations of the hadron spectrum by a lattice approximation of QCD. In the framework of Mackey's theory this problem is solved by the 'Subgroup Theorem', and 'Frobenius' Theorem'. We shall deal with our problem along this line [10].

First we make some remarks on the irreps of the continuum symmetry group  $\mathcal{G} \simeq \mathcal{SE} \times SU(4)$ . The irreps of  $\mathcal{G}$  can be constructed as products of the irreps of the spinorial Euclidean group  $\mathcal{SE}$  with the irreps of  $SU(4)$ . According to Wigner's construction, the irreps of  $\mathcal{SE}$  are characterized by an 'Euclidean mass shell':  $p^2 = M^2$ , and by a spin  $\sigma$  determined by an irrep of the 'little group'  $SU(2) \ni s \rightarrow D^\sigma(s)$ . The irreps of  $SU(4) : D^{F_c}(f)$  relevant for quark model considerations are for the  $q\bar{q}$ -system:  $(4) \otimes (\bar{4}) = (1) \oplus (15)$ , and for the  $qqq$ -system:  $(4) \otimes (4) \otimes (4) = (20) \oplus 2 \cdot (20') \oplus (4)$ .

The 'Subgroup Theorem' deals with the decomposition of  ${}_{\mathcal{G}}\mathcal{U}^{L(H_1)}|_{H_2}$ , the restriction to a subgroup  $H_2$  of a representation  ${}_{\mathcal{G}}\mathcal{U}^{L(H_1)}$  of a group  $\mathcal{G}$  induced by a representation  $L(H_1)$  of the little group of first kind  $H_1$ . It states that  ${}_{\mathcal{G}}\mathcal{U}^{L(H_1)}|_{H_2} \simeq \sum_k W^{x_k}$ . Here  $\mathcal{G} \ni x_k$  denote representatives of the  $H_1 x_k H_2$  double cosets,  $W^{x_k}$  is the representation of  $H_2$  induced by  $L^{x_k} : x_k^{-1} H_1 x_k \cap H_2 \ni y \mapsto L(x_k y x_k^{-1})$ . In our case we have to identify  $\mathcal{G}$  with  $G$ ,  $\mathcal{G}_L$  with  $H_2$ , and the little group of first kind of the continuum group  $S_j^{(1)} \simeq O(3) \times SU(4)$  with  $H_1$ . The double cosets might be represented by boosts  $x_k \sim \Lambda(\bar{p}_j) \in \mathcal{G}$  with  $\bar{p}_j = \Lambda(\bar{p}_j)\bar{p}$ ,  $\bar{p}$  reference momentum on the mass shell,  $\bar{p}_j$  the independent reference momenta of  $St_j$ .

$I(L_1, L_2 || \mathcal{G}) = \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \text{Trace}(L_1(g^{-1})) \text{Trace}(L_2(g))$  defines the intertwining

number between two representations  $L_1, L_2$  of a group  $G$ . If  $L_1$  is irreducible it gives the multiplicity with which  $L_1$  is contained in  $L_2$ . For induced representations Frobenius' theorem states:  $I(G U^{L(H)}, L_2 | G) = I(L, L_2 | H | H)$ . This formula allows the calculation of the multiplicity of the irreps of  $\mathcal{G}_L$  contained in  $W^{\mathbf{x}}$ . Adding up we get the final result:

PROPOSITION 4. The lattice restriction of the irrep  $U^{\mathbf{x}}$ ,  $\chi = (M, F_c, \sigma_c)$  of the continuum symmetry group  $\mathcal{G}$  contains the irrep  $U^{\mathbf{x}_L}$ ,  $\chi_L = (j, F, \sigma)$  of  $\mathcal{G}_L$  with the multiplicity:

$$I(U^{\mathbf{x}}|_L, U^{\mathbf{x}_L} | \mathcal{G}_L) = \frac{1}{|S_{j,F}|} \sum_{\rho \in S_{j,F}} \Xi_{\rho}^{F_c, F} \overline{\text{Trace} D^{\sigma_c}(\Lambda(\bar{p}_j) \rho \Lambda^{-1}(\bar{p}_j))} \text{Trace}(D^{\sigma}(\rho))$$

$$\text{with } \Xi_{\rho}^{F_c, F} = \frac{1}{|\kappa|} \sum_{\epsilon d^K \in \kappa} \overline{\text{Trace}(D^{\sigma_c}(\epsilon d^K \rho))} \text{Trace}(\Gamma^F(\epsilon d^K, \rho)), \text{ and } \bar{p}_j^2 = M^2.$$

We made the physical assumption  $M < \frac{\pi}{b}$ , otherwise the case gets too involved. Extensive tables cover most of the physical interesting cases [10].

5. QCD with DK fermions is supposed to describe a quark model with four flavours. Therefore it is a major problem for the calculation of the hadron spectrum in the framework of lattice QCD with staggered fermions to relate lattice states of a given lattice symmetry  $\chi_L$  to physical particles with quantum numbers  $\chi$  defined by the continuum symmetry. A natural condition is:  $I(U^{\mathbf{x}}|_L, U^{\mathbf{x}_L}) \neq 0$ . On the otherhand, one believes that lattice calculations approach the continuum, if they produce energy degenerate states with all the lattice quantum numbers making up for a complete continuum particle multiplet. There are very few dynamical lattice calculation which include a complete consideration of these symmetry aspects. We want to mention strong coupling approximations of the meson spectrum [11], and of the baryon spectrum [12], and some recent Monte Carlo calculations [13]. Of course the real physical significance of our considerations depends on the understanding of a possible physical meaning of the flavour structure of Dirac-Kähler fermions, a problem about which to speculate goes far beyond the scope of this report.

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## LATTICE FERMIONS WITHOUT SPECIES DOUBLING AND WITH AXIAL ANOMALY

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Using the method of finite differences for the Dirac field equations on the lattice, a new scheme is proposed that gives exact solutions at any time step. The hamiltonian of the Dirac field is translational invariant, hermitian, avoids fermion doubling, and, for the massless case, preserves global chiral symmetry. Coupling the fermion field to the electromagnetic vector potential we construct a gauge invariant vector current leading to the correct axial anomaly.

### 1. QUANTIZATION OF THE DIRAC FIELD

The discrete analog of the equal time anticommutation relations for the Dirac field  $\psi_{\alpha j}^n$  and its hermitian adjoint  $\psi_{\alpha j}^{\dagger n}$  read as follows:

$$[\psi_{\alpha j}^n, \psi_{\beta j'}^{\dagger n}]_+ = \delta_{\alpha\beta} \frac{1}{\epsilon} \delta_{jj'} \quad (1.1)$$

$$[\psi_{\alpha j}^n, \psi_{\beta j'}^n]_+ = [\psi_{\alpha j}^{\dagger n}, \psi_{\beta j'}^{\dagger n}]_+ = 0 \quad (1.2)$$

where  $\psi_{\alpha j}^n \equiv \psi_{\alpha}(j\epsilon, n\tau)$ ,  $\psi_{\alpha j}^{\dagger n} \equiv \psi_{\alpha}^{\dagger}(j\epsilon, n\tau)$  are defined on the lattice,  $\epsilon, \tau$  being space and time intervals, and  $j, n$  integer numbers.

As in the Klein Gordon field, we introduce the method of finite differences in the Heisenberg picture of the equations of motion

$$\frac{1}{8\tau} (\psi_{j+1}^{n+1} + 2\psi_j^{n+1} + \psi_{j-1}^{n+1} - \psi_{j+1}^{n-1} - 2\psi_j^{n-1} - \psi_{j-1}^{n-1}) = \frac{1}{4i} [\psi_j^{n+1} + 2\psi_j^n + \psi_j^{n-1}, H] \quad (1.3)$$

The time independent Hamiltonian can be constructed in the (1+1) dimensional lattice as follows:

$$H = \epsilon \sum_{i=0}^{N-1} \psi_i^{\dagger n} \left\{ \gamma_4 \gamma_1 \frac{1}{2\epsilon} (\psi_{i+1}^n - \psi_{i-1}^n) + M \gamma_4 \frac{1}{4} (\psi_{i+1}^n + 2\psi_i^n + \psi_{i-1}^n) \right\} \quad (1.4)$$

with

$$\gamma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad i \gamma_1 \gamma_4 = \gamma_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad (1.5)$$

Inserting (1.4) in (1.3) we find:

$$\tilde{\nabla}_j \tilde{\nabla}_n R_j^n = 0 \quad (1.6)$$

where

$$R_j^n \equiv \left( \gamma_1 \frac{1}{\epsilon} \Delta_j \tilde{\Delta}_n - i \gamma_4 \frac{1}{\tau} \Delta_n \tilde{\Delta}_j + M \tilde{\Delta}_j \tilde{\Delta}_n \right) \psi_j^n$$

and  $\Delta_j, \tilde{\Delta}_j$  are the difference (average) operators with respect to the space index:

$$\Delta_j f_j \equiv f_{j+1} - f_j, \quad \tilde{\Delta}_j f_j \equiv \frac{1}{2} (f_{j+1} + f_j). \quad \text{Similarly for the time index.}$$

The general solution of (1.6) for  $R_j^n$  is:

$$R_j^n \equiv (A_j + B_n) (-1)^{j+n} \quad (1.7)$$

with  $A_j$  and  $B_n$  arbitrary functions. Applying the operator

$$\gamma_1 \frac{1}{\epsilon} \nabla_j \tilde{\nabla}_n - i \gamma_4 \frac{1}{\tau} \nabla_n \tilde{\nabla}_j - M \tilde{\nabla}_j \tilde{\nabla}_n$$

to both sides of (1.7), we obtain the wave equation for  $\psi_j^n$  if  $A_j$  and  $B_n$  are const. Conservation of current (2.1) requires const = 0. Thus we obtain the discrete analog of Dirac equation

$$R_j^n = 0 \quad (1.8)$$

The same result was obtained by Bender, Milton and Sharp [Ref. 1, formula 18] applying the method of finite elements to the action.

Let us construct solutions to (1.8) of the form

$$\psi_j^n = w(k, E) f_j^n(k, E) \equiv w(k, E) \begin{pmatrix} 1 + \frac{1}{2} i \epsilon k \\ 1 - \frac{1}{2} i \epsilon k \end{pmatrix} \begin{pmatrix} 1 - \frac{1}{2} i \tau E \\ 1 + \frac{1}{2} i \tau E \end{pmatrix}^n \quad (1.9)$$

The four component spinors  $w(k, E)$  must satisfy

$$(i \gamma_1 k - \gamma_4 E + M) w(k, E) = 0 \quad (1.10)$$

Multiplying this equation from the left by  $(i \gamma_1 k - \gamma_4 E - M)$  we obtain the "dispersion relation" for the Dirac equation

$$E^2 - k^2 = M^2 \quad (1.11)$$

Imposing boundary conditions,  $f_0^n = f_N^n = 1$  we can construct a complete set of Dirac wave functions of the form

$$\frac{1}{\sqrt{N\epsilon}} \sqrt{\frac{M}{E_m}} u_m f_j^n(k_m, E_m) \quad \cdot \quad \frac{1}{\sqrt{N\epsilon}} \sqrt{\frac{M}{E_m}} v_m f_j^{*n}(k_m, E_m) \quad (1.12,13)$$

where

$$E_m = +\sqrt{k_m^2 + M^2}, \quad k_m = \frac{2}{\epsilon} \tan \frac{\pi m}{N}, \quad m = 0, \pm 1, \dots, N \quad (1.14)$$

and the spinors  $u_m$  and  $v_m$  are defined as usual:

$$u_m = w(k_m, E_m) = \left( \frac{E_m + M}{2M} \right)^{\frac{1}{2}} \begin{pmatrix} 1 \\ k_m \\ E_m + M \end{pmatrix} \quad (1.15)$$

$$v_m = w(-k_m, -E_m) = \left( \frac{E_m + M}{2M} \right)^{\frac{1}{2}} \begin{pmatrix} k_m \\ E_m + M \\ 1 \end{pmatrix} \quad (1.16)$$

With the aid of this orthonormal set we can expand the fields in the usual way

$$\psi_j^n = \frac{1}{\sqrt{N\epsilon}} \sum_{m=-N/2}^{N/2-1} \sqrt{\frac{M}{E_m}} \left( u_m c_m f_j^n(k_m, E_m) + v_m d_m^\dagger f_j^{*n}(k_m, E_m) \right) \quad (1.17)$$

$$\bar{\psi}_j^n = \frac{1}{\sqrt{N\epsilon}} \sum_{m=-N/2}^{N/2-1} \sqrt{\frac{M}{E_m}} \left( \bar{u}_m c_m^\dagger f_j^{*n}(k_m, E_m) + \bar{v}_m d_m f_j^n(k_m, E_m) \right) \quad (1.18)$$

where  $\bar{\psi}_j^n \equiv \psi_j^{\dagger n} \gamma_4$ . Inverting these expressions, with the help of the orthogonality conditions

$$N^{-1} \sum_{j=0}^{N-1} f_j^{*n}(k_m, E_m) f_j^n(k_{m'}, E_{m'}) = \delta_{mm'}$$

we derive the anticommutation relations

$$[c_m, c_{m'}^\dagger]_+ = \delta_{mm'}, \quad [d_m, d_{m'}^\dagger]_+ = \delta_{mm'} \quad (1.19,20)$$

with other anticommutation relations vanishing.

Finally the Hamiltonian can be written in terms of these operators

$$H = \sum_{m=-N/2}^{N/2-1} \frac{E_m}{\left(1 + \frac{1}{4} k_m^2 \epsilon^2\right)} \left( c_m^\dagger c_m + d_m^\dagger d_m - 1 \right) \quad (1.21)$$

Notice that the zero point energy is finite due to the lattice:

$$E_0 = \sum_{m=N/2}^{N/2-1} \frac{E_m}{\left(1 + \frac{1}{4} k_m^2 \epsilon^2\right)} < \infty \quad (1.22)$$

Our model for the fermion field satisfies the following conditions:

- i) the hamiltonian is translational invariant with respect to the space indices.
- ii) the hamiltonian is hermitian (1.4).
- iii) for  $M = 0$ , the wave equation (1.7) is invariant under global chiral transformations.
- iv) there is no "fermion doubling" as it can be seen in (1.14). In fact,  $E_m$  takes the value  $M$  at  $m = 0$  and nowhere else.
- v) the hamiltonian is non local. Using the finite Fourier transform the hamiltonian  $H(k_m) = i \gamma_4 \gamma_1 k_m + \gamma_4 M$ , given by (1.10), and consequently the dispersion relations (1.11) are smooth functions of  $k_m$ , except for  $m = N/2$ . Therefore our model escapes the no-go theorem by Nielsen and Ninomiya<sup>2</sup>.

## 2. ANOMALIES IN AXIAL VECTOR CURRENT

In order to construct vector and axial currents in terms of the fermions fields in the presence of external electromagnetic vector potential, we multiply (1.8) from the left by  $\bar{\Delta}_j \bar{\Delta}_n \bar{\psi}_j^n$  and then we multiply the adjoint equation of (1.8) from the right by  $\bar{\Delta}_j \bar{\Delta}_n \psi_j^n$ . Adding together both results we find:

$$\frac{1}{\epsilon} \Delta_j \left( \bar{\Delta}_n \bar{\psi}_j^n \gamma_1 \bar{\Delta}_n \psi_j^n \right) - \frac{1}{\tau} i \Delta_n \left( \bar{\Delta}_j \bar{\psi}_j^n \gamma_4 \bar{\Delta}_j \psi_j^n \right) = 0 \quad (2.1)$$

This equation can be considered the discrete version of the "conservation law" for the vector current

$$j_1 = i \left( \bar{\Delta}_n \bar{\psi}_j^n \right) \gamma_1 \left( \bar{\Delta}_n \psi_j^n \right), \quad j_4 = i \left( \bar{\Delta}_j \bar{\psi}_j^n \right) \gamma_4 \left( \bar{\Delta}_j \psi_j^n \right) \quad (2.2)$$

The same equation (6.1) can be applied to the axial current

$$j_1^5 = i \left( \bar{\Delta}_n \bar{\psi}_j^n \right) \gamma_1 \gamma_5 \left( \bar{\Delta}_n \psi_j^n \right), \quad j_4^5 = i \left( \bar{\Delta}_j \bar{\psi}_j^n \right) \gamma_4 \gamma_5 \left( \bar{\Delta}_j \psi_j^n \right) \quad (2.3)$$

Both currents are invariant under global chiral transformations but they are not invariant under  $U(1)$ -gauge transformations:

$$\psi_j^n \rightarrow \Omega_j^n \psi_j^n, \quad \bar{\psi}_j^n \rightarrow \bar{\psi}_j^n \Omega_j^{n\dagger} \quad (2.4)$$

with  $\Omega_j^n$  some unitary function of discrete variables. In order to have gauge invariance we define a gauge field on the lattice

$$U_j^{nn'} = \frac{1 + \frac{i}{4} \tau \left\{ (A_0)_j^n + (A_0)_j^{n'} \right\}}{1 - \frac{i}{4} \tau \left\{ (A_0)_j^n + (A_0)_j^{n'} \right\}}, \quad U_{jj'}^n = \frac{1 + \frac{i}{4} \epsilon \left\{ (A_1)_j^n + (A_1)_{j'}^n \right\}}{1 - \frac{i}{4} \epsilon \left\{ (A_1)_j^n + (A_1)_{j'}^n \right\}} \quad (2.5.6)$$

where  $(A_1, iA_0)$  are the two component electromagnetic vector potential, each of them satisfying the wave equation with  $M = 0$  namely,

$$\left( \frac{1}{\tau^2} \Delta_n \nabla_n \bar{\Delta}_j \tilde{\nabla}_j - \frac{1}{\epsilon^2} \Delta_j \nabla_j \bar{\Delta}_n \tilde{\nabla}_n \right) (A_1)_j^n = 0 \quad (2.7)$$

and similarly for  $(A_0)_j^n$ .

The gauge fields  $U_j^{nn'}$  are associated with the link between the points  $(j, n) \rightarrow (j, n')$  in the positive direction, and they transform under the gauge group as follows:

$$U_j^{nn'} \rightarrow \Omega_j^{n\dagger} U_j^{nn'} \Omega_j^{n'}, \quad U_{jj'}^n \rightarrow \Omega_j^{n\dagger} U_{jj'}^n \Omega_{j'}^n \quad (2.8.9)$$

Inserting these fields in the vector and axial currents between fermion fields at separated points, we get

$$j_1 = \frac{i}{4} \left\{ \bar{\psi}_j^n \gamma_1 \psi_j^n + \bar{\psi}_j^n \gamma_1 U_j^{n,n+1} \psi_j^{n+1} + \bar{\psi}_j^{n+1} \gamma_1 U_j^{n+1,n} \psi_j^n + \bar{\psi}_j^{n+1} \gamma_1 \psi_j^{n+1} \right\} \quad (2.10)$$

$$j_4 = \frac{i}{4} \left\{ \bar{\psi}_j^n \gamma_4 \psi_j^n + \bar{\psi}_j^n \gamma_4 U_{j,j+1}^n \psi_{j+1}^n + \bar{\psi}_{j+1}^n \gamma_4 U_{j+1,j}^n \psi_j^n + \bar{\psi}_{j+1}^n \gamma_4 \psi_{j+1}^n \right\} \quad (2.11)$$

and similarly for  $j_1^5$  and  $j_4^5$ . Using (2.8) and (2.9) we can prove that all these expressions for the vector and axial currents are invariant under the gauge transformations (2.4).

Now we want to calculate the vacuum expectation value of the divergence of the vector and axial current. We assume that this vacuum expectation value approaches the corresponding non interacting fields for vanishing spacial separation of the fields<sup>3</sup>. Thus we use solution of the free massless fermion field (1.7) with  $M=0$ , namely,

$$\psi_j^n = \frac{1}{\sqrt{N\varepsilon}} \sum_{m=-\frac{N}{2}}^{\frac{N}{2}-1} \left\{ u_m c_m f_j^n(k_m, \omega_m) + v_m d_m^\dagger f_j^n(k_m, -\omega_m) \right\} \quad (2.12)$$

with  $u_m = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ , and  $v_m = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ , and the operators  $c_m$  and  $d_m$  satisfying:

$$[c_m, c_m^\dagger]_+ = \delta_{mm}, \quad [d_m, d_m^\dagger]_+ = \delta_{mm}. \quad (2.13)$$

Applying these operators to the vacuum of the Fock space, we get

$$d_m^\dagger |0\rangle = c_m^\dagger |0\rangle = 0 \quad \text{for } m \geq 0 \quad (2.14)$$

$$d_m |0\rangle = c_m |0\rangle = 0 \quad \text{for } m < 0. \quad (2.15)$$

Collecting these properties we obtain for the axial current

$$\begin{aligned} \langle 0 | \frac{1}{\varepsilon} \Delta_j j_1^5 - \frac{1}{\tau} i \Delta_n j_4^5 | 0 \rangle &= \\ &= \frac{-1}{2N \frac{ig\kappa}{N}} \frac{\frac{1}{\varepsilon} \tilde{\Delta}_n \Delta_j (A_0)_j^n}{1 - i\varepsilon \tilde{\Delta}_j \tilde{\Delta}_n (A_0)_j^n - \frac{\varepsilon^2}{4} \tilde{\Delta}_j^2 (A_0)_j^{2n}} + \frac{e^{-i\pi/N}}{2N \frac{sen\kappa}{N}} \frac{\frac{1}{\tau} \tilde{\Delta}_j \Delta_n (A_1)_j^n}{1 - i\varepsilon \tilde{\Delta}_j \tilde{\Delta}_n (A_1)_j^n - \frac{\varepsilon^2}{4} \tilde{\Delta}_n^2 (A_1)_j^{2n}} + \text{c.c.} \end{aligned}$$

which in the limit,  $N \rightarrow \infty$ ,  $\varepsilon \rightarrow 0$ ,  $\tau \rightarrow 0$ , becomes

$$\langle 0 | \partial_1 j_1^5 + \partial_4 j_4^5 | 0 \rangle = -\frac{1}{\pi} (\partial_1 A_0 - \partial_0 A_1) = -\frac{1}{\pi} F_{10} \quad (2.16)$$

We come to the conclusion that our model leads to an interaction which is  $U(1)$ -gauge invariant, but the divergence of the axial current gives in the continuous limit a photon mass, as expected by the axial anomaly. (For schemes in one dimension see Ref.4).

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Noncommutative Geometry and the Standard Model in  
Elementary Particle Physics

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Abstract

An extension of the standard model of electroweak interactions which incorporates the usual gauge fields and the Higgs fields in one generalized Yang-Mills field (or superconnection) is discussed. It is shown that both this Yang-Mills field and the corresponding field strength (supercurvature) take their values in the real graded Lie algebra  $(SU(2|1))$ . The model is characterized by a constant background supercurvature which is invariant under arbitrary, constant  $SU(2|1)$  gauge transformations. The Higgs mechanism receives a new and geometrical interpretation.

I would like to discuss a model which is inspired by noncommutative geometry, see e.g. /1/, and its connection to the standard model (SM) in elementary particle physics. It is also connected to the model we have seen in the talk of Y. Ne'eman in this conference. But the point of view here is different. It contains no ghosts and no other particles but those which we have in the SM and the realization of supersymmetry in it is very different from the usual realizations. In particular, I would like to review some aspects of the work contained in /2-5/.

Our aim is to attain a better understanding and improvement of the SM which has, as is well known, a lot of parameters and some unsatisfactory aspects. The usual way to reduce the number of parameters is to introduce more symmetry. Up to the present, the theoretical efforts to increase the symmetry increase also the degrees of freedom and contain therefore de facto more parameters than before. This is the case with supersymmetry, Kaluza-Klein theory, grand unified theories and superstring theory. The model I am going to discuss introduces more symmetry without introducing new particles. Therefore it promises to reduce the number of parameters and certainly to give a deeper understanding of the SM. The essential ingredients of the model are

- i) a certain Kaluza-Klein ansatz without additional nonexistent fields,
- ii) a certain supersymmetry  $SU(2|1)$  without additional nonexistent fields,
- iii) all this in a noncommutative way.

This leads to a superconnection  $\mathcal{A}$  with values in  $\text{Lie}(SU(2|1))$  with a specific  $Z_2$  grading, so that the even (odd) part of  $\mathcal{A}$  takes values in the even (odd) part of  $\text{Lie}(SU(2|1))$  and contains an additional derivative  $d_M$  which acts on the values of  $\mathcal{A}$ . In what follows we shall discuss shortly 1. the superconnection, 2. the real  $SU(2|1)$  superalgebra, 3. the superderivative " $d = d_C + d_M$ " and the supercurvature, 4. the bosonic Lagrangian and Higgs mechanism (geometrical) and 5. comment on the realization of supersymmetry.

### 1. The Superconnection

We may start heuristically as in the Kaluza-Klein approach by splitting the connection in two parts  $A_M = (A_\mu, A_j)$  which take their values on some subspace  $N(n)$  of a matrix space  $M(n)$ . The new ansatz is to consider the "j-direction" discrete which leads to a specific  $Z_2$  grading /2/:  $A_M = (A_\mu, \phi)$  so that introducing also a  $Z_2$  grading in the matrix space  $M(n)$ , the  $A_\mu(\phi)$  takes its values on the even part  $M_0(n)$  (odd part  $M_1(n)$ ) of  $M(n)$ . The superconnection takes the form of a 2x2 block matrix which may be represented by

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{pmatrix} = \begin{pmatrix} A & i/\mu\phi \\ i/\mu\bar{\phi} & B \end{pmatrix} \quad (1)$$

$A$  and  $B$  are 1-forms, gauge fields and the  $\phi, \bar{\phi}$  are 0-forms, Higgs fields.  $\mu$  is a mass scale which we may put equal 1. Superconnections were recently studied also in /6/.

### 2. The Supersymmetry $SU(2|1)$

For  $N(n)$  we take the graded Lie algebra  $SU(2|1)$  /7/ the even part of which,  $SU(2) \times U(1)$ , corresponds to the electroweak part of the SM. The  $\text{Lie}(SU(2|1))$  with supercommutator  $\langle, \rangle$  is given by

$$\text{Lie}(SU(2|1)) = \{m \in M(3) | m^+ = -m, \text{str } m = 0\}. \quad (2)$$

With the grading matrix  $\Gamma = \text{diag}(1, 1, -1)$  we have for the even and odd projections of  $m, m_{0/1} = 1/2(m \pm m\Gamma)$ . For the reasons explained in /4/ we take the graded associative product in  $M(n)$ :

$$m \cdot n = m_0 n_0 + m_0 n_1 + m_1 n_0 + i m_1 n_1. \quad (3)$$

The graded commutator is given by

$$\langle m, n \rangle = [m_0 n_0] + [m_0, n_1] + [m_1, n_0] + i[m_1, n_1], \quad (4)$$

where e.g.  $m_0 n_1$  corresponds to the usual matrix multiplication and  $[,]$  ( $\{, \}$ ) to the commutator (anticommutator). The generators of  $SU(2|1)$  in the Cartan

basis is given by /3,4,8/  $J_3 = \text{diag } 1/2(1,-1,0)$ ,  $Y = \text{diag } (1,1,2)$  and

$$J_+ = 1/\sqrt{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Omega_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Omega_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (5)$$

and

$$J_- = J_+^\dagger, \quad \Omega_-^\dagger = \Omega_+^\dagger, \quad \Omega_+^\dagger = \Omega_-^\dagger.$$

The superconnection can be expressed in this basis /3/ as:

$$\mathcal{A} = i(\sqrt{2} \vec{J} \cdot \vec{W} + 1/\sqrt{6} Y W^8 + \Omega_+ \bar{\Phi}^0 + \Omega_- \Phi^- + \Omega_-^\dagger \phi^0 + \Omega_+^\dagger \phi^+) . \quad (6)$$

It can also be represented by the block matrix (1) with  $\bar{\Phi} := (\bar{\phi}^0, \bar{\phi}^+)$ .

### 3. Superderivative and Supercurvature

A new ansatz which has far-reaching consequences (Higgs mechanism) is the adjointed derivative  $d_M$  which acts on  $N(n) \subseteq M(n)$  /2/. In our case it can be constructed by use of the odd element  $n \in N_1(n)$ :

$$d_M m := \langle n, m \rangle = [n, m_0] + i\{n, m_1\} \quad (7)$$

with

$$n = i(c_1 \Omega_+ + c_2 \Omega_- + hc) = i \begin{pmatrix} 0 & C \\ \bar{C} & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} . \quad (8)$$

$C$  can be chosen  $c_1 = 1$ ,  $c_2 = 0$ . So we have in the space of matrices  $M(n)$  the graded associative algebra (multiplication as in eq.(3)) with the derivative  $d_M$ :  $\text{GDA1} = (M(n), \cdot, d_M)$ . Together with the De Rham algebra on the space-time  $X$ , with the usual Cartan differential  $d_C$ ,  $\text{GDA2} = (\Lambda^*(X), \wedge, d_C)$ . By use of the graded tensor product we may construct the space of matrix valued forms on  $X$ :

$$\text{GDA3} := \text{GDA1} \otimes \text{GDA2} = (M(n) \otimes \Lambda^*(X), \otimes, d) . \quad (9)$$

This leads immediately for the elements  $M, N$  of  $\text{GDA3}$  ( $M = m \otimes f$ ,  $N = n \otimes g$ ) to the total grading  $\partial(m \otimes f) = (\partial m + \partial f) \text{ mod } 2$ , to the total associative product:

$$m \otimes f \otimes n \otimes g = (-)^{\partial n \partial f} m \cdot n \otimes f \wedge g , \quad (10)$$

to the total derivative

$$d(m \otimes f) = (d_M m) \otimes f + (-)^{\partial m} m \otimes d_C f , \quad (11)$$

and to the graded commutator

$$[M, N]_g := M \otimes N - (-)^{\partial M \partial N} N \otimes M . \quad (12)$$

Since there is no danger of confusion we use  $\langle M, N \rangle = \text{Tr} M N$ . The superconnection  $\tilde{\mathcal{A}}$  may be recognized as an element of  $\text{Lie}(SU(2|1)) \otimes \Lambda^1(X)$  with  $\text{rank} \tilde{\mathcal{A}} = 1$ . The supercurvature is given canonically by the generalized structure equation

$$\tilde{\mathcal{F}} = d\tilde{\mathcal{A}} + 1/2 \langle \tilde{\mathcal{A}}, \tilde{\mathcal{A}} \rangle. \tag{13}$$

$\tilde{\mathcal{F}}$  is an element of  $\text{Lie}(SU(2|1)) \otimes \Lambda^2(X)$  and has of course the same block structure as the superconnection  $\tilde{\mathcal{A}}$ . It turns out that the following splitting is relevant /4,5/

$$\tilde{\mathcal{A}} = \mathcal{A}_3 + \mathcal{A}_0, \quad \mathcal{A}_0 = -\gamma. \tag{14}$$

It corresponds to defining the backshifted field  $\mathcal{F} := \tilde{\mathcal{F}} - C$  and leads also to a splitting of  $\tilde{\mathcal{A}}$ , given by

$$\tilde{\mathcal{F}} = \mathcal{F}_3 + \mathcal{F}_0 \quad \text{with} \quad \mathcal{F}_0 = -1/2 \gamma^2 \quad \text{and} \tag{15}$$

$$\mathcal{F}_3 = d_C \mathcal{A}_3 + 1/2 \langle \mathcal{A}_3, \mathcal{A}_3 \rangle. \tag{16}$$

It is remarkable that  $\mathcal{A}_0$  is invariant under constant  $SU(2|1)$  transformations.  $\mathcal{A}_0$  and  $\mathcal{F}_0$  are "absolute elements" in the theory.\*  $\mathcal{F}_0$  is proportional to the electric charge  $\mathcal{F}_0 = 1/2 \gamma^2 (J_3 + 1/2 Y)$ , and may be considered as acting as a constant magnetic field on a spherically symmetric atom /4,5/. The final Lagrangian has only the  $(U(1))_0$  symmetry. We may add a constant  $C_F = -1/2 \gamma^2 (J_3 + xY)$  and we have  $\mathcal{F} = \tilde{\mathcal{F}} + C_F = \mathcal{F}_3 + 1/2 aY$ , with  $(a = 1/2 - x)$ . In terms of  $\mathcal{F}$  (the backshifted field), the Lagrangian has the symmetry  $SU(2) \times U(1)$  of the SM.

4. The Bosonic Lagrangian and Higgs Mechanism (geometrical)

Given the supercurvature  $\tilde{\mathcal{F}}$ , the Lagrangian may be constructed canonically by the trace /2/:

$$\mathcal{L} = -\text{Tr} \langle \tilde{\mathcal{F}}^* \tilde{\mathcal{F}} \rangle. \tag{17}$$

Taking the trace leads immediately, as expected, to the breaking of the supersymmetry which is both theoretically and experimentally inevitable. This leads to /2-5/

$$g^2 \mathcal{L} = -1/4 (\mathcal{F}_{\mu\nu})^2 - 1/4 (\mathcal{F}_{\mu\nu}^8)^2 + 2 \overline{D_\mu \psi} \psi + D^\mu \psi \psi - V(\psi) \tag{18}$$

with

$$D_\mu \psi := D_\mu \psi + AC - CB; \quad D_\mu^\dagger := d_C^\dagger + A\psi - \psi B \tag{19}$$

and

$$2g^2 V(\psi) = 4[5\lambda - 3/2 a c_1^2 \psi^2 + 3a^2 c_2^4 (\psi = \psi + C)]. \tag{20}$$

\* Isospin transformation leads to the special choice  $c_1 = 0, c_2 = 0$  ( $\psi = \psi$ ).

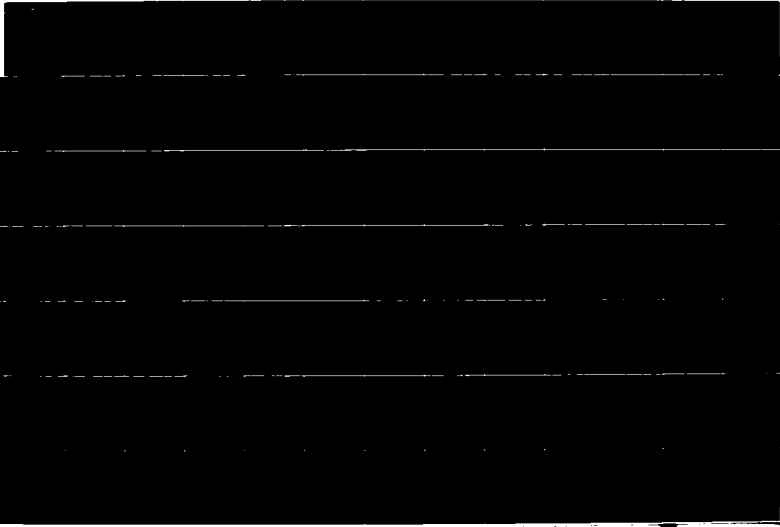
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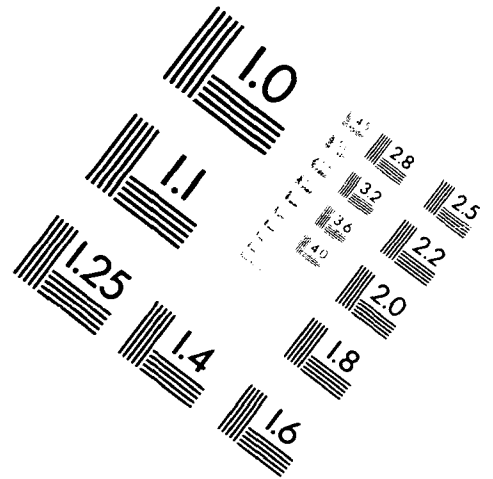
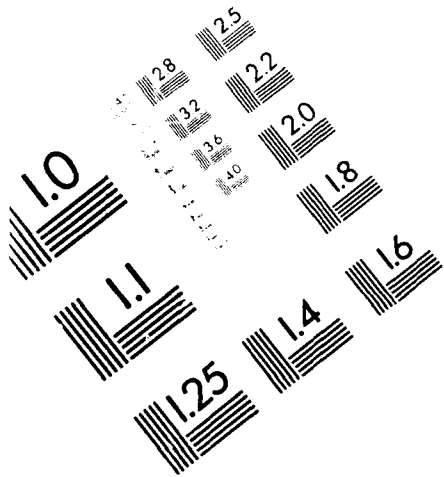




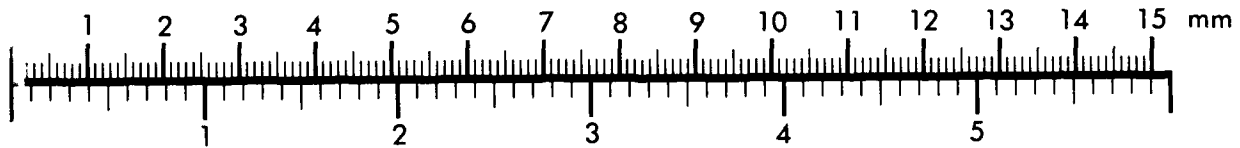
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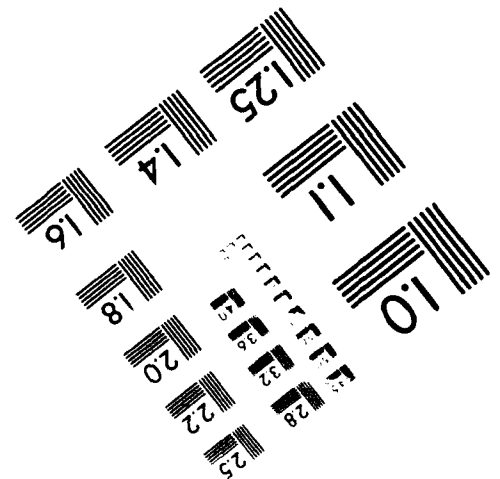
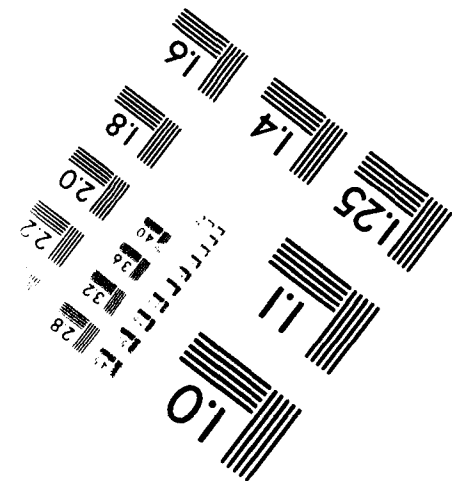
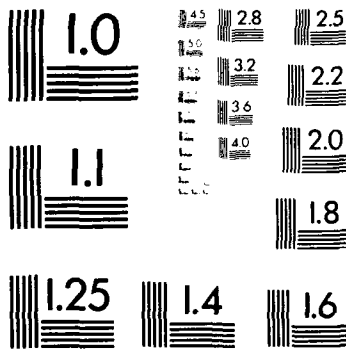
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$V$  is characterized by the center  $-C$  and the radius of the minimum in the  $\phi$ -space. These results are directly visible in the supercurvature  $\hat{\mathcal{F}}$ , too: In its block matrix structure, we have for the essential terms:

$$\hat{\mathcal{F}}_{11} = F^A + F^\phi, \quad \hat{\mathcal{F}}_{12} = -i \mathcal{D}_\mu \phi. \quad (21)$$

$F^A$  leads via (17) to the kinetic term in  $A$ , and  $F^\phi$ , given by

$$F^\phi = -i (\phi C + \bar{C} \phi + \phi \phi), \quad (22)$$

leads via (17) to the potential ( $V(\phi) \sim \text{Tr}(F^\phi)^2$ ).

It is important to note that the vacuum degeneracy arises from the term  $\phi C + \bar{C} \phi = d_M A^{\text{odd}}$  in  $F^\phi$  and the  $W$ - and  $Z$ -mass from the term  $AC - C B = d_M A^{\text{even}}$  in  $\mathcal{D}_\mu \phi$ . We realize that the adjoined matrix derivative  $d_M$  is responsible for the spontaneous symmetry breaking and the Higgs mechanism. This constitutes a new and geometrical derivation of the Higgs mechanism. We have furthermore obtained in (18) the Lagrangian of the standard electroweak model, with some parameters fixed by the above construction of  $\mathcal{F}$  and  $\mathcal{L}$ . This may fix the value of the Weinberg angle to  $\theta_w = \pi/6$  and lead to the ratio  $M_H/M_W = \sqrt{2}/3$ . There are however doubts whether this fixing is legitimate.

#### 5. On the Realization of Supersymmetry

I would like to terminate with a few remarks which at the moment reflect more my own feelings than imperturbable theorems. *Phenomenologically, the fact that leptons and quarks  $((\nu_L, e_L, e_R), (u_L, d_L, u_R, d_R))$  fit into irreducible representations of  $SU(2|1)$  is a most important observation [3,7,9].* If we try to interpret  $SU(2|1)$  as an inner symmetry, and if we are not willing to accept additional particles beside the experimental, existing ones, we are forced to interpret the odd transition, e.g.  $\Omega_-: e_R \rightarrow e_L$  ( $u_R \rightarrow u_L$ ), not as corresponding to the usual vector current induced by vectorlike particles, but as "scalar currents", corresponding to Yukawa couplings and induced by spin zero particles like the  $\phi$ . This leads us again back to the SM. We have a model which contains in an unusual way the  $SU(2|1)$  supersymmetry and it is not in contradiction to the experiments. The prize for this achievement is that only a few not very strong predictions (e.g. the Cabibbo-Kobayashi-Maskawa matrix) beyond the SM are possible [3]. Theoretically this model may contain a new mathematical structure, corresponding to a graded connection with adjoined derivative.

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# ON THE SOLUTION OF THE RELATIVISTIC TWO BODY EQUATION

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## 1 The Radial Equation

Recently there has been much interest in the relativistic dynamics of two (or more) interacting particles. In fact the dynamics of interacting fermions via the electromagnetic field is the basic problem for tests of Quantum Electrodynamics (QED) in low energy bound state problems.

A relativistic two body equation, for two matter fields  $\psi_1$  and  $\psi_2$  interacting via the electromagnetic field, has been derived from first principles of QED:

$$[(\gamma_\mu i\partial_\mu - m_1) \otimes \gamma^0 + \gamma^0 \otimes (\gamma_\mu i\partial_\mu - m_2) + V]\phi(x, y) = 0 \quad (1)$$

where  $x^0 \neq |\bar{x} - \bar{y}|$ ,  $\phi(x, y) = \psi_1(x) \otimes \psi_2(y)$  is a 16-component composed field. The relativistic potential  $V$  contains terms coming from minimal and pauli coupling, for the minimal coupling:

$$V = -\frac{e_1 e_2}{r} \gamma^\mu \otimes \gamma_\mu \quad (2)$$

In the tensor product  $\otimes$  we shall always mean particle 1 first and particle 2 second.

Equation (1) has many remarkable properties, among them the exact separability of the center of mass and relativistic coordinates. One then sees that it is a one-time equation. Moreover, equation (1) can be fully separated into angular and radial parts. The radial part consists of 16 radial equations which separates into two sets of eight. In each set four of the eight equations are algebraic. The algebraic equations are used to eliminate four component functions. Thus we arrive for the first set at four coupled first order differential equations:

$$\left. \begin{aligned} & \left[ \left( E + \frac{2\alpha}{r} \right) - \frac{M}{E} \left( E + \frac{2\alpha}{r} \right) - \frac{4J^2}{r^2} \right] u_1 + 2 \left( E + \frac{2\alpha}{r} \right) \frac{dz_2}{dr} + 2 \frac{\Delta m J}{r} y_0 = 0 \\ & \left[ \left( E - \frac{2\alpha}{r} \right) - \Delta m^2 \right] z_2 - 2E \frac{dy_0}{dr} - 2 \frac{\Delta m J}{r} v_{00} = 0 \\ & \left[ \left( E - \frac{2\alpha}{r} \right) + \left( E + \frac{2\alpha}{r} \right) - M - \frac{4J^2}{Er^2} \left( E - \frac{2\alpha}{r} \right) \right] v_{00} - 2 \left( E - \frac{2\alpha}{r} \right) \left( \frac{dy_0}{dr} + \frac{y_0}{r} \right) \\ & - 2 \frac{\Delta m J}{Er} \left( E - \frac{2\alpha}{r} \right) z_2 = 0 \\ & \left[ E \left( E + \frac{2\alpha}{r} \right) - \Delta m^2 \right] y_0 + 2 \left( E + \frac{2\alpha}{r} \right) \left( \frac{dv_{00}}{dr} - \frac{v_{00}}{r} \right) + 2 \frac{\Delta m J}{r} u_1 = 0 \end{aligned} \right\} \quad (3)$$

where  $E$  is the total energy in the Center of mass frame ( $M = m_1 + m_2$ ),  $\Delta m = m_1 - m_2$ ,  $\alpha = -\epsilon_1 \epsilon_2$ ,  $J^2 = j(j+1)$  and  $j$  is the total angular momentum. The indices of the function  $u_1, y_0, z_2$  and  $v_{00}$  denote the spin components. For the other set, we obtain identical equations to (3) with the replacement  $\Delta m \rightarrow M$  and changing sign of some component functions.

Due to the  $(1/r^2)$  singularities equations(3) do not possess simple power series solutions. In fact substitution into (3) with:

$$u_1 = \sum a_n r^n, \dots, v_{00} = \sum b_n r^n \quad (4)$$

leads to 3-term and 4-term recurrence relations which are difficult to solve for  $a_n, \dots, b_n$ .

One may try to eliminate more component functions by forming O.D.E. of the second order out of the set (3). We still obtain four equations in two sets. The first set is the one of two coupled equations:

$$\left. \begin{aligned} & \frac{du_1}{dr^2} + \frac{2\alpha E}{r^2 [E(E + \frac{2\alpha}{r}) - \Delta m^2]} \frac{du}{dr} + \left\{ \frac{[E(E + \frac{2\alpha}{r}) - \Delta m^2][E(E + \frac{2\alpha}{r}) - M^2]}{4\epsilon^2} - \frac{J^2}{r^2} \right\} u_1 \\ & + \frac{2\alpha \Delta m^2}{r^3 [E(E + \frac{2\alpha}{r}) - \Delta m^2]} v_{00} = 0 \\ & \frac{d^2 v_{00}}{dr^2} + \frac{2\alpha \Delta m^2}{r^2 (E + \frac{2\alpha}{r}) [E(E + \frac{2\alpha}{r}) - \Delta m^2]} \frac{dv_{00}}{dr} + \\ & + \left\{ \frac{[(E - \frac{2\alpha}{r})(E + \frac{4\alpha}{r}) - m^2][E(E + \frac{2\alpha}{r}) - \Delta m^2]}{4(E - \frac{2\alpha}{r})(E + \frac{2\alpha}{r})} - \frac{J^2}{r^2} - \frac{2\alpha \Delta m^2}{r^3 (E + \frac{2\alpha}{r}) [E(E + \frac{2\alpha}{r}) - \Delta m^2]} \right\} v_{00} + \\ & \frac{2\alpha \Delta m J}{r^3 [E(E + \frac{2\alpha}{r}) - \Delta m^2]} v_{00} = 0. \end{aligned} \right\} \quad (5)$$

The second set consists of another one of two coupled second order equations in the two functions  $y_0$  and  $z_2$  but with more complex structures. It is clear that these equations are more singular than (3) and simple power series solutions do not exist. In the following, functional series solutions are suggested and this seems to work. The basic idea is to replace the set  $\{r^n\}$  in (4) by an appropriate set of complete functions  $\{f_n\}$  and hope that such series terminate. In the next two sections we shall illustrate the method but with some limiting cases. These limits are obtained by expanding the potential terms as power series of  $(\alpha/r)$ . In the power counting  $(1/r)$  is counted as  $\alpha$ .

## 2 The free System ( $\alpha = 0$ )

For the first set of the radial equations we obtain:

$$\left. \begin{aligned} & \frac{d^2 u_1}{d\rho^2} + \left(1 - \frac{J^2}{\rho^2}\right) u_1 = 0, \quad \frac{d^2 v_{00}}{d\rho^2} + \left(1 - \frac{J^2}{\rho^2}\right) v_{00} = 0 \\ & \frac{d^2 z_2}{d\rho^2} - \frac{2J^2}{\rho^3 \left(\epsilon^2 - \frac{J^2}{\rho^2}\right)} \frac{dz_2}{d\rho} + \left[\epsilon^2 - \frac{J^2}{\rho^2}\right] z_2 - \frac{2\alpha J}{\rho^2 \left(\epsilon^2 - \frac{J^2}{\rho^2}\right)} y_0 = 0 \\ & \frac{d^2 y_0}{d\rho^2} - \frac{2J^2}{\rho^3 \left(\epsilon^2 - \frac{J^2}{\rho^2}\right)} \frac{dy_0}{d\rho} + \left[\epsilon^2 - \frac{J^2}{\rho^2} - \frac{2E^2}{\rho^2 \left(\epsilon^2 - \frac{J^2}{\rho^2}\right)}\right] y_0 - \frac{2\alpha J}{\rho^2 \left(\epsilon^2 - \frac{J^2}{\rho^2}\right)} z_2 = 0 \end{aligned} \right\} \quad (6)$$

where  $\rho = kr$ ,  $4k^2 = \frac{(E^2 - \Delta m^2)(E^2 - M^2)}{E^2} = \frac{E^2}{E^2 - \Delta m^2}$ , and  $a = \frac{\Delta m^2 \epsilon^2}{E}$ .

The first two equations have the well known regular (at  $\rho = 0$ ) solutions  $u_1 = \rho j_j(\rho)$  and  $v_{00} = \rho j_j(\rho)$ , where  $j_n(\rho)$  is the spherical Bessel function. The form of the other two coupled equations suggest the solutions  $z_2 = \sum_{n=0} A_n \rho j_{n+s}(\rho)$  and  $y_0 = \sum_{n=0} B_n \rho j_{n+s}(\rho)$ . Substituting into (6), it can be shown that the initial equations imply  $s = j - l$ . Moreover, if  $A_1 = 0 = b_1$ , then  $A_n = 0 = B_n$  for odd  $N$ . For even  $n$  (put  $n + 2 = 2m$ ) we obtain the recurrence relations:

$$-J^2 \frac{(2m+j-3)(2m+j-4)}{\{2(2m+j)-5\}\{2(2m+j)-3\}} A_{2m-2} - J^2 \frac{(2m+j+2)(2m+j+3) - J^2}{\{2(2m+j)+1\}\{2(2m+j)+3\}} A_{2m+2} \\ + \left[ \epsilon^2 \{(2m+j)(2m+j-1) - J^2\} - 2J^2 \frac{(2m+j-1)(2m+j) - J^2 + 3}{\{2(2m+j)+1\}\{2(2m+j)-3\}} \right] A_{2m} \\ - 2aJ B_{2m} = 0 \quad (7)$$

$$-J^2 \frac{(2m+j-3)(2m+j-4)}{\{2(2m+j)-5\}\{2(2m+j)-3\}} B_{2m-2} - J^2 \frac{(2m+j+2)(2m+j+3) - J^2}{\{2(2m+j)+1\}\{2(2m+j)+3\}} B_{2m+2} \\ + \left[ \epsilon^2 \{(2m+j)(2m+j-1) - J^2\} - 2J^2 \frac{(2m+j-1)(2m+j) - J^2 + 3}{\{2(2m+j)+1\}\{2(2m+j)-3\}} \right] A_{2m} \\ - 2aJ A_{2m} = 0 \quad (8)$$

Notice that for  $m = 2$  the coefficients of  $A_2$  in (7) and  $B_2$  in (8) vanish. Hence, if we assume that  $A_4 = 0 = B_4$ , then  $A_{2m} = 0 = B_{2m}$  for  $m \geq 2$ . For ( $A_{-n} = 0 = B_{-n}$ ) (7) and (8) lead to four equations, only two of them are independent in the coefficients  $A_0, B_0, A_2, B_2$ . Choosing  $A_0 = 0$ , then  $B_0 = 0$ , we obtain the two solutions:

$$z_2 = A_2 \rho j_{j+1}(\rho), \quad y_0 = B_0 \rho j_{j-1} + B_2 \rho j_{j-2} \quad \text{and} \quad z_2 = A_0 \rho j_{j-1} + A_2 \rho j_{j+1}, \quad y_0 = B_2 \rho j_{j+1},$$

which can be checked by direct substitution in (6).

### 3 The Interacting System ( $\alpha \neq 0$ )

Due to space limitation, we consider only two limiting cases for the set (5).

(i) Up to  $\alpha^2$ , for the set (5) we obtain:

$$\frac{d^2}{d\rho^2} \begin{pmatrix} u_1 \\ v_{00} \end{pmatrix} + \left[ -\frac{1}{4} + \frac{\lambda}{\rho} - \frac{J^2}{\rho^2} \right] \begin{pmatrix} u_1 \\ v_{00} \end{pmatrix} = 0 \quad (9)$$

where we put  $\rho = 2kr$ ,  $4k^2 = \frac{(m^2 - E^2)(E^2 - \Delta m^2)}{2}$  and  $\lambda = \frac{\alpha}{4kE}(2E^2 - M^2 - \Delta m^2)$ . The regular solutions (at  $\rho = 0$ ) of (9) are given by the hydrogenic wave functions  $u_1 \equiv R_{n_j}(\rho)$  and  $v_{00} \equiv R_{n_j}(\rho)$  provide that  $\lambda = n$  (integer) and  $n \geq j + 1$ . This implies the energy mass relation:

$$E^2 = \frac{M^2 + \Delta m^2}{2} \pm \frac{M^2 - \Delta m^2}{2} \left( 1 + \frac{\alpha^2}{n^2} \right)^{-\frac{1}{2}} \quad (10)$$

(ii) Up to  $\alpha^4$ , for the set (5) we obtain:

$$\begin{aligned} \frac{d^2 u_1}{d\rho^2} + \frac{a}{\rho^2} \frac{du_1}{d\rho} + \left[ -\frac{1}{4} + \frac{\lambda}{\rho} - \frac{J^2 - \alpha^2}{\rho^2} \right] u_1 - \frac{a\Delta m J}{E^3 \rho} v_{00} &= 0 \\ \frac{d^2 v_{00}}{d\rho^2} + \frac{\Delta m^2 a}{E^2 \rho^2} \frac{dv_{00}}{d\rho} + \left[ -\frac{1}{4} + \frac{\lambda}{\rho} - \frac{J^2 - \alpha^2 \delta_E}{\rho^2} - \frac{a\Delta m^2}{E^2 \rho^3} \right] v_{00} - \frac{a\Delta m J}{E^3 \rho} u_1 &= 0 \end{aligned} \quad (11)$$

where we put  $\lambda$ ,  $\rho$  and  $k$  as before,  $a = \frac{4\alpha k E}{2}$  and  $\delta_E = \frac{\Delta m^2 - M^2}{2} + \frac{\Delta m^2 M^2}{2}$ .

Again by comparison with (9) one may try the series solutions:  $u_1 = \sum_{l=0} A_l w_{l+s}(\rho)$  and  $\sum_{l=0} B_l w_{l+s}(\rho)$ , where we have put  $R_{nl}(\rho) = w_l(n, \rho)$ . The following recurrence relations (written for first time) are needed:

$$\begin{aligned} l \frac{dw_l}{d\rho} &= \left( \frac{n}{2} - \frac{l^2}{\rho} \right) w_l - \frac{1}{2} \sqrt{n^2 - l^2} w_{l-1} \\ (l+1) \frac{dw_{l+1}}{d\rho} &= \left( \frac{n}{2} - \frac{l^2}{\rho} \right) w_{l+1} - \frac{1}{2} \sqrt{n^2 - (l+1)^2} w_l. \end{aligned} \quad (12)$$

Again, following the same steps of section 2, we show that (11) have

$$u_1 = \sum_{l=0}^{n-j-1} A_l w_{l+j} \quad \text{and} \quad v_{00} = \sum_{l=0}^{n-j-1} B_l w_{l+j}$$

as solutions.

It is to be noted that to have these (regular at  $\rho = 0$ ) solutions, the same energy mass relation (10) is obtained.

Details of these considerations and their physical interpretations will be listed elsewhere. It is worth mentioning that for positronium ( $\Delta m = 0$ ) the radial equations and their solution simplify.

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## VACUUM FLUCTUATIONS IN PRISMATIC CAVITIES

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### INTRODUCTION.

Spectral features and energy level statistics of semiclassical and quantum systems have been a subject of growing interest. It is commonly accepted that a chaotic behavior is described by a Wigner statistics, while Poisson statistics is related to a completely random behavior<sup>1</sup>. Nevertheless, it has been shown that such a sharp distinction may be misleading<sup>2-6</sup>.

In this work, we calculate the energy spectrum of vacuum fluctuations for a massless scalar field which satisfies boundary conditions inside a rectangular box. The spectrum is similar to that of a quantum particle in a billiard, with the difference that all the energy levels are 'occupied'. The fluctuations do not obey a strictly Poisson distribution, and thus we conjecture that a realistic detector would not observe quantum vacuum as 'white noise', except in the unrealistic limit of infinite high frequency. This kind of study is important due to recent results, both theoretical and experimental, of quantum field effects on atomic systems in cavities.

### SPECTRAL DENSITY OF VACUUM FLUCTUATIONS.

Consider a prismatic rectangular cavity with infinite axis along the  $x_1$  direction and transverse lengths  $b$  and  $c$  along  $x_2$  and  $x_3$ , respectively. The Wightman correlation functions for a scalar field  $D^+(x, x') = \langle 0 | \phi(x) \phi(x') | 0 \rangle$ ,

and  $D^-(x, x') = \langle 0 | \phi(x') \phi(x) | 0 \rangle$  that satisfy Dirichlet boundary conditions at  $x_2 = 0, b$  and  $x_3 = 0, c$  are found using the method of images in a straightforward way:

$$D^\pm(x, x') = \frac{1}{4\pi^2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left\{ \frac{1}{(x_1 - x'_1)^2 + (x_2 - x'_2 - 2bm)^2 + (x_3 - x'_3 - 2an)^2 - (t - t' \mp i\epsilon)^2} \right. \\ - \frac{1}{(x_1 - x'_1)^2 + (x_2 + x'_2 - 2bm)^2 + (x_3 - x'_3 - 2an)^2 - (t - t' \mp i\epsilon)^2} \\ - \frac{1}{(x_1 - x'_1)^2 + (x_2 - x'_2 - 2bm)^2 + (x_3 + x'_3 - 2an)^2 - (t - t' \mp i\epsilon)^2} \\ \left. + \frac{1}{(x_1 - x'_1)^2 + (x_2 + x'_2 - 2bm)^2 + (x_3 + x'_3 - 2an)^2 - (t - t' \mp i\epsilon)^2} \right\} \quad (1)$$

Following the procedure outlined in Ref.7, we calculate the energy spectrum at a given point  $\mathbf{x} = (x, y, z)$  inside the cavity. This is achieved by taking the static observer limit  $\mathbf{x}' \rightarrow \mathbf{x}$ ,  $t - t' = \sigma$ , of the Wightman functions, and Fourier transforming with respect to the proper time  $\sigma$ . The double series may be evaluated in closed form with the aid of Poisson summation formula (for details of the calculation see Ref.7). The final result for the energy spectrum  $de/d\omega$  is:

$$\frac{de}{d\omega} = \frac{\omega^2}{2\pi a b} \sum_{k_2} \sum_{k_3} \left( \frac{(1 - e^{2\pi i k_2 y/b}) (1 - e^{2\pi i k_3 z/c})}{[\omega^2/\pi^2 - k_2^2/b^2 - k_3^2/c^2]^{\frac{1}{2}}} \right) \quad (2)$$

In this formula, the integers  $k_2$  and  $k_3$  take all the values such that

$$\omega^2 \leq \omega_k^2 \equiv \pi^2 (k_2^2/b^2 + k_3^2/c^2).$$

The resulting spectrum is highly irregular, with a series of resonances located at  $\omega = \omega_k$ . They can be determined by drawing a rectangular grid with

spacings  $b$  and  $c$ , the resonances are then given by all the possible values of the distance from the origin to each of the crossing points in the grid. In practice, this method permits to calculate a great number of resonances in order to perform a statistical analysis.

The distribution of resonance fluctuations may be formally identified once the distribution has been 'unfolded'<sup>6</sup>, with those of a particle in a two-dimensional rectangular box. According to Berry<sup>1</sup>, the energy levels for those systems should exhibit a Poisson uncorrelated statistics in the semiclassical limit. However, this particular spectrum has interesting properties which have been the subject of exhaustive statistical study in the last few years<sup>2-5</sup>. First of all, when the ratio  $b/c$  is a rational, the distribution cannot be Poissonian because the energy levels become infinitely degenerate. On the other hand, when the parameter  $b/c$  is taken to be irrational, unexpected fluctuations in the nearest neighbour spacing distribution occur. This result was discovered by Casati, Chirikov, and Guarnieri<sup>2</sup>, who found that the level spacing distribution exhibits a significant deviation from a Poisson distribution, and that the long-term correlations characterized by a  $\Delta_3$  Dyson-Metha statistics<sup>8</sup> differed considerably from the expected linear behavior. They also argued that this result should hold in the multidimensional case. These results gave rise to further studies: Berry<sup>3</sup> calculated the semiclassical limit of the  $\Delta_3$  statistics, clarifying the meaning of the deviations from the Poisson statistics. Feingold<sup>4</sup> found that a slow and irregular decay of the excessive fluctuations was obtained when millions of states were included.

What we want to stress in this work is that the scalar vacuum field spectrum is a definite physical example for which the above results apply, and that vacuum fluctuations inside a cavity are not strictly uncorrelated. The

point is that, although the deviations from Poisson statistics may decrease for large quantum numbers, in practice there is always a cutoff frequency above which the approximation of impenetrable walls fails to be reliable. Furthermore, a physical detector is susceptible only in a narrow frequency windows around its own resonant frequency, and thus measures  $\Delta_j$  in practice. As pointed out above, the detector does not observe a Poissonian distribution.

The conclusion is that the spectrum of vacuum fluctuations in a cavity is Poissonian only for high frequencies and in the limit of ideal conducting boundaries, but that less idealized conditions may open the way to chaos in quantum field theory.

A detailed analysis of the vacuum fluctuations and Casimir energy in rectangular closed cavities, both for a scalar and an electromagnetic field, will be considered in forthcoming articles.<sup>9-10</sup>

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## Delta Potentials on Planes in QED

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### Abstract

QED is considered in the presence of delta shaped external gauge potentials with support on one or two planes. Using the propagators determined in these special field configurations the parameter dependence of the vacuum energy (similar to the Casimir effect) is calculated. Thereby, it turns out that also in the case of massive fields nonrelativistic field theory is unable to approximate the results of relativistic field theory. Surprisingly, a parallel calculation using the zeta function method leads to a wrong result if one does not investigate the nonleading terms carefully. First loop calculations exhibit an unexpected renormalization behaviour which may be typical for certain singular background fields.

### INTRODUCTION

$\delta$ -functions are broadly used idealized elements of theoretical physics. With its help it is possible to formulate models which in many cases can be solved explicitly. In quantum mechanics quite a number of such investigations exists [1] whereas in quantum field theory investigations of this kind are just at the beginning. Here, we consider the case of  $\delta$ -functions with support on (parallel) planes so that they effectively depend on one coordinate only:

$$e\mathbf{A} = 0 \quad , \quad eA_0 = \sum_{i=1}^n a_i \delta(x_3 - d_i) \quad . \quad (1)$$

With such a procedure we in fact introduce a more general type of boundary conditions in field theory which generalizes the mostly used Dirichlet boundary condition. In physical terms, such a  $\delta$ -function may be seen as a model of a penetrable boundary. From another point of view, it can be considered as a generalized potential which contains at most one bound state for each degree of freedom.

### QUANTIZED FIELDS IN EXTERNAL DELTA POTENTIALS

The most simple case seems to be the charged scalar field, described by the Klein-Gordon equation  $[(\partial_\mu - ieA_\mu)^2 + m^2]\phi(x) = 0$ . Inserting (1) for the potential  $A_\mu$  there a difficulty connected with the product of two  $\delta$ -functions appears. One possible solution consists in the choice of

$$[\square + m^2 - 2 \sum_{i=1}^n b_i \delta(x_3 - d_i)]\phi(x) = 0 \quad (2)$$

as new field equation. The drawback of this equation is that due to the simple coupling  $2b_i \delta(x_3 - d_i)\phi(x)$  (resulting from the term  $(eA_0)^2$ ) the charge sensitivity has been lost. Nevertheless, we will study this equation because it is much simpler than the Dirac equation which will be considered later. The  $\delta$ -potential leads to the additional boundary condition for the scalar field at the position of the  $\delta$ -function

$$\partial_3 \phi|_{x_3=d_i+\epsilon} - \partial_3 \phi|_{x_3=d_i-\epsilon} = -2b_i \phi|_{x_3=d_i}. \quad (3)$$

The positive (negative) energy solutions of this field equation consist of one bound state, symmetric, and antisymmetric scattering states. The quantum field can be constructed with the help of a mode decomposition containing creation  $a^+$ ,  $b^+$  and destruction operators  $a^-$ ,  $b^-$ .

For later calculations we need the Feynman propagator. It can be written as follows:

$${}^s D^c(x, y) = D^c(x - y) + \bar{D}(x, y) \quad (4)$$

$$D^c(x - y) = \frac{i}{2} \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{1}{\Gamma} e^{i\vec{p}(\vec{x} - \vec{y}) + i\Gamma|x_3 - y_3|} \quad (5)$$

$$\bar{D}(x, y) = -\frac{b}{2} \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{1}{\Gamma - ib\Gamma} e^{i\vec{p}(\vec{x} - \vec{y}) + i\Gamma(|x_3| + |y_3|)} \quad (6)$$

(the unusual notations are  $\vec{p} = (p_0, p_1, p_2)$ ,  $\vec{x} = (x_0, x_1, x_2)$  and  $\Gamma = \sqrt{\vec{p}^2 - m^2 + i\epsilon}$ ), where  $D^c(x - y)$  is the standard propagator of free field theory and  $\bar{D}(x, y)$  an additional term containing the correction due to the  $\delta$ -function potential. This unusual representation is quite appropriate for all further calculations. The second part of the propagator explicitly contains the bound state  $\Gamma = ib$  for  $b > 0$  as pole in the physical sheet  $Im \Gamma > 0$ . For  $b \rightarrow -\infty$  the propagator satisfies the Dirichlet boundary condition.

### GENERALIZATIONS

If we want to discuss Casimir-like configurations with two planes represented by  $\delta$ -functions then we have to repeat the same construction like above for the field equation with  $i = 1, 2$  and  $b_1 = b_2$ . Here, the field modes are much more complicated. Again, they contain bound states, symmetric, and antisymmetric scattering states. Without going into detail [2],[3] we quote the result for the propagator only

$$D(x, y) = -\frac{b}{2} \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{e^{i\vec{p}(\vec{x} - \vec{y})}}{\Gamma} \frac{1}{(\Gamma - ib)^2 + b^2 e^{2i\Gamma d}}$$

$$\cdot \left\{ (\Gamma - ib)e^{i\Gamma(|x_3-d_1|+|y_3-d_1|)} + ibe^{i\Gamma(|x_3-d_1|+|y_3-d_2|+d)} + (d_1 \leftrightarrow d_2) \right\} \quad (7)$$

Note that also here the discrete eigenstates (bound states) appear as zeros of the denominator. In addition, there zeros of the denominator exist which do not lie on the real axis in the  $p_0$ -plane (so that they do not belong to the spectrum) and which could be interpreted as resonance states.

Let us now turn to the more interesting case of the Dirac equation which looks for the special potential as follows

$$[i\gamma^\mu \partial_\mu - m + \gamma^0 a \delta(x_3 - d_1)]\psi(x) = 0 \quad (8)$$

The substitution of the  $\delta$ -function by a boundary condition for the Dirac spinor is also nontrivial. We have to take into account that the field itself cannot be continuous at the position of the  $\delta$ -function. So, the following boundary condition can be derived [2],[3]:

$$\psi|_{x_3=d_1+\epsilon} = R\psi|_{x_3=d_1-\epsilon} \quad , \quad R = \exp(i\gamma^0\gamma^3\Theta) \quad , \quad \sin\Theta = \frac{a}{(1+a^2/4)} \quad (9)$$

Again, the energy eigenstates are found. In opposition to the approximated Klein-Gordon equation the charge sensitivity is preserved. As it should be, there are either bound states for the particle or for the antiparticle. For simplicity, we write down the propagator  $S^{cc}(x, y) = S^c(x - y) + \tilde{S}(x, y)$  corresponding to one  $\delta$ -function only where

$$S(x, y) = \frac{a}{4} \int \frac{d^3\tilde{p}}{(2\pi)^3} \frac{e^{i\tilde{p}(\tilde{x} - \tilde{y}) + i\Gamma(|x_3| + |y_3|)}}{\Gamma^2} (\tilde{p}\tilde{\gamma} + m - \epsilon(x_3)\gamma^3\Gamma) \frac{(\gamma^0\Gamma - (ia/2)(\tilde{p}\tilde{\gamma} - m)}{\lambda_- \Gamma - ia p^0} (\tilde{p}\tilde{\gamma} + m + \epsilon(y_3)\gamma^3\Gamma) \quad (10)$$

The energy eigenfunctions as well as the propagators for one or two delta functions can be found in [2],[3].

#### VACUUM ENERGY

As the simplest quantity of physical interest we calculate the vacuum energy per unit area corresponding to two  $\delta$ -potentials separated by the distance  $d$ . This is a slight generalization of the classical Casimir problem where the plates are now idealized by  $\delta$ -functions. We illustrate the procedure for the scalar field. The vacuum energy per unit area is given by

$$E_{vac} = \int_{-\infty}^{+\infty} dx_3 \langle 0|T_{00}|0 \rangle, \quad T_{00} = P(\partial_x, \partial_y)\phi(x)\phi(y)|_{x \rightarrow y} \quad (11)$$

where  $T_{\mu\nu}$  is the energy momentum tensor for the scalar field written here in a symbolic notation containing a point splitting procedure (useful for the regularization

process) and the differentiation operator  $P$ . So, by formally taking the vacuum expectation value we arrive at an expression containing the Feynman propagator as an essential element.

$$E_{vac} = -i \int_{-\infty}^{+\infty} dx_3 \partial_{x_0} \partial_{y_0} [D^c(x-y) + \bar{D}(x,y)] |_{x \rightarrow y} \quad (12)$$

The aim of our calculation is to determine the distance dependent part of the vacuum energy, therefore all other distance independent contributions will be omitted. Obviously, this concerns the contribution from the free field propagator  $D^c$  as well as further parameter independent contributions. After some algebra we obtain an expression which for large distances leads to

$$E_{vac} = \begin{cases} \frac{-b^2}{8(m-b)^2} \left(\frac{m}{\pi d}\right)^{3/2} e^{-2md} & , \quad m \neq 0, \quad b < m \\ -\frac{\pi^2}{720} \frac{1}{d^3} & , \quad m = 0 \end{cases} \quad (13)$$

The spinor case which corresponds to the field equation

$$[i\gamma^\mu \partial_\mu - m + \gamma^0 a(\delta(x_3 - d_1) + \delta(x_3 - d_2))] \psi(x) = 0 \quad (14)$$

can be treated in the same manner but the algebra is much more involved. The result is

$$E_{vac} = \begin{cases} \frac{1}{4} \frac{a^2}{\lambda_-} \left(\frac{m}{\pi d}\right)^{3/2} e^{-2md} & , \quad \lambda_- = 1 - \frac{a^2}{4} & , \quad m \neq 0 \\ \frac{1}{6\pi^2} \frac{a^2}{d^3} & , \quad m = 0 \end{cases} \quad (15)$$

The conclusions following from these calculations are: for large distances (which is the physically interesting limit in any case) the contributions of massive fields to the Casimir effect (electrons contained in metallic plates etc.) are exponentially suppressed. For massless scalar fields, the well-known Casimir result is recovered. In the spinor case, opposite to the scalar theory the resulting Casimir force is repulsive. One further interesting point concerns a corresponding nonrelativistic calculation. Usually one believes that the essential impact of metallic plates is to change the low energy spectrum of the fluctuations of the electromagnetic field, therefore the Casimir effect is considered as an infrared effect. If this would be true for the case of massive fields in the presence of delta functions then a nonrelativistic calculation should be possible. An explicit nonrelativistic calculation [3] shows that this is not the case, the distance dependent part of the vacuum energy (at least for  $a < 0$ ) vanishes. This means that the deformation of the energy spectrum caused by a nonrelativistic approximation is so serious that it leads to a wrong approximation for the Casimir energy.

### ZETA FUNCTION METHOD

The  $\zeta$ -function method is a very powerful method for calculating effective actions and vacuum energies. The mathematical background is as follows: Let  $K$  be a self-adjoint operator with a discrete spectrum  $K\phi_n = \lambda_n\phi_n$  and nonzero eigenvalues  $\lambda_n$  corresponding to the normalized eigenfunctions  $\int dx \bar{\phi}_n(x)\phi_m(x) = \delta_{nm}$ . Then, we define the  $\zeta$ -function of the operator  $K$  as

$$\zeta_K(s) = \text{Tr}[K]^{-s} = \int dx dy \delta(x-y) \sum_{n=1}^{\infty} \lambda_n^{-s} \bar{\phi}_n(x)\phi_n(y) \quad (16)$$

$$= \sum_{n=1}^{\infty} \lambda_n^{-s} < \infty \quad \text{for } \text{Re } s > s_0 \quad (17)$$

In physics, however, discrete eigenvalues of operators are not the rule. As an example, we study a complex scalar field under the influence of two  $\delta$ -potentials. First we have to turn to Euclidean field theory. The operator is  $K = -(\partial_4^2 + \Delta) + m^2 - 2b(\delta(x_3 - d_1) + \delta(x_3 - d_2))$  where with the help of the generalized boundary conditions the  $\delta$ -functions determine a self-adjoint operator. If we choose  $b < 0$  then this operator possesses a continuous spectrum with no discrete eigenvalue. So, we cannot expect to obtain a physically meaningful result using the  $\zeta$ -function method. To have discrete eigenvalues we introduce one further boundary condition in  $x_3$ -direction namely we are considering a finite interval of length  $L$  with Dirichlet boundary conditions at its ends. It turns out that this is sufficient for a successful application of the  $\zeta$ -function method in the present case. In [4] we obtained the following result for the  $\zeta$ -function:

$$\begin{aligned} \zeta_K(s) = & \frac{V_2 T_E}{2\pi^2} \frac{\Gamma(3/2)\Gamma(s-1/2)}{\Gamma(s)} \left\{ \frac{1}{\pi} \int_0^{\infty} d\kappa (\kappa^2 + m^2)^{(3/2-s)} \left[ 2L - \frac{b}{(b^2 + \kappa^2)} \right] \right. \\ & \left. + \frac{2b^2}{\pi} \int_m^{\infty} d\kappa (\kappa^2 - m^2)^{(3/2-s)} \frac{\left(d + \frac{1}{\kappa-b}\right) e^{-2d\kappa}}{(\kappa-b)^2 - b^2 e^{-2d\kappa}} \right\} \quad (18) \end{aligned}$$

The vacuum energy can be extracted using the formula

$$V_2 T_E E_{\text{vac}} = \text{Tr} \lg K = -\frac{d}{ds} \zeta_K(s)|_{s=0} \quad (19)$$

where the infinite quantities  $V_2$  (volume of a two-dimensional Euclidean space) and  $T_E$  (volume of a one-dimensional Euclidean space, imaginary time) reflecting the symmetries of the problem will drop out for the vacuum energy per unit area by definition. However, the first term in the curly bracket which contains one further infinite contribution ( $2L \rightarrow \infty$ ) is unexpected. This is an untypical contribution for  $\zeta$ -function calculations and it can be omitted by hand because it is parameter independent. This first term would be the result of a naive calculation without imposing additional boundary conditions. The second term describes the dependence of the vacuum energy on the coupling constant and the third term leads to

the distance dependent contribution to the vacuum energy already calculated earlier.

### INTERACTING QED

Here we discuss quantum electrodynamics at the one loop level containing a  $\delta$ -function as an external potential ( $a_1 = a$ ) [5]. In perturbation theory the standard Feynman rules are valid with the one exception that we have to use the more complicated spinor propagator  ${}^s S^c$ . Let us calculate the mass operator for this configuration. Because  ${}^s S^c$  is a summed up propagator we expect that besides the standard divergences of free field theory the self-energy diagram also contains contributions from the triangle diagram (with one external field insertion) which exhibit infinities. According to conventional wisdom that the inclusion of electromagnetic background fields does not change the divergences of QED we would expect no further divergences. This however is not the case here. A direct calculation of the divergent part of the mass operator (using Feynman gauge and a UV cut-off  $\Lambda$ ) yields the result that the self-energy part containing the second part  $\bar{S}$  of the spinor propagator leads to the expected structure of the divergency however with an unexpected complicated coefficient function of the dimensionless coupling constant  $a$

$$\begin{aligned} \bar{\Sigma}(x, y)|_{div.} &= -i \frac{e^2}{8\pi^2} \gamma^0 f(a) \delta(x_3) \delta^{(4)}(x - y) \ln \Lambda^2 & (20) \\ f(a) &= \frac{1}{a} \left[ 3 \left( \frac{\lambda_+}{a} \arctan \frac{a}{\lambda_-} - 1 \right) + \frac{a^3}{4} \left( \frac{\lambda_+}{a} \arctan \frac{a}{\lambda_-} + 1 \right) \right], \quad \lambda_{\pm} = 1 \pm \frac{a^2}{4}. \end{aligned}$$

Such a function can appear only if each insertion of the  $\delta$ -function in this diagram produces an additional divergent term. Loosely speaking, the reason is that the  $\delta$ -function fixes external lines (corresponding to the external field) onto the same point  $x_3 = 0$ . However, the theory remains renormalizable but one has to use some complicated nonlinear parameter renormalization.

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# General evolution equation for the distribution amplitude of exclusive virtual Compton scattering

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## 1 Introduction

So far the theoretical description of hadronic scattering processes within the framework of QCD is by no means complete since the phenomenon of confinement is not understood up to now. However, for processes at large momentum transfer  $q$  ( $q^2 = -Q^2 < 0$ ) the scattering amplitude may be split into a perturbatively calculable hard scattering (short distance) part and a phenomenological determined soft (long range) part.

In the light-cone dominated cases of deep inelastic lepton-hadron scattering and (virtual) photoproduction of a meson the corresponding soft parts are the "parton distribution functions" and the (meson) "wave functions", respectively. Both functions, however, fulfil well known evolution equations with respect to  $Q^2$ , namely the Altarelli-Parisi (AP) and the Brodsky-Lepage (BL) equation, whose kernels are also perturbatively calculable. Furthermore, these kernels are shown to be related to the anomalous dimensions of some nonlocal light-ray operators [1].

Here, we consider another light-cone dominated process, the virtual exclusive Compton scattering, which - as could be presumed - shows the same general behaviour, but - what is unexpected - whose corresponding evolution kernel  $K(T, T')$  contains the AP- and BL-kernels as limiting cases. Thereby, it is interesting to note that this new kernel may be obtained from the BL-kernel by an analytic continuation procedure; therefore, it may be denoted as "extended BL-kernel" [2].

## 2 Distribution amplitudes for exclusive virtual Compton scattering

The exclusive virtual Compton scattering will be considered in the generalized Bjorken region:  $Q^2 = -q^2 \rightarrow \infty$  with  $\xi = -q^2/Pq$  and  $\eta = \Delta q/Pq$  fix (Here, the following

variables have been used:  $q = (q_1 + q_2)/2$ ,  $P = P_1 + P_2$ ,  $\Delta = P_2 - P_1$  where  $q_1$  or  $q_2$  and  $P_1$  or  $P_2$  are the in- or outgoing photon and hadron momenta, respectively, with the restrictions  $q_1^2 < 0$ ,  $q_2^2 \geq 0$ . It may be shown that in this region the helicity amplitudes  $T(\lambda', \lambda) = \varepsilon_2^\mu(\lambda') T_{\mu\nu} \varepsilon_1^\nu(\lambda)$  asymptotically are given by  $(1/2)\varepsilon_2(\lambda')\varepsilon_1(\lambda) \cdot T_\mu^\mu$  for the transverse helicities, and vanish otherwise. Therefore, only the trace of the scattering amplitude

$$T_{\mu\nu}(P, \Delta, q) = i \int d^4x e^{iqx} \langle P_2 | RT(J_\mu(\frac{x}{2})J_\nu(\frac{-x}{2})S) | P_1 \rangle$$

has to be considered; here  $J_\mu(x) = (1/2) : \bar{\psi}(x)\gamma_\mu(\lambda^3 - \lambda^8/\sqrt{3})\psi(x) :$  is the electromagnetic current of the hadrons (for flavour  $SU(3)$ ),  $R$  the usual renormalization procedure, and  $S$  the renormalized  $S$ -matrix.

As usual, the product of the composite current operators will be expanded near the light-cone ( $x^2 \approx 0$ ) with respect to an appropriate operator basis. For the following consideration it is essential to use the *nonlocal* light-cone expansion due to Anikin and Zavalov [3]. In our special case it reads (in leading order):

$$R(TJ^\mu(\frac{x}{2})J_\mu(\frac{-x}{2})S) \approx \int_0^1 d^2\kappa F_a(x^2, \kappa; \mu^2) (\bar{R}TO^a(\hat{x}, \kappa)S)_{(\mu^2)}$$

with the light-ray operators

$$O^a(\hat{x}, \kappa) =: \bar{\psi}(\kappa_1 \hat{x})(\hat{x}\gamma)U(\kappa_1 \hat{x}, \kappa_2 \hat{x})\psi(\kappa_2 \hat{x}):$$

where

$$U(\kappa_1 \hat{x}, \kappa_2 \hat{x}) = \mathcal{P} \exp \left( -ig \int_{\kappa_2}^{\kappa_1} d\tau \hat{x}^\mu A_\mu(\tau \hat{x}) \right)$$

is the path ordered phase factor due to the gluon field  $A_\mu(x)$  projected onto a light-ray which is determined by some  $\hat{x}$ ,  $\hat{x}^2 = 0$ , related to  $x$ . Here,  $\bar{R}$  is a new well-defined renormalization procedure. The singular coefficient functions  $F_a$  may be determined perturbatively; in the Born approximation they are given by

$$F_a(x^2, \kappa) \approx i e_a (2\pi^2(x^2 - i\epsilon)^2)^{-1} \delta(\kappa_+) (\delta(\kappa_- - 1/2) - \delta(\kappa_- + 1/2))$$

with  $\kappa_\pm = (\kappa_2 \pm \kappa_1)/2$  and  $e_a = (2/9)\delta_{a0} + (1/6)\delta_{a3} + (1/6\sqrt{3})\delta_{a8}$ .

Putting all terms together we finally obtain after some integrations and change of variables:

$$T_\mu^\mu(P, \Delta, q) \approx 2 \int dt \left( \frac{1}{\xi + t} - \frac{1}{\xi - t} \right) \epsilon_a q^a(t, \eta; \mu^2 = Q^2),$$

where the distribution amplitudes

$$q^a(t, \tau; \mu^2) = \int \frac{d(\kappa_- \hat{x} P)}{2\pi \hat{x} P} e^{i\kappa_- \hat{x} P t} \langle P_2 | \bar{R}TO^a(\hat{x}, \kappa_-)S | P_1 \rangle \Big|_{\hat{x}\Delta = \tau \hat{x} P}$$



contain the long range behaviour of the process (which is related to the hadron states  $|P_i\rangle$ ).

It is quite remarkable that, contrary to the well known distributions for the hadron wave functions and the deep inelastic scattering, the above amplitudes depend on two independent parameters,  $t$  and  $\tau$ , the latter being related to the hadron momenta  $P_1$  and  $P_2$ . In fact, taking the limits  $\tau \rightarrow 0$  (i.e.  $P_1 = P_2$ ) and  $\tau \rightarrow -1$  (i.e.  $P_1 = 0$ ) we (formally) obtain the usual meson wave function  $\phi(x = \frac{1+t}{2}, Q^2) = \lim_{\tau \rightarrow -1} q^a(t, \tau, Q^2)$  and also the quark distribution function  $q^a(t, Q^2) = \lim_{\tau \rightarrow 0} q^a(t, \tau, Q^2)$ , respectively.

### 3 Evolution kernel of the distribution amplitude and its relation to the AP- and the BL-kernel

The generalized distribution amplitudes  $q^a(t, \tau, \mu^2)$  contain some perturbative aspects which are determined by the renormalization group equation of the corresponding light-ray operators,

$$\mu \frac{d}{d\mu} (\overline{RTO}^a(\tilde{x}, \underline{\kappa})S)_{(\mu^2)} = \int d^2 \underline{\kappa}' \gamma(\underline{\kappa}, \underline{\kappa}') (\overline{RTO}^a(\underline{\kappa}', \tilde{x})S)_{(\mu^2)}$$

where the anomalous dimension  $\gamma(\underline{\kappa}, \underline{\kappa}')$  is determined perturbatively. Using the  $\alpha$ -representation for the Feynman diagrams the following behaviour has been obtained [1]:

$$\gamma(\underline{\kappa}, \underline{\kappa}') = (1/\kappa_-^2) \gamma(w_+ \equiv (\kappa'_+ - \kappa_+)/\kappa_-, w_- \equiv \kappa'_-/\kappa_-),$$

with the support of the (new) function  $\gamma(w_+, w_-)$  restricted with respect to the new parameters according to

$$|w_{\pm}| \leq 1, \quad |w_+ \pm w_-| \leq 1.$$

This gives rise to a corresponding renormalization group equation for the distribution amplitudes which, if the renormalization point is chosen as  $\mu^2 = Q^2$ , can be expressed as an evolution equation,

$$Q^2 \frac{d}{dQ^2} q^a(t, \tau; Q^2) = \int_{-1}^1 \frac{dt'}{2|\tau|} K\left(\frac{t}{\tau}, \frac{t'}{\tau}\right) q^a(t', \tau; Q^2),$$

with the evolution kernel ( $t/\tau \equiv T$ )

$$K(T, T') = \int dw_- \gamma(w_+ = w_- T' - T, w_-).$$

This kernel is nothing else but the (partly integrated) anomalous dimension of the light-ray operators, and therefore does not depend on the hadron states  $|P_i\rangle$  explicitly. (For physical states  $|P_i\rangle$  the support of the amplitudes  $q^a(t, \tau; Q^2)$  is restricted to  $|\tau| \leq 1$  and  $|t| \leq 1$ ).

So far, evolution equations of the above form have not been considered in the literature. However, the AP- and the BL-equations may be obtained as limiting cases. The corresponding kernels are

$$P(z/z') = \frac{1}{2} \lim_{\tau \rightarrow 0} \frac{|z'|}{|\tau|} K\left(\frac{z}{\tau}, \frac{z'}{\tau}\right),$$

and

$$V(x, y) = K(2x - 1, 2y - 1) \Big|_{0 \leq x, y \leq 1}.$$

respectively. It is necessary to remark that, contrary to the limiting procedure for the distribution functions itself, the limits for the evolution kernel  $K(T, T')$  are well defined; for  $V(x, y)$  this is quite trivial.

As it is evident from the last equation, the BL-kernel coincides with  $K(t, t')$  in the restricted region  $-1 \leq t, t' \leq 1$  of the  $(t, t')$ -plane. Therefore, the question arises if it contains enough information to extend it into the whole  $(t, t')$ -plane. In fact, using again the  $\alpha$ -representation we have shown that, in every order of perturbation theory, this can be done by analytic continuation. From this it is obvious, that it is also possible to determine the AP-kernel via the "extended BL-kernel"  $K(t, t')$  from the usual BL-kernel. This result is new and somewhat surprising since both kernels are related to kinematical quite different physical processes; on the other hand, this shows the virtues of the nonlocal light-cone expansions. In addition, using this nontrivial relation we were able to show that the already computed 2-loop approximations of the AP- and the BL-kernels are consistent with each other.

A more detailed version of these results, indicating also the proofs, will be given elsewhere [4].

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## THE PERSISTENCE OF INTERACTIONS IN QUANTUM FIELD THEORIES

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The separation of observer and observations is one of the most perplexing problems to confront the Copenhagen interpretation of quantum mechanics. As is well known, several paradoxes arise from this question including that of the famous friend of Professor Wigner. Ultimately, the pursuit of these paradoxes leads to a discussion of the applicability of quantum mechanics to living organisms(1).

In a more immediate sense this putative dichotomy is encountered in the supposition that interactions can arise from the dynamics of a theory. By "arise" is meant here that there is a period when a system doesn't interact with a second system, then a period when interactions occur, then subsequently a period when no interactions are present. The second period is, somehow, supposed to develop or evolve through the field equations. For a very important class of field theories including quantum electrodynamics of spin 1/2 systems this supposition is false. In particular, the fine structure constant appearing in the field equations must also enter the boundary conditions. This fact can be established quite generally without recourse to specific assumptions about methods of solution of the field equations. Similar behavior is also exhibited in exact solutions of self interacting systems(2,3).

The basic field equations of quantum electrodynamics of spin 1/2 particles are:

$$(i\gamma^\mu \partial_\mu - m)\psi - eA^\mu \gamma_\mu \psi = 0 \quad (1)$$

$$\partial^\mu \partial_\mu A_\nu = e\bar{\psi} \gamma_\nu \psi \quad (2)$$

The commutation relations of the electromagnetic vector potential are:

$$\partial_t A_\mu(\vec{x}, t) A_\nu(\vec{y}, t) - A_\nu(\vec{y}, t) \partial_t A_\mu(\vec{x}, t) = P_{\mu\nu}(\vec{x}, t) \delta(\vec{x} - \vec{y}) \quad (3)$$

The anti-commutation relations of the spinor field and its hermitian adjoint are:

$$\psi_\alpha(\vec{x}, t) \psi_\beta^\dagger(\vec{y}, t) + \psi_\beta^\dagger(\vec{y}, t) \psi_\alpha(\vec{x}, t) = \delta(\vec{x} - \vec{y}) \delta_{\alpha\beta} \quad (4)$$

$P_{\mu\nu}(\vec{x}, t)$  is chosen to be consistent with the gauge conditions on  $A_\mu$ .

Multiply equations (1) and (2) on both sides by  $e$ . The results are:

$$(i\gamma^\mu \partial_\mu - m) e\psi - eA^\mu \gamma_\mu e\psi = 0 \quad (5)$$

$$\partial_\mu eA^\mu = e\tilde{\psi} \gamma_\mu e\psi \quad (6)$$

Rename the quantities  $eA^\mu = B^\mu$  and  $e\psi = \chi$ . These new named fields satisfy equations which do not contain  $e$ . Therefore, either (I) the fields  $B^\mu$  and  $\chi$  are independent of  $e$  or (II)  $e$  must enter the boundary conditions. In either case, as asserted, the interaction in the original fields  $A^\mu$  and  $\psi$  will be persistent since in case (I) they will have the form:

$$A_\mu = e^{-1} B_\mu \quad (7)$$

$$\psi = e^{-1} \chi \quad (8)$$

with  $\chi$  and  $B_\mu$  independent of  $e$  and in case (II), if  $e$  enters the boundary conditions the field interactions are, by definition, persistent.

Next, case (I) is inconsistent with the canonical commutation relations. To see this multiply equations (3) and (4) on both sides by  $e^2$  to get

$$\partial_t e A_\mu(\vec{x}, t) e A_\nu(\vec{y}, t) - e A_\nu(\vec{y}, t) \partial_t e A_\mu(\vec{x}, t) = e^2 P_{\mu\nu}(\vec{x}, t) \delta(\vec{x} - \vec{y}) \quad (9)$$

$$e \psi_\alpha(\vec{x}, t) e \psi_\beta^\dagger(\vec{y}, t) + e \psi_\beta^\dagger(\vec{y}, t) e \psi_\alpha(\vec{x}, t) = \delta(\vec{x} - \vec{y}) \delta_{\alpha\beta} e^2 \quad (10)$$

On the left side of equations (9) and (10) one has the new named fields  $B^\mu$  and  $\chi$  which, as has been shown in equations (5) and (6), are independent of  $e$ , while on the right side one has factors  $e^2$ . The result is inconsistent, therefore case (I) is eliminated by the canonical commutation relations. The conclusion is that only case (II) is acceptable on the basis of the field equations and the canonical commutation relations. The charge  $e$  must enter the boundary conditions.

This is not an empty conclusion. The dynamics as defined by the field equations and the commutation and anti commutation relations does not determine the coupling constant. This proof is complete and self-contained. It can be generalized to other theories including gauge theories. The arguments depend only upon the field equations and the canonical commutation relations(4). A more complete discussion will be given elsewhere.

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### Relativistic Quantum Mechanics on Fock Space

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Quantum field theory does not offer the only framework for describing hadronic interactions (including production reactions) in terms of underlying constituents. It is also possible to describe hadronic processes using a relativistic quantum mechanics in which the mass operator acts on a Fock space of underlying constituents. Using the Lie algebra of the Poincaré group, Dirac [1] showed that a relativistic dynamics could be formulated in terms of the so-called instant, front, or point forms. Although the instant and front forms have received considerable attention because of their relation to quantum field theory [2], the goal of this note is to show that the point form lends itself rather naturally to a Fock space formulation, making it possible to deal with phenomena such as production reactions in a relatively straightforward way.

We begin by quickly reviewing some relativistic kinematics, first for one and then  $n$ -particles. Let  $|pj\sigma\rangle$  be a relativistic state of four-momentum  $p$ , spin  $j$ , and spin projection  $\sigma$ . Under Lorentz transformations  $\Lambda \in SO(1,3)$  and four-translations  $a$ , such states transform as

$$U_\Lambda |pj\sigma\rangle = \sum_{\sigma'=-j}^{+j} D_{\sigma'\sigma}^{j,\sigma}(R_w) |\Lambda p, j\sigma'\rangle$$

$$U_a |pj\sigma\rangle = e^{i p \cdot a} |pj\sigma\rangle, \quad (1)$$

where  $p \cdot a := p^T g a$ ,  $p = (E, \vec{p})$ ,  $g = \text{diag}(1, -1, -1, -1)$  and  $R_w$  is the Wigner rotation,

$$R_w = B^{-1}(\Lambda v) \Lambda B(v), \quad (2)$$

with  $B(v)$  a canonical boost [3] (coset representative) satisfying  $p = B(v)p(st)$ ,  $p(st) = (m, \vec{0})$ ,  $p \cdot p = m^2$  ( $m$  is the mass of the constituent particle) and  $v = p/m$  the four velocity.

The four-momentum operator  $P_0^\mu$  and Lorentz generators  $J^{\alpha\beta}$  which result from infinitesimal transformations of Eq. (1) give rise to the following operators:

$$M_0^2 := P_0 \cdot P_0 \quad (\text{free mass operator})$$

$$V^\mu := P_0^\mu M_0^{-1} \quad (\text{four-velocity operator})$$

$$W_\mu := \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} J^{\alpha\beta} P^\nu \quad (\text{Pauli-Lubanski operator})$$

$$\widetilde{W}_\mu := \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} J^{\alpha\beta} V^\nu \quad (\text{modified Pauli-Lubanski operator}). \quad (3)$$

Acting on the states in Eq. (1),  $M_0$  gives the mass  $m$  of the constituent,  $V^\mu$  the four velocity,  $\widetilde{W} \cdot \widetilde{W}$  the spin  $j$ , and  $n \cdot \widetilde{W}$  the spin projection ( $n$  a c-number four vector) [3].

All of this is readily generalized to  $n$ -particle states, which are  $n$ -fold tensor products of one-particle states. To prepare for the point form of relativistic dynamics, define ( $n$ -particle) velocity states as

$$\begin{aligned} |v, \vec{k}_i, \mu_i\rangle &:= U_{B(v)} |k_1 j_1 \mu_1 \dots k_n j_n \mu_n\rangle \\ &= \prod_{i=1}^n \sum_{\sigma_i} |p_i j_i \sigma_i \dots p_n j_n \sigma_n\rangle D_{\sigma_i, \mu_i}^{j_i}(k_i, B(v)) \end{aligned} \quad (4)$$

with  $k_i = (E_i, \vec{k}_i)$ ,  $\Sigma \vec{k}_i = \vec{0}$ ,  $E_i = \sqrt{m_i^2 + \vec{k}_i \cdot \vec{k}_i}$  and  $p_i = B(v)k_i$ . Then

$$\begin{aligned} U_\Lambda |v, \vec{k}_i, \mu_i\rangle &= \prod_{i=1}^n \sum_{\mu'_i} |\Lambda v, R_u \vec{k}_i, \mu'_i\rangle D_{\mu'_i, \mu_i}^{j_i}(R_u) \\ U_a |v, \vec{k}_i, \mu_i\rangle &= e^{im_n v \cdot a} |v, \vec{k}_i, \mu_i\rangle. \end{aligned} \quad (5)$$

where  $m_n = \Sigma E_i = \sum \sqrt{m_i^2 + \vec{k}_i \cdot \vec{k}_i}$  is the  $n$ -particle invariant mass and  $(k_i, B(v))$  is a Wigner rotation. What is striking about Eq. (5) is that under Lorentz transformations the internal variables  $(\vec{k}_i, \mu_i)$  transform exactly as nonrelativistic variables, with Wigner rotations replacing ordinary rotations. This means that orbital and spin angular momentum can be coupled to give the total angular momentum, exactly as is done in nonrelativistic quantum mechanics, even though the total angular momentum is an eigenvalue of the relativistic invariant  $\vec{W} \cdot \vec{W}$ .

In the point form of dynamics, the free mass operator  $M_0$  is modified to become the interacting mass operator  $M$ , so that the four-momentum operator becomes

$$P^\mu := V^\mu M; \quad (6)$$

unlike the other forms of dynamics the Lorentz generators remain unchanged.  $P^\mu$  and  $J^{\alpha\beta}$  will satisfy the Poincaré group commutation relations provided

$$[M, V^\mu] = 0, \quad [M, U_\Lambda] = 0 \quad (\text{or } [M, J^{\alpha\beta}] = 0); \quad (7)$$

a simple way of satisfying Eq. (7) is to require that the kernel of  $M$  acting on velocity states, Eq. (4), be rotationally invariant and independent of the four velocity. From Eq. (6) it follows that the Hamiltonian is  $MV^0$  and the generator of spatial translations is  $M\vec{V}$ . Further, the Lorentz covariance of  $P^\mu$  follows from the covariance of the velocity operator  $V^\mu$ .

A Fock space of constituent particles can now be introduced as the direct sum of (antisymmetrized)  $n$ -particle spaces, which, when written as velocity states gives the following correspondence (suppressing the spin and mass variables  $j, m$ ):

|                |          |                                                             |                                                        |                     |          |
|----------------|----------|-------------------------------------------------------------|--------------------------------------------------------|---------------------|----------|
| vacuum         | :        | $ 0\rangle$                                                 |                                                        |                     |          |
| 1 particle     | :        | $ p\sigma\rangle \rightarrow$                               | $ v\mu_0\rangle$                                       | (degenerate) vacuum |          |
| 2 particle     | :        | $ p_1\sigma_1 p_2\sigma_2\rangle \rightarrow$               | $ v\mu_0 \vec{k}_1 \mu_1\rangle$                       | 1 particle          |          |
| $\vdots$       | $\vdots$ | $\vdots$                                                    | $\vdots$                                               | $\vdots$            | $\vdots$ |
| $n+1$ particle | :        | $ p_1\sigma_1 \dots p_{n+1}\sigma_{n+1}\rangle \rightarrow$ | $ v\mu_0 \vec{k}_1 \mu_1 \dots \vec{k}_n \mu_n\rangle$ | $n$ -particle       |          |

(8)

Thus, by extracting the overall velocity and spin component  $\mu_0$ , an ordinary  $(n+1)$  particle state becomes an  $n$ -particle velocity state. Creation operators are defined on these velocity states by writing

$$|v\mu_0 \vec{k}\mu\rangle := a^\dagger(\vec{k}\mu)|v\mu_0\rangle \quad (9)$$

since  $a^\dagger(\vec{k}\mu)$  acts only on internal variables, it is sufficient to require that  $a^\dagger$  transform as

$$U_R a^\dagger(\vec{k}\mu) U_R^{-1} = \sum a^\dagger(R\vec{k}, \mu') D_{\mu'\mu}^{1/2}(R), \quad R \in SO(3) \quad (10)$$

and, with the commutation relation  $\{a(\vec{k}\mu), a^\dagger(\vec{k}'\mu')\} = E\delta^3(\vec{k} - \vec{k}')\delta_{\mu\mu'}$ , it is possible to build up the entire "internal" Fock space (from which the overall velocity has been extracted).

Since  $M$  must commute with  $V^\mu$  and  $J^{\alpha\beta}$ , a general mass operator can be written in terms of creation and annihilation operators. A simple example is

$$M = \int M(\vec{k}\mu, \vec{k}'\mu') a^\dagger(\vec{k}\mu) a(\vec{k}'\mu') \quad (11)$$

where the kernel  $M(\vec{k}\mu, \vec{k}'\mu')$  must be rotationally invariant. An algebra of mass operators can be formed by requiring that the kernels satisfy the commutation relations of  $SL(2, \mathbb{R})$ ; the resulting relativistic harmonic oscillator eigenfunctions are then specified purely algebraically on the Fock space [4]. It is clear that more general mass operators, involving products like  $a^\dagger a^\dagger a, a^\dagger a^\dagger a a, \dots$  can also be given and these in turn can be used to generate interactions such as relativistic spin orbit or tensor forces [5]. Because of the fermionic nature of the creation and annihilation operators, it is also possible to generate superalgebras of mass operators. Finally, for some separable potentials, it is possible to solve exactly the relativistic Lippmann-Schwinger equations for particle production processes [6].

Though internal symmetries such as flavor  $SU(3)$  have not been discussed, it is worthwhile noting that the generators of internal symmetries—since they commute with  $V^\mu$  and  $J^{\alpha\beta}$ —can also be thought of as mass operators, so that it is straightforward to construct mass operators that give the Gell-Mann, Okubo mass formula [5]. And since the velocity "vacuum" state transforms under rotations (see Eq. (8)), introducing an  $SU(3)$  internal symmetry label makes it possible to enlarge the spin-internal symmetry transformations to  $SU(6)$  and thus build a fully relativistic  $SU(6)$  structure on the (internal) Fock space [5].

Finally, it is possible to give a point form formulation of hadronic current operators, in which the current operator transforms as an irreducible tensor operator under the



interacting Poincaré group, depending on whether the photon momentum transfer is time-like, light-like, or space-like. Write

$$J^\mu(x) = \sum_b \int d^4Q \epsilon^{iQ \cdot x} D^\mu_b(B(Q)) J_b(Q) \quad (12)$$

where

$$\begin{aligned} b = 1, 2, 3 & \quad Q^2 > 0 \\ & = 1, 2 \quad Q^2 = 0 \\ & = 0, 1, 2 \quad Q^2 < 0 \end{aligned}$$

and  $D^\mu_b(\Lambda) = \Lambda^\mu_{\nu} g_{\nu b}$ , with  $B(Q)$  a boost. Then if  $J_b(Q)$  transforms as

$$\begin{aligned} U_\Lambda J_b(Q) U_\Lambda^{-1} &= \sum_{b'} D_{b'b}(\Lambda) J_{b'}(\Lambda Q) \\ [P^\mu, J_b(Q)] &= Q^\mu J_b(Q) \end{aligned} \quad (13)$$

the hadronic current operator will automatically satisfy current conservation and transform as

$$\begin{aligned} U_\Lambda J^\mu(x) U_\Lambda^{-1} &= \Lambda^{-1\mu}_{\nu} J^\nu(\Lambda x) \\ U_a J^\mu(x) U_a^{-1} &= J^\mu(x+a) \end{aligned} \quad (14)$$

Moreover, it is possible—using the operator  $J_b(Q)$ —to formulate a relativistic impulse approximation, wherein the form factors of the constituents fix the form factors of the bound states of the mass operator [5].

We have sketched the framework of a relativistic quantum mechanics of constituent particles on Fock space, using Dirac's point form of relativistic dynamics. A number of issues remain to be investigated, such as under what circumstances a given mass operator  $M$  has the correct cluster properties on the Fock space, but the approach seems to be a promising alternative to quantum field theoretic approaches.

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## INTERNAL DYNAMICS OF MAJORANA AND DIRAC INFINITE COMPONENT WAVE EQUATIONS AND MAGNETIC BOUND STATES

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### Abstract

The connection between the internal dynamics of the Majorana's (and Dirac's) infinite component wave equation, and the new exact tightly bound solutions of the two-body problem of electrodynamics due to magnetic interactions is established.

### I. Introduction

In 1932 Majorana<sup>1</sup> established a wave equation based on a unitary infinite-dimensional representation of the Lorentz group in contrast to the Dirac equation which was based on the 4-dimensional non-unitary representation of the Lorentz group. As a result, Majorana equation does not have negative energy solutions, but albeit infinitely many excited states. Dirac,<sup>2</sup> in 1971, gave another wave equation without negative energy solutions. It turned out that Dirac's new equation is a projection to the ground state of the Majorana equation.<sup>3</sup> Both equations describe a composite system which in its rest frame is a two-dimensional oscillator, or a two-dimensional Kepler motion, depending on the choice of the internal coordinates.<sup>3,4</sup> The wave equation boosts this oscillator and makes a covariant description of the moving composite system possible. Later on more general infinite-component covariant wave equations with three-dimensional internal dynamics have been developed<sup>5</sup> and applied to account for many properties of hadrons as composite systems.<sup>6</sup>

### II. Infinite Component Wave Equations

The infinite component wave equations we are considering are of the form

$$(\Gamma^\mu P_\mu - K)\psi = 0 \tag{1}$$

where  $\Gamma^\mu$  is a vector operator in a unitary representation of the Lorentz group (or a more general dynamical group  $G$  containing the Lorentz group), and  $K$  is a constant or a scalar operator in that unitary representation. Dirac's new equation is obtained by

adding to (1) consistently the subsidiary condition  $\epsilon_{\mu\nu\lambda\rho} S^{\nu\lambda} P^\mu \psi = 0$ . Now in the rest frame of the particle, eq.(1) becomes

$$(\Gamma^0 M - K)\psi = 0 \quad (2)$$

The operator  $\Gamma^0$  can be written as a differential operator in a two-dimensional space and has two simple realizations in terms of the coordinates related to each other by a conformed transformation. Equation (2) can then be written in two different ways<sup>3</sup> ( $K = \text{const}$ ):

$$\left( -\frac{\hbar^2}{2\mu} \Delta_q^2 + \frac{1}{2} \mu \omega^2 q^2 - E \right) \psi = 0 \quad (3)$$

or

$$\left( -\frac{\hbar^2}{2\mu} \Delta_r^2 - \frac{\alpha}{r} - E \right) \psi = 0 \quad (3')$$

where  $q = (q_1, q_2)$ , and  $r = \sqrt{x^2 + y^2}$  are two-dimensional coordinates. Thus although eq.(1) is covariant, the rest frame equations look like nonrelativistic two-body equations, due to the choice of the internal coordinates.

The next problem naturally is to find the physical meaning and realization of the internal dynamics of the Majorana "particle". Because infinite component wave equations account in a simple way many characteristic properties of hadrons (e.g. dipole form factors, mass spectra, decay rates of excited states, structure functions, polarizabilities and scattering amplitudes), it is important to connect the internal dynamics of these equations with constitutive models of hadrons.

### III. Internal Dynamics

We show here that the two-dimensional internal dynamics of the Majorana equation, eq.(3), can be connected precisely to a relativistic two-body problem of electrodynamics. The system is described by the relativistic Lagrangian of Clausius<sup>7</sup> ( $c = 1$ )

$$L = -m_1 \sqrt{a - v_1^2} - m_2 \sqrt{1 - v_2^2} - \frac{\alpha}{r} (C - D \mathbf{v}_1 \cdot \mathbf{v}_2) \quad (4)$$

where  $C_1 = D = 1$  but we introduce these parameters to see the effect of the Coulomb ( $C$ ) and magnetic ( $D$ ) terms separately. Equation (4) can be derived as an exact equation of classical electrodynamics from an action principle, if an invariant center of mass time  $\tau$  is properly chosen (rather than proper times of individual particles).<sup>8</sup>

A remarkable special exact solution of the equations of motion resulting from (4) have been found recently.<sup>9</sup> This is obtained by putting the constraints

$$\frac{m_1}{\sqrt{1 - v_1^2}} + D \frac{\alpha}{r} = 0 \quad \text{and} \quad \frac{m_2}{\sqrt{1 - v_2^2}} + D \frac{\alpha}{r} = 0, \quad (5)$$

whence the center of mass momentum  $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$  is automatically zero. The equations of motion of the relative coordinates are then

$$\dot{\mathbf{p}} = -E \frac{\mathbf{r}}{r^2} + \alpha \frac{\mathbf{r}}{r^3} 2(C - D), \quad \dot{\mathbf{r}} = \frac{1}{\alpha D} \mathbf{r} \mathbf{p} \quad (6)$$

where  $E$  is the conserved energy. These equations can be solved exactly in two ways<sup>9</sup>, directly, or by going over to a new time  $T$  with  $dT/dr = r$ , in which case we get an equivalent 2-dimensional Coulomb motion (because angular momentum conservation)

$$\dot{r} = \frac{1}{\alpha D} p, \quad \dot{p} = -E \frac{r}{r^3} + 2(D - C)\alpha \frac{r}{r^4} \quad (7)$$

The orbits are ellipses. The system can now be quantized leading again to an effective Schrödinger equation of the form (3), although the theory is fully relativistic.

#### IV. The Mass Spectrum

The mass spectrum in the attractive case, is

$$M = \frac{(m_1^2 + m_2^2)^{1/2}}{|\alpha|D} \left[ n_r + 1/2 + \left( \ell^2 + 2D\alpha^2 \left(1 - \frac{C}{D}\right) \right)^{1/2} \right]$$

where  $n_r$  and  $\ell$  are the angular and radial quantum numbers.

The important feature of this solution is that it is nonperturbative ( $\alpha$  is in the denominator) and non-analytic in  $D$ , i.e. in the magnetic term in the Lagrangian (1). There is no Coulomb or nonrelativistic limit; it exists even if  $C = 0$ . For  $C = D = 1$  (physical case) we have the simpler form

$$M = \frac{(m_1^2 + m_2^2)^{1/2}}{|\alpha|} (n + \ell + 1/2).$$

The linearly rising mass spectrum is also a property of a generalization of Majorana equation (1) when  $K$  is a scalar operator.<sup>5</sup>

The existence of this surprising new solution is due to both relativity and to magnetic interactions. Although the Coulomb force has been studied for centuries the magnetic force  $\mathbf{v}_1 \cdot \mathbf{v}_2/r$  has been somehow neglected.

The magnetic term is even very important at macroscopic electrodynamics, and determines magnetic units in terms of the electric ones. In atomic physics it gives rise to small spin-orbit contributions. But now we see that at much shorter distances it dominates and gives rise to new states of matter. That the magnetic interactions can lead at short distances to new tightly bound states has been suggested earlier<sup>10,11,12</sup> and studied in a number of explicit models.<sup>13</sup> It is significant therefore that an exact solution exists in a fully relativistic realistic two-body system. The more so, if we look at the numerical values of the size and energies of the system after quantization. When applied to the electron-positron system,  $m_1 = m_2 = m_e$ , we get the mass spectrum

$$M = \frac{\sqrt{2}m_e}{\alpha} (n + \ell + 1/2)$$

compared with the mass of  $\pi^0$  of  $\frac{2m_e}{\alpha}$  which has the quantum numbers of  $(e^+e^-)$ .

The magnetic system has a number of other, one could say, revolutionary, features. It shows a new phase of electrodynamics from the Coulomb (electric) dominated long-distance phase, to the magnetic dominated short distance phase.<sup>13</sup> It also shows how to obtain large hadronic masses from almost massless leptonic constituents (the mass generation mechanism<sup>14</sup>). In the usual potential models the mass of the bound states is always less than the sum of the masses of the constituents. Here we have to have a new intuition about velocity dependent forces. The mass created comes from the tremendous kinetic energies of the constituents. It would be interesting to generalize the preceding theory to spinning particles.

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# Particles with Fractional Spin in (2+1)-Relativistic Quantum Theory

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### Abstract

Starting from unitary irreducible representations of the (2+1)-Poincaré group we construct a covariant wave equation for fractional spin particle with finite mass. The amplitude is shown to be of infinite-component with the frequency of definite sign. The second quantization is carried out by using the relation of the covariant amplitude to the original one that belongs to an irreducible representation of the Poincaré group. It is shown that no consistent set of commutation relations exists to describe free fractional spin particles in the framework of (2+1)-relativistic quantum field theory.

### I

Let  $\phi^{(\pm)}(k)$  be one-particle amplitudes belonging to those unitary irreducible representations (UIRs) of the (2+1)-Poincaré group \* which are characterized by  $k_\mu k^\mu = m^2$  and  $\text{sgn}(k^0) = \pm$ . Then following Wigner [1] we see that under the (2+1)-Poincaré group the amplitudes  $\phi^{(\pm)}(k)$  submit to transformations such that

$$\phi'^{(\pm)}(k) = \exp i(\mathbf{a}\mathbf{k} \mp a^0 \omega_k) \cdot \phi^{(\pm)}(k) \quad \text{for translation } a^\mu, \quad (1.1)$$

$$\phi'^{(\pm)}(k) = Q^{(\pm)}(\Lambda, k) \phi^{(\pm)}(\Lambda^{-1}k) \quad \text{for (2+1)-Lorentz transformation } \Lambda, \quad (1.2)$$

where  $\omega_k = \sqrt{|\mathbf{k}|^2 + m^2}$ , and the factor  $Q^{(\pm)}(\Lambda, k)$  (called Wigner rotation) is given by representing the transformation  $(\alpha_k^{(\pm)})^{-1} \Lambda \alpha_{\Lambda^{-1}k}^{(\pm)}$  in terms of a UIR of the little group [1]  $SO(2)$ , which in the present case is the set of Lorentz transformations leaving the 3-vector  $l^{(\pm)} = (\pm m, 0, 0)$  invariant. Here  $\alpha_k^{(\pm)}$  are the boost transformations defined by  $k^\mu = (\alpha_k^{(\pm)})^\mu_\nu l^{(\pm)\nu}$ .

To obtain an explicit form of  $Q^{(\pm)}(\Lambda, k)$  we shall use an infinitesimal Lorentz transformation

$$\Lambda^\mu_\nu = \delta^\mu_\nu - \omega^\mu_\nu, \quad (1.3)$$

where  $\omega^{\mu\nu} (= \omega^\mu_\lambda \eta^{\lambda\nu}) = -\omega^{\nu\mu}$  are infinitesimal parameters. Then it can be shown that the transformation  $(\alpha_k^{(\pm)})^{-1} \Lambda \alpha_{\Lambda^{-1}k}^{(\pm)}$  is just a rotation through the infinitesimal

\*Throughout this paper we use the Lorentz metric  $\eta^{\mu\nu}$  ( $\mu, \nu = 0, 1, 2$ ) defined by  $\eta^{00} = -\eta^{11} = -\eta^{22} = 1$ .

angle  $\bar{\theta} = \theta \mp (\tau^1 k^2 - \tau^2 k^1)/(\omega_k + m)$ . Here we have used the notations  $\tau^1 = \omega^{10}$ ,  $\tau^2 = \omega^{20}$  and  $\theta = \omega^{12}$ . Then the Wigner rotation is found to be of the form

$$Q^{(\pm)}(\Lambda, k) = 1 + i(\theta \mp \frac{\tau^1 k^2 - \tau^2 k^1}{\omega_k + m} S) \quad (1.4)$$

with an arbitrary real constant  $S$ , which uniquely specifies a UIR of the little group. Thus on account of (1.2) we are led to the Lorentz transformation of the following form:

$$\phi^{(\pm)}(k) = (1 - i\tau^1 K^1 - i\tau^2 K^2 + i\theta J)\phi^{(\pm)}(k), \quad (1.5)$$

where  $\phi^{(\pm)}(k)$  are of single component and the generators are given by

$$K^1 = k^0(i\frac{\partial}{\partial k^1} + \frac{k^2}{\omega_k(\omega_k + m)}S), \quad K^2 = k^0(i\frac{\partial}{\partial k^2} - \frac{k^1}{(\omega_k + m)}S) \quad (1.6)$$

and

$$J = \frac{1}{i}(k^1 \frac{\partial}{\partial k^2} - k^2 \frac{\partial}{\partial k^1}) + S,$$

with  $k^0 = \pm\omega_k$ . Defining  $G^{\mu\nu} (= -G^{\nu\mu})$  by  $G^{10} = K^1$ ,  $G^{20} = K^2$  and  $G^{12} = J$ , we see that they form the Lie algebra of the (2+1)-Poincaré group together with  $k^\mu$ . We may call the constant  $S$  the spin angular momentum in (2+1)-dimension.

The Casimir operators of the (2+1)-Poincaré group are seen to be  $k_\mu k^\mu$  and  $1/2 \cdot \epsilon_{\mu\nu\lambda} k^\mu G^{\nu\lambda}$ , the latter of which may be called the Pauli-Lubansky scalar by analogy with the (3+1)-dimensional case. By virtue of (1.6), in the UIR under consideration they are written as

$$k_\mu k^\mu = m^2, \quad W \equiv \frac{1}{2} \epsilon_{\mu\nu\lambda} 2k^\mu G^{\nu\lambda} = \text{sgn}(k^0) mS. \quad (1.7)$$

## II

We shall now rewrite the formalism given in the preceding section into a covariant form, in which we shall denote the amplitude by  $\psi(k)$  and the generators acting on it by

$$\tilde{G}^{\mu\nu} = i(k^\mu \partial / \partial k_\nu - k^\nu \partial / \partial k_\mu) + \Sigma^{\mu\nu}, \quad (2.1)$$

where  $\Sigma^{\mu\nu} (= -\Sigma^{\nu\mu})$  are numerical matrices satisfying the Lie algebra of  $SO(1, 2)$ . The Pauli-Lubansky scalar in this case is written as  $\tilde{W} = 1/2 \cdot \epsilon_{\mu\nu\lambda} k^\mu \tilde{G}^{\nu\lambda} = s_\mu k^\mu$  with  $s_\mu \equiv 1/2 \cdot \epsilon_{\mu\nu\lambda} \Sigma^{\nu\lambda}$  which obey the algebra

$$[s_1, s_2] = -is_0, \quad [s_0, s_1] = is_2, \quad [s_0, s_2] = -is_1. \quad (2.2)$$

Since the covariant amplitude  $\psi(k)$  linearly depends on the original amplitudes  $\phi^{(\pm)}(k)$ , it must submit to the relation  $\tilde{W}\psi(k) = W\psi(k)$ , that is,

$$s_\mu k^\mu \psi(k) = \text{sgn}(k^0) mS \psi(k), \quad \text{together with} \quad k_\mu k^\mu \psi(k) = m^2 \psi(k). \quad (2.3)$$

The set of these equations for  $\psi(k)$  provide us with conditions for the particle under consideration to have the squared mass  $m^2$  and the spin angular momentum  $S$ . Eq.(2.3) shows  $P$  and  $T$  violations for  $mS \neq 0$ .

In the rest system the first equation of (2.3) reduces to  $s_0\psi = S\psi$ , so that the spin  $S$  must be an eigenvalue of the matrix  $s_0$ . For massive particles with  $|S| = n/2$  ( $n$ ; nonnegative integer) we can obtain a covariant description in  $x$ -space in a similar manner to that in (3+1)-dimension [2], and through the second quantization procedure we can arrive at the ordinary spin-statistics relation [3].

In the following, however, we shall concentrate our arguments upon the fractional spin case. Since  $S$  is an eigenvalue of  $s_0$ , no finite dimensional representation is applicable to  $s_\mu$ . Furthermore it can be shown that for a fractional  $S$  the set of covariant equations  $s_\mu k^\mu \psi(k) = \lambda \psi(k)$  and  $k_\mu k^\mu \psi(k) = m^2 \psi(k)$  with a suitable constant  $\lambda$  has only solutions with  $k^0$  of definite sign. Thus, as for such  $s_\mu$  we employ, corresponding to  $S \geq 0$ , the hermitian generators of the UIRs  $D^{(\pm)}(\mp S)$  of  $SO(1, 2)$ , in which the  $j$ - $j'$  element of  $s_0$  is given by  $(s_0)_{j,j'} = (S \pm j \mp 1)\delta_{j,j'}$  ( $j, j' = 1, 2, \dots, \infty$ ), and examine the case of  $k^0 > 0$  starting with  $\phi^{(+)}(k)$ .

Now let us introduce an amplitude  $\chi(k)$  with infinite components, in which the  $j$ -th component is defined by

$$\chi_j(k) = \delta_{j,1} \phi^{(+)}(k) \quad (j = 1, 2, \dots, \infty). \quad (2.4)$$

On account of the above definition of  $s_0$ , Eq.(2.4) is seen to be equivalent to the set of equations [3]

$$s_0 \chi(k) = \text{sgn}(k^0) S \chi(k) \quad \text{and} \quad k_\mu k^\mu \chi(k) = m^2 \chi(k). \quad (2.5)$$

It is obvious that transformations of  $\chi(k)$  are

$$\chi'(k) = \exp i(\mathbf{a}\mathbf{k} - a^0 \omega_k) \cdot \chi(k) \quad \text{for translation,}$$

$$\chi'(k) = (1 - i\tau \mathbf{K} + i\theta S) \chi(k) \quad \text{for Lorentz transformation,} \quad (2.6)$$

where the generators  $K^1$ ,  $K^2$  and  $J$  are given by (1.6) with  $k^0 = \omega_k$  and with the replacement of  $S$  by  $s^0$ . Thus we are able to use this  $\chi(k)$  in place of the original amplitude  $\phi^{(+)}(k)$ . With these preparations we define the covariant amplitude  $\psi(k)$  by the relation

$$\psi(k) = U(k) \chi(k), \quad (2.7)$$

where  $U(k)$  stands for a unitary operator such that

$$U(k) = \exp[i \tanh^{-1}(|\mathbf{k}|/k^0) \cdot (s_1 k^2 - s_2 k^1) / |\mathbf{k}|]. \quad (2.8)$$

Then after some lengthy calculations we obtain, from (2.5) and (2.6), the following equations for  $\psi(k)$  [3]:

$$(s_\mu k^\mu - mS) \psi(k) = 0, \quad (k_\mu k^\mu - m^2) \psi(k) = 0, \quad (2.9)$$

$$\psi'(k) = (1 + \frac{i}{2} \omega_{\mu\nu} \Sigma^{\mu\nu}) \psi(\Lambda^{-1} k) = 0 \quad \text{for Lorentz transformation,} \quad (2.10)$$



where  $\Sigma^{\mu\nu} = \epsilon^{\mu\nu\lambda} s_\lambda$ . Hence we are led to

$$(s_\mu \partial^\mu + imS)\psi(x) = 0, \quad (\partial_\mu \partial^\mu + m^2)\psi(x) = 0, \quad (2.11)$$

$$\psi'(x) = (1 + \frac{i}{2} \omega_{\mu\nu} \Sigma^{\mu\nu}) \psi(\Lambda^{-1}x) \quad (2.12)$$

with

$$\psi(x) = \frac{1}{2\pi} \int \frac{d^2\mathbf{k}}{\sqrt{2}\omega_k} e^{-ik_\mu x^\mu} \psi(k) |_{k^0=\omega_k}. \quad (2.13)$$

Thus we have arrived at a covariant description of fractional spin particles.\*

Now we shall regard  $\psi(x)$  in the above as a field operator. Since  $\phi^{(+)}(k)$ 's are seen to describe a system consisting of independent harmonic oscillators, we may assume the commutation relations  $[\phi^{(+)}(k), \phi^{(+)\dagger}(k')]_{\mp} = \omega_k \delta^2(\mathbf{k} - \mathbf{k}')$  and  $[\phi^{(+)}(k), \phi^{(+)}(k')]_{\mp} = 0$  among them [2], where the indices  $-$  and  $+$  attached to the commutators correspond to Bose and Fermi statistics, respectively. Then using (2.4), (2.7) and (2.13) we can derive the following commutation relations in  $x$ -space [3]:

$$\begin{aligned} [\psi_j(x), \psi_{j'}^\dagger(y)]_{\mp} &= F_{jj'}(i s_\mu \partial^\mu / m - S) \Delta^{(+)}(x - y), \\ [\psi_j(x), \psi_{j'}(y)]_{\mp} &= 0, \end{aligned} \quad (2.14)$$

where  $F(z)$  stands for a function such that  $F(0) = 1$  and  $F(n) = 0$  for non-vanishing integer  $n$ , and  $\Delta^{(+)}(x)$  is the positive frequency part of the  $\Delta$ -function in (2+1)-dimension:

$$\Delta^{(+)}(x) = \frac{1}{(2\pi)^2} \int \frac{d^2\mathbf{k}}{2\omega_k} e^{-ik_\mu x^\mu} |_{k^0=\omega_k}. \quad (2.15)$$

Here it is noted that the right hand side of the first line of (2.14) does not vanish for space-like  $x^\mu - y^\mu$ , since we can actually show the relation

$$\sum_{j=1}^{\infty} F_{jj}(i s_\mu \partial^\mu / m - S) \Delta^{(+)}(x) |_{x^0=0, \mathbf{x} \neq 0} \neq 0. \quad (2.16)$$

Thus it is concluded that the commutation relations for the fractional spin fields lead to the violation of microcausality in any case of Bose and Fermi statistics. It is shown that the situation is unchanged for parastatistics.

Details of this article will be published elsewhere together with related topics [3].

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\*Incidentally, for  $k^0 < 0$  we have  $(s_\mu \partial^\mu - imS)\psi(x) = 0$  and  $(\partial_\mu \partial^\mu + m^2)\psi(x) = 0$  with the same  $s_\mu$  as those in (2.11).

## The connection between charge conjugation, unitarity and statistics

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### Abstract

Algebras of quantum field oscillators recently proposed to describe possible small violations of Pauli exclusion principle exhibit a connection between charge conjugation, unitarity and statistics; the connection can be seen in the corresponding quantum field theory through the appearance of ill-defined norm states and, at the level of current algebra, is suggested by the value of the Schwinger term in some related representations of Kac-Moody and Virasoro algebras.

New attempts have recently been made to formulate a quantum field theory in which small violations of Pauli exclusion principle are possible<sup>1,2</sup>. The usual theory of parafermions<sup>3</sup> describes possible violations of the principle but runs into difficulties<sup>4,5,6</sup> that are circumvented in the recently proposed theories which, after Grenberg and Mohapatra<sup>1</sup>, we call *paronic*. However, these paronic theories are also plagued by difficulties<sup>7-10</sup> and we here report some investigations confirming the rigidity of Pauli principle against any kind of violation<sup>8-10</sup>. These investigations point to an interesting connection between charge conjugation, unitarity and statistics in quantum field theory.

In quantum field theory we must have a unitary space cyclicly generated from a unique vacuum state  $|0\rangle$  which obeys the so called zero-particle and one-particle conditions given, respectively, by

$$a_i |0\rangle = 0, \quad a_i a_j^\dagger |0\rangle = p \delta_{ij} |0\rangle, \quad (1)$$

where  $a_i$  represents an annihilation operator,  $a_i^\dagger$  a creation operator. each index is used to represent particle variables, such as momentum, spin and flavor (the delta stands accordingly for products of deltas of Dirac and Kroenecker) and  $p$  is a number which is characteristic of the theory. The unitary structure of the theory is given by the products of bras and kets and its statistical signature is determined by algebraic relations among annihilation and creation operators.

Here we consider a paronic theory with trilinear relations<sup>1</sup>, based on a model due to Ignat'ev and Kuz'min<sup>11</sup>, and another one with some bilinear relations<sup>2</sup>

recently studied in the literature in connection with quantum groups<sup>12</sup> (for a review on quantum groups see reference 13). The paronic trilinear relations are given by<sup>1</sup>:

$$[c_1 a_i^\dagger a_j + c_2 a_j a_i^\dagger, a_k] = -\delta_{ik} a_j, \quad a_i^3 = 0, \quad (2)$$

where  $c_1 = (2\beta^2 - 1)/(\beta^4 - \beta^2 + 1)$ ,  $c_2 = (\beta^2 - 2)/(\beta^4 - \beta^2 + 1)$  and  $\beta$  is a real parameter that fixes the point of interpolation of the paronic statistics between the parafermionic statistics of order one and two, these two limiting cases being realized by the values zero and one of the parameter, respectively; a small  $\beta$  accounts for the smallness of the possible violation of Pauli principle. The value  $p = 1$  in the one-particle condition in (1) is determined, from the fundamental relations (2), by following a method devised by Greenberg and Messiah<sup>5</sup>. For  $\beta$  different of zero and one, as pointed out by Greenberg and Mohapatra<sup>7</sup>, a theorem due to Govorkov<sup>14</sup> implies that this paronic theory is plagued by states of negative squared norms, the simplest of them being a four-particle state of the form  $a_i^\dagger a_j^\dagger a_i^{\dagger 2} |0\rangle$  ( $i \neq j$ ). Govorkov's theorem does not rule out a theory which has no maximal occupancy number, as the occupancy number two given by the second trilinear relation in (2), and an example of such a theory was proposed by Mohapatra<sup>2</sup>; it is a paronic theory based on the bilinear relation:

$$a_i a_j^\dagger + (1 - \beta^2) a_j^\dagger a_i = \delta_{ij}. \quad (3)$$

For  $\beta = 0$  it obviously reduces to a fermionic relation and a small  $\beta$  is again supposed to describe small violations of Pauli principle.

Now we want to consider the relativistic version of those paronic theories by imposing their fundamental relations (2) or (3) to a relativistic half-spin field<sup>10</sup>. From the relation obeyed by the field it is easy to derive, by using the orthogonality of the Dirac spinors, the corresponding relations for the particle and antiparticle oscillators. From the paronic theory with trilinear relations (2) we obtain:

$$[c_1 b_i^\dagger b_j + c_2 b_j b_i^\dagger, b_k] = -\delta_{ik} b_j, \quad [c_2 d_i^\dagger d_j + c_1 d_j d_i^\dagger, d_k] = +\delta_{ik} d_j. \quad (4)$$

where we are using the notation of Bjorken and Drell<sup>15</sup>. As a consequence of the paronic statistics these relations are manifestly not invariant under charge conjugation. It is easy to calculate in this relativistic theory the following squared norms:  $\|b_i^\dagger b_j^\dagger b_i^\dagger |0\rangle\|^2 = 2\beta^2(1 - \beta^2)$ ,  $\|d_i^\dagger d_j^\dagger d_i^\dagger |0\rangle\|^2 = 2\beta^2(\beta^2 - 1)$  ( $i \neq j$ ). Obviously, they cannot be both non-negative, except at the trivial limits  $\beta = 0$  and  $\beta = 1$  in which the usual statistics and the charge symmetry are restored. If we consider the paronic theory with bilinear relations (3), which is not ruled out by Govorkov's theorem, a similar analysis starting with relations (3) leads to the squared norms:  $\|b_i^{\dagger 2} b_j^\dagger |0\rangle\|^2 = \beta^2$ ,  $(\beta^2 - 1)^4 \|d_i^{\dagger 2} d_j^\dagger |0\rangle\|^2 = -\beta^2$  ( $i \neq j$ ). For  $\beta$  small but not zero the negative squared norm of the three antiparticle state leads again to the connection between

charge conjugation, unitarity and statistics. Going back to the relativistic version of the paronic theory with trilinear relations (2), we can construct<sup>8,9</sup> from them (and some other relations) a module with cyclic vector  $|0\rangle$ . It is then possible to construct<sup>9</sup> representations of Kac-Moody and Virasoro algebras<sup>16</sup> on a 1+1 dimensional version of this module. Since the theory of Kac-Moody algebras provides an unitary structure for integrable representations with positive integer level<sup>16</sup> our construction can teach us something about the paronic theory itself. The result is that the central charge in the representations is the positive integer  $c_1\beta^2 - c_2 = 2$ , but since it is independent of  $\beta$  the possibility of gaining a unitary structure (at the level of current algebra) leads to the absence of paronic features, a fact also pointing to the relation between charge conjugation, unitarity and statistics.

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## ON KINEMATICAL ACAUSALITY IN WEINBERG'S EQUATIONS FOR ARBITRARY SPIN\*

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Abstract: It is shown that Weinberg's equations for arbitrary spin contain not only  $2(2j+1)$  physical solutions, but also kinematically acausal solutions. Despite the existence of these physically unacceptable solutions we are able to construct a fully relativistic and causal Feynman propagator for arbitrary spin.

Despite the fact that there remain fundamental difficulties<sup>1</sup> with quantum field theories of high-spin particles, an internally consistent relativistic phenomenology of these particles is urgently needed. This is due to the advent of higher energy nuclear facilities which will make possible the study of the mutual interactions of hadrons with high spin. Following Wigner<sup>2</sup>, Weinberg<sup>3</sup> and Ryder<sup>4</sup> we have recently proposed<sup>5</sup> such a relativistic framework.

While the dynamical acausality<sup>1</sup> in relativistic equations for spin  $j \geq 1$  are well known, there is a general belief that high-spin relativistic wave equations are free from difficulties at the free particle level. In this contribution to the "Second International Wigner Symposium" we report that the Weinberg equations satisfied by the  $(j, 0) \oplus (0, j)$  covariant spinors suffer from kinematical acausality.

Weinberg's equations<sup>3,5</sup> for arbitrary spin can be conveniently written in the following general form

$$\left( \gamma_{\{\mu\}} p^{[\mu]} - m^{2j} I \right) \psi(\vec{p}) = 0, \quad (1)$$

where  $\{\mu\}$  is a set of  $2j$  Lorentz indices and  $p^{[\mu]}$  is a product of  $2j$  contravariant energy-momentum vectors, i.e. for  $j = 1/2$ ,  $\gamma_{\{\mu\}} p^{[\mu]} = \gamma_{\mu} p^{\mu}$ ; for  $j = 1$   $\gamma_{\{\mu\}} p^{[\mu]} = \gamma_{\mu\nu} p^{\mu} p^{\nu}$ , and so on. For one time and three space dimensional spacetime, there are  $[4(4+1) \cdots (4+2j-1)] / (2j)!$  gamma matrices (of dimension  $2(2j-1) \times 2(2j+1)$ ) which are symmetric in the Lorentz indices. To be more specific we present as an example the case  $j = 3/2$ . For this case Eq. (1) becomes

$$(\gamma_{\mu\nu\lambda} p^{\mu} p^{\nu} p^{\lambda} - m^3 I) \psi^{(j=3/2)}(\vec{p}) = 0, \quad (2)$$

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with the  $8 \times 8$  chiral representation  $\gamma_{\mu\nu\lambda}$  given by

$$\gamma_{\mu\nu\lambda} p^\mu p^\nu p^\lambda = \begin{pmatrix} 0 & \left[ \eta_{\mu\nu} p^\mu p^\nu (p^0 + 2\vec{J} \cdot \vec{p}) - \frac{1}{6} \{ (2\vec{J} \cdot \vec{p})^2 - \vec{p}^2 \} \{ 2\vec{J} \cdot \vec{p} - 3p^0 \} \right] \\ \left[ \eta_{\mu\nu} p^\mu p^\nu (p^0 - 2\vec{J} \cdot \vec{p}) - \frac{1}{6} \{ (2\vec{J} \cdot \vec{p})^2 - \vec{p}^2 \} \{ 2\vec{J} \cdot \vec{p} - 3p^0 \} \right] & 0 \end{pmatrix}. \quad (3)$$

Eq. (1) is a set of homogenous coupled equations.

In order for a solution to exist a necessary condition is

$$\text{Determinant} \left( \gamma_{\{\mu\}} p^{\{\mu\}} - m^{2j} I \right) = 0. \quad (4)$$

For a given  $j$  this equation is a  $2j[2(2j+1)]$ th order equation in  $E$ . Solving this gives the dispersion relations  $E(\vec{p}, m)$  for the  $2j\{2(2j+1)\}$  solutions. Of these solutions there are  $N_C(j) = 2(2j+1)$  solutions which satisfy  $E = \pm \sqrt{p^2 + m^2}$ . There remain, however,  $N_A(j) = 2j\{2(2j+1)\} - 2(2j+1) = 2(2j-1)(2j-1)$  solutions which do not satisfy the correct dispersion relation for  $E(\vec{p}, m)$ . In our example case of  $j = 3/2$ , these results are given explicitly in Table I. Explicit results for the wave equations satisfied by the  $(1,0) \oplus (0,1)$  and  $(2,0) \oplus (0,2)$  covariant spinors can be found in Ref. [5].

**Table I** Dispersion relations  $E = E(\vec{p}, m)$  associated with Eq. (2).  $N_C(\frac{3}{2}) = 8$ , and  $N_A(\frac{3}{2}) = 16$ . Here  $p^2 = \vec{p} \cdot \vec{p}$ .

| (Multiplicity) | Dispersion Relation                                           | Interpretation                                                                  |
|----------------|---------------------------------------------------------------|---------------------------------------------------------------------------------|
| (4)            | $E = +\sqrt{p^2 + m^2}$                                       | Causal, "particle"<br>$u_{\pm\frac{3}{2}}(\vec{p}), u_{\pm\frac{1}{2}}(p')$     |
| (4)            | $E = -\sqrt{p^2 + m^2}$                                       | Causal, "antiparticle"<br>$v_{\pm\frac{3}{2}}(\vec{p}), v_{\pm\frac{1}{2}}(p')$ |
| (4)            | $E = +\left(\frac{2p^2 + i\sqrt{3}m^2 - m^2}{2}\right)^{1/2}$ | Kinematically Acausal                                                           |
| (4)            | $E = -\left(\frac{2p^2 + i\sqrt{3}m^2 - m^2}{2}\right)^{1/2}$ | Kinematically Acausal                                                           |
| (4)            | $E = +\left(\frac{2p^2 - i\sqrt{3}m^2 - m^2}{2}\right)^{1/2}$ | Kinematically Acausal                                                           |
| (4)            | $E = -\left(\frac{2p^2 - i\sqrt{3}m^2 - m^2}{2}\right)^{1/2}$ | Kinematically Acausal                                                           |

The solutions termed "kinematically acausal" do not satisfy the correct dispersion relation  $E^2 = p^2 + m^2$ . A similar study of the Weinberg's equations satisfied by the  $(1, 0) \oplus (0, 1)$  and  $(2, 0) \oplus (0, 2)$  covariant spinors reveals existence of "tachyonic" solutions with  $E = \pm \sqrt{p^2 - m^2}$ .

As a consequence of this kinematical acausality the Green function

$$\left( i^{2j} \gamma_{\{\mu\}} \partial^{i\{\mu\}} - m^{2j} I \right) G^{(j,0) \oplus (0,j)}(x - x') = \delta^4(x - x') , \quad (5)$$

associated with the Weinberg's equations, propagates not only the physically acceptable causal solutions but also the physically undesirable (at least in the context of our present understanding) acausal/tachyonic solutions.

However, the object which most naturally enters the calculations of the scattering amplitudes, via canonical perturbation theory, is

$$\langle x | S_F^j | y \rangle \equiv \langle 0 | T[\Psi^{(j,0) \oplus (0,j)}(x) \bar{\Psi}^{(j,0) \oplus (0,j)}(y)] | 0 \rangle , \quad (6)$$

with the matter field operator defined by

$$\Psi^{(j,0) \oplus (0,j)}(x) = \sum_{\sigma=-j}^{+j} \int \frac{d^3 p}{(2\pi)^3} \frac{\Omega}{2\omega_{\vec{p}}} \left[ u_{\sigma}(\vec{p}) a(\vec{p}, \sigma) e^{-ip \cdot x} - v_{\sigma}(\vec{p}) b^{\dagger}(\vec{p}, \sigma) e^{-ip \cdot x} \right] , \quad (7)$$

where  $\Omega$  is a normalization factor which can be chosen for convenience. In addition  $\bar{\Psi}(x) \equiv \Psi^{\dagger}(x) \gamma_{\sigma\sigma\cdots\sigma}^{C.A.}$ , with the canonical representation  $\gamma_{\sigma\sigma\cdots\sigma}$  as a diagonal matrix with  $(2j + 1) \times (2j + 1)$  identity matrix  $I$  in the upper left corner, and  $-I$  in the lower right corner.

The  $2(2j + 1)$  particle-antiparticle covariant spinors  $u_{\sigma}(\vec{p})$  and  $v_{\sigma}(\vec{p})$ ,  $\sigma = j, j-1, \dots, -j$ , associated with the kinematically causal solutions of the Weinberg's equations can be obtained via the action of the  $2(2j + 1) \times 2(2j + 1)$  boost matrix (in "canonical representation," defined in Ref. [5])

$$M_{cA}(\vec{p}) = \begin{pmatrix} \cosh(\vec{J} \cdot \vec{\varphi}) & \sinh(\vec{J} \cdot \vec{\varphi}) \\ \sinh(\vec{J} \cdot \vec{\varphi}) & \cosh(\vec{J} \cdot \vec{\varphi}) \end{pmatrix} , \quad (8)$$

on the  $2(2j + 1)$  rest spinors in the form of the  $2(2j + 1)$ -dimensional column vectors  $u_{+j}(\vec{0}) = (N(j), 0, 0, \dots, 0)$ ,  $\dots$ ,  $v_{-j}(\vec{0}) = (0, 0, \dots, 0, N(j))$ . Based on certain considerations<sup>5</sup> of the  $m \rightarrow 0$  limit of the covariant spinors, the simplest choice for the normalization factor is:  $N(j) = m^j$ . In equation (8)  $\vec{J}$  are the  $(2j + 1) \times (2j + 1)$  angular momentum matrices with  $J_z$  diagonal. The parameter

$\varphi$  is given by

$$\cosh(\varphi) = \gamma = \frac{1}{\sqrt{1-v^2}} = \frac{E}{m}, \quad \sinh(\varphi) = v\gamma = \frac{|\vec{p}|}{m}, \quad \hat{\varphi} = \frac{\vec{p}}{|\vec{p}|}. \quad (9)$$

A straight forward calculation yields the following simple expression for the configuration-space causal Feynman propagator

$$\begin{aligned} \langle x | S_F^j | y \rangle &= \sum_{\sigma=-j}^{+j} \int \frac{d^3p}{(2\pi)^3} \frac{\Omega^2}{2\omega_{\vec{p}}} \\ &\times \left[ u_{\sigma}(\vec{p}) \bar{u}_{\sigma}(\vec{p}) e^{-ip \cdot (x-y)} \theta(x^0 - y^0) + \eta v_{\sigma}(\vec{p}) \bar{v}_{\sigma}(\vec{p}) e^{+ip \cdot (x-y)} \theta(y^0 - x^0) \right], \end{aligned} \quad (10)$$

where  $\eta = +1$  for bosons and  $-1$  for fermions.

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## CLASSICAL EIGENSPINORS AND THE DIRAC EQUATION

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Many authors<sup>[1,2]</sup> have sought new insights into the Dirac equation and its classical limit. Here, an approach based on the covariant Pauli algebra<sup>[3]</sup>  $\mathcal{P}$  demonstrates close relationships between classical and quantum theories of "elementary" fermions.

The observed 4-velocity and orientation of a particle is given by the Lorentz transformation  $\Lambda$  which transforms properties of the particle from its rest frame  $K_0$  to the observer's "lab" frame  $K$ . Only "elementary" particles are considered here. A particle is said to be *elementary* if and only if its motion in  $K$  at any point  $x(\tau)$  on its world line can be described by a *single* Lorentz transformation  $\Lambda(\tau)$ . Such a particle can not contain independent structures (such as several elementary particles) since it would then require different transformations  $\Lambda$  for each component. A particle containing separate parts which move rigidly together could be "elementary" but would suffer a well-known conflict with causality when accelerated. Therefore an "elementary" particle may be characterized as *structureless*, although the existence of an intrinsic orientation, for example spin, is not precluded. Only positive-energy particles are considered here, even though negative-energy ones, representing antiparticles, can be consistently included in the classical theory.

The transformation  $\Lambda$  can be written as the product of a rotation  $\mathcal{R}$  and a boost  $\mathcal{B}$ :  $\Lambda = \mathcal{B}\mathcal{R}$ . In  $\mathcal{P}$ ,  $\Lambda$  may be any unimodular element ( $\Lambda \bar{\Lambda} = 1$ ); if unitary, it is a rotation ( $\Lambda \rightarrow \mathcal{R} = \bar{\mathcal{R}}^{\dagger}$ ); if hermitean, it is a boost ( $\Lambda \rightarrow \mathcal{B} = \mathcal{B}^{\dagger}$ ). The 4-velocity of the particle is found by applying  $\Lambda$  to the rest 4-velocity  $u_{rest} = 1$  (in units with  $c = 1$ ):

$$u = \Lambda u_{rest} \Lambda^{\dagger} = \Lambda \Lambda^{\dagger} = \mathcal{B}^2. \quad (1)$$

Note that  $u$  is independent of the rotation  $\mathcal{R}$ . If the particle is observed in a different frame  $K'$ ,  $\Lambda$  must be replaced by the transformed

$$\Lambda \rightarrow \Lambda' = L\Lambda, \quad (2)$$

where  $L \in \text{Sl}(2, C)$  transforms quantities from  $K$  to  $K'$ . This transformation behavior is one way to define *spinors*, and  $\Lambda$  may be called the *eigenspinor* of the particle.

The motion of an elementary particle is determined from its initial position and velocity if the evolution  $\Lambda(\tau) = L(\tau, \tau_0)\Lambda(\tau_0)$  of its eigenspinor is known. This is a spinor transformation (2) where  $L(\tau, \tau_0)$  is a Lorentz transformation obeying

$L(\tau_0, \tau_0) = 1$ , and  $L(\tau, \tau_0) = L(\tau, \tau_1)L(\tau_1, \tau_0)$ . Both  $\Lambda(\tau)$  and  $L(\tau, \tau_0)$  obey the same equation of motion, which takes the form of an identity (a dot indicates a derivative with respect to the proper time  $\tau$ )

$$\dot{\Lambda} = G\Lambda, \quad G = \dot{\Lambda}\bar{\Lambda}. \quad (3)$$

Since  $\Lambda$  is unimodular  $\Lambda\bar{\Lambda} = 1$ , the scalar part of  $G$  vanishes:  $1 \cdot G = \dot{\Lambda} \cdot \bar{\Lambda} = 0$ , and  $\Lambda$  is 4-orthogonal to  $\Lambda$ . From (2) and (3),  $G$  transforms as a 6-vector:  $G \rightarrow LG\bar{L}$ . In particular, if  $G_{rest}$  is  $G$  as seen in the rest frame of the particle, then

$$G = \Lambda G_{rest} \bar{\Lambda} \quad (4)$$

and the time evolution (3) of the eigenspinor is also given by  $\dot{\Lambda} = \Lambda G_{rest}$ .

With the help of  $\Lambda$ , any physical quantity can be transformed from  $K_0$  to  $K$ . Thus the set  $\{e_\mu\}$  of four constant basis 4-vectors of the rest frame, where  $e_0 = 1$  is the unit 4-vector along the rest-frame time axis and  $e_k = -\bar{e}_k$  are the spatial unit vectors, are seen in the lab frame to be the tetrad of 4-vectors known as *Frenet vectors*[1,4]

$$u_\mu = \Lambda e_\mu \Lambda^* \quad (5)$$

where the subscript here labels distinct 4-vectors, not components. In particular, the Frenet vector  $u_0$  is the 4-velocity of the particle in the lab frame. The time dependence of  $u_\mu$  is found directly from (3) and its hermitean conjugate:

$$\dot{u}_\mu = \dot{\Lambda} e_\mu \Lambda^* + \Lambda e_\mu \dot{\Lambda}^* = G u_\mu + u_\mu G^*. \quad (6)$$

With a particular defined orientation of the rest frame axes, this is exactly the Frenet-Serret equations[1,4], and  $G$  may be identified with the *Darboux* vector.

For example, the velocity of a charge  $e$  of mass  $m$  in an external electromagnetic field  $F = E + iB$ , is governed by (6) with  $\mu = 0$  and

$$G = \frac{e}{2m} F(\tau) - 2mS, \quad (7)$$

where  $S$  is an arbitrary 6-vector constrained to obey  $Su + uS^* = 0$ . This gives the usual Lorentz force.[5] The only effect of  $S$  is to add a rest-frame spin rate of  $4ms_0$ , where  $s_0$  is the hermitean rest-frame value of the 6-vector  $-iS$ .

The "classical Dirac equation" is simply the spinor form of the equation relating the 4-momentum and the 4-velocity:  $p = mu$ , and its spatial reverse  $\bar{p} = m\bar{u}$ . From (1) and the unimodularity of the eigenspinor  $\Lambda$ , these relations can be written

$$p\bar{\Lambda}^* = m\Lambda, \quad \bar{p}\Lambda = m\bar{\Lambda}^*. \quad (8)$$

Classically  $\rho = \rho^*$  and the two equations (8) are equivalent. Any spinor  $\Lambda$  which satisfies (8) must, within an arbitrary initial rotation and a real scalar multiplying factor, be the eigenspinor of the particle. In a  $(2 \times 2)$  matrix representation of (8), the two columns, say  $\eta$  and  $\xi$  of the eigenspinor  $\Lambda = (\eta, \xi)$  are acted upon independently. The spinor transformation (2) also holds independently for  $\eta$  and  $\xi$ : they are 2-spinors. Since  $\bar{\Lambda}^* = (-\bar{\xi}^*, \bar{\eta}^*)$  where  $\bar{\eta}^* = -i\sigma_2 \eta^*$  and similarly for  $\xi$ , each of the equations (8) is equivalent to the pair

$$\rho \bar{\eta}^* = m \xi, \quad \bar{\rho} \xi = m \bar{\eta}^*. \quad (9)$$

When the two 2-spinors are combined into the column "bispinor"

$$\psi = \begin{pmatrix} \bar{\eta}^* \\ \xi \end{pmatrix}, \quad (10)$$

the classical equations (9) take the usual quantum form

$$\gamma_\mu \rho^\mu \psi = m \psi \quad (11)$$

where  $\gamma_\mu$  are the usual Dirac matrices in the Weyl representation. The choice (10) is not unique; since the axes chosen to specify the initial orientation of the rest frame are arbitrary, the various representations related by an initial rotation are physically equivalent.

If the Frenet vectors (5) are expressed in terms of the 2-spinors, some are seen to be bilinear covariants of  $\psi$ . Thus components of the 4-velocity

$$u = \Lambda u_{\text{rest}} \Lambda^* = \Lambda \Lambda^* = \eta \eta^* + \xi \xi^* \quad (12)$$

are the bilinear covariants associated with the quantum current density:

$$u^\mu = \frac{1}{2} \bar{\psi} \gamma^\mu \psi \quad (13)$$

where  $\bar{\psi} = \psi^\dagger \gamma^0 = (\xi^*, \bar{\eta})$  and the scalar identity  $\bar{\eta} \eta^* = \eta^* \eta$  has been used. Similarly, the Frenet vector  $u_3$  is -2 times the spin dual  $s$ , and has components  $\bar{\psi} \gamma^4 \gamma^5 \psi$ . The six bilinear covariants which are components of the antisymmetric spin tensor correspond to the 6-vector  $S = i s \bar{u}$ , whereas the scalar and pseudoscalar bilinear covariants are determined by the unimodularity condition  $\Lambda \bar{\Lambda} = 1$  to be  $\bar{\psi} \psi = 2$  and  $\bar{\psi} \gamma^5 \psi = 0$ , respectively. This accounts for all 16 bilinear covariants; none are left to express the Frenet vectors  $u_1, u_2$ . Indeed,  $u_1$  and  $i u_2$  are the hermitean and antihermitean parts of  $2 \eta \xi^*$  which can not be expressed as a bilinear covariant of the form  $\bar{\psi} [\dots] \psi$ . The Dirac theory is evidently incapable of describing elementary particles with more than one preferred direction (the spin direction).

Since the spin is proportional to the Frenet vector  $u_3$ , its classical time dependence is given by (6). With  $G$  as in (7), and with  $S = i\Lambda s_0 \bar{\Lambda}$  where  $s_0$  is constant, (6) is equivalent to the BMT equation<sup>[6]</sup> for a spin with g-factor  $g = 2$  in an electromagnetic field  $F$ . The value  $g = 2$  is a consequence of assuming a *linear* time development for  $\Lambda$ . A more general, nonlinear equation can accommodate an arbitrary g-factor, but this is unsatisfactory for an elementary particle which is not an isolated point.

Expressions for the symmetry transformations P, T, and C can also be identified for the forms  $\psi$  and  $\Lambda$ . The  $\psi$  forms are identical to the usual quantum ones, but the corresponding forms for  $\Lambda$  are simpler and have transparent geometrical interpretations. The parity transformation P:  $\psi \rightarrow \gamma_0 \psi$  corresponds to spatial inversion  $\Lambda \rightarrow \bar{\Lambda}^T$ , whereas charge conjugation C corresponds to the reflection  $\Lambda \rightarrow \Lambda e_1$ , the PT transformation rotates the initial rest frame by  $\pi$  about  $-e_1$  and CPT multiplies  $\Lambda$  by  $i$ .

The classical Dirac equation for a free particle is just the momentum representation of the quantum Dirac equation. It can be used to derive the momentum-space free-fermion propagators used in standard QED treatments. If the free particle solutions are known, the full quantum perturbation formalism results from the superposition of eigenspinors. In terms of  $\Lambda$  the quantum plane-wave solutions  $\psi(x) = \psi(0) \exp(-ip \cdot \bar{x}/\hbar)$  are  $\Lambda = \Lambda(0) \exp(i e_3 p \cdot \bar{x}/\hbar) = \Lambda(0) \exp(i e_3 m \tau/\hbar)$  which describe a rest-frame spin about  $-e_3$  at the *Zitterbewegung frequency*  $2m/\hbar$ . In the presence of a potential energy  $V$  the spin frequency should probably be modified to  $2(m+V)$ .

For adiabatically changing potential energies of purely electromagnetic origin, this is sufficient to establish the operator relation

$$i\partial\psi(x) = (\rho + eA)\psi(x) \quad (14)$$

and hence all of standard quantum Dirac theory. A phase shift in the bispinor  $\psi$  is equivalent to a rotation about the spin axis, and local gauge invariance results as usual.

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# Smooth massless limit of spin-2 theories

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## Abstract

We find a set of auxiliary ghost fields for the massive spin-1 and spin-2 fields, such that the Lagrangian, the propagators, and the number of physical fields have a smooth massless limit.

## 1 Introduction

We know from Wigner's classification, that massive particles with spin  $s$  have  $2s + 1$  helicities (being defined as angular momentum in the direction of the momentum of the particle), while for massless particles of spin  $s$  there are two representations with helicities  $\pm s$ . So while for spin-0 particles we have one component in both cases, for spin-1 we have two components for photons, but 3 for Z-bosons, and a massive spin-2 particle would have 5 components, to be compared with the two of gravitons. For higher spin, a smooth massless limit seems to require more and more auxiliary physical fields. On the other side Gupta-Bleuler electrodynamics has four field components (two physical and two ghost), so we expect auxiliary ghost fields for the massive theory.

Such a smooth massless limit is necessary for—to say the least—the infrared problem. When we introduce a small mass in the massless propagators, we should obtain a massive propagator of the same spin. There are many possible auxiliary fields to achieve this; here we want to concentrate on two, which we think are of special interest. One scheme uses a minimal set, the other one has only massless auxiliary fields.

## 2 Spin-1: Minimal scheme

The propagator of the massive spin-1 Proca-field,

$$\langle \tilde{A}_\mu \tilde{A}_\nu \rangle = -\frac{\eta_{\mu\nu} - \frac{1}{m^2} k_\mu k_\nu}{k^2 - m^2} \quad (1)$$

diverges in the massless limit. To rectify this, we make a "gauge transformation"

$$A_\mu = \tilde{A}_\mu - \frac{ik_\mu}{m^2} B. \quad (2)$$

If the scalar field  $B$  commutes with the Proca field, and has the propagator of a ghost,

$$\langle \hat{A}_\mu B \rangle = 0, \quad \langle BB \rangle = -\frac{m^2}{k^2 - m^2} \quad (3)$$

then we obtain for the new vector field

$$\langle A_\mu A_\nu \rangle = -\frac{\eta_{\mu\nu}}{k^2 - m^2}, \quad \langle A_\mu B \rangle = -\frac{ik_\mu}{k^2 - m^2}. \quad (4)$$

In the massless limit, we get the propagator of the electromagnetic field in Feynman gauge, and the Nakanishi-Lautrup field  $B$ .

The corresponding Lagrangian in the old and new fields is

$$\begin{aligned} L &= -\frac{1}{4}F^2 + \frac{m^2}{2}\hat{A}^2 + \frac{1}{2m^2}(B\Box B + m^2 B^2) \\ &= -\frac{1}{4}F^2 + \frac{m^2}{2}A^2 + B(\partial A) + \frac{1}{2}B^2; \end{aligned} \quad (5)$$

expressed in the new field it has a smooth massless limit. This formulation is discussed e.g. by Nakanishi [1].

We begin with a massive vector field which carries a massive spin-1 representation of the Poincaré group, and a scalar ghost of equal mass, and end up with a Gupta-Bleuler triplet. If we denote physical helicity components by bold type, negative norm ghosts by italic and norm 0 ghosts by usual type we have the helicity components:

|                                                     |                                                               |     |
|-----------------------------------------------------|---------------------------------------------------------------|-----|
| massive<br>$(\pm\mathbf{1}, \mathbf{0}) \oplus (0)$ | massless<br>$(0) \rightarrow (\pm\mathbf{1}) \rightarrow (0)$ | (6) |
|-----------------------------------------------------|---------------------------------------------------------------|-----|

The number of physical fields is not constant. The helicity-0 component of the Proca-field and the ghost combine to the scalar and gauge modes of the Gupta-Bleuler triplet. The gauge freedom of the electromagnetic field appears only in the limit. This situation can be improved by adding further auxiliary fields.

### 3 Spin-1: Massless auxiliary fields

If we expand Eq. (1) in powers of mass, we get

$$\langle \hat{A}_\mu \hat{A}_\nu \rangle = \frac{1}{m^2} \frac{k_\mu k_\nu}{k^2} - \frac{\eta_{\mu\nu}}{k^2} + \frac{k_\mu k_\nu}{(k^2)^2} + O(m^2). \quad (7)$$

The "gauge transformation"

$$A_\mu = \hat{A}_\mu - ik_\mu \phi \quad (8)$$

gives in the limit the propagator in Feynman gauge for  $A_\mu$ , if the auxiliary scalar field satisfies

$$\langle \phi \phi \rangle = -\frac{1}{(k^2)^2} - \frac{1}{m^2} \frac{1}{k^2}. \quad (9)$$

This can be obtained from a dipole ghost  $\phi_0$ , which has with  $\phi_1 = \square\phi_0$

$$\langle\phi_0\phi_0\rangle = -\frac{1}{(k^2)^2}, \quad \langle\phi_0\phi_1\rangle = \frac{1}{k^2}. \quad (10)$$

The field

$$\phi = \phi_0 + \frac{1}{2m^2}\phi_1 \quad (11)$$

satisfies Eq. (9). From the dipole we can form an additional scalar field

$$\Phi = m\phi_0 + \frac{1}{2m}\phi_1 \quad (12)$$

which in the massless limit becomes a particle

$$\langle\Phi\Phi\rangle = \frac{1}{k^2} - \frac{m^2}{(k^2)^2}, \quad \langle\Phi A_\mu\rangle = -im\frac{k_\mu}{(k^2)^2}. \quad (13)$$

The corresponding Lagrangian in the old and new fields is

$$\begin{aligned} L &= -\frac{1}{4}F^2 + \frac{m^2}{2}\dot{A}^2 + \frac{1}{2}\phi_1^2 - \phi_1\square\phi_0 \\ &= -\frac{1}{4}F^2 + \frac{m^2}{2}A^2 + \frac{1}{2}\phi_1^2 - m\Phi(\partial A) - \frac{1}{2}\Phi\square\Phi. \end{aligned} \quad (14)$$

In the limit we get electrodynamics with gauge fixing and a scalar field  $\Phi$ . In the Abelian Higgs model it is the Goldstone mode [2].

If we denote the helicity components of the dipole ghost by  $(0) \rightarrow (0)$  we have

|                                                    |                                                                 |      |
|----------------------------------------------------|-----------------------------------------------------------------|------|
| massive                                            | massless                                                        | (15) |
| $(\pm 1, \mathbf{0}) \oplus [(0) \rightarrow (0)]$ | $[(0) \rightarrow (\pm 1) \rightarrow (0)] \oplus (\mathbf{0})$ |      |

So the number of physical fields remains constant. The gauge freedom of the dipole ghost,  $\phi_0 \rightarrow \phi_0 + \Lambda$ , with  $\square\Lambda = 0$  becomes in the limit the gauge freedom of the electromagnetic field.

## 4 Spin-2: Minimal scheme

To simplify the notation we multiply the propagator of the massive spin-2 Fierz-Pauli field with currents (energy momentum tensors) and write the Born-amplitude (see e.g. [3])

$$\langle t \cdot H H \cdot t \rangle = \frac{1}{k^2 - m^2} \left( t \cdot t - \frac{1}{2} t' t' - \frac{2}{m^2} (k \cdot t)^2 + \frac{1}{6m^4} (2kk \cdot t + m^2 t')^2 \right), \quad (16)$$

where  $t \cdot t = t_{\mu\nu} t^{\mu\nu}$  and  $t' = t^\mu{}_\mu$ . A new field

$$h_{\mu\nu} = H_{\mu\nu} + \frac{ia}{m^2} (k_\mu B_\nu + k_\nu B_\mu) + \frac{b}{m^2} (2k_\mu k_\nu + m^2 \eta_{\mu\nu}) D \quad (17)$$

with  $2a^2 = 6b^2 = 1$  and the propagators

$$\langle B_\mu B_\nu \rangle = m^2 \frac{\eta_{\mu\nu}}{k^2 - m^2}, \quad \langle DD \rangle = -\frac{1}{k^2 - m^2} \quad (18)$$

gives

$$\langle t \cdot h h \cdot t \rangle = \frac{1}{k^2 - m^2} \left( t \cdot t - \frac{1}{2} t' t' \right). \quad (19)$$

In the massless limit this is the propagator of linearized gravity. It propagates helicity  $\pm 2$  modes only; one role of the ghost  $D$  is to produce the nonvanishing trace of the massless field  $h$ . The two-point function  $\langle D h_{\mu\nu} \rangle$  still has a singular limit, which can be dealt with by introducing  $G = \frac{m a}{b} D + \frac{i}{m} k_\mu B^\mu$ .

$H_{\mu\nu}$  carries a massive spin-2 representation,  $B_\mu$  a spin-1 ghost and spin-0 representation of positive norm, and  $D$  a spin-0 ghost, all of equal mass. We have the following helicity components:

| massive                                                     | massless                                                                                        |
|-------------------------------------------------------------|-------------------------------------------------------------------------------------------------|
| $(\pm 2, \pm 1, 0) \oplus (\pm 1, 0) \oplus (0) \oplus (0)$ | $(0) \rightarrow (\pm 1) \rightarrow (\pm 2, 0) \rightarrow (\pm 1) \rightarrow (0) \oplus (0)$ |

(20)

The Lagrangian for this formulation, including further auxiliary ghosts which give a constant number of physical fields and a gauge freedom of the massive auxiliary fields can be found in [4], where it is used for the infrared regularization of quantum gravity.

As in section 3 we can also find massless auxiliary fields. Expanding the propagator (16) we find that a spin-1 dipole and a spin-0 tripole are sufficient to regularize the massless limit. As the central representation of the tripole has negative norm, a physical application of the minimal scheme seems to be more straightforward.

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## On the metaplectic representation in quantum field theory

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We give an account of the infinite dimensional metaplectic representation in the Segal-Bargmann framework: fully explicit formulae for the integral kernels of the representation are presented. Quantization of bosons interacting with external fields may be formulated with the help of the metaplectic representation, with a suitable recipe for the selection of complex structures: closed expressions for the vacuum persistence amplitude,  $S$ -matrix ... are obtained. Occurrence of Schwinger terms, for instance the central charge of the Virasoro algebra, is directly linked to the non-split central circle extension of the symplectic group.

The classical manifold underlying bosons fields is just a symplectic vector space, i.e., a real vector space  $V$  with a symplectic form  $S$ . Pick a complex structure  $J$ , i.e., a real-linear operator on  $V$  satisfying  $J^2 = -I$ , which leaves  $S$  invariant and such that  $S(v, Jv) > 0$ , for  $v \neq 0$ . To fix ideas, one can think of  $V$  as the space of real solutions for a Klein-Gordon equation with external interaction,  $S$  as the usual symplectic form on  $V$ , and  $J$  is chosen as the *unique* complex structure commuting with the time evolution generator for the *free* Klein-Gordon equation:

$$J = \begin{pmatrix} 0 & -(-\Delta + m^2)^{-1/2} \\ (-\Delta + m^2)^{1/2} & 0 \end{pmatrix}.$$

Then  $V$  can be regarded as a complex vector space with the obvious rule  $(\alpha + i\beta)v := \alpha v + \beta Jv$  for  $\alpha, \beta$  real, and the hermitian form

$$\langle u | v \rangle := S(u, Jv) + iS(u, v) = d_J(u, v) + id_J(Ju, v) \quad (1)$$

is a positive definite inner product on  $V$ . We shall henceforth assume that  $V$  is complete for the inner product (1) and separable.

Let  $\mathcal{B}(V)$  denote the space of antiholomorphic functions on  $V$  such that  $\|f\|_{\mathcal{B}}^2 := \sup_{V'} \int_{V'} |f(u)|^2 e^{-\frac{1}{2}\langle u|u \rangle} du$ , where  $V'$  ranges over the finite dimensional complex subspaces of  $V$  [1]. Denote  $E_v(u) := \exp(\frac{1}{2}\langle u|v \rangle)$ . Then  $E_v \in \mathcal{B}(V)$  and for any  $\Psi \in \mathcal{B}(V)$ , we have  $\langle E_v, \Psi \rangle_{\mathcal{B}} = \Psi(v)$ . This is a well-known property of the system of coherent states in Bargmann space, that generalizes to the infinite dimensional case. We write  $|0\rangle = E_0$  to denote the vacuum (i.e., the constant function 1).

Consider the group  $Sp$  of invertible continuous real-linear transformations leaving  $S$  invariant. One has  $g \in Sp$  iff  $gJ = Jg^{-t}$ , where the superscript 't' means transposition with respect to  $d_J$ , and  $g^{-t} := (g^t)^{-1}$ . We may decompose any real-linear operator  $g$  on  $V$  into linear and antilinear parts by  $p_g := \frac{1}{2}(g - JgJ)$ ,  $q_g := \frac{1}{2}(g + JgJ)$ . Then  $g \in Sp$  iff  $p_g = \frac{1}{2}(g + g^{-t})$ ,  $q_g = \frac{1}{2}(g - g^{-t})$ . If  $g \in Sp$ , then  $p_g$  is invertible; we define  $T_g := q_g p_g^{-1}$ . We can parametrize  $g \in Sp$  by the pair  $(p_g, T_g)$ ; for  $g$  can be written in a unique way as  $g = (I + T_g)p_g$ , where  $T_g$  is antilinear and symmetric and  $I - T_g^2$  is positive definite,  $p_g$  is linear and satisfies  $p_g^t(I - T_g^2)p_g = I$ .

Now, let  $\beta$  denote an irreducible Weyl system:

$$\beta(v)\beta(v') = \beta(v + v') \exp[-\frac{1}{2}S(v, v')]. \quad (2)$$

and consider the new Weyl system  $\beta_g(v) := \beta(gv)$ . This is unitarily equivalent to  $\beta$  iff  $T_g$  is a real Hilbert-Schmidt operator. This is Shale's theorem [2], which can be easily proved using Bargmann space techniques. Write  $Sp'(V)$  for the subgroup of elements of  $Sp(V)$  that verify Shale's criterium. By definition, the metaplectic representation of  $Sp'(V)$  intertwines  $\beta$  and  $\beta_g$ :

$$\nu(g)\beta(v)\nu(g)^{-1} = \beta(gv). \quad (3)$$

This defines  $\nu(g)$  up to multiplication by a complex number of absolute value 1. It is easily proved [3] that all operators on  $\mathcal{B}(V)$  whose domains contain the principal vectors  $E_v$ , have integral kernels:  $Af(u) = \langle \overline{K_A}(u, \cdot), f \rangle_B$ . In particular:

$$K_{\nu(g)}(u, v) = c_g \exp \frac{1}{4} \{ \langle u | T_g u \rangle + 2 \langle p_g^{-1} u | v \rangle + \langle T_{g^{-1}} v | v \rangle \}, \quad (4)$$

when the Weyl system has the standard form:

$$\beta(v)F(u) := \exp(\frac{1}{4} \langle 2u - v | v \rangle) F(u - v). \quad (5)$$

The constant  $c_g$  satisfies  $|c_g| = \det^{1/4}(I - T_g^2)$ .

It is straightforward, albeit tedious, to compute the kernel of  $\nu(g)\nu(h)$ , for  $g, h \in Sp'(V)$ , using Gaussian integral formulas. One arrives at  $\nu(g)\nu(h) = c(g, h)\nu(gh)$ , where the scalar  $c(g, h)$  must be a phase factor, in order each  $\nu(g)$  be unitary. It is found:

$$c(g, h) = \exp(i \arg \det^{-1/2}(I - T_h T_{g^{-1}})).$$

The group cocycle represented by  $c(g, h)$  is not trivial, i.e., it cannot be redefined away by changing the phase of the vacuum persistence amplitude  $c_g$ .

The space  $\mathcal{B}(V)$  is just a disguised form of the boson Fock space. To make the translation, consider the correspondence:

$$(u \mapsto 2^{-n/2} \langle u | v_1 \rangle \dots \langle u | v_n \rangle) \longleftrightarrow v_1 \vee \dots \vee v_n \in V^{\vee n} \quad (6)$$

between antiholomorphic polynomials and elements of the symmetric algebra  $S(V)$  of  $V$ . This extends to an isometry between  $\mathcal{B}(V)$  and boson Fock space  $\overline{S(V)}$ .

Introduce now the boson field  $\phi := -i d\beta$ . The annihilation and creation operators for the boson field  $\phi$  are the real-linear operators on  $\mathcal{B}(V)$ :

$$a(v) := \frac{1}{\sqrt{2}} [\phi(v) + i\phi(Jv)], \quad a^\dagger(v) := \frac{1}{\sqrt{2}} [\phi(v) - i\phi(Jv)]. \quad (7)$$

The coherent states are generated from the vacuum by  $E_v = \exp\left(\frac{i}{\sqrt{2}} a^\dagger(v)\right) |0\rangle$ . The effect of the metaplectic representation on the creation and annihilation operators is readily determined. Let us write

$$a_g(v) := \frac{1}{\sqrt{2}} [\phi(v) + i\phi(gJg^{-1}v)], \quad a_g^\dagger(v) := \frac{1}{\sqrt{2}} [\phi(v) - i\phi(gJg^{-1}v)]. \quad (8)$$

in accordance with (7), for any  $g \in Sp'(V)$ . Since  $gJv = (p_g + q_g)Jv = J(p_g - q_g)v$ , we obtain the Bogoliubov transformation  $a_g(gv) = a(p_g v) + a^\dagger(q_g v)$ ,  $a_g^\dagger(gv) = a(q_g v) + a^\dagger(p_g v)$ . We have immediately:  $\nu(g)a(v) = a_g(gv)\nu(g)$ ,  $\nu(g)a^\dagger(v) = a_g^\dagger(gv)\nu(g)$ , so that each  $a_g(gv)$  annihilates the "out-vacuum"  $|0_{\text{out}}\rangle := \nu(g)|0\rangle =: c_g f_{T_g}$ .

The classical scattering operator  $S_{c1}$  belongs to  $Sp$ . If, moreover,  $T_{S_{c1}}$  is of the Hilbert-Schmidt class, then its quantum counterpart  $\nu(S_{c1})$  is the  $S$ -matrix. We want now to factorize  $\nu(g)$  as  $\nu(g) = c_g S_1 S_2 S_3$ , where the  $S_i$  are operators on  $\mathcal{B}(V)$ , to be chosen so that

$$S_3 E_v = \exp \frac{1}{4} (\widehat{T}_g v | v) E_v, \quad S_2 E_v = E_{p_g^{-1} v}, \quad S_1 E_{v'} = f_{T_g} E_{v'}. \quad (9)$$

Since the  $a^\dagger(v)$  act as multiplication operators,  $S_1$  need only satisfy  $S_1|0\rangle = f_{T_g}$  if it is made up of creation operators only. We have  $a^\dagger(v_1)a^\dagger(v_2)|0\rangle(u) = -\frac{1}{2}\langle u | v_1 \rangle \langle u | v_2 \rangle$ , so we take  $S_1 := \exp(-\frac{1}{2}a^\dagger T_g a^\dagger)$ , with

$$a^\dagger T_g a^\dagger := \sum_{j,k} a^\dagger(f_k) \langle f_k | T_g \epsilon_j \rangle a^\dagger(\epsilon_j), \quad (10)$$

where  $\{\epsilon_j\}$ ,  $\{f_k\}$  are orthonormal bases for  $V$ ; this series converges (to an operator whose domain includes all  $E_v$ ) iff  $T_g$  is Hilbert-Schmidt; and the sum is independent of the orthonormal bases used. Similarly, one checks that with the definition:

$$a B a := \sum_{j,k} a(\epsilon_j) \langle B \epsilon_j | f_k \rangle a(f_k), \quad (11)$$

it is obtained  $S_3 = \exp(-\frac{1}{2}a T_g^{-1} a)$ , and with the definition:

$$\exp(a^\dagger C a) := \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{k_1 \dots k_n \\ l_1 \dots l_n}} a^\dagger(f_{k_1}) \dots a^\dagger(f_{k_n}) \langle f_{k_1} | C \epsilon_{l_1} \rangle \dots \langle f_{k_n} | C \epsilon_{l_n} \rangle a(\epsilon_{l_1}) \dots a(\epsilon_{l_n}),$$

we check that  $S_2 := \exp(a^\dagger(p_g^{-1} - I)a)$  is what we need. We have arrived this way to an explicit expression for the  $S$ -matrix in terms of the  $(p_g, T_g)$  parametrization of the element  $g = S_{c1}$  of the symplectic group on  $V$ .

Let us write  $C_X := \frac{1}{2}(X - JXJ)$ ,  $A_X := \frac{1}{2}(X + JXJ)$  to denote, respectively, the linear and antilinear parts of an element  $X$  of the Lie algebra  $sp'(V)$  of  $Sp'(V)$ . Then  $C_X$  is skewsymmetric and  $A_X$  is symmetric. The *derived representation* of  $\nu$  may be defined, for  $X \in sp'(V)$ , by:  $\dot{\nu}(X) := \frac{d}{dt}|_{t=0} e^{i\theta_X(t)} \nu(\exp tX)$ , where the phase factor  $\theta_X(t)$  is chosen so that  $t \mapsto \nu(\exp tX)$  is an homomorphism. We define the **anomaly** of the derived metaplectic representation as the bilinear form  $\alpha$  on  $sp'(V)$  determined by  $\alpha(X, Y) := [\dot{\nu}(X), \dot{\nu}(Y)] - \dot{\nu}([X, Y])$ . This is in fact the infinitesimal counterpart of the cocycle  $c(g, h)$  of the metaplectic representation  $\nu$ . After computation:

$$\alpha(X, Y) = \frac{1}{2} \text{Tr}_{\mathbb{C}}([A_X, A_Y]). \quad (13)$$

Here we defined, for  $A \in \text{End}_{\mathbb{R}}(V)$ ,  $\text{Tr}_{\mathbb{C}}[A] := \text{Tr}[P_+ A P_+]$  where  $P_+ := \frac{1}{2}(I - iJ)$  projects on the canonical polarization of the complexified of  $V$ .

Let us see how the anomaly which we have derived from the metaplectic representation allows us to compute the well-known Virasoro anomalous term. In this context  $V$  is the vector space  $\text{Map}(\mathbb{S}^1, \mathbb{R})/\mathbb{R}$  of smooth real-valued maps on the circle  $\mathbb{S}^1$ , modulo the constant maps. The symplectic form  $S$  is given by  $S(f, h) := \frac{1}{4\pi} \int_0^{2\pi} f'(\theta)h(\theta) d\theta$ , which is nondegenerate on  $V$  (in the weak sense). The operator  $J \in \text{End}_{\mathbb{R}}(V)$  determined by  $J(\sin k\theta) := \epsilon_k \cos(k\theta)$ ,  $J(\cos k\theta) := -\epsilon_k \sin(k\theta)$ , with  $\epsilon_k = +1$  or  $-1$  according as  $k$  is positive or negative, is the only positive compatible complex structure on  $V$  commuting with the rotations. Now  $f \in V$  can be expanded as a Fourier series without constant term:  $f(\theta) = \sum_{k \neq 0} a_k \epsilon^{ik\theta}$  with  $a_{-k} = a_k^*$ . The complex amplification of  $J$  on  $V_{\mathbb{C}} = \text{Map}(\mathbb{S}^1, \mathbb{C})/\mathbb{C}$  satisfies  $J(\epsilon^{ik\theta}) = i\epsilon_k \epsilon^{ik\theta}$ . After completing  $V$  in the inner product determined by  $J$ , we obtain the Hilbert space  $L^2(\mathbb{S}^1, \mathbb{R})/\mathbb{R}$ . The polarization  $W_0$  is just the Hardy space  $H^2(\mathbb{D})/\mathbb{C}$  of holomorphic functions on the unit disk which extend to square-integrable functions on the boundary  $\mathbb{S}^1$  and vanish at the origin.

The Virasoro group  $\text{Diff}^+(\mathbb{S}^1)$  of orientation-preserving diffeomorphisms of the circle acts on  $V$  by  $(g_\circ f)(\theta) := f(\circ^{-1}(\theta))$  for  $\circ \in \text{Diff}^+(\mathbb{S}^1)$ . It is seen that  $g_\circ \in Sp(V)$  for each  $\circ$ . In fact, the  $g_\circ$  belong to  $Sp'(V)$ , as has been shown by G. Segal [4]. The metaplectic representation of  $Sp'(V)$  thus gives rise to a projective unitary representation of the Virasoro group, and at the infinitesimal level the derived metaplectic representation will carry the Virasoro Lie algebra into operators on  $L^2(\mathbb{S}^1, \mathbb{R})/\mathbb{R}$ . In effect, the Virasoro Lie algebra consists of vector fields  $\xi(\theta) \frac{d}{d\theta}$  on the circle  $\mathbb{S}^1$  for which  $\xi(\theta)$  is smooth. A basis for the (complexified) Lie algebra is given by the vector fields  $X_k := i\epsilon^{-ik\theta} \frac{d}{d\theta}$ . It is clear that they verify the Lie algebra relations:  $[X_k, X_m] = (m-k)X_{k+m}$ . Write  $A_k := \frac{1}{2}(X_k + JX_kJ)$  to denote the anti-linear part of  $X_k$ . Then from the previous equations we get at once:

$$A_k(\epsilon^{in\theta}) = \frac{1}{2}n(-1 + \epsilon_n \epsilon_{n-k}) \epsilon^{i(n-k)\theta}. \quad (14)$$

(Notice that  $A_k$  is of finite rank.) We see that  $[A_m, A_k](\epsilon^{in\theta})$  is a multiple of  $\epsilon^{i(n-k-m)\theta}$ , and so  $\text{Tr}_{\mathbb{C}}([A_m, A_k]) = 0$  unless  $m = -k$ . Moreover,

$$[A_{-k}, A_k](\epsilon^{in\theta}) = \frac{1}{2}n\{2k + \epsilon_n(\epsilon_{n+k}(n+k) - \epsilon_{n-k}(n-k))\} \epsilon^{in\theta}. \quad (15)$$

The anomaly is now easy to compute. We note that  $\text{Tr}_{\mathbb{C}}([A_{-k}, A_k])$  is just the sum of the diagonal coefficients in (15) for  $n > 0$ ; and these coefficients vanish for  $n > |k|$ . Thus, if  $k > 0$ ,

$$\alpha(X_{-k}, X_k) = \frac{1}{4} \sum_{n=1}^{k-1} (2nk - n(n+k) - n(n-k)) = \frac{k^3 - k}{12}$$

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**Canonical quantization in chiral soliton  
model with a hidden local symmetry**

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**1. Purpose** — In the framework of the collective-coordinate quantization of the  $SU(2)$  nonlinear  $\sigma$ -model, two of the present authors (A.P.K. and N.M.C.)<sup>1</sup> has considered a soliton solution under a special Ansatz concerning the static configuration, and shown the possible existence of a **pion-like** quantum soliton. Here the word **quantum** means that such a soliton disappears when  $\hbar$  tends to zero; thus the quantum-mechanical structure of this model plays an important role. Our aims are i) to investigate quantum mechanical aspects of this model in connection with various procedures of canonical quantization, especially by noting a specific property of this model, i.e. the hidden local symmetry, and ii) to examine properties of the soliton.

**2. Ansatz** — The model Lagrangian is that of the  $SU(2)$   $\sigma$ -model given by

$$L = \frac{1}{4} \int_{\mathcal{R}} d^3x \operatorname{Tr}(U_{L,R}^R U_{L,R}), \quad (2.1)$$

where  $U_{L,R}(\vec{x}, t) = (\partial U(\vec{x}, t) / \partial x^B) U(\vec{x}, t)$ ;  $U(\vec{x}, t)$  is written as  $U(\vec{x}, t) = A(t) \cdot \sigma_0(\vec{x}) A(t)^\dagger$  with  $A \in SU(2)$ . **Ansatz** for  $\sigma_0(\vec{x})$  is to take

$$\sigma_0(\vec{x}) = \exp[i \tau_\mu n_0^\mu \theta(r)], \quad r = |\vec{x}|, \quad (2.2)$$

where  $(n_0^\mu)$  is a fixed unit vector in the isospin space; we take it as

$(n_0^\mu) = (0, 0, 1)$ . We have  $U(\vec{x}, t) = \exp[i \tau_\mu n^\mu(t) \theta(t)]$  with  $n^\mu(t) = D(A)_\rho n_0^\rho$  and  $A \tau_\rho A^\dagger = D(A)_\rho \tau_\mu$ . Then the classical form of 1.(2.1) is expressed as

$$L_{\text{eff}}^{\text{cl}} = \frac{1}{2} \{ \dot{\theta} \} \dot{n}^\mu \dot{n}_\mu - V[\theta], \quad |\dot{\vec{n}}|=1, \quad (2.3)$$

$$\text{where } I[\theta] = \int_{\pi}^2 \int d^3x \sin^2 \theta(r) \text{ and } V[\theta] = \frac{1}{2} \int_{\pi}^2 \int d^3x [d\theta/dr]^2 \quad (2.4)$$

Although this model is very simple, it has a noticeable aspect, **hidden local symmetry**;  $U(\vec{x}, t)$  is invariant under the **local** right transformation

$$A(t) \rightarrow A(t)' = A(t) \exp[-i\alpha(t)\tau_3/2]. \quad (2.5)$$

The present model, therefore, a quantum-mechanical example of the remark by Cremmer and Julia.<sup>2</sup>

**3. Canonical quantization — (a) Intrinsic formulation:**  $A(t) \text{CSU}(2)/U(1)$  is specified by a set of two real parameters  $\{q^b; b = 1, 2\}$ . Define

$$A^\dagger(\partial A/\partial q^b) = \frac{i}{2} \sum_{m=1}^2 \tau_m C(q)_b^m + \frac{i}{2} \tau_3 h(q)_b. \quad (3.1)$$

From the outset, all calculations are performed quantum-mechanically on the basis of the hypothesis

$$[q^b, \dot{q}^d] = i\hbar g(q)^{bd}; \quad (3.2)$$

$g^{bd}$  is an unknown function of  $q^d$ 's, which is to be determined after imposing the quantization conditions. Procedures are similar to the  $SU(3)/U(1)$  case.<sup>3</sup>

From Lagrangian (2.1), one obtains  $p_b = \{g_{bd}, \dot{q}^d\}/2$  with  $g_{bd} = I[\theta] \eta_{mn} C_b^m C_d^n$ . The canonical quantization conditions for  $q^b$  and  $p_b$  lead to  $g^{be} g_{ed} = \delta_d^b$ , from which  $g^{be}$  is determined. Then Lagrangian (2.1) is shown to have the same form as  $L_{\text{eff}}^{\text{cl}}$  (2.3) quantum-mechanically, where

$$\dot{n}^\mu(t) = \frac{1}{2} C_{3m}^n \{D(A)_n^\mu C_b^m, \dot{q}^b\}. \quad (3.3)$$

Defining  $L_\rho = I[\theta] \epsilon_{\rho\mu\lambda} n^\mu \dot{n}^\lambda$ , we obtain

$$[L_\mu, L_\rho] = i\hbar \epsilon_{\mu\rho}^\lambda L_\lambda. \quad (3.4)$$

$L_\rho$ 's are interpreted as the isospin operators, and the eigenvalues of  $L_\rho L^\rho$  are  $\hbar^2 \mathcal{Q}(\mathcal{Q} + 1)$ ,  $\mathcal{Q} = 0, 1, \dots$ . Our Lagrangian is expressed as

$$L_{\text{eff}} = (2 I[\theta])^{-1} L_\rho L^\rho - (\Delta V[\theta] + V[\theta]), \quad (3.5)$$

where  $\Delta V[\theta] = -\hbar^2/(2 I[\theta]) = \hbar^2 R/4$ ;  $R$  is the scalar curvature. The

first term in R.H.S. of (3.5) is equal to the covariant kinetic energy  $K$ ;

so we take Hamiltonian to be

$$H = K + \Delta V[\theta] + V[\theta]. \quad (3.6)$$

**(b) Embedding formulation:** We are to consider the problem of a particle motion with a mass  $I[\theta]$  and coordinates  $(n^\mu)$  in  $\mathbb{R}_3$ , subject to the constraint  $f(n) = n_\mu n^\mu - 1 = 0$ . The formulation on the basis of Dirac method has been described in our previous papers<sup>4</sup> (as to a more general case  $\mathbb{M}_n \subset \mathbb{R}_p$ , see ref. 5). After determining the fundamental Dirac brackets, we go to quantum mechanics and perform quantum-mechanical calculations from the outset; then we obtain the same algebraic relations and Hamiltonian as in (a).

**(c) Quantization in case of hidden local symmetry:** We write  $A \in SU(2)$  as

$$A = a^4 + i \sum_{\mu=1}^3 \tau_\mu a^\mu \quad \text{with } A^\dagger A = 1 = a^B a_B. \quad (3.7)$$

$$\text{Define } w^\mu, \mu = 1, 2, 3; \quad A^\dagger \dot{A} = \frac{i}{2} \tau_\mu w^\mu \quad (\text{in the classical level}). \quad (3.8)$$

We obtain from (2.1)

$$L^{cl} = \frac{1}{2} I[\theta] (4 \dot{a}_B \dot{a}^B - w w) - V[\theta], \quad (w = w^3). \quad (3.9)$$

Under the infinitesimal transformation of (2.5), we have

$$w' = w - \dot{\alpha}, \quad (3.10a)$$

$$[a^B]' = [1 + i\alpha T_3][a^B] \quad \text{with } T_3 = \frac{1}{2} \begin{bmatrix} \tau_2 & 0 \\ 0 & -\tau_2 \end{bmatrix}, \quad (3.10b)$$

$$[D_t a]' = [1 + i\alpha T_3][D_t a] \quad \text{with } D_t = d/dt + iw T_3. \quad (3.10c)$$

Thus, Lagrangian incorporating the hidden local symmetry is

$$L_{hid} = 2 I[\theta] [D_t a]^B [D_t a]_B - V[\theta] + \lambda(1 - a_B a^B). \quad (3.11)$$

Following Dirac method, we obtain five constraints;

$$\begin{aligned} \Phi_1 &= P_\lambda (= \partial L_{hid} / \partial \dot{\lambda}) \approx 0, & \Phi_2 &= P_w (= \partial L_{hid} / \partial \dot{w}) \approx 0 \\ \Phi_3 &= [T_3]_B^D a^B P_D = 0, & \Phi_4 &= a^B a_B - 1 = 0, & \Phi_5 &= a^B P_B = 0; \end{aligned} \quad (3.12)$$

the first three and the last two are belonging to the first and the second classes. Imposing three gauge-fixing conditions, we obtain the fundamental Dirac brackets among  $\{a^B, P_B\}$ ; then the same algebraic relations among  $n^\mu$ 's and  $\dot{n}^\mu$ 's as before. Going back to the starting Lagrangian, we obtain

through quantum-mechanical calculations the same Hamiltonian as (3.6).

**4. Quantum soliton and final remarks** — The soliton of our model has, if exists, the following properties: 1) The baryon number  $N_B = 0$ ; 2) the spin  $J = 0$ ; 3) the isospin  $I = Q$  (non-negative integer).<sup>6</sup> The eigenvalue of  $H$  (3.6) is

$$E_Q = \kappa^2 \{Q(Q+1) - 1\} / (2 I[\theta]) + V[\theta] + V_\pi[\theta]. \quad (4.1)$$

Here  $V_\pi[\theta]$  is the pion-mass contribution; its concrete form is  $V_\pi[\theta] = m_\pi^2 I[\theta] / 2$  when we take<sup>7</sup>  $L_\pi = m_\pi^2 f_\pi^2 \text{Tr}(\tau_\mu U^\dagger \tau^\mu U - 3) / 16$ . The necessary conditions (Derrick) for a stable soliton to exist, obtained for  $E_Q(\rho) = E_Q(\theta(r) \rightarrow \theta(r/\rho))$ , and the condition for the soliton solution of the integrodifferential equation  $\delta E_Q(\rho=1) / \delta \theta(r) = 0$  to exist are shown to be inconsistent with each other.<sup>6</sup>

One of possible ways to get a quantum soliton is to take account of the breathing mode, on which we have examined in ref.1 on the basis of the scale symmetry.<sup>7</sup> Details of the present report are developed in ref.6., and detailed considerations of the above remark will be appeared elsewhere.

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## SELF-DUALITY CONDITION IN CHERN-SIMONS HIGGS

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1. In (2+1) dimensional spacetime the possibility of including in abelian Higgs model the Chern-Simons (CS) term<sup>1</sup> has generated a great deal of interest. It was noted<sup>2</sup> that in the Chern-Simons-Higgs (CSH) system, the energy functional obeys a Bogomol'nyi-type<sup>3</sup> lower bound for a special choice of the Higgs potential. The bound is achieved if the Higgs's scalar field  $a$  satisfies the following first order self-duality condition<sup>2</sup>.

$$\mathcal{D}_1 a = -i\mathcal{D}_2 a, \quad \text{or} \quad \mathcal{D}_i a = -i\epsilon^{ij}\mathcal{D}_j a, \quad (1)$$

where  $\mathcal{D}_m = \partial_m + i\epsilon v_m$ ,  $m = 0, 1, 2$  are the spacetime indices while  $i = 1, 2$ . Our metric is  $\eta_{mn} = \text{diag}(-1, 1, 1)$  with  $\epsilon^{12} \equiv \epsilon^{012} = 1$ . We set  $a = \epsilon^{i\omega} \rho^{\frac{1}{2}}$  and substitute in (1). The spatial part of  $v_m$  is then found to be  $v_i = -\partial_i \omega - \frac{1}{2\epsilon} \epsilon^{ij} \partial_j \ln \rho$ . The electromagnetic field is then given with the aid of  $v_m$  as

$$B = f_{12} = \frac{1}{2\epsilon} \nabla^2 \ln \rho$$

$$E^i = -\partial^i v^0 + \partial^0 v^i.$$

The critical form of the potential can also be obtained by directly solving<sup>4,5</sup> the eqs. of motion with the aid of the self-duality condition. This procedure can also be extended to the scalar superfield, a supersymmetric self-duality condition<sup>4</sup> postulated, and the eqs. of motion solved for the supersymmetric potential (Sec. 2).

The Lagrangian for the *Bosonic Chern-Simons Higgs* system is

$$\mathcal{L} = -(\tilde{\mathcal{D}}^l a^*)(\mathcal{D}_l a) - V(|a|^2) - \frac{\kappa}{4} \epsilon^{lmn} v_l f_{mn} - \frac{1}{4} f_{lm} f^{lm}, \quad (2)$$

where  $\tilde{\mathcal{D}}_m = \partial_m - i\epsilon v_m$ . The equations of motion are found to be

$$\mathcal{D}^l \mathcal{D}_l a = V'(|a|^2) a \quad \text{and} \quad -\partial_m f^{ml} + \frac{\kappa}{2} \epsilon^{lmn} f_{mn} = j^l. \quad (3)$$

Here  $V'(|a|^2) = \partial V / \partial |a|^2$  and  $j^l = i\epsilon(a^* \mathcal{D}^l a - a \tilde{\mathcal{D}}^l a^*)$  is the Noether current,  $\partial_{ij}^l(v) = 0$ . For static configurations and on using the self-duality condition (1), eq.(3) leads to

$$\partial_i \partial_i v_0 + \kappa f_{12} = 2\epsilon^2 v_0 |a|^2, \quad \epsilon^2 v_0^2 + \epsilon f_{12} = V'(|a|^2). \quad (4)$$

We also obtain

$$j_1 = \epsilon \partial_2 |a|^2, \quad j_2 = -\epsilon \partial_1 |a|^2 \quad \text{and} \quad f_{12} + \kappa v_0 = \epsilon(|a|^2 - C^2), \quad (5)$$

where  $C$  is a constant. From (4) and (5) we derive the general result

$$V'(|a|^2) = \epsilon^2 v_0^2 + \frac{\epsilon}{\kappa} (2\epsilon^2 |a|^2 - \partial_i^2) v_0 \quad (6)$$

where  $v_0$  is given by

$$\left[ \frac{1}{\kappa^2} (2\epsilon^2 |a|^2 - \partial_i^2) + 1 \right] v_0 = \frac{\epsilon}{\kappa} (|a|^2 - C^2).$$

Consider first  $\kappa \rightarrow 0$  (no Chern-Simons term). From eqs. (4) and (5) we are led to  $(2\epsilon^2 |a|^2 - \partial_i^2) v_0 = 0$ ,  $V'(|a|^2) = \epsilon^2 (|a|^2 - C^2) + \epsilon^2 v_0^2$ . For the choice  $v_0 = 0$  we obtain then

$$V = (\epsilon^2/2)(|a|^2 - C^2)^2.$$

In the limit  $\epsilon \rightarrow \infty$ ,  $\kappa \rightarrow \infty$  such that  $(\epsilon^2/\kappa) \rightarrow$  finite, the terms originating from the Maxwell term in the eqs. of motion drop out. We find from eqs. (4) and (6)  $\epsilon f_{12} = 2(\epsilon^2/\kappa)(\epsilon v_0)|a|^2$ ,  $\epsilon v_0 = (\epsilon^2/\kappa)(|a|^2 - C^2)$  and  $V'(|a|^2) = (\epsilon^2/\kappa)^2 (|a|^2 - C^2)(3|a|^2 - C^2)$  leading to

$$V(|a|^2) = (\epsilon^2/\kappa)^2 (|a|^2 - C^2)^2 |a|^2. \quad (7)$$

which is seen to saturate the lower bound of the energy functional. Setting  $\rho = \epsilon x p(\chi/2)$  we find that  $\chi$  satisfies the Liouville equation<sup>7</sup>.

It may be worth remarking that in the general case if we impose in addition to the self-duality condition the ansatz  $(2\epsilon^2 |a|^2 - \partial_i^2) v_0 = 0$  then from eqs. (4) and (5) we are led to  $\epsilon v_0 = (\epsilon^2/\kappa)(|a|^2 - C^2)$ ,  $f_{12} = 0$  and

$$V(|a|^2) = \frac{1}{3} (\epsilon^2/\kappa)^2 (|a|^2 - C^2)^3. \quad (8)$$

which, however, does not saturate the lower bound. A different and more complicated potential is obtained if we impose, say, the ansatz  $\partial_i^2 v_0 = 0$ .

**2.** Consider next the *Supersymmetric Chern-Simons-Higgs system*. The gauge vector potential in the case of 2+1 spacetime dimensions is contained in a Majorana spinor connection superfield (for notation see ref. 5)

$$\Gamma^\alpha(x, \theta) = \chi^\alpha(x) + \bar{\theta}_\beta \left( \frac{1}{2} \epsilon^{\beta\alpha} v(x) + \gamma_i^{\beta\alpha} v'(x) \right) + i \bar{\theta} \theta \eta^\alpha(x). \quad (9)$$

where  $\eta^\alpha = \lambda^\alpha(x) - \frac{1}{2} (\gamma^i \partial_i \chi(x))^\alpha$ . Here the Majorana 2-spinor field  $\lambda(x)$  is the superpartner of the gauge field  $v_i(x)$  while the spinor  $\chi(x)$  and scalar  $v(x)$  are auxiliary fields. The covariant spinorial derivative is  $D^\alpha = (\partial/\partial \bar{\theta}_\alpha) + i(\gamma^i \theta)^\alpha \partial_i$  and  $\bar{D}_\alpha = \epsilon_{\alpha\beta} D^\beta$ . They satisfy  $\{D_\alpha, D^\beta\} = -2i \gamma^{i\beta}_\alpha \partial_i$ . The field strength superfield is defined by

$$\begin{aligned}
W^\alpha(x, \theta) &= \frac{i}{2} \bar{D}_\beta D^\alpha \Gamma^\beta, \\
&= \lambda^\alpha(x) + \frac{1}{2} \bar{\theta}_\beta (\epsilon^{lmn} f_{lm}) \gamma_n^{\beta\alpha} + \frac{i}{2} \bar{\theta} \theta (\gamma^l \partial_l \lambda)^\alpha.
\end{aligned} \quad (10)$$

where  $f_{lm} = (\partial_l v_m - \partial_m v_l)$ .

The matter superfield is a complex scalar superfield

$$\Phi(x, \theta) = a(x) + i\bar{\theta}\psi(x) + i\bar{\theta}\theta f(x). \quad (11)$$

where  $a(x)$  is a complex scalar,  $\psi^\alpha(x)$  its complex superpartner and  $f(x)$  an auxiliary complex scalar. The gauge covariant spinorial derivatives may be defined to be

$$\nabla^\alpha \Phi = (D^\alpha + \epsilon \Gamma^\alpha) \Phi, \quad \bar{\nabla}^\alpha \Phi^* = (D^\alpha - \epsilon \Gamma^\alpha) \Phi^*. \quad (12)$$

The closure relation,  $\{\bar{\nabla}_\alpha, \nabla^\beta\} = -2i\gamma^{l\beta}{}_\alpha \nabla_l$  where  $\nabla_l = (\partial_l + \epsilon \Gamma_l)$  and  $\Gamma_l = \frac{i}{2} \bar{D} \gamma_l \Gamma$  is easily established.

From the total action

$$I = \int d^3x d^2\theta \left\{ \left( \frac{1}{4} \bar{\nabla}_\alpha \Phi^* \nabla^\alpha \Phi + iV(|\Phi|^2) \right) + \frac{1}{8} \bar{W}_\alpha W^\alpha - \frac{\kappa}{8} \bar{\Gamma} W \right\} \quad (13)$$

where  $V$  is the superpotential, we obtain the following eqs. of motion

$$\frac{1}{4} \bar{\nabla}_\alpha \nabla^\alpha \Phi(x, \theta) = iV'(|\Phi|^2) \Phi, \quad (14)$$

$$(\gamma^l \partial_l W)^\alpha - \kappa W^\alpha = \epsilon (\Phi^* \nabla^\alpha \Phi - \Phi \bar{\nabla}^\alpha \Phi^*) \quad (15)$$

and the conservation of Noether's current requires  $\bar{D}_\alpha (\Phi^* \nabla^\alpha \Phi - \Phi \bar{\nabla}^\alpha \Phi^*) = 0$ .

We adopt the supersymmetric gauge  $\bar{D}\Gamma = 0$  and consider static configurations. The self-duality constraint on the matter superfield now takes the form<sup>4</sup>

$$\nabla^\alpha \Phi = i(\gamma^0 \nabla)^\alpha \Phi, \quad \bar{\nabla}^\alpha \Phi^* = -i(\gamma^0 \bar{\nabla})^\alpha \Phi^*. \quad (16)$$

We derive from eq.(14)

$$\Gamma^0 = \frac{2i}{\epsilon} V'(|\Phi|^2), \quad (17)$$

where  $\Gamma^l = \frac{i}{2} \bar{D} \gamma^l \Gamma$  with  $l = 0, 1, 2$  and the supersymmetric gauge corresponds to  $\partial_l \Gamma^l = 0$ . The Noether current is also conserved.

In the absence of the (super) Maxwell term we derive from eq.(15)

$$\kappa F_{12} = -\frac{\epsilon}{2} \bar{D} D |\Phi|^2, \quad (18)$$

$$\kappa \Gamma_0 = i\epsilon (|\Phi|^2 - C^2). \quad (19)$$

where  $F_{12} = (\partial_1 \Gamma_2 - \partial_2 \Gamma_1)$ . Comparing the two expressions for  $\Gamma_0$  we derive immediately the specific superpotential

$$V(|\Phi|^2) = -\frac{\epsilon^2}{4\kappa}(|\Phi|^2 - C^2)^2. \quad (20)$$

For the case of vanishing  $\kappa$  the superpotential corresponding to the self-dual solutions is found by following a similar procedure. In both cases the supersymmetric actions contain the results of the purely bosonic theory as is easily shown by integrating the superfield action over  $\theta$  and eliminating the auxiliary fields by using their eqs. of motion. The same is true of the supersymmetric self-duality condition when analysed in terms of the component fields. We obtain these results without the arguments for invoking an explicit N=2 supersymmetry of the action<sup>7</sup>.

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EXACT RESULTS IN TWO PARTICLE SCATTERING  
IN 2+1 DIMENSIONAL GRAVITY

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ABSTRACT

By a judicious choice of dynamical variables, we couple sources to Chern-Simons-Witten gravity in 2+1 dimensions. We find that 't Hooft's scattering amplitude is obtained exactly if the invariant associated with the momentum  $p^a = p_1^a - p_2^a$  is an integer in units of  $4\pi/G$ , where  $G$  is the gravitational constant. The general case is also given.

The description of two particle scattering in gravity in 2+1 dimensions has so far been achieved by coupling the dynamical variables associated with the particles to the gravity field.<sup>1,4</sup> A more satisfactory scheme would be to represent the particles with fields also. But it is not yet known how to do this. In Witten's approach,<sup>4,6</sup> one begins with the Chern-Simons action for the Poincaré group, which for a manifold  $M$ , with topology  $R \times \Sigma$ , where  $R$  represents the time, can be written as<sup>4</sup>

$$I_{CS} = \int dt \int_{\Sigma} \left[ e^j e^a \partial_i \omega_a - \eta_{ab} e^a \omega^b - \eta_{ab} \omega^a \omega^b \right] e^i \quad (1)$$

$$i, j = 1, 2 \quad ; \quad a, b = 0, 1, 2$$

Here  $e^a_{\mu}$  and  $\omega^a_{\mu}$  are components of the connection

$$A_{\mu} = e^a_{\mu} P_a + \omega^a_{\mu} J_a \quad (2)$$

where  $P^a$  and  $J^a$  are, respectively, the momentum and the angular momentum generators of the Poincaré group. Also,

$$F^a[A] = \epsilon^{ij} F^a_{ij}[A] \quad (3)$$

where  $F^a_{ij}[A]$  are the spatial components of the field strength tensor for the connection  $A$ , and  $\epsilon^{ij}$  is the

antisymmetric tensor in two space dimensions. Here, we largely follow the notation of reference [6]. To couple a point-like source characterized by phase space coordinates  $(p^a, q^a)$  to the Poincaré gravity, one can supplement the action (1) by the source action<sup>4</sup>

$$I = \int_C dt \eta_{ab} p^a [\partial_t q^b + t^\mu (e^b{}_\mu + \varepsilon^b{}_{bc} q^c \omega^d{}_\mu)] + \lambda (p^2 + m^2) \quad (4)$$

where the path  $C$  is the particle trajectory,  $\lambda$  is a Lagrange multiplier, and  $t^\mu = \frac{dx^\mu}{dt}$  is tangent to the path  $C$ .

To couple two or more sources to the Chern-Simons theory one can add a term of the form (4) for each of the point sources involved. This method is not unique, however, to the extent that the choice of canonical variables in a phase space corresponding to more than one particle is not unique. We take advantage of this non-uniqueness to write down an action<sup>7</sup> which leads to 't Hooft's version of the two-particle scattering amplitude for certain quantized values of one of the Wilson loop observables. Let  $(p^a_1, q^a_1)$  and  $(p^a_2, q^a_2)$  be the phase space variables of the two particles. We shall loosely refer to  $(q^a)$  and  $(p^a)$  as coordinates and momenta of the particles, although, as explained in detail in references [6], these quantities are related to space-time coordinates and momenta by a multivalued gauge transformation. Consider the combinations

$$P^a = p^a_1 + p^a_2, \quad p^a = p^a_1 - p^a_2 \quad (5)$$

and let  $Q^a$  and  $q^a$  be the corresponding canonically conjugate variables. Then, consider the following action for the two particle (source) system coupled to gravity:

$$I_S = \int_{C_Q} dt \eta_{ab} P^a [\partial_t Q^b + T^\mu (e^b{}_\mu + \varepsilon^b{}_{cd} Q^c \omega^d{}_\mu)] \\ + \int_{C_q} dt \eta_{ab} p^a [\partial_t q^b + t^\mu (e^b{}_\mu + \varepsilon^b{}_{cd} q^c \omega^d{}_\mu)] \\ + \lambda_1 \left( \frac{1}{2} P^2 + m_1^2 \right) + \lambda_2 \left( \frac{1}{2} p^2 + m_2^2 \right) \quad (6)$$

This action is invariant under the Poincaré gauge transformations

$$\delta e^a{}_\mu = -\partial_\mu \rho^a - \varepsilon^a{}_{bc} \omega^b{}_\mu \rho^c - \varepsilon^a{}_{bc} e^b{}_\mu \tau^c \quad (7)$$

$$\delta \omega^a{}_\mu = -\partial_\mu \tau^a - \varepsilon^a{}_{bc} \omega^b{}_\mu \tau^c \quad (8)$$

$$\delta q^a = \rho^a - \varepsilon^a{}_{bc} \tau^b q^c; \quad \delta Q^a = \rho^a - \varepsilon^a{}_{bc} \tau^b Q^c \quad (9)$$

$$\delta p^a = -\varepsilon^a{}_{bc} \tau^b p^c; \quad \delta P^a = -\varepsilon^a{}_{bc} \tau^b P^c \quad (10)$$

It is important to note that in (6) the paths  $C_Q$  and  $C_q$  correspond not to the individual particle trajectories but to the trajectories specified by the conjugate canonical variables,  $Q^a$  and  $q^a$ , respectively. As discussed in reference [6] for the standard case, these coordinates belong to a manifold  $M_Q$  the spacial part of which is the plane punctured by the two sources. Also the topology  $R \times \Sigma$  allows us to identify  $Q^0$  and  $q^0$  with the

time,  $t$ . The variation of the combined actions  $I_{CS}$  given by (1) and  $I_g$  given by (6) leads to the field equations

$$F^a(\omega) = P^a \delta^2(x, x_Q) + p^a \delta^2(x, x_q) \quad (11)$$

$$F^a(e) = L_{q^a} \delta^2(x, x_Q) + L_{p^a} \delta^2(x, x_q) \quad (12)$$

$$\text{where} \quad L_{q^a} = \epsilon^a_{bc} Q^c P^b \quad \text{and} \quad L_{p^a} = \epsilon^a_{bc} q^c p^b.$$

To gain more insight into the dynamics of two-particle scattering, consider the Wilson loop enclosing both sources. Let

$$s = P \cdot P = (p_1 + p_2)^2 \quad ; \quad p^2 = p \cdot p = (p_1 - p_2)^2 \quad (13)$$

Then, we can use a method which is technically similar to that used by Carlip [5] to obtain the angle addition formula when the sources are associated with individual particles. In our case, where the sources carry the charges  $P^a$  and  $p^a$ , respectively, the angle addition formula takes the form

$$\cos(H/2) = \cos(\sqrt{s}/2) \cos(p/2) - \sin(\sqrt{s}/2) \sin(p/2) [P \cdot p / p \sqrt{s}] \quad (14)$$

Thus, in general, the Hamiltonian describing the two-particle system is fairly complicated. But this expression simplifies dramatically when the observable  $p$  takes on quantized values according to

$$p = 4\pi n / G \quad ; \quad n = 1, 2, \dots \quad (15)$$

where we have reinserted the gravitational constant,  $G$ , which up until now had been set equal to unity. For these values of  $p$ , we then have

$$H = \sqrt{s} \quad , \quad \text{mod } 2\pi \quad (16)$$

The ambiguity can be removed by requiring  $H$  to vanish when  $s$  vanishes. In a frame in which the conjugate coordinate  $Q^a$  is at rest at the origin, we get, from the definition of  $s$ ,

$$H = E_1 + E_2 \quad (17)$$

We emphasize that for quantized values of  $p$  given by (15) this is an exact expression.

Let  $\Psi(Q, q)$  be the wave function describing the two-body problem. Viewed from the frame in which  $Q^a = 0$  at the origin,  $\Psi = \Psi(q)$ , effectively, and is determined by the Hamiltonian (17). To be able to give a space-time interpretation, we recall that  $q^a$  belongs to the manifold  $M_q$  the spacial part of which is the plane punctured by two sources. By requiring that  $Q^a$  remain at rest at the origin, we have fixed four of the six Poincaré gauge transformation at our disposal. We are still free to make spacial rotations and time translations. So, consider the gauge transformation

$$q'(r(t), \phi(t)) = (\exp \tau^0 J_0) q(r(t), \phi(t)) \quad (18)$$

It corresponds to a spacial rotation  $\tau^0$  given by

$$\tau^0 = (1 - |E_1 + E_2| / 2\pi) \phi \quad (19)$$

This transformation which is clearly not  $2\pi$  periodic leaves  $s$ , and hence  $H$ , invariant. But it is easy to see that the transformed coordinates  $q^a$  acquire a phase under the rotation  $\phi \rightarrow \phi + 2\pi$ :

This transformation which is clearly not  $2\pi$  periodic leaves  $s$ , and hence  $H$ , invariant. But it is easy to see that the transformed coordinates  $\mathbf{q}'$  acquire a phase under the rotation  $\phi \rightarrow \phi + 2\pi$ :

$$\mathbf{q}'(r, \phi + 2\pi) = \exp 2\pi - (E_1 + E_2) \mathbf{q}(r, \phi) \quad (20)$$

That is, they satisfy the matching conditions for coordinates on a cone characterized by the deficit angle  $\beta = E_1 + E_2$ .

We therefore conclude that by fixing the gauge in the manner that we have, the two particle scattering problem for the quantized relative momenta given by (15) is equivalent to the motion of the relative coordinate  $\mathbf{q}'$  on a cone with deficit angle determined by the free Hamiltonian (17). It is important to note that the cone describing the reduced two-body problem, which we are discussing here need not be a solution of the equations of motion (11). All of these conclusions are in agreement with those of 't Hooft, except that in our case the free Hamiltonian (17) is valid only when the quantization condition (15) holds.

When the quantization condition (15) does not hold, the two particle scattering problem can again be reduced to an equivalent one body problem in which the relative coordinate  $\mathbf{q}'$  moves on a cone. This is because none of the above arguments depended on the value of the deficit angle being  $E_1 + E_2$ . For the general case, in the frame  $Q^a = 0$  the expression (14) reduces to

$$\cos H/2 = \cos[(E_1 + E_2)/2] \cos p/2 - \sin(E_1 + E_2)/2 \sin(p/2) \cdot (m_1^2 - m_2^2)/p (E_1 + E_2) \quad (21)$$

Clearly,  $H$  is a complicated function of the energies and momenta, and there will be corrections to the simple expression  $H = E_1 + E_2$ . But, in all cases the parameter  $\tau^0$  of the gauge transformation (18) is given by  $\tau^0 = (1 - H/2\pi)\phi$ , and the transformed coordinates  $\mathbf{q}'$  will satisfy the matching conditions expected of the coordinates on a cone with a deficit angle given by  $H$ .

In closing, we note that a number of our arguments, in particular the last one, also applies to the case where the two sources carry individual particle charges, but our choice of dynamical variables is more natural for the description of scattering. Moreover, the extension of these results to describe the scattering of particles with spin is straight-forward and will be given elsewhere.

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# On Gauge Dependence of Perturbative Predictions in QCD<sup>1</sup>

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## Abstract

We review the gauge dependence of perturbative computations in non-abelian gauge field theory in particular in QCD. Perturbative results for the running coupling constant  $\alpha$ , for various physical quantities  $R$  and for the QCD scale parameter  $\Lambda$  are in general renormalization scheme and gauge dependent. In abelian gauge field theory models – due to the existence of renormalization scheme and gauge invariant effective coupling constant – such a dependence may be avoided, in nonabelian case it is generic. We discuss computational and principle difficulties connected with this fact and show in addition that known QCD representations for mesons and baryons are also gauge dependent.

## 1 Introduction

Perturbative approximations in nonabelian gauge field theory (GFT), especially in QCD, depend generically on the gauge and on the chosen renormalization scheme (RS), in contrast to abelian group gauge field theory, like QED. This difference is connected with the fact that in abelian GFT a unique renormalization scheme (RS) and a gauge invariant effective coupling constant exists [1], whereas in a nonabelian GFT many RS-invariant effective coupling constants are possible which are all gauge dependent [2]; hence perturbative expansion are also RS and/or gauge dependent. This is a computational and may be also conceptual difficulty of nonabelian GFT and of QCD since there is no known inbuilt mechanism which tells us the meaning of a perturbative calculation for a physical quantity in connection with a given RS and a fixed gauge.

We analyse in Sec. 2 the gauge dependence of the running coupling constant (rcc)  $\alpha = g^2/4\pi$  in QCD. The momentum subtraction RS is from the physical point of view a distinguished one but in this schemes and all the others the rcc is gauge dependent, which yields qualitative effects, e.g. a rcc  $\alpha(\mu)$  finite for all  $\mu$ .

In Sec. 3 we discuss the gauge dependence of some perturbative predictions of QCD and demonstrate that the gauge can significantly change this predictions.

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In Sec. 4 we consider the gauge and RS dependence of the QCD scale parameter  $\Lambda$  and show that its values in two different RS can differ e.g. by a factor of 2.

We analyse in Sec. 5 the gauge dependence of hadron representations in QCD [5]. We show explicitly that representations for proton,  $\pi$ -meson, K-meson or  $\rho$ -meson in terms of quarks and gluons used in [5] depend on the gauge.

Sec. 6 deals with the gauge dependence of effective coupling constant (cc).

We complete our work with a discussion of the obtained results and mention an alternative model for electro-weak and strong interactions based on a nonabelian gauge field theory model which is spontaneously broken to the abelian  $U(1)$  subgroup [6], [7]; one should expect that in this model the gauge dependence problem of perturbative predictions will not appear.

## 2 Gauge dependence of QCD coupling constant

The QCD cc  $\alpha(\mu)$  is discussed mostly in the modified minimal subtraction ( $\overline{\text{MS}}$ ) renormalization scheme [8], which scheme is distinguished from other RS by the fact that the  $n$ -loop calculations of various physical quantities are relatively simple and more specifically the renormalization group functions like  $\beta$  and  $\gamma$  are gauge independent. One should stress however that  $\overline{\text{MS}}$  scheme has difficulties. In fact

(a) The mass parameter  $\mu$  in the  $\overline{\text{MS}}$  scheme has no physical meaning;  $\mu$  is introduced to have the action integral in  $4 - \epsilon$  dimensional space-time dimensionless: the coupling constant  $g$  (which in  $4 - \epsilon$  space has a dimension) is replaced by  $g\mu^{\epsilon/2}$  (see e.g. [9]). Consequently  $\mu$  is not directly connected with a physical scale and its interpretation as energy or momentum transfer scale parameter [8] in QCD calculations is not evident.

(b) In the  $\overline{\text{MS}}$  scheme  $\alpha(\mu)$  in the low energy region blows up when  $\mu \rightarrow \Lambda$ , where  $\Lambda$  is the QCD scale parameter [8]. For instance in the simplest case of one-loop approximation one has ( $c$  is a constant)

$$\alpha(\mu) = \frac{c}{\ln \frac{\mu^2}{\Lambda^2}} \underset{\mu \rightarrow \Lambda}{\rightarrow} \infty \quad (1)$$

Some authors interpret this as a hint that perturbation theory breaks down in the low energy region, whereas others connect this phenomenon with QCD confinement properties [8]. However since  $\mu$  has no connection to the energy scale both interpretations are doubtful. In fact we show below that in some momentum subtraction (MOM) renormalization schemes (1) does not hold even in low energy region.

In order to overcome the mentioned computational difficulties of the  $\overline{\text{MS}}$  scheme several authors advocated another scheme, the MOM scheme, as the physically motivated one [3]. In the MOM scheme particle propagators assume their free value at  $k^2 = -\mu^2$ ,  $\mu > 0$ ; for instance the ghost propagator  $D^{ab}(k^2)$  satisfies [3]

$$D^{ab}(k^2)|_{k^2=-\mu^2} = \delta^{ab} \frac{1}{\mu^2} \quad (2)$$

Thus here the concept of a scale given by  $\mu$  is justified [3]. The renormalization group equation for the rcc  $\alpha(Q^2)$  in the MOM scheme, based on quark-gluon vertex, has the form ( $t \equiv \frac{1}{2} \ln Q^2/\mu^2$ )

$$\frac{d}{dt} \left( \frac{\alpha}{\pi} \right) = - \left( \frac{\alpha}{\pi} \right)^2 \left( \beta_0 + \beta_1 \left( \frac{\alpha}{\pi} \right) + \beta_2 \left( \frac{\alpha}{\pi} \right)^2 + \dots \right) \quad (3)$$

The  $\beta$ -function coefficients depend on the quark mass  $m$ ,  $\mu$  and the gauge parameter  $a$  and have a rather complex form (see [10], Eq.(18)); e.g the asymptotic form of  $\beta_0$  for  $\lambda = m/\mu^2 \rightarrow \infty$  and  $m_i/m \neq 0$  is [10]

$$\beta_0 \sim \frac{3}{4} \left( \frac{13}{3} - a + \frac{\ln \lambda}{\lambda} \left( \frac{5}{6} - \frac{10}{3}a + \frac{1}{2}a^2 \right) \right) + O \left( \frac{\ln \lambda}{\lambda^2} \right). \quad (4)$$

i.e.  $\beta_0$  is second order polynomial of gauge parameter  $a$ . Furthermore, the rcc  $\alpha(Q^2)$  given by a solution of (3) is, in the MOM scheme, gauge dependent. One can consider this as a difficulty of the MOM scheme [9] or as an indication that the MOM scheme is more physical than other RS since exploiting the gauge dependence of  $\alpha(Q^2)$  one can get a rcc which is finite for all  $Q^2$  [10]. In order to illustrate this we present in Fig. 2.1. the gauge dependence of  $\alpha(Q^2)$

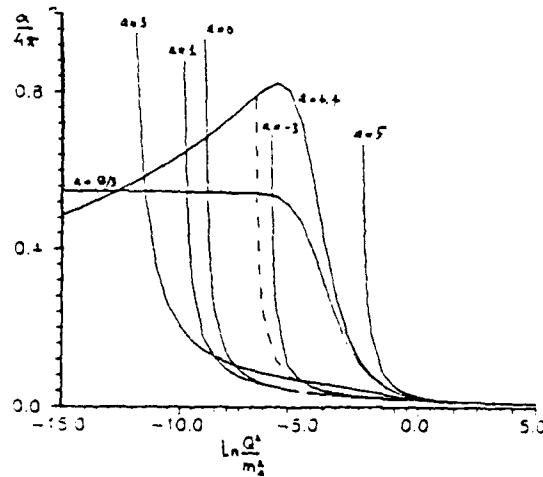


Fig. 2.1. The gauge dependence of  $\alpha(Q^2)$  for  $m = m_c$  - the mass of the charm quark. The dotted line corresponds to massless case  $m_c = 0$  which is usually considered.

For a complete analysis of  $\beta_0$  see [10].

We stress that there are others MOM-type RS in which the rcc  $\alpha(Q^2)$  is gauge dependent. For instance in so called asymmetric momentum subtraction (AMOM) RS we have [11] a

$$\beta_0 = -3.5, \quad \beta_1 = -3.25 + 1.1063a - 0.375a^2 - 0.2813a^4 \quad (5)$$

In this scheme  $\beta_0$  is a constant and  $\beta_1$  is a third order polynomial in  $a$ . The comparison of MOM and AMOM scheme shows that there are no similarities in the behaviour of  $\beta_1$  as function of  $a$ . In Fig 2.2 the evolution of the rcc  $\alpha(t)$  at  $t = \frac{1}{2} \ln \frac{\mu^2}{\Lambda^2}$  variable [11]  $a$  is shown.

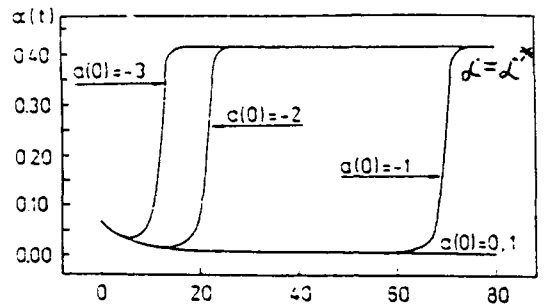


Fig. 2.2 The rcc  $\alpha(t)$  for different gauges  $a = 1, 0, -1, -2, -3$  and  $\alpha(0) = 0.06$ .

For  $a \geq 0$  (say for the Landau gauge  $a = 0$  or the Feynman gauge  $a = 1$ )  $\alpha(t) \rightarrow 0$  as  $t \rightarrow \infty$ . For  $a < 0$  we obtain a different qualitative behaviour: the rcc decreases (as for  $a \geq 0$ ) but for larger  $t > t_a$  (the value of  $t_a$  depends on  $a$ ) it sharply increases until it reaches the fixed-point value  $\alpha^*$ . Therefore the evolution of the rcc in the two-loop approximation in the AMOM scheme is different from the  $\overline{\text{MS}}$  scheme. Notice that even asymptotic freedom is lost in AMOM scheme for some values of the gauge parameter e.g. for  $a = -1, -2$  or  $-3$ . We refer our original work [11] where these problems are discussed in some detail.

### 3 Gauge dependence of QCD perturbative predictions for physical quantities

It is generally accepted that the perturbation series for a physical quantity  $R$

$$R = \lim_{N \rightarrow \infty} R^{(N)}; \quad R^{(N)} = \frac{\alpha}{\pi} \sum_{k=0}^{\infty} r_k \left( \frac{\alpha}{\pi} \right)^k \quad (6)$$

is gauge independent. However if we work in the MOM scheme the cc  $\alpha(\mu)$  is gauge dependent and consequently the Feynman expansion coefficients  $r_k$ . Therefore the gauge independence of  $R$  in (6) is achieved by a subtle cancellation of the gauge dependence between many terms of perturbative expansion. Hence the truncated perturbation series of  $R^{(N)}$  (which is the only numerical value available in applications of QCD) for a physical quantity  $R$  is in general gauge dependent. In order to illustrate this fact we consider the pure QCD contribution to the physical quantity  $R$

$$R = \frac{\sigma_T(\epsilon^+\epsilon^- \rightarrow \text{hadrons})}{\sigma(\epsilon^+\epsilon^- \rightarrow \mu^+\mu^-)} = 3 \sum_f Q_f^2 (1 + \hat{R}) \quad (7)$$

where  $\tilde{R}$  - in the three-loop approximation - is given by

$$\tilde{R} = \frac{\alpha}{\pi} \left( 1 + r_1(a) \frac{\alpha}{\pi} + r_2(a) \left( \frac{\alpha}{\pi} \right)^2 \right). \quad (8)$$

In general  $r_1$ ,  $r_2$  and  $\alpha$  depend on the gauge parameter  $a$ : they are calculated for the AMOM and  $\overline{\text{MS}}$  renormalization scheme in our work [11] b. We give in Fig. 3.1 and Fig. 3.2 the energy and  $a$  dependence for various values of  $a$  and the energy respectively. The RS and the gauge dependence is obvious, e.g.  $\tilde{R}$  for  $a = 3$  both scheme differ by a factor of 2 in the region around  $\sqrt{s} = 30$  GeV.

Note that changing the gauge parameter moderately from  $a = 0$  (Landau gauge) to  $a = 3$  (Yennie gauge) one changes the  $\tilde{R}$  value at  $\sqrt{s} = 25$  GeV by 30 %. Fig. 3.2 shows that for  $a > 3$  the dependence of  $\tilde{R}$  on  $a$  is strong and taking the gauge slightly larger than Yennie gauge one can change  $\tilde{R}$  by 100 % or more.

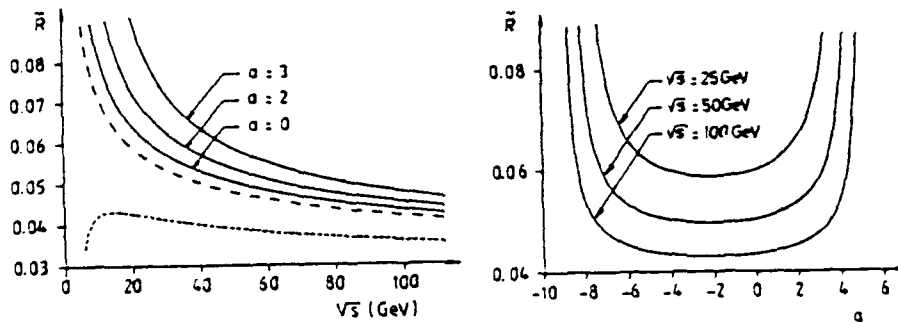


Fig. 3.1 The energy dependence of  $\tilde{R}$  for  $a = 0.2$  and  $3$  in AMOM scheme. The dashed line represents  $\tilde{R}$  in  $\overline{\text{MS}}$  scheme.

Fig. 3.2 The gauge dependence of  $\tilde{R}$  in the AMOM scheme for  $\sqrt{s} = 25, 50$  and  $100$  GeV

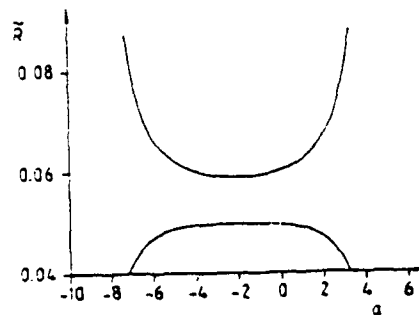


Fig. 3.3 The  $\tilde{R}$  gauge dependence in AMOM scheme for  $\sqrt{s} = 25$  GeV; two loop approximation (lower curve) and three loop approximation (upper curve).

One could expect that including higher order corrections would soften the gauge dependence in general. To check this, we present in Fig 3.3 the gauge dependence of  $\tilde{R}$  in two and three loop approximation in the AMOM scheme.

Passing from two to three loop approximation a qualitatively different behaviour of  $\tilde{R}$  as a function of gauge parameter  $a$  is obtained. This is partly because in the three loop approximation the three loop  $\beta$ -function coefficient is the fourth order polynomial in the gauge parameter [11] b.

#### 4 Gauge dependence of the QCD scale parameter $\Lambda$

The QCD scale parameter  $\Lambda$  is considered as a fundamental quantity in QCD. It was introduced as renormalization group invariant through

$$\Lambda = \mu \exp\left(-\int^{\alpha} \frac{d\alpha'}{\beta(\alpha')}\right) \quad (9)$$

Celmaster and Gonsalves observed, that passing from a cc  $\alpha_X$  in  $X$  RS to a cc  $\alpha_Y$  in  $Y$  RS by

$$\left(\frac{\alpha_Y}{\pi}\right) = \frac{\alpha_X}{\pi} \left(1 + Q_1 \frac{\alpha_X}{\pi} + Q_2 \left(\frac{\alpha_X}{\pi}\right)^2 + \dots\right)$$

one has in all perturbative orders

$$\Lambda_Y = \Lambda_X \exp\left(\frac{Q_1}{\beta_0}\right).$$

In MOM RS  $Q_1$  is gauge dependent [11], e.g. in the AMOM RS

$$Q_1(a) = \frac{1}{4} \left( \frac{233}{12} + 3a + \frac{3}{4}a^2 - \frac{10}{9}n_f \right)$$

whereas in RS based on the gluon-quark vertex

$$Q_1(a) = \frac{1}{4} \left( \frac{89}{4} + \frac{25}{6}a - \frac{3}{4}a^2 + C \left( -\frac{85}{36} - \frac{25}{9}a + \frac{1}{4}a^2 \right) - \frac{10}{9}n_f \right)$$

holds, with  $n_f$  as number of quark flavours and  $C = 2.344\dots$

We present in Fig. 4.1 the gauge dependence of QCD scale parameter in various RSs.

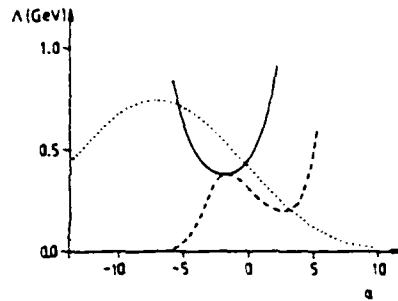


Fig. 4.1 The gauge dependence of QCD scale parameter  $\Lambda$  for the RSs in which  $\alpha$  is defined by means of gluon-ghost vertex  $\Gamma_{A\bar{\eta}\eta}(0, -\mu^2, -\mu^2)$  (continuous line), the three-gluon vertex  $\Gamma_{AAA}(-\mu^2, -\mu^2, -\mu^2)$  (dashed line), and the gluon-quark vertex  $\Gamma_{A\bar{u}u}$  (dotted line).

We see a rather strong gauge dependence of QCD scale parameter. It seems therefore that the frequently quoted value of QCD scale  $\Lambda \approx 300$  MeV determined in various experiments using  $\overline{MS}$  scheme is of little principle interest since in other schemes  $\Lambda$  may be 2 or more times bigger or smaller.

## 5 Gauge dependence of hadron representations in QCD

Hadrons are represented in QCD in terms of quarks and gluons. It is interesting to ask whether this representations for proton,  $\pi$ -meson,  $\rho$ -meson etc. utilized in QCD calculations are gauge dependent.

Consider a proton state-vector  $|P\Lambda\psi\rangle$  with the momentum  $P$  and helicity  $\Lambda$  in the Fock space of quarks and gluons. The state-vector  $|P\Lambda\psi\rangle$  is defined in the form [5]

$$|P\Lambda\psi\rangle = \sum_{\lambda_i, f_i, \alpha_i} \int \prod_{l=1}^3 d\rho(p_l) \psi_{P\Lambda}(p_1, \dots, \alpha_3) \times |p_1 \lambda_1 f_1 \alpha_1\rangle |p_2 \lambda_2 f_2 \alpha_2\rangle |p_3 \lambda_3 f_3 \alpha_3\rangle \quad (10)$$

Here the first sum runs over helicities  $\lambda_i$ , flavours  $f_j$  and colours  $\alpha_k$  of a quark with momentum  $p_i$ .  $d\rho(p_i)$  is the invariant measure over the quark momentum.  $|p_i \lambda_i f_i \alpha_i\rangle$  is the ket in 1-particle quark subspace and  $\psi_{P\Lambda}(P_1, \dots, \alpha_3)$  is the projection of proton vector-state on the Fock subspace generated by three quarks.

$$\psi_{P\Lambda}(P_1, \dots, \alpha_3) = \left( \prod_{i=1}^3 \langle p_i \lambda_i f_i \alpha_i | \right) P\Lambda\psi \rangle \quad (11)$$

One could add in principle to r.h.s. of (10) contributions from higher particle Fock subspaces e.g. 3 quark + quark - antiquark subspace, three quarks +  $n$  gluons subspace etc. . . . ; however following [5] we limit ourself to the representation (10).

The expansion (10) is gauge invariant, because all product vector-states of the different quark states transform as the tensor product of the fundamental vector representation of the gauge group  $SU(3)$  whereas the corresponding coefficient function (11) transforms as the tensor product of adjoint fundamental representation. However the analytic form of  $\psi_{P\Lambda}$ -function is up to now unknown. Hence in order to do numerical QCD calculations of hadron interactions one has to assume some concrete expression for  $\psi_{P\Lambda}$ . In all expressions known to us the following factorization is assumed [5]

$$\psi_{P\Lambda}(p_1, \dots, \alpha_3) = \psi_{P\Lambda}(p_1, \dots, f_3) \epsilon_{\alpha_1 \alpha_2 \alpha_3} \quad (12)$$

$\psi_{P\Lambda}(p_1, \dots, f_3)$  is chosen often of gaussian type (see e.g. [5] iii, Eq.(1.8)).

We show now that the factorization (12) implies a gauge dependence of proton vector-state  $|P\Lambda\psi\rangle$ . To see this one has to find transformation properties of the 1-quark state  $|p\lambda f\alpha\rangle$  in  $p$ -space under the gauge group  $SU(3)$  which acts in the  $x$ -space, i.e. on the quark field  $\psi_{\lambda f\alpha}(x)$  as

$$\psi'_{\lambda f\alpha}(x) = U_{\alpha\beta}(x) \psi_{\lambda f\beta}(x) \quad (13)$$

where  $U(x)$  is  $3 \times 3$  unitary unimodular matrix with  $x$ -dependent entries  $U_{ij}(x)$ . The 1-quark state  $|x\lambda f\alpha\rangle$  in  $x$ -space is

$$|x\lambda f\alpha\rangle = \psi_{\lambda f\alpha}^+(x) |0\rangle \quad (14)$$

where  $\psi_{\lambda f\alpha}^+(x)$  is the creation operator with helicity  $\lambda$ , flavour  $f$  and colour  $\alpha$  and  $|0\rangle$  the Fock vacuum. Under gauge transformations (14) transforms with (13) as

$$|x\lambda f\alpha\rangle' = U_{\alpha\beta}(x) |x\lambda f\beta\rangle \quad (15)$$

The corresponding transformation of the states in  $p$ -space is obtained through a Fourier transform of (15)

$$|p\lambda f\alpha\rangle' = (U_{\alpha\beta}(0) * |0\lambda f\beta\rangle)(p) = \int d^4 p' U_{\alpha\beta}(p - p') |p'\lambda f\alpha\rangle \quad (16)$$

where

$$U_{\alpha\beta}(p) = \frac{1}{(2\pi)^2} \int \epsilon^{ipx} U_{\alpha\beta}(x) d^4 x. \quad (17)$$

For product states, we have e.g.

$$\begin{aligned} & (|p_1 x_1 f_1 \alpha_1\rangle |p_2 \lambda_2 f_2 \alpha_2\rangle)' = \\ & \int dp'_1 dp'_2 U_{\alpha_1 \beta_1}(p_1 - p'_1) U_{\alpha_2 \beta_2}(p_2 - p'_2) |p'_1 \lambda_1 f_1 \beta_1\rangle |p'_2 \lambda_2 f_2 \beta_2\rangle. \end{aligned}$$

A local gauge transformation in  $x$ -space implies a nonlocal gauge transformation in  $p$ -space.



Now the representation (10) of the proton with  $\psi_{P\Lambda}$  given by (12) would be gauge invariant if

$$\sum_{\alpha_i} \epsilon_{\alpha_1 \alpha_2 \alpha_3} \left( \prod_{i=1}^3 |p_i \lambda_i f_i \alpha_i\rangle \right)' = \sum_{\alpha_i} \epsilon_{\alpha_1 \alpha_2 \alpha_3} \prod_{i=1}^3 |p_i \lambda_i f_i \alpha_i\rangle. \quad (18)$$

(18) implies with (16)

$$\prod_{i=1}^3 U_{\alpha_i \beta_i}(p_i - q_i) = \prod_{i=1}^3 \delta_{\alpha_i \beta_i} \delta(p_i - q_i) \quad (19)$$

Using the well known parametrization of  $U(x)$ -matrices in terms of the Wigner  $D_{MN}^J(\varphi(x), \nu(x), \gamma(x))$ -matrices [13] one easily verifies that (19) cannot be satisfied. Consequently  $|P\Lambda\psi\rangle$  with the coefficient function (12) is gauge dependent.

The same arguments apply in case of the quark representation of other hadrons e.g.  $\pi$ ,  $K$  or  $\rho$  mesons. In this case e.g. for the  $K$ -meson, the function  $\psi_K(p_1, \dots, \alpha_2)$  is usually taken in the following factorized form [5]

$$\psi_K(p_1, \dots, \alpha_2) = \psi_K(p_1, \dots, f_2) \delta_{\alpha_1 \alpha_2} \quad (20)$$

Again the obtained representation for  $K$  meson is gauge dependent.

Because of the gauge dependence of the state-vectors it would be interesting to analyse the possible gauge dependences of the expectation values of a given physical observable in the given representations.

## 6 Gauge dependence of abelian and nonabelian QFT in comparison

We mention that the question why the problem of gauge dependence in nonabelian GFT does not show up for abelian GFT like QED. One of the reasons is that here a RS exists and also a gauge invariant effective charge  $\alpha_{inv}$ , which can serve as expansion parameters. An expansion for a physical quantity  $R$

$$R = \left( \frac{\alpha_{inv}}{\pi} \right) \sum_{k=0}^{\infty} r_k \left( \frac{\alpha_{inv}}{\pi} \right)^k \quad (21)$$

is term by term RS and gauge invariant, and thus also the truncated series. Consequently the numerical values of perturbative predictions for  $R^{(N)}$  are meaningful. The RS and gauge invariant effective rcc is given by

$$\alpha_{inv}(q^2) = \frac{\alpha_B}{1 + \Pi_B(q^2)} = \frac{\alpha^X}{1 + \Pi^X(q^2)} \quad (22)$$

where  $\alpha_B$  and the photon self-energy  $\Pi_B$  are bare quantities and  $\alpha^X$  and  $\Pi^X$  are quantities considered in a RS denoted by  $X[1]$ . In addition, the Thompson limit  $q^2 \rightarrow 0$  is

$$\alpha = \lim_{q^2 \rightarrow 0} \alpha_{inv}(q^2) = \frac{1}{137} \quad (23)$$

Since  $\alpha_{inv}$  depends on bare quantities only its numerical value for a given  $q^2$  is the same in all RS.  $\alpha_{inv}$  used as the expansion parameter gives a truncated series for  $R$  which is RS and gauge invariant.

In nonabelian GFT models this construction for a RS and a gauge invariant coupling constant  $\alpha_{inv}$  is not possible. In QCD there are four (general classes) RS invariant coupling constants based on the quark-gluon vertex, the triple gluon vertex, the quartic gluon vertex and the ghost-gluon vertex [2], but all are gauge dependent and hence is no perturbative expansion (6) which would be term by term RS and gauge invariant. Consequently all physical or other quantities  $R^{(N)}$  represented by truncated perturbation series are in general RS and/or gauge dependent.

## 7 Discussion

Non-abelian gauge (local) field theories as a description of physical systems should allow on one side mathematical methods to understand their structure and on the other side computational methods and results, which can be compared with experiments. The mathematical framework of such theories is geometrically convincing, the concept of gauge invariance is somehow understood and one has learned to solve the difficulty that the (quantized) theory has to be rendered at least by a choice of a renormalization scheme RS. The observable quantities should be gauge independent and if they will depend on the RS a physically preferred or at least a consistent RS should exist.

To get results one needs approximation methods. Perturbation theory uses the effective coupling constant as gauge independent expansion parameter. In the special case of an abelian gauge theory, like QED, there exists a gauge invariant effective coupling constant which can serve as such a parameter and a unique RS is at hand. For the non-abelian case this is not so. Different authors refer to different RS with different physical justifications, there is no generally accepted consistent RS which covers many and various types of experiments. Any RS invariant coupling constant of QCD is gauge dependent, one can only speculate that there is a preferred gauge for each RS, such that truncated perturbation expansion has "good convergent" properties.

Concerning the RS dependence we find considerable changes from one RS to the other; this is expected, but seldom demonstrated. Furthermore there is in general a gauge dependence if one calculates physical quantities with truncated perturbation expansions; this is certainly also trivial but it is in some sense surprising that it is really strong. The effective coupling constant is gauge dependent.

Alternative computational methods e.g. non-perturbative approaches are needed.

which give gauge independent results for observable quantities, even approximately. We mentioned one further possibility. It seems that a spontaneously broken non-abelian gauge theory which reduces the non-abelian gauge group to its abelian subgroup would be a suitable choice in connection with computability. Such a model based on spontaneously broken SU(6) gauge invariance was proposed for strong interactions and extended to strong and electro-weak interactions in [6, 7]. Here computational difficulties and the problems of RS and gauge dependence of perturbative predictions are similar in QED.

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## ON SYMMETRIES INSIDE COLOR SUPERALGEBRAS

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Parasupersymmetric quantum mechanics, initially proposed by Rubakov and Spiridonov [0], has recently been reformulated [0] in order to support standard [0] as well as spin-orbit coupling [0] procedures of super-symmetrization as it is the case in supersymmetric quantum mechanics [0]. These two possible superpositions of bosonic and parafermionic [0] degrees of freedom can be studied in the simplest 1- and 2-dimensional space contexts respectively for oscillatorlike systems and can be generalized to arbitrary n-dimensions [0].

For the general case we are dealing with two parasupercharges defined by

$$Q_{\mp} = \mp \frac{i}{2} [a_{\mp,j}(\xi_{\pm,j} + \eta_{\pm,j}) - a_{\pm,j}(\xi_{\pm,j} - \eta_{\pm,j})], \quad (1)$$

where summations on repeated indices are understood. Here  $j = 1, 2, \dots, n$  and the even bosonic variables  $a_{-,j} \equiv a_j(a_{+,j} \equiv a_j^{\dagger})$  coincide with the corresponding annihilation (creation) operators while the  $\xi_{\pm,j}$  and  $\eta_{\pm,j}$  are the odd parafermionic variables characterizing each of the abovementioned supersymmetrization procedures [0]. We then simply get the commutation relations

$$\begin{aligned} [Q_-, [Q_+, Q_-]] &= 2Q_-H, & [Q_+, [Q_-, Q_+]] &= 2Q_+H, \\ [H, Q_-] &= 0, & [H, Q_+] &= 0 \end{aligned} \quad (2)$$

and the idempotence relations

$$Q^3 = Q_+^3 = 0$$

leading to a new (Lie) structure that we call (for evident reasons in connection with Witten's work [0] in susersymmetric quantum mechanics) the parasuperalgebra  $\text{Psqm}$  (2). This new mathematical object is characterized as a  $\mathbb{Z}_2$ -graded structure which in terms of even ( $\mathbb{E}$ ) and odd ( $\mathbb{O}$ ) operators contains the following products

$$[\mathbb{E}, [\mathbb{E}, \mathbb{E}]], [\mathbb{E}, [\mathbb{E}, \mathbb{O}]], [\mathbb{E}, [\mathbb{O}, \mathbb{O}]], [\mathbb{O}, [\mathbb{O}, \mathbb{O}]]. \quad (3)$$

It can be shown, in particular, that the relations (2) are also equivalent to double commutation relations [0].

The general Lie parasuperalgebras appear at this stage as unknown Lie structures but which can be related, in some specific cases, to ternary algebras [0] and to color superalgebras [0, 0] including Lie superalgebras in particular. In fact, we propose to include the Green-Cusson [0] Ansätze inside our parastatistical characteristics in order to identify some of our parasuperalgebras with color superalgebras belonging to the so-called  $C(2, S)$ -family [0] by referring to two-dimensional grading vectors. Here we can identify Rittenberg-Wyler classes of generators with our even and odd ones, so that their  $\mathbb{Z}_2$ -graduation (for further details, see [0]). We thus recover color superalgebras as particular parasuperalgebras and the general study of the latter is now going on.

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## $N = 1$ SUPERGRAVITY AS A NON-LINEAR $\sigma$ -MODEL

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### Abstract

This talk is based on the papers by E.A. Ivanov and the author [1, 2] according to which the  $N = 1$  supergravity in superspace is found to be a non-linear  $\sigma$ -model. More precisely  $N = 1$  supergravity (resp. the minimal Einstein version of it) is consistently reformulated as a theory of simultaneous non-linear realizations of two complex finite-dimensional supergroups generating via their closure the whole infinite-dimensional supergroup  $G$  of  $N = 1$  supergravity. The only independent Goldstone superfield accompanying a spontaneous breaking of the infinite-dimensional supersymmetry  $G$  down to the rigid  $N = 1$  supersymmetry appears to be an axial vector superfield  $H^{\mu\dot{\mu}}(x, \theta, \bar{\theta})$  identified with the  $N = 1$  supergravity prepotential. All the other Goldstone superfields are expressed in terms of  $H^{\mu\dot{\mu}}$  and its derivatives by imposing appropriate covariant constraints on the corresponding Cartan superforms (the inverse Higgs effect). Possible implications of the proposed formulation of  $N = 1$  supergravity are discussed. In particular the intriguing analogy between  $N = 1$  supergravity and the (super)  $p$ -branes theories is pointed out.

## 1 Introduction

It is well known that Einstein's theory of gravity not only exhibits a beautiful structure but also predicts with a great success all known gravitation experimental results. Notwithstanding there are many attempts to reformulate it. They are stimulated by problems of gravity theory itself and, in particular, by a recent development in particle physics in which most of the theories of current interest are viewed as generalized  $\sigma$ -models. These models as all theories possessing spontaneously broken symmetry, can be universally constructed in terms of corresponding non-linear realizations. The main ideas of such approaches can be illustrated on gravitation theory itself and summarized as follows:

1. Gravitational interaction like all other interactions has a dynamical symmetry group  $G$ .
2. The group  $G$  is obtained from a symmetry group  $H$  yielding the physical conservation laws of the theory and forming a sub-group of  $G$ .
3. The corresponding Lagrangian is constructed by means of the dynamical group  $G$ .

In other words the non-linear realizations of  $G$  consist of identification of Goldstone fields with those parameters of  $G$  which are not connected with generators yielding the physical conservation laws, i. e., with the parameters of the factor space  $G/H$ . From geometrical structure of  $G/H$  the gravitational Lagrangians (as well as conserved currents) are explicitly constructed in terms of group invariants by methods of differential geometry.

In the Borisov-Ogievetsky theory of gravitation [3] the dynamical group  $G$  is the group  $DiffR^{3,1}$  and the symmetry group  $H$  generating the physical conservation laws of the theory - the Poincar group  $P$ . Then the gravity theory itself appears to be a non-linear realization of  $DiffR^{3,1}$  and the gravitational field nothing else than the Goldstone field identified with parameters of  $DiffR^{3,1}/P$ .

Non-linear realization technique was first developed for finite-dimensional Lie groups [4] and then for some infinite-dimensional ones [5]. In the case of non-linear realizations of infinite-dimensional group  $DiffR^{3,1}$  the situation is greatly simplified due to the Ogievetsky theorem [6] according to which the algebra of  $DiffR^{3,1}$  can be regarded as a closure of two finite-dimensional algebras: the 20-dimensional affine algebra  $A(4)$  and the 15-dimensional conformal algebra  $C$ . Thus non-linear realizations of  $DiffR^{3,1}$ ,  $DiffR^{3,1}/P$  can be constructed by taking simultaneous non-linear realizations of  $A(4)$  and  $C$ , namely  $A(4)/P$  and  $C/P$ . The non-linear realizations on  $A(4)/P$  give rise to a 10-component Goldstone field  $h_{\mu\nu}(x)$  and that on  $C/P$  yield the scalar Goldstone field  $\varphi(x)$  as well as its gradient  $\varphi_{,\mu}(x)$ . The requirement that both non-linear realizations should be realized simultaneously leads to the following identifications of the Goldstone fields:  $\varphi(x) = \frac{1}{4}h_{\mu\mu}(x)$ . Thus the resulting theory has only the 10-components gravitation field  $h_{\mu\nu}(x)$ . Since its Lagrangian gives rise to the same equations of motion as appear in the Einstein theory of gravitation, the resulting theory is equivalent to that of Einstein (for details see [3]).

In the paper we shall briefly discuss  $N = 1$  minimal Einstein supergravity in the same spirit [1, 2]. First, in Section 2, we shall recall its geometrical formulation [7] and, using the results of [1], a theorem according to which the infinite-dimensional gauge superalgebra  $A$  of  $N = 1$  supergravity gauge supergroup  $G$  can be regarded as a closure of two finite-dimensional superalgebras  $A_I$  and  $A_{II}$  corresponding to supergroups  $G_I$  and  $G_{II}$  respectively. Then in Section 3 and 4, non-linear realizations of  $G_I$  and  $G_{II}$  are described and redundant Goldstone fields are eliminated by the inverse Higgs effect. Thus singling out the invariants with respect to  $G_{II}$  which are simultaneously also covariants with respect to  $G_I$  we, in the end get one essential Goldstone superfield  $H^{\mu\dot{\mu}}(x, \theta, \bar{\theta})$ . In Section 5 we show that  $N = 1$  minimal Einstein supergravity described in detail in [7] is equivalent to the above mentioned theory of simultaneous non-linear realizations on  $G_I/P$  and  $G_{II}/P$  with gravitational axial superfield  $H^{\mu\dot{\mu}}(x, \theta, \bar{\theta})$  playing the role of the Goldstone superfield and with the Lagrangian being the simplest invariant with respect to  $G_I$  and  $G_{II}$ . Finally, in Section 6, we discuss analogies of the non-linear treatment of  $N = 1$  supergravity with the p-brane type theories and indicate some of the further developments.

## 2 Gauge Supergroup of $N = 1$ Minimal Einstein Supergravity and its Structure

The Ogievetsky-Sokatchev formulation [7] of  $N = 1$  minimal Einstein supergravity is based on the  $(4+2)$ -dimensional complex superspace

$$\begin{aligned} C^{4/2} &= C^{4/4}/C^{0/2} = \{(x_L^{\rho\dot{\rho}}, \theta_L^\mu)\} \\ &= \{(x_R^{\rho\dot{\rho}}, \theta_R^\mu)\} \end{aligned} \quad (1)$$

with  $\{(x_L^{\rho\dot{\rho}}, \theta_L^\mu)\}$  and  $\{(x_R^{\rho\dot{\rho}}, \theta_R^\mu)\} = \{(x_L^{\rho\dot{\rho}}, \theta_L^\mu)\}$  being its left and right-handed parametrizations. In superspace  $C^{4/2}$  the infinite-parameter complex gauge supergroup  $G$  acts which is infinitesimally defined by

$$\begin{aligned} \delta x_i^{\rho\dot{\rho}} &= \lambda^{\rho\dot{\rho}}(x_L, \theta_L), \\ \delta \theta_L^\mu &= \lambda^\mu(x_L, \theta_L). \end{aligned} \quad (2)$$

Here,  $\lambda^{\rho\dot{\rho}}$  and  $\lambda^\mu$  are arbitrary superfunction-parameters satisfying the condition

$$\frac{\partial \lambda^{\rho\dot{\rho}}}{\partial x_L^{\rho\dot{\rho}}} - \frac{\partial \lambda^\mu}{\partial \theta_L^\mu} = 0 \quad (3)$$

which expresses infinitesimally the preservation of the supervolume on  $C^{4/2}$  corresponding to the minimal version of the  $N = 1$  supergravity (for details see [7, 8, 9, 1]).

Using the results of our previous paper [1] the following theorem is true:

**Theorem:** The infinite-parameter gauge superalgebra corresponding to transformations (2) restricted by (3) can be obtained by taking the closure of two finite-dimensional subalgebras:

$$\begin{aligned} A_I = \left\{ Q_\mu = -i \frac{\partial}{\partial \theta_L^\mu} \equiv -i \partial_\mu; \quad P_{L\rho\dot{\rho}} = -i \frac{\partial}{\partial x_L^{\rho\dot{\rho}}}; \right. \\ \left. Q_{\rho\dot{\rho}}^\mu = \theta^\mu \partial_{L\rho\dot{\rho}}; \quad K_{\rho\dot{\rho}} = -i (\theta_L^\mu \theta_{L\mu}) \partial_{\rho\dot{\rho}} \right\} \end{aligned} \quad (4)$$

and

$$\begin{aligned} A_{II} = \left\{ Q_\mu; P_{L\rho\dot{\rho}}; Q_{\rho\dot{\rho}}^\mu; R_{\beta\dot{\beta}}^{\alpha\dot{\alpha}} = -i \left[ x^{\alpha\dot{\alpha}} \partial_{\beta\dot{\beta}} - \frac{1}{4} \delta_{\beta\dot{\beta}}^{\alpha\dot{\alpha}} (x\partial) \right]; \right. \\ \left. T_{(\mu\nu)} = \frac{1}{2} \theta_{(\mu} \theta_{\nu)}; I_\mu^{\rho\dot{\rho}} = x^{\rho\dot{\rho}} \partial_\mu; D = -i (x^{\rho\dot{\rho}} \partial_{\rho\dot{\rho}} + 2\theta^\mu \partial_\mu) \right\}. \end{aligned} \quad (5)$$

The structure relations of superalgebras  $A_I$  and  $A_{II}$  are specified in [2]. The meaning of  $A_I$  and  $A_{II}$  is rather simple.  $A_I$  is the superalgebra of all Grassmann vector



fields (i.e. each element of  $A_I$  can be obtained by gauging ordinary 1-translations in purely Grassmann directions) and  $A_{II}$  is the special linear superalgebra in  $\epsilon^{4/2}$ .

Remarks:

1. The Theorem can be proved by taking it to account the following facts: First, all the generators of superalgebras  $A_I$  and  $A_{II}$  belong to the five types of generators listed in [1] which yield transformations (2.2) satisfying (2.3). Second, all the lowest dimensional generators of the superalgebra  $A$  can be obtained from the generators of  $A_I$  and  $A_{II}$  by using their structure relations and relations of the type  $[A_I, A_{II}]$ . Finally, the higher dimensional generators of  $A$  can be derived step by step from the previous ones by successively commuting the latter with each other and using the induction technique.
2. In [1] the superalgebra  $A$  was obtained by taking the closure of two superalgebras that differ from  $A_I$  and  $A_{II}$ . The choice of  $A_I$  and  $A_{II}$  has the advantage for constructing non-linear realizations in the next sections since  $A_{II}$  (in contradistinction to the superalgebras used in [1] involves the generators  $R_{\alpha\beta}^{\gamma\delta}$  and  $I_{\mu}^{\rho\sigma}$  which will be shown to have as their associate Godstone superfields those including as the lowest components gauge fields of graviton and gravitino.
3. The superalgebra  $A_{II}$  contains the Lorentz generators  $M_{\alpha\beta}$  and  $M_{\dot{\alpha}\dot{\beta}}$  given by

$$M_{\alpha\beta} = R_{(\alpha\beta)} + T_{(\alpha\beta)}, \quad M_{\dot{\alpha}\dot{\beta}} = R_{(\dot{\alpha}\dot{\beta})}, \quad (6)$$

where

$$R_{(\alpha\beta)} = -ix_{L(\alpha\beta)L(\rho\sigma)} \cdot R_{(\rho\sigma)} = -ix_{L(\dot{\alpha}\dot{\beta})L(\rho\sigma)} \cdot R_{(\rho\sigma)},$$

$$R_{(\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta})} = \frac{1}{4} \left\{ R_{(\dot{\alpha}\dot{\beta})(\dot{\gamma}\dot{\delta})} + \epsilon_{\dot{\alpha}\dot{\beta}} R_{(\dot{\gamma}\dot{\delta})} + \epsilon_{\dot{\gamma}\dot{\delta}} R_{(\dot{\alpha}\dot{\beta})} \right\}.$$

These Lorentz generators form a semi-direct sum with superalgebra  $A_I$ . Thus, without losing generality, they can be added to  $A_I$ .

4. The generators (4), (5) are essentially complex and so the corresponding group elements will in general be defined on the complex parameter manifold. Factorization over one or another real subgroup (i.e. passing to a coset) will then amount to leaving only imaginary parts in the corresponding group parameters.

### 3 Non-Linear Realization of $G_I$

Let  $G_I$  denote the complex supergroup the superalgebra of which is  $A_I$  defined in (4). Each element  $G_I$  of supergroup  $G_I$  can be parametrized in the following way convenient for constructing the non-linear realization of  $G_I$

$$g_I = g_1 g_2 g_3, \quad (7)$$

where

$$\begin{aligned} g_1 &= \exp i(\theta^\alpha Q_\alpha + x_L^{\alpha\dot{\alpha}} P_{L\alpha\dot{\alpha}}), \\ g_2 &= \exp i(\psi_\mu^{\rho\dot{\rho}}), \\ g_3 &= \exp i(a^{\rho\dot{\rho}} K_{\rho\dot{\rho}}). \end{aligned} \quad (8)$$

Transformation properties of the group parameters  $\theta^\alpha$ ,  $x_L^{\alpha\dot{\alpha}}$ ,  $\psi_\mu^{\rho\dot{\rho}}$  and  $a^{\rho\dot{\rho}}$  follow from the group multiplication law

$$g_I^0 g_I = g_I', \quad g_I^0 \text{ is a fixed element of } G_I \quad (9)$$

Assuming  $g_I^0$  of the form

$$g_I^0 \approx I + i(\epsilon^\alpha Q_\alpha + \epsilon^{\alpha\dot{\alpha}} P_{L\alpha\dot{\alpha}} + \beta_\mu^{\nu\dot{\nu}} Q_{\nu\dot{\nu}}^\mu + \gamma^{\alpha\dot{\alpha}} K_{\alpha\dot{\alpha}}) \quad (10)$$

we obtain the infinitesimal transformation laws of supergroup parameters

$$\begin{aligned} \delta\theta^\alpha &= \epsilon^\alpha, \\ \delta x_L^{\rho\dot{\rho}} &= \epsilon^{\rho\dot{\rho}} + i\theta^\mu \beta_\mu^{\nu\dot{\nu}} + (\theta^\delta \theta_\delta) \gamma^{\rho\dot{\rho}}, \\ \delta\psi_\mu^{\rho\dot{\rho}} &= \beta_\mu^{\rho\dot{\rho}} - 2i\theta_\mu \gamma^{\rho\dot{\rho}}, \\ \delta a^{\rho\dot{\rho}} &= \gamma^{\rho\dot{\rho}}. \end{aligned} \quad (11)$$

Now, following the general routine [4, 5] we introduce left-invariant Cartan 1-forms  $\omega_I^\alpha$ ,  $\omega_I^{\dot{\alpha}\dot{\beta}}$ ,  $\omega_{I\mu}^{\rho\dot{\rho}}$  and  $k_I^{\dot{\alpha}\dot{\beta}}$  via

$$g_I^{-1} dg_I = i \left\{ \omega_I^\alpha Q_\alpha + \omega_I^{\dot{\alpha}\dot{\beta}} P_{L\alpha\dot{\alpha}} + \omega_{I\mu}^{\rho\dot{\rho}} S_{\nu\dot{\nu}}^\mu + k_I^{\dot{\alpha}\dot{\beta}} K_{\dot{\alpha}\dot{\beta}} \right\}. \quad (12)$$

By a direct computation we obtain

$$\omega_I^\alpha = d\theta^\alpha \quad (13)$$

$$\omega_{IL}^{\dot{\alpha}\dot{\beta}} = dx_L^{\dot{\alpha}\dot{\beta}} + i\psi_\mu^{\dot{\alpha}\dot{\beta}} d\theta^\mu \quad (14)$$

$$\omega_{I\mu}^{\rho\dot{\rho}} = d\psi_\mu^{\rho\dot{\rho}} + 2ia^{\rho\dot{\rho}} d\theta_\mu \quad (15)$$

$$k_I^{\dot{\alpha}\dot{\beta}} = da^{\dot{\alpha}\dot{\beta}} \quad (16)$$

We shall use these left-invariant Cartan 1-forms to eliminate some Goldstone superfields which are associated with the group parameters. More precisely, we shall identify the group parameters associated with  $P^{\mu\dot{\mu}} = P_L^{\mu\dot{\mu}} + P_R^{\mu\dot{\mu}} = P_L^{\mu\dot{\mu}} + \overline{P_L^{\mu\dot{\mu}}}$  with the bosonic coordinates  $x^{\mu\dot{\mu}}$  of the real superspace, while that for  $P_A^{\mu\dot{\mu}} = i(P_L^{\mu\dot{\mu}} - P_R^{\mu\dot{\mu}})$  with the Goldstone superfield  $H^{\mu\dot{\mu}}(x, \theta, \bar{\theta})$ . The group parameters  $\theta^\mu \equiv \theta_L^\mu$ ,  $\bar{\theta}^\mu \equiv \theta_R^\mu = (\theta_L^\mu)^\dagger$  are interpreted as grassmannian coordinates of the real superspace  $R^{4|4}$ . The complex (4+2)-dimensional superspace  $\mathbb{1}$  can be regarded as an (8+4) dimensional real superspace  $\{(x_L^{\dot{\alpha}\dot{\beta}}, x_R^{\rho\dot{\rho}}, \theta_L^\mu, \bar{\theta}_R^\mu)\}$ . In this superspace a real physical superspace

$\{(x^{\rho\dot{\rho}}, \theta^\mu, \bar{\theta}^{\dot{\mu}})\}$  is imbedded as a (4+4)-dimensional hypersurface determined by four equations  $x_L^{\rho\dot{\rho}} - x_R^{\rho\dot{\rho}} = 2iH^{\rho\dot{\rho}}(x, \theta, \bar{\theta})$  [7]. The remaining group parameters  $\zeta_\mu^{\rho\dot{\rho}}$  and  $a^{\rho\dot{\rho}}$  will be identified with the corresponding Goldstone superfields given on  $R^{4/4}$  which will be expressed in terms of  $H^{\rho\dot{\rho}}(x, \theta, \bar{\theta})$  by exploiting the inverse Higgs effect and thus eliminated from the theory.

Namely expanding the appropriate Cartan 1-forms (13)-(16) in the covariant differentials  $\nabla x^{\alpha\dot{\alpha}}$ ,  $d\theta^\mu$  and  $d\bar{\theta}^{\dot{\mu}}$  we can define covariant derivatives and spinor covariant derivatives of  $H^{\alpha\dot{\alpha}}$  and  $\zeta_\mu^{\alpha\dot{\alpha}}$  (for detail see [2]). Equating these covariant derivatives to zero we obtain

$$\begin{aligned} \zeta_\mu^{\alpha\dot{\alpha}} &= 2\nabla_\mu H^{\alpha\dot{\alpha}}, & \zeta_{\dot{\mu}}^{\alpha\dot{\alpha}} &= 2\nabla_{\dot{\mu}} H^{\alpha\dot{\alpha}} \\ \text{and} & & & \\ a^{\alpha\dot{\alpha}} &= -\frac{i}{2}(\nabla^\mu \nabla_\mu) H^{\alpha\dot{\alpha}}. \end{aligned} \quad (17)$$

Thus the non linear realization of  $G_I$  can be entirely formulated in terms of the superfield  $H^{\mu\dot{\mu}}(x, \theta, \bar{\theta})$  and the basic building blocks of the Ogievetsky-Sokatchev formulation of minimal  $N = 1$  supergravity ( $H^{\mu\dot{\mu}}(x, \theta, \bar{\theta})$ ,  $\nabla_\beta H^{\mu\dot{\mu}}$ ,  $\nabla_{\dot{\beta}} H^{\mu\dot{\mu}}$ ) naturally emerge already at the rigid supersymmetry level. The problem now is to select those of the  $G_I$ -covariants which are simultaneously covariant with respect to the supergroup  $G_{II}$ .

#### 4 Non-Linear Realization of $G_{II}$

Analogously to the previous case  $G_{II}$  denotes a complex supergroup the superalgebra of which is  $A_{II}$  defined in (5). Each element  $G_{II}$  of supergroup  $G_{II}$  can be parametrized as

$$g_{II} = \tilde{g}_{II} l, \quad (18)$$

where  $\tilde{g}_{II}$  denotes the element of the coset space  $G_{II}/L$  with  $L$  being the Lorentz group

$$L = \exp(i l^{\alpha\dot{\beta}} M_{\alpha\dot{\beta}}) \exp(i \bar{l}^{\dot{\alpha}\beta} M_{\dot{\alpha}\beta}), \quad \bar{l}^{\dot{\alpha}\beta} \equiv \overline{l^{\alpha\dot{\beta}}}. \quad (19)$$

Here  $M_{\alpha\dot{\beta}}$  and  $M_{\dot{\alpha}\beta}$  are defined in (6) and  $l$  is an element of the Lorentz group.

The element  $\tilde{g}_{II}$  of the coset space  $G_{II}/L$  can be parametrized in the following way

$$g_{II} = g_1 g_2 g_3 g_4 g_5 \quad (20)$$

where  $g_1$  and  $g_2$  are defined in (8) and

$$\begin{aligned} g_3 &= \exp(i \lambda_{\rho\dot{\rho}}^\mu I_\mu^{\rho\dot{\rho}}), \\ g_4 &= \exp(i \pi_{\alpha\dot{\alpha}}^{\beta\dot{\beta}} R_{\beta\dot{\beta}}^{\alpha\dot{\alpha}}), \\ g_5 &= \exp(i \varphi D), \end{aligned} \quad (21)$$

The Cartan 1-forms are defined once again according to the general rules [4, 5] by

$$g_{II}^{-1}dg_{II} = i \left\{ \omega_{II}^{\alpha} Q_{\alpha} + \omega_{LII}^{\alpha\dot{\alpha}} P_{L\alpha\dot{\alpha}} + \omega_{II\mu}^{\alpha\dot{\alpha}} Q_{\alpha\dot{\alpha}}^{\mu} + \Omega_{\rho\dot{\rho}}^{\mu} I_{\mu}^{\rho\dot{\rho}} + \omega_{R\alpha\dot{\alpha}}^{\beta\dot{\beta}} R_{\beta\dot{\beta}}^{\alpha\dot{\alpha}} + \omega_T^{\alpha\beta} T_{(\alpha,\beta)} + \omega_D D \right\}. \quad (22)$$

By comparing both sides of (22) we obtain

$$\begin{aligned} \omega_{II}^{\alpha} &= (d\theta^{\alpha} + i\lambda_{\rho\dot{\rho}}^{\alpha}\omega_{LI}^{\rho\dot{\rho}})\epsilon^{2\varphi}, \\ \omega_{LII}^{\alpha\dot{\alpha}} &= dx_L^{\rho\dot{\rho}} + i\psi_{\mu}^{\rho\dot{\rho}}d\theta^{\mu} B_{\rho\dot{\rho}}^{\alpha\dot{\alpha}}\epsilon^{\varphi} = \omega_{LI}^{\rho\dot{\rho}} B_{\rho\dot{\rho}}^{\alpha\dot{\alpha}}\epsilon^{\varphi}, \\ \omega_{II\mu}^{\alpha\dot{\alpha}} &= d\psi_{\mu}^{\lambda\dot{\lambda}} B_{\lambda\dot{\lambda}}^{\alpha\dot{\alpha}}\epsilon^{-\varphi}, \\ \Omega_{\rho\dot{\rho}}^{\mu} &= (d\lambda_{\beta\dot{\beta}}^{\mu} + \lambda_{\beta\dot{\beta}}^{\sigma}\lambda_{\gamma\dot{\gamma}}^{\mu}d\psi_{\sigma}^{\gamma\dot{\gamma}})(B^{-1})_{\rho\dot{\rho}}^{\beta\dot{\beta}}\epsilon^{\varphi}, \\ \omega_{R\alpha\dot{\alpha}}^{\beta\dot{\beta}} &= -\lambda_{\rho\dot{\rho}}^{\gamma}\psi_{\gamma}^{\rho\dot{\rho}}(B^{-1})_{\alpha\dot{\alpha}}^{\rho\dot{\rho}} B_{\nu\dot{\nu}}^{\beta\dot{\beta}} + (B^{-1})_{\alpha\dot{\alpha}}^{\tau\dot{\tau}} dB_{\tau\dot{\tau}}^{\beta\dot{\beta}}, \\ \omega_T^{\alpha\beta} &= \frac{1}{4}\lambda_{\gamma\dot{\gamma}}^{(\alpha}d\psi^{\beta)\gamma\dot{\gamma}}, \\ \omega_D &= d\varphi - \frac{1}{4}\lambda_{\rho\dot{\rho}}^{\mu}d\psi_{\mu}^{\rho\dot{\rho}}, \end{aligned} \quad (23)$$

where  $B$  is a function of the Goldstone superfields associated with  $R_{\beta\dot{\beta}}^{\alpha\dot{\alpha}}$  and defined by

$$g_s^{-1}R_{\beta\dot{\beta}}^{\alpha\dot{\alpha}}g_s = (B^{-1})_{\tau\dot{\tau}}^{\alpha\dot{\alpha}} B_{\beta\dot{\beta}}^{\rho\dot{\rho}} R_{\rho\dot{\rho}}^{\tau\dot{\tau}}. \quad (24)$$

All these 1-forms, except those associated with  $M_{\alpha\dot{\alpha}}$ ,  $M_{\beta\dot{\beta}}$  which are hidden in  $\omega_R$  and  $\omega_T$ , undergo the induced Lorentz transformation with respect to their spinor indices when  $G_{II}$  acts on  $\tilde{g}_{II}$  by left shifts

$$g_{II}^0\tilde{g}_{II} = g_{II}'L^{\text{ind}}, \quad (25)$$

where

$$L^{\text{ind}} \approx I + i\delta h^{\alpha\dot{\alpha}}(x, \theta, \bar{\theta})M_{\alpha\dot{\alpha}} + i\delta \bar{h}^{\dot{\alpha}\alpha}(x, \theta, \bar{\theta})M_{\dot{\alpha}\alpha}$$

and  $g_{II}^0$  is a fixed element of  $G_{II}$ .

Applying the general formula (25) the transformation properties of the coset parameters  $\theta^{\mu}$ ,  $x_L^{\rho\dot{\rho}}$ ,  $\psi_{\mu}^{\gamma\dot{\gamma}}$ ,  $\lambda_{\rho\dot{\rho}}^{\mu}$ ,  $h^{\rho\dot{\rho}}$ ,  $\bar{h}^{\dot{\rho}\rho}$  and  $B_{\tau\dot{\tau}}^{\sigma\dot{\sigma}}$  under  $G_{II}$  can be obtained (for detail see [2]).

Now we shall eliminate extra Goldstone superfields and single out the covariants of  $G_{II}$  which are simultaneously covariant w.r.t.  $G_I$ .

Looking at eqs. (23) and (24) we see that  $\omega_{LII}^{\alpha\dot{\alpha}}$  is covariant also w.r.t.  $G_I$  since  $R$  and  $D$  can be added to  $G_I$  as extra automorphism generators and thus  $G_I$  does not transform the superfields  $B$  and  $\varphi$  at all.

Next we shall decompose  $\omega_{LII}^{\alpha\dot{\alpha}}$  into the covariant differentials of  $x^{\alpha\dot{\alpha}}$  and  $H^{\alpha\dot{\alpha}}$  once again and then extract covariant derivatives of  $H^{\alpha\dot{\alpha}}$ . It turns out that conditions

for elimination of  $\psi_\mu^{\rho\dot{\rho}}$  in the non-linear realization of  $G_I$  (i.e.  $\mathcal{D}_\mu H^{\alpha\dot{\alpha}} = \mathcal{D}_\mu H^{\alpha\dot{\alpha}} = 0$ ) are simultaneously covariant w.r.t.  $G_{II}$  so that  $\psi_\mu^{\alpha\dot{\alpha}}$  given by (17) possesses correct transformation properties with respect to both,  $G_I$  and  $G_{II}$  (for the proof see [2]).

Eliminating  $\psi_\mu^{\alpha\dot{\alpha}}$ ,  $\psi_\mu^{\alpha\dot{\alpha}}$  by (17) the covariant differentials of  $X^{\alpha\dot{\alpha}}$  and  $H^{\alpha\dot{\alpha}}$  acquire the forms

$$\Delta x^{\rho\dot{\rho}} = \nabla x^{\alpha\dot{\alpha}}(b_{\alpha\dot{\alpha}}^{\rho\dot{\rho}} + \partial_{\alpha\dot{\alpha}} H^{\gamma\dot{\gamma}} c_{\gamma\dot{\gamma}}^{\rho\dot{\rho}}) \quad (26)$$

$$\Delta H^{\rho\dot{\rho}} = -\nabla x^{\gamma\dot{\gamma}}(c_{\gamma\dot{\gamma}}^{\rho\dot{\rho}} - \partial_{\gamma\dot{\gamma}} H^{\lambda\dot{\lambda}} b_{\lambda\dot{\lambda}}^{\rho\dot{\rho}}) \quad (27)$$

with

$$b_{\rho\dot{\rho}}^{\alpha\dot{\alpha}} \equiv \frac{1}{2}(B_{\rho\dot{\rho}}^{\alpha\dot{\alpha}} \epsilon^\tau + \bar{B}_{\rho\dot{\rho}}^{\alpha\dot{\alpha}} \epsilon^{\bar{\tau}})$$

$$c_{\rho\dot{\rho}}^{\alpha\dot{\alpha}} \equiv \frac{i}{2}(B_{\rho\dot{\rho}}^{\alpha\dot{\alpha}} \epsilon^\tau + \bar{B}_{\rho\dot{\rho}}^{\alpha\dot{\alpha}} \epsilon^{\bar{\tau}})$$

Their structure is completely specified by expressing the remaining Goldstone superfields  $\lambda$ ,  $\pi$  (or  $B$ ) and  $\varphi$  in terms of  $H^{\mu\dot{\mu}}$ .

We begin with  $B_{\tau\dot{\tau}}^{\sigma\dot{\sigma}}$ . By inspecting the structure of the Cartan forms 23 we conclude that  $B_{\tau\dot{\tau}}^{\sigma\dot{\sigma}}$  can be eliminated by imposing appropriate constraint on one of the spinor covariant derivatives of the Goldstone field  $\psi_\mu^{\alpha\dot{\alpha}}$ . These are defined as the coefficients in front of  $\omega_{II}^\sigma, \bar{\omega}_{II}^{\dot{\sigma}}$  in the  $G_{II}$ -covariant Cartan form  $\omega_{II\mu}^{\rho\dot{\rho}}$

$$\omega_{II\mu}^{\rho\dot{\rho}} = \Delta x^{\lambda\dot{\lambda}} \Delta_{\lambda\dot{\lambda}} \psi_\mu^{\rho\dot{\rho}} + \omega_{II}^\sigma \Delta_\sigma \psi_\mu^{\rho\dot{\rho}} - \bar{\omega}_{II}^{\dot{\sigma}} \bar{\Delta}_{\dot{\sigma}} \psi_\mu^{\rho\dot{\rho}} \quad (28)$$

where

$$\Delta_\sigma \psi_\mu^{\rho\dot{\rho}} = \nabla_\sigma \psi_\mu^{\lambda\dot{\lambda}} B_{\lambda\dot{\lambda}}^{\rho\dot{\rho}} \epsilon^{-2\tau}, \quad \bar{\Delta}_{\dot{\sigma}} \psi_\mu^{\rho\dot{\rho}} = \bar{\nabla}_{\dot{\sigma}} \psi_\mu^{\lambda\dot{\lambda}} B_{\lambda\dot{\lambda}}^{\rho\dot{\rho}} \epsilon^{-3\bar{\tau}}.$$

It turns out that only  $\bar{\Delta}_{\dot{\sigma}} \psi_\mu^{\rho\dot{\rho}}$  can be used for implementing the sought covariant constraint since it is covariant w.r.t.  $G_I$  and  $G_{II}$  (and hence w.r.t. the whole infinite-dimensional  $N = 1$  supergravity group). It is meaningless to equate  $\bar{\Delta}_{\dot{\sigma}} \psi_\mu^{\rho\dot{\rho}}$  to zero because it would contradict to the flat-superspace limit

$$\varphi = 0, \quad \lambda_{\rho\dot{\rho}}^\nu = 0, \quad B_{\lambda\dot{\lambda}}^{\rho\dot{\rho}} = \delta_\lambda^\rho \delta_{\dot{\lambda}}^{\dot{\rho}}, \quad H^{\mu\dot{\mu}} = \frac{1}{2} \theta^\mu \bar{\theta}^{\dot{\mu}}, \quad \psi_\mu^{\lambda\dot{\lambda}} = 2 \nabla_\mu H^{\lambda\dot{\lambda}} = \delta_\mu^\lambda \bar{\theta}^{\dot{\lambda}}.$$

Thus one should equate  $\bar{\Delta}_{\dot{\sigma}} \psi_\mu^{\rho\dot{\rho}}$  to a proper Lorentz-covariant constant matrix consistent with the flat limit namely

$$\bar{\Delta}_{\dot{\sigma}} \psi_\mu^{\lambda\dot{\lambda}} = -\delta_{\dot{\sigma}}^{\dot{\lambda}} \delta_\mu^\lambda. \quad (29)$$

From here by taking into account that  $\det B = 1$  we can express superfields  $B_{\beta\dot{\beta}}^{\gamma\dot{\gamma}}$  and  $\varphi, \bar{\varphi}$  in terms of  $H^{\mu\dot{\mu}}$ .

Let us explain now how to eliminate the Goldstone superfield  $\lambda_{\beta\dot{\beta}}^\mu$ . The corresponding constraints arise from the requirement that in the  $d\bar{\theta}$ -projections of the Cartan forms standing in front of the generators  $R$ ,  $D$  and  $T$  only the inhomogeneously transformed components associated with the Lorentz generators  $M_{\alpha\beta}$ ,  $M_{\dot{\alpha}\dot{\beta}}$  survive. These constraints are again manifestly  $G_{II}$ - and  $G_I$ -covariant and give  $\lambda_{\rho\dot{\rho}}^\mu$  in terms of  $H^{\mu\dot{\mu}}$  (for detail see [2]). Hence we are eventually left with a single Goldstone superfield  $H^{\mu\dot{\mu}}(x, \theta, \bar{\theta})$  which alone supplies non-linear realizations of  $G_I$  and  $G_{II}$ . This confirms from another point of view than in [7] its role as the fundamental geometric object of the minimal  $N = 1$  supergravity.

It remains to show how the minimal  $N = 1$  supergravity action reappears within the present framework.

## 5 The Invariant Action

After employing the inverse Higgs effect constraints, the remaining simultaneous  $G_I$ - and  $G_{II}$ -covariants are reduced to the covariant differentials of the  $N = 1$  superspace coordinates, the covariant differential of  $H^{\mu\dot{\mu}}(x, \theta, \bar{\theta})$  and to the covariant derivative of the Goldstone field  $\lambda_{\rho\dot{\rho}}^\mu$  (the projection of the Cartan form  $\Omega_{\rho\dot{\rho}}^\mu$  onto the covariant differential  $\Delta\bar{\theta}^{\dot{\mu}}$ ). Since the last covariant rise to a higher derivative invariant and the covariant differential  $\Delta H^{\rho\dot{\rho}}$  identically vanishes we are left with the covariant differentials  $\Delta x^{\mu\dot{\mu}}$ ,  $\Delta\theta^\mu$ ,  $\Delta\bar{\theta}^{\dot{\mu}}$ . An obvious simplest invariant is the supervolume of  $N = 1$  superspace  $(x^{\mu\dot{\mu}}, \theta^\mu, \bar{\theta}^{\dot{\mu}})$  constructed as an integral of the Berezinian of the corresponding vielbeins over  $d^4x d^2\theta d^2\bar{\theta}$ . As shown in [2] it coincides up to a renormalization factor with the minimal Einstein  $N = 1$  supergravity superspace action in the form given in [8].

## 6 Concluding Remarks

i) The presented non-linear realization approach allows an algorithmic construction of  $N = 1$  supergravity based on the universal method of Cartan forms augmented with the inverse Higgs phenomenon. The  $N = 1$  supergravity prepotential  $H^{\mu\dot{\mu}}(x, \theta, \bar{\theta})$  appears from the beginning as a Goldstone superfield describing the simultaneous spontaneous breaking of  $G_I$  and  $G_{II}$  supersymmetries. Many objects and relations introduced "by hand" or postulated in the Ogietyvsky-Sokatchev approach acquire a clear group-theoretical meaning. For instance objects  $F$  and  $\bar{F}$  playing the crucial role in [7, 8] turn out to be related to the Goldstone superfield associated with the spontaneously broken generator  $D_{II}$  form  $G_{II}$ . The relations (4.25) in [8] prove to be a particular case of the inverse Higgs effect.

ii) It is worth mentioning that the inverse Higgs effect constraints are purely algebraic, in contradistinction to the standard  $N = 1$  supergravity constraints which are reduced to certain differential equations (vanishing of some components of the

torsion), the prepotential being a solution of the latter. In the present formulation these latter constraints are secondary, they can be shown to be a consequence of the Maurer-Cartan structure equations for  $G_I$  and  $G_{II}$ .

iii) It is interesting to see how the complex geometry of  $N = 1$  supergravity [7] (the preservation of chirality) reappears in the manifests of the non-linear realization description. Before all, it manifests itself in that one deals with the complex supergroups  $G_I$  and  $G_{II}$  in a holomorphic parametrization. The  $C^{4/2}$ -coordinates  $x_L^{\mu\dot{\mu}}$ ,  $\theta^\mu$  naturally come out as the parameters of the relevant complex coset spaces. The constraints of the inverse Higgs effect in the present case can also be interpreted as a kind of the covariant chirality conditions stating the absence of the  $d\bar{\theta}$ -projections in the corresponding Cartan forms.

iv) Let us stress the defining role of the non-linear realization of the linear supergroup  $G_{II}$ . The structure of the basic building blocks of  $N = 1$  supergravity, the covariant differentials  $\Delta x^{\mu\dot{\mu}}$ ,  $\theta^\mu$ , is completely specified by this non-linear realization (together with the inverse Higgs effect). The role of  $G_I$  is in a sense subsidiary – it provides very simple criterias in which cases the  $G_{II}$ -covariant quantities and relations are covariant under the whole  $N = 1$  supergravity group.

v) The construction of  $N = 1$  supergravity as a non-linear realization of the complex supergroup  $G_{II}$  in the coset supermanifold  $G_{II}/L$ , with  $N = 1$  superspace  $(x^{\mu\dot{\mu}}, \theta^\mu, \bar{\theta}^{\dot{\mu}})$  as a real subspace and the  $N = 1$  supergravity action as a  $G_{II}$ -invariant supervolume of this subspace suggests an interesting analogy of  $N = 1$  supergravity with the (super) $p$ -branes (strings, membranes, ...) in the treatment of [10]. Actually, the minimal  $N = 1$  supergravity is recognized as a kind of "spinning" super  $p$ -brane of dimension  $(4/4)$  moving in the complex coset  $G_{II}/L$  as the target space. the Goldstone superfields eliminated by the inverse Higgs effect are direct analogs of the Goldstone fields which parametrize in ordinary  $p$ -branes the cosets of the relevant Lorentz groups and are expressed there in terms of the translation Goldstone fields by the same procedure [10]. This similarity raises some questions, in particular, whether  $N = 1$  supergravity can be reproduced as an effective "low-energy" limit of some higher-dimensional superfield supersymmetric theories, by analogy with condensation of (super) $p$ -branes in a field theory [11].

vi) Closely related to the latter remark is the problem of existence of theories with a "linearly realized"  $N = 1$  supergravity group. Such theories could be related to the non-linear realization formulation of  $N = 1$  supergravity much like linear sigma models with associated internal symmetries are related to the corresponding non-linear sigma models, via appearance of non-zero vacuum expectation values of some fields. Our construction gives a hint that these linear realizations should operate with linear representations of the supergroup  $G_{II}$ . An analogous problem for the Einstein gravitation theory has been settled in [3]. As suggested by Witten [12], the linear sigma model of this kind describes the phases with unbroken local symmetries in gauge theories and can be presumably understood as topological field theories.

vii) Finally we note that the non-linear realization treatment of the non-minimal  $N = 1$  supergravity theories can seemingly be constructed in an analogous way. However, it is a much more ambitious problem to find a general principle allowing us to construct higher  $N$  supergravities by the non-linear realization techniques. One might hope to obtain in this way the geometric prepotential formulations of supergravities with  $N \geq 3$  which are unknown at present.

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## Hilbert Superspace, Unitary Representation of Lie Supergroup and Supersymmetric Quantum Field Theory

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### 1. Introduction

In the supersymmetric quantum field theory, a unitary representation of a Lie supergroup or a Lie superalgebra plays an important role. If we are content with a representation of a Lie superalgebra, Hilbert superspaces are not necessary. But if we want to consider a representation of a Lie supergroup, especially the induced representation<sup>1</sup>, one of the most powerful tool to construct a unitary representation, the theory of Hilbert superspaces<sup>2</sup> is indispensable.

### 2. Algebra $\Lambda$ of supernumbers

Let  $\Lambda = \Lambda_0 \hat{\oplus} \Lambda_1 = \mathcal{C} \hat{\oplus} \mathcal{N}$  be a Grassmann algebra over  $\mathcal{C}$  with countable number of generators  $v_i$ , where  $\mathcal{N}$ : set of nilpotent elements (soul). Define the projection  $b: \Lambda \rightarrow \mathcal{C}$  (body map) and the subalgebra  $\Lambda^{(n)}$  of  $\Lambda$  generated by  $v_1, \dots, v_n$ . Then  $\Lambda^{(n)}$  is a Banach algebra with the norm

$$\|\lambda\| = \sum_M |c_M|, \text{ for } \lambda = \sum_M c_M v_M.$$

where  $v_M = v_{i_1} \cdots v_{i_m}$ ,  $c_M \in \mathcal{C}$ . Let  $I_n$  be the ideal of  $\Lambda^{(n)}$  generated by  $v_n$ , then we have  $\Lambda^{(n)} = \Lambda^{(n-1)} \hat{\oplus} I_n$  and every element of  $\Lambda$  is uniquely written as

$$\lambda = \sum_{n \geq 0} \lambda_n, \quad \lambda_n \in I_n, \quad (I_0 = \Lambda^{(0)} = \mathcal{C}) \quad (1)$$

Let  $\omega = \omega_n$  be an arbitrary increasing sequence of positive integers and define a norm  $p_\omega(\lambda)$  on  $\Lambda$  by

$$p_\omega(\lambda) = \sum_n \omega_n \|\lambda_n\|,$$

where  $\lambda$  is expressed as (1). Then  $p_\omega$  gives the inductive limit topology of  $\Lambda^{(n)}$ . Moreover, each norm satisfies

$$p_\omega(\lambda\mu) \leq p_\omega(\lambda)p_\omega(\mu)$$

for  $\lambda, \mu \in \Lambda$ , and  $\Lambda$  is a topological algebra. The algebra  $\Lambda$  has the following fundamental properties<sup>3</sup>.

- (1)  $\Lambda$  is a complete and nuclear space.
- (2) Any bounded set of  $\Lambda$  is contained in  $\Lambda^{(n)}$  for some  $n$ .
- (3) The soul of any element of  $\Lambda$  is nilpotent.

### 3. Hilbert superspace

#### A. Definitions

Assume that  $\Lambda$  has a continuous grade-preserving involution  $*$  satisfying  $p_\omega(\lambda^*) =$

$p_\omega(\lambda)$ .

Definition 1. A  $\mathbb{Z}_2$ -graded  $\Lambda$ -module  $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$  is called a *inner product superspace* if it has a  $\mathbb{Z}_2$ -graded inner product  $(\cdot, \cdot): \mathcal{H}_\alpha \times \mathcal{H}_\beta \rightarrow \Lambda_{\beta-\alpha}$  ( $\alpha, \beta \in \mathbb{Z}_2$ ) satisfying the following conditions: For  $f, g, h \in \mathcal{H}$  and  $\lambda \in \Lambda$ ,

- (1)  $(g, f) = (f, g)^*$  (symmetric).
- (2)  $(f, g + h) = (f, g) + (f, h)$  (biadditive).
- (3)  $(f, g\lambda) = (f, g)\lambda$  (sesquilinear).
- (4)  $b((f, f)) \geq 0$  (positive definite).

The locally convex topology on  $\mathcal{H}$  called the  $\sigma$ -topology is defined by the following system of open neighbourhoods of 0:

$$U(f, \omega, \epsilon) = \{h \in \mathcal{H} \mid p_\omega((h, f)) < \epsilon\},$$

where  $\epsilon > 0$  and  $f \in \mathcal{H}$ .

Definition 2. A  $\mathbb{Z}_2$ -graded subspace  $H$  of  $\mathcal{H}$  over  $\mathcal{C}$  is a *base Hilbert space* of  $\mathcal{H}$  if the following conditions are satisfied.

- (1)  $H$  is a Hilbert space with the inner product  $(\cdot, \cdot)$  of  $\mathcal{H}$ . This means that  $(f, g) \in \mathcal{C}$  for all  $f, g \in H$  and  $H$  is complete.
- (2) The norm topology of the Hilbert space  $H$  is stronger than the induced topology from the  $\sigma$ -topology of  $\mathcal{H}$ .
- (3)  $(H, h) = 0$  implies  $h = 0$  for  $h \in \mathcal{H}$ .

Definition 3. A inner product superspace  $\mathcal{H}$  is called a *Hilbert superspace* if it has a base Hilbert space  $H$ .

#### B. Structure

Let  $h \in \mathcal{H}$ , then  $(h, \cdot)$  is a continuous  $\mathcal{C}$ -linear mapping from  $H$  to  $\Lambda$ . Since  $(h, f) \mid f \in H, \|f\| \leq 1$  is a bounded set in  $\Lambda$ ,  $(h, f) \mid f \in H$  is contained in a finite dimensional subspace  $\Lambda^{(n)}$  of  $\Lambda$ . Then we have by the Riesz theorem  $(h, \cdot) = \sum_{i=1}^k \lambda_i (h_i, \cdot)$ ,

where  $\lambda_i \in \Lambda^{(n)} \subset \Lambda$ . Thus we have  $h = \sum_{i=1}^k h_i \lambda_i^*$ .

Theorem 4. Let  $\mathcal{H}$  be a Hilbert superspace with base Hilbert space  $H$ . Then the  $\mathbb{Z}_2$ -graded tensor product  $H \otimes \Lambda$  of  $H$  and  $\Lambda$  over  $\mathcal{C}$  is isomorphic to  $\mathcal{H}$ . Conversely, let  $H$  be a  $\mathbb{Z}_2$ -graded Hilbert space over  $\mathcal{C}$ , then  $H \otimes \Lambda$  is a Hilbert superspace with base Hilbert space  $H \otimes 1$ .

#### C. Topologies

The system  $P_\omega$  of the norms  $P_\omega$  defined by

$$P_\omega(h) = \sup p_\omega((f, h)) \mid f \in H, \|f\| \leq 1,$$

defines a topology called the  $\epsilon$ -topology, and the  $\pi$ -topology is defined by the system  $Q_\omega$  of the norms

$$Q_\omega(h) = \inf \sum_{i=1}^k \|h_i\| p_\omega(\lambda^i),$$

for  $h = \sum_{i=1}^k h_i \lambda^i \in \mathcal{H}$ .

Since  $\Lambda$  is a nuclear space  $\epsilon$ - and  $\pi$ -topologies coincide<sup>4</sup>. The inequality

$$p_\omega((g, h)) \leq Q_\omega(g)P_\omega(h)$$

follows from the inequalities

$$p_\omega((g, h)) \leq \sum_{i=1}^k p_\omega((g, \lambda_i, h)) \leq \sum_{i=1}^k \|g_i\| p_\omega(\lambda_i) P_\omega(h).$$

for  $g = \sum_{i=1}^k g_i \lambda_i \in \mathcal{H}$ , with  $g_i \in H$  and  $\lambda_i \in \Lambda$ .

#### D. Stability of the $\epsilon$ -topology

Theorem 5. The  $\epsilon$ -topology does not depend on the base Hilbert space.

Proof. Let  $H_i$  be a base Hilbert space with the norm  $\|\cdot\|_i$  and  $P_\omega^i$  a system of norms of  $\mathcal{H}$  corresponding to  $H_i$  for  $i = 1, 2$ . Since  $H_1$  and  $H_2$  are Fréchet spaces and  $(\cdot, \cdot) : H_1 \times H_2 \rightarrow \Lambda$  is separately continuous (from the definition of base Hilbert space)  $(\cdot, \cdot)$  is continuous, i.e., we have

$$p_\omega((f_1, f_2)) \leq C_\omega \|f_1\|_1 \|f_2\|_2$$

for  $f_i \in H_i$  ( $i = 1, 2$ ). Thus we have

$$P_\omega^1(f_2) \leq C_\omega \|f_2\|_2, \quad P_\omega^2(f_1) \leq C_\omega \|f_1\|_1$$

and

$$p_\omega((h, f)) \leq P_\omega^1(h) Q_\omega^1(f) \leq C_\omega P_\omega^1(h) P_\omega^1(f) \leq C_\omega'' P_\omega^1(h) \|f\|_2,$$

for  $h \in H$  and  $f \in H_2$ . Consequently, we have

$$P_\omega^2(h) \leq C_\omega' P_\omega^1(h), \quad P_\omega^1(h) \leq C_\omega'' P_\omega^2(h)$$

for  $h \in \mathcal{H}$ .

#### 4. $\Lambda$ -linear operators of Hilbert superspaces

Definition 6. Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be two Hilbert superspaces. Then a continuous mapping  $T$  from  $\mathcal{H}_1$  to  $\mathcal{H}_2$  is called a continuous  $\Lambda$ -linear operator if it satisfies the following conditions:

$$T(x + y) = Tx + Ty, \quad T(x\lambda) = (Tx)\lambda$$

for  $x, y \in \mathcal{H}_1$  and  $\lambda \in \Lambda$ . Moreover,  $T$  is called *grade-preserving* if  $T\mathcal{H}_\alpha \subset \mathcal{H}_\alpha$  for  $\alpha \in \mathbf{Z}_2$ .

Definition 7. A continuous  $\Lambda$ -linear operator  $U$  from  $\mathcal{H}_1$  to  $\mathcal{H}_2$  is called *unitary* if it satisfies  $(Ux, Uy)_2 = (x, y)_1$  for every  $x, y \in \mathcal{H}_1$  and the range of  $U$  is the whole space  $\mathcal{H}_2$ .

Theorem 8. Let  $H$  be a base Hilbert space of a Hilbert superspace  $\mathcal{H}$  and  $U$  be a grade-preserving continuous unitary operator of  $\mathcal{H}$  (from  $\mathcal{H}$  to  $\mathcal{H}$ ). Then  $U(H)$  is also a base Hilbert space. Conversely, if  $H_1$  and  $H_2$  are two base Hilbert spaces of a Hilbert superspace  $\mathcal{H}$  then there exists a grade-preserving continuous unitary operator  $U$  of  $\mathcal{H}$  such that  $U(H_1) = H_2$ .

#### 5. Orthogonal complements of Hilbert subsuperspaces

Definition 9. A  $\mathbf{Z}_2$ -graded submodule  $\mathcal{K}$  of  $\mathcal{H}$  is called a Hilbert subsuperspace of  $\mathcal{H}$ , if the following conditions are satisfied.

(i)  $\mathcal{K}$  is a Hilbert superspace with the inner product  $(\cdot, \cdot)$  of  $\mathcal{H}$ .

(ii) The topology of  $\mathcal{K}$  coincides with the induced topology from  $\mathcal{H}$ .

Theorem 10. Let  $\mathcal{H}$  be a Hilbert superspace and let  $\mathcal{K}$  be a Hilbert subsuperspace of  $\mathcal{H}$ . Then the orthogonal complement  $\mathcal{K}^\perp = \{h \in \mathcal{H} \mid (\mathcal{K}, h) = 0\}$  of  $\mathcal{K}$  is a Hilbert subsuperspace of  $\mathcal{H}$  and we have

$$\mathcal{H} = \mathcal{K} \dot{+} \mathcal{K}^\perp \text{ (orthogonal sum).}$$

#### 6. Unitary induced representation of Lie supergroup

Let  $G$  be a Lie supergroup with even generators  $X_i$  ( $1 \leq i \leq p$ ) and odd generators  $Y_j$  ( $1 \leq j \leq q$ ). Let  $G_0$  be the group generated by the elements of the form  $\exp(\sum_{i=1}^p x_i X_i)$  with  $x_i \in \Lambda_0$  and  $\Theta = \{g \in G \mid g = \exp(\sum_{j=1}^q \xi_j Y_j), \xi_j \in \Lambda_1\}$  then every element  $g \in G$  is uniquely expressed as  $g = g_0 \theta$  for  $g_0 \in G_0$  and  $\theta \in \Theta$ . Thus the homogeneous superspace  $X = G_0 \backslash G$  is homeomorphic to  $\Theta \cong (\Lambda_1)^q$ .

Let  $(G_0, L, \mathcal{H})$  be a unitary representation of  $G_0$ . For  $\exp \xi \in \Theta$  and  $g \in G$ , define functions  $\tilde{g}_0(g, \xi) \in G_0$  and  $\tilde{\xi}(g, \xi) \in \Theta$  by  $(\exp \xi)g = \tilde{g}_0(\exp \xi)$ . Define  $U^L$  by

$$g \rightarrow U^L(g)\phi(\xi) = L(\tilde{g}_0(g, \xi)) \left| \frac{\partial \tilde{\xi}^j}{\partial \xi^k} \right|^{1/2} \phi(\tilde{\xi}(g, \xi))$$

for  $\mathcal{H}$ -valued function  $\phi(\xi)$ . Then  $(G, U^L, \mathcal{H} \otimes \Lambda(\xi))$  is a unitary representation induced by  $(G_0, L, \mathcal{H})$  if we define the inner product by  $(\phi, \psi) = \int (\phi(\xi), \psi(\xi)) d\xi$  for  $\phi, \psi \in \mathcal{H} \otimes \Lambda(\xi)$ , where  $(\phi(\xi), \psi(\xi))$  is the inner product of  $\mathcal{H}$ .

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# Phonons as Boson particles\*

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## 1 Introduction

The role played by phonons in condensed matter physics is well known. Phonons mediate various interactions between quasi-particles and phonons interact with external fields, phonons carry an energy and a quasi-momentum [1,2,3].

From the literature one learns that phonons are collective or macroscopic observables, and that their dynamics is induced by the microdynamics of the system.

Here we report about work in which we were able to define phonons as bona fide Boson fields or Boson particles, and to study their dynamics.

## 2 Phonons

We consider a  $\nu$ -dimensional cubic lattice  $\mathbb{Z}^\nu$  with in each lattice point  $i \in \mathbb{Z}^\nu$  a matrix algebra  $\mathcal{A}_i = M_n$  of site observables.

Take any state  $\rho$  which is lattice translation invariant and time invariant. Consider the fluctuation of an observable  $A \in M_n$  in the state  $\rho$ :

$$F_{\Lambda,\rho}(A) = \frac{1}{\sqrt{|\Lambda|}} \sum_{i \in \Lambda} (A_i - \rho(A))$$

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If the state is enough clustering one shows that

$$\lim_{\Lambda \rightarrow \mathbb{Z}^{\nu}} F_{\Lambda, \rho}(A) \equiv F_{\rho}(A)$$

exists in the topology of convergence in the distributional sense even for operators [4,5].

Now we observe the remarkable fact that  $F_{\rho}$  yields a representation of canonical commutation relations, i.e.  $F_{\rho}(A)$  for all local observables  $A$  is a Boson field. It turns out that these macroscopic fluctuations coincide with the phonons in solid state physics.

### 3 Phonon dynamics

If one has a microscopic Hamiltonian dynamics  $A \rightarrow A_t, t \in \mathbb{R}$ , then, at least for short range interactions, the fluctuation  $F_{\rho}(A_t)$  of  $A_t$  also exists. In that case we are able to define a macroscopic dynamics  $\tilde{\alpha}_t$  by the formula:

$$\tilde{\alpha}_t F_{\rho}(A) = F_{\rho}(A_t)$$

The dynamics  $\tilde{\alpha}_t$  is then the phonon dynamics, and the study of phonon physics amounts to the study of this dynamics.

It turns out that the phonon spectrum (i.e. of  $\tilde{\alpha}_t$ ) is quite different in nature from the microscopic dynamics of the system. We worked out many models and observed that although the microscopic spectrum is absolutely continuous, the phonon spectrum of the system has many discrete points. This is observed in the Schwinger model [6], the Overhauser model [7], and a spin-density wave model [8].

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## IX. $C^*$ -Algebraic Methods



## The modular group as physical symmetry

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Since several years I am interested in explaining the appearance of symmetries in physics. It is my opinion that only the invariance under Euclidean transformations is a consequence of general principles. All other symmetries need an explanation. Even the Lorentz-boosts are broken in charged superselection sectors which means that this symmetry has to be derived if present in the vacuum sector.

In this representation we want to show that in quantum field theory the Lorentz-boosts do appear as symmetry in the vacuum sector and have a natural explanation. We show that the modular group for the algebra of the wedge domain can always be interpreted as the group of Lorentz-boosts, provided the cyclic and separating vector is the vacuum-vector. In order to obtain this result one must show that the modular group induces outer automorphisms of the translation group. To prove this three properties are essential. The locality condition, the spectrum condition for the translations, and the structure of the wedge domain, which is mapped by a semi-group of translations into itself. By the same method we show that the CPT-transformation can be identified with the modular conjugation. If the wedge is fixed then one is dealing with a two-dimensional problem. Therefore, in all of the investigations we shall only deal with field theories in two space-time dimensions.

The C.P.T.-theorem has been proved in relativistic quantum field theory by R.Jost [Jo] in the frame of Wightman-field theory. In this proof he revealed the connection of the C.P.T.-symmetry with the assumptions of positivity of the energy, Lorentz-invariance, and the standard locality assumptions. In this proof the existence of a vacuum state was essential. But up to now there is no proof of the C.P.T.-theorem in the theory of local observables in the sense of Araki, Haag, and Kastler.

A different aspect of the C.P.T.-operator has been realized in connection with the Tomita [To] Takesaki [Ta]-theory of modular Hilbert-algebras. Bisognano and Wichmann [BW1], [BW2] observed that the C.P.T.-operator and the Lorentz-boosts are related to the modular conjugation and to the modular group of the wedge-algebra, provided the theory of local observables is generated from a Wightman-field which is covariant under Lorentz-transformations. Here again one is dealing with the vacuum-sector.

### Notation:

The symbol  $\{\mathcal{M}, \mathcal{H}, \Omega, \Delta, J\}$  means an algebra  $\mathcal{M}$  acting on a Hilbert space  $\mathcal{H}$  such that  $\Omega$  is cyclic and separating for  $\mathcal{M}$ . The modular operator and modular conjugation given by this situation is denoted by  $\Delta$  and  $J$ .

In the following we are looking at endomorphisms of  $\mathcal{M}$  induced by unitary operators which leave  $\Omega$  fixed. We are first looking at one single endomorphism. Next we are

treating the case of a continuous semi-group with positive generator. Finally we apply the results to the wedge algebra in the theory of local observables.

**First Result:**

Let  $\{\mathcal{M}, \mathcal{H}, \Omega, \Delta, J\}$  be a von Neumann algebra and let  $U$  be unitary with  $U\Omega = \Omega$  and assume  $U$  induces an endomorphism of  $\mathcal{M}$  i.e.  $U\mathcal{M}U^* \subset \mathcal{M}$  then we have:

(1) With  $S = J\Delta^{\frac{1}{2}}$  the relation

$$US = SU.$$

(2) The operator-function

$$\Delta^{it}U\Delta^{-it}$$

has an analytic extension into the strip,

$$S(-\frac{1}{2}, 0) = \{z; -\frac{1}{2} < \Im z < 0\},$$

and it is continuous on the boundary, and it fulfills the estimate

$$\|\Delta^{iz}U\Delta^{-iz}\| \leq 1.$$

If  $U$  induces an automorphism then  $U$  commutes as well with  $\Delta$  as with  $J$  and there is no interplay between  $U$  and the modular group. Next we are turning to the case where we have a semi-group of unitaries with positive generator and which induce a semi-group of endomorphisms.

**Second Result:**

Let  $\{\mathcal{M}, \mathcal{H}, \Omega, \Delta, J\}$  be a von Neumann algebra and let  $U(a)$  be a one-parametric group of unitaries with  $U\Omega = \Omega$  and assume  $U(a)$  induce for  $a \geq 0$  endomorphisms of  $\mathcal{M}$  i.e.  $U(a)\mathcal{M}U^*(a) \subset \mathcal{M}$ ,  $a \geq 0$  then we have:

1. the expression

$$\Delta^{it}U(\epsilon^\xi)\Delta^{-it} =: V(t, \xi)$$

has an analytic extension into the tube  $T$  and is continuous on the closure, where  $T$  is defined as follows:

$$T = \{(z, \zeta); (\Im z, \Im \zeta) \in B\},$$

where  $B$  is the interior of the quadrangle, the corners of which are given by the points

$$\{(0, 0), (-\frac{1}{2}, -\pi), (-\frac{1}{2}, 0), (0, \pi)\}.$$

2. The operator  $V(z, \zeta)$  is bounded

$$\|V(z, \zeta)\| \leq 1, \quad \text{for } (z, \zeta) \in T.$$

3.  $V(t - \frac{i}{2}, \xi - it)$  has the value

$$V(t - \frac{i}{2}, \xi - it) = \Delta^{it}JU(-\epsilon^\xi)J\Delta^{-it}.$$

$U(a)$  has a positive generator, therefore it has an analytic continuation into the upper half-plane. Since it induces endomorphisms of  $\mathcal{M}$  for positive  $a$ , one maps the upper

half-plane by  $a = e^\xi$  onto the strip  $(0, \pi)$ . For real  $\xi$  we have an analytic continuation into  $t \in S(-\frac{1}{2}, 0)$  and for real  $t$  an analytic continuation into  $\xi \in S(0, \pi)$ . This gives analyticity in a tube with triangular base. Looking now at the set  $\Im m t = -\frac{1}{2}$  one finds that one can analytically continue in the variable  $\xi$  into the strip  $S(-\pi, 0)$ . Altogether this gives the second result.

In order to find further analytic continuations one has to make use of the cyclicity of  $\Omega$  for  $\mathcal{M}$  and for  $\mathcal{M}'$ . To this end one introduces two functions.

**Notation:**

Let  $\{\mathcal{M}, \mathcal{H}, \Omega, \Delta, J\}$  be as before and let  $A \in \mathcal{M}$  and  $B' \in \mathcal{M}'$  then we define:

$$F^+(t, \xi) = (\Omega, B' \Delta^{it} U(e^\xi) \Delta^{-it} A \Omega)$$

and

$$F^-(t, \xi) = (\Omega, A \Delta^{it} U(-e^\xi) \Delta^{-it} B' \Omega).$$

As one sees from the construction of the two functions they coincide for real values of  $t$  and of  $\xi$ . However the third statement of the second result implies much more analyticity.

**Third Result:**

*The two functions  $F^+(t, \xi)$  and  $F^-(t, \xi)$ , which are holomorphic in the tubes  $T$  and  $-T$  respectively, are different representations of one holomorphic function  $H(z, \zeta)$ , which is holomorphic in the domain*

$$\{\Re z, \Re \zeta \in \mathbf{R}^2\} \cap \{-\pi < 2\pi \Im m z - \Im m \zeta < \pi\}. \quad (*)$$

*In this domain  $H(z, \zeta)$  is bounded. One has*

$$|H(z, \zeta)| \leq \max\{\|A\Omega\| \|B'^*\Omega\|, \|A^*\Omega\| \|B'\Omega\|\}.$$

*This latter statement implies that  $H(z, \zeta)$  is constant along the direction given by the strip (\*) i.e.*

$$H(z, \zeta) = H(z + \frac{w}{2}, \zeta + w\pi) \quad \text{for } w \in \mathbf{C}.$$

Using now the fact that  $\mathcal{M}\Omega$  and  $\mathcal{M}'\Omega$  are both dense in  $\mathcal{H}$  we obtain:

**Main Result:**

*Let  $\{\mathcal{M}, \mathcal{H}, \Omega, \Delta, J\}$  be as before. Let  $U(a)$  be a one-parametric group of unitaries with  $U\Omega = \Omega$  and assume  $U(a)$  induces for  $a \geq 0$  endomorphisms of  $\mathcal{M}$  i.e.  $U(a)\mathcal{M}U^*(a) \subset \mathcal{M}$ ,  $a \geq 0$  then we have for  $(t, a) \in \mathbf{R}^2$*

(a)

$$\Delta^{it} U(a) \Delta^{-it} = U(e^{-2\pi t} a)$$

and (b)

$$JU(a)J = U(-a).$$

If we assume, instead, that  $U(a)$  induces endomorphisms for  $a \leq 0$  then (b) remains unchanged but (a) is replaced by

$$\Delta^{it} U(a) \Delta^{-it} = U(e^{2\pi t} a).$$

### Application: Two-dimensional Q.F.T.

Let  $\{\mathcal{M}, \mathcal{H}, \Omega, \Delta, J\}$  be as before. Now we assume that we have a continuous representation  $U(a), a \in \mathbf{R}^2$  of the vector group of  $\mathbf{R}^2$ . Furthermore, we assume

- (a)  $U(a)\Omega = \Omega$
- (b) The spectrum of  $U(a)$  is contained in the forward light cone  $V^+ := \{a \in \mathbf{R}^2; a_0 \geq |a_1|\}$
- (c) Let  $W$  be the set

$$W := \{a \in \mathbf{R}^2; a_1 \geq |a_0|\}$$

then we assume

$$U(a)\mathcal{M}U^*(a) \subset \mathcal{M} \quad \text{for } a \in W.$$

Now we introduce the lightcone coordinates

$$a^+ = \frac{a_0 + a_1}{2}, a^- = \frac{a_0 - a_1}{2} \quad \text{or} \quad a_0 = a^+ + a^-, a_1 = a^+ - a^-.$$

Since  $U(a)$  fulfills the spectrum condition it follows that  $U(a^+)$  and  $U(a^-)$  both have positive generators and, moreover, we have

$$U(a^+)\mathcal{M}U^*(a^+) \subset \mathcal{M} \text{ for } a^+ \geq 0 \quad \text{and} \quad U(a^-)\mathcal{M}U^*(a^-) \subset \mathcal{M} \text{ for } a^- \leq 0.$$

Therefore, we apply the main result and obtain:

#### Fourth Result:

With the assumptions described above and with

$$\Lambda(t) = \begin{pmatrix} \cosh 2\pi t & -\sinh 2\pi t \\ -\sinh 2\pi t & \cosh 2\pi t \end{pmatrix}$$

and

$$U(\Lambda(t)) = \Delta^t$$

then  $\{U(\Lambda), U(a)\}$  induce a representation of the two-dimensional Poincaré-group which fulfills the spectrum condition and which have  $\Omega$  as invariant vector. Furthermore, one has

$$JU(a)J = U(-a)$$

All these transformations act local on the net constructed from the algebra of the wedge and its commutant.

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## Modular Structure of Unbounded Operator Algebras and Wightman Quantum Field Theory

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Algebras of unbounded operators ( $Op^*$ -algebras) arise naturally in the Wightman formulation of axiomatic quantum field theory. Tomita-Takesaki theory plays an important role for a study of structures of von Neumann algebras and for a study of quantum field theory [1~3], and so it is desirable to extend the results of Tomita-Takesaki theory to  $Op^*$ -algebras. Such a study has been done in [4~7] on the physical situations and in [8~11] for the general theory. In this report we introduce standard systems and modular systems which are able to develop the Tomita-Takesaki theory in  $Op^*$ -algebras, and apply them to the Wightman quantum field theory.

### 1. Standard systems and modular systems for general $Op^*$ -algebras

Let  $(\mathcal{M}, \xi_0, \mathcal{A})$  be given, where

- (i)  $\mathcal{M}$  is an  $Op^*$ -algebra on a dense subspace  $\mathcal{D}$  in a Hilbert space  $\mathcal{H}$  [12~14];
- (ii)  $\xi_0 \in \mathcal{D}$  and  $\mathcal{M}\xi_0$  is dense in  $\mathcal{H}$ ;
- (iii)  $\mathcal{A}$  is a von Neumann algebra on  $\mathcal{H}$  such that  $\mathcal{A}'\xi_0$  is dense in  $\mathcal{H}$  and  $\mathcal{A}'$  is contained in the weak commutant  $\mathcal{M}'_w$  of  $\mathcal{M}$ .

In general, the weak commutant  $\mathcal{M}'_w$  of  $\mathcal{M}$  is not necessarily a von Neumann algebra, but by (iii) there exists the induced extension  $\iota_{\mathcal{A}'}(\mathcal{M})$  of  $\mathcal{M}$  by  $\mathcal{A}'$ , that is, it is the closure of an  $Op^*$ -algebra  $\mathcal{M}_1 \equiv \{X_1; X \in \mathcal{M}\}$  on the linear span of  $\overline{\mathcal{A}'\mathcal{D}}$  defined by  $X_1(C\xi) = CX\xi$  for  $X \in \mathcal{M}$ ,  $C \in \mathcal{A}'$  and  $\xi \in \mathcal{D}$ . [14,15]. Since  $\iota_{\mathcal{A}'}(X)$  is affiliated with  $\mathcal{A}$  for each  $X \in \mathcal{M}$  and  $\mathcal{M}\xi_0$  is dense in  $\mathcal{H}$ , it follows that  $\mathcal{A}\xi_0$  is dense in  $\mathcal{H}$ . Hence, both the map:  $X\xi_0 \rightarrow X^t\xi_0$ ,  $X \in \mathcal{M}$  and  $\mathcal{A}\xi_0 \rightarrow \mathcal{A}^t\xi_0$ ,  $\mathcal{A} \in \mathcal{A}$  are closable in  $\mathcal{H}$  and their closures are denoted by  $S_{\mathcal{M}\xi_0}$  and  $S_{\mathcal{A}\xi_0}$ , respectively. Let  $S_{\mathcal{M}\xi_0} = J_{\mathcal{M}\xi_0}\Delta_{\mathcal{M}\xi_0}^{1/2}$  and  $S_{\mathcal{A}\xi_0} = J_{\mathcal{A}\xi_0}\Delta_{\mathcal{A}\xi_0}^{1/2}$  be the polar decompositions of  $S_{\mathcal{M}\xi_0}$  and  $S_{\mathcal{A}\xi_0}$ , respectively. Then we see that  $S_{\mathcal{M}\xi_0} \subset S_{\mathcal{A}\xi_0}$ , and  $J_{\mathcal{A}\xi_0}\mathcal{A}J_{\mathcal{A}\xi_0} = \mathcal{A}'$  and  $\Delta_{\mathcal{A}\xi_0}^t \mathcal{A} \Delta_{\mathcal{A}\xi_0}^{-t} = \mathcal{A}$  for all  $t \in \mathbf{R}$  by the Tomita fundamental theorem [2]. But, we don't know how the unitary group  $\{\Delta_{\mathcal{A}\xi_0}^t\}_{t \in \mathbf{R}}$  acts on the  $Op^*$ -algebra  $\mathcal{M}$ , and so we define a system which has the best conditions.

**Definition 1.1.** A system  $(\mathcal{M}, \xi_0, \mathcal{A})$  is said to be standard if the above conditions (i), (ii), (iii) and the following condition (iv) hold:

$$(iv) \quad \Delta_{\mathcal{A}\xi_0}^t \mathcal{D} = \mathcal{D} \text{ and } \Delta_{\mathcal{A}\xi_0}^t \mathcal{M} \Delta_{\mathcal{A}\xi_0}^{-t} = \mathcal{M}, \quad \forall t \in \mathbf{R}.$$

We have the following results for standard systems.

**Theorem 1.2.** Suppose that  $\Gamma = (\mathcal{M}, \xi_0, \mathcal{A})$  is a standard system. Then,  $S_{\mathcal{M}\xi_0} = S_{\mathcal{A}\xi_0}$  and the vector state  $\omega_{\xi_0}$  on  $\mathcal{M}$  satisfies the KMS-condition with respect

to a one-parameter group  $\{\sigma_t^\Gamma\}_{t \in \mathbf{R}}$  of  $*$ -automorphisms of  $\mathcal{M}$  defined by  $\sigma_t^\Gamma(X) \equiv \Delta_{\mathcal{M}\xi_0}^t X \Delta_{\mathcal{M}\xi_0}^{-t} = \Delta_{\mathcal{M}\xi_0}^t X \Delta_{\mathcal{M}\xi_0}^{-t}$  for  $X \in \mathcal{M}$  and  $t \in \mathbf{R}$ .

Important examples of standard systems are appeared in the Wightman quantum field theory as seen in Section 3. The notion of standard systems is useful, but the relation (iv) in Definition 1.1 doesn't hold in general, and so by relaxing this requirement we define the notion of modular systems which is able to develop unbounded Tomita-Takesaki theory and is more applicable to examples.

**Definition 1.3.** A system  $\Gamma = (\mathcal{M}, \xi_0, \mathcal{A})$  is said to be modular if the above conditions (i),(ii),(iii) and the following condition (iv)' hold:

(iv)' There exists a subspace  $\mathcal{E}$  of  $\mathcal{D}(t_{\mathcal{A}'}(\mathcal{M}))$  such that  $\xi_0 \in \mathcal{D}$ ,  $t_{\mathcal{A}'}(\mathcal{M})\mathcal{E} \subset \mathcal{E}$  and  $\Delta_{\mathcal{M}\xi_0}^t \mathcal{E} \subset \mathcal{E}$  for all  $t \in \mathbf{R}$ .

Let  $\Gamma = (\mathcal{M}, \xi_0, \mathcal{A})$  be a modular system and  $\mathcal{F}$  the set of all subspaces  $\mathcal{E}$  of  $\mathcal{D}(t_{\mathcal{A}'}(\mathcal{M}))$  satisfying the conditions in (iv)'. We put  $\mathcal{D}_\Gamma = \bigcup_{\mathcal{E} \in \mathcal{F}} \mathcal{E}$ . Then  $\mathcal{D}_\Gamma$  is a subspace of  $\mathcal{D}(t_{\mathcal{A}'}(\mathcal{M}))$  containing  $\mathcal{M}\xi_0$  and  $\mathcal{A}'\mathcal{D}_\Gamma = \mathcal{D}_\Gamma$ , and so  $\mathcal{U}(\Gamma) \equiv \{Y \in \mathcal{L}^+(\mathcal{D}_\Gamma); \bar{X}\eta\mathcal{A}\}$  is a generalized von Neumann algebra on  $\mathcal{D}_\Gamma$  [9] and the  $Op^*$ -algebra  $\mathcal{L}(\Gamma)$  on  $\mathcal{D}_\Gamma$  generated by  $\{\Delta_{\mathcal{M}\xi_0}^t t_{\mathcal{A}'}(\mathcal{M}) \Delta_{\mathcal{M}\xi_0}^{-t} \mathcal{D}_\Gamma; t \in \mathbf{R}\}$  is an  $Op^*$ -subalgebra of  $\mathcal{U}(\Gamma)$ . We have the following result.

**Theorem 1.4.** Suppose that  $\Gamma = (\mathcal{M}, \xi_0, \mathcal{A})$  is a modular system. Then  $(\mathcal{L}(\Gamma), \xi_0, \mathcal{A})$  and  $(\mathcal{U}(\Gamma), \xi_0, \mathcal{A})$  are standard systems.

$\mathcal{L}(\Gamma)$  is said to be a left  $Op^*$ -algebra of the modular system  $\Gamma$  and  $\mathcal{U}(\Gamma)$  is said to be a left generalized von Neumann algebra of  $\Gamma$ . We remark that if  $\Gamma = (\mathcal{M}, \xi_0, \mathcal{A})$  is a standard system, then  $\mathcal{D}_\Gamma = \mathcal{D}(t_{\mathcal{A}'}(\mathcal{M}))$  and  $\mathcal{L}(\Gamma) = t_{\mathcal{A}'}(\mathcal{M})$ .

## 2. Standard systems in the Wightman quantum field theory

We give some examples of standard systems in the Wightman quantum field theory, and consider the connection between standard systems obtained from wedge regions in Minkowski space and the association of a local net of von Neumann algebras with a Wightman field. The almost results stated here are obtained by making use of the works of Bisognano and Wichmann [4,5] and Driessler, Summers and Wichmann [7]. Let  $\varphi$  be one scalar, hermitian Wightman field with a cyclic vacuum  $\Omega$ . It is regarded as a linear map of the Schwartz space  $\mathcal{S}(\mathbf{R}^4)$  into an  $Op^*$ -algebra  $\mathcal{L}^+(\mathcal{D}_1)$  such that  $\varphi[f]^\dagger = \varphi[f^*]$  for  $f \in \mathcal{S}(\mathbf{R}^4)$  adhering the standard assumptions [16,17]. For any subset  $R$  of Minkowski space  $M$  let  $\mathcal{P}_0(R)$  be the  $Op^*$ -algebra generated by  $\{\varphi[f]; f \in \mathcal{S}(\mathbf{R}^4), \text{supp } f \subset R\}$ , and  $\Lambda \rightarrow U(\Lambda)$  be a strongly continuous unitary group of the Poincaré group  $P$  on the Hilbert space  $\mathcal{H}$  obtained by the completion of  $\mathcal{D}_1$ . Bisognano and Wichmann [4] determined the modular group and the modular involution for the right wedge region  $W_R = \{x \in M; x^3 > |x^4|\}$  and the left wedge region  $W_L = \{x \in M; x^3 < -|x^4|\}$  as follows: Since  $\Omega$  is a cyclic and separating vector for  $\mathcal{P}_0(W_R)$  and  $\mathcal{P}_0(W_L)$ ,  $X\Omega \rightarrow X^\dagger\Omega$ ,  $X \in \mathcal{P}_0(W_R)$  (resp.  $\mathcal{P}_0(W_L)$ ) is closable and its closure is denoted

by  $\mathcal{S}_{\mathcal{P}_0(W_R)\Omega}$  (resp.  $\mathcal{S}_{\mathcal{P}_0(W_L)\Omega}$ ). Let  $\mathcal{S}_{\mathcal{P}_0(W_R)\Omega} = J_{\mathcal{P}_0(W_R)\Omega} \Delta_{\mathcal{P}_0(W_R)\Omega}^{1/2}$  be the polar decomposition. Then  $J_{\mathcal{P}_0(W_R)\Omega}$  equals the antiunitary involution  $J = U(\pi_3, 0)\Theta_0$ , where  $\pi_3$  denotes the rotation by angle  $\pi$  about the 3-axis and  $\Theta_0$  denotes the canonical TCP-operator, and  $\Delta_{\mathcal{P}_0(W_R)\Omega}^{1/2}$  equals a positive self-adjoint operator  $V(i\pi)$  obtained by analytic continuation of a one-parameter unitary group  $\{V(t)\}_{t \in \mathbf{R}}$  of velocity transformations in the 3-direction.  $\mathcal{S}_{\mathcal{P}_0(W_L)\Omega} = \mathcal{S}_{\mathcal{P}_0(W_R)\Omega}^* = JV(-i\pi)$ . Furthermore, the pair  $(J, \{V(t)\}_{t \in \mathbf{R}})$  satisfies the relations:  $J\mathcal{P}_0(W_R)J = \mathcal{P}_0(W_L)$ ,  $V(t)\mathcal{P}_0(W_R)V(t)^{-1} = \mathcal{P}_0(W_R)$  and  $V(t)\mathcal{P}_0(W_L)V(t)^{-1} = \mathcal{P}_0(W_L)$  for all  $t \in \mathbf{R}$ .

It is natural to consider when there exists a von Neumann algebra  $\mathcal{A}(W_R)$  such that  $(\mathcal{P}_0(W_R), \Omega, \mathcal{A}(W_R))$  is a standard system. We have the following result for this question.

**Theorem 2.1.**  $(\mathcal{P}_0(W_R), \Omega, \mathcal{A}(W_R))$  is a standard system if and only if there exists a von Neumann algebra  $\mathcal{A}(W_R)$  on  $\mathcal{H}$  such that  $\mathcal{A}(W_R)' \subset \mathcal{P}_0(W_R)'_{\omega}$  and  $\mathcal{A}(W_R) \subset \mathcal{P}_0(W_L)'_{\omega}$ .

We next consider the connection between the standardness of  $(\mathcal{P}_0(W_R), \Omega, \mathcal{A}(W_R))$  and the association of a local net of von Neumann algebras with a Wightmann field. A local net is an assignment  $R \rightarrow \mathcal{A}(R)$  of regions  $R$  of the Minkowski space  $M$  with von Neumann algebras  $\mathcal{A}(R)$  satisfying the conditions of isotony, i.e.  $\mathcal{A}(R_1) \subset \mathcal{A}(R_2)$  if  $R_1 \subset R_2$ , locality, i.e.  $[\mathcal{A}(R_1), \mathcal{A}(R_2)] = 0$  if  $R_1$  and  $R_2$  are spacelike separated, and covariance, i.e.  $U(\Lambda)\mathcal{A}(R)U(\Lambda)^{-1} = \mathcal{A}(\Lambda R)$  for all  $\Lambda \in P$  [1.3]. A Wightman field  $\varphi$  is associated to a local net  $\mathcal{A}$  of von Neumann algebras if each field operator  $\varphi[f]$  has an extension to a closed operator,  $\varphi[f]_e \subset \varphi[f^*]_e^*$ , that is affiliated with  $\mathcal{A}(R)$  if the support of the test function  $f$  is contained in the interior of  $R$ . We have the following results:

**Theorem 2.2.** Let  $\mathcal{W}$  be the set  $\{\Lambda W_R; \Lambda \in P\}$  and  $\mathcal{K}$  the set of all closed double cones with a non-empty interior. Then,  $\varphi$  is associated to some local net  $W \in \mathcal{W} \rightarrow \mathcal{A}(W)$  of von Neumann algebras iff there exists a standard system  $(\mathcal{P}_0(W_R), \Omega, \mathcal{A}(W_R))$  such that

- (a)  $U(\Lambda)\mathcal{A}(W_R)U(\Lambda)^{-1} = \mathcal{A}(W_R)$  for each  $\Lambda \in P$  s.t.  $\Lambda W_R = W_R$ ;
- (b)  $U(\Lambda)\mathcal{A}(W_R)U(\Lambda)^{-1} \subset \mathcal{A}(W_R)$  for each  $\Lambda \in P$  s.t.  $\Lambda W_R \subset W_R$ ;
- (c)  $U(\Lambda)\mathcal{A}(W_R)U(\Lambda)^{-1} \subset \mathcal{A}(W_R)'$  for each  $\Lambda \in P$  s.t.  $\Lambda W_R \subset W_L$ .

Furthermore,  $\varphi$  is associated to some local net  $K \in \mathcal{K} \rightarrow \mathcal{B}(K)$  of von Neumann algebras iff there exists a standard system  $(\mathcal{P}_0(W_R), \Omega, \mathcal{A}(W_R))$  satisfying the above conditions (a), (b), (c) and the condition

- (d):  $(\bigcup\{U(\Lambda)\mathcal{A}(W_R)U(\Lambda)^{-1}; \Lambda W_R \subset K^c\})'' \subset \mathcal{P}_0(K)'_{\omega}$  for each  $K \in \mathcal{K}$ , where  $K^c$  is the causal complement of  $K$ .

By Theorem 2.2 we have the following results under the additional assumptions.

**Corollary 2.3.** Suppose that  $\mathcal{P}_0(W_R)'_{\omega}$  is a von Neumann algebra. Then,  $(\mathcal{P}_0(W_R), \Omega, (\mathcal{P}_0(W_R)'_{\omega})')$  is standard, iff  $(\mathcal{P}_0(W_R)'_{\omega})' = \mathcal{P}(W_L)'_{\omega}$ , iff  $\varphi$  is associated to a local net of  $W \in \mathcal{W} \rightarrow (\mathcal{P}_0(W)'_{\omega})'$ .

**Corollary 2.4** Suppose that  $\mathcal{P}_0(K)'_{\mathcal{W}}$  is a von Neumann algebra for all  $K \in \mathcal{K}$ . Then the following statements are equivalent. (1)  $(\mathcal{P}_0(W_R), \Omega, (\mathcal{P}_0(W_R)'_{\mathcal{W}})')$  is a standard system. (2)  $\varphi$  is associated to a local net  $W \in \mathcal{W} \rightarrow (\mathcal{P}_0(W)'_{\mathcal{W}})$  of von Neumann algebras. (3)  $\varphi$  is associated to some local net  $K \in \mathcal{K} \rightarrow \mathcal{B}(K)$  of von Neumann algebras. (4)  $\varphi$  is associated to a local net  $K \in \mathcal{K} \rightarrow (\mathcal{P}_0(K)'_{\mathcal{W}})$  of von Neumann algebras. (5)  $\varphi$  is associated to a local net  $K \in \mathcal{K} \rightarrow \mathcal{P}_0(K)'_{\mathcal{W}}$  of von Neumann algebras.

Borchers and Yngvason have showed that the condition (5) in Corollary 2.4 is also equivalent to a certain positivity property of the Wightman distributions in ([6] Theorem 3.1).

**Corollary 2.5.** Suppose that  $\mathcal{P}_0(W_R)$  is essentially self-adjoint. Then the following statements are equivalent. (1)  $(\mathcal{P}_0(W_R), \Omega, (\mathcal{P}_0(W_R)'_{\mathcal{W}})')$  is standard. (2)  $\varphi$  is associated to a local net  $W \in \mathcal{W} \rightarrow (\mathcal{P}_0(W)'_{\mathcal{W}})$  of von Neumann algebras. (3)  $\varphi$  is associated to some local net  $K \in \mathcal{K} \rightarrow \mathcal{B}(K)$  of von Neumann algebras. (4)  $\varphi$  is associated to a local net  $K \in \mathcal{K} \rightarrow \mathcal{P}_0(K)'_{\mathcal{W}}$  of von Neumann algebras.

In this section we have investigated the standardness of systems  $(\mathcal{P}_0(W), \Omega, \mathcal{A}(W))$  for wedge-regions  $W$ . But, it is difficult to give examples of standard systems for domains except wedge-regions. By the Hislop and Longo result ([18] Theorem 2) we see that for a massless free field  $(\mathcal{P}_0(O), \Omega, (\mathcal{P}_0(O)'_{\mathcal{W}})')$  is a modular system: for each open double cone  $O$  in  $M$ .

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## C\*-algebraic mean-field systems and geometric quantization

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Quantum lattices are described by means of quasilocal algebras: Let  $\mathcal{R}$  be a lattice and  $\Lambda$  an arbitrary finite subset of points in the lattice, i.e.,  $\Lambda = \{1, \dots, k, \dots, m\}$ . At each point  $k$  there is an algebra  $\mathbb{M}_k^{(n)}$  of  $n \times n$ -matrices. The tensor product  $\mathcal{A}^\Lambda = \otimes_{k \in \Lambda} \mathbb{M}_k^{(n)}$  is called the local algebra related to  $\Lambda$ . A system of sets  $\Lambda$  which is directed by means of the inclusion relation allows the construction of the inductive limit [1]  $\mathcal{A} = \lim_{\Lambda \rightarrow \mathcal{R}} \mathcal{A}^\Lambda$ .  $\mathcal{A}$  is a quasilocal C\*-algebra [2], [3].

The set  $\mathcal{S}(\mathcal{A})$  of states is constructed in dual manner but we are not interested in the most general states. Instead we start from a state space  $\mathcal{S}^P$  of states which are invariant under permutations of lattice points and are suited for the following.  $\mathcal{S}^P$  is weakly closed and a simplex and so is its extremal boundary  $\partial_e \mathcal{S}^P$  which is given by states  $\varphi = \otimes_k \varphi^k$  where all  $\varphi^k$  are described by the same element  $\varphi \in \mathcal{S}^k \cong \mathcal{S}(\mathbb{M}^{(n)})$ . The central decomposition of each state  $\omega \in \mathcal{S}^P$  decomposes into states of  $\partial_e \mathcal{S}^P$ . Note that to each  $\omega$  there corresponds a central measure  $\mu_\omega$ .

The central decomposition of a state carries over to the GNS-representation where it induces a spatial decomposition which is described by a direct integral:  $(\Pi_\omega, \mathcal{H}_\omega, \Omega_\omega) = \int_E^{\oplus} (\Pi_\varphi, \mathcal{H}_\varphi, \Omega_\varphi) d\mu_\omega(\varphi)$  with  $E = \text{supp}(\mu_\omega) \subset \partial_e \mathcal{S}^P \cong \mathcal{S}(\mathbb{M}^{(n)})$ .

Let  $\mathcal{M}_\omega = \Pi_\omega(\mathcal{A})''$  (= weak closure of  $\Pi_\omega(\mathcal{A})$ ) be the von Neumann algebra which corresponds to the GNS-representation  $\Pi_\omega$ . The decomposition of  $\Pi_\omega$  implies a decomposition  $\mathcal{M}_\omega = \int_E^{\oplus} \mathcal{M}_\varphi d\mu_\omega(\varphi) = \mathcal{M}_{\varphi_0} \otimes L^\infty(E, \mu_\omega)$ . The algebra  $\mathcal{M}_\omega$  has in general a non-trivial center  $\mathcal{Z}_\omega = \mathcal{M}_\omega \cap \mathcal{M}'_\omega = \int \mathbb{C}_\varphi d\mu_\omega(\varphi)$ . This is in contrast to the original quasilocal algebra  $\mathcal{A}$ , which is simple.

Also the construction of mean-field operators depends on the state  $\omega$ . Let  $\{\sigma_i\}_1^m$  be a basis in  $\mathbb{M}^{(n)}$  and  $m^\Lambda(\sigma_i) = \frac{1}{|\Lambda|} \sum_{k \in \Lambda} \sigma_i^k = m_i^\Lambda$  ( $|\Lambda|$  is the size of  $\Lambda$ ) be the local mean of  $\sigma_i$ . The limit  $\lim_{\Lambda} \Pi_\omega(m^\Lambda(\sigma_i)) = m(\sigma_i)$  exists in the weak operator topology and belongs to the center  $\mathcal{Z}_\omega$  of  $\mathcal{M}_\omega$ . This abelian algebra  $\mathcal{Z}_\omega \cong L^\infty(E, d\mu_\omega)$  is very large but since  $E$  is compact it contains the algebra  $\mathcal{C}(E)$  of continuous functions on  $E$ .  $\mathcal{C}(E)$  is generated by the mean field operators.

With  $\mathcal{C} = \Pi_\omega(\mathcal{A}) \otimes \mathcal{C}(E)$  we have  $\Pi_\omega(\mathcal{A}) \subset \mathcal{C} \subset \mathcal{M}_\omega$ . The gradual construction of the algebra of operators  $\mathcal{M}_\omega$  depending on the state  $\omega$  in which the infinite system is, will also reflect itself in the dynamics of the system. The polynomial  $Q^\Lambda := Q(m_1^\Lambda, \dots, m_r^\Lambda)$  depends on local operators and determines the local Hamiltonian  $H_\Lambda^Q := |\Lambda|Q^\Lambda$  (which is an extensive quantity). The local dynamics of  $a \in \mathcal{A}^\Lambda$  is described by the one-parameter group  $\tau_t^{Q^\Lambda}(a) = \exp(itH_\Lambda^Q) a \exp(-itH_\Lambda^Q)$ . The thermodynamical limit can be taken only if a suitable state is chosen and the GNS representation is constructed, that is:  $\tau_t^Q(\Pi_\omega(a)) := \text{stop} - \lim_{\Lambda} \Pi_\omega(\tau_t^{Q^\Lambda}(a))$ . The mapping  $\tau_t^Q$  does not leave  $\Pi_\omega(\mathcal{A})$  invariant. But extension of  $\tau_t^Q$  to mean fields (in general a non-trivial construction) leads to a norm-continuous automorphism group contained in  $\text{Aut}(\mathcal{C})$ . The algebra  $\mathcal{C}(E)$  generated by the mean fields is then mapped into itself [4], [5], [6].

Finally if the state  $\omega$  is invariant under the group  $\tau_t^Q$  the GNS-construction implies a unitary one-parameter group  $U_t^Q$  on  $\mathcal{H}_\omega = \mathcal{H}_0 \otimes L^2(E, \mu_\omega)$ . If  $\delta^Q$  is the generator of  $\tau_t^Q$  we have the following result [7], [8].

**Theorem:** *There is a unique selfadjoint operator on the GNS Hilbert space.  $H_\omega^Q : D(H_\omega^Q) \rightarrow \mathcal{H}_\omega$  such that*

- 1)  $\tau_t^Q(A) = e^{itH_\omega^Q} A e^{-itH_\omega^Q}$ ;  $t \in \mathbb{R}$ ,  $A \in \mathcal{M}_\omega$ ;
- 2)  $\delta^Q(A) = i[H_\omega^Q, A]$  on  $D(H_\omega^Q)$   $A \in D(\delta^Q)$ ;
- 3)  $\exp(itH_\omega^Q)\Omega_\omega = \Omega_\omega \forall t \in \mathbb{R}$  i.e.  $H_\omega^Q\Omega_\omega = 0$ ;
- 4)  $D = \{A\Omega_\omega, A \in \Pi_\omega(\mathcal{A}) \otimes \mathcal{C}(E)\} \subset D(H_\omega^Q) \subset \mathcal{H}_\omega$  is a core of  $H_\omega^Q$ .
- 5) For  $\mathcal{H}_\omega \ni A\Omega_\omega$ ;  $A = a \cdot f$ ,  $a \in \Pi_\omega(\mathcal{A}^\lambda)$  and  $f \in \mathcal{C}^2(E)$  one has

$$H_\omega^Q(A\Omega_\omega) = \frac{1}{i}\delta^Q(A)\Omega_\omega = (i[H_\omega^Q|_\Lambda, a] \cdot f)\Omega_\omega + (a \cdot \frac{1}{i}\{Q, f\})\Omega_\omega.$$

where

$$H_\omega^Q|_\Lambda = \sum_{i=1}^r \frac{\partial Q}{\partial x_i} \Pi_\omega(m^\lambda(\sigma_i)).$$

The second term in this quantum mechanical generator of the time translations in the Heisenberg picture contains a Poisson bracket. This structure goes back to the (quantum) commutator in  $\mathbb{M}^{(n)}$  as follows: Choose an hermitian sub-basis  $\{\sigma_i\}_1^k$ ,  $k \leq n^2$ , which contains all relevant operators for  $H_\omega^Q$  and which constitutes in the form  $\{i\sigma_i\}_1^k$  the basis for a Lie algebra  $\mathcal{G}$ . The corresponding Lie group  $G \subset SU(n)$  acts via  $\star$ -automorphisms  $\alpha_g, g \in G$ , in  $\mathcal{C}$ . Choose  $\omega$  not only time invariant but also  $\alpha_g^*$  invariant (with  $\alpha_g^*$  invariant central support  $E$ ). Every  $\varphi \in E \subset \partial_r S^P$  constitutes an element  $x$  in the dual  $\mathcal{G}^*$  of the Lie algebra  $\mathcal{G}$  by the prescription  $x(\sigma_i) := \langle \varphi, m^\lambda(\sigma_i) \rangle$ , which is independent of  $\Lambda$ , and this shows  $E \subset \mathcal{G}^*$ . For  $f, g \in \mathcal{C}^\infty(E)$ , the differentials  $d_x f, d_x g$  are in  $(\tan_x E)^* = \mathcal{G}^{**} \cong \mathcal{G}$ . Thus one sets [5], [9]

$$\{f, g\}(x) := -[d_x f, d_x g](x), \quad x \in E \subset \mathcal{G}^*.$$

We have there the interesting situation that the quantum time generator contains a classical time generator  $\{Q, \cdot\}$ , which is by construction a self-adjoint operator in  $L^2(E, \mu_\omega)$ . Since according to the basic principles of quantum mechanics the generator  $H_\omega^Q$  is an energy observable, also its part with the Poisson bracket should have a physical meaning. It seems that in the field of macroscopic quantum phenomena [10], [11] generators of this type are crucial. On the other hand it contradicts completely the philosophy of the quasi local algebra [2], even in its extension to weak closures.

For a theoretical justification for extending the concept of a quantum observable to the Poisson bracket operators we want to employ the formalism of geometric quantization [9], [12]. For simplicity let us assume that  $E$  is just one orbit of the coadjoint representation of  $G$  in  $\mathcal{G}^*$ . Then the symplectic form  $\sigma$  belonging to the Poisson bracket is non-degenerate and  $(E, \sigma)$  is a symplectic manifold. If the cohomology class of  $\sigma$  in  $H^2(E, \mathbb{C})$  fulfills a certain integrality condition, then according to [9], [12] there exists a quantum manifold  $Y$  over  $E$ , that is a principal  $U(1)$  fiber bundle over  $E$  with contact structure. Let us write  $Y = U_{x \in E} T_x$ , where  $T_x \cong T \cong U(1)$  is the one-dimensional

torus for all  $x \in E$ . The contact structure on  $Y$  is constituted by a certain  $C^\infty$  one-form  $\bar{\sigma}$ , the exterior derivative of which gives  $\sigma$ .

In order to introduce quantum observables and quantum states one considers the group of quantomorphisms  $\text{Quant}(Y)$  which consists of all diffeomorphisms of  $Y$  which respect  $\bar{\sigma}$  and commute with the action of the torus on  $Y$ . A one parametric subgroup of  $\text{Quant}(Y)$  is by definition the action of  $\mathbb{R}$  on  $Y$  by quantomorphisms. One can show that it is completely determined by a function  $Q \in C^\infty(E)$ . Given such a  $Q$  we denote the corresponding finite time symplectomorphism by  $\varphi_t^Q$  (acting on  $\omega$ ) and the quantomorphism by  $\bar{\varphi}_t^Q$  (acting on  $Y$ ). The generator of the latter (a  $C^\infty$ -vector field on  $Y$ ) is denoted by  $Z_Q$ . One finds for their Lie bracket

$$[Z_Q, Z_{Q'}] = Z_{\{Q, Q'\}},$$

(where we have set  $\hbar = 1$  in the formalism of [9], [12]).

Now one can define a quantum state as a mapping  $F : \text{Quant}(Y) \rightarrow \mathbb{C}$  which satisfies

- (i)  $F(z) = z$ , where  $z \in T$  denotes here also its action on  $Y$ ;
- (ii)  $F$  is positive definite on  $\text{Quant}(Y)$ ;
- (iii)  $F$  is continuous.

Conditions (ii) and (iii) give rise to a unique probability measure  $\mu$  on  $\mathbb{R}$  by the Bochner formula

$$F(e^{tZ_Q}) = \int_{\mathbb{R}} e^{ist} d\mu(s).$$

Thus  $\mu$  gives the distribution of the  $Z_Q$ -values in the state  $F$ .

We claim now, that the Poisson bracket part of the generator of the limiting mean field dynamics is a quantum observable in the sense of Souriau. Since we have the limiting dynamics strictly speaking only for polynomials  $Q \in C^\infty(E)$ , we restrict the following to this case. (There are strong indications, that the mean field dynamics can be introduced for arbitrary  $Q \in C^\infty(E)$ , cf. [13] for certain analytic functions for  $Q$ .)

The transformed measure  $\mu_\omega^t := \mu_\omega \circ \varphi_t^Q$  is equivalent (observe:  $\mu_\omega$  is faithful) to  $\mu_\omega$  with Radon-Nikodym derivative  $d\mu_\omega^t/d\mu_\omega =: D_t$ . Now set

$$U_t^Q f \Omega_\omega := z_Q(f \circ \varphi_{-t}^Q) D_t^{-1/2} \Omega_\omega, \quad (*)$$

where  $f \in C^\infty(E)$ , and  $z_Q \in T$  is independent of  $x \in E$ . Equation (\*) gives rise to a continuous family of unitary operators on  $\overline{C^\infty(E)\Omega_\omega}^{\|\cdot\|} \cong L^2(E, \mu_\omega)$ , which has a Poisson bracket generator. The vector  $\Omega_\omega$  is used up to now as a state at most for the algebra  $\mathcal{M}_\omega$ , the largest we have mentioned. Considering  $U_t^Q$  as a GNS- (resp. Kolmogorov-) representation of  $\varphi_t^Q$  we have the Souriau state

$$F(\varphi_t^Q) := (\Omega_\omega | U_t^Q \Omega_\omega), \quad \forall t \in \mathbb{R}, \quad \forall Q \in C^\infty(E).$$

Thus, by this ansatz we have a prescription how to extend  $\omega$  to the von Neumann algebra created by  $\mathcal{M}_\omega$  and all  $U_t^Q$ . To this the Poisson bracket operators are affiliated.

If  $E$  is the union of several  $G$ -orbits, then one has to perform interesting reduction procedures to apply the above mentioned ideas. There may be some relationships to [14]. That in general a mean field dynamics in the previously described sense involves more coordinates than is physically meaningful is illustrated for the BCS-model in terms of the full CAR-algebra (instead of the pair algebra) in [15]. Such a reduction of variables seems to be also essential for the theory of macroscopic quantum phenomena [11].

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## Quasi Averages for Mean Field Models in Algebraic Quantum Statistical Theory

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### 1 Introduction

An important method in quantum statistical mechanics is the calculation of quasi averages for the identification of condensation phenomena. Let us report shortly the main ideas (see e.g. [1]): modify a given microscopic Hamiltonian  $H$  by a small (but finite) perturbation  $\varepsilon h$ . Then calculate the thermodynamical limit of the equilibrium states at temperature  $T$  and finally turn off the strength of the perturbation. The resulting state is called a *quasi average* (QA) in contrast to the *regular average* (RA), where the original Hamiltonian is not modified. In certain cases, namely if phase transitions appear, QAs and the RAs differ from one another. In terms of Bogoliubov, the state shows *degeneracy*, usually being connected with a symmetry of the model.

We will imitate this procedure in the rigorous frame of operator algebraic quantum statistical theory (see e.g. [2]). This clarifies many aspects of the above explained procedure: What is the meaning of the QA? What is the degeneracy? Which states can be calculated as QA? What is the nature of the microscopic perturbation?

### 2 Mathematical Frame

The  $C^*$ -algebra  $\mathcal{A} := \bigotimes_{i \in \mathbb{N}} \mathcal{B}$ , with  $\mathcal{B} = M_n(\mathbb{C})$ , is used as *observable algebra* of the quantum lattice system, with state space  $\mathcal{S}(\mathcal{A}) := \{\omega \in \mathcal{A}^* \mid \langle \omega : \mathbb{1} \rangle = 1, \omega \text{ positive}\}$ . Because only mean field models will be investigated, consider especially the set of *permutation invariant or homogeneous states*

$$\mathcal{S}^P(\mathcal{A}) := \{\omega \in \mathcal{S}(\mathcal{A}) \mid \omega = \omega \circ \Theta_\sigma, \forall \sigma \in P\}$$

with the finite permutations  $P := \bigcup_{n \in \mathbb{N}} S_n$  and  $\Theta_\sigma \bigotimes_{i \in \mathbb{N}} x_i := \bigotimes_{i \in \mathbb{N}} x_{\sigma(i)} \in \mathcal{A}$ .  $\mathcal{S}^P(\mathcal{A})$  is a *Bauer simplex* with extremal boundary  $\partial_e \mathcal{S}^P(\mathcal{A}) := \{\Pi_\varphi \mid \varphi \in \mathcal{S}(\mathcal{B})\}$  and  $\langle \Pi_\varphi : \bigotimes_{i \in \mathbb{N}} x_i \rangle := \prod_{i \in \mathbb{N}} \langle \varphi : x_i \rangle, \forall \bigotimes_{i \in \mathbb{N}} x_i \in \mathcal{A}$ . [3]. The *extremal decomposition* of  $\omega \in \mathcal{S}^P(\mathcal{A})$  coincides with the *central decomposition* into factorial states. The *central measure*  $\mu_\omega$  gives the classical probability for the occurrence of a *pure phase*  $\Pi_\varphi$ , if a physical system is prepared in the state  $\omega$ . The states  $\Pi_\varphi$ , lying in the support of the central measure  $\mu_{\omega_\beta}$  of a *limiting Gibbs state*  $\omega_\beta$  at inverse temperature  $\beta$ , are QAs in the above described context. The limiting Gibbs state is an

$w^*$ -accumulation point of the sequence  $(\omega_j^{H_n})_{n \in \mathbf{N}}$  with  $(H_n)_{n \in \mathbf{N}}$  being selfadjoint elements in  $\mathcal{A}_n := \otimes_{i=1}^n \mathcal{B} \subset \mathcal{A}$  for each  $n \in \mathbf{N}$ , [4].  $\omega_j^{H_n}$  is the usual Gibbs state at inverse Temperature  $\beta$ :  $\text{tr}(\epsilon^{-\beta H_n}) / \text{tr}(\epsilon^{-\beta H_n})$ , with  $\text{tr}$  as the usual trace on  $\mathcal{A}_n$  respectively the trace state on  $\mathcal{A}$ .

As model class we specify the family of all microscopic Hamiltonians  $(H_n)_{n \in \mathbf{N}} = (nh_n)_{n \in \mathbf{N}}$ , with  $h := (h_n)_{n \in \mathbf{N}}$  being a (selfadjoint) *approximately symmetric sequence*, abbreviated by  $\dot{\mathcal{Y}}$  respectively  $\dot{\mathcal{Y}}_{sa}$ , [5]. This can be considered as the largest class of mean field models on  $\mathcal{A}$ .

The reason is that these models possess the important property, that the mapping  $j : \dot{\mathcal{Y}} \rightarrow \mathcal{C}(\mathcal{S}(\mathcal{B}), \mathbf{C})$ ,  $h \rightarrow [j(h)](\varphi) := \lim_{n \rightarrow \infty} \langle \Pi_\varphi : h_n \rangle$ ,  $\forall \varphi \in \mathcal{S}(\mathcal{B})$  is well defined and surjective ( $\mathcal{C}(\mathcal{S}(\mathcal{B}), \mathbf{C})$  are the continuous functions on  $\mathcal{S}(\mathcal{B})$ ). By this, each element in  $\mathcal{C}(\mathcal{S}(\mathcal{B}), \mathbf{C})$  can be connected with a class of approximately symmetric sequences. Furtheron, there is a minimal principle of the free energy density for the limiting Gibbs states of the mean field model  $h$ : *Every limiting Gibbs state minimizes the functional  $f(\beta, h, \cdot) : \mathcal{S}^P(\mathcal{A}) \rightarrow \mathbf{R}$ , this means there is an element of the Bauer-Simplex  $\mathcal{S}(\beta, h) := \{\omega \in \mathcal{S}^P(\mathcal{A}) \mid f(\beta, h, \omega) = f(\beta, h)\}$  with  $f(\beta, h) := \inf\{f(\beta, h, \omega') \mid \omega' \in \mathcal{S}^P(\mathcal{A})\}$ . [4, 5].*

### 3 Symmetries and Limiting States

To discuss generalized QAs, we define *internal symmetries* on the set of approximately symmetric sequences. In the following,  $H$  is a closed (and therefore compact) subgroup of the unitary operators in  $\mathcal{B}$ .  $\mu_H$  is the unique Haar measure on  $\mathcal{C}(H, \mathbf{C})$ . There exists a canonical representation  $\Theta$  of  $H$  as automorphisms on  $\mathcal{A}$ :  $\Theta : u \rightarrow \Theta_u$  is defined by linear and continuous extension of

$$\Theta_u(\otimes_{i \in \mathbf{N}} x_i) := \otimes_{i \in \mathbf{N}} \text{Ad}_u x_i = \otimes_{i \in \mathbf{N}} u x_i u^* \quad \forall \otimes_{i \in \mathbf{N}} x_i \in \mathcal{A}, \forall u \in H.$$

**Definition 3.1** A function  $f \in \mathcal{C}(\mathcal{S}(\mathcal{B}), \mathbf{R})$  is called *invariant with respect to  $H$* , if

$$f \circ (\text{Ad}_u)^* = f \text{ for all } u \in H.$$

The following types of symmetries may occur in a general mean field model:

- (i)  $H$  is a global internal symmetry of the system  $h \in \dot{\mathcal{Y}}_{sa}$ , if  $j(h)$  is invariant with respect to  $H$ .
- (ii)  $H$  is a strict internal symmetry of the system  $h \in \dot{\mathcal{Y}}_{sa}$  if there exists an  $n_0 \in \mathbf{N}$ , such that for all  $n \geq n_0$  holds:  $\Theta_u h_n = h_n, \forall u \in H$ .

In terms of these definitions, we can formulate the main result:

**Theorem 3.2** Choose an approximately symmetric sequence  $h \in \dot{\mathcal{Y}}_{sa}$  with the global internal symmetry  $H$ . Then for all  $\Pi_\varphi \in \mathcal{S}(\beta, h)$ , there exists a  $h' \in \dot{\mathcal{Y}}_{sa}$  with  $H$  as a strict internal symmetry and  $\lim_{n \rightarrow \infty} \|h_n - h'_n\| = 0$  such that

$$\omega^{\beta h'} := \int_H \Theta_u^* \Pi_\varphi d\mu_H(u)$$

is the limiting Gibbs state of the system  $h'$ . The support of the central measure  $\mu_{\omega, \beta, h'}$  is concentrated on the orbit  $\mathcal{O}_H(\Pi_{\varphi, \omega}) := \{\Pi_{\text{Ad}_u^* \varphi, \omega} \mid u \in H\}$ .

PROOF:

We will give a sketch of the proof, for details see [5, 6]:

- (i) For every model  $h \in \dot{\mathcal{Y}}_{sa}$  with global internal symmetry, there exists a model  $\tilde{h}$  with this symmetry as a strict one and  $\lim_{n \rightarrow \infty} \|h_n - \tilde{h}_n\| = 0$ , [6]. Note that  $\mathcal{S}(\beta, h) = \mathcal{S}(\beta, \tilde{h})$ .
- (ii) Choose  $\Pi_{\varphi} \in \mathcal{S}(\beta, \tilde{h}) \cap \partial_e \mathcal{S}^P(\mathcal{A}) = \partial_e \mathcal{S}(\beta, \tilde{h})$ . Such a state exists, since  $\mathcal{S}(\beta, \tilde{h})$  is a Bauer-Simplex [4].  $H$  is an global internal symmetry and therefore it follows, that  $\int_H \Theta_u^* \Pi_{\varphi} d\mu_H(u) \in \mathcal{S}(\beta, \tilde{h})$  [4, Proposition 3.9].
- (iii) Finally take a strict symmetric perturbation that prepares the limiting Gibbs state  $\int_H \Theta_u^* \Pi_{\varphi} d\mu_H(u) \in \mathcal{S}(\beta, h)$ . For this use the methods elaborated by Raggio and Werner [5] and the constructions in [6]. The model  $h'$  is constructed with the help of the principle of minimal free energy density for the limiting Gibbs state and the separability of  $\mathcal{S}(\mathcal{A})$ .  $\square$

**Corollary 3.3** *Note, that Theorem 3.2 is valid for every group  $H$ , which is a global internal symmetry of the model  $h \in \dot{\mathcal{Y}}_{sa}$ , especially for every subgroup of the 'maximal global internal symmetry' group of  $h$ .*

## 4 Conclusions

With the presented results, it is possible to calculate every pure phase state in the central decomposition of a limiting Gibbs state (use Theorem 3.2, with  $H = \{1\}$ ). By the help of resymmetrization, more complicated limiting Gibbs states can be calculated, namely all states, which can be decomposed with the Haar-measure of an internal symmetry into pure phases, Corollary 3.3. This is relevant for models, with  $\mathcal{S}(\beta, h)$ , consisting of more than one orbit of pure phases with minimal free energy, where in general no explicit information on the limiting Gibbs state is available [7].

In contrast to the original way of calculating QAs, the two limiting procedures (thermodynamic limit and then the change to the zero value of the perturbation) are now combined to one limit and the microscopic perturbations differ from these ones, used by Bogoliubov. These are only technical questions. A detailed analysis of the proof in [6] shows, that the limiting procedure can be decomposed into the two original ones. Furtheron in special models all terms can be chosen explicitly as magnetic fields or particle sources, etc. . The main aspects: Correct size and action of the perturbation on the original system are fulfilled. In view of  $\lim_{n \rightarrow \infty} \|h_n - h'_n\| = 0$ , the perturbation  $h_n - h'_n$  can be considered as a purely microscopic one, which leaves the thermodynamical density functionals of internal energy and free energy density unchanged. An upper bound for the strength of a perturbation, which changes the Limiting state in a non continuous manner, is given.

The new definition of strict and global internal symmetries is an instrument to classify a given system with respect to symmetries appearing on different levels of the description. It is obvious that the influence from the microscopic model up to macroscopic features strongly depends on the strict symmetry aspects. There exist model discussions with both types of internal symmetries and the consequences on limiting Gibbs states, [8]. In this paper the influence of a bounded coupling interaction between two BCS-superconductors is analyzed. In the uncoupled case, the strict internal symmetry consists of the gauge transformations in the two superconductors. In the coupled case, this is only a global internal symmetry. As consequence the decomposition of the limiting Gibbs state becomes weighted over the phase differences between the two superconductors, that allows to proof the Josephson-relations for the coupled model.

Finally some interesting problems will be noted: Do elements  $h \in \tilde{\mathcal{Y}}_{sa}$  exist, with a global but not a strict internal symmetry  $H$  and a limiting Gibbs state  $\omega$  with  $\omega \circ \Theta_u = \omega, \forall u \in H$ ? This is connected with the search for elements  $h' \in \tilde{\mathcal{Y}}_{sa}$  with arbitrary  $\omega \in \mathcal{S}(\beta, h)$  as limiting Gibbs state and  $\lim_{n \rightarrow \infty} \|h_n - h'_n\| = 0$ .

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## GENERALIZED QUANTUM MECHANICS AND CLASSICAL OBSERVABLES

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States of a finite quantum system are considered also in a role of classical macroscopic field. Quantum mechanics (QM) is reformulated and generalized in terms of a Poisson system. It contains the nonlinear QM (NLQM) as well as the dynamics of mean-field theory (MFT), and also the dynamics on submanifolds of coherent states as well as classical mechanics (CM). The double role of traditional states in QM leads to possible interpretation of the newly introduced nonlinear quantum observables: the operator representation of a given observable can depend on classical macroscopic background of the microsystem. Two types of quantum mixed states are distinguished to describe correlations with macroscopic background. The mentioned models described in the same simple framework differ mutually only by subsets of observables used for their description.

QM can be reformulated *equivalently* [1, 2] in terms of (infinite dimensional) classical Hamiltonian mechanics on the phase space  $P(\mathcal{H})$  consisting of one-dimensional complex subspaces  $\mathbf{x}, \mathbf{y}, \dots$  of the complex Hilbert space  $\mathcal{H}$ . Linear operators  $X = X^*$  on  $\mathcal{H}$  then correspond to the functions  $h_X: \mathbf{x} \mapsto h_X(\mathbf{x}) := \text{Tr}(P_{\mathbf{x}}X) \equiv \langle x|X|x \rangle / \langle x|x \rangle$  on  $P(\mathcal{H})$ , where  $P_{\mathbf{x}} \equiv P_x$  ( $0 \neq x \in \mathbf{x}$ ) is the orthogonal projection onto  $\mathbf{x}$ . The Poisson bracket is defined by

$$\{h_X, h_Y\}(\mathbf{x}) = i \text{Tr}(P_{\mathbf{x}}[X, Y]) =: h_{i[X, Y]}(\mathbf{x}), \quad (1)$$

where  $[X, Y] := XY - YX$  is the commutator. The Schrödinger equation is then equivalent to Hamilton equations corresponding to (1): If  $H$  is the Hamiltonian operator of a QM system, then the evolution of the "observables"  $f := h_X$  is described by the Heisenberg-Hamilton (resp. von Neumann-Liouville) equations

$$\frac{d}{dt} f(\varphi_t^h \mathbf{x}) = \{h, f\}(\varphi_t^h \mathbf{x}), \quad \mathbf{x} \in P(\mathcal{H}), \quad t \in \mathbb{R} \quad (2)$$

where  $h \equiv h_H$ , and  $\varphi_t^h \equiv \varphi_t^H$  is the "Hamiltonian" (resp. "Poisson") flow on  $P(\mathcal{H})$  corresponding to the unitary evolution  $t \mapsto \exp(-itH)x$  of vectors  $x \in \mathcal{H}$ , i.e. a one-parameter group of transformations of  $P(\mathcal{H})$  conserving Poisson brackets which can be determined from (2). This immediate rewriting of QM differs from an "ordinary Hamiltonian CM" on  $P(\mathcal{H})$  by a specific restriction of the set  $\mathcal{F}(P(\mathcal{H}))$  of real-valued differentiable functions used as "observables" and "generators": QM uses only those  $f \in \mathcal{F}(P(\mathcal{H}))$  that have the form  $f \equiv h_X$  ( $X = X^*$ ). Let us call these  $h_X$  *affine functions* (or also "Kählerian functions", [1]): They can be considered as affine functions defined on all convex combinations  $\rho := \sum \lambda_j P_j \in \mathcal{S}_*$  ( $:=$  the set of all normal states on  $\mathcal{L}(\mathcal{H})$ ) of the pure states  $P_j \in P(\mathcal{H})$ . Other  $f$  will be called *nonlinear functions* on  $P(\mathcal{H})$ . The

"equation of motion" for general  $f, h \in \mathcal{F}(P(\mathcal{H}))$  has the form (2), where the Poisson bracket is the unique extension of (1) to generally nonlinear  $h, f \in \mathcal{F}(P(\mathcal{H}))$ :

$$\{h, f\}(\mathbf{x}) := i \operatorname{Tr}(P_{\mathbf{x}}[d_{\mathbf{x}}h, d_{\mathbf{x}}f]). \quad (3)$$

The differential  $d_{\mathbf{x}}f$  is defined here in the Fréchet sense, cf. [2]. The formal transition from QM to NLQM consists (in our transcription) in the addition to affine "generators" of QM of also some nonlinear ones. Inclusion of any additional (nonlinear) symmetry generator  $Q$  into this formulation of QM leads to a considerable extension of the theory. The flow  $\varphi^Q$  on  $P(\mathcal{H})$  does not conserve the "transition probabilities", i.e. there is  $t \in \mathbb{R}$  such that for  $\varphi := \varphi_t^Q$  one has

$$\operatorname{Tr}(P_{\mathbf{x}}P_{\mathbf{y}}) \neq \operatorname{Tr}(P_{\varphi\mathbf{x}}P_{\varphi\mathbf{y}}). \quad (4)$$

This is a consequence of the Wigner theorem: the conservation by  $\varphi^Q$  of  $|\langle x|y \rangle|^2$  for all  $x, y \in \mathcal{H}$  means that  $Q$  can be chosen as an affine function. The inequality (4) implies that:

- (i) Affine functions  $f$  are generally transformed into nonlinear functions  $\varphi^*f := f \circ \varphi$ ; hence, the introduction of nonlinear generators requires also introduction of nonlinear observables into the theory.
- (ii) Transformation (under  $\varphi$ ) of density matrices  $\rho := \sum_j \lambda_j P_{\mathbf{x}_j}$ , depends on their (different possible) decompositions into extremal elements  $P_{\mathbf{x}_j}$ . We conclude from this that one has to distinguish between probability measures  $\mu$  on  $\Gamma(\mathcal{H})$  (representing density matrices in the traditional QM), and the states described by density matrices themselves: The former are called here *genuine mixtures*, and the later are *elementary mixtures*; the genuine mixtures describe states of physical systems corresponding to classical probability distributions of the "elementary quantum states" ( $\equiv$  density matrices), and can be interpreted as describing correlations of the quantal system with a "macroscopic background". [3, 2].
- (iii) Distinction of the density matrices from their convex decompositions leads to necessity of definition of evolutions ( $\equiv$  groups of symplectic transformations) of density matrices which is independent of the evolution of elements  $\mathbf{x}_j \in P(\mathcal{H})$ . The corresponding Poisson bracket on the set of real-valued functions  $f, h, \dots$  defined on the set  $\mathcal{S}_*$  of all density matrices  $\rho$  is

$$\{h, f\}(\rho) := i \operatorname{Tr}(\rho[d_{\rho}h, d_{\rho}f]), \quad (5)$$

where  $d_{\rho}f \in \mathcal{L}(\mathcal{H})$  for "sufficiently nice"  $f$ .

Technical problems connected with manifold structure of  $\mathcal{S}_*$  and a definition of Poisson flows on  $\mathcal{S}_*$  corresponding to (also unbounded and not everywhere defined) nonlinear functions on  $\mathcal{S}_*$  are partly solved in [2]. Restrictions (or "projections") of these flows to submanifolds of coherent states determined by group representations  $U(G)$  lead to standard approximation schemes in QM, [4, 2], e.g. to the time-dependent Hartree-Fock approximation.

We propose an interpretation scheme of nonlinear functions  $f: \mathcal{S}_* \rightarrow \mathbb{R}$  occurring in the role of "observables", which is inspired by the specific formulation [3] of the quantum MFT. Interpretation of  $f$  is not specified uniquely by  $f$  itself:  $f$  can be written in a form  $f(\nu) \equiv \operatorname{Tr}(\nu f(\mathbf{F}(\nu)))$ , where  $\mathbf{f}$  is a selfadjoint operator-valued function on

the dual  $\text{Lie}(G)^*$  of the Lie algebra of a Lie group  $G$ , and the affine mapping  $\mathbf{F}: \mathcal{S}_*(\ni \nu) \rightarrow \text{Lie}(G)^*$  is determined by a continuous unitary representation  $U(G)$  of  $G$  in  $\mathcal{H}$ . Hence, the observable is specified either by  $\mathfrak{f}$ , or by the function  $\hat{f}: \mathcal{S}_* \times \mathcal{D}(\mathbf{F}) \rightarrow \mathbb{R}$ ,  $(\rho; \nu) \mapsto \hat{f}(\rho; \nu) := \text{Tr}(\rho \mathfrak{f}(\mathbf{F}(\nu)))$ , where  $\mathcal{D}(\mathbf{F}) \subset \mathcal{S}_*$  is a dense domain. The first variable  $\rho$  is called the *quantum variable*, and the second one  $\nu$  is the *classical variable*. The classical variable describes a state of an (possibly fictitious) infinite ensemble of equal quantal systems (with the corresponding value  $\mathbf{F}(\nu)$  of the classical selfconsistent "mean-field" generated by them), whereas the quantum variable corresponds to the actual value  $\rho$  of microscopic state, which can be for a single system different from  $\nu$ , [3]. Hence the state of a microsystem is described not only by a genuine mixture  $\mu$ , but also by a function  $\hat{\rho}: \mathcal{S}_* \rightarrow \mathcal{S}_*$  describing what specific quantum state  $\hat{\rho}(\nu)$  is "occupied" by the considered microsystem, provided the "classical state" (representing the infinite ensemble) is  $\nu \in \mathcal{S}_*$ . A simultaneous restriction of the theory to an orbit of  $U(G)$ , and to a set of scalar-valued observables  $\mathfrak{f}$  gives CM on a homogeneous phase space of  $G$ .

The expectation of  $\hat{f}$  in the state  $\omega_{\mu, \rho}$  described by a genuine mixture  $\mu$ , and the "quantum state deviation"  $\hat{\rho}$  is expressed by

$$\omega_{\mu, \rho}(\hat{f}) \equiv \omega_{\mu, \rho}(\mathfrak{f}) := \int \hat{f}(\hat{\rho}(\nu), \nu) \mu(d\nu). \quad (6)$$

Time evolution, and other continuous groups of transformations of the observables  $\hat{f}$  (resp. of the states  $\omega_{\mu, \rho}$ ) can be canonically defined, [2].

The described scheme leads naturally in many cases, by a use of the group representations  $U(G)$ , to reduction of solution of nonlinear Schrödinger equations [5] to solutions of classical finite dimensional ordinary differential equations, [2].

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## Gauge Covariant Squeezing of the Vacuum

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In the Coulomb gauge the canonical field variables for the electro-magnetic field are the vector potential  $A(x)$  and the transverse electric field  $\Pi(x) := -\epsilon_0 E^T(x)$ . They are smeared by means of complex test functions  $f = f^1 + i f^2 \in \mathcal{E} \subset L^2(\Lambda) \otimes \mathbb{C}^3$ , where  $\mathcal{E}$  is a pre-Hilbert space of complex 3-component functions localized in the cavity  $\Lambda \subset \mathbb{R}^3$ . The field operator is formally

$$\Phi(f) = \int_{\Lambda} \sum_{i=1}^3 [A_i(x) f_i^1(x) + \Pi_i(x) f_i^2(x)] d^3x,$$

which leads to the Weyl operator  $W(f) = \exp[i\Phi(f)]$ . The rigorous theory starts with the canonical commutation relations in Weyl form:

$$W(f)^* = W(-f), \quad W(f)W(g) = \exp\left[-\frac{i}{2} \operatorname{Im}(f|g)\right] W(f+g), \quad f, g \in \mathcal{E}. \quad (1)$$

The field algebra is the smallest (abstract)  $C^*$ -algebra [1], which contains all Weyl operators, and is denoted by  $\mathcal{W}(\mathcal{E})$  [2]. As  $C^*$ -algebra  $\mathcal{W}(\mathcal{E})$  is uniquely given by (1) and  $\mathcal{E}$ . It is simple, has trivial center, is nuclear, and is anti-liminary [3]. From the last property it follows that  $\mathcal{W}(\mathcal{E})$  has — beside the Fock representation, given by the GNS-triple  $(\Pi_F, \mathcal{H}_F, \Omega_F)$  over the bare vacuum state  $\omega_F$ , which is in the Fock space represented by the cyclic vector  $\Omega_F$  — over-countably many (quasi-) inequivalent (irreducible) representations.

The states  $\varphi$  (also the non-regular ones) are uniquely given by their characteristic function

$$C_{\varphi}(f) := \langle \varphi; W(f) \rangle, \quad f \in \mathcal{E},$$

where the r.h.s. denotes the expectation of  $W(f)$  in the state  $\varphi$ . Especially for the Fock vacuum state  $\omega_F$  holds  $C_F(f) := \langle \omega_F; W(f) \rangle = \exp\left[-\frac{1}{4} \|f\|^2\right]$ ,  $f \in \mathcal{E}$ . A state  $\varphi$  is classical, if  $C_{\varphi} = C_F P_{\varphi}$ , where  $P_{\varphi}$  is positive definite over  $\mathcal{E}$  [4] (and exhibits no anti-bunching).

The starting point for a squeezing procedure (theoretically and experimentally) is mostly a coherent laser state. Since to these states there is always attributed a classical (not necessarily sharp) phase, we have to deal with non-Fock coherent states [5], [6]. One finds in fact [7], that in the Dicke laser model with infinitely many atoms (appropriate for a macroscopic radiating material) the state for the time asymptotic quantized radiation is coherent (in the sense of the factorization condition of [8]) but not representable by a density operator in Fock-space. Replacing the coherence function of [8] (into which the normally ordered expectations factorize) by an arbitrary linear form  $L : \mathcal{E} \rightarrow \mathbb{C}$  we have the following recent result [6], [9]:

**1. Theorem:** *A first order coherent state  $\omega$  is non-Fock, iff its coherence function  $L$  is unbounded (in the norm-topology of  $\mathcal{E}$ ). In this case its characteristic function has the general form*

$$C_{\omega}(f) = C_F(f) \int_{\mathbb{C}} \exp\left[\frac{i}{\sqrt{2}} (zL(f) + \bar{z}\overline{L(f)})\right] d\mu(z) \quad (2)$$

with a probability measure  $\mu$  on  $\mathbb{C}$ , showing  $\omega$  to be classical (positive  $P$ -representation).

A non-Fock first order coherent state is second order coherent, iff it is  $n^{\text{th}}$ -order coherent for all  $n \geq 2$ , iff  $\mu$  in (2) is concentrated on the unit circle in  $\mathbb{C}$ .

Since a coherent state with unbounded linear form  $L : \mathcal{E} \rightarrow \mathbb{C}$  has divergent expectation values of the (unrenormalized, Fock-) particle number operator, it describes physically a situation with a macroscopic number of photons.

**2. Proposition:** Let  $\omega$  be a first order coherent state with unbounded  $L$ . Then its GNS-triple consists of the representation Hilbert space  $\mathcal{H}_\omega = \mathcal{H}_F \otimes L^2(\mathbb{C}, \mu)$ , where  $\mathcal{H}_F$  is the Boson Fock-space over  $\bar{\mathcal{E}}^{\|\cdot\|}$  and  $\mu$  is from (2); of the cyclic vector  $\Omega_\omega = \Omega_F \otimes 1$ , where  $\Omega_F$  is the Fock vacuum and 1 the ( $\mu$ -integrable) function on  $\mathbb{C}$  with constant value 1; and of the  $\star$ -homomorphism  $\Pi_\omega : \mathcal{W}(\mathcal{E}) \rightarrow \mathcal{B}(\mathcal{H}_\omega)$  defined by  $\Pi_\omega(W(f)) = W_F(f) \otimes W_{cl}(f)$ ,  $f \in \mathcal{E}$ , where  $W_F(f)$  is the Fock-Weyl operator and  $W_{cl}(f)$  the multiplication by the complex function  $z \in \mathbb{C} \mapsto \exp[\frac{i}{\sqrt{2}}(zL(f) + \bar{z}\bar{L}(f))]$  in  $L^2(\mathbb{C}, \mu)$ .

The associated von Neumann algebra is  $\mathcal{M}_\omega := \overline{\Pi_\omega(\mathcal{W}(\mathcal{E}))}^{\text{st}} = \mathcal{B}(\mathcal{H}_F) \otimes L^\infty(\mathbb{C}, \mu)$  with center  $\mathcal{Z}_\omega \cong L^\infty(\mathbb{C}, \mu)$ . In  $\mathcal{Z}_\omega$  is the macroscopic phase operator  $\Theta_\omega$  given by the function  $z \mapsto \vartheta(z) \equiv \text{Arg}(z)$  in  $L^\infty(\mathbb{C}, \mu)$ . If  $\mu$  is quasi-invariant under phase rotation, the gauge transformations of the first kind are unitarily implemented in  $\mathcal{H}_\omega$  with the renormalized particle number operator  $N_\omega = N_F \otimes \mathbb{1} + \mathbb{1} \otimes \frac{\partial}{i\partial\vartheta}$  as self-adjoint generator. It holds

$$[\Theta_\omega, N_\omega] \subset i\mathbb{1}. \quad (3)$$

Since  $\omega$  in the foregoing Proposition is regular (in fact it is analytic) we have  $\Pi_\omega(W(f)) = \exp[i\Phi_\omega(f)]$  with the self-adjoint field operators  $\Phi_\omega(f)$ ,  $f \in \mathcal{E}$ , satisfying

$$[\Phi_\omega(f), \Phi_\omega(g)] \subset i \text{Im}(f|g) \mathbb{1}. \quad (4)$$

We obtain here from Proposition 2

$$\Phi_\omega(f) = \Phi_F(f) \otimes \mathbb{1} + \mathbb{1} \otimes \Phi_{cl}(f),$$

where  $\Phi_{cl}(f)$  is the multiplication by  $z \in \mathbb{C} \mapsto \sqrt{2} \text{Re}(zL(f))$ . The corresponding annihilation operators are

$$a_\omega(f) = \frac{1}{\sqrt{2}} [\Phi_\omega(f) + i\Phi_\omega(if)] = a_F(f) \otimes \mathbb{1} + \mathbb{1} \otimes a_{cl}(f), \quad (5)$$

with  $a_{cl}(f)$  the multiplication by  $z \in \mathbb{C} \mapsto \bar{z}\bar{L}(f)$ .

Let be  $J : \mathcal{E} \rightarrow \mathcal{E}$  an involution (which is anti-linear with  $J = J^* = J^{-1}$ ) and  $\mathcal{E}_r$  its fixed point-space. Then  $\Phi_\omega(f)$  and  $\Phi_\omega(if)$  are canonically conjugate for  $f \in \mathcal{E}_r$ , with  $\|f\| = 1$ , in virtue of (4), and constitute the "quadrature components" of  $a_\omega(f)$  by (5) (cf. [10] and references therein).

The field fluctuations are

$$(\Omega_\omega | \Delta \Phi_\omega^2(f) \Omega_\omega) = \frac{1}{2} \|f\|^2 + (1 | \Delta \Phi_{cl}^2(f) 1),$$

where the first term of the r.h.s. are vacuum fluctuations for all  $f \in \mathcal{E}$ .

In order to squeeze  $\omega$  we consider the general Bogoliubov transformation  $\gamma_T$ , based on the real-linear symplectic transformation  $T : \mathcal{E} \rightarrow \mathcal{E}$ , i.e.  $\text{Im}(Tf|Tg) = \text{Im}(f|g)$ .

for all  $f, g \in \mathcal{E}$ . By definition we have  $\gamma_T(W(f)) := W(Tf)$ ,  $f \in \mathcal{E}$ , which extends to a  $\star$ -automorphism of  $\mathcal{W}(\mathcal{E})$  [2]. The field fluctuations of the transformed state  $\omega \circ \gamma_T$  may be calculated in terms of the old state with the transformed field operators  $\gamma_T(\Phi_\omega(f)) := \Phi_\omega(Tf)$ ,  $f \in \mathcal{E}$ , and are given by

$$(\Omega_\omega | \Delta \Phi_\omega^2(Tf) \Omega_\omega) = \frac{1}{2} \|Tf\|^2 + (1 | \Delta \Phi_{\Omega_1}^2(Tf) 1).$$

Observing the unique decomposition  $T = T_1 + T_2$  into the (complex) linear and anti-linear parts,  $T_1$  resp.  $T_2$ , we find the following necessary and sufficient conditions for  $T$  to be symplectic

$$T_1^* T_2 = T_2^* T_1, \quad T_1^* T_1 - T_2^* T_2 \subset \mathbb{1}. \quad (6)$$

It holds

$$\gamma_T(a_\omega(f)) = a_\omega(T_1 f) + a_\omega^*(T_2 f). \quad (7)$$

**3. Proposition:** *The Bogoliubov transformed macroscopic coherent state  $\omega \circ \gamma_T$  has (at least) one mode  $f_- \in \mathcal{E}$  with squeezed vacuum fluctuations  $\frac{1}{2} \|Tf_-\|^2 < \frac{1}{2} \|f_-\|^2$ , iff  $T_2 \neq 0$  ( $\Rightarrow T_1 \neq 0$  by (6)), and this holds iff the transformed vacuum  $\omega_F \circ \gamma_T$  is non-classical, that is, iff  $P_{\omega_F \circ \gamma_T}(f) = \exp[\frac{1}{4}(\|f\|^2 - \|Tf\|^2)]$  is not positive definite.*

In [11] a multimode squeezing transformation is investigated, where the transformed vacuum has infinitely many squeezed modes and is a special case of the above situation. In this and other works [12] the counting statistics of squeezed photons are treated. Strictly speaking only those field excitations can be counted, however, which behave correctly under gauge transformations of the first kind,  $\gamma_\vartheta(W(f)) = W(e^{i\vartheta} f)$ ,  $f \in \mathcal{E}$ ,  $\vartheta \in [0, 2\pi)$ . Just if  $T_2 \neq 0$ , (7) does not behave under  $\gamma_\vartheta$  like an annihilation operator and we have

$$\gamma_T \circ \gamma_\vartheta \neq \gamma_\vartheta \circ \gamma_T.$$

We, therefore, propose the improved squeezing transformation

$$\hat{\gamma}_T(a_\omega(f)) := a_\omega(T_1 f) + e^{-2i\Theta_\omega} a_\omega^*(T_2 f), \quad (8)$$

involving the macroscopic phase operator  $\Theta_\omega$ . Only if the phase has a sharp value one has the conventional case, in which, however, there is no renormalized particle number operator  $N_\omega$ , cf. (3). Just if  $N_\omega$  exists (that is, if the phase fluctuates over all of  $[0, 2\pi)$ ) one has

$$\hat{\gamma}_T \circ \gamma_\vartheta = \gamma_\vartheta \circ \hat{\gamma}_T, \quad \forall \vartheta \in [0, 2\pi).$$

and the squeezed photons have nearly a particle structure.

The transformation (8) has the analogous form as the gauge covariant Bogoliubov transformation in the BCS-theory (cf. [13] and references therein), which was crucial for charge conservation and the definition of the tunneling supercurrent. The term with the doubled phase operator in the exponential corresponds there to the annihilation operator of condensed Cooper pairs. From (5) and the definition of  $\Theta_\omega$  we can see, that it is also in our case the main part of the squared annihilation operator for the classical (= phase correlated) photons. The coherent two-photon structure may here possibly be connected with the two-photon structure of squeezed light.

In any experimental realization of the squeezing transformation (cf., e.g. [14]) one finds strong indications that the involved phase is in fact macroscopic in that it couples directly to macroscopic devices (and does not refer to the microscopic phase of a one-photon wave function). Applying the dual transformation  $\hat{\gamma}_T^*$  to  $\omega$ , the quantum mechanical part constitutes the gauge covariantly squeezed vacuum. Since this transformed vacuum is non-classical and coupled to a macroscopic phase one should here look for macroscopic quantum phenomena in the realm of quantum optics [15], [16].

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# QUANTIZED RADIATION FROM COLLECTIVELY ORDERED ATOMS

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## 1 Summary

For infinite mean-field quantum lattice systems [1], [2], [3] coupled to the boson field by means of cocycle equations it is constructed a general class of global quantum dynamics. In physical applications such systems are due to collectively ordered  $N$ -level atoms or the Josephson junction weakly interacting with the electromagnetic field. Restricting to the photons in the infinite time limit ( $t \rightarrow \infty$ ) for very general initial states we obtain macroscopic classical states on the  $C^*$ -Weyl algebra [4] associated with the quantized radiation, in which one partially refinds the collective ordering of the atoms. In the special case of the Dicke model [5], [6], [7], [8] these photon states show quantum optical coherence of first (and higher) order [9], [10], [11], [12].

## 2 The dynamics

Let us first consider the mean-field system. As  $C^*$ -algebra we have the infinite tensor product  $\mathcal{A} = \bigotimes_{n \in \mathbb{N}} \mathbb{M}_m$ ,  $\mathbb{M}_m$  denoting the  $m \times m$ -matrices. In the representation  $\Pi_\lambda$  associated with the folium  $\mathcal{F}_\lambda$  generated by the permutation invariant states on  $\mathcal{A}$  the limits  $m_\lambda(x) = s\text{-}\lim_{\Lambda \rightarrow \infty} \Pi_\lambda(m_\Lambda(x))$  of the local mean-field operators ( $\Lambda \subset \mathbb{N}$ ,  $|\Lambda| < \infty$ )

$$m_\Lambda(x) := \frac{1}{|\Lambda|} \sum_{n \in \Lambda} \mathbb{1} \otimes \cdots \otimes \underbrace{x}_{n\text{-th place}} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots, \quad x \in \mathbb{M}_m$$

exist in the strong operator topology, and the  $m_\lambda(x)$ ,  $x \in \mathbb{M}_m$ , are elements of the center  $\mathcal{Z}_\lambda$  of the von Neumann algebra  $\mathcal{M}_\lambda := \Pi_\lambda(\mathcal{A})''$ . There exists a projection valued measure  $\mathcal{E}_\lambda : \mathbb{R}^{m^2-1} \rightarrow \mathcal{Z}_\lambda$  with  $\text{supp}(\mathcal{E}_\lambda) =: E_\lambda$  being convex, such that  $\xi \mapsto \int_{E_\lambda} \xi(x) d\mathcal{E}_\lambda(x)$  defines a  $*$ -isomorphism from the continuous functions  $\mathcal{C}(E_\lambda)$  onto the  $C^*$ -subalgebra  $\mathcal{N}^\lambda$  of  $\mathcal{Z}_\lambda$  generated by  $\{m_\lambda(x) \mid x \in \mathbb{M}_m\}$ .

For each polynomial  $Q \equiv Q(m_\lambda)$  in the local mean-field operators  $m_\lambda(x)$ ,  $x \in \mathbb{M}_m$ , there exists a limiting dynamics  $\alpha_t^Q(\cdot) = \lim_{\Lambda \rightarrow \infty} e^{it\Lambda Q(m_\lambda)}, e^{-it\Lambda Q(m_\lambda)}$  on  $\mathcal{M}_\lambda$ . It is  $\alpha_t^Q(\mathcal{N}^\lambda) = \mathcal{N}^\lambda \forall t \in \mathbb{R}$  and the restriction  $\alpha_t^Q|_{\mathcal{N}^\lambda}$  is given within the isomorphism  $\mathcal{N}^\lambda \cong \mathcal{C}(E_\lambda)$  by a classical flow  $\varphi_t^Q$  on the classical phase space  $E_\lambda$ . For literature, see [1], [2], [3].

The boson system is described by the Weyl algebra  $\mathcal{W}(E)$  [13] over the one-boson testfunction space  $E$ , and some quasi-free dynamics,  $\gamma_t^S(\mathcal{W}(f)) = \mathcal{W}(e^{itS}f) \forall f \in E$ .



$W(f)$ ,  $f \in E$ , denoting the Weyl operators and  $S$  the one-boson hamiltonian. Let be  $\tau$  a locally convex topology on  $E$ , stronger than the scalar product such that  $e^{itS}$  is  $\tau$ -continuous, and denote by  $\Pi_b$  the representation of  $\mathcal{W}(E)$  associated with the folium  $\mathcal{F}_b$  of the  $\tau$ -continuous states on  $\mathcal{W}(E)$ ,  $\Pi_b(\mathcal{W}(E))'' =: \mathcal{M}_b$ ,  $\Pi_b(W(f)) =: W_b(f)$ . By continuity  $W_b(g) \in \mathcal{M}_b$  is also well-defined for  $g$  in the  $\tau$ -completion  $\overline{E}^\tau$  of  $E$ .

According to hamiltonians of the form

$$H = A_a \otimes \mathbb{1}_b + \mathbb{1}_a \otimes G_b + \sum_{k=1}^l \left( \int_{E_a} \xi_k(x) d\mathcal{E}_a(x) \otimes a_b^*(\phi_k) + \int_{E_a} \overline{\xi_k(x)} d\mathcal{E}_a(x) \otimes a_b(\phi_k) \right)$$

the dynamics of the interacting systems, which we study here, in the representation  $\Pi_a \otimes \Pi_b$  of  $\mathcal{A} \otimes \mathcal{W}(E)$  is given by

$$\tau_t^w(Z) = Q(\nu_t) (\alpha_t^Q \otimes \gamma_t^S)(Z) Q(\nu_t)^* \quad \forall Z \in \mathcal{M}_a \overline{\otimes} \mathcal{M}_b \quad \forall t \in \mathbb{R}$$

with the spectral integral  $Q(\nu_t) := \int_{E_a} \mathbb{1}_a \otimes W_b(\nu_t(x)) d(\mathcal{E}_a(x) \otimes \mathbb{1}_b) \in \mathcal{Z}_a \overline{\otimes} \mathcal{M}_b$ , where

$$\nu_t(x) = \int_{\lambda=0}^t e^{i\lambda S} \phi(\varphi_\lambda^Q x) d\lambda \in \overline{E}^\tau \quad \forall t \in \mathbb{R} \quad \forall x \in E_a. \quad (1)$$

and the coupling function  $\phi(x) = \sqrt{2} \sum_{k=1}^l \xi_k(x) \phi_k \forall x \in E_a$ , where  $\xi_k \in \mathcal{C}(E_a)$  and  $\phi_k \in \overline{E}^\tau$ . The function  $(t, x) \mapsto \nu_t(x)$  satisfies the cocycle equations

$$\nu_{s+t}(x) = \nu_s(x) + e^{isS} \nu_t(\varphi_s^Q x) \quad \forall x \in E_a \quad \forall s, t \in \mathbb{R}.$$

and (1) denotes some solution.

In the Schrödinger picture the dynamics is given by affine bijections  $\nu_t^w$  on the folium  $\mathcal{F}_a \otimes \mathcal{F}_b$  of  $\Pi_a \otimes \Pi_b$ -normal states on  $\mathcal{A} \otimes \mathcal{W}(E)$  satisfying

$$\langle \nu_t^w(\omega); Z \rangle = \langle \omega; \tau_t^w(Z) \rangle \quad \forall \omega \in \mathcal{F}_a \otimes \mathcal{F}_b \quad \forall Z \in \mathcal{M}_a \overline{\otimes} \mathcal{M}_b.$$

### 3 Time asymptotic boson states

If  $\omega|_b$  denotes the restriction of the state  $\omega$  from  $\mathcal{A} \otimes \mathcal{W}(E)$  to  $\mathcal{W}(E)$ ,  $\langle \omega|_b; Y \rangle := \langle \omega; \mathbb{1}_a \otimes Y \rangle \forall Y \in \mathcal{W}(E)$ , in the present section we investigate the existence of time asymptotic states  $\lim_{t \rightarrow \infty} \nu_t^w(\omega)|_b$  for some  $\omega \in \mathcal{F}_a \otimes \mathcal{F}_b$ .

Assume  $S$  to have pure absolutely continuous spectrum and the existence of the limits  $\lim_{t \rightarrow \infty} \frac{1}{\sqrt{2}} \langle \psi_{-t}(x) | f \rangle =: L_x(f)$  uniformly in  $x \in E_a$  for each  $f \in E$ . Clearly the  $L_x: E \rightarrow \mathbb{C}$  are linear forms, which in general may be unbounded with respect to the norm on  $E$ . Assume  $\varphi \in \mathcal{F}_b$  fulfilling the asymptotic product property  $\lim_{t \rightarrow \infty} \langle \varphi; W(e^{itS}f + g) \rangle = \langle \varphi; W(f) \rangle \langle \varphi; W(g) \rangle \forall f, g \in E$ , and denote by  $\mathcal{F}_b^\varphi$  the smallest subfolium of  $\mathcal{F}_b$  containing  $\varphi$  (GNS-folium associated with  $\varphi$ ;  $\varphi$  is a so-called  $\gamma^S$ -abelian state [13]).

Under the previous circumstances for every  $\omega \in \mathcal{F}_a \otimes \mathcal{F}_b^f$  exists time asymptotic states  $R_t(\omega) \in \mathcal{W}(E)$ ,  $t \in \mathbb{R}$ .

$$\langle R_t(\omega); W(f) \rangle = \langle \varphi; W(f) \rangle \int_{E_a} \exp\{i\sqrt{2} \operatorname{Re}(L_x(f))\} d\langle \omega; \mathcal{E}_a \circ \varphi_{\pm t}^Q(x) \otimes \mathbb{1}_b \rangle$$

for all  $f \in E$ , in the sense of  $\operatorname{weak}^* \lim_{t \rightarrow \infty} (\nu_t^k(\omega)|_b - R_t(\omega)) = 0 \forall \omega \in \mathcal{F}_a \otimes \mathcal{F}_b^f$ . The macroscopic ordering of the mean-field system, which is expressed by the classical phase space  $E_a$ , the flow  $\varphi_t^Q$ , and by the statistics of the initial state  $\omega \in \mathcal{F}_a \otimes \mathcal{F}_b^f$ , one partially refinds in the formula of  $R_t(\omega)$ , that is in the emitted boson field (radiation), more exactly, by means of their central decomposition.

Obviously, by definition we have  $\nu_t^k(\omega)|_b \in \mathcal{F}_b^f \forall \omega \in \mathcal{F}_a \otimes \mathcal{F}_b^f$ . But the limiting states  $R_t(\omega)$  may leave the folium  $\mathcal{F}_b^f$ .

Taking for  $\varphi \in \mathcal{F}_b$  the Fock state  $\omega_F$ ,  $\langle \omega_F; W(f) \rangle = \exp\{-\frac{1}{4}\|f\|^2\} \forall f \in E$ , all the time asymptotic states  $R_t(\omega)$  are classical.

## 4 The Dicke model

The Dicke model consists of a system of (infinite) two-level atoms interacting with the radiation field. As  $C^*$ -algebra one has  $\mathcal{A} = \otimes_{n \in \mathbb{N}} \mathbb{M}_2$ , that is  $m = 2$ , where  $\mathbb{M}_2$  is the observable algebra of a single two-level atom. It is assumed level-splitting  $\varepsilon > 0$  for each atom.  $E_a$  is given by  $\{x \in \mathbb{R}^3 \mid \|x\| \leq \frac{1}{2}\}$ , the polynomial  $Q$  is linear, and the flow  $\varphi_t^Q$  is the rotation around the  $x_3$ -axis with phase-velocity  $\varepsilon$ .

Regarding only one direction of polarization the testfunction space  $E$  is chosen to be a suitable dense subspace of  $L^2(\mathbb{R}^3)$ . The one-photon hamiltonian  $S := \sqrt{-\Delta}$  with the usual Laplacian  $\Delta$  on  $\mathbb{R}^3$ . Because at time  $t = 0$  there should be only a few photons we work with the folium  $\mathcal{F}_b^f$  associated with the Fock state  $\omega_F$ .

The time asymptotic states for  $\omega \in \mathcal{F}_a \otimes \mathcal{F}_b^f$  are given by integrals over the complex plane  $\mathbb{C}$ , the projection of  $E_a$  onto the  $\{x_1, x_2\}$ -plane of  $\mathbb{R}^3$ .

$$\langle R_t(\omega); W(f) \rangle = e^{-\frac{1}{4}\|f\|^2} \int_{\mathbb{C}} \exp\{i\sqrt{2} \operatorname{Re}(zG(f))\} d\rho_\omega \circ \varphi_{\pm t}^Q(z) \quad \forall f \in E,$$

where  $G: E \rightarrow \mathbb{C}$  is a fixed linear form which in terms of the Fourier transformation  $f \mapsto \hat{f}$  on  $L^2(\mathbb{R}^3)$  is expressed by

$$G(f) = i\pi \int_{\|k\|=\varepsilon} \overline{\hat{\phi}_1(k)} \hat{f}(k) dS(k) - \operatorname{pv} \int_{\mathbb{R}^3} \frac{\overline{\hat{\phi}_1(k)} \hat{f}(k)}{\|k\| - \varepsilon} d^3k.$$

Here  $\hat{\phi}_1(k)$ ,  $\phi_1 \in L^2(\mathbb{R}^3)$ , denotes the coupling constant of each two-level atom to the mode  $k \in \mathbb{R}^3$  of the radiation field. The first summand of  $G$  represents the resonance ( $\|k\| = \varepsilon$ ) between the photon field and the two levels of each atom (remember the level-splitting  $\varepsilon$ ), whereas the second one picks up in the surrounding of the resonant modes. The time asymptotic states here show macroscopic quantum optical coherence.

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## X. Differential Geometric Methods

# Quantization on the Gauge Orbit Space

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## Abstract

We propose a programme for quantization on the gauge orbit space and apply it to a nonabelian Higgs model as well as to an  $SU(3)$  - gauge model with Weyl fermions.

## 1 The gauge fixing problem

In 1977 Gribov [1] observed that the Coulomb gauge fixing does not make sense globally: Denote by  $C$  the affine space of gauge potentials (configuration space), by  $G$  the group of local gauge transformations and by  $N = C/G$  the gauge orbit space. Consider the subspace  $S$ , defined by the Coulomb (background) gauge condition

$$D^*(\bar{A})(A - \bar{A}) = 0, \quad (1)$$

with  $D^*(\bar{A})$  denoting the co-covariant derivative with respect to a fixed background gauge potential  $\bar{A}$ . It turns out that the orbit of  $G$  through  $\bar{A}$  intersects  $S$  in general several times (Gribov ambiguity) - showing that  $N$  cannot be parametrized by gauge potentials fulfilling (1) globally. In 1978 Singer [2] showed that in some cases there does not exist any (global) gauge, e.g. for pure  $SU(n)$  - gauge models on  $S^4$ . This is due to the nontrivial topological nature of the fibration  $C \rightarrow N$ , or - more precisely - of the principal fibre bundle  $C^{irr} \rightarrow C^{irr}/\hat{G}$ ,  $\hat{G} = G/\text{center}(G)$ , with  $C^{irr}$  denoting the (dense) subset of irreducible gauge potentials. The gauge orbit space is a complicated stratified set with the generic stratum given by the above principal bundle. For basic mathematical investigations concerning these structures we refer to [3].

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At one hand Gribov's observation stimulated interesting mathematical investigations, on the other hand there were attempts to circumvent this problem, see [4] and references therein. The basic strategy of these papers was to restrict the functional integration to gauge potentials lying within the Gribov horizon  $S_0 \subset S$ ,

$$S_0 = \{A \in S : -D^*(\bar{A})D(A) \geq 0\} , \quad (2)$$

(with  $-D^*(\bar{A})D(A)$  denoting the Faddeev - Popov operator). Unfortunately, until now all attempts to make this idea rigorous failed. Finally, we note that also simple algebraic gauges, like the axial gauge, do not solve the problem [5].

We conclude that one should define the functional integral (or any other quantization procedure) on the gauge orbit space. This way we were led to formulate the following programme:

1. Parametrization of  $N$  in terms of gauge invariant quantities.
2. Formulation of field dynamics in terms of invariants.
3. Calculation of the functional measure on  $N$ .
4. Regularization of the functional integral on the lattice.

(In a final step one would like to construct the continuum theory rigorously - an extraordinarily difficult problem, which we have not even touched.)

Before passing to a discussion of examples we must underline that  $N$  has cone-like singularities [6] - due to the existence of non-generic strata. Recent investigations on finite dimensional models simulating this situation [7] show that those singularities might play an important role in the functional integral. Nevertheless, for the time being, we neglect them in our considerations.

## 2 A nonabelian Higgs model

We consider the theory of an  $SU(2)$  gauge field  $A$  interacting with a matter field  $\Phi$  in the adjoint representation on Minkowski space  $M$ , described by the following classical Lagrangian

$$L = -V(\|\Phi\|^2) + \frac{1}{2} D_\mu \Phi^a D^\mu \Phi_a - \frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} , \quad (3)$$

with  $D_\mu \Phi^a$  and  $F_{\mu\nu}^a$  denoting the covariant derivative of  $\Phi$  and the curvature of  $A$  respectively.

One can show [8] that every equivalence class  $[(A, \Phi)]$  of generic configurations of this model is in 1-1 correspondence with a set of invariants  $(R, r, v, [\chi]; h)$ , where  $R$  and  $r$  are  $\mathbf{R}_+$ -valued,  $v$  is a  $\mathbf{R}$ -valued covector field,  $[\chi]$  is a  $\mathbf{Z}_2$ -class of  $SL(2, \mathbf{C})$ -valued fields and  $h$  is a magnetic vortex current. The fields  $v, [\chi]$  and  $h$  have to fulfill certain topological compatibility conditions. For a deeper discussion of geometrical

and topological aspects we refer to [9] and further references therein. We see that a hydrodynamical picture of field matter emerges; for the abelian Higgs model this was discussed earlier [10]. Also one can show that the magnetic vortex current fulfils

$$dh = k \quad , \quad (4)$$

with  $k$  denoting the magnetic monopole current, see [11]. We conclude that the magnetic monopoles contained in this model are confined by magnetic vortices - showing an interesting relation to ideas of 't Hooft and Mandelstam [12] concerning the solution of the quark confinement problem.

To formulate field dynamics in terms of the above invariants in an elegant way, we use the vector space isomorphism of the exterior algebra and the Clifford algebra

$$i : \Lambda M^C \longrightarrow \text{Cliff}(M^C) \quad . \quad (5)$$

For 1- and 2-forms we have  $i(\alpha_\mu dx^\mu) = \alpha_\mu \gamma^\mu$ ,  $i(\frac{1}{2}\beta_{\mu\nu} dx^\mu \wedge dx^\nu) = \frac{1}{4}\beta_{\mu\nu}[\gamma^\mu, \gamma^\nu]$ , with  $(\gamma^\mu)$  - Dirac matrices. The first term in (3) takes the form

$$L_1 = -V(R^2) \quad , \quad (6)$$

for the second term we get

$$L_2 = \frac{1}{2}\partial_\mu R \partial^\mu R + \frac{1}{8}g^2 R^2 r^2 \text{Tr} \{i(\chi) \overline{i(\chi)}\} \quad , \quad (7)$$

and for the last term we obtain

$$L_3 = \frac{1}{8}\text{Tr}(G\overline{G}) + \frac{1}{8}\text{Tr}(W\overline{W}) \quad . \quad (8)$$

with

$$\begin{aligned} G &= \overline{G} = i(dv + h) + \frac{1}{4}ig r^2 [i(\chi), \overline{i(\chi)}] \quad , \\ W &= \frac{1}{2}[i(dr), i(\chi)] + r D_v i(\chi) \quad , \\ D_v &= \frac{1}{2}(\partial_\mu + ig v_\mu)[\gamma^\mu, \cdot] \quad . \end{aligned}$$

In the above formulae  $g$  denotes the gauge coupling constant.

Concerning point 3. of the programme mentioned in section 1, we start with the (formal) measure

$$d\mu = \prod d\Phi \prod dA \quad . \quad (9)$$

One can show [8] that for topologically trivial configurations  $d\mu$  naturally decomposes into

$$d\mu = d\tilde{\mu} \cdot d\varrho \quad , \quad (10)$$

with

$$d\tilde{\mu} = \prod R^2 dR \prod r^7 dr \prod dv \prod d\nu(\chi) \quad . \quad (11)$$

Here  $d\mathcal{G}$  denotes the measure on the group of local gauge transformations and  $d\nu(\chi)$  the Haar measure on  $SL(2, \mathbb{C})$ . Thus,  $d\hat{\mu}$  is the (formal) measure on the gauge orbit space of topologically trivial configurations. Finally, one has to incorporate topologically nontrivial configurations. For that purpose one needs a measure  $dh$  on the space of magnetic vortex currents. Such a measure was proposed in [13].

Now we can write down the (formal) generating functional on the gauge orbit space

$$F = \int d\hat{\mu}(R, r, v, [\chi]) dh \exp \left\{ i \int L(R, r, v, [\chi]; h) \right\} . \quad (12)$$

One possibility to make this expression rigorous consists in approximating it on a lattice. For attempts to realize point 4. of the above programme we refer to [14] and further references therein.

### 3 An $SU(3)$ -model with Weyl fermions

We consider the theory of an  $SU(3)$ -gauge potential  $A$  interacting with a triplet  $\Psi$  of Weyl spinors in the fundamental representation of  $SU(3)$ , described by the following "classical" Lagrangian:

$$L = \frac{1}{2} \text{ImTr} \left( \sigma^\mu (D_\mu \Psi) \cdot \Psi^\dagger \right) - \frac{1}{4} \text{Tr} (F_{\mu\nu} \cdot F^{\mu\nu}) . \quad (13)$$

We denote the natural Hermitean metric on  $\mathbb{C}^3$  by  $(g_{AB})$ ,  $A, B = 1, 2, 3$ , and use the two-component spinor language, in which  $\Psi = (\Psi_A^K)$ ,  $K = 1, 2$ ,  $A = 1, 2, 3$ . The following considerations are valid for arbitrary (curved) space time  $M$ . In that case the covariant derivative  $D_\mu \Psi$  is given by

$$D_\mu \Psi_A^K = \partial_\mu \Psi_A^K + \omega_\mu{}^K{}_L \Psi_A^L + i A_\mu{}^B{}_A \Psi_B^K . \quad (14)$$

with  $(\omega_\mu{}^K{}_L)$  denoting the spin connection on  $M$ . The following quantities are obviously gauge invariant:

$$j^{\dot{K}L} \doteq \bar{\Psi}_A^{\dot{K}} g^{AB} \Psi_B^L , \quad (15)$$

$$a_\mu{}^{\dot{K}L} \doteq \frac{1}{2i} (B_\mu{}^{\dot{K}L} - \bar{B}_\mu{}^{L\dot{K}}) . \quad (16)$$

with

$$B_\mu{}^{\dot{K}L} \doteq (D_\mu \Psi_A^{\dot{K}}) g^{AB} \Psi_B^L , \quad (17)$$

$$V_\mu{}^K \doteq \frac{1}{2} \epsilon_{LM} \epsilon^{ABC} (D_\mu \Psi_A^K) \Psi_B^L \Psi_C^M . \quad (18)$$

We observe that  $j^{\dot{K}L}$  and  $a_\mu{}^{\dot{K}L}$  describe matter of mesonic and  $V_\mu{}^K$  of barionic type. We have

$$B_\mu{}^{\dot{K}L} = \frac{1}{2} D_\mu j^{\dot{K}L} + i a_\mu{}^{\dot{K}L} . \quad (19)$$



Moreover, we define

$$J^2 = \frac{1}{2} j^{\dot{K}\dot{L}} j_{\dot{K}\dot{L}} , \quad (20)$$

and notice that  $j_\alpha = \frac{1}{2} j^{\dot{K}\dot{L}} (\sigma_\alpha)_{\dot{K}\dot{L}}$ ,  $\sigma_\alpha$  - Pauli matrices, belongs to the forward light cone, (because we deal with Weyl spinors).

Generically, we have  $\Psi_A^1(x) \neq 0$  and  $\Psi_A^2(x) \neq 0$  for all  $x \in M$ . Still there are special configurations such that both  $C^3$ -vectors are proportional to each other. Generically, this happens on a set  $\square$  of isolated points. We denote  $M_0 \doteq M \setminus \square$  and notice that  $J > 0$  on  $M_0$ .

Theorem :

Every class of gauge equivalent generic configurations  $[(A, \Psi)]$  is in 1-1 correspondence with a triple  $(j, a, V)$ .

Sketch of the proof :

First we show that there exists a continuous gauge on  $M_0$  such that

$$\Psi_3^K = 0 . \quad (21)$$

Due to the decomposition  $C^3 \setminus \{0\} = \mathbf{R}_+ \times S^5$ , the vectors  $\Psi_A^1(x)$  and  $\Psi_A^2(x)$  define two points  $\hat{\Psi}_A^1(x)$  and  $\hat{\Psi}_A^2(x)$  on  $S^5 \cong \text{SU}(3)/\text{SU}(2)$ . Since  $\text{SU}(3)$  acts transitively on  $S^5$ , we can gauge  $\hat{\Psi}^1$  in such a way that it coincides with the first element of the canonical basis  $(e_A)$  in  $C^3$ . The stabilizer of the new  $\hat{\Psi}^1$  is equal to  $\text{SU}(2)$  and acts freely on the orbit  $S^3$  through  $\hat{\Psi}^2$ . The orbit intersects with the  $C^2$ -subspace  $\Sigma$  spanned by  $\epsilon_1$  and  $\epsilon_2$  at two points. There is, therefore, an  $\text{SU}(2)$ -gauge transformation, which does not move  $\hat{\Psi}^1$  and sends  $\hat{\Psi}^2$  into  $\Sigma$ .

Obviously,  $\Psi$  fulfilling (21) can be represented by a complex  $2 \times 2$ -matrix  $\hat{\Psi}$ , with  $\det \hat{\Psi} \neq 0$  for all  $x \in M_0$ , and  $\text{SU}(3)$ -gauge transformations are restricted to the subgroup

$$H = \left\{ \left( \begin{array}{c|c} g & 0 \\ \hline 0 & (\det g)^{-1} \end{array} \right) : g \in U(2) \right\} . \quad (22)$$

The polar decomposition of  $\text{GL}(2, \mathbf{C})$  gives

$$\hat{\Psi} = u \cdot \rho , \quad (23)$$

with  $u \in U(2)$  and  $\rho$  being positive Hermitian. Obviously, we can "gauge away"  $u$ . Finally, a simple calculation shows that there exists an  $\text{SL}(2, \mathbf{C})$ -(spin) gauge such that  $\rho$  can be represented by

$$\rho = \sqrt{J} \mathbf{1} . \quad (24)$$

Summarizing, we have shown that there exist  $\text{SU}(3)$ - and  $\text{SL}(2, \mathbf{C})$ -gauge transformations such that  $\Psi$  can be represented as follows:

$$\begin{aligned} \Psi_A^1 &= \sqrt{J} (1, 0, 0) \\ \Psi_A^2 &= \sqrt{J} (0, 1, 0) \\ \Psi_A^3 &= 0 . \end{aligned} \quad (25)$$

In this gauge we get (for simplicity, for the case of flat space time):

$$\begin{aligned} j^{\dot{K}L} &= J(\sigma_0)^{\dot{K}L} \\ a_\mu^{\dot{K}L} &= -J A_\mu^{\dot{K}L} \\ V_\mu^{\dot{K}} &= iJ^2 A_\mu^{\dot{K}3} , \end{aligned} \quad (26)$$

with  $A_\mu^{\dot{K}L} = \Psi_B^L A_\mu^{BD} \bar{\Psi}_D^{\dot{K}}$ . Formulae (25) and (26) show that given  $(j, a, V)$ , a representative  $(A, \Psi)$  - and, consequently, the class  $[(A, \Psi)]$  - can be reconstructed.  $\square$

Next we have to calculate the Lagrangian in terms of invariants. Using

$$F_{\mu\nu}^{AB} \Psi_B^{\dot{K}} = \frac{1}{i} [D_\mu, D_\nu] \Psi_B^{\dot{K}} g^{AB} , \quad (27)$$

and a decomposition property of the Hermitean metric,

$$2J^2 g_{AB} = \chi_{AB} + \gamma_{AB} , \quad (28)$$

with

$$\begin{aligned} \chi_{AB} &= \chi_A \bar{\chi}_B , \\ \chi_A &= \frac{1}{\sqrt{2}} \epsilon_{KL} \epsilon_A^{BC} \Psi_B^{\dot{K}} \Psi_C^{\dot{L}} , \\ \gamma_{AB} &= 2\bar{\Psi}_A^{\dot{K}} j_{\dot{K}L} \Psi_B^{\dot{L}} , \end{aligned}$$

we obtain the following result:

$$\frac{1}{2} \text{ImTr} (\sigma^\mu (D_\mu \Psi) \Psi^\dagger) \quad (29)$$

$$= a_\mu^\mu , \quad (30)$$

$$= G_{\mu\nu}^{\dot{K}L} (\bar{G}^{\mu\nu})^{NM} j_{\dot{K}N} j_{\dot{M}L} + 4 G_{\mu\nu}^{\dot{K}L} (\bar{G}^{\mu\nu})^{NM} j_{\dot{K}L} j_{\dot{M}N} + 32 W_{\mu\nu}^{\dot{K}} (\bar{W}^{\mu\nu})^{\dot{K}} j_{\dot{K}L} ,$$

with

$$\begin{aligned} G_{\mu\nu}^{\dot{K}L} &= J^2 D_{[\mu} B_{\nu]}^{\dot{K}L} - \frac{1}{2} V_{[\mu}^{\dot{K}} V_{\nu]}^{\dot{L}} - 2 \bar{B}_{[\mu}^{LM} B_{\nu]}^{\dot{K}N} j_{\dot{M}N} , \\ W_{\mu\nu}^{\dot{K}} &= J^2 D_{[\mu} V_{\nu]}^{\dot{K}} - 2 \left( \bar{B}_{[\mu}^{NM} V_{\nu]}^{\dot{L}} + \bar{B}_{[\mu}^{LM} V_{\nu]}^{\dot{N}} \right) j_{\dot{M}N} . \end{aligned}$$

Using (19) this result can be rewritten in terms of  $(j, a, V)$ .

It turns out that invariant quantities (15) - (18) can be defined for the case, when the  $\Psi_A^{\dot{K}}$  are treated as anticommuting variables. However, a complete description of the (Berezin) functional integral in terms of those quantities is still under investigation.

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## Constant Yang-Mills Potentials

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Big classes of solutions  $A_\alpha = A_\alpha(x)$  of the Yang-Mills equations

$$D^j F_{\alpha\beta} = 0, \quad F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]$$

are distinguished by conditions of selfduality (instantons, monopoles, dyons, ...), external symmetry (spherical, cylindrical, ... symmetry) or constancy (constant or covariantly constant potential or field strength). Let us discuss Yang-Mills potentials  $A = A_\alpha dx^\alpha$  with constant components  $A_\alpha$  in some gauge and choice of coordinates. The partial differential equations  $D^j F_{\alpha\beta} = 0$  collapse to the purely algebraic equations

$$[A^j, [A_\alpha, A_\beta]] = 0. \quad (1)$$

Here  $\alpha, \beta, \dots = 1, 2, \dots, n$  are tensor indices with respect to a vector space  $E$ , the  $A_\alpha$  are elements of some Lie algebra  $L$ , and  $[ \cdot ]$  denotes the commutator in  $L$ . The external space or spacetime  $E$  is equipped with a Euclidian or Minkowskian scalar product respectively, which is described by constant components  $g_{\alpha\beta}$  of a metric  $g$ . Greek indices are lowered or raised by means of the matrices  $(g_{\alpha\beta})$  or  $(g^{\alpha\beta}) := (g_{\alpha\beta})^{-1}$  respectively. There is an alternative representation in terms of tensor indices  $i, j, \dots = 1, 2, \dots, N$  with respect to the structure constants  $c_{ij}^k$  of  $L$  and 1-forms  $a^i$  on  $E$  as the components of  $A$ :

$$c_{ij}^k a^i \wedge a^j = 0, \quad f^k = c_{ij}^k a^i \wedge a^j, \quad (2)$$

where  $\wedge$  denotes the outer (alternating) product and  $\lrcorner$  the inner product of differential forms on  $E$  with respect to  $g$ . The problem (1), (2) looks simple, but it is *unsolved* as yet, that means a characterization of its general solution is not known. All one can do is to restrict to special classes of Lie algebras. The following negative result has been shown in [1]:

**Theorem 1.** A constant Yang-Mills potential on a Euclidian or Minkowski space with values in a compact Lie algebra is flat, i. e. gauge-equivalent to the zero potential.

If, for physical reasons, only compact Lie algebras are accepted for Yang-Mills theory, than we are done. But there are mathematical reasons [1] to discuss other

types of Lie algebras too. The following positive result is easy to show:

**Theorem 2.** A non-Abelian nilpotent Lie algebra admits non-flat constant Yang-Mills potentials.

Next we note some structural theorems:

**Theorem 3.** Let  $L = L_I \oplus L_{II}$  be the direct sum of Lie algebras  $L_I, L_{II}$  and let the  $L$ -potential  $A = A_I \oplus A_{II}$  be correspondingly decomposed into an  $L_I$ -potential  $A_I$  and an  $L_{II}$ -potential  $A_{II}$ . The Yang-Mills equation for  $A$  is equivalent to the separate Yang-Mills equations for  $A_I$  and  $A_{II}$ .

**Theorem 4.** Let  $L = L_I \circ L_{II}$  be the semidirect sum of an ideal  $L_I$  and a Lie subalgebra  $L_{II}$  and let  $A = A_I \oplus A_{II}$  be correspondingly decomposed. The Yang-Mills equation for  $A$  implies that for  $A_{II}$ .

**Theorem 5.** If a Lie subalgebra  $M$  of  $L$  admits a non-flat constant Yang-Mills potential then so does  $L$ .

In fact, theorem 2 describes a subcase of theorem 5, because every non-Abelian nilpotent Lie algebra contains the 3-dimensional Heisenberg algebra  $H(3)$  as a subalgebra.

Owing to the preceding theorems, low-dimensional Lie algebras  $L$  are to be discussed inductively with respect to their dimension  $N$ . G. M. Mubarakzyanov [2] classified all isomorphy types of real Lie algebras up to the dimension 5. Following him, the notation for a Lie algebra is  $L_{N,k}^{h,\dots}$  which means the  $k$ th non-decomposable algebra of dimension  $N$ . Eventual superscripts  $h, \dots$  stand for the continuous parameters on which the Lie algebra depends. The set of  $N$ -dimensional Lie algebras for  $N \leq 5$  decomposes into five classes with respect to the problem (1) or (2):

1. Decomposable Lie algebras. These can be let aside because for them the Yang-Mills equations are reduced to dimensions smaller than  $N$ .
2. Non-decomposable Lie algebras for which is every constant Yang-Mills potential is trivial:  
 $L_{1,1}, L_{2,2}, L_{3,2}, L_{3,3}^h, L_{3,4}^p, L_{3,5}, L_{3,6}, L_{4,2}, L_{4,4}, L_{4,5}^{\alpha,\beta}, L_{4,6}^{\alpha,\beta}, L_{5,7}^{\alpha,\beta}, L_{5,9}^{\alpha,\beta}, L_{5,11}, L_{5,12}, L_{5,13}^{\alpha,\beta}, L_{5,14}^p, L_{5,15}^{\alpha}(\alpha = 0), L_{5,16}^p, L_{5,17}^{\alpha,\beta}, L_{5,18}^p, L_{5,30}^{\alpha}$
3. Non-decomposable nilpotent Lie algebras which admit non-flat Yang-Mills potentials according to theorem 2:  
 $L_{3,1} \cong H(3), L_{4,1}, L_{5,1}, L_{5,2}, L_{5,3}, L_{5,4}, L_{5,5}, L_{5,6}$
4. Non-decomposable non-nilpotent Lie algebras  $L$  admitting a non-flat constant Yang-Mills potential with values in a non-trivial subalgebra  $M$  of  $L$ :  
 $L_{4,3}, L_{4,7}, L_{4,8}^h, L_{4,9}^p, L_{5,8}^{\alpha}, L_{5,10}, L_{5,13}^{\alpha}(\alpha \neq 0), L_{5,19}^{\alpha,\beta}, L_{5,20}^{\alpha}, L_{5,21}, L_{5,22}, L_{5,23}^{\alpha}, L_{5,24}^{\alpha}, L_{5,25}^{\alpha,\beta}, L_{5,26}^{\alpha,\beta}, L_{5,27}, L_{5,28}^{\alpha}, L_{5,29}, L_{5,30}^{\alpha}, L_{5,31}, L_{5,32}, L_{5,34}, L_{5,35}^{\alpha}, L_{5,36}, L_{5,37}, L_{5,38}, L_{5,39}, L_{5,40}$
5. None of the above:  $L_{4,10}$ .

The structure of  $L_{4,10}$  is given by the specialization of (2) to

$$f^1 = a^1 \wedge a^3 + a^2 \wedge a^4, f^2 = a^2 \wedge a^3 + a^4 \wedge a^1, f^3 = f^4 = 0.$$

An example of an  $L_{4,10}$ -valued Yang-Mills potential reads

$$A = (\epsilon^1 + \epsilon^2) \otimes (X_1 + X_2) + \epsilon^3 \otimes X_3 + \epsilon^4 \otimes X_4.$$

It is properly  $L_{4,10}$ -valued, because all non-trivial subalgebras of  $L_{4,10}$  belong to the class 2.

Few classical Lie algebras  $so(N)$  or  $sl(N, R)$  to the special orthogonal group  $SO(N)$  or the special linear group  $SL(N, R)$  respectively appear in our list:  $L_1 \cong so(2)$ ,  $L_{3,5} \cong so(3)$ ,  $L_{3,6} \cong sl(2, R)$ .

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# PHASE SPACE STRUCTURE IN GAUGE THEORIES AND QUANTUM DYNAMICS

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## Abstract

A phase space (PS) structure in minisuperspace cosmological models with gauge fields is investigated. It is shown for the  $SO(n)$ ,  $n > 3$ , gauge group that the physical PS differs from an ordinary plane. It is argued that the wormhole size quantization should change due to a non-trivial physical PS structure of gauge fields. Some physical consequences of the found phenomenon are briefly reviewed.

1. It is well-known that the main feature of gauge theories is the existence of the first-class constraints [1], i.e. some relations between canonical coordinates  $q^i$  and momenta  $p_i$ ,  $i = 1, 2, \dots, n$ ,  $\varphi_a(q, p) = 0$ , which have to be valid during evolution and satisfy the following conditions  $\{\varphi_a, \varphi_b\} = C_{ab}^c \varphi_c$  where  $\{ \cdot \}$  means the Poisson brackets.

Let the total phase space (PS) of a first-class constrained system be an even-dimensional Euclidean space  $\mathbf{R}^{2n}$ . For constrained systems, a physical trajectory  $q^i = q^i(t)$ ,  $p_i = p_i(t)$  must lie on a manifold in  $\mathbf{R}^{2n}$  determined by the equations  $\varphi_a = 0$ . However, the surface of the first-class constraints does not form a physical PS because solutions of both Hamiltonian equations of motion and constraints depend on the gauge arbitrariness [1]. Indeed, due to the equalities  $\varphi_a = 0$ , we may add to a Hamiltonian  $H$  a linear combination of constraints, i.e., change  $H$  in Hamiltonian equations of motion by the generalized Hamiltonian  $H_E = H + \lambda_a \varphi_a$  where  $\lambda_a$  are arbitrary functions of time [1]. Therefore, solutions  $q^i(t)$ ,  $p_i(t)$  depend on these functions. Variations of  $\lambda_a$  mean a gauge transformation of the solution because the first-class constraints are generators of gauge transformations [1],  $p_i \rightarrow p_i + \delta p_i$ ,  $q^i \rightarrow q^i + \delta q^i$ ,  $\delta q^i = \int \omega_a \varphi_a, q^i$ , analogously for  $\delta p_i$ , where  $\omega_a$  are infinitesimal arbitrary functions of time. So, a choice of a concrete form of  $\lambda_a$  is equivalent to gauge fixing.

Points of the total PS connected by a gauge transformation correspond to the same physical state. By definition there should be a one-to-one correspondence between physical states of a system and points of a physical PS. Thus, in order to determine a physical PS in a first-class constrained system, one should identify all points connected by gauge transformations on the surface of constraints  $\varphi_a = 0$ .

It turns out that many gauge theories including the Yang-Mills ones have a non-trivial structure of the physical PS, i.e. it is not an even-dimensional Euclidean space [2]–[4]. Below we demonstrate this using as an example the so-called Einstein-Yang-Mills minisuperspace models and briefly discuss some dynamical consequences of this phenomenon.

2. Dynamical variables in the Einstein-Yang-Mills system are gauge potentials  $A_\mu(\mathbf{x})$  and a metric tensor  $g_{\mu\nu}(\mathbf{x})$ . This system is rather difficult by itself. Following, however, Ref.[5] one may introduce a set of simplifying assumptions and consider closed cosmologies with an  $\mathbf{R} \odot \mathbf{S}^3$  topology. In this case gauge fields on homogeneous space are described by the  $SO(4)$ -invariant ansatz [6]–[8]. The reduced system (the minisuperspace model) contains only a finite number of degrees of freedom corresponding to gravitational and gauge fields. Namely, gauge fields with the  $SO(n)$  group,  $n > 3$ , are described by a scalar  $\chi = \chi(t) \in \mathbf{R}$ , a vector  $\mathbf{x} = \mathbf{x}(t) \in \mathbf{R}^l$ ,  $l = n - 3$ , and a real antisymmetric  $l \times l$  matrix  $y = y(t)$ , i.e.  $y = y_a T^a$ ,  $y_a \in \mathbf{R}$ ,  $T^a$  are generators of  $SO(l)$ , so that the effective minisuperspace action of the Einstein-Yang-Mills system reads [8],[9]

$$S = \frac{1}{2} \int_{t_1}^{t_2} dt \frac{N}{\varrho} \left[ - \left( \frac{\varrho}{N} \partial_t \varrho \right)^2 + \varrho^2 - \lambda^2 \varrho^4 + \left( \frac{\varrho}{N} \partial_t \chi \right)^2 + \left( \frac{\varrho}{N} D_t \mathbf{x} \right)^2 - 2V \right] \quad (1)$$

where  $N = N(t)$  and  $\varrho = \varrho(t)$  describe gravitational degrees of freedom;  $D_t = \partial_t + y$  is the covariant derivative,  $\lambda^2 = \text{const}$  (it goes from the cosmological term);  $V = V(\chi, \mathbf{x}^2)$  is a potential induced by self-interaction of the Yang-Mills fields, its explicit form is not essential for what follows.

The action (1) is invariant under two local groups, the reparametrization one

$$t \rightarrow t'(t), \quad N(t) \rightarrow N(t') \frac{dt'}{dt} \quad (2)$$

and the  $SO(l)$  gauge group, under which only variables  $\mathbf{x}$  and  $y$  transform as follows

$$\mathbf{x} \rightarrow \Omega \mathbf{x}, \quad y \rightarrow \Omega y \Omega^T + \Omega \partial_t \Omega^T, \quad (3)$$

where  $\Omega = \exp \omega_a T^a \in SO(l)$ ,  $\Omega \Omega^T = \Omega^T \Omega = 1$  and  $\omega_a = \omega_a(t)$  are arbitrary functions of time; other variables remain unchanged.

Varying the action  $S$  with respect to the velocities  $\dot{N}$ ,  $\dot{y}_a$ ,  $\dot{\varrho}$ ,  $\dot{\mathbf{x}}$  and  $\dot{\chi}$ , we find canonical momenta  $p_N$ ,  $p_a$ ,  $p_\varrho$ ,  $\mathbf{p}$  and  $p_\chi$ , respectively, and then the canonical Hamiltonian

$$H = \frac{N}{2\varrho} \left[ -p_\varrho^2 - \varrho^2 + \lambda^2 \varrho^4 + p_\chi^2 + \mathbf{p}^2 + 2V - y_a \sigma^a \right] \equiv \frac{N}{\varrho} [H_{WD} - y_a \sigma^a] \quad (4)$$

where  $\sigma^a = \mathbf{p} T^a \mathbf{x}$ . Due to two local symmetries of  $S$  (2) and (3), the system has the primary constraints [1]  $p_N = p_a = 0$  ( $S$  is independent of  $\dot{N}$  and  $\dot{y}_a$ ) and the secondary ones [1] which are equivalent to the following

$$H_{WD} = 0, \quad (5)$$

$$\sigma^a = 0. \quad (6)$$

All the constraints are the first-class ones. As a consequence, the Hamiltonian vanishes, which is always the case for systems with a reparametrization symmetry.



Eq.(5) is the classical Wheeler-DeWitt equation for the minisuperspace model. The constraints (6) generate the  $SO(l)$  gauge transformations of the canonically conjugated variables  $\mathbf{x}$  and  $\mathbf{p}$ . However, not all of them are independent. The number of independent constraints is  $l - 1$  since any vector in  $\mathbf{R}^l$  has a stationary subgroup  $SO(l - 1)$ .

**3.** Consider now the physical PS structure of gauge field degrees of freedom. Obviously, the total PS of these variables consists of points  $(\mathbf{x}, \mathbf{p}) \in \mathbf{R}^{2l}$  and  $(\chi, p_\chi) \in \mathbf{R}^2$  (we ignore the pure unphysical degrees of freedom  $(y_a, p_a = 0)$ ). The physical PS is a subspace of the total PS picked out by the constraints  $\sigma_a = 0$  and an identification of all points connected by gauge transformations on the surface of the constraints.

Variables  $\chi$  and  $p_\chi$  are gauge-invariant therefore their PS is a usual plane,  $\mathbf{R}^2$ . The general solution of Eq.(8) is  $\mathbf{p} = \xi \mathbf{x}$  where a function of time  $\xi$  is determined by dynamics (by the potential  $V$ ). This solution means that only radial excitations are admissible. Further, one may always direct a vector  $\mathbf{x}$  along one of the coordinate axes with the help a gauge transformation (the unitary gauge), for example, we put  $x_i = \delta_{i1}x$ . As a consequence,  $p_i = \delta_{i1}p$ ,  $p = \xi x$ . However, this is not the end. There remain residual gauge transformations forming the  $\mathbf{Z}_2$  gauge group with the help of which one may change the sign of  $x$  :  $x \rightarrow \pm x$  (the gauge rotations through the angle  $\pi$ ).

The residual gauge group cannot decrease a number of physical degrees of freedom, but it reduces their PS. Indeed, the sign of  $p$  should change simultaneously with that of  $x$  due to the equality  $p = \xi x$ . Hence the points  $(x, p)$  and  $(-x, -p)$  on the phase plane  $\mathbf{R}^2$  are gauge equivalent and should be identified. The phase plane turns into a cone unfoldable into a half-plane that is just the physical PS because the gauge arbitrariness is exhausted.

**4.** The PS structure modification leads to some dynamical consequences in classical and quantum theories. In particular, periods of periodic motion depend on the PS structure, which is easy to see in our model.

It was shown in Ref.[9] that in the unitary gauge  $x_i = \delta_{i1}x$  there exist periodic solutions  $x(\eta)$ ,  $\varrho(\eta)$  and  $\chi(\eta)$  with periods  $T_x$ ,  $T_\varrho$  and  $T_\chi$ , respectively. If we interpret the solution  $\varrho(\eta)$  as a wormhole connecting two points in the same space, the gauge fields should be the same at both the sides. Since  $\chi(\eta)$  and  $x(\eta)$  are periodic, the period  $T_\varrho$  (the time between two  $\varrho$ -maxima) should be an integer multiple of their periods [6], i.e.  $T_\varrho = nT_\chi = mT_x$  where  $n$  and  $m$  are numbers. The last relation leads to the exponential quantization of a wormhole size [6],[9].

However, the PS of  $x$  is a cone unfoldable into a half-plane. It means that for  $x = x(\eta)$  oscillating around  $x = 0$  [4], the physical period is  $T_x^{ph} = T_x/2$  because points  $x < 0$  are gauge equivalent to points  $x > 0$ ;  $T_x^{ph}$  is the time during which the system returns to an initial physical state. Therefore the real quantization rule of wormholes reads

$$T_\varrho = nT_\chi = mT_x^{ph} = \frac{m}{2}T_x. \quad (7)$$

As a consequence, the quantization of the wormhole size is also modified.

For theories with an arbitrary gauge group, periods of oscillating physical degrees of freedom are defined by powers of the independent Casimir operators for a given representation because the residual gauge group identifying some points in a physical PS is the Weyl group [3],[4]. As a consequence, the physical PS structure influences directly the WKB-quantization method since frequencies of periodic motion determine quantum energy levels in this case [3],[4],[10].

In quantum theory, the path integral approach for an evolution operator kernel depends on the PS structure [3],[4],[11], which leads to modifications of a quasiclassical approximation [3],[4],[10]. It turns out that a solution of the problem itself of a path integral construction for the first-class constrained systems with the reduced physical PS gives automatically an approach for correct solving Gribov's problem [12],[4]. Further, quantum Green functions have unusual analytical properties so that elementary excitations of degrees of freedom with the reduced physical PS cannot normally propagate [13]. For the considered minisuperspace model, the found PS reduction gives rise also to a modification of the path integral representation for the ground-state wave function of the Universe and, as a consequence, its quasiclassical calculations should be modified [14]. In conclusion, we would like to note that the phenomenon of the physical PS reduction may take place for fermionic (Grassman) degrees of freedom too [4]. However, there are specific dynamical features in this case [4],[15].

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## Quantized Fiber Dynamics for Extended Elementary Objects

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### ABSTRACT

A dynamical feedback mechanism between quantized matter and an underlying fiber bundle geometry with  $SO(4,1)$  as structural group is investigated in the presence of gravitation. Spinless matter, described in a quantum mechanical manner, is represented in terms of generalized wave functions (sections),  $\Psi_x^{(\rho)}(\xi, \zeta)$ , defined on a Hilbert bundle  $\mathcal{H}$  over space-time carrying a local phase space representation of  $SO(4,1)$  related to the principal series of UIR determined by  $\rho$ . Bilinear currents in the matter fields are introduced which act as source currents for the bundle geometry inducing further geometric quantities (torsion, de Sitter boost-fields) which specify the geometry beyond the classical metric determined through Einstein's equations.

In order to describe quantized matter in the presence of gravitational fields generated by distant macroscopic classical masses one introduces a Hilbert bundle  $\mathcal{H}$  over a curved space-time base manifold  $B$ . (See [1] and the earlier references quoted there.) For the description of leptons one would consider a bundle  $\mathcal{H}$  carrying a system of covariance of the Poincaré group. The typical fiber of  $\mathcal{H}$  would in this case be a Hilbert space providing a unitary irreducible phase space representation of the inhomogeneous Lorentz group,  $ISO(3,1)$ , yielding a quantum mechanical description of (spinless or spin  $\frac{1}{2}$ ) particles of mass  $m$  in terms of square-integrable functions depending on the local fiber variables  $(q, p)$  with  $q \in T_x(B)$  and  $p \in V_m^\pm \subset T_x^*(B)$ , where  $V_m^\pm$  is the positive (+) or negative (-) energy shell in momentum space, with  $p^2 = m^2$ , and  $T_x(B)$  denotes the affine tangent space of  $B$  at  $x$  on which the Poincaré group acts as a group of motions and  $T_x^*(B)$  denotes the cotangent space at  $x$  (compare [2]). In a geometric framework intended for the description of hadrons which are characterized by an intrinsic length parameter  $R$  of the order of  $10^{-13}$  cm and a corresponding spacial extension one would have to change the phase space bundle  $E'(B, F' = M_4 \times V_m^\pm, G' = ISO(3,1))$  used in the Poincaré case with  $M_4$  denoting Minkowski space (viewed as an affine space), by "curling up" the flat fiber  $M_4$  isomorphic to the homogeneous space  $ISO(3,1)/SO(3,1)$ , and go over to a curved fiber  $G/H$  isomorphic to a space-time  $V_4'$  of constant curvature with curvature radius  $R \approx 10^{-13}$  cm. Here  $H = SO(3,1)$  denotes the Lorentz group, required to be a closed subgroup of  $G$ , with respect to which  $G$  contracts to the Poincaré group in a formal limit  $R \rightarrow \infty$ . Hence  $G$  is one of the ten parameter de Sitter groups  $SO(4,1)$  or  $SO(3,2)$ . Let us choose on physical grounds (spacial compactness and absence of closed timelike geodesics) the (4,1)-de Sitter group as the relevant group for the description of extended hadrons and introduce the de Sitter phase space bundle  $\tilde{E} = \tilde{E}(B, \tilde{F} = V_4' \times C^\pm, G = SO(4,1))$  over  $B$ , which is a soldered bundle [3] associated to the de Sitter frame bundle  $P(B, G = SO(4,1))$  over  $B$ . Here the soldering is made through the de Sitter subspace  $V_4'$  of the local phase space  $\tilde{F}_x$  isomorphic to  $\mathcal{N}^\pm = V_4' \times C^\pm$  (compare [4]) with  $V_4'(x)$  and  $B$  having first order contact at each  $x \in B$ .  $C^\pm$  denotes the cone in the

embedding five-dimensional Lorentzian space  $R_{4,1}$  (in which  $V'_4$  is a one-shell hyperboloid of radius  $R$ ) with the fifth coordinate normalized to  $1/R$ , i.e.

$$\begin{aligned} C^\pm &: [\zeta, \zeta] = \zeta^a \zeta^b \eta_{ab} = 0 \quad \text{with } \zeta^5 = \frac{1}{R}, \\ V'_4 &: [\xi, \xi] = \xi^a \xi^b \eta_{ab} = -R^2, \end{aligned}$$

where  $\eta_{ab} = \text{diag}(1, -1, -1, -1, -1)$  and the summations run over  $a, b = 0, 1, 2, 3, 5$ . The superscript of  $C^\pm$  stands for sign  $\zeta^0 = \pm$  with the vector  $\zeta = (\zeta^i, \zeta^5 = \frac{1}{R})$ ;  $i = 0, 1, 2, 3$ , characterizing a so-called horosphere or horocycle through the origin  $\xi = (0, 0, 0, 0, -R)$  of  $V'_4$  determining thereby a direction of a whole wave field given by an infinite set of horospheres parallel to the one through  $\xi$ . (Compare [1] and [5] and the meaning of a horosphere as a wave surface of constant phase in a space of constant curvature.) Hence  $\zeta \in C^\pm$  parametrizes the space of horospheres and thus the space of horospherical wave solutions of the  $G$ -invariant wave equation on  $V'_4$  used in [1] and [5]. In the Inönü-Wigner contraction limit  $SO(4, 1) \xrightarrow{R \rightarrow \infty} ISO(3, 1)$  sign  $\zeta^0$  goes over into sign  $p^0$  labelling the positive or negative mass shell in the Poincaré case [5].

Let us now construct a *phase space representation* of the de Sitter group for spinless particles,  $\tilde{U}(A_g) = \tilde{U}^{(\rho)}(A_g)$ , related to the unitary irreducible representation (UIR) of the principal series of  $SO(4, 1)$  characterized by the parameter  $\rho$ ;  $0 \leq \rho < \infty$ , with  $\rho$  determining the mass of the particle in question [see below]. The Hilbert space  $L^2(\tilde{\Sigma}_x^\pm)$  of square-integrable functions in the variables  $(\xi, \zeta) \in \tilde{\Sigma}^\pm \subset \mathcal{N}^\pm$  with respect to a  $G$ -invariant measure,  $d\tilde{\Sigma}(\xi, \zeta)$  [compare Eq. (4) below], is denoted by  $\mathcal{H}_\eta^{(\rho)}$ . Here  $\tilde{\Sigma}^\pm = h \times C^\pm$  is a six-dimensional horospherical submanifold [6], [1], where  $h$  denotes a horosphere being a spacelike hypersurface in  $V'_4$ . (Compare also [7].) One can construct a coherent state basis of  $\mathcal{H}_\eta^{(\rho)}$  in terms of horospherical waves from  $SO(3)$ -invariant resolution generators  $\tilde{\eta}(\zeta')$  yielding a parametrization of the basis of  $\mathcal{H}_\eta^{(\rho)}$  in terms of the coset space  $SO(4, 1)/SO(3)$ . In [1] it is shown that  $\mathcal{H}_\eta^{(\rho)}$  is a one-particle resolution kernel Hilbert space with  $\mathcal{H}_\eta^{(\rho)} = \mathcal{H}^+ \oplus \mathcal{H}^-$ , where the superscripts  $+$  and  $-$  denote the sign of  $\zeta^0$ , with  $\mathcal{H}^+$  and  $\mathcal{H}^-$  denoting the one-particle and one-antiparticle Hilbert spaces, respectively.

As a geometric arena for the description of spinless quantized matter we now introduce the soldered (first quantized) Hilbert bundle  $\mathcal{H} = \mathcal{H}(B, \mathcal{F} = \mathcal{H}_\eta^{(\rho)}, \tilde{U}(A_g))$ , associated to  $P(B, SO(4, 1))$ , with standard fiber  $\mathcal{H}_\eta^{(\rho)}$  carrying a system of covariance of the  $(4, 1)$ -de Sitter group. Denoting the generalized coherent state basis of the local fiber  $\mathcal{H}_\eta^{(\rho)}(x)$  which is related to a particular choice of gauge  $\sigma = u(x)$  on  $P$  by  $\Phi_{\xi, \zeta}^{u(x)}$ , with  $x \in B$ ;  $(\xi, \zeta) \in \tilde{\Sigma}_x^\pm$ , constituting a *local de Sitter quantum frame*, one has the following resolution of unity at  $x \in B$  [parameter  $\rho$  is suppressed]:

$$\int_{\tilde{\Sigma}_x^\pm} |\Phi_{\xi, \zeta}^{u(x)}\rangle d\tilde{\Sigma}(\xi, \zeta) \langle \Phi_{\xi, \zeta}^{u(x)}| = \mathbf{1}_x^\pm, \quad (1)$$

and the expansion of a state vector  $\Psi_x^{(\rho)\pm}$  belonging to the principal series of UIR's of  $SO(4,1)$  with respect to the local basis  $\Phi_{\xi,\zeta}^{u(x)}$ :

$$\Psi_x^{(\rho)\pm} = \int_{\tilde{\Sigma}^\pm} d\tilde{\Sigma}(\xi, \zeta) \Psi_x^{(\rho)}(\xi, \zeta) \Phi_{\xi,\zeta}^{u(x)}. \quad (2)$$

Here  $d\tilde{\Sigma}(\xi, \zeta)$  is the  $G$ -invariant measure [1]

$$d\tilde{\Sigma}(\xi, \zeta) = \frac{1}{R^2} \frac{1}{[\xi, \zeta]^2} \delta(|[\xi, \zeta]| - c) d\mu(\xi) \delta([\zeta, \xi]) d^4\zeta \quad (3)$$

obeying  $d\tilde{\Sigma}(A_g\xi, A_g\zeta) = d\tilde{\Sigma}(\xi, \zeta)$  with  $A_g \in SO(4,1)$ . In (3) the quantity  $d\mu(\xi) = (R/|\xi^5|) d\xi^0 d\xi^1 d\xi^2 d\xi^3$  denotes the invariant measure on  $V_4'$  and  $c$  is a positive constant determining a particular horosphere  $H_\zeta^c$  in  $V_4'(x)$  parallel to the horosphere  $H_\zeta^1$  through  $\xi$  characterized by  $\zeta$ .

The coefficient  $\Psi_x^{(\rho)}(\xi, \zeta)$  in the expansion (2) is the *scalar de Sitter coordinate wave function*, or for short, the *generalized matter wave function* which may be regarded as a section on the first quantized de Sitter bundle  $\mathcal{H}$  over space-time with standard fiber  $\mathcal{H}_{\tilde{\eta}}^{(\rho)}$ . The variables  $(\xi, \zeta)$  in the local fiber of the bundle  $\tilde{E}$  play the rôle of local stochastic variables on  $\mathcal{H}$ . [Second quantized bundles with Fock-space fibers constructed with tensor products of one-particle spaces  $\mathcal{H}^+$  and one-antiparticle spaces  $\mathcal{H}^-$  were considered in [1]. For simplicity we shall, however, base the definition of the bilinear currents in the fields  $\Psi_x^{(\rho)}(\xi, \zeta)$  and their adjoints on the first quantized bundle  $\mathcal{H}$  introduced above.] Eq. (2) can be reversed and yields using a bracket notation for the integration over the hypersurface  $\tilde{\Sigma}_x^\pm$  with measure (3):

$$\Psi_x^{(\rho)}(\xi, \zeta) = \langle \Phi_{\xi,\zeta}^{u(x)} | \Psi_x^{(\rho)\pm} \rangle_{\tilde{\Sigma}_x^\pm}. \quad (4)$$

The transformation rule for  $\Psi_x^{(\rho)}(\xi, \zeta)$  under de Sitter gauge transformations  $A_{g(x)}$  on  $\mathcal{H}$  is  $(\tilde{U}(A_{g(x)})\Psi_x^{(\rho)})(\xi, \zeta) = \Psi_x^{(\rho)}(A_{g(x)}^{-1}\xi, A_{g(x)}^{-1}\zeta)$ .

From the construction of the coherent state basis described in more detail in [1] it is apparent that  $\Psi_x^{(\rho)}(\xi, \zeta)$  is, for fixed  $x \in B$  and  $\zeta \in C^\pm$ , an eigenfunction of the Laplace-Beltrami operator<sup>1)</sup>,  $\square_\xi = \frac{1}{2R^2} L_{ab}(\xi) L^{ab}(\xi)$ , on  $V_4'(x)$  with eigenvalue  $\kappa(\kappa+3)/R^2$  where  $\kappa = -\frac{3}{2} + i\rho$ ;  $0 \leq \rho < \infty$ , i.e. [5]

$$\left( \square_\xi + \frac{2}{R^2} + \left[ \frac{mc}{\hbar} \right]^2 \right) \Psi_x^{(\rho)}(\xi, \zeta) = 0. \quad (5)$$

This yields the following relation between the constant  $R$ , the mass  $m$  of the particle and the value of  $\rho$ :  $\left[ \frac{mc}{\hbar} \right]^2 R^2 = \rho^2 + \frac{1}{4}$ . For fixed  $R$  the parameter  $\rho$  is thus a variable determining a particular mass value.

We, finally, quote the kernel for the propagation from the point  $(\xi, \zeta)$  to the point

<sup>1)</sup>  $L_{ab}(\xi) = i \left( \xi_a \frac{\partial}{\partial \xi^b} - \xi_b \frac{\partial}{\partial \xi^a} \right)$ ;  $L^{ab}(\xi) = \eta^{ac} \eta^{bd} L_{cd}(\xi)$ .

$(\xi', \zeta')$  in the local fiber over  $x \in B$  in  $\mathcal{H}$ :

$$\tilde{K}_{\tilde{\eta}, x}^{(\rho)}(\xi', \zeta'; \xi, \zeta) = \langle \Phi_{\xi', \zeta'}^{\mathbf{u}(x)} | \Phi_{\xi, \zeta}^{\mathbf{u}(x)} \rangle_{\tilde{\Sigma}_x^\pm}, \quad (6)$$

and the reproducing property following from (1)

$$\tilde{K}_{\tilde{\eta}, x}^{(\rho)}(\xi', \zeta'; \xi, \zeta) = \int_{\tilde{\Sigma}_x^\pm} \tilde{K}_{\tilde{\eta}, x}^{(\rho)}(\xi', \zeta'; \xi'', \zeta'') \tilde{K}_{\tilde{\eta}, x}^{(\rho)}(\xi'', \zeta''; \xi, \zeta) d\tilde{\Sigma}(\xi'', \zeta''). \quad (7)$$

So far we have prepared the ground for a quantum *kinematical* description of spinless matter described in terms of generalized wave functions defined on the first quantized de Sitter bundle  $\mathcal{H}$  constructed over curved space-time in the presence of gravitation. Let us now introduce a dynamical feed back between quantized matter present in the geometry and the imprint this leaves on the underlying bundle geometry. To this end we introduce bilinear source currents, constructed in terms of the quantum fields  $\Psi_x^{(\rho)}(\xi, \zeta)$  and their adjoints, and use them as material source currents determining the bundle geometry beyond the metric part which is determined by the energy-momentum distribution of classical matter as in Einstein's theory. This then yields a *quantum fiber dynamics* (QFD) on  $\mathcal{H}$  determined through covariant current-curvature field equations [8] characterizing the short distance behaviour of quantized matter in the presence of gravitation in a framework based on the (4,1)-de Sitter group as a gauge group.

To recover gravitation in a de Sitter gauge invariant manner as the gauge theory of the Lorentz subgroup  $H$  one has now first to go over to a *nonlinear realization* of the transformations of the de Sitter group in terms of "Wigner rotations" being in this case Lorentz transformations belonging to the stability subgroup  $H$  of the point  $\xi$  in the fiber. This is done by introducing a section  $\xi^a(x)$ ;  $a = 0, 1, 2, 3, 5$ , on the (soldered) de Sitter bundle  $E = E(B, F = G/H \simeq V_4, SO(4, 1))$  and reducing the connection  $\omega^R$  in  $P$  with the help of the boost transformation  $A(\xi(x)) : \xi \rightarrow \xi(x)$  to a nonlinearly transforming form. We call this the nonlinear (N.L.) gauge on  $P$  denoted by

$$[\sigma_{N.L.}^* \omega^R]^a_b = \begin{pmatrix} [\omega^R]_j^i & \theta^{Ri} \\ \theta^R_j & 0 \end{pmatrix} \quad \text{for } a = (i, 5); b = (j, 5). \quad (8)$$

The r.h. side of (8) remains form-invariant under gauge transformations on  $E : \xi(x) \rightarrow \xi^l(x) = A_{g(x)}\xi(x)$  with the Latin indices suffering a transformation with a matrix  $\Lambda(\xi^l(x), \xi(x)) \in H$ . The one-forms  $\theta_j^R$ ;  $j = 0, 1, 2, 3$ , in (8) are the soldering forms of the Cartan connection in  $P$  in the N.L. gauge. The de Sitter frame bundle  $P$  introduced here and the Lorentz frame bundle appearing in a vierbein formulation of gravitation may, in the N.L. gauge, be viewed as *interlocked bundles* with the Lorentz part  $[\omega^R]_{ij}$  given by  $[\omega^R]_{ij} = \bar{\omega}_{ij} + \tau_{ij}^R$ , where  $\bar{\omega}_{ij}$  is the usual Lorentz connection appearing in a purely metric theory and  $\tau_{ij}^R$  is the torsion addition resulting from the embedding of the Lorentz gauge symmetry into a bigger (here de Sitter) gauge symmetry. The additional geometric quantities characterizing the geometry in  $P(B, SO(4, 1))$  beyond the metric are thus the torsion induced in the base, turning  $B$  into a Riemann-Cartan space-time  $U_4$ , and the coefficients of the soldering forms of the bundle connection representing de Sitter boost degrees of freedom.

In [8] a set of covariant bilinear hermitean currents of the type

$$J_{kab}^{(\rho)}(x) = \langle \Psi_x^{(\rho)}(\xi, \zeta) | \hat{O}_{kab} | \Psi_x^{(\rho)}(\xi, \zeta) \rangle_{\tilde{\Sigma}_x^\pm} \quad (9)$$

are introduced using  $G$ -invariant integration over the local fibers of  $\mathcal{H}$ , where  $\hat{O}_{kab}$ ;  $k = 0, 1, 2, 3$ ;  $a, b = 0, 1, 2, 3, 5$ , are operators constructed with the infinitesimal generators  $\tilde{M}_{ab}$  of the phase space representation  $\tilde{U}(A_g)$  of  $SO(4, 1)$  and the operators  $D_k^R$  for covariant differentiation on  $\mathcal{H}$ . These currents are now used as material source currents in a set of de Sitter gauge covariant nonlinear field equations for the torsion and the de Sitter boost fields specifying the geometry beyond the metric determined from Einstein's equations as in general relativity. In [8] the axial vector torsion case is treated together with the restriction that the de Sitter boost fields  $[\sigma_{N.L.}^* \omega^R]_{i5} = \theta_i^R = \theta^k \Gamma_{ki5}^R$  are antisymmetric  $[\Gamma_{ki5}^R = -\Gamma_{ik5}^R]$ , where  $\theta^k$  are the fundamental one-forms on  $B$ . For this particular case the field equations together with the Bianchi identities for the bundle curvature are investigated and discussed and the consequences worked out in conjunction with Einstein's equations for the metric.

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# Conformal Geometry and Spacetime Gauge Theories

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## Abstract

Geometric properties of classical gauge theories of  $C = O_0(4,2)$  are investigated within the framework of first and second order  $G$ -structures. In particular we discuss difficulties in conformal gauge theories arising from the non-reductivity of the isotropy subgroup in  $C$ , in comparison with the reductive Poincaré- and deSitter/anti-deSitter case.

## 0 Introduction

Consider a relativistic metric or conformal transitive spacetime group of at least 1. order, i.e. a group  $S$  of automorphisms of a Lorentz- or (Lorentz-) conformal manifold  $M$ , acting transitively on  $M$  ('homogeneity') and containing the Lorentzgroup  $L := O_0(3,1)$ <sup>1</sup> ('space and time isotropy'). Then one knows that  $S$  belongs, up to local isomorphisms, to the following list

| metric case                      | conformal case                                            |
|----------------------------------|-----------------------------------------------------------|
| $P := T^4 \otimes L$ (Poincaré-) | $P^{11} := T^4 \otimes (L \oplus D)$ (Poincaré-Weyl-) (1) |
| $O_0(4,1)$ (deSitter-)           | $C := O_0(4,2)$ (conformal group)                         |
| $O_0(3,2)$ (anti-deSitter-)      |                                                           |

Taken as candidates for spacetime *gauge* groups, there is a large number of investigations for the first four of these groups, and a remarkably smaller one in the case of  $C$ . From the mathematical point of view, two main structural differences between  $C$  and the remaining groups in (1) might be responsible for this fact :

- i)  $C$  is the only 'second order' group in (1)
- ii)  $C$  is the only group in (1) with *non-reductive* isotropy subgroup.

The first point leads to 2. order conformal  $G$ -structures (see sect.1), which, compared with 1. order conformal structures (gauge group  $P^{11}$ ), might be seen as unnecessarily complicated tools to represent conformal geometry; and non-reductivity

<sup>1</sup> $O_0(n,m) :=$  connected component of  $O(n,m)$ .

generates problems concerning the mathematical representation of gravitative potentials via connection forms (see sect.2).

On the other hand,  $C$  has the richest structure of the groups in (1) : all of them are subgroups of  $C$ . Hence, for a systematic investigation of spacetime gauge theories, a natural starting point seems to be the choice of  $C$  as a gauge group, possibly with appropriate symmetry breaking to one of its subgroups. We indicate and motivate a geometrical framework of a gauge scheme for  $C$ , i) exploiting fully the second order property of  $C$  and ii) accepting the non-reductivity in  $C$  as a hint for a deeper analysis of the nature of conformal gauge potentials.

Sects.1 and 2 sketch the role of the group  $C$  as a background structure which in an obvious way generates a conformal gauge bundle with Cartan connections  $\tau$  as natural candidates for gauge potentials. In sects.3, 4 and 5 a geometrical interpretation of these potentials is indicated and a symmetry breaking from conformal to Lorentz structures is performed which results in a *Riemann-Cartan-spacetime* (proofs and details of these constructions will be given elsewhere).

## 1 Conformal geometry (1. and 2. order structures)

We sketch the occurrence of 2. order conformal structures with the help of 'conformal coordinate systems' :

Consider the group  $C$  (locally diffeomorphic to  $T^4 \times (L \otimes D) \times K^4$ , with translations  $T^4$ , dilatations  $D$ , and special conformal transformations  $K^4 \cong T^4$ ) as a fibre bundle with 'extended Weyl group'  $W^{11} := (L \otimes D) \otimes K^4$  as structure group. The factorization via *right* multiplication of  $W^{11}$  on  $C$  gives the homogeneous space  $M^4 := C/W^{11}$ . As an intermediate step we get the  $W^7$ -bundle  $C/K^4$  over  $M^4 (= (C/K^4)/W^7)$  with  $W^7 := L \otimes D$ , the 'Weyl group',

$$C \xrightarrow{\pi_2} C/K^4 \xrightarrow{\pi_1} M^4. \quad (2)$$

*Left* multiplication of  $W^{11}$  on  $C$  then defines on  $M^4$  for each  $a \in W^{11}$  the 'conformal coordinate system'

$$\alpha_a : \mathbb{R}^4 = T^4 \rightarrow aT^4 \xrightarrow{\pi} M^4, \quad \pi := \pi_1 \circ \pi_2. \quad (3)$$

centered at  $x_0 := \pi(\mathbf{1}) = \alpha_a(0) \in M^4$ . Distinguishing 1. and 2. order contact classes, each  $\alpha_a$  induces a 'conformal linear frame' at  $x_0$ , the 1. order jet  $j_0^1(\alpha_a)$  of  $\alpha_a$  (Lorentz frame at  $x_0$  up to conformal factor) and a 'conformal 2. order frame' at  $x_0$ , the 2. order jet  $j_0^2(\alpha_a)$ . Obviously, the collection of all  $j_0^1(\alpha_a)$ ,  $a \in W^{11}$ , induces a conformal structure (light cone structure) on the tangent space  $T_{x_0}M^4$ . Performing this procedure for all  $a \in C$ , we get a conformal structure  $[\eta]$  on all of  $M^4$ , i.e. a Lorentz metric up to positive functions on  $M^4$ , and, via  $C \ni a \mapsto \alpha_a$  a natural identification of the  $W^{11}$ -bundle  $C$  with the 2. order conformal frame

bundle  $L_{[\eta]}^2 M^4$  [6], [2]. Similarly to (3), one defines (local) 'conformal coordinate transformations'

$$\varphi_a := \alpha_1^{-1} \circ \alpha_a : \mathbb{R}^4 = T^4 \rightarrow \mathbb{R}^4 = T^4, \quad a \in W^{11}. \quad (4)$$

For  $a = (a^i) \in K^4 \subset W^{11}$ ,  $\varphi_a$  locally is given by the usual special conformal transformation  $x^i \mapsto \bar{x}^i = (x^i + a^i x^2)/(1 + a_k x^k + a^2 x^2)$ , hence with trivial 1. order action in  $0 \in \mathbb{R}^4$ . This implies a natural identification of the  $W^7$ -bundle  $C/K^4$  with the 1. order conformal frame bundle  $L_{[\eta]} M^4$ ,

$$\begin{array}{ccc} C & \equiv & L_{[\eta]}^2 M^4 \subset L^2 M^4 \quad (2. \text{ order frame bundle}) \\ \downarrow & & \downarrow \pi_2 \\ C/K^4 & \equiv & L_{[\eta]} M^4 \subset LM^4 \quad (\text{linear frame bundle}) \\ \downarrow & & \downarrow \pi_1 \\ C/W^{11} & \equiv & M^4 \quad M^4 \end{array} \quad (5)$$

(A corresponding discussion for the Poincaré group leads to the identification of  $P$ , considered as an  $L$ -bundle over  $\mathbb{R}^4 = T^4 = P/L$ , with the 1. order bundle  $L_\eta \mathbb{R}^4$  of orthonormal frames in Minkowskispac ( $\mathbb{R}^4, \eta$ ). Similarly for the '1. order groups'  $O_0(3,2)$ ,  $O_0(4,1)$  and  $P^{11}$ .)

## 2 Cartan connections

The application of conformal coordinate systems  $\alpha_a$  for the interpretation of  $C$  as a 2. order  $G$ -structure extends if one introduces 'geometry':

Consider first the case of  $P \equiv L_\eta \mathbb{R}^4$ , with 'orthonormal coordinate systems'  $\alpha_a : \mathbb{R}^4 = T^4 \rightarrow aT^4 \rightarrow P/L$ ,  $a \in P$ , instead of (3). The Maurer-Cartan-form on  $P$  then splits into a translational and a Lorentzian part,  $\overset{\circ}{\tau} = \overset{\circ}{\theta} + \overset{\circ}{\lambda}$ , and the tangent space in  $a \in P$  of the submanifold  $aT^4 \subset P$  coincides with the kernel  $\ker \overset{\circ}{\lambda}_a$  of the 1-form  $\overset{\circ}{\lambda}_a$  on  $T_a P$ . Because of

$$T^4 l = l T^4 \quad \text{for } l \in L, \quad (6)$$

this means right- $L$ -equivariance of the 'horizontal subspaces'  $\ker \lambda_l$ : hence  $\overset{\circ}{\lambda}$  is a connection form on  $P \equiv L_\eta \mathbb{R}^4$ . In Lie algebra terms (6) implies  $[l, \mathfrak{t}^4] \subset \mathfrak{t}^4$ , i.e. the Lorentz algebra  $l$  is reductive in the Poincaré algebra  $\mathfrak{p}$ .

In contrast to  $P$ , the 'isotropy part'  $\overset{\circ}{\lambda} + \overset{\circ}{\delta} + \overset{\circ}{\kappa}$  of the Maurer-Cartan-form  $\overset{\circ}{\tau} = \overset{\circ}{\theta} + \overset{\circ}{\lambda} + \overset{\circ}{\delta} + \overset{\circ}{\kappa}$  on  $C$ , with dilatation part  $\overset{\circ}{\delta}$  and special conformal part  $\overset{\circ}{\kappa}$ , is not a connection form on  $C \equiv L_{[\eta]}^2 M^4$ , because of the non-reductivity of  $\mathfrak{w}^{11}$  in  $\mathfrak{c}$  ( $[\mathfrak{w}^{11}, \mathfrak{t}^4] = l \oplus \mathfrak{d} \not\subset \mathfrak{t}^4$ ). However, extending  $L_{[\eta]}^2 M^4$  to the 'affine' 2. order frame bundle  $A_{[\eta]}^2 M^4 := L_{[\eta]}^2 M^4 \times_{W^{11}} C$  (associated bundle with respect to the  $W^{11}$ -left-multiplication on  $C$  and, moreover, a principal  $C$ -bundle over  $M^4$ ) with natural embedding

$$k : L_{[\eta]}^2 M^4 \rightarrow A_{[\eta]}^2 M^4, \quad (7)$$

$\overset{\circ}{\tau}$  extends uniquely to a (flat) connection form  $\overset{\circ}{\tau}^-$  on  $A_{[\eta]}^2 M^4$ , s.th.  $\overset{\circ}{\tau} = k^* \overset{\circ}{\tau}^-$ ; i.e.  $\overset{\circ}{\tau}$  is a Cartan connection [6] on  $L_{[\eta]}^2 M^4$ , invariant with respect to automorphisms of  $L_{[\eta]}^2 M^4$  given by left multiplications of  $a \in C$ . Observe the nonsingularity of the  $\mathfrak{t}^4$ -valued 1-form  $\overset{\circ}{\theta}$ , which coincides with the pull back via  $\pi_2$  of the canonical 1-form (soldering form) on  $L_{[\eta]} M^4 \subset LM^4$ .

### 3 The 2. order gauge bundle

The structures mentioned above,  $M^4 = C/W^{11}$ ,  $[\eta]$ ,  $L_{[\eta]} M^4$ ,  $L_{[\eta]}^2 M^4$ ,  $\overset{\circ}{\tau}$ ,  $A_{[\eta]}^2 M^4$ ,  $\overset{\circ}{\tau}^-$ , all are extracted solely from the Lie group structure of  $C$ ; in this sense they are given canonically. In particular, the existence of the natural connection form  $\overset{\circ}{\tau}^-$  on the  $C$ -bundle  $A_{[\eta]}^2 M^4$  might suggest to use  $A_{[\eta]}^2 M^4$  as a 'gauge bundle' for  $C$  with background structures  $[\eta]$  and  $\overset{\circ}{\tau}^-$ , and to introduce 'gauge potentials' via the generalization of  $\overset{\circ}{\tau}^-$  to arbitrary connections  $\bar{\tau}$ . Since there is no natural nonsingular 4-form on  $A_{[\eta]}^2 M^4$  ( $\frac{1}{4!} \epsilon_{ijkl} \overset{\circ}{\theta}^i \wedge \overset{\circ}{\theta}^j \wedge \overset{\circ}{\theta}^k \wedge \overset{\circ}{\theta}^l$  is nonsingular on  $L_{[\eta]}^2 M^4$ , but not on  $A_{[\eta]}^2 M^4$ ) for the construction of Langrangians on  $A_{[\eta]}^2 M^4$  without introducing further objects, one is forced to pass to the 'equivalent' structures  $L_{[\eta]}^2 M^4$ ,  $\overset{\circ}{\tau}$ . Gauging  $C$  then is realized via the interpretation of arbitrary Cartan connections  $\tau = \theta + \lambda + \delta + \kappa$  on  $L_{[\eta]}^2 M^4$  as gauge potentials. This is in complete accordance with the (1. order) case of Poincaré gauge theory [3], where the  $P$ -bundle  $A_\eta R^4$  of affine  $\eta$ -orthonormal frames is reduced to  $L_\eta R^4 \equiv P \subset A_\eta R^4$ , and Cartan connections  $\tau = \theta + \lambda$  on  $L_\eta R^4$  are split into tetrad fields  $\theta$  (cf. sect.4) and connections  $\lambda$ .

The critical point in the conformal case is the fact, that in the corresponding splitting of  $\tau$  into 'tetrad field' and  $\lambda + \delta + \kappa$ , the latter fails to be a connection form, with consequences e.g. concerning the possibilities to define covariant derivatives on  $L_{[\eta]}^2 M^4$  and to construct invariant Lagrangians.

Our aim here is to ignore this fact and to proceed to indicate a gauge formalism on  $L_{[\eta]}^2 M^4$  consequently applying the notion of Cartan connections in formal analogy with the case of the Poincaré group (and the remaining 1. order groups in (1)).

### 4 Geometrical interpretation of $\tau$

The typical distinction between internal and (1. order) spacetime gauge theories is the occurrence of the canonical 1-form  $\overset{\circ}{\theta}$  on the bundle of linear frames. This is intimately related to the interpretation of  $\theta$  in  $\tau = \theta + \lambda$  as the tetrad field in Poincaré gauge theories, i.e. as a geometric object defining the spacetime metric. In bundle terms, this interpretation is simply stated by

**Proposition 1** Given a Cartan connection  $\tau = \theta + \lambda$  on  $L_\eta \mathbb{R}^4$ . Then  $\theta$  fixes uniquely an embedding

$$i : L_\eta \mathbb{R}^4 \rightarrow L\mathbb{R}^4 \quad (8)$$

s.th.  $i^*(\overset{\circ}{\theta}) = \theta$ .

The  $L$ -bundle  $i(L_\eta \mathbb{R}^4) \subset L\mathbb{R}^4$  thus gives a Lorentz metric  $g$  on  $\mathbb{R}^4$ , s.th.  $i(L_\eta \mathbb{R}^4) = L_g \mathbb{R}^4$ , the bundle of  $g$ -orthonormal frames. As a result, each Cartan connection  $\tau$  on  $L_\eta \mathbb{R}^4$  determines a Riemann-Cartan-spacetime  $(\mathbb{R}^4, g, i_*\lambda)$ , i.e. a Lorentz manifold  $(\mathbb{R}^4, g)$  with  $g$ -compatible nonsymmetric connection  $i_*\lambda$ .

An analogous 2. order mechanism in the conformal case requires some further investigations :

We first define the 'Cartan derivative'  $\nabla$  corresponding to the Cartan connection  $\tau = \theta + \lambda + \delta + \kappa$  on  $L_{[\eta]}^2 M^4$  via pull back of the covariant derivative  $\bar{\nabla}$  corresponding to the connection  $\bar{\tau}$  on  $A_{[\eta]}^2 M^4$  'associated' with  $\tau$  :

$$\nabla \psi^s := d\psi^s + \tau^a \wedge r_{i^a}{}^s \psi^i \quad (9)$$

for fields  $\psi = (\psi^s)$  transforming under  $C$  (with Lie algebra representation  $r$  of  $o(4, 2)$ ), and

$$\nabla \tau := d\tau + \frac{1}{2}[\tau, \tau].$$

Calling

$$\nabla \theta^i = d\theta^i + \lambda^a{}_k \wedge \theta^k + \delta \wedge \theta^i \quad (10)$$

the torsion of  $\tau$  (or of  $\theta + \lambda + \delta$ ), one proves

**Proposition 2** Let  $\tau = \theta + \lambda + \delta + \kappa$  be torsionfree ( $\nabla \theta = 0$ ). Then there is a unique embedding

$$j : L_{[\eta]}^2 M^4 \rightarrow L^2 M^4, \quad (11)$$

s.th.  $j^*(\overset{\circ}{\theta}^{(2)}) = \theta + \lambda + \delta$  (with canonical 1-form  $\overset{\circ}{\theta}^{(2)}$  on  $L^2 M^4$  [6]).

Moreover, there is a unique conformal structure  $[g]$  on  $M^4$ , s.th.  $j(L_{[\eta]}^2 M^4) = L_{[g]}^2 M^4$  (conformal 2. order structure corresponding to  $[g]$ ).

To generalize for arbitrarily given  $\tau = \theta + \lambda + \delta + \kappa$  on  $L_{[\eta]}^2 M^4$  one shows: There is exactly one Lorentz-valued 1-form  $\lambda^s$ , s.th.  $\theta + \lambda^s + \delta$  is torsionfree and extendible to a Cartan connection. Writing  $\overset{\circ}{\theta}^{(2)} = \overset{\circ}{\theta} + \overset{\circ}{\lambda} + \overset{\circ}{\delta}$  for the canonical 1-form on  $L_{[\eta]}^2 M^4 \subset L^2 M^4$  ( $\equiv$  Maurer-Cartan-form if  $[g] = [\eta]$ ) we eventually conclude : Each Cartan connection  $\tau = \theta + \lambda + \delta + \kappa$  on  $C = L_{[\eta]}^2 M^4$  uniquely determines a conformal structure  $[g]$  on  $M^4$  and a bundle isomorphism  $j : L_{[\eta]}^2 M^4 \rightarrow L_{[g]}^2 M^4$ , s.th.  $j^*(\overset{\circ}{\theta} + \overset{\circ}{\lambda} + \overset{\circ}{\delta}) = \theta + \lambda^s + \delta$ . Correspondingly, the final geometrical interpretation of  $\tau$  is that of a potential which fixes  $[g]$  together with a Cartan connection

$$j_*\tau = \overset{\circ}{\theta} + j_*\lambda + \overset{\circ}{\delta} + j_*\kappa \quad (12)$$

on  $L_{[g]}^2 M^4$ , where  $\overset{\circ}{\theta}$  and  $\overset{\circ}{\delta}$  are canonical.

## 5 Symmetry breaking $W^{11} \rightarrow L$

Symmetry breaking 'conformal structure  $\rightarrow$  Lorentzstructure' usually is performed on 1. order level, i.e. as a reduction  $W^7 = L \otimes D \rightarrow L$ , using equivariant functions  $\varphi : L_{[g]}M^4 \rightarrow \mathbb{R}$ ,  $\varphi > 0$ , with nontrivial weight. We indicate a 'prolongation' of this mechanism to 2. order to be able to link up with the result of sect.4 :

Define the 'prolongation'  $\overset{(2)}{\varphi} : L_{[g]}^2M^4 \rightarrow \mathbb{R}^6$  of  $\varphi$  (where  $\varphi$  transforms as  $\varphi \mapsto (\det a)^{-\frac{1}{2}}\varphi$  for  $L_{[g]}M^4 \ni e \mapsto ea, a \in W^7$ ) by  $\overset{(2)}{\varphi} := (\partial^\mu \phi \partial_\mu \phi / 2\phi, -\partial^\mu \phi, \phi)^T$ , with pull back  $\phi$  of  $\varphi$  to  $L_{[g]}^2M^4$  and proper interpretation of  $\partial^\mu$  in  $\overset{(2)}{\varphi}$ . Take the  $W^{11}$ -invariant Lagrangian  $L = (\frac{1}{2} \overset{(2)}{\varphi} k^i \nabla_i \overset{(2)}{\varphi}) \phi^{-4} \epsilon$ ,  $\epsilon := \frac{1}{4!} \epsilon_{ijkl} \theta^i \wedge \theta^j \wedge \theta^k \wedge \theta^l$ , with standard generators  $k^i$  of  $K^4 \subset C$  in the  $\mathbb{R}^6$ -basis  $\xi = (y^0 + y^5)/\sqrt{2}, y^1, \dots, y^4, \phi := (y^0 - y^5)/\sqrt{2}$  (metric signature on  $\mathbb{R}^6 : -, - - - +, +$ ). Then variation with respect to  $\psi := 1/\phi$  yields the eqn.

$$D^i D_i \psi + j_* \kappa^i_i \psi = 0, \quad (13)$$

which is valid on each 'prolonged'  $L$ -bundle in  $L_{[g]}^2M^4$ , i.e. on each subbundle of the form  $L_{\bar{g}}^2M^4 \subset L_{[g]}^2M^4, \bar{g} \in [g]$ , and where  $D$  is the covariant derivative corresponding to the Lorentz connection  $j_*\lambda$  on  $L_{\bar{g}}^2M^4$  in each case. Eqn. (13) includes the special case  $D^i D_i \psi + (R/6)\psi = 0$  for  $j_*\tau = \overset{\circ}{\theta} + \overset{\circ}{\lambda} + \overset{\circ}{\delta} + \overset{\circ}{\kappa}$  (cf. [5]),  $R$  being the scalar curvature of  $\bar{g}$ , resp..

The isotropy subbundle  $(\overset{(2)}{\varphi})^{-1}(0,0, \phi = 1)$  now turns out to be a prolonged  $L$ -bundle, i.e. equals  $L_{g'}^2M^4$  for a unique  $g' \in [g]$ . Hence, having exploited the full structure of the conformal group, we end up with a *Riemann-Cartan-spacetime*  $(M^4, g', (\pi_2 \circ j)_*\lambda)$ .

## 6 Generalizations

i. The formalism indicated in sects.1., ..., 5 can be generalized for arbitrary base manifolds  $M$  admitting Lorentzstructures (or, equivalently, conformal structures). Moreover, the (*naturally* 'soldered') gauge bundle  $L_{[g]}^2M^4$  may be replaced by an *abstract*  $W^{11}$ -(gauge)-bundle  $B$  over  $M$ , thus stressing the formal analogy with 'internal' gauge theories. Given  $M$ , then  $B$  is fixed up to strong bundle isomorphisms, due to the following observation :

For any two (orientated and time orientated) Lorentz metrics  $g, g'$  on  $M$  there exists a strong automorphism of  $LM$ , s.th.  $L_g M \rightarrow L_{g'} M$  [4]; correspondingly for conformal 1. and 2. order structures.

ii. Using an invariant 4-form on  $B$  similar to  $\phi^{-4}\epsilon$  in sect.5, and Cartan derivatives on  $B$  to construct  $W^{11}$ -invariant Lagrangians, it is straightforward to get the general field equations for matter fields and Cartan connections  $\tau$  on  $B$ .

The geometrical interpretation of  $\tau$  then is given via an induced embedding  $j : B \rightarrow L^2M$  replacing  $j$  in (11).

iii. There is a clear meaning of first and second order conformal *spin* structures.

iv. As an example for an invariant matter Lagrangian on  $B$  without use of the extra field  $\phi$  in ii., we mention

$$L = \sqrt{2}(\Psi^* \gamma^{\circ} \dot{k}^i (\nabla_i \Psi + n \delta_i \Psi) - \Psi^* \gamma^{\circ} \Psi) \epsilon + \text{herm. conj.}$$

with 4-spinor  $\Psi =: (\psi^1, \psi^2)^T$ , generators  $\dot{k}^i$  of  $K^4$  in the standard  $SU(2, 2)$ -representation and 'degree of homogeneity'  $n = -2$  of  $\Psi$ . This yields an  $W^{11}$ -invariant equation of Weyl type (cf. [1]) for the 2-spinor  $\psi^1$  on  $B$  with torsion term  $Q^i_{ki} \theta^k \wedge \theta^i := D\theta^i$ ,

$$\sigma^i (D_i + \frac{1}{2} Q^k_{ki}) \psi^1 = 0,$$

and the gauge freedom to choose  $\psi^2$  arbitrarily.

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## DIFFERENTIAL GEOMETRY OF SPACETIME TANGENT BUNDLE

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Both string theory and the quantum mechanics of the vacuum polarization in accelerated frames determine a universal upper limit on allowable proper acceleration relative to the vacuum.<sup>1-6</sup> If the limiting acceleration is universal, then it must apply invariantly for all observers. The latter requirement defines maximal-acceleration invariant phase space as a fiber bundle in which spacetime is the base manifold and four-velocity space is the fiber manifold.<sup>4</sup> In a coordinate basis, the implied structure of the bundle metric,  $G_{MN}$ , is that of the diagonal lift<sup>7</sup> of the spacetime metric,  $g_{\mu\nu}$ , namely,

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} + g_{\alpha\beta} A_{\mu}^{\alpha} A_{\nu}^{\beta} & A_{n\mu} \\ A_{m\nu} & g_{mn} \end{pmatrix}, \quad (1)$$

where  $A_{\nu}^{\mu} = \rho_0 v^{\lambda} \Gamma_{\lambda\nu}^{\mu}$  is the gauge potential, and  $\Gamma_{\lambda\nu}^{\mu}$  is the spacetime affine connection. A point in the bundle manifold has coordinates  $\{x^M; M=0,1,\dots,7\} = \{x^{\mu}, x^m; \mu=0,1,2,3; m=4,5,6,7\} \equiv \{x^{\mu}, \rho_0 v^{\mu}; \mu=0,1,2,3\}$ , where  $x^{\mu}$  and  $v^{\mu}$  are the spacetime and four-velocity coordinates, respectively. Greek indices referring to spacetime range from 0 to 3, lower case Latin indices referring to four-velocity space range from 4 to 7, and upper case Latin indices referring to a point in the bundle range from 0 to 7. Any lower case Latin index,  $n$ , appearing in a canonical spacetime tensor or connection is defined to be  $n - 4$  implicitly. The length  $\rho_0$  is of the order of the Planck length, namely,  $\rho_0 = c^2/a_0 = (\hbar G/c^3)^{1/2}/2\pi\alpha$ , where  $a_0$  is the maximal proper acceleration relative to the vacuum,  $c$  is the velocity of light in vacuum,  $\hbar$  is Planck's constant divided by  $2\pi$ ,  $G$  is the universal gravitational constant, and  $\alpha$  is a dimensionless number of order unity.<sup>1,4-6</sup>

In an anholonomic basis adapted to the affine connection,<sup>4,7</sup> the bundle metric has a simple block diagonal form with the spacetime metric in both the base and the fiber, and the Levi-Civita connection coefficients,  $(8)\Gamma_{AB}^M$ , of the bundle manifold are given by<sup>4</sup>

$$(8)\Gamma_{\alpha\beta}^{\mu} = \left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\} = \frac{1}{2} g^{\mu\lambda} (g_{\alpha\lambda,\beta} + g_{\beta\lambda,\alpha} - g_{\alpha\beta,\lambda}), \quad (2)$$

$$(8)\Gamma_{ab}^{\mu} = (8)\Gamma_{ba}^{\mu} = \frac{1}{2} (F_{ba}^{\mu} + \Pi_{ba}^{\mu} + \Pi_{ab}^{\mu}), \quad (3)$$

$$(8)\Gamma_{ab}^{\mu} = -\frac{1}{2} (T_{ab}^{\mu} + T_{ba}^{\mu}), \quad (4)$$

$$(8)\Gamma_{\alpha\beta}^m = \frac{1}{2} (F_{\alpha\beta}^m - \Pi_{\alpha\beta}^m - \Pi_{\beta\alpha}^m), \quad (5)$$



$$(8) \Gamma_{\alpha b}^m = \frac{1}{2}(T_{b\alpha}^m + T_{\alpha b}^m) , \quad (6)$$

$$(8) \Gamma_{b\alpha}^m = \{^m_{b\alpha}\} + \frac{1}{2}(T_{b\alpha}^m - T_{\alpha b}^m) , \quad (7)$$

$$(8) \Gamma_{ab}^m = \Pi_{ab}^m = \frac{1}{2} \rho_0^{-1} g^{mn} (\partial/\partial v^a g_{nb} + \partial/\partial v^b g_{na} - \partial/\partial v^n g_{ab}) . \quad (8)$$

Here  $\{^{\mu}_{\alpha\beta}\}$  and  $\Pi^{\mu}_{\alpha\beta}$  are the Christoffel symbols in spacetime and four-velocity space, respectively, and in the anholonomic basis, the following notation is implicit:  $\{^{\mu}_{\alpha\beta}\} \equiv \partial/\partial x^{\mu} - \rho_0^{-1} A^{\beta}_{\mu} \partial/\partial v^{\beta}$ . Also  $F^{\alpha}_{\mu\nu} = \rho_0 v^{\lambda} R^{\alpha}_{\lambda\mu\nu}$  is the gauge curvature field, where  $R^{\alpha}_{\lambda\mu\nu}$  is the spacetime Riemann curvature tensor. The field  $T^{\beta}_{\mu\nu}$  is given by

$$T^{\beta}_{\mu\nu} = \{^{\beta}_{\mu\nu}\} - \rho_0^{-1} \partial/\partial v^{\nu} A^{\beta}_{\mu} . \quad (9)$$

In general, the spacetime base manifold of the maximal-acceleration invariant fiber bundle is non-Riemannian. Recently, as a very special case, a Riemannian spacetime manifold was considered for the base manifold, and it was shown that in this case the bundle manifold is the associated tangent bundle, and relationships were investigated between the natural lift of spacetime geodesics and geodesics in the spacetime tangent bundle.<sup>7-9</sup> Also, a Riemannian Schwarzschild-like spacetime was considered which is a solution following from an appropriate action defined on the spacetime tangent bundle.<sup>5,6</sup> Possible modifications were calculated to the canonical red shift formula for a Schwarzschild spacetime. It is of interest to consider more general base manifolds, such as Finsler spacetime.<sup>10</sup>

If the spacetime manifold is a Finsler manifold, then

$$g_{\mu\nu}(x, v) = \frac{1}{2} \partial^2 / \partial v^{\mu} \partial v^{\nu} L^2(x, v) , \quad (10)$$

where  $L(x, v)$  is the fundamental function, a scalar on the spacetime tangent bundle.<sup>7,10</sup> The Finsler spacetime metric is also homogeneous of degree zero, and it follows that

$$v^{\alpha} (\Pi_{\mu\alpha\nu} + \Pi_{\nu\alpha\mu}) = v^{\alpha} (\Pi_{\alpha\mu\nu} + \Pi_{\nu\mu\alpha}) = 0 . \quad (11)$$

If the spacetime affine connection,  $\Gamma^{\mu}_{\alpha\beta}$ , is of the Levi-Civita form, and the spacetime manifold is Finslerian, then Eqs.(2)-(8) are readily shown to be of the same form as the well-known Levi-Civita connection coefficients for a generic tangent bundle of a Finsler manifold (Eqs.3.12a-h of Ref.9). The connection coefficients are consistent with Cartan's theory of Finsler space, provided the gauge curvature field,  $F^{\mu}_{\alpha\beta}$ , is vanishing. Furthermore, if the spacetime metric is independent of the four-velocity, then the coefficients reduce to the form corresponding to a tangent bundle of a Riemannian base manifold.<sup>7-9</sup>

To further characterize the differential geometry of the spacetime tangent bundle, it is of interest to consider the following exterior differential of a one-form  $\omega$ :

$$d\omega \equiv d(\rho_{\sigma} v_{\mu} dx^{\mu}) \equiv \frac{1}{2} J_{AB} dx^A \wedge dx^B . \quad (12)$$

If the spacetime connection has the Levi-Civita form, and the spacetime is Finslerian, then using Eqs.(1) and (11), one obtains

$$J_A^B = \begin{pmatrix} A_{\alpha}^{\beta} & -\delta_{\alpha}^b - A_{\kappa\alpha} A^{b\kappa} \\ \delta_a^{\beta} & -A_a^b \end{pmatrix} , \quad (13)$$

and one verifies that

$$J_A^D J_D^B = -\delta_A^B . \quad (14)$$

Thus, the spacetime tangent bundle of a Finsler spacetime manifold is almost complex, with almost complex structure given by Eq.(13).<sup>7,9,11</sup>

Next, it is of interest to consider  $\nabla_E J_A^B$ , where  $\nabla_E$  is the covariant derivative involving the Levi-Civita bundle connection,  ${}^{(8)}\Gamma_{BC}^A$ . Using Eqs.(2)-(8), the components of  $\nabla_E J_A^B$ , in the anholonomic frame adapted to the spacetime affine connection, reduce to

$$\nabla_{\epsilon} J_{\alpha}^{\beta} = \frac{1}{2} (F_{\alpha\epsilon}^{\beta} - F_{\alpha\epsilon}^{\beta} + \Pi_{\alpha\epsilon}^{\beta} - \Pi_{\alpha\epsilon}^{\beta}) , \quad (15)$$

$$\nabla_{\epsilon} J_{\alpha}^b = \frac{1}{2} (T_{\epsilon\alpha}^b - T_{\alpha\epsilon}^b) , \quad (16)$$

$$\nabla_{\epsilon} J_a^{\beta} = -\frac{1}{2} (T_{\epsilon a}^{\beta} - T_{a\epsilon}^{\beta}) , \quad (17)$$

$$\nabla_{\epsilon} J_a^b = -\frac{1}{2} (F_{a\epsilon}^b - F_{a\epsilon}^b + \Pi_{a\epsilon}^b - \Pi_{a\epsilon}^b) , \quad (18)$$

$$\nabla_e J_{\alpha}^{\beta} = \frac{1}{2} (T_{e\alpha}^{\beta} - T_{e\alpha}^{\beta} + T_{\alpha e}^{\beta} - T_{\alpha e}^{\beta}) , \quad (19)$$

$$\nabla_e J_{\alpha}^b = \frac{1}{2} (F_{e\alpha}^b + \Pi_{\alpha e}^b - \Pi_{e\alpha}^b) , \quad (20)$$

$$\nabla_e J_a^{\beta} = \frac{1}{2} (F_{ea}^{\beta} + \Pi_{a e}^{\beta} - \Pi_{ea}^{\beta}) , \quad (21)$$

$$\nabla_e J_a^b = -\frac{1}{2} (T_{e a}^b - T_{e a}^b + T_{a e}^b - T_{a e}^b) . \quad (22)$$

For the Finsler spacetime manifold, all possible contributions in Eqs.(15)-(22) involving combinations of  $\Pi_{\alpha\beta}^{\mu}$  and  $T_{\alpha\beta}^{\mu}$  can be shown to be vanishing. It follows that if the spacetime manifold is Finslerian, and the gauge curvature field  $F_{\alpha\beta}^{\mu}$  is vanishing, then Eqs.(15)-(22) are also vanishing, and one concludes that<sup>7,9</sup>

$$\nabla_E J_A^B = 0 . \quad (23)$$

Eq.(23) is the condition that the spacetime tangent bundle be Kählerian.

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## Applications of sections along a map in Classical Mechanics

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### 1. INTRODUCTION, DEFINITIONS AND BASIC NOTATION

We "rediscovered" the concept of vector field along a map when studying the converse of Noether's Theorem in time-independent Lagrangian mechanics as a way of taking into account non-point transformations [1] and it had also been used when looking for a geometric interpretation of the time evolution  $K$ -operator relating Hamiltonian and Lagrangian constraints for singular systems [2]. In a recent paper [3] it was also proven to be very useful for an alternative introduction of basic objects in Classical Mechanics, and giving a geometric meaning for the "total time derivative" or to objects like the Legendre transformation, the 1-form  $\theta_L$  and the Euler 1-form  $\delta L$  in the Lagrangian formalism. It allows to work in a geometric way with non-point transformations and is the appropriate tool for the geometric version of the Second Noether's Theorem [4] allowing a geometric definition of the evolution  $K$ -operator. It also provides the generalization for dealing with (even singular) Lagrangian containing higher order derivatives [5] and has also application in geometry as in the theory of Ehresmann nonlinear connections [6].

Let  $\pi: E \rightarrow M$  be a fibre bundle and  $\phi: N \rightarrow M$  a differentiable map. A section along  $\phi$  is a map  $\sigma: N \rightarrow E$  such that  $\pi \circ \sigma = \phi$ . The set of sections along  $\phi$  will be noted  $\Sigma_\phi(\pi)$ . When  $E$  is a vector bundle the set  $\Sigma_\phi(\pi)$  is endowed with a  $C^\infty(N)$ -module structure. Of a particular interest are the vector bundles  $\tau_M: TM \rightarrow M$ ,  $\pi_M^p: (T^*M)^{\wedge p} \rightarrow M$ , and  $\rho_M^p: (T^*M)^{\wedge p} \otimes TM \rightarrow M$ , and in these cases we will denote these sets  $\mathfrak{X}(\phi) = \Sigma_\phi(\tau_M)$ ,  $\Lambda^p(\phi) = \Sigma_\phi(\pi_M^p)$ , and  $V^p(\phi) = \Sigma_\phi(\rho_M^p)$ , respectively. When  $N = M$  and  $\phi = \text{id}$  the set  $\mathfrak{X}(\text{id})$  coincides with  $\mathfrak{X}(M)$  and the set  $\Lambda^p(\text{id})$  reduces to  $\Lambda^p(M)$ .

Interesting examples are the following: Let  $\gamma: \mathbb{R} \rightarrow M$  be a curve in  $M$ . The tangent vectors  $\dot{\gamma}$  define a section  $\dot{\gamma}: \mathbb{R} \rightarrow TM$  of  $\tau_M$  along  $\gamma$ . The restriction of  $X \in \mathfrak{X}(M)$  on the curve  $\gamma$  is also a vector field along  $\gamma$ . The generalization of these examples are: Let  $\phi$  be a map from  $N$  to  $M$ . A vector field  $Y \in \mathfrak{X}(N)$  defines a vector field along  $\phi$  by composing it with the tangent map:  $T\phi \circ Y \in \mathfrak{X}(\phi)$ . Similarly, when  $X \in \mathfrak{X}(M)$  the restriction  $X \circ \phi$  of  $X$  on the image by  $\phi$  is a vector field along  $\phi$ . Similar things can be told for forms. If  $\beta$  is a  $p$ -form in  $M$ , the restriction  $\beta \circ \phi$  of  $\beta$  on the image by  $\phi$  is a  $p$ -form along  $\phi$ . Given  $\alpha \in \Lambda^p(\phi)$ ,  $T^*\phi \circ \alpha$  is a  $p$ -form in  $N$ . The pull-back by  $\phi$  of  $\beta \in \Lambda^p(M)$  is obtained by iteration of both processes  $\phi^*(\beta) = T^*\phi \circ \beta \circ \phi$ .

When  $E$  is a vector bundle and  $\{\sigma_\alpha\}$  a local basis of  $\Sigma(\pi)$ , then  $\{\sigma_\alpha \circ \phi\}$  is a local basis of  $\Sigma_\phi(\pi)$ , and  $\sigma \in \Sigma(\pi)$  can be written as  $\sigma = \zeta^\alpha(\sigma_\alpha \circ \phi)$  with  $\zeta^\alpha \in C^\infty(N)$ .

In the above case, taking local coordinates  $(z^A)$  in  $N$  and  $(x^i)$  in  $M$  we have

$$X \in \mathfrak{X}(\phi) \quad X = X^i \left( \frac{\partial}{\partial x^i} \circ \phi \right) \quad \text{and} \quad \alpha \in \Lambda^1(\phi) \quad \alpha = \alpha_i(dx^i \circ \phi)$$

where  $X^i$  and  $\alpha_i$  are functions in  $N$ .

There exists a canonical section along  $\pi$  in each vector bundle  $\pi: E \rightarrow M$ , which is given by the identity map in  $E$ . When choosing local coordinates  $(x^i, y^\alpha)$  in  $E$  and a local basis  $\{\sigma_\alpha\}$  of sections for  $\pi$  such that  $y^\alpha(\epsilon) = \sigma_\alpha(\pi(\epsilon))$ , for  $\epsilon \in E$ , then the local expression of  $C$  is  $C = y^\alpha(\sigma_\alpha \circ \pi)$ . The most important cases in Classical Mechanics are those of  $E = TM$  or  $E = T^*M$ . Then  $C$  reduces in these cases to the "total time derivative"  $\mathbf{T}$  (in the time-independent formalism) and  $\theta_0^\vee$ , the 1-form along  $\pi$  corresponding to the  $\pi$ -semibasic Liouville 1-form  $\theta$ . Notice that when  $\phi$  is a submersion, every  $\phi$ -semibasic  $p$ -form in  $N$  may be identified to a  $p$ -form along  $\phi$ ,  $\alpha^\vee$ . The coordinate expressions are

$$\mathbf{T} = v^i \left( \frac{\partial}{\partial x^i} \circ \tau_M \right) \quad \text{and} \quad \theta_0^\vee = p_i(dx^i \circ \pi_M).$$

Vector fields along  $\phi$  act on functions on  $M$  giving rise to functions on  $N$ . If  $X \in \mathfrak{X}(\phi)$  and  $n \in N$  then  $X(n)$  is a tangent vector to  $M$  at the point  $\phi(n)$  which acts on a function  $h \in C^\infty(M)$  by  $(Xh)(n) = X(n)h$ . The Leibnitz rule for tangent vectors implies that  $X(hl) = \phi^*h Xl + \phi^*l Xh$ . A map satisfying this property is called a  $\phi^*$ -derivation (of degree 0). Pidello and Tulczijev [7] generalized the theory of Frölicher and Nijenhuis for these new derivations (see also [6]).

In the time-dependent formalism it is usually considered  $\pi: E = \mathbb{R} \times Q \rightarrow \mathbb{R}$  and the  $k$ -jet bundle  $J^k\pi$  is  $\mathbb{R} \times T^kQ$ . The vector fields along  $\pi_{k+1,k}$ ,  $\mathbf{T}^{(k)}$  are defined by  $\mathbf{T}^{(k)} \circ j^{k+1}\sigma = (j^k\sigma)_* \circ \frac{d}{dt}$ ,  $\forall \sigma \in \Sigma(\pi)$ . In local coordinates the local expression of  $\mathbf{T}^{(k)}$  is:  $\mathbf{T}^{(k)} = \frac{\partial}{\partial t} \circ \pi_{k+1,k} + \sum_{l=0}^k q_{(l+1)}^i \left( \frac{\partial}{\partial q_{(l)}^i} \circ \pi_{k+1,k} \right)$ .

If  $Z \in \mathfrak{X}(\pi_{1,0})$  for every integer number  $k$  there exists  $Z^{(k)} \in \mathfrak{X}(\pi_{k+1,k})$  projectable onto  $Z^{(l)}$ ,  $l = 0, 1, \dots, k-1$ , namely,  $\pi_{k,l*} \circ Z^{(k)} = Z^{(l)} \circ \pi_{k+1,l+1}$  and characterized by  $Z^0 \equiv Z$  and for any contact 1-form  $\vartheta \in \mathcal{C}_1(\pi_k)$ ,  $d_{Z^{(k)}}\vartheta \in \mathcal{C}_1(\pi_{k+1})$ . If  $Z = Z_0 \frac{\partial}{\partial t} + Z^i \frac{\partial}{\partial q^i}$  is the local expression of  $Z \in \mathfrak{X}(\mathbb{R} \times Q)$ , then  $Z^{(1)} = \left[ Z_0 \frac{\partial}{\partial t} + Z^i \frac{\partial}{\partial q^i} \right] \circ \pi_{1,0} + (d_{\mathbf{T}^{(0)}}Z^i - v^i d_{\mathbf{T}^{(0)}}Z^0) \frac{\partial}{\partial v^i}$ .

## 2. SYMMETRY IN TIME-DEPENDENT LAGRANGIAN MECHANICS

Time dependent Lagrangian Mechanics is based on  $\delta L = i_{\mathbf{T}^{(1)}} d\Theta_L \in \wedge^1(\mathbb{R} \times T^2Q)$ , with local coordinate expression

$$\delta L = \left[ \frac{d}{dt} \left( \frac{\partial L}{\partial v^i} \right) - \pi_{2,1}^* \left( \frac{\partial L}{\partial q^i} \right) \right] (dq^i - v^i dt)$$

and  $\Theta_L \in \wedge^1(\mathbb{R} \times TQ)$  given by  $\Theta_L = dL \circ S + Ldt \in \wedge^1(\mathbb{R} \times TQ)$ , which in local coordinates reads  $\Theta_L = \frac{\partial L}{\partial v^i} (dq^i - v^i dt) + L dt$ .

We recall that a Cartan symmetry is a vector field  $Z \in \mathfrak{X}(\mathbb{R} \times TQ)$  such that there exists a function  $F \in C^\infty(\mathbb{R} \times TQ)$  satisfying  $\mathcal{L}_Z \Theta_L = dF$ , or in an equivalent way if  $i(Z)d\Theta_L = dG$ , where  $G = F - i(Z)\Theta_L$ ; then  $G$  is a constant of the motion and  $Z$  is a dynamical symmetry (i.e.,  $\mathcal{L}_Z \Gamma = h\Gamma$ ). Cartan symmetries which are 1-jet prolongation of a vector field in  $\mathbb{R} \times Q$ , i.e.  $Z = X^1$ , are called Noether symmetries. We aim to express the symmetry properties in terms of the Lagrangian. So we introduce the

following definition: A  $\pi$ -vertical  $X \in \mathfrak{X}(\pi_{1,0})$  is said to be a generalised symmetry of the Lagrangian  $L$  iff there exists a function  $F \in C^\infty(\mathbb{R} \times TQ)$  such that  $d_{X^1}L = d_{T(1)}F$ , where  $X^1 \in \mathfrak{X}(\pi_{2,1})$  is the 1-prolongation of  $X$ . Then, it is possible to show [8]:

**Theorem:** If  $X \in \mathfrak{X}(\pi_{1,0})$   $\pi$ -vertical is a generalised symmetry of a regular Lagrangian  $L$ , then the function  $G = F - \Theta_L^Y(X)$  is a constant of the motion of the system defined by  $L$ . And, conversely, if  $G$  is a constant of the motion of such a system, then there will exist a generalised symmetry  $X$  of  $L$  with associated function  $F = G + \Theta_L^Y(X)$ .

This generalised concept of symmetry leads in a natural way to a geometric version of the Second Noether's Theorem: let  $\{X_k\}_{k=0,\dots,R}$  be a family of  $\pi$ -vertical vector fields along  $\pi_{1,0}$  and let also  $\varepsilon$  be an indeterminate function in  $\mathbb{R}$ . Then,  $X_\varepsilon$  will denote the  $\pi$ -vertical vector field along  $\pi_{1,0}$   $X_\varepsilon = \sum_k \pi_1^*(d_T^k \varepsilon) X_k \in \mathfrak{X}(\pi_{1,0})$ , where  $d_T^k \varepsilon$  means  $\frac{d^k \varepsilon}{dt^k}$  if  $k \neq 0$  and  $d_T^0 \varepsilon \equiv \varepsilon$ .

The  $\pi$ -vertical vector field along  $\pi_{1,0}$ ,  $X_\varepsilon \in \mathfrak{X}(\pi_{1,0})$ , is said to be a gauge symmetry of  $L \in C^\infty(J^1\pi)$  if  $X_\varepsilon$  is a generalized symmetry of  $L$  no matter of the choice of the arbitrary functions  $\varepsilon$ , i.e. there exists a function  $F_\varepsilon$  such that  $d_{X_\varepsilon^1}L = d_{T(1)}F_\varepsilon$ . This is equivalent to say that  $\delta L^V(X_\varepsilon \circ \pi_{2,1}) = -d_{T(1)}G_\varepsilon$  with  $G_\varepsilon \in C^\infty(\mathbb{R} \times TQ)$  given by  $G_\varepsilon = F_\varepsilon - \theta_L^Y(X_\varepsilon)$ . It can then be proved that:

**The Second Noether's Theorem:** If the above vector field along  $\pi_{1,0}$ ,  $X_\varepsilon$  is a gauge symmetry of a Lagrangian  $L$ , then  $L$  is singular. Moreover, the identity

$$\mathcal{N}_R = \sum_{k=0}^R \pi_{R+2,k+2}^*(-d_T)^k \alpha_k \in C^\infty(J^{R+2}\pi) = 0,$$

holds if and only if  $X_\varepsilon$  is a gauge symmetry and the corresponding function  $F_\varepsilon$  can be written as a sum  $F_\varepsilon = \sum_{k=0}^R (\pi_1^* d_T^k \varepsilon) F_k$ .

### 3. OTHER APPLICATIONS

Given a Lagrangian function  $L \in C^\infty(TM)$  we associate to it a Poincaré-Cartan 1-form  $\theta_L = S^*(dL)$ , the Legendre map  $\mathcal{F}L : TM \rightarrow T^*M$  and the time evolution operator  $K_L : C^\infty(T^*M) \rightarrow C^\infty(TM)$ . They can also be redefined as sections along maps: The 1-form  $\theta_L$  is semibasic and can be seen as a 1-form along  $\tau$ ,  $\theta_L \in \Lambda(\tau)$ ,  $\theta_L = (\partial L / \partial v^i)(dq^i \circ \tau)$ . It is identified in this way with the Legendre map.

Let  $\chi$  be the natural diffeomorphism between  $T^*(TM)$  and  $T(T^*M)$ , with coordinate expression  $\chi(x, v, p_x, p_v) = (x, p_v, v, p_x)$ . Then  $K = \chi \circ dL$  maps  $TM$  in  $T(T^*M)$  in such a way that  $\tau_{T^*M} \circ K = \mathcal{F}L$ , say,  $K$  is a vector field along  $\mathcal{F}L$ . In coordinates,  $K$  is given by

$$K = v^i \left( \frac{\partial}{\partial x^i} \circ \mathcal{F}L \right) + \frac{\partial L}{\partial x^i} \left( \frac{\partial}{\partial p_i} \circ \mathcal{F}L \right)$$

and it is very useful to relate constraint functions arising in the Hamiltonian and Lagrangian formulations respectively.

Finally we want to point out that vector fields along maps are also relevant in the study of degenerate systems: Let  $(N, \omega)$  be a symplectic manifold and  $\phi: P \rightarrow N$  of constant rank. Given a 1-form  $\alpha$  in  $P$ , we look for the set of points in which a solution of  $i_\Gamma(\phi^*\omega) = \alpha$  exists, where  $\Gamma$  is a vector field in  $P$ , i.e., we are interested in the

submanifold  $i_C: C \rightarrow P$  of  $P$  in which such a solution  $\Gamma' \in \mathfrak{X}(C)$  does exist, namely  $i_{\Gamma'}((\phi \circ i_C)^*\omega) = i_C^*\alpha$ .

The above problem may be splitted in two. First we study the conditions for the existence of  $X \in \mathfrak{X}(\phi)$  such that  $i_X\omega = \alpha$ , and then we determine the conditions for  $X$  to be image under  $T\phi$  of a vector field in  $P$ . This is equivalent to the original problem because of the relation  $i_{T\phi \circ \Gamma}\omega = i_\Gamma(\phi^*\omega)$ . The second step is but the condition for the solution to be tangent to  $P$ .

Using a well-known result of Linear Algebra we see that the equation  $i_X\omega = \alpha$  has a solution with  $X \in \mathfrak{X}(\phi)$  in  $p \in P$  iff it satisfies  $\langle z, \alpha(p) \rangle = 0$  for all  $z \in T_pP$  such that  $T_p\phi(z) = 0$ . Remark that if  $X$  is a solution and  $Z \in \mathfrak{X}(\phi)$  is such that  $\hat{\omega}(\phi(p))(Z(p)) \in \text{Ker } T_p^*\phi$ ,  $\forall p \in P$  then  $X + Z$  is a solution too. When  $\alpha$  is exact,  $\alpha = dF$ , if for  $n \in \text{Imag } \phi \subset N$  the submanifold  $\phi^{-1}(n)$  is connected, then the above condition is equivalent to  $F$  to be  $\phi$ -projectable, namely there exists  $\tilde{F} \in C^\infty(N)$  such that  $\phi^*(\tilde{F}) = F$ .

The generalization to the case of a presymplectic manifold is: the equation  $i_X\omega = \alpha$  admits a solution in  $p \in P$  iff  $\langle z, \alpha(p) \rangle = 0$  for any  $z \in T_pP$  such that  $T_p\phi(z) \in \text{rad}(\omega)$ , where  $\text{rad}(\omega) = \{v \in TN \mid \omega(v, w) = 0 \quad \forall w \in TN\}$ .

Once the condition holds in  $P$  we look for the existence of a vector field  $\Gamma$  in  $P$  such that  $T\phi \circ \Gamma = X$ , which has a solution iff the equation  $T_p\phi(\Gamma(p)) = X(p)$  has solution for any  $p \in P$ . This is equivalent to  $\langle X(p), \lambda \rangle = 0$  for any  $\lambda$  of  $T_{\phi(p)}^*N$  such that  $T_p^*\phi(\lambda) = 0$ , or in other words iff  $\langle \delta, X \rangle = 0$  for all  $\delta \in \bigwedge^1(\phi)$  such that  $T^*\phi \circ \delta = 0$ . This gives rise to an immersed submanifold  $i_1: P_1 \rightarrow P$  of  $P$  and we repeat the preceding steps.

If the image by  $\phi$  is an immersed submanifold  $j: N_0 \rightarrow N$  of  $N$ , then a similar algorithm is used for finding a solution in  $N_0$ . If  $\zeta$  is a constraint function for  $N$ , then  $\phi^*\zeta$  is a constraint function for  $P$ . This is a generalization of what happens with the theory defined by a singular Lagrangian when  $\phi$  is the Legendre transformation.

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## A POINCARÉ GAUGE THEORY ON REGGE SIMPLEXES

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### 1. Introduction

Regge calculus [1] has been around for more than 30 years, still its applications have been rather limited. We enlarge Regge's original description of the metric field by the addition of a connection, in order to be able to set up a first order formalism, where the metric and the connection are varied independently. This formalism would be evidently closer to normal gauge theories of the Yang-Mill type, although there is of course the n-bein field which makes Einstein's theory very different from Yang-Mill theories. We choose as the gauge group the Poincaré group as the most prominent example among the numerous attempts to formulate Einstein's theory as a gauge theory [2]. In this theory we encounter in general torsion, a subject which has so far been neglected in Regge calculus (see Ref. [3] for an exception). If torsion becomes important at the subatomic level, as it is sometimes suggested, then it is of course interesting to study approximate solutions to the field equations on a small scale lattice, and this can be conveniently done by using Regge calculus.

### 2. Teleparallel Theory

First we want to show how the pure translational gauge theory arises on simplexes. A translational gauge theory can be regarded as a special case of the full Poincaré gauge theory, where one chooses the Lorentz gauge connection to be trivial everywhere [4]. The frame field  $e$  is now no longer locally determined only up to local Lorentz transformation but rather fixed up to an equivalence class of global Lorentz transformation, which adds in the 4 dimensional case six degrees of freedom to the frame. One defines the teleparallel connection  $w_T$  by requiring that the frame is parallel transported along itself,  $De=0$ . This condition induces  $w_T$  through

$$w_T = e^{-1} \partial e \quad (1)$$

which has no curvature, because from  $De=0$  follows that  $D^2e=0$ , hence  $w_T$  is integrable. To state it simpler, the components of the vector relative to one

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frame stay the same in any frame and define in such a way an absolute parallelism [5]. For the simplicial manifold one has to define an orthonormal frame  $e$  inside each simplex. The frame determines the edgelengths of each simplex by

$$l^2_{ij} = e^a_{ij} e^b_{ij} \eta_{ab} \quad (2)$$

where we define

$$e^a_{ij} := \int_1^j e^a.$$

The condition that the edgelengths on the common boundary of two simplexes  $\sigma_1$  and  $\sigma_2$  agree can be written as

$$l^2_{ij}(e(\sigma_1)) = l^2_{ij}(e(\sigma_2)) \quad \text{for all } (ij) \text{ in } \sigma_1 \cap \sigma_2 := \sigma_{12}.$$

The knowledge of the 10 edgelengths is not sufficient to define the 16 components of the frame. In [6] we held  $e$  constant on each simplex  $\sigma$ , but this can be regarded as a special case of the more general assumption that  $e$  is allowed to vary linearly in  $\sigma$ . This will give rise to a constant  $w_\Gamma$  according to (1). If we consider only metric compatible connections, then frames on a point of the boundary of two adjacent simplexes are related by a linear, isometric, orientation preserving mapping. These connections have been investigated by us before [6]. Let us recall that two vectors  $v(x\sigma_1)$  and  $v(x\sigma_2)$  are called *natural parallel* if they can be obtained by parallel transport from the same vector in  $T(x\sigma_{12})$ , the tangent space at  $x$ . The subscript  $x$  refers here to a common point of the specified simplexes, i.e. a point on  $\sigma_{12}$ . The parallel transport LC of the Levi-Civita connection  $\Gamma$  can then be defined as a linear, isometric, orientation preserving map from  $T(x\sigma_1)$  to  $T(x\sigma_2)$  that respects natural parallelism. The parallel transport  $w$  of a general metric connection MC, that in general carries torsion, is a linear, isometric, orientation preserving map. It is related to an orthogonal matrix  $O^a_b$  in the following way:

$$MC(e^a(x\sigma_1)) = O^a_b(LC(e^b(x\sigma_1))) \quad (3)$$

This somewhat awkward expression results from the fact, that while the metrics in  $\sigma_1$  and  $\sigma_2$  agree on  $\sigma_{12}$ , they still can differ in the normal direction to  $\sigma_{12}$ . In the case when we cover  $\sigma_1$  and  $\sigma_2$  with a common metric, LC is represented by the identity map and we can simply write

$$e^a(x\sigma_1) = O^a_b e^b(x\sigma_2). \quad (4)$$

The matrix  $O^a{}_b$  can at most vary linearly on  $\sigma_{12}$ , due to the initial assumption that  $e$  changes linearly on every simplex. A constant matrix  $O^a{}_b$  on  $\sigma_{12}$ , as in reference [6], can be obtained by imposing either the condition that  $e(\sigma)=\text{constant}$ , or requiring that  $w_T(\sigma_1)=w_T(\sigma_2)$  on  $\sigma_{12}$ .  $O^a{}_b$  represents the integrated contortion one-form  $K$ . Because MC has no curvature, the Einstein-Hilbert Lagrangian  $L_{E-H}$  vanishes identically and is not suitable for a teleparallel theory. Because schematically  $R = R_T + K \wedge K + (\text{total divergence})=0$ , where  $R_T$  is the curvature of the Levi-Civita connection, one alternative is to use instead  $R_T = -K \wedge K$  in the Lagrangian to obtain an Einstein-Cartan analog of  $L_{E-H}$ .

### 3. Poincaré Gauge Theory

The second way to generate torsion is via the connection, this time regarded as independent rotational degree of freedom of the Lorentz part of the Poincaré group. Inside each simplex, where we have a differentiable manifold, we augment the Levi-Civita connection  $\Gamma$  with a metric compatible contortion one-form  $K$ , which has therefore values in  $SO(3,1)$ . To preserve the piecewise linear structure we require that  $K$  is constant in each simplex. This time we impose no boundary condition because this would be too restrictive. We make use of the simplicial affine coordinates of Sorkin [7]. Given are  $n+1$  barycentric basis vectors  $e_i$ , together with a barycentric dual basis  $e^i$ , which satisfy  $\sum_i e_i = \sum_i e^i = 0$ , and  $\sum_i e_i \otimes e^i = 1$ ,  $i=0 \dots n$ .

The inner product is normalized according to

$$\langle e_i, e^j \rangle = \bar{\delta}_{ij} := \begin{cases} \frac{n}{n+1} & i=j \\ -\frac{1}{n+1} & i \neq j. \end{cases}$$

Sorkin has shown that the metric  $\bar{g}_{ij}$  is expressible in affine coordinates as

$$\bar{g}_{ij} = -\frac{1}{2} \delta_{mn} \bar{\delta}^m{}_i \bar{\delta}^n{}_j. \quad (5)$$

The affine coordinates define an affine frame  $e$  as an invertible map from the affine space  $A^n$  to  $R^n$ . The inverse relations are  $e^a{}_j e^j{}_b = \delta^a{}_b$ ,  $e^i{}_a e^a{}_j = \bar{\delta}^i{}_j$ , with  $a, b = 1, \dots, n$ , and  $i, j = 0, 1, \dots, n$ . Also, like usual,  $\bar{g}_{ij} = \eta_{ab} e^a{}_i e^b{}_j$ . For the torsion free case  $\Gamma$  vanishes in each simplex  $\sigma$ . The general connection one-form can be written as  $w = \Gamma + K$ . In each simplex we express  $K$  in affine coordinates by applying it to the affine vector basis,  $K(e_i) = \bar{K}_i$ . Because of linearity we find that  $\sum_i \bar{K}_i = 0$ . In each simplex the curvature two-form is  $R = dK + K \wedge K = K \wedge K$ .

The contribution  $S_K$  of the contortion to the Regge action  $S_R$  takes therefore the following form:

$$S_K = \int_M K^a_f \wedge K^b_\lambda \wedge e^c_\mu \wedge e^d_\nu \epsilon_{abcd} = \sum_\sigma \text{Vol}(\sigma) \bar{K}^a_i \bar{K}^b_j \bar{e}^c_m \bar{e}^d_n \bar{\epsilon}^{ijmn} \epsilon_{abcd} \\ = \sum_\sigma \text{Vol}(\sigma) [\bar{K}^i_j \bar{K}^m_n \bar{g}^{nm} - \bar{K}^m_i \bar{K}^j_n \bar{g}^{nm}] . \quad (6)$$

One can show that the contribution of  $K$  on the  $n-1$  simplex boundary is a complete divergence, so that we only have to consider contributions from the interior of the simplex. The variation of  $S_K$  with respect to  $\bar{K}^i_j$  leads in the source free case to  $\bar{K}^i_j = 0$ , whereas  $\delta S / \delta l^2_{ij}$  ( $S := S_R + S_K$ ) leads, after insertion of  $K=0$ , to the standard Regge equation [3], as is expected from the continuum result. The use of the simplicial coordinates simplifies the variation of  $S_K$  with respect to  $l_{ij}$ , which will become important for treating nontrivial spin-matter distributions. One should add, that it is possible to go over to a true first order formalism and use the affine  $n$ -beins  $e^a_i(\sigma)$  instead of the edgelengths  $l_{ij}^2$ . This would be closer to the gravitational continuum formalism of exterior forms. The inclusion of Dirac fields as spin sources is necessary in order to study realistic effects of torsion on space-times. In this respect it might be also interesting to formulate the higher derivative Lagrangians of Hehl [8] on the Regge lattice in order to obtain dynamical torsion. Although curvature square terms are difficult to formulate on simplexes, this is not so for terms which are quadratic in the contortion. Further studies are under investigations.

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## A symplectic derivation of the dynamics of any smoothly deformable medium

E. Binz

### Introduction

Let  $M$  and  $N$  both be smooth orientable manifolds.  $M$ , thought of a body consisting of a deformable medium, is supposed to be compact.  $N$  shall be Riemannian. A configuration of the medium is a smooth embedding of  $M$  into  $N$ . We will show that the equation of motion of such a medium moving and deforming in  $N$  can be derived in the realm of symplectic geometry on the configuration space  $E(M, N)$ , as in the Hamiltonian case. Moreover from this equations naturally splits off a generalized wave term. Generalized means here that the Laplacian is configuration dependent. External force densities are neglected here.

### 1. Geometric preliminaries

Let  $M$  be a compact, oriented, connected, smooth manifold and  $N$  be a connected, smooth and oriented manifold with a Riemannian metric  $\langle \cdot, \cdot \rangle$ , assumed to be fixed. A configuration  $j$  in the smooth Fréchet manifold  $E(M, N)$  of all smooth embeddings of  $M$  in  $N$  (cf. [Bi, Sn, Fi]) defines a Riemannian metric  $m(j)$  on  $M$  by setting

$$m(j)(X, Y) := \langle TjX, TjY \rangle, \quad \forall X, Y \in \Gamma(TM).$$

(More customary is the notation  $j^* \langle \cdot, \cdot \rangle$  instead of  $m(j)$ .) We use  $\Gamma(\mathbf{E})$  to denote the collection of all smooth sections of any smooth vector bundle  $\mathbf{E}$  over a manifold  $Q$ . Let  $\nabla$  be the Levi-Civita connection of the Riemannian manifold  $(N, \langle \cdot, \cdot \rangle)$ . The Levi-Civita connection  $\nabla(j)$  of  $m(j)$  on  $M$  is obtained as follows: If  $Y \in \Gamma(TM)$  then we set

$$Tj(\nabla(j)_X Y) := \nabla_X(TjY) - (\nabla_X(TjX))^\perp$$

for all  $X$ . Here  $\perp$  means the normal component with respect to  $\langle \cdot, \cdot \rangle$ . Instead of  $(\nabla_X(TjX))^\perp$  we write  $S(j)(X, Y)$  and call  $S(j)$  the second fundamental tensor of  $j$ .

The metric  $\langle \cdot, \cdot \rangle$  on  $N$  induces a "Riemannian structure"  $\mathcal{G}$  on the configuration space  $E(M, N)$  as follows: For  $j \in E(M, N)$ , let  $\mu(j)$  be the Riemannian volume defined on  $M$  by the given orientation and the metric  $m(j)$ . We set

$$\mathcal{G}(j)(l_1, l_2) := \int_M \langle l_1, l_2 \rangle \mu(j)$$

for any two tangent vectors  $l_1, l_2 \in T_j E(M, N) = \{l \in C^\infty(M, TN) \mid \pi_N \circ l = j\}$  abbreviated by  $C^\infty(M, TN)$ , where  $\pi_N : TN \rightarrow N$  is the canonical projection. It is clear that

$\mathcal{G}(j)$  is a continuous, symmetric, positive-definite bilinear form on  $C_j^\infty(M, TN)$  for each  $j \in E(M, N)$ .

## 2. A metric $\mathcal{B}$ on $E(M, N)$ and its associated one- and two-forms

The dynamics of deformable media to be introduced later relies on the metric  $\mathcal{B}$  on the configuration space  $E(M, N)$ , a *Diff* $M$ -principal bundle (cf. [Bi, Sn, Fi]). This metric will be based on a **density map**  $\rho: E(M, N) \rightarrow C^\infty(M, \mathbb{R})$  supposed to satisfy  $\rho(j)(p) > 0$  for any  $j \in E(M, N)$  and any  $p \in M$  as well as a continuity equation

$$d\rho(j)(k) = -\frac{\rho(j)}{2} \operatorname{tr}_{m(j)} dm(j)(k) \quad \forall j \in E(M, N) \text{ and } \forall k \in C_j^\infty(M, TN)$$

with  $\operatorname{tr}_{m(j)}$  being the trace formed with respect to  $m(j)$  (for the existence of such maps cf. [Bi]). The symbol  $d$  denotes the differential of maps of which the domain is a Fréchet manifold and which assume values in a Fréchet space. The metric  $\mathcal{B}$  is then defined by

$$\mathcal{B}(j)(l_1, l_2) := \int_M \rho(j) \langle l_1, l_2 \rangle \mu(j)$$

for each  $j \in E(M, N)$  and for each pair  $l_1, l_2 \in C_j^\infty(M, TN)$ . This metric depends smoothly on all of its variables.

The Levi-Civita connection and the one- and two-forms associated with  $\mathcal{B}$  are based on

$$d\mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)(h) = \mathcal{B}(j)((TK_1(h))^{vert}, \mathcal{K}_2) + \mathcal{B}(j)((TK_2(h))^{vert}, \mathcal{K}_1)$$

holding for any two  $\mathcal{K}_1, \mathcal{K}_2 \in \Gamma(TE(M, N))$  and any  $h \in C_j^\infty(M, TN)$ ; *vert* means the vertical component formed in  $T^2N$ . This shows that the covariant derivative

$$\nabla: \Gamma(TE(M, N)) \rightarrow \Gamma(TE(M, N))$$

given by

$$\nabla_k \mathcal{L}(p) = \left( T_l \mathcal{L}(k(p)) \right)^{vert} \quad \forall l, k \in T_j E(M, N), \forall p \in M \text{ and } \forall j \in E(M, N)$$

for any choice of  $\mathcal{L} \in \Gamma TE(M, N)$  is the Levi-Civita connection of  $\mathcal{B}$ . Here  $T_l \mathcal{L}$  denotes the tangent map of the vector field  $\mathcal{L}$  on  $E(M, N)$  at  $l$ . The one form  $\Theta_{\mathcal{B}}$  on  $TE(M, N)$  is given by

$$\Theta_{\mathcal{B}}(l)(k) := -\mathcal{B}(j)(l, T\pi_E(k)) = -\mathcal{B}(j)(l, T\pi_N \circ k).$$

Here  $\pi_E: TE(M, N) \rightarrow E(M, N)$  is the canonical projection. The two-form  $\omega_{\mathcal{B}}$  associated with  $\mathcal{B}$  reads as

$$\begin{aligned}\omega_{\mathcal{B}}(l)(k_1, k_2) &:= d\Theta_{\mathcal{B}}(l)(k_1, k_2) = \mathcal{B}(j)(k_2^{\text{vert}}, T\pi_N \circ k_1) - \mathcal{B}(j)(k_1^{\text{vert}}, T\pi_N \circ k_2) \\ &= \int_M \rho(\pi_N \circ l) \omega^b(k_1, k_2) \mu(\pi_N \circ l)\end{aligned}$$

for any two  $k_1, k_2 \in C_1^\infty(M, T^2N)$  and any  $l \in C_j^\infty(M, TN)$ . Here  $\omega^b$  is the pullback of the canonical two-form on the cotangent bundle  $T^*N$  of  $N$  by the metric  $\langle, \rangle$  on  $N$ .

### 3. One-forms on $E(M, N)$ as constitutive laws

The sorts of constitutive laws describing the quality of a deformable medium we have in mind, will be special one-forms  $F$  on the configuration space  $E(M, N)$ , in accordance with the definition as given e.g. in [E,S] and [Bi]. They are precisely those which yield a well defined force density  $\Phi : E(M, N) \rightarrow TE(M, N)$  on  $M$ .

On the Fréchet manifold  $\mathcal{A}_E^1(M, TN)$  of all smooth  $TN$ -valued one-forms covering embeddings we have a natural metric  $\mathcal{g}$ , called the dot metric (cf. [Bi] and [Bi,Fi]), which is nothing else but the Dirichlet integral. The one-form  $F$  on  $E(M, N)$  is said to be  $\mathcal{g}$ -representable if there exists a smooth section  $\alpha : E(M, N) \rightarrow \mathcal{A}_E^1(M, TN)$  of the bundle  $\mathcal{A}_E^1(M, TN)$ , such that

$$F(j)(l) = \int_M \alpha(j) \cdot \nabla l \mu(j) \equiv \mathcal{g}(j)(\alpha(j), \nabla l)$$

for  $j \in E(M, N)$  and  $l \in C_j^\infty(M, TN)$ . The  $\mathcal{g}$ -density  $\alpha$  of  $F$  is called the *stress form*. For the definition of the dot product in the integrand and for the proof of the following theorem we refer to [Bi], [Bi,Fi] and [Hö].

#### 3.1 Theorem:

$F$  is  $\mathcal{g}$ -representable iff it admits a smooth vector field  $\mathcal{H} : E(M, N) \rightarrow TE(M, N)$ , called a *constitution field*, for which

$$F(j)(l) = \int_M \langle \Delta(j)\mathcal{H}(j), l \rangle \mu(j)$$

holds for all variables of  $F$ . The map  $\Delta(j)\mathcal{H}(j) \in T_jE(M, N)$  is the (*internal*) *force density* at the configuration  $j \in E(M, N)$ . Vica versa, if  $\Phi : E(M, N) \rightarrow TE(M, N)$  is a smooth vector field satisfying the integrability condition for the problem

$$\Delta(j)\mathcal{H}(j) = \Phi(j) \quad \forall j \in E(M, N)$$

then  $F$  determined by  $\mathcal{H}$  is  $\mathcal{g}$ -representable with stress form  $\nabla\mathcal{H}$ . If  $\dim M = \dim N$ , then  $T$  given by  $T(X, Y) := \langle \nabla(j)_X \mathcal{H}(j), dJ \circ Y \rangle$  for any  $X, Y \in \Gamma TM$  is a stress tensor of

which  $\Phi(j)$  is the divergence for any  $j \in E(M, N)$ . Thus  $\mathcal{H}$  and  $T$  describe the medium equivalently.

As an example of a constitutive law we consider the derivative  $d\mathcal{A}$  of

$$\mathcal{A}(j) := \int_M \mu(j) \quad \forall j \in E(M, N),$$

called the volume of  $M$  determined by  $m(j)$ . The derivative can be represented as

$$d\mathcal{A}(j)(l) = \int_M Tj \cdot \nabla(j)l \mu(j) = \mathcal{G}(j)(\Delta(j)\bar{j}, l) \quad \forall j \in E(M, N). \quad (3.1)$$

$\Delta(j)\bar{j}$  is defined as follows: Let  $\nabla^*$  denote the covariant divergence, then

$$\Delta(j)\bar{j} = -\nabla^*(Tj) \quad (3.2)$$

for some  $\bar{j} \in C_j^\infty(M, TN)$  determined up to a harmonic field along  $j$  (the integrability condition of (3.2) is satisfied). Clearly  $\Delta(j)\bar{j}$  is pointwise normal to  $Tj(M) \subset TN$  as one immediately deduces from the theorem of Gauss.  $\bar{j}$  can be chosen such that it depends smoothly on  $j$ . Since  $S(j)(X, Y) := (\nabla_X(Tj)Y)^\perp$  is symmetric in  $X$  and  $Y$ , we find

$$\nabla^*(Tj) = -\text{tr } S(j) = - \sum_{i=1}^{\dim M} S(j)(E_i, E_i) .$$

$E_1, \dots, E_{\dim M}$  being a moving orthonormal frame on  $M$ . The vector field  $\text{tr } S(j)$  along  $j$  is called the mean curvature field. Clearly  $Tj$  and  $\nabla\bar{j}$  are identical.

Let  $F$  be any constitutive law with constitutive field  $\mathcal{H}$ . We will split off  $d\mathcal{A}$  from  $F$ , based on (3.1): Recall that  $L_j^2(M, TN)$  is the space of all vector fields  $l$  of  $M$  along  $j$  for which  $\int_M \langle l, l \rangle \mu(j) = \|l\|_{\mathcal{G}}^2$  is finite. Then taking the component of  $\Delta(j)\mathcal{H}(j)$  along  $\bar{j}$  in  $L_j^2(M, TN)$  for each  $j$  yields

$$\Delta(j)\mathcal{H}(j) = \mathbf{a}(j) \cdot \Delta(j)\bar{j} + \Delta(j)\mathcal{H}_1(j) \quad (3.3)$$

for a well defined  $\mathbf{a}(j) \in \mathbb{R}$  and some  $\mathcal{H}_1(j) \in C_j^\infty(M, N)$  for which  $\mathcal{H}_1(j)$  is orthogonal to  $\Delta(j)\bar{j}$  in  $L_j^2(M, TN)$ . By looking at (3.3) we have immediately the following:

### 3.2 Theorem:

For each constitutive law  $F$ , the constitutive field  $\mathcal{H}$  determines uniquely a smooth map, the capillarity  $\mathbf{a} : E(M, N) \rightarrow \mathbb{R}$  given for each  $j \in E(M, N)$  by

$$\mathbf{a}(j) := F(j) (\bar{j} / \|\nabla\bar{j}\|_{\mathcal{G}}^2)$$

and splits uniquely into

$$\mathcal{H}(j) = \mathbf{a}(j) \cdot \bar{j} + \mathcal{H}_1(j) \quad \forall j \in E(M, N).$$

Here  $\mathcal{H}_1$  has vanishing volume sensitive part, is  $L_2$ -orthogonal to  $\Delta(j)\bar{j}$  and varies smoothly with  $j$ .

#### 4. The dynamics determined by a constitutive law

Let  $F : TE(M, N) \rightarrow \mathbb{R}$  be constitutive law with a smooth constitutive vector field  $\mathcal{H} \in \Gamma(TE(M, N))$ . The (not necessarily exact) **work form**  $W_F : T^2E(M, N) \rightarrow \mathbb{R}$  is given by

$$W_F(l)(k) := d\mathcal{E}_{kin}(l)(k) - (\pi_E^* F)(l)(k)$$

for any  $l \in TE(M, N)$  and for any  $k \in T^2E(M, N)$  with  $\mathcal{E}_{kin}(j)(l) := \frac{1}{2}\mathcal{B}(j)(l, l)$ . The dynamics determined by  $F$  is defined by the unique vector field  $\mathcal{X}_F$  for which

$$\omega_B(\mathcal{X}_F, \mathcal{X}) = W_F(\mathcal{X}) \quad \forall \mathcal{X} \in \Gamma T^2E(M, N).$$

Its existence is established in the following which is easily verified:

##### 4.1 Theorem:

Given a constitutive law  $F$  on  $E(M, N)$  with constitutive field  $\mathcal{H}$  then

$$\mathcal{X}_F(l) = \mathcal{S} \circ l + \frac{1}{\rho(\pi_N \circ l)} \cdot (\Delta(\pi_N \circ l)\mathcal{H}(\pi_N \circ l))^{\text{vert}} \quad \forall l \in TE(M, N)$$

where *vert* denotes the pointwise formed vertical lift on  $N$  determined by  $\nabla$  and where  $\mathcal{S}$  is the spray of  $\langle, \rangle$  on  $TN$ .

The theorems below are also immediate, the second one is based on theorem 3.1.

##### 4.2 Theorem:

The equation of a motion  $\sigma : (-\lambda, \lambda) \rightarrow E(M, N)$  subjected to  $F$  with constitutive field  $\mathcal{H} \in \Gamma(TE(M, N))$  and with any initial data is given by

$$\nabla_{\frac{d}{dt}} \dot{\sigma}(t) = \frac{1}{\rho(\sigma(t))} \Delta(\sigma(t))\mathcal{H}(\sigma(t)).$$

The motion  $\sigma$  is free i.e a geodesic iff  $F = 0$ .



**4.3 Theorem:**

Let  $F$  be a constitutive law with constitutive field  $\mathcal{H}$ . Any motion  $\sigma : (-\lambda, \lambda) \rightarrow E(M, N)$  (with any initial condition) is subjected to  $F$  iff

$$\nabla_{\frac{d}{dt}} \dot{\sigma}(t) = \frac{\mathbf{a}(\sigma(t))}{\rho(\sigma(t))} \cdot \Delta(\sigma(t))\bar{\sigma}(t) + \frac{1}{\rho(\sigma(t))} \Delta(\sigma(t))\mathcal{H}_1(\sigma(t)) \quad \forall t \in (-\lambda, \lambda).$$

Moreover the balance law

$$d\mathcal{E}_{kin}(\sigma(t))(\dot{\sigma}(t)) = \mathbf{a}(\sigma(t)) \cdot d\mathcal{A}_F(\sigma(t))\dot{\sigma}(t) + \pi_N^* F_1(\sigma(t))\dot{\sigma}(t) \quad \forall t \in (-\lambda, \lambda)$$

holds true.  $F_1$  is the constitutive law associated with  $\mathcal{H}_1$ , the volume insensitive part of  $\mathcal{H}$ . If  $N$  is Euclidean then  $\bar{\sigma}(t) = \sigma(t)$ ; if in addition  $1 + \dim M = \dim N$  and  $\mathcal{N}(\sigma(t))$  is the positively oriented unit normal field along  $\sigma(t)$  then

$$\Delta(\sigma(t))\sigma(t) = H(\sigma(t))\mathcal{N}(\sigma(t))$$

with  $H(\sigma(t))$  being the non normalized mean curvature of  $\sigma(t)$ .

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**XI. Nonlinear Partial  
Differential Equations,  
Dynamical Systems  
and Neural Networks**

**On the role of symmetries in PDE – problems**

E. Brüning

0. Introduction: It is fairly well understood that and how by various mechanisms the presence of a compact symmetry group in a PDE – problem typically induces a considerable simplification. This happens for instance in the symmetric Ljusternik – Schnirelman multiplicity theory (reference [1] and references there), the principle of symmetric critically [6], and the symmetric bifurcation theory [4].

In contrast to this standard role it has been discovered and partially understood recently that and how the presence of a non compact symmetry group can add a new complexity to the problem leading to a considerable complication [2, 5, 3, 1]. This we want to explain here for the following class of eigenvalue – problems for systems of global semilinear elliptic operators, i.e. for the problems to determine a real number  $\lambda$  such that the equation

$$-\Delta u(x) = \lambda g(x, u(x))$$

$$g : \mathbb{R}^n \times \mathbb{R}^N \rightarrow \mathbb{R}^N, \quad n \geq 3, \quad N \geq 1, \quad g(x, 0) = 0 \quad \text{a.e.} \quad (0.1)$$

admits a solution  $u = u_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^N$ .

We decide to look for solutions in the space  $E$  of finite energy functions  $u : \mathbb{R}^n \rightarrow \mathbb{R}^N$ ,

$$K(u) = \frac{1}{2} \|Du\|_2^2 < \infty. \quad (0.2)$$

$E$  is defined as the completion of  $\mathcal{C}_0^\infty(\mathbb{R}^n; \mathbb{R}^N)$  with respect to the energy norm

$$f \mapsto \|Df\|_2. \quad \text{Note that } E \subset L^{2^*}(\mathbb{R}^n; \mathbb{R}^N), \quad 2^* = \frac{2n}{n-2}.$$

We rely on a variational approach (constrained minimization) and assume accordingly

$$g(x, y) = \frac{\partial G}{\partial y}(x, y), \quad G = \text{potential of } g, \quad G(x, 0) = 0 \quad \text{a.e.} \quad (0.3)$$

$g, G$  are Carathéodory – functions.

On the domain

$$D(V) = \{u \in E \mid \hat{G}(u) \in L^1(\mathbb{R}^n)\}, \quad \hat{G}(u)(x) = G(x, u(x)) \quad (0.4)$$

we define

$$V(u) = \int \hat{G}(u) \, dx. \quad (0.5)$$

1. Constrained minimization: The symmetry group.

For a nonempty level surface  $V^{-1}(c) = \{u \in D(V) \mid V(u) = c\}$  denote

$$I = I_c(K, V) = \inf \{K(u) \mid u \in V^{-1}(c)\} \quad (1.1)$$

$$M = M_c(K, V) = \{u \in V^{-1}(c) \mid K(u) = I\} \quad (1.2)$$

$$S = S_c(K, V) = \{\underline{v} = (v_j) \in V^{-1}(c)^{\mathbb{N}} \mid \lim_{j \rightarrow \infty} K(v_j) = I\}$$

(1.3)

The symmetry group  $\mathcal{G} = \mathcal{G}(K, V)$  for (0.1) is defined by

$$\mathcal{G} = \{\phi: D(V) \rightarrow D(V) \mid V(\phi u) = V(u), K(\phi u) \leq K(u) \quad \forall u \in D(V)\} \quad (1.4)$$

and it follows immediately

$$u \in M \Rightarrow \mathcal{G}u = \{\phi u \mid \phi \in \mathcal{G}\} \subseteq M \quad (1.5)$$

$$\underline{v} = (v_j) \in S, \phi = (\phi_j) \in \mathcal{G}^{\mathbb{N}} \Rightarrow \phi(\underline{v}) = (\phi_j v_j) \in S. \quad (1.6)$$

Case studies show that  $\mathcal{G}$  can contain a compact subgroup (e.g. rotations), a noncompact subgroup (e.g. translations), and some discrete elements (e.g. spherically symmetric rearrangement of functions). In the local version of (0.1) one shows by means of compact Sobolev embeddings

$$S = C_1 = \{\underline{v} \in S \mid v = w - \lim_{i \rightarrow \infty} v_{j(i)} \in V^{-1}(c) \text{ for some subsequence}\}$$

but in our global version because of the action (1.6) of the symmetry group the following subclasses of the space  $S$  of minimising sequences for (1.1) can occur:

$$C_2 = \{\underline{v} \in S \mid v = w - \lim_{i \rightarrow \infty} v_{j(i)} \notin V^{-1}(c) \text{ for every subsequence}\}$$

$$C_3 = \{\underline{v} \in S \mid \exists \phi = (\phi_j) \in \mathcal{G}^{\mathbb{N}}: \phi(\underline{v}) \in C_1\}$$

$$C_4 = \{\underline{v} \in S \mid \phi(\underline{v}) \in C_2 \forall \phi \in \mathcal{G}^{\mathbb{N}}\}.$$

The occurrence of these subclasses  $C_2, C_3, C_4$ , adds a new complexity to this minimisation problem. The subclass  $C_2$  occurs whenever the symmetry  $\mathcal{G}$  has a noncompact subgroup corresponding to unbounded orbits in the underlying Euclidean space  $\mathbb{R}^n$ .

Recall that by the Lagrange multiplier theorem every minimiser  $u \in M$  which is a regular point of  $V^{-1}(c)$  is a weak solution of (0.1), i.e. it satisfies

$$K'(u) = \lambda V'(u) \quad (1.8)$$

for some  $\lambda = \lambda(u) \in \mathbb{R}$ , if  $K', V'$  denote the Fréchet derivatives.

2. The concentration functional: Our growth restrictions for the potential  $G$  are that  $\forall y \in \mathbb{R}^n$ , almost all  $x \in \mathbb{R}^n$

$$|G(x, y)| \leq A(x) + B_1(x)|y|^{2^*/p} [|y| \leq 1] + B_2(x)|y|^{2^*/q} [|y| > 1] \quad (2.1)$$

with  $1 \leq p \leq 2^*$ ,  $1 < q \leq 2^*$ ;  $A \in L^1(\mathbb{R}^n)$ ,  $B_1 \in L^{p'}(\mathbb{R}^n)$ ,

$B_2 \in L^{q'}_{loc}(\mathbb{R}^n)$ , and if  $p = 1$ :  $\|1_{B(r)}^+\|_{\infty} \rightarrow 0$  for  $r \rightarrow \infty$  where

$B(r) = \{x \in \mathbb{R}^n \mid \|x\| \leq r\}$  and where the prime denotes the Hölder - conjugate exponent. And similar bounds for  $g$  but with exponents lowered by 1. It follows

$$D(V) = E \quad \text{and} \quad V \in \mathcal{C}^1(E) \quad (2.2)$$

Lemma:  $F : S \rightarrow \mathbb{R}$  is well defined by

$$F_c(\underline{v}) = \lim_{r \rightarrow \infty} (\lim_{i \rightarrow \infty} (\inf_{\phi \in \mathcal{F}} \|1_{B(r)}^+ \hat{G}(\phi v_{j(i)})\|_1)) \quad (2.3)$$

and satisfies

$$0 \leq F_c(\underline{v}) \leq \lambda(\underline{v}) \equiv \sup_j \|\hat{G}(v_j)\|_1 \quad (2.4)$$

$$F_c(\phi \underline{v}) = F_c(\underline{v}) \quad \forall \underline{v} \in S \quad \forall \phi \in \mathcal{F}^{\mathbb{N}} \quad (2.5)$$

and one shows

Theorem:  $F_c(\underline{v}) = 0$  iff  $\exists \phi \in \mathcal{F}^{\mathbb{N}}$  such that  $\underline{u} = \phi(\underline{v})$

satisfies the concentration condition (C)

$$\forall \epsilon > 0 \exists r > 0 : \sup_i \|1_{B(r)}^+ \hat{G}(u_{j(i)})\|_1 \leq \epsilon \quad (C)$$

Therefore  $F_c$  is called the concentration functional. It follows

Lemma:  $F_c(\underline{v}) = 0 \Rightarrow \underline{v} \in C_3$

and thus

Theorem: There are minimizers for the constrained minimization problem (1.1), i.e.,  $M \neq \emptyset$ . If  $V^{-1}(c)$  contains only regular points every  $u \in M$  is a weak solution of (0.1), i.e.,  $u$  satisfies (1.4).

**Remarks :** (a) It is known how to implement that  $V^{-1}(c)$  contains only regular points [1, 2]. (b) By an extended version of elliptic regularity theory [1] one can also obtain classical solutions of (0.1). (c) In the presence of a symmetry group of scale transformations  $T_\sigma : E \rightarrow E$  leaving  $\mathcal{G}_0^\infty$  invariant and satisfying for all  $\sigma > 0$  and all  $u \in D(V)$

$$\begin{aligned} V(T_\sigma u) &= \sigma^s V(u) && \text{for some } s \in \mathbb{R} \\ K(T_\sigma u) &= \sigma^\gamma K(u) && \text{for some } 0 < \gamma < 1 \end{aligned}$$

one can proceed similarly to show that every  $\lambda > 0$  is an eigenvalue of (0.1) even under much weaker growth restrictions on  $G$  allowing  $D(V) \not\subseteq E$ . Then in particular the existence of a solution of the system of global classical field equations ( $\lambda = 1$  in 0.1)) follows.

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## SECOND-ORDER DIFFERENTIAL INVARIANTS FOR THE POINCARÉ, GALILEI AND CONFORMAL ALGEBRAS IN MANY-DIMENSIONAL SPACES

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We investigate differential invariants in the framework of symmetry analysis of differential equations. Knowledge of differential invariants of a certain algebra or group facilitates classification of equations invariant with respect to this algebra or group. There are also some general methods for investigation of differential equations which need the explicit form of differential invariants for these equations' symmetry group. Differential invariants were considered by S. LIE [3]. TRESSE [3] had proved the theorem on the existence and finiteness of a functional basis of differential invariants for a Lie algebra.

Speaking about differential invariants we mean absolute ones. Necessary definitions can be found e.g. in [3, 3].

We adduce here a basis of second-order differential invariants for the invariance group of the free wave equation and describe sets of invariants for the invariance group of the free Schrödinger equation and for some of its subalgebras. The conformal algebra  $AC(1, n)$  is defined by its basis operators

$$p_\mu = i g_{\mu\nu} \frac{\partial}{\partial x_\nu}, \quad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu,$$

$$D = x_\mu p_\mu + u_r p_{u^r}, \quad \left( p_{u^r} = -i \frac{\partial}{\partial u^r}, \quad r = 1, 2, \dots, m \right)$$

$$K_\mu = 2x_\mu D - x_\nu x_\nu p_\mu.$$

Here  $i$  is the imaginary unit,  $\mu, \nu$  take the values  $0, 1, \dots, n$ ; the summation is implied over the repeated indices, if they are small Greek letters, in the following way:

$$x_\nu x_\nu = x_\nu x^\nu = x_0^2 - x_1^2 - \dots - x_n^2, \quad g_{\mu\nu} = \text{diag}(1, -1, \dots, -1).$$

We put that  $x_\nu$  and  $x^\nu$  are equivalent with respect to summation not to mix up signs of derivatives and numbers of functions.

**Statement 1.** Any absolute differential invariant for the Poincaré algebra  $AP(1, n) = \langle p_\mu, J_{\mu\nu} \rangle$  of order  $\leq 2$  for a set of  $m$  scalar functions  $u = (u^1, u^2, \dots, u^m)$  is a function of the following expressions:

$$u^r, \quad R_k(u_\mu^r, u_{\mu\nu}^1), \quad S_{jk}(u_{\mu\nu}^r, u_{\mu\nu}^1).$$

$$k = 1, \dots, n+1; \quad j = 0, \dots, k; \quad r = 1, \dots, m.$$

We designate as  $R_k$ ,  $S_{jk}$  invariants of rotational group in Minkowski space

$$R_k(u_\mu, u_{\mu\nu}) \equiv u_{\mu_1} u_{\mu_2} u_{\mu_1 \mu_2} u_{\mu_2 \mu_3} \cdots u_{\mu_k \mu_{k-1}}.$$

$$S_{jk}(u_{\mu\nu}, v_{\mu\nu}) \equiv u_{\mu_1 \mu_2} u_{\mu_2 \mu_3} \cdots u_{\mu_{j-1} \mu_j} v_{\mu_j \mu_{j+1}} \cdots v_{\mu_k \mu_1}.$$

$$v_\mu \equiv \frac{\partial u}{\partial x_\mu}, \quad u_{\mu\nu} \equiv \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}.$$

The rule of summation for Greek indices is as assumed above.

**Statement 2.** A basis of second order differential invariants for algebra  $AC(1, n)$  and for a scalar function  $u$  contains  $n+1$  elements e.g. of the form

1. when  $\lambda = 0$ :

$$u, \quad S_k(w_{\mu\nu})(u_\alpha u_\beta)^{-2k}, \quad k = 1, \dots, n;$$

2. when  $\lambda \neq 0$ :

$$\hat{S}_k(\theta_{\mu\nu}) u^{k(2/\lambda-1)}, \quad k = 1, \dots, n;$$

where

$$\theta_{\mu\nu} = \lambda u_{,\mu\nu} + (1-\lambda) \frac{u_\mu u_\nu}{u} - g_{\mu\nu} \frac{u_\alpha u_\alpha}{2u},$$

$$w_{\mu\nu} = u_\alpha u_\alpha (u_{\mu\nu} + \frac{g_{\mu\nu}}{1-n} u_{,33}) - u_\alpha (u_\mu u_{\alpha\nu} + u_\nu u_{\alpha\mu}),$$

$$\hat{S}_k(\theta_{\mu\nu}) \equiv \theta_{\mu_1 \mu_2} \theta_{\mu_2 \mu_3} \cdots \theta_{\mu_k \mu_1}.$$

**Definition.** Tensors  $\theta_a$  and  $\theta_{ab}$  of order 1 and 2 are called covariant with respect to some algebra  $L$  if

$$X_j \theta_a = \rho_j \theta_a + \rho'_{a'} \theta_{a'},$$

$$X_j \theta_{ab} = \sigma_j \theta_{ab} + \sigma'_{a'} \theta_{cb} + \sigma'_{b'} \theta_{ac},$$

$$L = \langle J_{ab}, X_j \rangle, \quad a, b, c = 1, \dots, n.$$

$X_j$  are first-order differential operators,  $\rho_j, \sigma_j$  are some functions,  $\rho'_{a'}, \sigma'_{a'}$  are skew-symmetric tensors.

It is easy to show that the expressions  $S_k(\theta_{j^k}), R_k(\theta_j, \theta_{ab})$ , where  $\theta_a, \theta_{ab}$  are  $L$ -covariant tensors, are relative invariants of this algebra. Further we adduce tensors for which  $\rho_j, \sigma_j = 0$  and  $\hat{S}_k, S_{jk}, R_k$  are absolute invariants.



Such covariant tensors for the conformal algebra  $AC(1, n)$  and a set of  $m$  scalar functions  $u^r$  are of the form

1. when  $\lambda \neq 0$ :

$$\theta_\mu^r = (u_\mu^r/u^r - u_\mu^1/u^1)(u^1)^{1/\lambda}$$

$$\theta_{\mu\nu}^r = (u^1)^{2/\lambda-1}(\lambda u_{\mu\nu}^r + (1-\lambda)u_\mu^r u_\nu^r/u^r - g_{\mu\nu}u_\alpha^r u_\beta^r/2u^r);$$

2. when  $\lambda = 0$ :

$$u_{\mu\nu}^r = (u_\alpha^1 u_\beta^1)^{-2}(u_\alpha^r u_\beta^r(u_{\mu\nu}^r - \frac{g_{\mu\nu}}{1-n}u_\alpha^r u_\beta^r) - u_\alpha^r(u_\mu^r u_{\beta\nu}^r + u_\nu^r u_{\beta\mu}^r)).$$

Here we do not assume summation over  $r$ .

The Galilei algebra  $AG_2(1, n)$  which is the symmetry algebra of the free Schrödinger equation is defined by basis operators [3]

$$\begin{aligned} p_0 &= i\frac{\partial}{\partial t}, & p_a &= i\frac{\partial}{\partial x_a}, & J_{ab} &= x_a p_b - x_b p_a, & G_a &= t p_a - m x_a J, \\ J &= i(v\partial_v - v^* \partial_{v^*}), & D &= 2t p_0 - x_a p_a + \lambda I & (I &= v\partial_v + v^* \partial_{v^*}) \\ A &= t^2 p_0 - t x_a p_a + \lambda t I + \frac{m x^2}{2} J; & v &= v(x, t), & x &= (x_1, \dots, x_n) \\ AG(1, n) &= \langle p_0, p_a, J, J_{ab}, G_a \rangle, & AG_1(1, n) &= \langle p_0, p_a, J, J_{ab}, g_a, D \rangle. \end{aligned}$$

To simplify the form of invariants we introduce the change of dependent variable:

$$v = \exp \Phi, \quad \text{Im}(\Phi) = \tan^{-1}(\text{Im}v/\text{Re}v).$$

For the algebras listed below we adduce only invariants depending on  $\Phi$ ,  $\Phi_i$ ,  $\Phi_{ii}$  and absolutely covariant tensors:

1.  $AG(1, n)$ ,  $m \neq 0$ :

$$\begin{aligned} \Psi + \Phi^*, & \quad M_1 = 2im\Phi_t + \Phi_a \Phi_a, & M_1^*, \\ M_2 &= -m^2 \Phi_{tt} + 2im\Phi_a \Phi_{at} + \Phi_a \Phi_b \Phi_{ab}, & M_2^*, \\ \Phi_{ab}, & \quad \Phi_{ab}^*, & \theta_a = im\Phi_{at} + \Phi_a \Phi_{ab}, & \theta_a, & \Phi_a + \Phi_a^*. \end{aligned}$$

2.  $AG_1(1, n)$ ,  $m \neq 0$ :

$$\begin{aligned} \frac{M_1^*}{M_1}, & \quad \frac{M_2}{M_1^2}, & \frac{M_2^*}{M_1^2}, & \quad \Phi + \Phi^* & (\lambda = 0), & M_1 \exp \frac{2}{\lambda}(\Phi + \Phi^*) & (\lambda \neq 0), \\ \Phi_{ab} M_1^{-1}, & \quad \Phi_{ab}^* M_1^{-1}, & \theta_a M_1^{-3/2}, & \quad \theta_a^* M_1^{-3/2}, & (\Phi_a + \Phi_a^*) M_1^{-1/2} \end{aligned}$$

3.  $AG_2(1, n)$ ,  $m \neq 0$ ;  $\lambda = -n/2$ :

$$\frac{N_1}{\exp \frac{4}{n}(\Phi + \Phi^*)}, \quad \frac{N_1^*}{N_1}, \quad \frac{N_2}{N_1^2}, \quad \frac{N_2^*}{N_1^2}, \quad \hat{\theta}_a = \theta_a M_1^{-3/2}, \quad \hat{\theta}_a^*,$$

$$\theta_{ab} = (\Phi_{ab} - \frac{2\delta_{ab}}{n} M_1) M_1^{-1}, \quad \theta_{ab}^*.$$

where summation over lowercase Latin indices is as follows:

$$x_a x_a = x_1^2 + x_2^2 + \dots + x_n^2; \quad N_1 = 2im\Phi_t + \Phi_{aa} + \Phi_a \Phi_a,$$

$$N_2 = \frac{1}{n} \Phi_{aa} N_1 + M_2.$$

Examples of various Poincare and Galilei invariant equations for scalar fields can be found in [3]. With the described bases of invariants it is possible to construct wide classes of new invariant equations.

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ON COMPLEX NONLINEAR RELATIONS FOR THE WAVE MECHANICAL DESCRIPTION OF  
THE DYNAMICS AND ENERGETICS OF CONSERVATIVE AND DISSIPATIVE SYSTEMS

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In wave mechanics, the information about the state and dynamics of a system is contained in a generally complex state vector or wave function and can be extracted by suitably chosen operators. The time evolution, as well as the energy of the system, are determined by the Hamiltonian operator  $H$ , which can be obtained from the classical Hamiltonian function via so-called canonical quantization. Thus, the differential equation governing the dynamics of the system can be written in the form

$$i\hbar \dot{\Psi} = H \Psi = -\frac{\hbar^2}{2m} \Delta \Psi + V\Psi \quad (1)$$

which is the well-known linear time-dependent Schrödinger equation (SE), containing the conservative potential  $V$ . The energy of a system described by this equation is a constant of motion and the evolution is reversible in time.

Attempts to include dissipative forces, such as frictional forces which transfer mechanical energy into other forms, like e.g. heat, thus opening the possibility of having an irreversible time evolution, face several difficulties. In particular, a corresponding wave equation cannot simply be obtained via canonical quantization, because, for dissipative systems, a classical Hamiltonian with the same physical meaning as in the reversible theory has not been known until recently [1].

In earlier works [2-5], we have shown that it is possible to obtain a satisfactory wave equation, starting from Newton's form of classical mechanics and using three axioms taken from experimental experience: (1) the uncertainty principle or complementarity, (2) the occurrence of interference phenomena in experiments with material systems, and (3) the correspondence principle, specified in the form of Ehrenfest's theorem. This formalism was first proposed by Madlung and Mrowka [6,7] in order to "rederive" the SE for didactical reasons. We extended the method by adding a diffusion term to the differential equation for a distribution function  $\rho(\vec{r},t)$ , thus changing the reversible continuity equation

$$\dot{\rho} + \text{div } \vec{j} = \dot{\rho} + \text{div}(\rho \vec{v}) = 0 \quad (2)$$

into the irreversible Fokker-Planck-type equation

$$\dot{\rho} + \text{div}(\vec{j} + \vec{j}_D) = \dot{\rho} + \text{div}(\rho \vec{v}) - D \Delta \rho = 0. \quad (3)$$

This equation for the wave-intensity-like quantity  $\rho$  can be separated into equations for the complex wave-amplitude-like quantity  $\Psi$  and its conjugate complex, using the additional condition

$$-D \frac{\Delta \rho}{\rho} = \gamma (\ln \rho - \langle \ln \rho \rangle) \quad , \quad (4)$$

where  $\langle \dots \rangle = \int \Psi^* \dots \Psi d\vec{r}$  denotes mean values. The resulting wave equation is the nonlinear Schrödinger-type equation (NLSE)

$$i \hbar \dot{\Psi} = \left\{ -\frac{\hbar^2}{2m} \Delta + V + \gamma \frac{\hbar}{i} (\ln \Psi - \langle \ln \Psi \rangle) \right\} \Psi \quad , \quad (5)$$

where the logarithmic nonlinearity is connected with a linear-velocity-dependent frictional force with friction constant  $\gamma$ . Properties of this equation, special consequences of the additional nonlinear term, and exact solutions are described in detail in Refs. [2-5].

Using the most simple, but also most important examples, namely, the one-dimensional free motion and the one-dimensional harmonic oscillator, it will be shown that, for the usual reversible as well as for the frictionally damped irreversible case, it is possible to describe the dynamical properties contained in the time-dependent (linear and nonlinear) SEs equally well by a set of Newtonian equations of motion. However, these equations are coupled in a unique way, which is connected with a rather unusual "conservation of angular momentum" in a complex plane.

For the potentials  $V=0$  and  $V = \frac{m}{2} \omega^2 x^2$  it is possible to obtain Gaussian shaped wave-packet-type (WP) functions as solutions of the usual SE as well as of our NLSE. The *particle* aspect is expressed by the fact that the maximum of the WP follows the classical trajectory; the *wave* aspect is contained in the WP width. The equations of motion for these two dynamical properties are not independent, but uniquely connected.

A Gaussian WP solution of the reversible SE (1) can be written in the form

$$\Psi_L(x,t) = N_L(t) \exp \left\{ i \left[ y(t) \bar{x}^2 + \frac{1}{\hbar} \langle p \rangle \bar{x} + K(t) \right] \right\}, \quad (6)$$

where  $\bar{x} = x - \langle x \rangle = x - \eta(t)$ , i.e. the maximum of the WP is at the position of

the classical trajectory,  $\langle x \rangle = \eta(t)$ .  $N_L$  is a normalization factor,  $\langle p \rangle = m\dot{\eta}$ , and the purely time-dependent term  $K(t)$  can be considered as a phase factor that will not be relevant for the further discussion. The width of the WP,  $\langle \tilde{x}^2 \rangle^{1/2}$ , is connected with the complex quantity  $y(t) = y_R(t) + iy_I(t)$  in front of the quadratic term  $\tilde{x}^2$  in the exponent via

$$\frac{2\hbar}{m} y_I = \frac{\hbar}{2m \langle \tilde{x}^2 \rangle} = \frac{1}{\alpha^2(t)} \quad (7)$$

Inserting WP (6) into Eq. (1) yields the usual Newtonian equation for the mean value of position, determining the path of the WP maximum (for  $V=0$ , in all following equations the terms containing  $\omega$  disappear),

$$\ddot{\eta} + \omega^2 \eta = 0 \quad (8)$$

where overdots denote time-derivatives.

In order to determine the time dependence of the WP width, the complex (quadratic) nonlinear equation of Riccati-type

$$\frac{2\hbar}{m} \dot{y} + \left( \frac{2\hbar}{m} y \right)^2 + \omega^2 = 0 \quad (9)$$

has to be solved. This means that a coupled pair of differential equations has to be solved. From the imaginary part of Eq. (9) follows

$$\frac{2\hbar}{m} y_R = -\frac{1}{2} \frac{\dot{y}_I}{y_I} = \frac{\dot{\alpha}}{\alpha} \quad (10)$$

Inserting this into the real part of Eq. (9) finally leads to the (real) Newton-type equation

$$\ddot{\alpha} + \omega^2 \alpha = \frac{1}{\alpha^3} \quad (11)$$

for  $\alpha = [(2m/\hbar)\langle \tilde{x}^2 \rangle]^{1/2}$ , which is up to a constant factor equivalent to the WP width.

Thus, Eq. (11), which formally differs from Eq. (8) for the WP maximum only by an additional inverse cubic term  $1/\alpha^3$ , determines the time dependence of the WP width. Equations (11) and (8) can be solved analytically.

The corresponding equations for the WP solutions of our NLSE are

$$\ddot{\eta} + \gamma \dot{\eta} + \omega^2 \eta = 0 \quad (12)$$

for the WP maximum (particle aspect) and

$$\ddot{\alpha} + \left( \omega^2 - \frac{\gamma^2}{4} \right) \alpha = \frac{1}{\alpha^3} \quad (13)$$

for the WP width (wave aspect).

In the case of the reversible SE, the connections between the dynamics of the particle and wave aspects can be found by linearizing the Riccati equation (9) with the help of the ansatz

$$\frac{2\hbar}{m} y = \frac{\dot{\lambda}}{\lambda} \quad , \quad (14)$$

yielding the linear Newtonian equation

$$\ddot{\lambda} + \omega^2 \lambda = 0 \quad (15)$$

for the complex quantity  $\lambda = \hat{u} + i \hat{z}$  (similar relations for the NLSE are given in Ref. [8]).

It can be shown (for details see e.g. Ref. [8]) that the imaginary part of  $\lambda$  is directly proportional to the classical trajectory and thus to the position of the WP maximum,

$$\frac{\hat{z} p_0 \alpha_0}{m} = \frac{z p_0}{m} = \langle x \rangle = \eta(t) \quad . \quad (16)$$

Furthermore, real and imaginary parts of  $\lambda$  are uniquely connected via the relation

$$\hat{z} \hat{u} - \hat{u} \hat{z} = 1. \quad (17)$$

A simple way to show the connection of  $\hat{z}$  and  $\hat{u}$ , or  $\lambda$ , with the width of the WP can be given, if  $\lambda$  is written in polar coordinates in the complex plane,

$$\lambda = \hat{u} + i \hat{z} = \alpha e^{i\varphi} \quad , \quad (18)$$

with  $\alpha = (\hat{u}^2 + \hat{z}^2)^{1/2}$ . The logarithmic time derivative of this quantity yields

$$\frac{\dot{\lambda}}{\lambda} = \frac{\dot{\alpha}}{\alpha} + i \dot{\varphi} \quad . \quad (19)$$

Comparison with Eqs. (14), (10) and (7) shows that the absolute value of  $\lambda$  is identical with the quantity  $\alpha$  that is proportional to the square root of  $\langle \tilde{x}^2 \rangle$  and that

$$\dot{\varphi} = \frac{1}{\alpha^2} \quad (20)$$

is valid. Thus, via  $\alpha^2 = \hat{u}^2 + \hat{z}^2 = (2m/\hbar) \langle \tilde{x}^2 \rangle$  knowledge of  $\hat{z}$  (from the classical trajectory) and thus  $\hat{u}$  (from Eq. (17)) also yields  $\langle \tilde{x}^2 \rangle$ . On the other hand, knowledge of  $\alpha^2$  yields  $\dot{\varphi}$ , and thus via integration the phase  $\varphi$  of  $\lambda$ . Separation of  $\lambda$  in real and imaginary parts yields  $\hat{z}$  and thus the classical path. The important relation (20) can be proved again, using the relation connecting

$\hat{z}$  and  $\hat{u}$ . Inserting  $\hat{u} = \alpha \cos \varphi$  and  $\hat{z} = \alpha \sin \varphi$  in relation (17) yields

$$\dot{\hat{z}}\hat{u} - \hat{u}\dot{\hat{z}} = \alpha^2 \dot{\varphi} = 1, \quad (21)$$

thus proving Eq. (20).

A remarkable similarity exists between the motion of the quantity  $\lambda$  in a complex plane, where  $\lambda$  characterizes the dynamics of a *one-dimensional* problem and the motion of a particle in a *real two-dimensional* plane under the influence of a central force.

Relation (20),  $\dot{\varphi} = 1/\alpha^2$ , corresponds to *conservation of angular momentum* in a complex plane, a property which has *no classical analogue*. The (real) Newtonian equation (11) for  $\alpha$  corresponds to the radial equation of the two-dimensional motion in real space, where the  $1/\alpha^3$  term corresponds to a centrifugal force!

Regarding the energetics of the discussed systems, it can be shown (for details see forthcoming publication) that the difference between the mean value of the Hamiltonian operator and the corresponding classical energy, the energy fluctuation

$$\langle \hat{E} \rangle_L = \langle H \rangle - E_{\text{class}} = \frac{1}{2m} \langle \hat{p}^2 \rangle_L + \frac{m}{2} \omega^2 \langle \hat{r}^2 \rangle_L = \text{const.}, \quad (22)$$

can be considered as a Hamiltonian function for the fluctuation of position  $\langle \tilde{x}^2 \rangle$  or  $\alpha$ , i.e. the WP width, respectively.

The corresponding Lagrangian, expressed in terms of  $\alpha$ ,  $\varphi$ ,  $\dot{\alpha}$  and  $\dot{\varphi}$  reads

$$\tilde{Z}_L(\alpha, \varphi, \dot{\alpha}, \dot{\varphi}) = \frac{\hbar}{4} (\dot{\alpha}^2 + \alpha^2 \dot{\varphi}^2 - \omega^2 \alpha^2). \quad (23)$$

The canonically conjugate momentum for the radial part is given by

$$P_\alpha = \frac{\partial \tilde{Z}_L}{\partial \dot{\alpha}} = \frac{\hbar}{2} \dot{\alpha} \quad (24)$$

and the Euler-Lagrange equations yield the radial equation (11).

The canonically conjugate momentum for the angular part, given by

$$P_\varphi = \frac{\partial \tilde{Z}_L}{\partial \dot{\varphi}} = \frac{\hbar}{2} \alpha^2 \dot{\varphi} \quad (25)$$

is not only a constant of motion, but due to the validity of relation (20) it has the specific value

$$P_\varphi = \frac{\hbar}{2}. \quad (26)$$

So, the angular momentum for the motion of  $\lambda$  in the complex plane is not only conserved, but it is also a half-integer of  $\hbar$ , whereas orbital angular momenta in quantum mechanics usually are multiple integers of  $\hbar$ . However, half-integers of  $\hbar$  are known from the spin, a quantity that has no classical analogue, similar to the angular momentum of the *complex* quantity  $\lambda$ .

For our NLSE, an energetic invariant similar to (22) exists, that allows to set up an equivalent Lagrange-Hamilton formalism for the position and momentum fluctuations including dissipation.

The equations of motion describing the wave aspect are rather similar for the SE and the NLSE, however, one important difference should be pointed out. Considering the equation for  $\ddot{\alpha}$ , it becomes obvious that due to the coefficient  $(\omega^2 - \gamma^2/4)$  of the term linear in  $\alpha$  for our NLSE, it is possible to compensate the effect of the external potential, if the "perturbation", expressed by the parameter  $\gamma$ , fulfills  $\omega = \gamma/2$ . In this case,  $\alpha$  and thus  $\langle \tilde{x}^2 \rangle$  and the current in the density equation (3) for the damped oscillating system behave like the corresponding quantities of a free particle without any friction. So, the addition of a dissipative term cannot only be destructive, but on the contrary it can also create a new kind of ordered coherent phenomenon, like a current in the density equation, that, given the same external conditions, would not be possible without the perturbation. Possible connections with macroscopic quantum effects will be discussed elsewhere.

#### ACKNOWLEDGEMENTS

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GRADIENT STRUCTURE OF GLOBAL ATTRACTORS FOR  
DISSIPATIVE NONLINEAR NONAUTONOMOUS  
PARTIAL DIFFERENTIAL EQUATIONS

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### 1. Introduction

In a recent paper [3], we proved existence and finite-dimensionality of global attractors for some classes of dissipative nonlinear nonautonomous partial differential equations (DNLNAPDE). While some results are known for attractors in nonautonomous cases, very little is known about the detailed structure of global attractors in this situation compared with the state of knowledge for autonomous situations. In the latter cases, the simplest structure for global attractors is obtained for so-called gradient systems. For these systems, Lyapunov functionals exist and, under suitable conditions, the global attractor is the union of the unstable manifolds of the equilibrium points. In the present contribution, we discuss some aspects of a gradient structure for some classes of DNLNAPDEs. Examples illustrating the theory presented herein can be given in terms of the classes of reaction-diffusion equations and nonlinear wave equations discussed in [3].

### 2. Definitions and preliminary results

Following the formulation in [3], let  $u$  denote a solution of a DNLNAPDE in a real Banach space  $B$  such that  $u(t+s)$  represents a solution at time  $t+s$  ( $t \geq 0, s \in \mathbf{R}$ ) corresponding to given  $u(s) = \phi \in B$  at time  $s$ . We consider a two-parameter family of maps, called *processes* [2],  $V(t,s)$  with the properties  $V(0,s)\phi = \phi$ ,  $V(t+\theta, s) = V(t, s+\theta)V(t,s)\phi$  for  $s \in \mathbf{R}, t, \theta \geq 0, \phi \in B$ . We shall be especially interested in distinguished processes  $W$  related to solutions of the DNLNAPDE by  $W(t,s)\phi = u(t+s), u(s) = \phi$ . Let  $V$  be a process on  $B$  and let  $\tau \in \mathbf{R}$ . We call the  $\tau$  *translate* of  $V$  the process  $V_\tau(t,s)\phi = (V(\tau)V)(t,s)\phi = V(t, \tau+s)\phi$ , for  $t \geq 0, s \in \mathbf{R}$ . Denote by  $C_b(\mathbf{R}, B)$  the Banach space of all bounded continuous functions from  $\mathbf{R}$  to  $B$ . A process  $V$  on  $B$  is called *almost periodic* if the set  $\{V_\tau(t,s)\phi, \tau \in \mathbf{R}\}$  is precompact in  $C_b(\mathbf{R}, B)$  (as a function of the parameter  $s \in \mathbf{R}$ ) pointwise in  $t \geq 0$  and  $\phi \in B$ . The closure in  $C_b(\mathbf{R}, B)$  of the set of translates of an almost periodic process  $V$  with the above sense of convergence is called the *hull* of  $V$ ,  $H(V)$ .

In [3,4] invariant sets were defined in terms of a skew-product structure for PDEs in an analogous fashion to earlier work on nonautonomous ordinary differential equations. To define this structure, consider the mappings  $\pi(t,s)(\phi,V) = (V(t,s)\phi, \sigma(t)V)$ ,  $V \in H(W)$ ,  $\phi \in B$ ,  $s \in \mathbf{R}$ ,  $t \geq 0$ .

In order to discuss nonautonomous gradient systems, we need three fundamental concepts: unstable sets, backward extensions of almost periodic processes, and an analogue for nonautonomous systems of an equilibrium point of an autonomous system. The existence of backward extensions was proved in [3]. It is convenient to define *nonautonomous equilibrium points* (as we shall call them) in terms of properties of Lyapunov functionals. We define these in a similar manner to those of Dafermos[2] for compact processes, a class of processes closely related to our almost periodic processes.

**DEFINITION 2.1.** Let  $V$  be an almost periodic process on  $B$ . A map  $L: \mathbf{R} \times B \rightarrow \mathbf{R}$  is called a *Lyapunov functional for  $V$*  if:

- (i) the one-parameter family of maps  $L(s, \cdot): B \rightarrow \mathbf{R}$ ,  $s \in \mathbf{R}$ , is equicontinuous,
- (ii) for fixed  $s \in \mathbf{R}$ ,  $L(t+s, V(t,s)\phi)$  is a continuous nonincreasing function of  $t$  which is bounded from below and  $L(t+s, V(t,s)\phi) \leq L(s,\phi)$  for all  $t \geq 0$ ,  $s \in \mathbf{R}$ , and  $\phi$  belonging to an appropriate dense subspace  $B$  of  $B$ ,
- (iii) if  $\{t_n\} \subset \mathbf{R}^+$  is a sequence such that  $\{V_{t_n}\}$  is convergent in the topology defined on the previous page, then the sequence  $\{L(t+s+t_n, \phi)\}$  is also convergent for all  $s \in \mathbf{R}$  and all  $\phi \in B$ .

A *limiting Lyapunov functional generated by  $L$*  is a map  $\tilde{L}: \mathbf{R} \times B \times H(V) \rightarrow \mathbf{R}$  constructed as follows: for  $Z \in H(V)$ ,  $s \in \mathbf{R}$ ,  $\phi \in B$ ,  $\tilde{L}(Z,s,\phi) = \lim L(s+t_n, \phi)$  as  $n \rightarrow +\infty$ , where  $\{t_n\} \subset \mathbf{R}^+$  is any sequence such that  $V_{t_n} \rightarrow Z$  as  $n \rightarrow +\infty$  in the topology defined at the beginning of this section.

**DEFINITION 2.2.** Given  $V, L$  and  $\tilde{L}$  as defined above, we define the subset  $P$  of  $B$  to be the set of all extensions  $U$  of  $V$  as the process  $Z$  associated with the extension  $U$  as in the relation  $U(t+\theta, s; \eta) = Z(t, s+\theta; U(\theta, s; \eta))$  for  $t \geq 0$ ,  $\theta, s \in \mathbf{R}$ , runs over  $H(V)$  such that  $\tilde{L}(Z, \sigma, U(\sigma, s; \beta)) = 0$  for all  $\sigma \in \mathbf{R}$  with  $\tilde{L}(Z, s, \phi) = \limsup 1/t [\tilde{L}(Z, t+s, Z(t, s; \phi)) - \tilde{L}(Z, s, \phi)]$ .

$P$  is nonempty[2] and is the analogue for nonautonomous gradient systems of the set of equilibrium points for autonomous systems. We will apply the preceding definitions to the distinguished process  $W$ . We say that a forcing function  $f$  is *admissible* if  $H(f)$  is compact in  $C_b(\mathbf{R}, B)$ . Examples of functions  $f$  which satisfy this condition are given in [3,4]. We only note here that this class includes functions that are almost periodic from  $\mathbf{R}$  to  $B$ .

The backward extensions  $U$  need not be unique. In [3] we obtained backward uniqueness from the injectivity of the maps  $S(t,s;h)\phi = W(t,s;\phi,h)$ ,  $h \in H(f)$ . The utility of these maps is a consequence of the fact that, if the solutions of the DNLNAPDE are unique and depend on the forcing function  $f$  in a Lipschitz continuous manner, then there exists a one-to-one correspondence between  $h \in H(f)$  and  $V \in H(W)$  such that  $W(t,s;\phi,h) = V(t,s;\phi,f)$  for all  $t \geq 0$  and  $s \in \mathbf{R}$ . In [3], we proved existence of global attractors for  $\pi(t,s)$  assuming, among other things, that the maps  $S$  have similar properties.

### 3. Nonautonomous gradient systems

We now indicate that, under appropriate hypotheses, the global attractor of a nonautonomous gradient system coincides with the union of unstable sets of elements of  $\mathbf{P}$ . Our results will be seen to be generalizations of corresponding results for autonomous gradient systems [1].

We first consider the structure of the mappings  $S$  in neighborhoods of nonautonomous points. For the distinguished process  $W$  and one of its backward extensions  $U$ , define the *unstable set* of  $\chi$  by  $M(\chi) = \{\psi \in B : U(\theta,s;\psi,f) \text{ is defined for } \theta \leq 0 \text{ and } \|U(\theta,s;\psi,f) - \chi\| \rightarrow 0 \text{ as } \theta \rightarrow -\infty\}$  and a *lifted unstable set* of  $\chi$  by  $\tilde{M}(\chi) = \{(\psi,U) : \psi \in M(\chi)\}$ .

**DEFINITION 3.1.** Let  $\chi \in \mathbf{P}$  and fix  $Y \in H(W)$  or, equivalently,  $h \in H(f)$ . We say that  $S(s,h)$ , for fixed  $s \in \mathbf{R}$ , is *hyperbolic at*  $\chi$  if the following conditions are satisfied for each  $t \geq 0$ :

- 1) in some neighborhood  $O$  of  $\chi$ ,  $S(s,u,h)$  has a Frechet differential  $S'(s,u,h)$  for  $u \in O$ ,  $S'(s,u,h) : \mathcal{B} \rightarrow \mathcal{B}$ ,
- 2) the linear operator  $S'(s,u,h)$  is locally Hölder continuous in  $u$ ,
- 3) the spectrum of  $S'(s,\chi,h)$ ,  $\sigma(S'(s,\chi,h))$ , does not intersect the unit circle with center at the origin.

$\chi \in \mathbf{P}$  is called *hyperbolic* if for any  $t \geq 0$ ,  $s \in \mathbf{R}$ , and  $h \in H(f)$ , the mapping  $S(t,s;h)$  is hyperbolic at  $\chi$  and:

- 4) the invariant linear subspaces  $B_+$ ,  $B_-$  corresponding to subsets of  $\sigma(S'(t,s;h)(\chi))$  in the domains  $\{|\lambda| > 1\}$ ,  $\{|\lambda| < 1\}$ , respectively, are independent of  $t$ ,  $s$ , and  $h$  and  $\dim B_+ < +\infty$ .

**THEOREM 3.1** Suppose that  $\{\pi(t,s), t \geq 0\}$  satisfies the following conditions:

- 1)  $\pi(t,s) : B \times H(W) \rightarrow B \times H(W)$  ( $t \geq 0, s \in \mathbf{R}$ ) is continuous,
- 2) there exists a continuous Lyapunov functional with the properties listed in Definition 2.2,

- (3.1) for any  $t \in \mathcal{D}$ ,  $\pi(t, s)$  maps any bounded set of  $B \times H(W)$  into a precompact set,  $\pi(t, s)$  is uniformly bounded for  $0 \leq t \leq +\infty$ ,  
 the set  $\{\chi_i\} \subset P$  is finite and each  $\chi_i$  is hyperbolic.

Then the set  $\tilde{A} = \bigcup_{\chi \in P} \tilde{M}(\chi)$  for  $\chi \in P$  is the global attractor of  $\{\pi(t, s)\}$ .

Theorem 3.1 holds for a large class of nonlinear parabolic equations. However, it is not valid for hyperbolic equations as it stands because (3.1) is generally not satisfied in that case. We have formulated a modified version of Theorem 3.1 which covers such equations. A proof of the continuity of  $\{\pi(t, s), t \geq 0\}$  has been given in [3].

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## Linearizing a Nonlinear Paradigm: Universal Description of KAM-Tori and Cantori

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### 1 Introduction

In 1887 Poincaré formulated the first ideas about chaos in nonlinear dynamical systems. A decisive step towards understanding such systems was achieved by Kolmogorov, Arnold and Moser through their celebrated KAM-Theorem (1954-1963), which describes the complementary organization of chaos and order. Since that time a lot of work has enlarged our knowledge of the subtle details of nonlinear dynamics. In this article we present a new approach to infinite invariant sets, which paves the way to a universal description of KAM-Tori and their leaky counterparts, the so-called Cantori.

### 2 Phase Transitions in Twist Maps and Frenkel-Kontorova Models (FKM)

Twist maps provide us with some kind of mathematical laboratory, which helps us to study KAM-Tori and Cantori. An area-preserving twist map has the form

$$r_{n+1} = r_n + \kappa f(\theta_n), \quad (1)$$

$$\theta_{n+1} = \theta_n + r_n + \kappa f(\theta_n), \quad (2)$$

where  $\kappa$  is the nonlinearity parameter and  $f(\theta + 1) = f(\theta)$ . A wellknown example is the "Standard Map" with  $f(\theta) = \frac{1}{2\pi} \sin(2\pi\theta)$ . There the most stable KAM-Torus is the "Golden Torus" with rotation number

$$\omega := \lim_{n \rightarrow \infty} \left( \frac{\theta_n - \theta_0}{n} \right) = \frac{1}{2}(\sqrt{5} - 1) \equiv \omega_G. \quad (3)$$

Greene [1] calculated the critical value  $\kappa_c = 0.9716\dots$ , where the torus breaks up to become a Cantorus via a second-order phase transition.

A physical model that is directly linked to area-preserving twist maps is the "Frenkel-Kontorova Model". It consists of an infinite chain of harmonically coupled particles in an external periodic potential. The energy is given by

$$E(\{u_n\}) = \sum_{n \in \mathbf{Z}} \frac{1}{2} (u_{n+1} - u_n)^2 + \kappa V(u_n), \quad (4)$$

where  $V(u + 1) = V(u)$ .

Using the substitutions:  $\theta_n \equiv u_n \pmod{1}$ ;  $r_n \equiv u_n - u_{n-1}$  and  $V' \equiv f$ , (1) and (2)

can be interpreted as the stationarity condition of the FKM. KAM-Tori and Cantori correspond to incommensurate groundstates of this model, having irrational mean lattice constant

$$\omega := \lim_{(N-N') \rightarrow \infty} \left( \frac{u_N - u_{N'}}{N - N'} \right). \quad (5)$$

Here the "Lock-in Transition" [2] of the physical system with increasing  $\kappa$  corresponds to the torus break-up mentioned above.

We recently extended the FKM by adding higher harmonics in the potential and found several novel phenomena [3] like "recurrence of KAM-Tori". Furthermore, second-order phase transitions between different Cantorus phases can be observed and some potentials even support quasi-first-order Cantorus-Cantorus transitions. But the technical problems in calculating and identifying groundstates numerically are almost forbidding, since there exist vast numbers of metastable states. So we tried to construct an analytical method to deal with groundstates in general zed FKM.

### 3 Cantorus Configurations in Piecewise Parabolic Models

The first nontrivial analytically solvable FKM was discussed by Aubry [4], and Percival [5]. It is a piecewise parabolic model, whose formal solution can be found using a "Lattice Green's Function". We extended this model to an "M-parabolic model" with potential

$$V_M = \min_{m \in \mathbb{Z}} \{p(m; u)\}; \quad (6)$$

$$p(m; u) = \frac{\kappa}{2}(u - m)^2 + \frac{1}{2}h_m. \quad (7)$$

and the periodicity condition  $h_{m+M} = h_m$ . In the case  $M = 2$  [6] we have a model with one order parameter  $\psi$  (fraction of particles in odd wells), and the energy per particle of an  $\omega$ -Cantorus configuration is given by

$$\epsilon_2 = \frac{\kappa}{2}h(1 - \psi) + \mathcal{G}(\kappa, \omega; \psi) + \text{const.} \quad (8)$$

$\mathcal{G}$  is a convex, continuous, highly nondifferentiable function, which turns out to be universal for all M-parabolic models!

Minimizing  $\epsilon_2$  with respect to  $\psi$  directly yields the incommensurate groundstate. This model allows analytical calculation of the sequence of discontinuous Cantorus-Cantorus transitions, occurring when  $h$  (or  $\kappa$ ) is varied (see Fig.1).

Furthermore, the existence of novel phenomena like incommensurate defects and an infinite number of metastable Cantorus configurations can be proved.

In the general case,  $M > 2$ , the corresponding energy per particle is

$$\epsilon_M(\kappa, \omega; \psi_1, \dots, \psi_{M-1}) = \frac{\kappa}{2} \sum_{m=1}^{M-1} h_m \psi_m + \sum_{s=1}^{M-1} \sum_{t=s}^{M-1} \mathcal{G}(\kappa, \omega; \sum_{\tau=s}^t \psi_\tau) + \text{const.} \quad (9)$$

where the order parameters  $\{\psi_m\}$  are the fractions of particles in the  $m$ -th type of parabola. Calculation of the Cantorus groundstate boils down to searching the unique minimum of a convex  $M$ -dimensional surface.

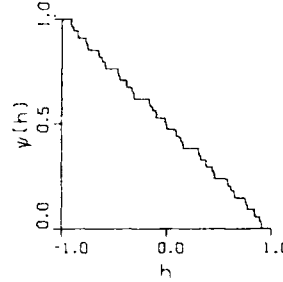


Fig.1: The “devil’s staircase”-dependence of  $\psi$  on  $h$  in the 2-parabola model ( $\kappa = 0.03$ ).

#### 4 “Parabolization” of Arbitrary Potentials

An arbitrary periodic potential  $V(x)$  can be approximated with an  $M$ -parabolic potential  $P(x)$  using the nonlinear “parabolic contact transformation” (see Fig.2)

$$\xi(x) = x - \frac{1}{\kappa} V'(x), \quad (10)$$

$$h(\xi(x)) = \frac{2}{\kappa} V(x) - \left(\frac{V'(x)}{\kappa}\right)^2. \quad (11)$$

Here  $\xi(x)$  describes the parabola vertex,  $h(\xi)$  is related to the parabola height and  $\kappa$  is a free parameter. It is possible to construct an arbitrary close  $M$ -parabolic approximation; thus we can handle the groundstate problem with the method described in the previous section.

Remarkably, the limit  $M \rightarrow \infty$  can be performed rigorously and leads to an integral representation for the energy. With  $h_m \rightarrow h(\xi)$ ,  $\psi_m \rightarrow \psi(\xi)$  (particle density function) and  $\theta(\xi) := \int_0^\xi \psi(\eta) d\eta$  we get

$$\epsilon(\kappa, \omega; \psi) = \frac{\kappa}{2} \int_0^1 h(\xi) \psi(\xi) d\xi + \frac{1}{2} \int_0^1 \int_0^1 \mathcal{G}(\kappa, \omega; \theta(\eta) - \theta(\xi)) d\eta d\xi + const. \quad (12)$$

The first integral on the right side represents the linear potential term depending on the individual model. The second term with the universal “devil’s-bowl functional”  $\mathcal{G}$  describes the nonlinear elastic energy.

Defining  $\mathcal{A}(x) := -\frac{2}{\kappa} \mathcal{G}'(x)$  and

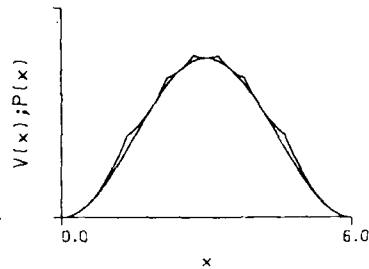
$$\Delta(\xi) := h(\xi) - \int_0^\xi \int_\xi^1 \mathcal{A}(\kappa, \omega; \theta(\eta) - \theta(\xi)) d\eta d\xi, \quad (13)$$

we can derive a precise criterion for identifying Torus and Cantorus groundstates:

$$\Delta(\xi) \begin{cases} = 0 & , \psi(\xi) > 0 \\ \geq 0 & , \psi(\xi) = 0. \end{cases} \quad (14)$$

$(\Delta(\xi) = 0) \wedge (\psi(\xi) = 0)$  describes criticality, i.e., Torus-Cantorus or Cantorus-Cantorus transitions.

Fig.2: Parabolic contact transformation:  
A potential  $V(x)$  and the corresponding  
approximating 6-parabola potential  $P(x)$ .



## 5 Conclusion

We have shown that the problem of determining the incommensurate groundstate in arbitrary FKM can be solved, in principle, by "Linearization", i.e., using a sufficiently good "Parabolization" (parabolic contact transformation) of the given potential. The resulting convex problem has to be treated by an optimization algorithm. The inverse problem is directly solvable, i.e., given a Can/Torus groundstate, we can immediately calculate the corresponding M-parabolic potential.

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## INTEGRABLE SYSTEMS RELATED TO MEMBRANES

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The dynamics of a relativistically invariant bosonic membrane moving in  $D$  dimensional space-time can be described (in the "light-cone gauge") by [0]

$$H = \frac{1}{2} \int_M d\Omega \left( \sum_{i=1}^{D-2} p_i^2(\Omega) + k^2 \sum_{i \neq j} \{x_i, x_j\}^2 \right) \quad (1)$$

where  $x_i$  and  $p_i$  are time dependent functions on some two dimensional compact manifold  $M$ .  $\Omega$  denotes a parameterization on  $M$  (for simplicity, we will restrict ourselves to  $M = T^2$ : so  $\Omega = (\varphi_1, \varphi_2) \in [0, 2\pi]^2$ ), and

$$\{x_i, x_j\} := \epsilon^{rs} \partial_r x_i \partial_s x_j \quad r, s = 1, 2 \quad (2)$$

Actually, (1) must be supplemented by the constraint

$$K := \sum_{i=1}^{D-2} \{x_i, p_i\} = 0 \quad (3)$$

in order to really coincide with the (Mass)<sup>2</sup> of the membrane, but we will forget about (3) from now on. The equations of motion following from (1) are

$$\ddot{x}_i = k^2 \{x_j, \{x_j, x_i\}\} \quad (4)$$

In 4 dimensions ( $D - 2 = 2$ ),

$$H = \frac{1}{2} \int d\Omega (p_x^2 + p_y^2 + k^2 \{x, y\}^2) \quad (5)$$

If one treats further one of this fields as non-dynamical, one obtains

$$H = \frac{1}{2} \int d\Omega (p^2 + k^2 \{x, \omega\}^2) \quad (6)$$

where  $\omega$  is now an external time independent field. Eq. (6) is a particular case of a class of  $2 + 1$  dimensional field theories,

$$H = \frac{1}{2} \int d\Omega (p^2 + \lambda F^2(\{x, \omega\})) \quad (7)$$

$$F F'' = \alpha F'^2 \quad \alpha \in \mathbb{R} \quad (8)$$

which are natural generalizations of a class of  $1 + 1$  dimensional integrable theories [0], and for which one may easily write down an infinite set of Poisson commuting conserved charges,

$$Q_n = \sum_{i=0}^n \binom{n}{i} \lambda_i \int d\Omega (p^{n-i} F^i) \quad n \in \mathbb{N} \quad (9)$$

$$\lambda_i = \begin{cases} 0 & \text{if } i \text{ is odd} \\ \frac{(2k-1)(2k-3)\dots(1)\cdot\Lambda^k}{(2k-1+\alpha)(2k-3+\alpha)\dots(\alpha+1)} & \text{if } i = 2k \end{cases} \quad \Lambda = \lambda(\alpha + 1). \quad (10)$$

One simply has to note that the equations of motion (corresponding to (7)),

$$\ddot{x} = \lambda \{ \{x, \omega\}, \omega \} F'^2(\{x, \omega\}) (\alpha + 1) \quad (11)$$

are equivalent to

$$\dot{L} = \epsilon_{abc} \partial_a L \partial_b \omega \partial_c M \quad a, b, c = 1, 2, 3 \quad (12)$$

$$L = p(\varphi_1 \varphi_2) + \epsilon(\varphi_3) F(\{x, \omega\}) \quad (13)$$

$$M = -f(\varphi_3) F'(\{x, \omega\}) \quad (14)$$

$$f'^2 - \alpha f f'' = \lambda_2 \quad , \quad \epsilon = -f' \quad (15)$$

So

$$Q_n = \int d\Omega \int_0^{2\pi} \frac{d\varphi_3}{2\pi} (L(\varphi_1 \varphi_2 \varphi_3))^n \quad (16)$$

is automatically time independent. It is also not difficult to show that they are in involution:

$$\begin{aligned} [Q_m, Q_n]_{P.B.} &= \sum_{i=0}^m \sum_{j=0}^n \binom{m}{i} \binom{n}{j} \lambda_i \lambda_j \left[ \int p^{m-i} F^i \cdot \int p^{n-j} f^j \right] \\ &= \sum \sum \binom{m}{i} \binom{n}{j} \int d\Omega p^{m+n-i-j-1} \cdot \{ F^{i+j-1}, \omega \} \\ &\quad \frac{f^i}{(i+j-1)(m+n-i-j-1)} \\ &\quad \cdot (m-i)(m-i-1)j(j+\alpha-1) - (n-j)(n-j-1)i(i+\alpha-1) \\ &= 0 \end{aligned} \quad (17)$$

Finally, one should note that in fact

$$Q_{kl} := \int d\Omega \int_0^{2\pi} \frac{d\varphi_3}{2\pi} L^k \omega^l \quad (18)$$

is conserved for all  $k, l \in \mathbb{N}$

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## A Complex Formulation of Hamiltonian (or Birkhoffian) Theory

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### Abstract

A Complex form of Hamiltonian (Birkhoffian) Theory is introduced and is based on a dual pair of  $n$  dimensional unitary spaces with a corresponding dual pair of hermitean metric tensors,  $h_{ab}$  and  $h^{ab}$  such that  $h_{ac}h^{bc} = \delta_a^b$ . A  $2n$  dimensional complex phase space is defined with anti-hermitean metric tensor  $M = \begin{pmatrix} ih & -h \\ h & ih \end{pmatrix}$ . The  $n \times n$  block structure of  $M$  and the relation between  $M$  and  $h$  seem to be new, and only the special case of real antisymmetric  $M$  has been used previously to define a generalised phase space (or dynamic space), and to define a generalised Poisson bracket in Birkhoffian theory. Unlike the most general form of Birkhoffian theory, not just one pair but two pairs of canonically conjugate variables are defined in a generalised Hamiltonian (or Birkhoffian) sense for the most general complex form of Hamiltonian (or Birkhoffian) theory.

Some remarks and speculations are made about the role of  $h$  in the description of constrained dynamics and the curvatures of configuration space and phase space.

## 1 Introduction

Linear dynamical systems are well-known to be simpler in complex, rather than real, phase space [1]. Analytic extensions of Lagrangian and Hamiltonian functions of complex variables of the forms  $(q^a + iq'^a)$ ,  $(\dot{q}^a + i\dot{q}'^a)$  and  $(p^a + ip'^a)$  have been defined [2], but the effect of complex variables of the form  $(p^a + iq^a)$  in the Poisson bracket formulation of Hamilton's equations, does not seem to be well-known [3].

## 2 Complex Phase Space

A complex form of Hamilton's equation for a function  $F(z)$  and extended Hamiltonian  $K(z, \bar{z}, t)$  in complex phase space is

$$\dot{F}(z) = 2(\partial_a F)\dot{z}^a = (\partial_a F)4i\delta^{ab}(\overline{\partial_b K}) \equiv [F, K]_c \quad (1)$$

where  $z^a \equiv p^a + iq^a$  for  $a = 1, 2, 3, \dots, n$ ; the Cauchy-Riemann operator is  $\bar{\partial}_a \equiv \frac{1}{2} \left( \frac{\partial}{\partial p^a} + i \frac{\partial}{\partial q^a} \right)$  and  $[ , ]_c$  is a sesquilinear product which satisfies

$$[F, G]_c = -[\bar{G}, \bar{F}]_c \quad (2)$$

and

$$[[F, G]_p, H]_p + (\text{cyclic perms of } F, G, H) = 0. \quad (3)$$

In (3), the Poisson bracket can be expressed in terms of the sesquilinear product as follows  $[F, G]_p = \frac{1}{2}([F, G]_c - [\bar{G}, \bar{F}]_c)$  by use of (2) and

$$[F, G]_c = \left( \frac{\partial F}{\partial x^\mu} \right) (\omega^{\mu\nu} + i\delta^{\mu\nu}) \left( \frac{\partial G}{\partial x^\nu} \right), \quad (4)$$

where  $x = (p; q) \in \mathbb{R}^{2n}$  (real phase space);  $\mu, \nu = 1, 2, 3, \dots, 2n$ ; and

$$\omega = \begin{pmatrix} 0 & \vdots & -1 \\ \cdots & & \cdots \\ 1 & \vdots & 0 \end{pmatrix}$$

is the fundamental symplectic tensor; also  $\det(\omega^{\mu\nu} + i\delta^{\mu\nu}) = 0$ .

Hence (2, 3) are complex forms in the special case  $\Omega^{\mu\nu} = \omega^{\mu\nu}$ ,  $T^{\mu\nu} = \delta^{\mu\nu}$  of the Lie-admissible algebra axioms [2] as follows:

$$\Omega^{\mu\nu}(x) = -\Omega^{\nu\mu}(x), \quad T^{\mu\nu}(x) = T^{\nu\mu}(x) \quad (5)$$

and

$$\begin{aligned} & [[F, G]_s, H]_s - [F, [G, H]_s]_s + (\text{cyclic perms of } F, G, H) \\ &= [[H, G]_s, F]_s - [H, [G, F]_s]_s + (\text{cyclic perms of } H, G, F), \end{aligned} \quad (6)$$

where

$$F(x) = \left( \frac{\partial F}{\partial x^\mu} \right) \dot{x}^\mu = \left( \frac{\partial F}{\partial x^\mu} \right) (\Omega^{\mu\nu}(x) + T^{\mu\nu}(x)) \left( \frac{\partial B}{\partial x^\nu} \right) \equiv [F, B]_s \quad (7)$$

is a Birkhoffian generalisation of Hamilton's dynamical equation for a function  $F(x)$  and Birkhoffian  $B(x, t)$  in real phase space.

### 3 Dual Unitary Phase Spaces $\mathcal{U}^n$ and $\mathcal{U}^{*n}$

A more general form of a Birkhoffian dynamical equation corresponds to the generalisation of (1) to

$$\dot{F}(z) = 2(\partial_a F) \dot{z}^a = (\partial_a F) 4i h^{ab}(x) (\overline{\partial_b K}) \equiv [F, K]_{cg} \quad (8)$$

where (2) is satisfied for  $h^{ab}(x) = g^{ab}(x) + i f^{ab}(x)$  if  $g^{ab} = g^{ba}$  and  $f^{ab} = -f^{ba}$ , also (3) requires

$$h^{ab} (\overline{\partial_a h^{cd}}) = h^{ad} (\overline{\partial_a h^{cb}}). \quad (9)$$

It follows that

$$[F, K]_{cg} = \left( \frac{\partial F}{\partial x^\mu} \right) (\Omega^{\mu\nu}(x) + iT^{\mu\nu}(x)) \left( \frac{\partial K}{\partial x^\nu} \right) \quad (10)$$

where  $\Omega = \begin{pmatrix} -f & \vdots & -g \\ \cdots & & \cdots \\ g & \vdots & -f \end{pmatrix}$ ,  $T = -\omega\Omega$ , and hence

$$M \equiv \Omega + iT = \begin{pmatrix} ih & \vdots & -h \\ \cdots & & \cdots \\ h & \vdots & ih \end{pmatrix} \quad (11)$$

is singular.

If  $z^a = p^a + iq^a \in \mathcal{U}^n$ ,  $\zeta_a = P_a + iQ_a \in \mathcal{U}^{*n}$  and  $dz^a = h^{ab} d\zeta_b$ , then

$$[F, K]_{cg} = \left( \frac{\partial F}{\partial z^a} \right) 4i \left( \frac{\overline{\partial K}}{\partial \zeta_a} \right) = \left( \frac{\partial F}{\partial \zeta_a} \right) 4i \left( \frac{\overline{\partial K}}{\partial z^a} \right). \quad (12)$$

Hence Darboux's reduction of  $\Omega^{\mu\nu}$  to  $\omega^{\mu\nu}$  can be realised locally by either one of the two pairs of variables  $(p^a, Q_a)$ ,  $(P^a, q_a)$  which are canonically conjugate in the senses  $\dot{\zeta}_a = 2i \left( \frac{\overline{\partial K}}{\partial z^a} \right)$ ,  $\dot{z}^a = 2i \left( \frac{\overline{\partial K}}{\partial \zeta_a} \right)$ , and  $[\zeta_a, x^b]_{cg} = 2i\delta_a^b = [z^a, \zeta_b]_{cg}$ .

## 4 Dynamical Systems with a Constraint

For a dynamical system with flat configuration space variables  $\{z^a; a = 1, 2, 3, \dots, n\}$  and a single constraint  $\phi(x) = 0$ , then  $[\phi, \phi]_c = 4i \sum_a |\phi_a|^2 \neq 0$  in general. A well-known method[4], gives the singular tensor

$$\begin{aligned} 4i h^{ab} &= [z^a, z^b]_{cg} = [z^a, z^b]_c - [z^a, \phi]_c [\phi, z^b]_c / [\phi, \phi]_c \\ &= 4i \left( \delta^{ab} - \frac{(\partial_a)(\partial_b \phi)}{\sum_c |\phi_c|^2} \right) \\ &= h^{aa'} h^{bb'} ([\zeta_{a'}, \zeta_{b'}]_c - [\zeta_{a'}, \phi]_c [\phi, \zeta_{b'}]_c / [\phi, \phi]_c). \end{aligned}$$

Hence  $h_{ab} = \frac{1}{4i} ([\zeta_a, \zeta_b]_c - [\zeta_a, \phi]_c [\phi, \zeta_b]_c / [\phi, \phi]_c)$  is also a singular tensor. The choice  $\zeta_n \equiv \phi$  then gives  $h_{ab} = \delta_{ab} - \delta_{an} \delta_{bn}$  so that a dimensional reduction of phase space can be carried out to define an unconstrained dynamical system which is non-singular in the sense that

$$h_{ac} h^{bc} = \delta_a^b \text{ for } a, b, c = 1, 2, 3, \dots, n-1.$$

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## Bifurcation of Periodic Systems of Singularly Perturbed Dynamic Systems

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Consider the singularly perturbed nonlinear system

$$\begin{aligned} \dot{x} &= f(x, y, \varepsilon), \\ \varepsilon \dot{y} &= g(x, y, \varepsilon). \end{aligned} \quad (1)$$

where  $\varepsilon$  is a small real parameter,  $x$  and  $y$  are  $n$  and  $m$  dimension vectors respectively,  $f$  and  $g$  are  $n$  and  $m$  dimensional vector functions respectively. We suppose that (H1) when  $\varepsilon = 0$ , the system

$$\begin{aligned} \dot{x} &= f(x, y, 0), \\ 0 &= g(x, y, 0) \end{aligned} \quad (2)$$

has a closed orbit  $\Gamma_0: x = u(t), y = v(t)$ , where  $u(t)$  and  $v(t)$  are continuously periodic vector functions with period  $T$  (we say, they are  $T$ -periodic).

In this paper we regard  $\varepsilon$  as a bifurcation parameter and discuss under what conditions system (1) <sub>$\varepsilon$</sub>  can bifurcate a limit cycle from closed orbit  $\Gamma_0$ .

We denote  $f(u(t), v(t), 0)$  by  $f(t)$ ,  $\frac{\partial}{\partial x} f(u(t), v(t), 0)$  by  $f_x(t)$  and similar meanings are attached to  $f_y(t)$ ,  $f_\varepsilon(t)$ ,  $g_x(t)$ ,  $g_y(t)$  and  $g_\varepsilon(t)$ . Then, the variation system of (2) with respect to closed orbit  $\Gamma_0$  has the form

$$\frac{dz}{dt} = A(t)z, \quad (3)$$

where  $A(t) = f_x(t) - f_y(t)g_y^{-1}(t)g_x(t)$ .

We also assume that

(H2) vector functions  $f, g$  and their Jacobian matrices  $f_x, f_y, f_\varepsilon, g_x, g_y, g_\varepsilon$  are all uniformly continuous and of  $C^{(2)}$  in all arguments.

(H3) every eigenvalue of the matrix function  $g_y(t)$  has nonzero real part for all  $t$  ( $0 \leq t \leq T$ ).

(H4) the inverse matrix function  $g_y^{-1}(t)$  exists and  $g_y^{-1}(t)g_x(t)$  has continuous and bounded first derivatives.

**Theorem 1** *Suppose that (H1)-(H4) hold. If system (3) has  $n-1$  characteristic exponents with negative real parts, then for  $\varepsilon$  sufficiently small, system (1) has a*

stable limit cycle  $\Gamma_{\xi_0(\varepsilon)}$  near the closed orbit  $\Gamma_0$  of system (2), which tends to  $\Gamma_0$  as  $\varepsilon \rightarrow 0$ .

For proving our theorem, we are going to use a new local coordinate system near closed orbit  $\Gamma_0$ . We claim that

**Lemma 1** *Let  $f(\vartheta)$  be a continuously periodic  $n$ -vector function and belong to  $C^{(2)}$ . For all  $\vartheta$ ,  $\|f(\vartheta)\| \geq \delta > 0$ . Then there is a  $T$ -periodic  $n \times (n-1)$  matrix function  $S(\vartheta)$  such that  $Q(\vartheta) = (f(\vartheta), S(\vartheta))$  is a regular matrix function, and  $\frac{d}{d\vartheta}S(\vartheta)$  exists, it is also  $T$ -periodic.*

In fact, put  $f(\vartheta) = \text{col.}(f_1(\vartheta), f_2(\vartheta), \dots, f_n(\vartheta))$ . Since  $f(\vartheta)$  is a continuously periodic function, we can assume that  $\|f(\vartheta)\| \leq M$ , where  $M$  is a positive constant.

If  $n = 2$ , we take  $S(\vartheta) = \begin{pmatrix} -f_2(\vartheta) \\ f_1(\vartheta) \end{pmatrix}$ . Then Lemma 1 follows.

If  $n \geq 3$ , since  $f \in C^{(2)}$ , the curve  $x = f(\vartheta)$  given by parameter  $\vartheta$  in space  $\mathbb{R}^n$  can not fill the entire space. Therefore, there is a ray starting from the origin which does not intersect with the curve  $x = f(\vartheta)$ . So we can suppose that this ray coincides with  $X_1$ -axis, and hence we have  $\sum_{j=2}^n f_j^2(\vartheta) \geq \delta_0 > 0$ , where  $\delta_0$  is a constant. Let

$$S(\vartheta) = \begin{pmatrix} \nu f_2(\vartheta) & \nu f_3(\vartheta) & \dots & \nu f_n(\vartheta) \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix},$$

where  $\nu > \frac{M}{\delta_0} > 0$  is a constant. Then we have

$$\begin{aligned} \det Q(\vartheta) &= \det(f(\vartheta), S(\vartheta)) = f_1(\vartheta) - \nu \sum_{j=2}^n f_j^2(\vartheta) \\ &< M - \nu \delta_0 < 0, \end{aligned}$$

that is,  $Q(\vartheta)$  is a regular matrix function.

Because of the construction of  $S(\vartheta)$ , we know that it is  $T$ -periodic and  $\frac{d}{d\vartheta}S(\vartheta)$  exists, it is also  $T$ -periodic.

When Lemma 1 holds, we set  $Q^{-1}(\vartheta) = P(\vartheta) = \begin{pmatrix} p_1(\vartheta) \\ p_2(\vartheta) \end{pmatrix}$ , where  $p_1(\vartheta)$ ,  $p_2(\vartheta)$  are respectively  $1 \times n$  and  $(n-1) \times n$  matrix function. They are also  $T$ -periodic.

**Lemma 2** *Under the hypotheses (H1)-(H4) the following curve coordinate transformation*

$$\begin{aligned} x &= u(\vartheta) + S(\vartheta)\xi, \\ y &= v(\vartheta) - g_y^{-1}(\vartheta)g_x(\vartheta)S(\vartheta)\xi + \eta \end{aligned} \quad (4)$$

defines a set of new coordinates  $(\vartheta, \xi_1, \dots, \xi_{n-1}, \eta_1, \dots, \eta_m)$  at least in a neighborhood  $U_0$  of  $\Gamma_0$ .



In fact, from (4) we see that  $\xi = 0$ ,  $\eta = 0$  and  $\vartheta = t$  on closed orbit  $\Gamma_0$ . By Lemma 1, there is a matrix function  $S(\vartheta)$  such that  $Q(\vartheta) = (f(\vartheta), S(\vartheta))$  is regular. There, the Jacobian of transformation (4) on closed orbit  $\Gamma_0$  satisfies the inequality

$$\left| \det \begin{pmatrix} \frac{\partial x}{\partial \vartheta} & \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \vartheta} & \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix} \right| = \left| \det \begin{pmatrix} f(\vartheta) & S(\vartheta) & 0 \\ * & * & I_m \end{pmatrix} \right| = |\det Q(\vartheta)| > 0.$$

By continuity, it follows that transformation (4) can be satisfactorily performed at least in a neighborhood  $U_0$  of  $\Gamma_0$ .

**Lemma 3** In  $U_0$ , transformation (4) carries system (1) into the following system

$$\begin{aligned} \frac{d\xi}{d\vartheta} &= \frac{B(\vartheta)\xi + B_1(\vartheta)\eta + G(\xi, \eta, \vartheta, \varepsilon)}{1 + F(\xi, \eta, \vartheta, \varepsilon)}, \\ \varepsilon \frac{d\eta}{d\vartheta} &= \frac{\gamma(\vartheta)\eta + H(\xi, \eta, \vartheta, \varepsilon)}{1 + F(\xi, \eta, \vartheta, \varepsilon)}. \end{aligned} \quad (5)$$

where  $B(\vartheta) = P_2(\vartheta)(A(\vartheta)S(\vartheta) - \frac{d}{d\vartheta}S(\vartheta))$ ,  $A(\vartheta) = f_x(\vartheta) - f_y(\vartheta)g_y^{-1}(\vartheta)g_x(\vartheta)$ ,  $B_1(\vartheta) = P_2(\vartheta)f_y(\vartheta)$ ,  $C(\vartheta) = g_y(\vartheta)$ ,  $F(\xi, \eta, \vartheta, \varepsilon) = O(|\xi| + |\eta| + |\varepsilon|)$ ,  $G(\xi, \eta, \vartheta, \varepsilon) = O(|\xi|^2 + |\eta|^2 + |\varepsilon|)$ ,  $H(\xi, \eta, \vartheta, \varepsilon) = O(|\xi|^2 + |\eta|^2 + |\varepsilon|)$ . Moreover, functions  $F$ ,  $G$  and  $H$  are all continuously differentiable in all arguments,  $T$ -periodic in  $\vartheta$ , and for any  $\varepsilon > 0$ , there are constants  $M_1, M_2, M_3 > 0$  such that  $\|F(0, 0, \vartheta, \varepsilon)\| < M_1\varepsilon$ ,  $\|G(0, 0, \vartheta, \varepsilon)\| < M_2\varepsilon$ ,  $\|H(0, 0, \vartheta, \varepsilon)\| < M_3\varepsilon$  for all  $\vartheta$  ( $0 \leq \vartheta \leq T$ ).

Now according to the center integral manifold theory (See[0, 0]) and the hypotheses of Theorem 1, we have

**Lemma 4** Under hypotheses (H1)-(H4) there are a neighborhood  $U_1 \in U_0$  of  $\Gamma_0$  and a manifold  $M$  satisfying that

(i)  $M$  can be represented in  $U_1$  by

$$\eta = \phi(\xi, \vartheta, \varepsilon) \quad (6)$$

where  $\phi \in C^{(2)}$ , and

$$\lim_{\varepsilon \rightarrow 0} \phi(\xi, \vartheta, \varepsilon) = 0 \quad \text{uniformly in } \xi;$$

(ii)  $M$  is invariant with respect to system(5);

(iii) all bounded solutions of system(5) which belong entirely to  $U_1$  lie on  $M$ .

Thus, we can substitute (6) into the first equation of (5) and deal only with the following regular differential equation

$$\frac{d\xi}{d\vartheta} = \frac{B(\vartheta)\xi + B_1(\vartheta)\phi(\xi, \vartheta, \varepsilon) + H(\xi, \phi(\xi, \vartheta, \varepsilon), \vartheta, \varepsilon)}{1 + F(\xi, \gamma(\xi, \vartheta, \varepsilon), \vartheta, \varepsilon)} \stackrel{\text{def}}{=} X(\xi, \vartheta, \varepsilon) \quad (7)$$

**Lemma 5** *If system (3) has  $n-1$  characteristic exponents with negative real parts, then all characteristic exponents of the following system*

$$\frac{d\zeta}{d\vartheta} = B(\vartheta)\zeta, \quad (8)$$

where  $B(\vartheta)$  was given in Lemma 3, have real parts less than zero.

Now let  $\xi = \xi(\xi_0, \vartheta, \varepsilon)$  be a solution of (7) satisfying the initial condition

$$\xi_0 = \xi(\xi_0, 0, \varepsilon) \quad (9)$$

and define the Poincaré mapping  $\Psi(\xi_0, \varepsilon): \mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}^{n-1}$  as follows:

$$\begin{aligned} \Psi(\xi_0, \varepsilon) &= \xi(\xi_0, T, \varepsilon) - \xi_0 \\ &= \exp \int_0^T X(\xi(\xi_0, \vartheta, \varepsilon), \vartheta, \varepsilon) d\vartheta. \end{aligned}$$

Notice that  $\frac{\partial}{\partial \xi_0} \xi(\xi_0, \vartheta, 0)$  is a solution of the variation equation system

$$\frac{d}{d\vartheta} \left( \frac{\partial}{\partial \xi_0} \xi(\xi_0, \vartheta, 0) \right) = \frac{\partial}{\partial \xi} X(\xi, \vartheta, 0) \Big|_{\xi=\xi(\xi_0, \vartheta, 0)} \cdot \left( \frac{\partial}{\partial \xi_0} \xi(\xi_0, \vartheta, 0) \right).$$

meanwhile,

$$\begin{aligned} \frac{\partial}{\partial \xi} X(\xi, \vartheta, 0) \Big|_{\xi=0} &= \frac{\partial}{\partial \xi} \left( \frac{B(\vartheta)\xi + B_1(\vartheta)\phi(\xi, \vartheta, 0) + G(\xi, \phi(\xi, \vartheta, 0), \vartheta, 0)}{1 + F(\xi, \phi(\xi, \vartheta, 0), \vartheta, 0)} \right) \Big|_{\xi=0} \\ &= B(\vartheta), \\ D\Psi(\xi_0, \varepsilon) \Big|_{\xi_0=0, \varepsilon=0} &= \exp \left( \int_0^T \frac{\partial}{\partial \xi} X(\xi, \vartheta, 0) \Big|_{\xi=0} d\vartheta \right) = \exp \left( \int_0^T B(\vartheta) d\vartheta \right). \end{aligned}$$

And combining with Lemma 5, we have

**Lemma 6** *If the hypotheses of Theorem 1 hold, then for  $\varepsilon$  sufficiently small, there exists  $\bar{\xi}_0 = \bar{\xi}_0(\varepsilon)$  such that  $\bar{\xi}_0(0) = 0$  and  $\Psi(\bar{\xi}_0(\varepsilon), \varepsilon) \equiv 0$ , i.e. the orbit  $\Gamma_{\bar{\xi}_0(\varepsilon)}$  of (7) started from point  $(\bar{\xi}_0(\varepsilon), \varepsilon)$  near  $\Gamma_0$  of (2) is a closed orbit, and  $\Gamma_{\bar{\xi}_0(\varepsilon)}$  tends to  $\Gamma_0$  as  $\varepsilon \rightarrow 0$ .*

*Moreover, for  $\varepsilon$  sufficiently small, the moduli of all eigenvalues of the differential operator at point  $(\bar{\xi}_0(\varepsilon), \varepsilon)$  for Poincaré mapping are less than 1, and hence, the closed orbit  $\Gamma_{\bar{\xi}_0(\varepsilon)}$  is a stable limit cycle.*

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## STABILITY OF REAL STARS AND CATASTROPHE THEORY

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We propose to investigate the stability of real stars with the catastrophe theory, which shows that the real stars may have a mass like the neutron star. The latter depends on the charge of a spin-zero particle. A difference and a role of short-range self interaction and long-range Coulomb interaction in the forming of a star are discussed.

Recently we proposed to study the stability of boson and neutron stars[1], using the method developed in the theory of solitons by one of authors[2]. This application of the nonelementary catastrophe theory based on the classification of singularities of the smooth mappings. The mapping has been created by integrals of motion of Einstein equations, which are the star mass and the star charge. The procedure of the catastrophe theory, which we are applying, is practically reduced to the finding of *bifurcation diagram*, which describes the functional dependence of the conserved quantities with respect to each other [2.1]. For the common case of two values  $M$  and  $N$  the bifurcation diagram is represented by cuspidal curve. Each of these cusps is associated with some surface, so called the Whitney surface [2]. The bifurcation diagram corresponds to critical points of some catastrophe manifold. One of authors has shown[2] that the critical points of the minimum correspond to the stable soliton and that the critical points of the maximum correspond to unstable soliton. In [1] this ideology has been applied to cases of boson, neutron stars and white dwarfs. In the early universe, scalar particles played an important role. In that time it could be possible that clouds of these particles created stars under their gravitation field, so-called *boson stars* [3]. The largest part of suggested dark matter consists of boson stars[3]. The boson star may consist of many particles and have very heavy mass like the neutron star. The latter depends upon self interaction between bosons [3.1]. The boson star is stable at small densities. The instability will appear at some critical density. This result has been obtained with the aid of perturbation theory [3]. One may show that the point of instability corresponds to a coalescence of maximum and minimum in some mass-central density-charge surface [1] (catastrophe manifold).

Recently, the Higgs particles interacting with gauge field have been studied [4]. The gauge field may be considered as electromagnetic field trapped by star. In this case the bosons having a charge will interact via electromagnetic forces. In four dimensional universe this interaction may have attractive or repulsive character. Since the star corresponds to stationary solutions of Einstein equations, the effective interaction between bosons in the star will be repulsive as for nonrelativistic case. Because of this repulsive interaction between bosons there exist a critical charge of bosons, which corresponds to Coulomb instability of the star[4]. This instability is simply an expansion or a dispersion of the star. The problem of collapse calls for a special attention. The studying of charged stars allows to understand a nature and a role of different interactions in the forming of real stars. The special question is what kind of forces (long-range

Coulomb or short-range strong interactions) is more important in the increase of the star mass beyond the Chandrasecar limit. A charged boson star is described by a self-interacting scalar field  $\Phi$  coupled self-consistently to their gravitational field and to  $U(1)$ -gauge fields having the Lagrangian

$$\mathcal{L} = \frac{1}{2\kappa} \sqrt{|g|} R + \frac{1}{2} \sqrt{|g|} [g^{\mu\nu} (D_\mu \Phi)^* (D_\nu \Phi) - U((\Phi)^* \Phi)] - \frac{1}{4} \sqrt{|g|} F_{\mu\nu} F^{\mu\nu}, \quad (0.1)$$

where  $\kappa = 8\pi G$  is the gravitational constant in natural units,  $g$  the determinant of the metric  $g_{\mu\nu}$ ,  $\mu, \nu = (0, 1, 2, 3)$ ,  $R$  the curvature scalar, and  $U((\Phi)^* \Phi)$  the self-interaction potential. Because of the coupling between the scalar field and the  $U(1)$  Maxwell field  $A_\mu$ , it is convenient to introduce the gauge and generally covariant derivative for the Higgs field  $D_\mu \Phi = \partial_\mu \Phi + ie A_\mu \Phi$ , where  $g$  is the coupling constant. Further, we have the field strength tensor  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ .

From the principle of minimal action, we obtain the coupled Einstein-Maxwell-Klein-Gordon equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa T_{\mu\nu}(\Phi, A_\sigma), \quad (0.2)$$

$$\square_M \Phi + \frac{\partial U}{\partial \Phi^*} = 0, \quad (0.3)$$

$$\frac{1}{\sqrt{|g|}} \partial_\nu (\sqrt{|g|} F^{\mu\nu}) = -\frac{ie}{2} g^{\mu\nu} [(D_\nu \Phi) \Phi^* - (D_\nu \Phi)^* \Phi], \quad (0.4)$$

where

$$T_{\mu\nu}(\Phi, A_\sigma) = (D_\mu \Phi)^* (D_\nu \Phi) - F_\mu^\kappa F_{\nu\kappa} - \frac{g_{\mu\nu}}{\sqrt{|g|}} \mathcal{L}(\Phi, A_\sigma) \quad (0.5)$$

is the energy-momentum tensor and  $\square_M \Phi$  is a covariant  $U(1)$ -d'Alembertian.

The static, spherically symmetrical metric

$$ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (0.6)$$

in which the functions  $\nu = \nu(r)$  and  $\lambda = \lambda(r)$  depend on the Schwarzschild type radial coordinate  $r$ . For the boson field, we make the stationary ansatz

$$P(r, t) = S(r) e^{i\omega t}, \quad (0.7),$$

which describes a spherically symmetrical bound state with frequency  $\omega$ . To have only electric charges for the scalar field, we make the following choose for the gauge field

$$A_\mu(r, t) = (C_0(r), 0, 0, 0). \quad (0.8)$$

Because we have a coupled system of ordinary differential equations.

Analogous to the uncharged boson star, we use the Tolman mass

$$M := 4r \int_0^\infty [2(\omega + eC_0)^2 S^2 e^{-\nu} + e^{-\lambda-\nu} C_0'^2 - U] e^{(\nu+\lambda)/2} r^2 dr. \quad (0.9)$$

For the charged boson star, this expression does involve derivatives, in contrast to the boson star [3,1].

A second "integral of motion" arises from  $U(1)$  symmetry.

$$Q = eN = 4\pi\epsilon \int_0^{\infty} e^{(\lambda-\nu)/2} r^2 S^2(\omega + eC_0) dr. \quad (0.10)$$

The diagram  $(M, N)$  is the bifurcation diagram (see, Fig. 2 in Ref.[1]). This diagram plays the decisive role in the determination of the stability of the charged boson stars. In catastrophe theory this diagram is a skeleton of a catastrophe manifold (a multidimensional surface). The bifurcation diagram shows critical points (minima and maxima) of this surface. At the cusp, the minimum and the maximum coalesce. According to Whitney theorem, we may distinguish three type of objects on the surface: 1. Regular points 2. Fold points 3. Cusp points. Every cusp is produced through a projection of a Whitney surface on some plane  $(M, N)$ . The lower (upper) branch of the cusp represents the projection of minima (maxima) of the Whitney surface. We connect the stable solution of the Einstein equation with the lower branch, which is associated with a minimum. Intuitively this solution will be stable under small perturbation. On the other hand the solutions, associated with the upper branch will be unstable under some perturbation, which grow exponentially and destroy the star. We may say that at the cusp's point the one instability appears or disappears. Therefore, if we found a stable solution the all instabilities may be classified. Thus the method allows to find the star stability without consideration complicated equations created by perturbation theory.

Probably, the discovery of the catastrophe theory method [6,5] allows to consider the stability of multicomponent realistic stars. Here the complication of Einstein equations increases. The number of motion integrals increases. We should also consider a very complicated equation of state. But the catastrophe theory method may solve this problem and therefore it opens a new era for using the superpower of transcomputers.

Notice, there should exist *oscillating charged boson stars*. A variable electric field in such stars produces a magnetic field with creating a star radiation. Therefore it exists a significant bigger chance to discover the oscillating charged boson stars.

In [3] it was shown the importance of boson stars which is so massive as neutron stars provided the self-interaction between the spin-zero constituents is short-range. This fact follows from an asymptotic behavior of the maximal mass from the self-interaction constant  $\alpha$  [3]

$$M_{max} \sim \frac{0.22}{\sqrt{4\pi}} \alpha^{1/2} \frac{M_{Pl}^3}{m^2}. \quad (0.11)$$

For  $\alpha \sim 1$  and  $m = m_N$  ( the boson mass equals the mass of a neutron) we have an upper limit  $M_{max} \sim M_{Chandrasekhar} = M_{Pl}^3/m_N^2$ . The same dependence was found for the charged boson star but for the particle charge  $e$

$$M_{max} \sim \frac{1}{\sqrt{(\epsilon_{crit} - \epsilon)}} \alpha \frac{M_{Pl}^2}{m}. \quad (0.12)$$

$\epsilon_{crit}$  depends from the additional self-interacting strength. Hence, for  $\epsilon = \epsilon_{crit} - (m/M_{Pl})^2$  and  $m = m_N$  we get  $M_{max} = \alpha M_{Chandrasekhar}$  with the same answer like in the case of the uncharged boson star. Hence for  $\alpha \sim 1$  the mass of charged boson stars equals the mass of neutron stars. There may exist so gigantic stars as they are found in [3] if we choose  $\epsilon = \epsilon_{crit} - (m/M_{Pl})^4$ . We conclude that the choice of the electric charge produces the same effect like the self-interaction written through  $\alpha$ . Table 1 shows the analogies.

| Electric field                                                                  | self-interaction  |
|---------------------------------------------------------------------------------|-------------------|
| $\epsilon = 0$                                                                  | $\alpha = 0$      |
| $\epsilon = \epsilon_{crit}$                                                    | $\alpha = \infty$ |
| $\epsilon \in [\epsilon_{crit} - (m/M_{Pl})^2, \epsilon_{crit} - (m/M_{Pl})^4]$ | $\alpha \sim 1$   |
| $\epsilon \in [\epsilon_{crit} - (m/M_{Pl})^4, \epsilon_{crit} - (m/M_{Pl})^6]$ | $\alpha \sim 1$   |

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# Generalized Coarse-Graining and Irreversibility in Classical Dynamical Systems

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**Abstract** The operational approach in terms of POV measures and instruments is applied to the statistical description of classical dynamical systems, e.g., ergodic systems. Coarse graining of dynamical systems is defined by means of a class of POV measures which induce stochastic mappings (linear state transformations). Some different models of coarse graining for dynamical systems are introduced and the possibility of characterizing the instabilities of mixing dynamical systems is analyzed.

This paper is a summary of [1]. Starting-point of our discussion is Ruch's fundamental work [2], which develops basic concepts for the description of irreversibility.

## 1. Notion of Observable, Instrument and Coarse graining

Let  $(\Omega, \Sigma, \mu)$  be a measure space with phase space  $\Omega$ ,  $\sigma$ -algebra  $\Sigma$  and measure  $\mu$ . States are represented by positive norm-1 functions in  $L^1(\Omega, \Sigma, \mu)$ , that is the space of absolute integrable functions on  $\Omega$ .  $L^\infty(\Omega, \Sigma, \mu)$  denotes the space of all  $\mu$ -essentially bounded functions on  $\Omega$ .

An observable  $\epsilon$  on some measurable space  $(\tilde{\Omega}, \tilde{\Sigma})$  is a normalized positive-operator-valued (POV) measure on  $\tilde{\Sigma}$ , that is, an observable is a mapping  $\epsilon : \tilde{\Sigma} \rightarrow L^\infty(\Omega, \Sigma, \mu)_{\leq 1}^+$  with the properties:

$$- \epsilon(0) = 0, \quad \epsilon(\Omega) = \lambda_\Omega$$

$$- \{\tilde{\Delta}_i\}_{i \in I}, \quad \tilde{\Delta}_i \cap \tilde{\Delta}_j = 0 \text{ for } i \neq j : \quad \epsilon\left(\bigcup_{i \in I} \tilde{\Delta}_i\right) = \sum_{i \in I} \epsilon(\tilde{\Delta}_i)$$

The  $\sigma$ -algebra  $\tilde{\Sigma}$  is the set of all possible outcomes of  $\epsilon$ -measurement. A state  $\rho \in L^1(\Omega, \Sigma, \mu)_1^+$  and an observable  $\epsilon$  define a probability measure  $p_\rho^\epsilon : \tilde{\Sigma} \rightarrow [0, 1]$ ,  $\tilde{\Delta} \mapsto p_\rho^\epsilon(\tilde{\Delta}) = \int_\Omega \rho \epsilon(\tilde{\Delta}) d\mu_\epsilon$  ( $\epsilon$  absolutely continuous w.r.t.  $\mu_\epsilon$ ).  $p_\rho^\epsilon(\tilde{\Delta})$  is the probability for the outcome  $\tilde{\Delta}$ , if the system is in state  $\rho$ .

Ordinary observables in classical mechanics are represented by measurable functions on phase space  $\Omega$ . Every measurable function  $f : \Omega \rightarrow \mathbf{R}$  defines uniquely a projection-operator-valued (PV) measure, that is a special POV measure,  $p_f : \mathcal{B}(f(\Omega)) \mapsto \Xi = \{\text{all characteristic functions in } L^\infty(\Omega, \Sigma, \mu)\}$ . A POV measure  $f$  is a PV measure if and only if  $\epsilon(\tilde{\Delta})^2 = \epsilon(\tilde{\Delta})$  for all  $\tilde{\Delta} \in \tilde{\Sigma}$ .

Coarse graining interpreted as a restriction of the measurement of phase space volumes, i.e., not all phase space cells can be measured accurately, is defined by

means of a class of POV measures  $\epsilon : \Sigma \rightarrow L^\infty(\Omega, \Sigma, \mu)_{\leq 1}^+$  ( $\epsilon$  abs. cont. w.r.t.  $\mu$ ).  $\epsilon(\Delta)$  represents the smeared phase space cell  $\Delta$ , resp.  $\chi_\Delta \in \Xi$ . In this case the probability measure  $p_\rho^\epsilon : \Sigma \rightarrow [0, 1], \Delta \mapsto p_\rho^\epsilon(\Delta) = \int_\Omega \rho \epsilon(\Delta) d\mu = \int_\Delta \tilde{\rho} d\mu$  induces a positive, linear, trace preserving (in short stochastic) mapping  $\Phi_\epsilon : L^1(\Omega, \Sigma, \mu) \rightarrow L^1(\Omega, \Sigma, \mu), \rho \mapsto \Phi_\epsilon \rho = \tilde{\rho}$ .

On the other hand the dual  $\Phi^*$  of a stochastic mapping  $\Phi$  defines a POV measure  $\epsilon_{\Phi^*} : \Sigma/\Sigma_\mu \rightarrow L^\infty(\Omega, \Sigma, \mu)_{\leq 1}^+$  with  $\Sigma_\mu := \{\Delta \in \Sigma | \mu(\Delta) = 0\}$ .

An instrument is an operation-valued measure  $\mathcal{I} : \tilde{\Sigma} \rightarrow \mathcal{L}(L^1(\Omega, \Sigma, \mu)_{\leq 1}^+)$  on some measurable space  $(\tilde{\Omega}, \tilde{\Sigma})$  defined such that

- $\mathcal{I}(0) = 0, \mathcal{I}(\Omega) = \Phi_T, \Phi_T$  is a stochastic mapping
- $\{\tilde{\Delta}_i\}_{i \in I}, \tilde{\Delta}_i \cap \tilde{\Delta}_j = 0$  for  $i \neq j : \mathcal{I}\left(\bigcup_{i \in I} \tilde{\Delta}_i\right) = \sum_{i \in I} \mathcal{I}(\tilde{\Delta}_i)$ .

Two simple examples of instruments are given by (i)  $\mathcal{I}(\tilde{\Delta}) : \mathcal{I}(\tilde{\Delta})\rho = \rho \Phi^*(\chi_{\tilde{\Delta}})$  and (ii)  $\mathcal{I}(\tilde{\Delta}) : \mathcal{I}(\tilde{\Delta})\rho = \Phi_T \rho \chi_{\tilde{\Delta}}$ . In the case (i) one observes the POV measure  $\epsilon_{\Phi^*}$  and the initial state  $\rho$  is not changed ( $\Phi_T = \text{id}$ ), in the case (ii)  $\epsilon_{\Phi_T}$  is observed and the state  $\rho$  is changed into  $\Phi_T \rho$ , i.e., the system is open.

In the special case of measurable space  $(\Omega, \Sigma)$  the measurement performed by an instrument  $\mathcal{I}$  can be interpreted as coarse graining.

Consider now two concrete POV measures  $\Phi^*$ , which are important in further discussions:

- a) Let  $\{\Delta_i\}_{i \in I}$  be a partition:  $\Delta_i \cap \Delta_j = 0$  for  $i \neq j, \bigcup_{i \in I} \Delta_i = \Omega, \mu(\Delta_i) < \infty :$

$$\Phi^* : \Xi \rightarrow L^\infty(\Omega, \Sigma, \mu)_{\leq 1}^+, \quad \chi_\Delta \mapsto \Phi^*(\chi_\Delta) = \sum_{i \in I} \frac{\mu(\Delta \cap \Delta_i)}{\mu(\Delta_i)} \chi_{\Delta_i}$$

- b) Let  $(\Omega, \mathcal{B}(\Omega), \mu_H)$  be a measure space, where  $\Omega$  denotes a locally compact abelian group,  $\mathcal{B}(\Omega)$  the Borel algebra and  $\mu_H$  the Haar measure:

$$\Phi_j^* : \Xi \rightarrow L^\infty(\Omega, \mathcal{B}(\Omega), \mu_H)_{\leq 1}^+, \quad \chi_\Delta \mapsto \Phi_j^*(\chi_\Delta),$$

$$\Phi_j^*(\chi_\Delta)(x) = \int_\Omega f(x-y) \chi_\Delta(y) d\mu_H(y) = (\chi_\Delta * f)(x), f \in L^1(\Omega, \mathcal{B}(\Omega), \mu_H)_1^+.$$

## 2. Instability Measures for Mixing Dynamical Systems

Given a dynamical group  $\{i_t\}_{t \in \mathbb{R}}$  ( $i_t$  is an isomorphism),  $i_t \rho^* = \rho^*$  for a fixed state  $\rho^*$ .

$\{i_t\}_{t \in \mathbb{R}}$  is  $\rho^*$ -mixing, if

$$\forall \rho \in L^1(\Omega, \Sigma, \mu)_1^+ \quad \forall g \in L^\infty(\Omega, \Sigma, \mu) : \lim_{t \rightarrow \infty} \langle i_t \rho, g \rangle = \langle \rho^*, g \rangle.$$



A family  $\{\Phi_\alpha\}_{\alpha \in I}$  of stochastic mappings is called statistically complete, if the span of the joined ranges of the POV measures is  $\sigma$ -weakly dense in  $L^\infty(\Omega, \Sigma, \mu)$ .

A single  $\Phi$  is statistically complete iff  $\Phi$  is injective.

**Theorem** Let  $\{\Phi_\alpha\}_{\alpha \in I}$  be a statistically complete family,  $\Phi_\alpha \rho^* = \rho^*$  and  $\{i_t\}_{t \in \mathbf{R}}$  a dynamical group. Define the family  $\{\Phi_t^\alpha\}_{t \geq 0}$  by  $\Phi_t^\alpha := \Phi_\alpha \circ i_t$ . Then:

$$\forall \alpha \in I \quad \forall \rho \in L^1(\Omega, \Sigma, \mu)_1^+ : \lim_{t \rightarrow \infty} \|\Phi_t^\alpha \rho - \rho^*\| = 0 \implies \{i_t\}_{t \in \mathbf{R}} \text{ is } \rho^* \text{-mixing.}$$

Any measurable mapping  $K : \Omega \times \Omega \rightarrow \mathbf{R}^+$ ,  $\int_\Omega K(x, y) d\mu(x) = 1$ , defines a stochastic kernel and thereby a stochastic mapping  $\Phi$  by  $\Phi \rho(x) = \int_\Omega K(x, y) \rho(y) d\mu(y)$  for  $\rho \in L^1(\Omega, \Sigma, \mu)$ . Furthermore, if the functions  $g_x(y) := K(x, y)$  for fixed  $x \in \Omega$  are bounded (a.e.), one can prove the converse:

**Theorem** Let  $\{i_t\}_{t \in \mathbf{R}}$  be a  $\rho^*$ -mixing dynamical system,  $\Phi$  a stochastic mapping defined by a bounded stochastic kernel,  $\Phi \rho^* = \rho^*$ , and  $\{\Phi_t\}_{t \geq 0}$  given by  $\Phi_t := \Phi \circ i_t$ . Then

$$\forall \rho \in L^1(\Omega, \Sigma, \mu)_1^+ : \lim_{t \rightarrow \infty} \|\Phi_t \rho - \rho^*\| = 0.$$

By the first theorem every injective stochastic mapping  $\Phi$ ,  $\Phi \rho^* = \rho^*$ , can be interpreted as an instability measure for mixing dynamical systems  $\{i_t\}_{t \in \mathbf{R}}$ . Furthermore such a  $\Phi$  establishes a semigroup

$$\{W_t\}_{t \geq 0}, \quad W_t := \Phi \circ i_t \circ \Phi^{-1}, \quad W_t : \Phi(L^1(\Omega, \Sigma, \mu)) \rightarrow \Phi(L^1(\Omega, \Sigma, \mu)).$$

$\{W_t\}_{t \geq 0}$  represents the macro dynamics on the coarse grained state space  $\mathcal{M}_\Phi := \Phi(L^1(\Omega, \Sigma, \mu))$ . The semigroup property ensures the monotonic decrease of the function  $\|W_t \rho - \rho^*\|$ ,  $\rho \in \mathcal{M}_\Phi^+$ , which therefore is a  $H$  function, as any other convex function of  $\rho_t = W_t \rho$ , too.

Examples:

(i) Let  $\{\Delta_i^\alpha\}_{i \in I}$  be a partition:  $\Delta_i^\alpha \cap \Delta_j^\alpha = \emptyset$  for  $i \neq j$ ,  $\bigcup_i \Delta_i^\alpha = \Omega$ ,  $\mu(\Delta_i^\alpha) < \infty$ .

Thus one defines  $\Phi_\alpha : L^1(\Omega, \Sigma, \mu) \rightarrow L^1(\Omega, \Sigma, \mu)$ ,  $\Phi_\alpha(\rho) = \sum_{i \in I} \frac{(\rho, \chi_{\Delta_i^\alpha})}{\mu(\Delta_i^\alpha)} \chi_{\Delta_i^\alpha}$

with stochastic kernel  $K^\alpha(x, y) = \sum_{i \in I} \frac{1}{\mu(\Delta_i^\alpha)} \chi_{\Delta_i^\alpha}(x) \chi_{\Delta_i^\alpha}(y)$ . Then the family  $\{\Phi_\alpha\}_{\alpha \in I}$ , where  $\alpha$  runs over all partitions, is statistically complete.

(ii) Let  $(\Omega, \mathcal{B}(\Omega), \mu_H)$  be a measure space with locally compact abelian group  $\Omega$ . Define  $\Phi_f : \rho \mapsto \Phi_f \rho$ ,  $(\Phi_f \rho)(x) = \int_\Omega \rho(y) f(y-x) d\mu(y)$  with

$$K(x, y) := f(y-x) \text{ for } f \in L^1(\Omega, \mathcal{B}(\Omega), \mu_H)_1^+.$$

$\Phi_f$  becomes injective for such  $f \in L^1(\Omega, \mathcal{B}(\Omega), \mu_H)_1^+$ , for which the "Fourier" transform  $\hat{f}$  does not vanish on the character group of  $\Omega$ .  $\hat{f}$  is exactly the Fourier transform in the case of measure spaces  $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), \mu_L)$  and  $(X_{i=1}^n[0, 1], \mathcal{B}, \mu_L)$ . The Gaussian distribution for infinite- and  $f(x) = \gamma e^{-x}$  for finite measure spaces are examples for such functions. For finite measure spaces any injective  $\Phi_f$  defines an instability measure. The inverse  $\Phi_f^{-1}$  is not positive, i.e., an "unphysical" mapping.

### 3. Relation to other Entropy Functions

In the following let  $(\Omega, \Sigma, \mu)$  be a finite measure space with  $\mu(\Omega) = 1$ . Let  $\eta : \mathbf{R}^+ \rightarrow \mathbf{R}$  be a strictly convex function, satisfying  $\eta(1) = 0$ ,  $\lim_{t \rightarrow \infty} \frac{1}{t} \eta(t) = \infty$  and  $\mathcal{D}_\eta := \{f \in L^1(\Omega, \Sigma, \mu)_1^+ \mid \int_\Omega \eta(f) d\mu < \infty\}$ . Define  $S_\eta : \mathcal{D}_\eta \rightarrow \mathbf{R}$ ,  $\rho \mapsto S_\eta(\rho) = \int_\Omega \eta(\rho) d\mu$ . Then one can define for a fixed stochastic mapping  $\Phi$  the coarse grained entropy  $S_{\eta, \Phi} : \rho \mapsto S_{\eta, \Phi}(\rho) = \int_\Omega \eta(\Phi \rho) d\mu$ .

**Theorem** Let  $\{\Phi_t\}_{t \geq 0}$  be a family of bistochastic mappings,  $\eta$  a strictly convex function satisfying the assumptions introduced above. Then:

$$\forall \rho \in \mathcal{D}_\eta : \left( \lim_{t \rightarrow \infty} \|\Phi_t \rho - \mathbf{1}\| = 0 \Leftrightarrow \lim_{t \rightarrow \infty} \int_\Omega \eta(\Phi_t \rho) d\mu = 0 \right).$$

This establishes:

**Corollary** Let  $\Phi$  be an injective stochastic mapping,  $\Phi \mathbf{1} = \mathbf{1}$ ,  $\{i_t\}_{t \in \mathbf{R}}$  a dynamical group and  $\eta$  any fixed entropy function. Then:

$$\forall \rho \in \mathcal{D}_\eta : \lim_{t \rightarrow \infty} S_{\eta, \Phi}(i_t \rho) = 0 \Rightarrow \{i_t\}_{t \in \mathbf{R}} \text{ is uniform mixing.}$$

In that case every coarse grained entropy function  $S_{\eta, \Phi}$  can be interpreted an instability measure for mixing dynamical systems.

### 4. Generalized H-theorems

In general the semigroup property fails in case of  $\{\Phi_t\}_{t \geq 0}$ ,  $\Phi_t = \Phi_{c_g} \circ i_t$ . Hence one is interested to introduce a sufficient criterion which ensures the monotonic decrease of entropy functions. Monotonic decrease of mixing distance furnishes in a natural way a sufficient criterion:

A given family  $\{\Phi_t\}_{t \geq 0}$ ,  $\Phi_t \rho^* = \rho^*$ , of stochastic mappings is monotone for a fixed  $\rho \in L^1(\Omega, \Sigma, \mu)_1^+$ :

$$\forall \alpha, \beta \in \mathbf{R}^+ \forall t' \geq t \quad \|\alpha \Phi_{t'} \rho - \beta \rho^*\| \geq \|\alpha \Phi_t \rho - \beta \rho^*\|$$

$$\Leftrightarrow \exists \psi_\rho(t', t) \in ST(L^1) : \left( \rho_{t'} = \Phi_{t'} \rho = \psi_\rho(t', t) \rho_t = \psi_\rho(t', t) \circ \Phi_t \rho \ \& \ \psi_\rho(t', t) \rho^* = \rho^* \right).$$

The equivalence follows from [3]. The existence of the stochastic mapping  $\psi_\rho(t', t)$  guarantees in the case of finite measure spaces that every convex function induces a H function. An example is given in the following theorem:

**Theorem**  $(\Omega, \mathcal{B}(\Omega), \mu_H)$ ,  $\mu_H(\Omega) = 1$ , where  $\Omega$  is a locally compact abelian group,  $\{\Phi_t\}_{t \geq 0}$ ,  $\Phi_t := \Phi_f \circ i_t$ , where  $(\Phi_f \rho)(x) = \int_\Omega \rho(y) f(y-x) d\mu(y)$ , and  $(i_t \rho)(x) = \rho(S_t^{-1} x)$  for a group  $\{S_t\}_{t \in \mathbf{R}}$  with  $\mu_H(S_t^{-1} \Delta) = \mu_H(\Delta)$ ,  $\Delta \in \mathcal{B}(\Omega)$ . Then:

$$\forall \rho \in L^1(\Omega, \mathcal{B}(\Omega), \mu_H)_1^+ \forall t' \geq t \exists \psi_\rho(t', t) \in ST(L^1) : \\ \rho_{t'} = \Phi_{t'} \rho = \psi_\rho(t', t) \circ \Phi_t \rho, \quad \psi_\rho(t', t) \mathbf{1} = \mathbf{1}.$$

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# **Equilibrium Statistical Mechanics of Terrestrial Ecosystems \*)**

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## **1. Introduction**

The biogeophysical ecosphere is the union of the earth's ecosystems and is divided into soil-bound or terrestrial ecosystems, in aquatic, limnological, marine or cryogenic ecosystems, in atmospheric and finally in ecosystems related to volcanic nature. Ecosystems are catalytic feedback nets of mutual dependences. We consider here terrestrial ecosystems only, with main emphasis on forests in diverse types and specifications. Our results should be applicable as well to fields, cultivated by tillage and pasture.

A forest is usually composed of many living and also of anorganic constituents, for example the subsystem of trees of different species. The observables of these subsystems are biotic ones, like functions of the biomasses of certain species, as well as abiotic factors, like thermodynamic quantities

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(temperature, humidity etc.) and other macroscopic physical or chemical observables, like concentration of reactants, wind pressure and velocity or solar radiation intensity.

To the best of the author's knowledge, it was first E.H. Kerner [ 1 ], who noticed that the interaction potential, if dependent on the biotic observables, instead of the positions of particles in a gas, can be treated as in physics, leading to a useful statistical mechanics of ecosystems, where dynamics or equilibrium of probability distributions on the phase space of those subsystems is considered.

## 2. Subsystem Dynamics

The interaction of the abovementioned observable functions of the biomasses or of related biotic observables, and the abiotic factors, all abbreviated by  $q_i$ ,  $i = 1, \dots, 2n$ , is postulated to be given by an equation of motion of first order

$$\frac{d}{dt} q_i = - \sum_{j=1}^{2n} (\Gamma^{-1})_{ij} \frac{\partial U}{\partial q_j} = F_i \quad (1)$$

where  $U$  is the interaction energy from many-body potentials, and  $\Gamma$  is an antisymmetric, non-singular matrix. The equations of motion can be derived as Euler-Lagrange equations, from well-defined Lagrangean [ 2 ], containing the interaction energy  $U$ , which turns out to be an integral of motion, and a Hamiltonian flow does exist [ 2 ], is, however, not straightforward to deduce from the Lagrangean. It is important to note, that the usual Volterra - Lotka eco-subsystem-dynamics [ 1 ] is a particular example of an equation of type (1).

### 3. Equilibrium States and the Thermodynamic Limit

The canonical ensemble is defined by the Gibbs state of the Hamiltonian. The Gibbs measure is associated to the normalized weight function

$$Z^{-1} \exp(-\beta \mathfrak{H}((p),(q))) \quad (2)$$

where  $\mathfrak{H}((p),(q))$  is one of the previously introduced Hamilton functions and the  $p$ 's are the canonically conjugated momenta.

In [ 2 ], and [ 3 ], we consider, partially based on [ 1 ] and [ 4 ], finally three different Hamilton functions, which are consistent with the principles of classical mechanics. It turns out, that the equilibrium states are all unitarily equivalent, and the free energies are equivalent either. This gives with (2) a unique notion of what is known as ecological equilibrium.

Furthermore, using the theory of metric cones of interactions [ 5.6 ], we are able to prove rigorously the existence of the thermodynamic limit for the free energy and the mean correlation functions in a weak, measure-theoretic sense, almost everywhere with respect to the metric on a very large class of interactions. For the particular Volterra-Lotka one-body potentials, we give explicit expressions for the limit free energy and the limit Gibbs state [ 2.3 ].

A comparison of the equilibrium Volterra-Lotka probability distribution with biomass measurements in a beech forest is given in [ 3 ].

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## COMBINATORIAL GROUP-THEORETICAL APPROACH IN HIGH-ORDER NEURAL NETWORKS

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### INTRODUCTION

Dynamical models which capture the primary features of information processing and adaptive behaviour in living nervous systems have often been described by irreducible multi-neuron interactions of multiplicative character [1,2]. In fact, it is well known that synapses not only modify the membrane potentials of dendrites but also those of other synapses. Moreover, models with high-order interactions have impressive storage capacities increasing with a power of the order of the interactions. Hence, the stability of the stored information is expected to increase dramatically if high-order effects are taken into account.

### 1. THE MULTI-CONNECTED NETWORK MODEL

The model network consists of a set of  $N$  interacting binary threshold units  $\sigma_i$  which are only capable to take the value  $+1$  and  $-1$  for unit  $i$  "active" or "non active", respectively. We assume that each neuron  $i$  can interact with  $K_i$  other units of the network with  $1 \leq K_i \leq N - 1$ , while self-interactions are excluded. The state of neuron  $i$  at time  $t$  is then specified according to the deterministic threshold rule

$$\sigma_i(t+1) = \text{sgn}[h_i(t)] \quad i = 1, \dots, N \quad (1.1)$$

The net internal stimulus  $h_i(t)$  is defined in terms of a polynomial  $K_i$ -th order expansion of the commonly used linear superposition of the weighted input states

$$\begin{aligned} h_i(t) &= c_{i0} + \sum_{(j_1)} c_{ij_1} \sigma_{j_1}(t) + \dots + \sum_{(j_1) < \dots < (j_{K_i})} c_{ij_1 \dots j_{K_i}} \sigma_{j_1}(t) \dots \sigma_{j_{K_i}}(t) \\ &= C_0 + C_1(t) + \dots + C_{K_i}(t) \end{aligned} \quad (1.2)$$

where the sums are only taken over those  $K_i$  neurons that interact with neuron  $i$ .

In order to achieve faithful storage of  $p$  arbitrary patterns  $\underline{S}^1, \dots, \underline{S}^p \in \{-1, 1\}^N$  as fixed points of the dynamics (1.1) the coupling coefficients can be determined by the natural extension of the classical Hebbian learning rule to order  $K_i$  via

$$c_{ij_1 \dots j_{K_i}} = \sum_{\mu=1}^p S_i^\mu S_{j_1}^\mu S_{j_2}^\mu \dots S_{j_{K_i}}^\mu \quad (1.3)$$



Note however, that an  $s$ -order contribution in eq.(1.2) contains  $\binom{K_i}{s}$  constants. Due to this rapidly proliferating parametrization high-order networks have often been believed to be impracticable for real-world applications. In fact, optimizing a network with all possible high-order terms is clearly unfeasible, although the optimization problem has been successfully attacked by admitting only a strongly reduced pattern-specific interconnectivity [2,3]. In this report we will demonstrate how suitable summation techniques for the inclusion of high-order terms can completely eliminate the proliferation problem.

## 2. HIGH-ORDER CONTRIBUTIONS AND POLYA POLYNOMIALS

It is straightforward to evaluate a modified  $s$ -order contribution in eq.(2.2)

$$\bar{C}_s = \sum_{(j_1) \dots (j_s)} c_{ij_1 \dots j_s} \sigma_{j_1}(t) \dots \sigma_{j_s}(t) \quad , \quad (2.1)$$

which consists of  $K_i^s$  terms. The auxiliary quantity  $\bar{C}_s$  defined in (2.1) contains "diagonal" terms specified as those of which at least two indices  $j_1, \dots, j_s$  are the same. These terms are redundant since they already appear in lower-order contributions. Inserting (1.3) in (2.1) and interchanging the order of the summations leads to the simple form

$$\begin{aligned} \bar{C}_s &= \sum_{(j_1) \dots (j_s)} c_{ij_1 \dots j_s} \sigma_{j_1} \dots \sigma_{j_s} = \sum_{(j_1) \dots (j_s)} \sum_{\mu=1}^p S_i^\mu S_{j_1}^\mu \dots S_{j_s}^\mu \sigma_{j_1} \dots \sigma_{j_s} \\ &= \sum_{\mu=1}^p S_i^\mu \left( \sum_{(j_1)} S_{j_1}^\mu \sigma_{j_1} \right) \dots \left( \sum_{(j_s)} S_{j_s}^\mu \sigma_{j_s} \right) = \sum_{\mu=1}^p S_i^\mu \left[ \sum_{(j_1)} S_{j_1}^\mu \sigma_{j_1} \right]^s \quad . \end{aligned} \quad (2.2)$$

The corresponding desired  $s$ -order contribution  $C_s$  can then be evaluated via

$$C_s = \sum_{(j_1) < \dots < (j_s)} c_{ij_1 \dots j_s} \sigma_{j_1} \dots \sigma_{j_s} = \frac{1}{s!} \bar{C}_s \text{Det}_{\alpha, \beta=1, \dots, s} (\delta_{j_\alpha j_\beta}) \quad (2.3)$$

with  $\bar{C}_s$  taken from eq. (2.2). The  $s \times s$  determinant in (2.3) eliminates all "diagonal" terms with two or more indices coincident, while the statistical factor  $s!$  takes care of symmetric terms. Defining "generalized" overlaps of a current net configuration  $\underline{\sigma}(t)$  with one of the prescribed patterns  $\underline{S}^\mu$

$$m_\alpha^\mu(t) = \sum_{(j_1)} [S_{j_1}^\mu \sigma_{j_1}(t)]^\alpha \quad , \quad (2.4)$$

direct evaluation of the r.h. side of eq. (2.3) yields the following results for small  $s$ :

$$C_1 = \sum_{\mu=1}^p S_i^\mu [m_1^\mu] \quad (2.5)$$

$$C_2 = \sum_{\mu=1}^p S_i^\mu \frac{1}{2!} [(m_1^\mu)^2 - m_2^\mu] \quad (2.6)$$

$$C_3 = \sum_{\mu=1}^p S_i^\mu \frac{1}{3!} [(m_1^\mu)^3 - 3m_1^\mu m_2^\mu + 2m_3^\mu] \quad (2.7)$$

and

$$C_4 = \sum_{\mu=1}^p S_i^\mu \frac{1}{4!} [(m_1^\mu)^4 - 6(m_1^\mu)^2 m_2^\mu + 8m_1^\mu m_3^\mu + 3(m_2^\mu)^2 - 6m_4^\mu] \quad (2.8)$$

For arbitrary  $s$  it can be easily shown that the quantity  $C_s$  can be written as

$$C_s = \sum_{\mu=1}^p S_i^\mu \bar{P}_s(m_1^\mu, \dots, m_s^\mu) \quad (2.9)$$

with

$$\bar{P}_s(m_1, \dots, m_s) = \frac{1}{s!} \sum_{(\underline{\alpha})} \prod_{l=1}^s \gamma(\alpha_1, \dots, \alpha_s) m_l^{\alpha_l} \quad (2.10)$$

The sums in (2.10) are performed only over those  $s$ -dimensional vectors  $\underline{\alpha} = (\alpha_1, \dots, \alpha_s) \in \mathbb{N}_0^s$  whose components are solutions of the partitioning relation

$$\sum_{l=1}^s l\alpha_l = s \quad (2.11)$$

while the coefficients  $\gamma(\alpha_1, \dots, \alpha_s)$ , satisfying the sum rules

$$\sum_{(\underline{\alpha})} \gamma(\alpha_1, \dots, \alpha_s) = 0 \quad \text{and} \quad \sum_{(\underline{\alpha})} |\gamma(\alpha_1, \dots, \alpha_s)| = s! \quad (2.12)$$

are given by

$$\gamma(\alpha_1, \dots, \alpha_s) = s! / \left[ \prod_{l=1}^s (-1)^{\alpha_l+1} (l^{\alpha_l} \alpha_l!) \right] \quad (2.13)$$

The quantity  $\bar{P}_s(m_1, \dots, m_s)$  is a generalized Polyà polynomial [4] of the symmetric group  $S_s$ , where the signs of the corresponding cyclic permutations  $(-1)^{\alpha_l+1}$  have been included.

For arbitrary  $s$  the total number of solutions  $P(s)$  of (2.11) can be given in terms of the recurrence relation [6]

$$P(s) = \frac{1}{s} \sum_{l=1}^s \sigma(l) P(s-l) \quad (2.14)$$

where the divisor function  $\sigma(l)$  is the sum of the first powers of the divisors of  $l$ . Note that in the large  $s$ -limit  $P(s)$  behaves asymptotically [6] like

$$P(s) = \frac{1}{4s\sqrt{3}} e^{\pi\sqrt{\frac{2s}{3}}} \quad (2.15)$$

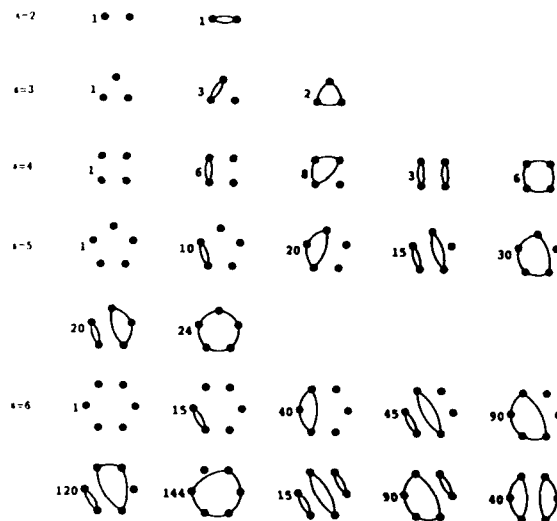
Finally, the generating function of the Polyà polynomials allows to calculate the sum of all individual  $s$ -order contributions, i.e. the total net internal stimulus  $h_i(t)$  eq.(1.2), in the thermodynamic limit:

$$\sum_{s=0}^{\infty} C_s = \sum_{\mu=1}^p S_i^{\mu} \exp\left\{ \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l} m_l^{\mu} \right\} . \quad (2.16)$$

Note however, that, though eq.(2.16) is a beautiful formal result, practical neural network applications often work within a fixed single order adapted to the "order" of the problem [7] such that the corresponding  $s$ -order contribution (2.3) is of practical value.

### 3. HIGH-ORDERS AND FERMION DIAGMAMTICS

Combinatorial group-theoretical considerations reveal also that there is a one-to-one correspondence between an  $s$ -order contribution  $C_s$  (2.3) and certain  $s$ -cell diagrams obeying fermion statistics which may be drawn in a plane. (Substituting the determinant in (3.9) by a permanent would correspond to Boson statistics). Each of these diagrams is uniquely defined by an  $s$ -dimensional vector  $\underline{\alpha} = (\alpha_1, \dots, \alpha_s)$  satisfying (2.11) which specifies the grouping of the  $s$  cells into a product of exchange clusters, consisting of  $\alpha_1$  1-cycles,  $\alpha_2$  2-cycles, ... and  $\alpha_s$   $s$ -cycles. The magnitude of the statistical weight factor  $\gamma(\alpha_1, \dots, \alpha_s)$  equals the number of ways in which  $s$  cells can be distributed into  $\alpha_l$  exchange clusters, containing  $l$  cells each, for  $l = 1, \dots, s$ . Figure 1 shows all possible cluster diagrams up to order  $s = 7$  described by  $s$  filled dots and the corresponding exchange lines. The latter only appear in closed polygons such that those cells belonging to a definite exchange cluster are connected through a closed loop. These cluster diagrams have also been used in the description of non-interacting fermions or bosons within the recently developed correlated density matrix formalism [5]. The exchange lines reflect the statistical correlations imposed by the Fermi- or Bose symmetries of the wavefunctions with respect to particle exchange.



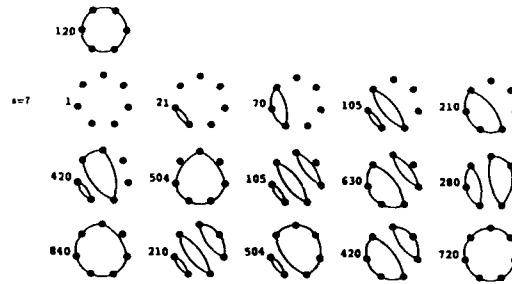


Fig.1: All possible cluster diagrams for  $s=2, s=3, \dots, s=7$ .

#### 4. CONCLUSION

We have shown that the problem of controlling high-order contributions in networks equipped with multi-cell interactions can be successfully solved with combinatorial group-theoretical tools which have long been known in Polyà's theory and the theory of boson and fermion diagrammatics.

Computer simulations reveal convincingly that the retrieval performance of the network, especially its discrimination capability, increases substantially if high order contributions are *consistently* included, i.e. if redundant "diagonal" terms are eliminated.

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# Geometry of feed-forward networks

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## Abstract

A geometric approach to the generalizing ability and representational properties of feed-forward networks is established. Feed-forward networks are considered as mapping networks. It is shown, that any set of training examples induces a foliation of their weight spaces. This intrinsic geometrical structure allows deeper insight into the role of different training patterns and the structure of the solution set.

## 1 Introduction

The intention of our approach is to extract those mathematical structures inherent in feed-forward networks, which lead to deeper insight into their generalizing properties, the organization of the solution set, the formation of internal representations, etc. For these networks the foliated structure of their weight spaces and the 'orbit' structure of internal and other symmetry groups are of relevance. Its only in a second step, that these results will be applied to practical questions like the development of more effective learning algorithms. In this paper we will reveal the foliated structure of the weight spaces.

We will concentrate on two-layer feed-forward networks with continuous, non-linear units. The case of more than one hidden layer will be tractable in a straight forward way. We consider these nets as mapping networks [1]. Correspondingly, in section 2 we describe their behavior in terms of input/output maps which are parametrized by elements of the weight space.

In section 3 it is shown that associated with a given set of training examples there is a foliated structure of the weight space. The set of all possible generalizations of the training set is characterized by a specific leaf, the solution set of this foliation. The concept of (almost) independent training examples is introduced. For an (almost) independent training set the solution leaf is (generically) a submanifold of the weight space. It is shown that a feed-forward network with a  $q$ -dimensional weight space can generalize at most  $q$  independent training examples. In section 4 the relevance of the foliated structure with respect to effective learning dynamics is shortly discussed.

## 2 Mapping Networks

In the following  $\mathbf{N}(l, h, m)$  denotes a two layer feed-forward network with  $l$  input units,  $m$  output units and  $h$  hidden units, i.e. a total of  $n = l + h + m$  neurons. We choose sigmoidal transfer functions and the standard activation function; i.e. the discrete activation dynamics is given by

$$a_i(t+1) := \sum_{j=1}^n w_{ij} o_j(t) + \theta_i, \quad i = l+1, \dots, n, \quad (1)$$

where  $w_{ij}$  denotes the weight from unit  $j$  to unit  $i$ ,  $\theta_i$  the bias,  $a_i(t)$  the activation at time  $t$  of unit  $i$ , and  $o_i(t)$  its output given by

$$o_i(t) := \sigma(a_i(t)), \quad (2)$$

with transfer function

$$\sigma(a) := \frac{1}{1 + e^{-a}}. \quad (3)$$

For convenience we denote the bias  $\theta_i$  in (1) by  $w_{i0}$ , i.e. by the weight from unit 0, which denotes the bias unit (constant output 1), to unit  $i$ . These equations define the standard setup for backpropagation networks [2], but the explicit form of the learning rule is not of interest here. Networks of this type can be described as 'mapping networks' [1]: If the configuration of the net is fixed by a weight vector  $w \in W$  where  $W \approx \mathbf{R}^q$  denotes the weight space, the behavior of the net can be understood in terms of an input/output function  $f_w : \mathbf{R}^l \supset D \rightarrow \mathbf{R}^m$  from a subset  $D$  of  $l$ -dimensional Euclidean space to a bounded subset  $f_w(D)$  of  $m$ -dimensional Euclidean space. The output of the net will be denoted by  $z = f_w(x) \in \mathbf{R}^m$ .

Since the behavior of the net will depend on its configuration; i.e. on the weight vector  $w \in W$ , it is represented by a function  $f : W \times \mathbf{R}^l \rightarrow \mathbf{R}^m$ , the characteristic function of the net, and we have  $f_w(x) := f(w, x)$ . Here  $f(w, x)$  is a  $C^\infty$ -function of both  $x$  and  $w$ .

Under supervised learning the net is trained on a training set  $\tau := \{(x^\mu, y^\mu)\}, \mu = 1, \dots, r$  of  $r$  desired input/output pairs. Thus the training set is a set of points  $(x^\mu, y^\mu) \in \mathbf{R}^l \times \mathbf{R}^m$ , and the task of a learning procedure is to adjust the weights  $w_{ij}$  of the net in such a way that the graph of the function  $f = f(w, x)$  matches the points  $(x^\mu, y^\mu)$  of the training set. From this point of view learning is nothing but a type of curve fitting [3]. In the following we assume that  $\tau$  can be perfectly learned, and we stress the following geometric view:

Definition: Given a net with characteristic function  $f$  and a training set  $\tau$ . A generalization of  $\tau$  is the graph of an i/o function  $f_w$  matching the points of  $\tau$ ; i.e. it is a submanifold  $\text{graph}(f_w) \subset \mathbf{R}^l \times \mathbf{R}^m$  with  $\tau \subset \text{graph}(f_w)$ . The corresponding  $w \in W$  is called a solution for  $\tau$ . The solution set  $S_\tau$  is the subset in  $W$  given by

$$S_\tau := \{w \in W \mid \tau \subset \text{graph}(f_w)\}. \quad (4)$$

In general a given network will be able to produce different generalizations of a given training set. They correspond to different weight vectors  $w \in S_\tau$  inducing different i/o maps  $f_w$ , all satisfying

$$f_w(x^\mu) = y^\mu, \quad (x^\mu, y^\mu) \in \tau. \quad (5)$$

### 3 Foliation of the weight space

For the sake of notational simplicity, we restrict our description to networks  $\mathbf{N}(l, h, 1)$ , i.e. to nets with only one output unit. The results will be strictly generalizable to nets with  $m$  output units, replacing  $\mathbf{R}$  valued functions on  $W$  by  $\mathbf{R}^m$  valued maps on  $W$ . Let  $\tau$  denote a given training set and assume, that the network  $\mathbf{N}(l, h, 1)$  is able to find a solution for  $\tau$ . Any training example  $(x^\mu, y^\mu) \in \tau$  induces a map  $f^\mu : W \rightarrow I \subset \mathbf{R}^1$  given by

$$f^\mu(w) := f(w, x^\mu). \quad (6)$$

with  $I := (0, 1)$ . Observe that the map  $f^\mu$  depends only on the input vector  $x^\mu$  of the example  $(x^\mu, y^\mu) \in \tau$ .

From the network model (1)-(3) it follows, that the functions  $f^\mu$  induced by the training examples are submersions. This follows from the fact that  $df^\mu(w) \neq 0$  for all  $w \in W$ . Here  $df^\mu$  denotes the exterior derivative of  $f^\mu$  (i.e.  $df^\mu$  may be viewed as the metric dual of the gradient vector field  $\nabla f^\mu$  on  $W$  [4]). So every  $z \in I$  is a regular value of  $f^\mu$ . Since  $df^\mu$  is a closed one-form, it defines an involutive distribution  $\mathcal{D}^\mu$  on  $W$  given by the kernel of  $df^\mu$  and generates a regular, codimension-1 foliation  $\mathcal{F}^\mu$  on  $W$  [5]. The leaves of  $\mathcal{F}^\mu$  are denoted by  $N_z, z \in I$ . Recall that the vector fields  $X$  belonging to the distribution  $\mathcal{D}^\mu$  are tangential to the leaves  $N_z$ .

Furthermore, it follows, that the training example  $(x^\mu, y^\mu) \in \tau$  defines a distinguished leaf of  $\mathcal{F}^\mu$ , i.e. a codimension-one submanifold  $N^\mu$  of  $W$ , given by

$$N^\mu := N_{y^\mu} = (f^\mu)^{-1}(y^\mu), \quad (7)$$

and we have the following

**Statement:** Every training example  $(x^\mu, y^\mu) \in \tau$  generates a codimension-1 foliation  $\mathcal{F}^\mu$ , and defines a distinguished leaf  $N^\mu$  of this foliation.

Of course, to every  $w \in N^\mu$  there corresponds an i/o map  $f_w : \mathbf{R}^l \rightarrow \mathbf{R}^m$  whose graph matches the training example  $(x^\mu, y^\mu) \in \mathbf{R}^l \times \mathbf{R}^m$ .

Now, consider a second training example  $(x^\nu, y^\nu) \in \tau, \nu \neq \mu$ . The corresponding function  $f^\nu$  generates a regular foliation  $\mathcal{F}^\nu$  and a distinguished leaf  $N^\nu$ . Suppose  $N^\mu \cap N^\nu$  is not empty; then every  $w \in N^\mu \cap N^\nu$  defines an i/o map  $f_w$  whose graph matches both examples  $(x^\mu, y^\mu), (x^\nu, y^\nu) \in \mathbf{R}^l \times \mathbf{R}^m$ .

Because of dimensional reduction, an interesting situation occurs if the two submanifolds  $N^\mu$  and  $N^\nu$  intersect transversally, i.e.

$$T_w N^\mu + T_w N^\nu = T_w W, \quad w \in N^\mu \cap N^\nu. \quad (8)$$

Here  $T_w N^\mu$  denotes the tangent space of the submanifold  $N^\mu$  at the point  $w \in N^\mu$ . In this case  $\text{codim}(N^\mu \cap N^\nu) = \text{codim}(N^\mu) + \text{codim}(N^\nu) = 2$  [4].

The condition that the leaves  $N^\mu$  and  $N^\nu$  intersect transversally means, that  $df^\mu(w)$  and  $df^\nu(w)$  are linearly independent for all  $w \in N^\mu \cap N^\nu$ . Or, differently stated, the differential 2-form  $df^\mu \wedge df^\nu$  on  $W$  is non-vanishing on the intersection, i.e.  $df^\mu \wedge df^\nu(w) \neq 0$  for  $w \in N^{\mu\nu} := N^\mu \cap N^\nu$ . Here  $\alpha \wedge \beta$  denotes the wedge product [4] of differential forms  $\alpha$  and  $\beta$  on  $W$ .

Correspondingly, with respect to all  $r$  training examples of  $\tau$  we define the  $r$ -form  $\omega_\tau$  associated with  $\tau$  by

$$\omega_\tau := df^1 \wedge \dots \wedge df^r. \quad (9)$$

If all the leaves of the  $r$  foliations  $\mathcal{F}^\mu$  intersect transversally the  $r$ -form  $\omega_\tau$  will be non-vanishing on  $W$ . We will use this differential form to characterize the training set  $\tau$ .

Definition: A training set  $\tau = \{(x^\mu, y^\mu)\}$  of size  $r$  is called independent, iff the corresponding one forms  $df^1, \dots, df^r$  are linearly independent in every point  $w \in W$ ; i.e. iff  $\omega_\tau(w) \neq 0$  for all  $w \in W$ . It is called almost independent, iff  $\omega_\tau(w) \neq 0$  for almost all  $w \in W$ , and dependent, iff the corresponding 1-forms  $df^1, \dots, df^r$  are linearly dependent in every point  $w \in W$ , i.e. iff  $\omega_\tau(w) = 0$  for all  $w \in W$ .

Definition: A point  $w \in W$  is called a critical point for  $\tau$ , iff  $\omega_\tau(w) = 0$ . The critical set  $C_\tau$  for  $\tau$  is given by

$$C_\tau := \{w \in W \mid \omega_\tau(w) = 0\}. \quad (10)$$

This means of course, that on the critical set  $C_\tau$  the gradients  $df^\mu$  of  $\tau$  are linearly dependent. If  $\tau$  is independent the critical set is empty; if  $\tau$  is almost independent,  $C_\tau$  has Lebesgue measure zero in  $W$ . If the size  $r$  of the training set  $\tau$  is greater than the dimension  $q$  of the weight space  $W$ , the critical set is the whole manifold  $W$ .

Let  $\tau = \{(x^\mu, y^\mu)\}$  denote an independent training set of size  $r$ . Since  $\omega_\tau$  is a closed  $r$ -form; i.e.  $d\omega_\tau = 0$ , it defines an involutive distribution  $\mathcal{D}_\tau$  and correspondingly generates a regular codimension  $r$  foliation  $\mathcal{F}_\tau$  of  $W$  [5]; i.e. the leaves are then codimension- $r$  submanifolds of  $W$ .

If  $\tau$  is almost independent, the distribution  $\mathcal{D}_\tau$  will not be involutive, and the corresponding foliation  $\mathcal{F}_\tau$  will be singular; i.e. its leaves are not of equal dimension. In fact some of the leaves may not even be submanifolds of  $W$  [5].

The leaves  $N_{z_1, \dots, z_r}$  of the foliation  $\mathcal{F}_\tau$  are parametrized by  $r$  values  $z_1, \dots, z_r \in I$ . In fact a leaf  $N_{z_1, \dots, z_r}$  is just the intersection of the leaves  $N_{z_1}, \dots, N_{z_r}$  of the codimension-1



foliations  $\mathcal{F}^\mu$  [5] corresponding to the  $r$  training examples  $\{(x^\mu, y^\mu)\}$ ; i.e.

$$N_{z_1 \dots z_r} = N_{z_1} \cap \dots \cap N_{z_r} \quad . \quad (11)$$

In particular, if  $N^1, \dots, N^r$  denote the codimension-one submanifolds (7) associated with the  $r$  training examples of  $\tau$ , then the solution set  $S_\tau$  for  $\tau$  will be given by the intersection of these submanifolds, i.e.

$$S_\tau := N^1 \cap \dots \cap N^r. \quad (12)$$

To show, that  $S_\tau$  is generically a codimension- $r$  submanifold of  $W$ , we define the map  $f_\tau : W \rightarrow I^r$  by

$$f_\tau(w) := (f^1(w), \dots, f^r(w)) \quad . \quad (13)$$

The solution set  $S_\tau$  is then given by

$$S_\tau := f_\tau^{-1}(y^1, \dots, y^r) \quad . \quad (14)$$

If  $\tau$  is independent, then  $f_\tau$  is a submersion and generates the regular foliation  $\mathcal{F}_\tau$ . In this case the solution set  $S_\tau$  is a leaf of this foliation, i.e. a codimension- $r$  submanifold of  $W$ .

If  $\tau$  is almost independent, then the set of critical points of  $f_\tau$  is of measure zero in  $I^r$  by Sard's theorem [4]; i.e. the generic situation is, that the  $r$  target values  $(y^1, \dots, y^r) \in I^r$  correspond to a regular value of  $f_\tau$ . But then  $S_\tau$  is again a codimension- $r$  submanifold of  $W$ .

**Statement:** If  $\tau$  is an (almost) independent training set, then its solution set  $S_\tau$  is (generically) a codimension- $r$  submanifold of  $W$ .

If  $\tau$  is an independent training set of maximal size  $r = q$ , then the corresponding regular foliation  $\mathcal{F}_\tau$  is of codimension  $q$ ; i.e. the leaves of  $\mathcal{F}_\tau$  are the points of  $W$ . This gives us an upper bound for the 'capacity' of the net under consideration:

**Lemma:** A feed-forward network with a  $q$ -dimensional weight space  $W$  can generalize at most  $q$  independent training patterns.

In general, this maximal storage capacity will not be available, since internal symmetries of the net and external symmetries inherent in the problem (the training set) will reduce the dimension of the effective weight space (see the last section for an example).

## 4 Foliations and learning

The process of learning can be described by a dynamical system, i.e. a vector field  $Y$  on  $W$  [6]. The problem is, that  $W$  is of very high dimension. We may use the foliated structure of  $W$  to reduce the dynamics to an effective weight space of smaller

dimension. The argument goes as follows: An effective dynamics shall move from leaf to leaf, finally reaching the solution set  $S_\tau$  on the solution set  $S_\tau$ ; i.e. the flow of the vector field  $Y$  shall move across the leaves. An interesting dynamics then will be one on the quotient space

$$A_\tau := W/\mathcal{F}_\tau \quad (15)$$

of  $W$  with respect to the foliation  $\mathcal{F}_\tau$ . The space  $A_\tau$  is defined by the following equivalence relation  $R_\tau \subset W \times W$ :

$$(w, w') \in R_\tau \quad \text{iff} \quad w, w' \in N_{z_1, \dots, z_r} \quad (16)$$

Assume in the following that  $\tau$  is independent. The corresponding foliation  $\mathcal{F}_\tau$  is then regular and the quotient space  $A_\tau$  can be given a quotient manifold structure [5]. Furthermore the foliation  $\mathcal{F}_\tau$  then can be represented by a submersion  $\varphi : W \rightarrow M$ , where  $M$  denotes a  $r$ -dimensional manifold and  $r$  is the size of  $\tau$ . Together with the canonical projection  $\pi : W \rightarrow A_\tau$ , given by  $\pi : w \rightarrow [w]$ , where  $[w]$  denotes the equivalence class of  $w$  with respect to  $R_\tau$ , we get the following commutative diagram for the bijection  $\eta$ :

$$\begin{array}{ccc} & W & \\ \pi \swarrow & & \searrow \varphi \\ A_\tau & \xrightarrow{\eta} & M \end{array}$$

The submersion  $\varphi$  is in our case given by the map  $f_\tau : W \rightarrow \mathbf{R}^r$  defined in (13). The quotient space  $A_\tau$  corresponds to the set of indices for the leaves of  $\mathcal{F}_\tau$  and it is isomorphic to the output space  $M = I^r \subset \mathbf{R}^r$ .

A learning procedure is then effectively given by a vector field  $Y$  on  $M$ . This dynamical system  $(Y, M)$  should be convergent, and the point  $m = (y^1, \dots, y^r)$  of  $M$  given by the  $r$  targets  $y^\mu$  must be an asymptotically stable equilibrium point of this dynamics. Although this representation might not be of practical use (there are non-linear coordinate transformations involved), it will help to describe the training dynamics in analytical terms.

## 5 Conclusion

Although the foliated structure of the weight space gives considerable geometric insight into the general structure of the solution set and the critical set with respect to the given training data, it will develop its full strength when combined with

symmetry arguments, i.e. with group theoretical techniques. The foliated structure will be useful for the construction of an effective learning dynamics for almost independent training sets.

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| Umirbekova, Liza           | Moscow                | USSR          |
| Ushveridze, A. G.          | Tbilisi               | USSR          |
| Vaia, Ruggero              | Firenze               | Italy         |
| van der Jeugt, Joris       | Gent                  | Belgium       |
| Varro, Sandor              | Budapest              | Hungary       |
| Verbeure, Andre            | Leuven                | Belgium       |
| Verch, Rainer              | Berlin                | Germany       |
| Villarreal, Carlos         | México                | México        |
| Vinet, Luc                 | Montréal              | Canada        |

|                                |                            |            |
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| Vogt, Andrew                   | Washington                 | USA        |
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| Watamura, Satoshi              | Karlsruhe                  | Germany    |
| Wehrhahn, Rodolfo              | Hamburg                    | Germany    |
| Werle, Jozef                   | Warsaw                     | Poland     |
| Wieczorek, E.                  | Zeuthen                    | Germany    |
| Wilson, Raj                    | San Antonio                | USA        |
| Winternitz, Pavel              | Montreal                   | Canada     |
| Wulff, Michael                 | Köln                       | Germany    |
| Yamamoto, Hiroshi              | Fukui                      | Japan      |
| Yarunin, V. S.                 | Moscow                     | USSR       |
| Yegorchenko, Irina             | Kiev                       | USSR       |
| Yuanshi, Huang                 | Fuzhou                     | China P.R. |
| Zaccharia, F.                  | Napoli                     | Italy      |
| Zacharias, Joachim             | Berlin                     | Germany    |
| Zachary, Woodford W.           | Washington DC              | USA        |
| Zaidi, H. R.                   | Fredericton, New Brunswick | Canada     |
| Zambrini, J. Claude            | Lisbon                     | Portugal   |
| Zanzinger, Stefan              | Tübingen                   | Germany    |
| Zeh, H. D.                     | Heidelberg 1               | Germany    |
| Zheludev, I. S.                | Moscow                     | USSR       |
| Zukowski, Marek                | Gdansk                     | Poland     |
| Zylka, Christian               | Leipzig                    | Germany    |