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A. H. Nuttall

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Foreword

This collection of technical reports addresses the following topics: threshold setting and required signal-to-noise ratios for combined normalization and or-ing; spectra and covariances for nonlinear processing in class A noise; exact performance of filtered and weighted energy detectors with mismatch; comparison of kernels for the modified Wigner function; solution of difference equation for wavenumber response of plate; detection performance of FSK in a partially-correlated fading medium.

Some of the material presented here is heavily based on the author's earlier work, which can be found in the following volumes in addition to the referenced technical reports:

***Performance of Detection and Communication Systems,**
NUSC Scientific and Engineering Studies, 1974;

Spectral Estimation,
NUSC Scientific and Engineering Studies, 1977;

Coherence Estimation,
NUSC Scientific and Engineering Studies, 1979;

Receiver Performance Evaluation and Spectral Analysis,
NUSC Scientific and Engineering Studies, 1981;

Signal Processing Studies,
NUSC Scientific and Engineering Studies, 1983;

Signal Processing Studies,
NUSC Scientific and Engineering Studies, 1985;

Signal Processing Studies,
NUSC Scientific and Engineering Studies, 1986;

Signal Processing Studies,
NUSC Scientific and Engineering Studies, 1987;

Signal Processing Studies,
NUSC Scientific and Engineering Studies, 1989;

Signal Processing Studies,
NUSC Scientific and Engineering Studies, 1990.

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NUSC Technical Report 8865
10 April 1991

Required Threshold Settings and Signal-to-Noise
Ratios for Combined Normalization and Or-ing

Albert H. Nuttall

ABSTRACT

The system of interest here includes the combined nonlinear transformations of normalization and or-ing in the data processing chain. The required threshold settings for specified false alarm probabilities are determined, as well as the required input signal-to-noise ratios for specified detection probabilities. Specializations to normalization or or-ing alone are available by particular choices of parameter values built into the models.

When the sizes of the transformations are changed, that is, the amount of normalization and/or or-ing, the thresholds will also have to be changed if the earlier probabilities are to be maintained; these options are included in the first program listed. In addition, the possibility of changing the number of thresholds, at the same time that the sizes of the nonlinear transformations are changed, is included in the second program listed. Furthermore, four alternatives are allowed for selecting the spacing of the thresholds, namely, equispaced in power, equispaced in decibels, preset normalized thresholds, or preset probabilities.

Approved for public release; distribution is unlimited.

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LIST OF SYMBOLS

RV	random variable
IID	independent identically distributed
PDF	probability density function
CDF	cumulative distribution function
CF	characteristic function
Prob	probability
x_m	m-th noise-only random variable, (1)
M	M+1 is the number of random variables $\{x_m\}$
$p_x(u)$	probability density function of x_m , (1)
$C_x(X)$	cumulative distribution function of x_m , (1)
$f_x(\xi)$	characteristic function of x_m , (2)
$\mu_x(j)$	j-th moment of x_m
$\chi_x(j)$	j-th cumulant of x_m
$\tilde{C}_x(P)$	inverse cumulative distribution function for x , (3)
f	nonlinear transformation, (5)
y	output of nonlinear transformation, (5)
w_{km}	inputs to normalizer and or-ing device, (8)
x_k	output from or-ing device, (8)
K	size of or-ing operation, (8)
$C_y(Y)$	cumulative distribution function of y , (9)
Y_1	first threshold for application to y , (9)
P_1	specified cumulative probability at Y_1 , (9)
$\tilde{C}_y(P)$	inverse cumulative distribution function for y , (9)
Y_T	last threshold for application to y , (10)
T	total number of thresholds, (10)

A_y	scale factor, (10)
dB_y	decibel ratio between Y_T and Y_1 , (10)
Y_t	t-th threshold for application to y , (11)
$\underline{Y}_t$	t-th normalized threshold for y , (12)
μ_y	mean of random variable y , (12)
σ_y	standard deviation of random variable y , (12)
P_t	cumulative probability value at threshold Y_t , (13)
$1-P_t$	false alarm probability, (13)
D_t	t-th threshold in decibels, (14), (16)
N	new size of nonlinear transformation, (20)
z	output of new nonlinear transformation, (20)
$C_z(z)$	cumulative distribution function of z , (21)
Z_t	t-th threshold for application to z , (21)
$\tilde{C}_z(P)$	inverse cumulative distribution function for z , (21)
$\underline{Z}_t$	t-th normalized threshold for z , (22)
μ_z	mean of random variable z , (22)
σ_z	standard deviation of random variable z , (22)
dB_z	decibel ratio between Z_T and Z_1 , (23)
U	new number of thresholds, (24)
Z'_u	u-th new threshold, (24)
P'_u	cumulative probability at Z'_u , (25)
$\underline{Z}'_u$	u-th normalized threshold for z , (26)
$C_y(Y;R)$	cumulative distribution of y with signal present, (27)
R	input power signal-to-noise ratio, (27)
S_1	specified probability, (27)
R_t	required signal-to-noise ratio at threshold Y_t , (27)

Pd_j	j-th detection probability, (30)
a	average of M random variables, (31)
$f_a(\xi)$	characteristic function of a, (32), (36)
$p_a(u)$	probability density function of a, (32), (37)
$p_y(u)$	probability density function of y, (34), (39)
$\overline{y^j}$	j-th moment of random variable y, (41)
$(b)_j$	$b(b+1)\cdots(b+j-1)$, (41)
$\chi_y(j)$	j-th cumulant of y, (55)
a_k	k-th denominator average, (62)
F_1, F_2	auxiliary constants, (79)
b_k	k-th denominator average, (81)
L	new size of or-ing device, (82)
G_1, G_2	auxiliary constants, (86)
Q_t	auxiliary function, (95), (100)

REQUIRED THRESHOLD SETTINGS AND SIGNAL-TO-NOISE
RATIOS FOR COMBINED NORMALIZATION AND OR-ING

INTRODUCTION

When weak signals of unknown strength and location have to be detected in the presence of noise of unknown and varying level, it is necessary to make estimates of the intensity of the interfering background. These estimated (noise) levels are then compared with that for a candidate signal level and location for purposes of making statements about the likelihood of signal presence or absence in that particular data segment under investigation. Here, we will investigate the performance capability of such a normalizer, in terms of the false alarm and detection probabilities, and determine the thresholds and input signal-to-noise ratios required to attain these probabilities.

Furthermore, when large amounts of multichannel data have to be processed with limited computational facilities in reasonable or short intervals of time, it is necessary to resort to shortcuts or data reduction in order to avoid overload. One possible approach is to employ or-ing, in which only the largest of a set of quantities is retained for further data processing and decision making.

Finally, in practice, it is often necessary to utilize both normalization and or-ing together. This combination of nonlinear processors requires a resetting of the thresholds that would have been appropriate for use of the normalizer alone. Here, we shall

investigate all three situations, namely normalization, or-ing, and a combination of normalization and or-ing, in terms of the probabilities and thresholds stated above.

When the size of the nonlinear transformation, whether it be normalization, or-ing, or both, is changed, the required thresholds will have to be changed if the previously realized (false alarm) probabilities are to be maintained. For example, suppose we had been or-ing 8 random variables and decided to change the or-ing size to accept 16 random variables instead, for purposes of further data reduction. Then, the required threshold settings would have to be modified to maintain specified false alarm probabilities, as would the required input signal-to-noise ratios for specified detection probabilities. This maintenance of probabilities under a change of size of transformation will be investigated here.

At the same time that the size of a nonlinear transformation is changed, it may be desired to subject its output to a different number of thresholds than were utilized previously. This possibility is allowed and analyzed in addition.

The particular physical situation considered here is that of multiple simultaneous beamformer outputs, each with large banks of narrowband filters subject to envelope-squared detection. For Gaussian noise inputs, the probability density function of each of these device outputs is exponential. Furthermore, when a Gaussian signal is also present at the input of one of these narrowband filters, the corresponding probability density

function of the detected output is still exponential, but with a level governed by the signal-to-noise ratio at that particular filter output. This scenario will be the mainstay of the analysis here.

The physical motivation for this study is to be able to set requantization levels on a display, in order to achieve constant marking density independent of the signal processing parameters such as the amount of or-ing and normalizer size. Such displays occur in active as well as passive sonar systems.

DEFINITIONS OF FUNCTIONS

The random variables (RV) for noise-only, $\{x_m\}$ for $0 \leq m \leq M$, are independent and identically distributed (IID) with common probability density function (PDF) $p_x(u)$ and cumulative distribution function (CDF) $C_x(X)$, where

$$C_x(X) = \text{Prob}(x_m < X) = \int_{-\infty}^X du p_x(u) . \quad (1)$$

The corresponding characteristic function (CF) of RV x_m is

$$f_x(\xi) = \int du \exp(i\xi u) p_x(u) , \quad (2)$$

where integrals without limits are over $-\infty, +\infty$. The moments and cumulants of order j of RV x_m are denoted by $\mu_x(j)$ and $\chi_x(j)$, respectively. The inverse function to CDF $C_x(X)$ in (1) is denoted as $\tilde{C}_x(P)$; that is,

$$\text{if } P = C_x(X), \text{ then } X = \tilde{C}_x(P) \text{ for } 0 < P < 1 . \quad (3)$$

As an example, consider narrowband filter outputs for which

$$\begin{aligned} p_x(u) &= \exp(-u) \text{ for } u > 0 , \\ C_x(X) &= 1 - \exp(-X) \text{ for } X > 0 , \\ f_x(\xi) &= (1 - i\xi)^{-1} \text{ for all } \xi , \\ \tilde{C}_x(P) &= -\ln(1 - P) \text{ for } 0 < P < 1 , \\ \mu_x(j) &= j! , \quad \chi_x(j) = (j-1)! \text{ for } j \geq 1 . \end{aligned} \quad (4)$$

The scaling of RV x_m in (4) has been taken such that its mean is 1. This is done solely for notational convenience; it will not affect the probabilities realized herein nor the required signal-to-noise ratios. However, it does influence the threshold settings calculated, which would have to be scaled for a different input noise level.

THRESHOLD RESETTING

A collection of IID RVs, $\{x_m\}$ for $0 \leq m \leq M$, is subject to a nonlinear transformation f , yielding output

$$y = f(x_0, x_1, \dots, x_M) . \quad (5)$$

For example, a normalizer is characterized by

$$y = \frac{x_0}{\frac{1}{M}(x_1 + x_2 + \dots + x_M)} , \quad M \geq 1 , \quad (6)$$

while an or-ing device yields

$$y = \max(x_1, x_2, \dots, x_M) , \quad M \geq 1 . \quad (7)$$

A combination of a normalizer and or-ing device will require a more general formulation; then we would use

$$x_k = \frac{w_{ko}}{\frac{1}{M} \sum_{m=1}^M w_{km}} \quad \text{for } 1 \leq k \leq K ,$$

$$y = \max(x_1, x_2, \dots, x_K) , \quad (8)$$

where we need two parameters, M and K , and $\{w_{km}\}$ are $K(M+1)$ IID RVs. Of course, then the K RVs $\{x_k\}$ are also IID.

Let $C_y(Y)$ be the CDF of RV y at the output of general nonlinearity f in (5). Choose threshold Y_1 such that specified cumulative probability P_1 is realized there; that is,

$$P_1 = C_Y(Y_1) , \text{ or } Y_1 = \tilde{C}_Y(P_1) , \quad (9)$$

where $\tilde{C}_Y$ is the inverse function to C_Y . $1-P_1$ is the false alarm probability at threshold Y_1 . Also, choose additional thresholds $\{Y_t\}$ such that $Y_1 < Y_2 < \dots < Y_T$, for a total of T thresholds, with the largest one being

$$Y_T = Y_1 A_Y , \text{ where } dB_Y = 10 \log_{10}(A_Y) \quad (10)$$

is a specified decibel value. Then take the remaining thresholds according to equal spacing rule

$$Y_t = Y_1 + \frac{Y_T - Y_1}{T - 1}(t - 1) \text{ for } 1 \leq t \leq T . \quad (11)$$

The normalized thresholds for RV y are defined as

$$\underline{y}_t = \frac{Y_t - \mu_Y}{\sigma_Y} \text{ for } 1 \leq t \leq T . \quad (12)$$

where μ_Y and σ_Y are the mean and standard deviation of RV y .

Then compute the cumulative probabilities realized at these thresholds $\{Y_t\}$, namely

$$P_t = C_Y(Y_t) \text{ for } 1 \leq t \leq T , \quad (13)$$

and print out M , dB_Y , μ_Y , σ_Y , T , $\{Y_t\}$, $\{\underline{Y}_t\}$, $\{P_t\}$. The false alarm probabilities are $\{1-P_t\}$.

An alternative choice is to space the thresholds $\{Y_t\}$ equally in decibels. That is, defining

$$D_t = 10 \log_{10}(Y_t) \quad \text{for } 1 \leq t \leq T \quad (14)$$

as the threshold in decibels, we take D_1 as given in terms of Y_1 and we take

$$D_T = D_1 + dB_y . \quad (15)$$

The intermediate decibel thresholds are then selected according to the equal spacing rule

$$D_t = D_1 + \frac{D_T - D_1}{T - 1}(t - 1) \quad \text{for } 1 \leq t \leq T . \quad (16)$$

The power thresholds can then be evaluated as

$$Y_t = 10^{D_t/10} \quad \text{for } 1 \leq t \leq T . \quad (17)$$

The cumulative probabilities at these latter power thresholds follow from (13) as before.

Another possibility is to simply set the thresholds according to

$$Y_t = \mu_y + \sigma_y \underline{Y}_t \quad \text{for } 1 \leq t \leq T , \quad (18)$$

where normalized thresholds $\{\underline{Y}_t\}$ are preset constants determined by the user. In this latter case, probability P_1 in (9) would not be realized at initial threshold $Y_1 = \mu_y + \sigma_y \underline{Y}_1$. In any event, the desired printouts are the quantities listed under (13).

Finally, we could set the thresholds such that preset values of probabilities $\{P_t\}$ are realized for all $1 \leq t \leq T$. That is, solve (13) for the thresholds according to

$$Y_t = \tilde{C}_y(P_t) \quad \text{for } 1 \leq t \leq T. \quad (19)$$

This amounts to setting T different false alarm probabilities $\{1-P_t\}$.

ALTERNATIVE SIZE TRANSFORMATION

Now consider the new RV z obtained by changing the parameter value from M to N in the given nonlinear transformation in (5):

$$z = f(x_0, x_1, \dots, x_N) . \quad (20)$$

N can be larger or smaller than M . Let the CDF of RV z be $C_z(z)$. We now choose new thresholds, $\{z_t\}$ for $1 \leq t \leq T$, such that the probabilities $\{P_t\}$ in (13) are maintained for RV z ; that is, we choose thresholds $\{z_t\}$ for RV z in (20) such that

$$C_z(z_t) = P_t , \quad \text{or } z_t = \tilde{C}_z(P_t) \quad \text{for } 1 \leq t \leq T . \quad (21)$$

These thresholds $\{z_t\}$, to be employed for RV z , will not necessarily have equal spacing, as did the thresholds $\{y_t\}$ in (11), for example, for RV y . The normalized thresholds for RV z are

$$\underline{z}_t = \frac{z_t - \mu_z}{\sigma_z} \quad \text{for } 1 \leq t \leq T , \quad (22)$$

where μ_z and σ_z are the mean and standard deviation of RV z .

Then print out N , dB_z , μ_z , σ_z , T , $\{z_t\}$, $\{\underline{z}_t\}$, $\{P_t\}$, where

$$dB_z \equiv 10 \log_{10}(z_T/z_1) . \quad (23)$$

DIFFERENT NUMBER OF THRESHOLDS

If RV z is to be subject to a different number, U , of thresholds than RV y was, it is not always reasonable to try to maintain the set of T probabilities $\{P_t\}$ realized in (13). (U can be larger or smaller than T .) One alternative is to maintain the edge probabilities P_1 and P_T according to (21), thereby determining values for Z_1 and Z_T . Then choose a different complete set of thresholds $\{Z'_u\}$ for RV z according to equal spacing rule

$$Z'_u = Z_1 + \frac{Z_T - Z_1}{U - 1}(u - 1) \quad \text{for } 1 \leq u \leq U. \quad (24)$$

We can then evaluate the cumulative probabilities at these latter threshold values as

$$P'_u = C_z(Z'_u) \quad \text{for } 1 \leq u \leq U. \quad (25)$$

Of course, $P'_1 = P_1$ and $P'_U = P_T$, since $Z'_1 = Z_1$ and $Z'_U = Z_T$. This also means that dB_z is still given by (23).

It must be noted that this change in philosophy, namely to maintain only edge probabilities P_1 and P_T , will not reduce to the earlier results when we set $U = T$ here. The new thresholds in z , given by (24), are equally spaced even if $U = T$, whereas the former thresholds given by (21) are not.

Print out N , dB_z , μ_z , σ_z , U , $\{Z'_u\}$, $\{\underline{Z}'_u\}$, $\{P'_u\}$, where $\{\underline{Z}'_u\}$ are the normalized thresholds

$$\underline{Z}'_u = \frac{Z'_u - \mu_z}{\sigma_z} \quad \text{for } 1 \leq u \leq U. \quad (26)$$

SIGNAL-TO-NOISE RATIO REQUIREMENTS

Let us return to the original nonlinear transformation (5) characterized by parameter M . The CDF of RV y , with signal present in just one of the input RVs $\{x_m\}$, is denoted by $C_y(Y;R)$, where R is the input signal-to-noise ratio (SNR) in that particular x_m RV containing a signal. If we want this new CDF to realize a specified probability value S_1 at all the thresholds $\{Y_t\}$ in (11), the required SNRs $\{R_t\}$ must satisfy

$$S_1 = C_y(Y_t;R_t) \quad \text{for } 1 \leq t \leq T. \quad (27)$$

From (13) and (27), we know that

$$P_t = C_y(Y_t) = C_y(Y_t;0) \quad \text{for } 1 \leq t \leq T. \quad (28)$$

Therefore, specified probability S_1 in (27) must satisfy

$$S_1 \leq P_t \quad \text{for } 1 \leq t \leq T, \quad (29)$$

in order that nonnegative SNRs $\{R_t\}$ can result as solutions to (27). That is, each cumulative distribution value must be decreased from P_t to S_1 , meaning that each exceedance probability has been increased from false alarm probability value $1-P_t$ to detection probability value $Pd_1 = 1-S_1$. The actual determination of CDF $C_y(Y;R)$ will have to wait until after we have specified and analyzed the nonlinear transformations (5) of interest, for the case of noise-only present.

If several detection probabilities, like $Pd_1 = .5$, $Pd_2 = .9$, $Pd_3 = .99$, are specified, there will be a set of equations like (27) for each case, namely

$$1 - Pd_j = S_j = C_y(Y_t; R_t) \quad \text{for } 1 \leq t \leq T ; \quad j = 1, 2, 3 . \quad (30)$$

STATISTICS OF NORMALIZER

Let RV y be obtained by means of a normalization procedure from IID RVs $\{x_m\}$, $0 \leq m \leq M$, according to transformation

$$y = \frac{x_0}{\frac{1}{M}(x_1 + \dots + x_M)} \equiv \frac{x_0}{a}, \quad (31)$$

where we assume that $x_m \geq 0$ for all m . The average RV, a , in the denominator of (31) has, respectively, CF and PDF

$$f_a(\xi) = [f_x(\xi/M)]^M,$$

$$p_a(u) = \frac{1}{2\pi} \int d\xi \exp(-iu\xi) [f_x(\xi/M)]^M. \quad (32)$$

The CDF of RV y in (31) is, using the statistical independence of x_0 and a ,

$$\begin{aligned} C_y(Y) &= \text{Prob}(y < Y) = \text{Prob}(x_0 < Ya) = \\ &= \int_0^\infty dt p_a(t) \int_0^{Yt} du p_x(u) = \int_0^\infty dt p_a(t) C_x(Yt). \end{aligned} \quad (33)$$

The corresponding PDF of RV y is, upon differentiation,

$$p_y(u) = \int_0^\infty dt t p_a(t) p_x(ut). \quad (34)$$

The moments of RV y can be found from a couple of forms:

$$\begin{aligned}\mu_Y(j) &= \overline{y^j} = \int_0^{\infty} du u^j p_Y(u) = \\ &= \overline{x_0^j} \overline{a^{-j}} = \mu_X(j) \mu_A(-j) = \int du u^j p_X(u) \times \int dt t^{-j} p_A(t) .\end{aligned}\quad (35)$$

Convergence of the last integral in (35) may not occur for larger values of j ; that is, due to the division in (31), RV y may not have finite higher-order moments.

EXAMPLE

The example presented in (4) is utilized here. From (4) and (32), the CF and PDF of average RV a in (31) are

$$f_a(\xi) = \frac{1}{(1 - i\xi/M)^M} \quad (36)$$

and

$$p_a(u) = \frac{M^M u^{M-1} \exp(-Mu)}{(M-1)!} \quad \text{for } u > 0 , \quad (37)$$

respectively. The CDF of RV y then follows from (33) and (4) as

$$\begin{aligned}C_Y(Y) &= \int_0^{\infty} dt \frac{M^M t^{M-1} \exp(-Mt)}{(M-1)!} [1 - \exp(-Yt)] = \\ &= 1 - \left(1 + \frac{Y}{M}\right)^{-M} \quad \text{for } Y > 0 .\end{aligned}\quad (38)$$

The PDF of RV y is therefore

$$p_Y(u) = \left(1 + \frac{u}{M}\right)^{-M-1} \quad \text{for } u > 0. \quad (39)$$

The inverse function to CDF $C_Y(Y)$ in (38) is

$$\tilde{C}_Y(P) = M \left[(1 - P)^{-1/M} - 1 \right] \quad \text{for } 0 < P < 1. \quad (40)$$

The j -th moment of RV y is given by

$$\begin{aligned} \mu_Y(j) &= \overline{y^j} = \int du u^j p_Y(u) = \int_0^\infty du u^j \left(1 + \frac{u}{M}\right)^{-M-1} = \\ &= M^{j+1} B(j+1, M-j) = M^{j+1} \frac{\Gamma(j+1) \Gamma(M-j)}{\Gamma(M+1)} = \frac{j! M^j}{(M-j)_j} \quad \text{for } j < M, \quad (41) \end{aligned}$$

where we used (39) and [1; 3.194 3 and 8.384 1]. In particular,

$$\overline{y} = \frac{M}{M-1}, \quad \overline{y^2} = \frac{2M^2}{(M-2)(M-1)}, \quad \sigma_Y^2 = \frac{M^3}{(M-2)(M-1)^2} \quad \text{for } M > 2. \quad (42)$$

The condition $M > 2$ is necessary for RV y to possess a finite variance. We now have all the quantities required for application in the previous section on threshold resetting.

For $M = \infty$, $f_a(\xi)$ in (36) equals $\exp(i\xi)$; that is, RV a in (31) is equal to the constant 1. This corresponds to no normalization and, in fact, (31) reduces to $y = x_0$. Also, (38) - (41) reduces to (4), as expected.

CHANGE IN SIZE OF NORMALIZER

If we now change from size M to N in normalizer (31), we obtain RV z as considered in (20) and sequel. Its CDF follows, by similarity of form to (38), as

$$C_z(Z) = 1 - \left(1 + \frac{Z}{N}\right)^{-N} \quad \text{for } Z > 0. \quad (43)$$

Its inverse function is

$$\tilde{C}_z(P) = N \left[(1 - P)^{-1/N} - 1 \right] \quad \text{for } 0 < P < 1. \quad (44)$$

Reference to (42) also reveals that the mean and standard deviation of RV z are

$$\mu_z = \frac{N}{N-1}, \quad \sigma_z = \frac{N}{N-1} \left(\frac{N}{N-2} \right)^{1/2} \quad \text{for } N > 2. \quad (45)$$

The new thresholds are given by (21) and (22).

STATISTICS OF OR-ING DEVICE

Let $\{x_m\}$, $1 \leq m \leq M$, be IID RVs with common PDF $p_x(u)$ and CDF $C_x(X)$. The CDF and PDF of the maximum RV

$$y = \max\{x_1, x_2, \dots, x_M\} , \quad (46)$$

yielded by an or-ing device, are then

$$C_y(Y) = [C_x(Y)]^M , \quad (47)$$

$$p_y(u) = M [C_x(u)]^{M-1} p_x(u) , \quad (48)$$

respectively. The inverse function to CDF C_y in (47) is

$$\tilde{C}_y(P) = \tilde{C}_x(P^{1/M}) \quad \text{for } 0 < P < 1 . \quad (49)$$

Here, again, $\tilde{C}_x$ is the inverse function to CDF C_x of RV x_m .

The CF of RV y follows from (48) as

$$f_y(\xi) = \int du \exp(i\xi u) M [C_x(u)]^{M-1} p_x(u) . \quad (50)$$

The moments of RV y may be obtained from (48) in the form

$$\mu_y(j) = \int du u^j M [C_x(u)]^{M-1} p_x(u) . \quad (51)$$

Alternatively, the cumulants can be obtained by a power series expansion of the natural logarithm of CF $f_y(\xi)$ in (50).

EXAMPLE

We again use the results given in (4). Substitution into (50) yields

$$\begin{aligned}
 f_y(\xi) &= \int_0^{\infty} du \exp(i\xi u - u) M [1 - \exp(-u)]^{M-1} = \\
 &= M \int_0^1 dt t^{-i\xi} (1-t)^{M-1} = \frac{\Gamma(1-i\xi) \Gamma(M+1)}{\Gamma(M+1-i\xi)} = \\
 &= \left[\prod_{m=1}^M \left(1 - \frac{i\xi}{m} \right) \right]^{-1}, \tag{52}
 \end{aligned}$$

where we let $t = \exp(-u)$ and used [1; 8.380 1 and 8.384 1]. This is a very compact form and is easily computed numerically, if necessary. This result illustrates that RV y has the same statistics as RV w defined by

$$w \equiv \sum_{m=0}^M \frac{x_m}{m} = x_1 + \frac{1}{2}x_2 + \dots + \frac{1}{M}x_M. \tag{53}$$

In order to determine the cumulants of RV y , we consider

$$\ln f_y(\xi) = - \sum_{m=1}^M \ln \left(1 - \frac{i\xi}{m} \right) = \sum_{m=1}^M \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{i\xi}{m} \right)^k. \tag{54}$$

The cumulants follow immediately as

$$\chi_Y(j) = (j-1)! \sum_{m=1}^M \frac{1}{m^j} \quad \text{for } j \geq 1. \quad (55)$$

In particular, the mean and variance of RV y are

$$\mu_Y = \chi_Y(1) = \sum_{m=1}^M \frac{1}{m}, \quad \sigma_Y^2 = \chi_Y(2) = \sum_{m=1}^M \frac{1}{m^2}, \quad (56)$$

respectively. It is seen that the mean of RV y increases without limit, in fact, logarithmic with M . On the other hand, the variance saturates at $\pi^2/6$, meaning that the standard deviation σ_Y of RV y cannot exceed $\pi/6^{1/2} = 1.283$.

The CDF of y follows from (47) and (4) as

$$C_Y(Y) = [1 - \exp(-Y)]^M \quad \text{for } Y > 0, \quad (57)$$

with corresponding inverse function

$$\tilde{C}_Y(P) = -\ln(1 - P^{1/M}) \quad \text{for } 0 < P < 1. \quad (58)$$

For $M = 1$, there is no or-ing, and (46) reduces to $y = x_1$. Also, (52), (55), (57), and (58) reduce to (4), as expected.

CHANGE IN SIZE OF OR-ING DEVICE

If we change from size M to N in or-ing device (46), we obtain RV z considered in (20) and sequel. Its CDF is, by similarity in form to (57),

$$C_z(z) = [1 - \exp(-z)]^N \quad \text{for } z > 0. \quad (59)$$

The corresponding inverse function is

$$\tilde{C}_z(p) = -\ln(1 - p^{1/N}) \quad \text{for } 0 < p < 1. \quad (60)$$

Reference to (56) reveals that the mean and variance of RV z are given by

$$\mu_z = \sum_{n=1}^N \frac{1}{n}, \quad \sigma_z^2 = \sum_{n=1}^N \frac{1}{n^2}. \quad (61)$$

The new thresholds for RV z are given by (21) and (22).

STATISTICS OF NORMALIZER AND OR-ING DEVICE

Here, we consider a set of $K(M+1)$ IID RVs $\{w_{km}\}$ subject to both normalization and or-ing, according to

$$a_k = \frac{1}{M} \sum_{m=1}^M w_{km}, \quad x_k = \frac{w_{ko}}{a_k} \quad \text{for } 1 \leq k \leq K, \quad (62)$$

$$y = \max(x_1, x_2, \dots, x_K). \quad (63)$$

The statistics of the normalization portion, namely $\{x_k\}$ for $1 \leq k \leq K$, were previously considered in (31) - (35) for the general case, and then specialized to an example in (36) - (42). Also, the analysis of the or-ing portion was conducted in (46) - (51) and then specialized to an example in (52) - (58). We will rely heavily on those results in order to minimize the presentation in this section.

EXAMPLE

We presume that all the input RVs $\{w_{km}\}$ in (62) have the common exponential PDF used in example (4) for all the earlier results here. Then, by reference to (31) and (39), we can state that the common PDF of RVs $\{x_k\}$ defined in (62) is

$$p_x(u) = \left(1 + \frac{u}{M}\right)^{-M-1} \quad \text{for } u > 0. \quad (64)$$

Similarly, the CDF of RVs $\{x_k\}$ follows, by reference to (38), as

$$C_X(X) = 1 - \left(1 + \frac{X}{M}\right)^{-M} \quad \text{for } X > 0. \quad (65)$$

The CDF of RV y defined in (63) is then

$$C_Y(Y) = [C_X(Y)]^K = \left[1 - \left(1 + \frac{Y}{M}\right)^{-M}\right]^K \quad \text{for } Y > 0. \quad (66)$$

Its inverse function follows readily as

$$\tilde{C}_Y(P) = M \left[\left(1 - P^{1/K}\right)^{-1/M} - 1 \right] \quad \text{for } 0 < P < 1. \quad (67)$$

The PDF of RV y can be obtained by differentiation of (66):

$$\begin{aligned} p_Y(u) &= K [C_X(u)]^{K-1} p_X(u) = \\ &= K \left[1 - \left(1 + \frac{u}{M}\right)^{-M}\right]^{K-1} \left(1 + \frac{u}{M}\right)^{-M-1} \quad \text{for } u > 0. \end{aligned} \quad (68)$$

The j -th moment of RV y can be found from the integral

$$\mu_Y(j) = \int_0^{\infty} du u^j p_Y(u) \quad \text{for } j < M. \quad (69)$$

where we have taken advantage of the fact that RV y can never be negative. However, a useful alternative in some cases is afforded by employing integration by parts on (69), with the result, for $j \geq 1$, that

$$\mu_Y(j) = j \int_0^{\infty} du u^{j-1} [1 - C_Y(u)] , \quad (70)$$

where we assume that

$$\lim_{u \rightarrow +\infty} u^j [1 - C_Y(u)] = 0 . \quad (71)$$

This requirement is tantamount to presuming that the j -th moment $\mu_Y(j)$ exists, that is, $j < M$.

When we use CDF (66) for RV y , then j -th moment (70) becomes

$$\mu_Y(j) = j \int_0^{\infty} du u^{j-1} \left\{ 1 - \left[1 - \left(1 + \frac{u}{M} \right)^{-M} \right]^K \right\} . \quad (72)$$

In order to evaluate this integral, let, for the moment,

$$Q = \left(1 + \frac{u}{M} \right)^{-M} . \quad (73)$$

Then, the bracketed term in (72) can be expanded according to

$$[1 - Q]^K = \sum_{k=0}^K \binom{K}{k} (-Q)^k = 1 + \sum_{k=1}^K (-1)^k \binom{K}{k} \left(1 + \frac{u}{M} \right)^{-Mk} . \quad (74)$$

Employment of this result in (72) yields

$$\begin{aligned}
\mu_Y(j) &= j \int_0^{\infty} du u^{j-1} \sum_{k=1}^K (-1)^{k-1} \binom{K}{k} \left(1 + \frac{u}{M}\right)^{-Mk} = \\
&= j \sum_{k=1}^K (-1)^{k-1} \binom{K}{k} \int_0^{\infty} du u^{j-1} \left(1 + \frac{u}{M}\right)^{-Mk} = \\
&= j! M^j \sum_{k=1}^K \binom{K}{k} \frac{(-1)^{k-1}}{(Mk-j)_j} \quad \text{for } 1 \leq j < M, \quad (75)
\end{aligned}$$

where we employed [1; 3.194 3 and 8.384 1] and simplified the end result. For $K = 1$, (75) reduces to (41), as it must.

The first two moments for RV y follow readily from (75) as

$$\mu_Y(1) = M \sum_{k=1}^K \binom{K}{k} \frac{(-1)^{k-1}}{Mk-1} \quad (76)$$

and

$$\mu_Y(2) = 2M^2 \sum_{k=1}^K \binom{K}{k} \frac{(-1)^{k-1}}{(Mk-2)(Mk-1)} \quad \text{for } M > 2. \quad (77)$$

Both of these results are easily computed by means of recurrences; however, a bad feature of these two sums is that they are alternating series and lose significance for large K due to the binomial coefficient which gets very large. A backup procedure is to revert to numerical integration of (68) - (69) or (70) - (72), both of which integrands can never go negative and which decay as u^{j-1-M} for large u .

However, better alternatives to the first and second moments have been found, that are not subject to cancellation and loss of significance. Namely, it is shown in appendix A that

$$\mu_Y(1) = M[F_1 - 1] , \quad \sigma_Y = M(F_2 - F_1^2)^{1/2} , \quad (78)$$

where

$$F_1 = \prod_{k=1}^K \left\{ \frac{k}{k - \frac{1}{M}} \right\} , \quad F_2 = \prod_{k=1}^K \left\{ \frac{k}{k - \frac{2}{M}} \right\} \quad \text{for } M > 2 . \quad (79)$$

These finite products are very useful and retain significance even for large K, where (76) and (77) are useless. The very compact BASIC program listed below computes both quantities in (78). The program has been written so as to avoid overflow, even when K is large.

```

K=
M=          ! M > 2
A=1/M
B=2/M
F1=F2=1
FOR Ks=1 TO K
F1=F1*Ks/(Ks-A)
F2=F2*Ks/(Ks-B)
NEXT Ks
Muy=M*(F1-1)
Sigy=M*SQR(F2-F1*F1)

```

(80)

By using the techniques in appendix A and laboriously expanding out (75) for $j = 3$, it has been found that

$$\mu_Y(3) = M^3(F_3 - 3F_2 + 3F_1 - 1) ,$$

where we define products

$$F_m = \prod_{k=1}^K \left\{ \frac{k}{k - \frac{m}{M}} \right\} \quad \text{for } 0 \leq m < M .$$

Notice that $F_0 = 1$. Based on this result and (78), we conjectured that the j -th moment is generally given by

$$\mu_Y(j) = M^j \sum_{n=0}^j (-1)^n \binom{j}{n} F_{j-n} \quad \text{for } j < M .$$

In fact, this has been confirmed numerically for several values of M , K , and j . For large j , the alternating character of this series would also suffer from loss of significance; however, for the low order moments of general interest, this is not a problem.

The third and fourth cumulants of RV y were also computed in terms of sequence $\{F_n\}$; they turned out to be, for $M > 4$,

$$\chi_Y(3) = M^3 [F_3 - 3F_2F_1 + 2F_1^3] ,$$

$$\chi_Y(4) = M^4 [F_4 - 4F_3F_1 - 3F_2^2 + 12F_2F_1^2 - 6F_1^4] .$$

But these rules for obtaining cumulants $\{\chi_Y(j)\}$ from products $\{F_n\}$ are identical to the general rules for obtaining cumulants from moments, within the factor M^j ; see, for example, [4; page 70, (3.41)]. Thus, we have a very efficient and accurate method for obtaining the low-order cumulants directly from the finite products $\{F_n\}$. The case for $\chi_Y(1)$ in (78) differs slightly from the usual rule, in the need to subtract 1. (See appendix C for analytic corroboration of these conjectures.)

ANALYTICAL CHECKS

Numerous checks on the results above are possible. When $M = \infty$ (no normalization), the CDF of y in (66) reduces to (57), where it must be remembered that K in this section on combined normalization and or-ing corresponds to M in the section on or-ing alone. On the other hand, if $K = 1$ (no or-ing) in (66), the result in (38) correctly emerges.

With regard to the inverse CDF in (67), it reduces to (58) for $M = \infty$, whereas it reduces to (40) for $K = 1$. The PDF of RV y , given in (68), reduces to the derivative of (57) for $M = \infty$, whereas it reduces to (39) for $K = 1$. Finally, the first two moments in (78) - (79) reduce, after some manipulations, to (56) for $M = \infty$; on the other hand, the j -th moment for $K = 1$ is best handled from form (75) which correctly reduces to (41).

EXTENSIONS

The case where the normalizer and or-ing device are followed by an averager is discussed in the summary below, and the method of determining the performance of that system is outlined. Another alternative with practical merit is that of normalization followed by averaging and or-ing. Since the CF of the normalizer output is available by a Fourier transform of (39), it can be raised to a power to find the CF at the averager output. Then another Fourier transform can yield the CDF. At this point, we could utilize (47) to find the CDF of the system output.

CHANGES IN SIZES OF NORMALIZER AND OR-ING DEVICE

We now address the change in size of the normalizer in (62) from size M to N and the change in size of the or-ing device in (63) from K to L . There are no restrictions on the relative sizes of the parameters. The new equations are

$$b_k = \frac{1}{N} \sum_{n=1}^N w_{kn}, \quad y_k = \frac{w_{ko}}{b_k} \quad \text{for } 1 \leq k \leq L, \quad (81)$$

$$z = \max(y_1, y_2, \dots, y_L). \quad (82)$$

The CDF of RV z follows, by similarity of form to (66), as

$$C_z(z) = \left[1 - \left(1 + \frac{z}{N} \right)^{-N} \right]^L \quad \text{for } z > 0. \quad (83)$$

The inverse function is easily shown to be

$$\tilde{C}_z(p) = N \left[\left(1 - p^{1/L} \right)^{-1/N} - 1 \right] \quad \text{for } 0 < p < 1. \quad (84)$$

The first two moments of RV z in (82) are given, by comparison with (78) and (79), as

$$\mu_z(1) = N[G_1 - 1], \quad \sigma_z = N \left(G_2 - G_1^2 \right)^{1/2}, \quad (85)$$

where

$$G_1 = \prod_{k=1}^L \left\{ \frac{k}{k - \frac{1}{N}} \right\}, \quad G_2 = \prod_{k=1}^L \left\{ \frac{k}{k - \frac{2}{N}} \right\}. \quad (86)$$

INPUT SIGNAL-TO-NOISE RATIO REQUIREMENTS

In this section, we will determine the CDFs of the outputs of the three nonlinear systems considered above, namely (31), (46), and (62) - (63), for the case where a signal is present in one of the input RVs. This will enable us to use the results given in (27) - (29) for determination of required input SNRs for a specified system detection probability $Pd_1 = 1 - S_1$.

NORMALIZER

The nonlinear transformation of immediate interest is the normalizer given by (31). The PDF of denominator average RV a is given by (37), while the PDF of numerator RV x_0 with signal present will be modified from (4) to the form

$$p_x(u;R) = \frac{1}{1+R} \exp\left(\frac{-u}{1+R}\right) \quad \text{for } u > 0, \quad (87)$$

where R is the input power SNR. The corresponding CDF is

$$C_x(X;R) = 1 - \exp\left(\frac{-X}{1+R}\right) \quad \text{for } X > 0. \quad (88)$$

By reference to (33), (38), and (88), we find the CDF of RV y in (31) for signal present to be

$$\begin{aligned} C_y(Y;R) &= \int_0^{\infty} dt \frac{M^M t^{M-1} \exp(-Mt)}{(M-1)!} \left[1 - \exp\left(\frac{-Yt}{1+R}\right) \right] = \\ &= 1 - \left(1 + \frac{Y/M}{1+R} \right)^{-M} \quad \text{for } Y > 0. \end{aligned} \quad (89)$$

We now employ this result in (27) to obtain

$$S_1 = 1 - \left(1 + \frac{Y_t/M}{1+R_t} \right)^{-M} \quad \text{for } 1 \leq t \leq T. \quad (90)$$

The solution for the required input SNR R_t is then given by

$$R_t = \frac{Y_t/M}{(1 - S_1)^{-1/M} - 1} - 1 \quad \text{for } 1 \leq t \leq T. \quad (91)$$

We must repeat here the caution mentioned in (29), namely that specified probability S_1 at threshold Y_t must be less than or equal to probabilities $\{P_t\}$ in order that nonnegative SNRs $\{R_t\}$ result in (91). This is reasonable since it corresponds physically to demanding a larger detection probability when signal is present than when absent.

OR-ING DEVICE

Here, we are interested in the or-ing device characterized by (46) when signal is present in one of the RVs $\{x_m\}$. The CDF of RV y is then given by

$$\begin{aligned} C_Y(Y;R) &= C_X(Y;R) [C_X(Y)]^{M-1} = \\ &= \left[1 - \exp\left(\frac{-Y}{1+R}\right) \right] [1 - \exp(-Y)]^{M-1} \quad \text{for } Y > 0, \end{aligned} \quad (92)$$

where we used (88) and (4).

If we now employ (92) in (27), we have to satisfy

$$S_1 = \left[1 - \exp\left(\frac{-Y_t}{1+R_t}\right) \right] [1 - \exp(-Y_t)]^{M-1} \quad \text{for } 1 \leq t \leq T. \quad (93)$$

The solution for the required input SNR is given by

$$R_t = \frac{Y_t}{-\ln(1 - S_1/Q_t)} - 1 \quad \text{for } 1 \leq t \leq T, \quad (94)$$

where

$$Q_t = [1 - \exp(-Y_t)]^{M-1} \quad \text{for } 1 \leq t \leq T. \quad (95)$$

Again, (29) must be satisfied.

NORMALIZER AND OR-ING DEVICE

The nonlinear transformation to be investigated here is the combination of normalization and or-ing, as characterized by (62) and (63), when signal is present only in RV w_{10} . Therefore, only RV x_1 in (63) contains signal.

The CDFs of RVs $\{x_k\}$, for $2 \leq k \leq K$, are given by (65). On the other hand, the CDF for x_1 is available by reference to (89), in the form

$$C_x(X;R) = 1 - \left(1 + \frac{X/M}{1+R}\right)^{-M} \quad \text{for } X > 0. \quad (96)$$

Therefore, the CDF for RV y in (63) is given by

$$\begin{aligned} C_y(Y;R) &= C_x(Y;R) [C_x(Y)]^{K-1} = \\ &= \left[1 - \left(1 + \frac{Y/M}{1+R}\right)^{-M}\right] \left[1 - \left(1 + \frac{Y}{M}\right)^{-M}\right]^{K-1} \quad \text{for } Y > 0, \end{aligned} \quad (97)$$

where we used (96) and (65).

When we equate this result to the specified probability S_1 according to (27), we obtain

$$S_1 = \left[1 - \left(1 + \frac{Y_t/M}{1+R_t}\right)^{-M}\right] \left[1 - \left(1 + \frac{Y_t}{M}\right)^{-M}\right]^{K-1} \quad \text{for } 1 \leq t \leq T. \quad (98)$$

The solution for the required input SNRs $[R_t]$ is given by

$$R_t = \frac{Y_t/M}{(1 - S_1/Q_t)^{-1/M} - 1} - 1 \quad \text{for } 1 \leq t \leq T, \quad (99)$$

where

$$Q_t = \left[1 - \left(1 + \frac{Y_t}{M} \right)^{-M} \right]^{K-1} \quad \text{for } 1 \leq t \leq T. \quad (100)$$

Restriction (29) must be satisfied here also.

Finally, if several detection probabilities $Pd_1, Pd_2, Pd_3,$ are of interest, we must satisfy (30), where

$$\max\{1-P_t\} \leq \min\{Pd_j\}. \quad (101)$$

As checks on the results in this subsection on combined normalization and or-ing, we observe for $M = \infty$ (that is, no normalization), (100) reduces to (95), where K here corresponds to M there for or-ing alone. Also, (99) reduces to (94). On the other hand, for finite M , but with $K = 1$ (that is, no or-ing), then (100) reduces to $Q_t = 1$, in which case (99) reduces to (91).

SUMMARY

We have determined the false alarm and detection probabilities for three different nonlinear transformations, namely a normalizer, an or-ing device, and a normalizer followed by or-ing. In particular, results are given for the following statistics of the outputs of each device: the PDF, the CDF, the inverse CDF, and either the moments or the cumulants (depending on their relative tractability). These results are sufficient to compute the receiver operating characteristics (ROCs) of the processors, if desired. However, we have also been able to solve explicitly for the input SNR required to realize specified false alarm and detection probabilities; this largely circumvents the need to compute ROCs.

Two programs, with numerical examples of their execution, are listed in appendix B. The first corresponds to the case where the number T of thresholds is kept fixed as the size of the nonlinear transformation is changed from M to N ; see (20) - (23). On the other hand, the second program allows the number of thresholds to change from T to U as the size of the nonlinear transformation is changed from M to N ; see (24) - (26).

Both programs are written for the general case where there is both normalization (of size M) and or-ing (of size K) included in the data processing; see (62) - (63). By making M infinite, the program will handle the case of or-ing alone; on the other hand, by setting $K = 1$, the program addresses the case of normalization alone. Thus, these two programs cover all the cases addressed in

this investigation. (Since it is not possible to actually set M infinite in a program, this situation is handled by setting M to any value less than or equal to 2, in order to flag this case in the program, and then branching appropriately at various points in the program. The substitute equations for this case of infinite M come, of course, from the earlier analysis for or-ing alone. For finite variance, normalization requires $M > 2$; see, for example, (77) or (79).)

We have not included the effects of averaging after the normalization and/or or-ing in this study. Hence, the required input SNRs calculated here sometimes turn out to be rather large. The exact analysis including averaging would be rather involved, since the new decision variable would have a characteristic function given by a power of the characteristic function $f_y(\xi)$ of current output variable y ; that is, from (68),

$$f_y(\xi) = K \int_0^{\infty} du \exp(i\xi u) \left[1 - \left(1 + \frac{u}{M} \right)^{-M} \right]^{K-1} \left(1 + \frac{u}{M} \right)^{-M-1}. \quad (102)$$

This is probably best handled through the use of fast Fourier transforms. The integrand decays as u^{-M-1} for large u , which is attractive since M , the normalization size, is generally fairly large in order to guarantee decent performance.

In the meantime, in order to get a rather rough idea of the improvement attainable by employment of averaging, it is suggested that the rule of thumb [3; (C-10)] for the input signal-to-noise ratio improvement in dB, $-5 \log A$, be used, where A is the number of independent quantities averaged.

APPENDIX A. SIMPLIFICATION OF SUMS (76) AND (77)

Here, we will convert the alternating series in (76) and (77) into finite products that retain significance, even for large values of K . We begin with the first line of (52) and expand the power term in a binomial series, obtaining

$$\begin{aligned} f_y(\xi) &= M \int_0^{\infty} du \exp(i\xi u - u) \sum_{k=0}^{M-1} (-1)^k \binom{M-1}{k} \exp(-ku) = \\ &= M \sum_{k=0}^{M-1} \binom{M-1}{k} \frac{(-1)^k}{k + 1 - i\xi} . \end{aligned} \quad (A-1)$$

Now, equate (A-1) to its alternative expression in the last line of (52), and then replace M everywhere by $K+1$, getting

$$(K+1) \sum_{k=0}^K \binom{K}{k} \frac{(-1)^k}{k + 1 - i\xi} = \prod_{k=1}^{K+1} \left(\frac{k}{k - i\xi} \right) . \quad (A-2)$$

Now let $z = 1 - i\xi$ in (A-2) and simplify; there follows

$$\sum_{k=0}^K \binom{K}{k} \frac{(-1)^k}{k + z} = \frac{K!}{(z)_{K+1}} . \quad (A-3)$$

Thus, the alternating series has been converted into a finite product which is useful for all values of K without loss of significance.

The use of (A-3) on (76) yields the result quoted in (78) and (79) for the first moment $\mu_y(1)$. On the other hand, in order to convert (77), it is necessary to take the preliminary step of breaking the rational function into two parts according to

$$\frac{1}{(k-b)(k-a)} = \frac{1}{b-a} \left[\frac{1}{k-b} - \frac{1}{k-a} \right], \quad (\text{A-4})$$

and then to use (A-3) once with $z = -1/M$ and once with $z = -2/M$. After manipulation, simplification, and cancellation of common terms with the square of $\mu_y(1)$, the end result for the standard deviation of RV y is found to be just the second term in (78). The results in (78) - (79) have been numerically checked with the original defining integral (72), for $j = 1$ and $j = 2$, for several values of M and K ; they agree within the accuracy of the computer employed.

A more general result than (A-3) is available by means of a different approach. First, for general values of a , note that

$$\binom{a}{k} = \frac{(-1)^k (-a)_k}{k!}, \quad \frac{1}{k+z} = \frac{1}{z} \frac{(z)_k}{(z+1)_k}. \quad (\text{A-5})$$

Then, the following alternating sum can be manipulated into a more useful form, namely

$$\sum_{k=0}^{\infty} \binom{a}{k} \frac{(-1)^k}{k+z} = \frac{1}{z} \sum_{k=0}^{\infty} \frac{(-a)_k (z)_k}{(z+1)_k k!} = \frac{1}{z} F(-a, z; z+1; 1) = \frac{\Gamma(1+a) \Gamma(z)}{\Gamma(1+a+z)}, \quad (\text{A-6})$$

where we used [2; 15.1.1 and 15.1.20]. If we set $a = K$ in (A-6), it reduces to (A-3). (A-6) is also equal to $B(1+a, z)$.

APPENDIX B. PROGRAMS

There are two programs listed in this appendix, both in BASIC for the Hewlett-Packard 9000 computer. None of the variables are declared INTEGER; thus, for example, input parameters M, K, N, L are all treated as REAL variables throughout.

The first program, listed on pages 42 - 44, requires that the number of thresholds T be maintained the same when the sizes, M and K, of the normalizer and or-ing device, respectively, are changed to N and L. On the other hand, the second program, listed on pages 46 - 49, allows the number of thresholds to change from T to U, which can be either larger or smaller.

The listings are heavily keyed to the explicit equations in the main text; this should enable the user to identify and modify particular manipulations if desired. It should be noted that, in the programs, the or-ing size begins at value K and gets changed to L. If these results are to be compared with the or-ing only results in the text, where the parameter M was used, it is necessary to make the switch from K in the program to M in the text. Example outputs from both programs are presented after the listing for each case.

```

10 ! NORMALIZER & ORING, EQUISPACED IN POWER, SAME NO. OF THRESHOLDS
20 P1=.85 ! SPECIFIED PROBABILITY, (9)
30 Dby=10 ! DECIBEL RATIO OF THRESHOLDS, (10)
40 T=7 ! NUMBER OF THRESHOLDS, (10)
50 M=8 ! INITIAL NORMALIZER SIZE > 2, (62)
60 ! FOR NO INITIAL NORMALIZATION, THAT IS, M INFINITE, SET M <= 2
70 K=12 ! INITIAL OR-ING SIZE > 0, (63)
80 N=16 ! NEW NORMALIZER SIZE > 2, (81)
90 ! FOR NO NEW NORMALIZATION, THAT IS, N INFINITE, SET N <= 2
100 L=24 ! NEW OR-ING SIZE > 0, (82)
110 Pd1=.5 ! SPECIFIED
120 Pd2=.9 ! DETECTION
130 Pd3=.99 ! PROBABILITIES, (30)
140 PRINT "P1 =";P1;" Dby =";Dby;" T =";T
150 PRINT "M =";M;" K =";K;" N =";N;" L =";L
160 REDIM Y(1:T),Yb(1:T),P(1:T),Z(1:T),Zb(1:T)
170 DIM Y(99),Yb(99),P(99),Z(99),Zb(99)
180 Rk=1/K ! (67)
190 Rl=1/L ! (84)
200 IF M>2 THEN 300
210 F1=F2=0 ! M INFINITE
220 FOR Ms=1 TO K ! (56), m
230 R1=1/Ms
240 F1=F1+R1
250 F2=F2+R1*R1
260 NEXT Ms
270 Muy=F1 ! (56)
280 Sigy=SQR(F2) ! (56)
290 GOTO 390
300 R1=1/M ! M > 2
310 R2=2/M
320 F1=F2=1
330 FOR Ks=1 TO K ! (79), k
340 F1=F1*Ks/(Ks-R1) ! (79)
350 F2=F2*Ks/(Ks-R2) ! (79)
360 NEXT Ks
370 Muy=M*(F1-1) ! (78)
380 Sigy=M*SQR(F2-F1*F1) ! (78)
390 IF N>2 THEN 490
400 G1=G2=0 ! N INFINITE
410 FOR Ns=1 TO L ! (61), n
420 R1=1/Ns
430 G1=G1+R1
440 G2=G2+R1*R1
450 NEXT Ns
460 Muz=G1 ! (61)
470 Sigz=SQR(G2) ! (61)
480 GOTO 580
490 R1=1/N ! N > 2
500 R2=2/N
510 G1=G2=1
520 FOR Ks=1 TO L ! (86), k
530 G1=G1*Ks/(Ks-R1) ! (86)
540 G2=G2*Ks/(Ks-R2) ! (86)
550 NEXT Ks
560 Muz=N*(G1-1) ! (85)
570 Sigz=N*SQR(G2-G1*G1) ! (85)
580 PRINT "Muy =";Muy;" Sigy =";Sigy
590 PRINT "Muz =";Muz;" Sigz =";Sigz
600 PRINT

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610 | EQUISPACED IN DECIBELS: REMOVE 880-970 AND INSERT 620-710
620 | R=1-P1^Rk | (9) & (67)
630 | IF M>2 THEN 660
640 | Y1=-LOG(R) | (58)
650 | GOTO 670
660 | Y1=M*(R^(-1/M)-1) | (67)
670 | D1=10*LGT(Y1) | (14)
680 | DelD=Dby/(T-1) | (15) & (16)
690 | FOR Ts=1 TO T | (16), t
700 | Dt=D1+DelD*(Ts-1) | (16)
710 | Y(Ts)=Y=10^(Dt/10) | (17)
720 | PRESET CONSTANTS: REMOVE 880-980 AND INSERT 730-760
730 | DATA 1,3,5,7,9,10,11 | USER MUST INPUT T NUMBERS
740 | READ Yb(*) | NORMALIZED Y THRESHOLDS
750 | FOR Ts=1 TO T | (18), t
760 | Y(Ts)=Y=Muy+Sigy*Yb(Ts) | (18)
770 | PRESET PROBABILITIES: REMOVE 880-1030 AND INSERT 780-870
780 | DATA .9,.99,.999,.9999,.99999,.999999,.9999999
790 | READ P(*) | (19)
800 | FOR Ts=1 TO T | (19), t
810 | P=P(Ts)
820 | R=1-P^Rk | (58)
830 | IF M>2 THEN 860
840 | Y(Ts)=Y=-LOG(R) | (58)
850 | GOTO 870
860 | Y(Ts)=Y=M*(R^(-1/M)-1) | (9) & (67)
870 | Yb(Ts)=(Y-Muy)/Sigy | (12)
880 | R=1-P1^Rk | (9) & (67)
890 | IF M>2 THEN 920
900 | Y1=-LOG(R) | (58)
910 | GOTO 930
920 | Y1=M*(R^(-1/M)-1) | (67)
930 | Ay=10^(Dby/10) | (10)
940 | Ytc=Y1*Ay | (10)
950 | DelY=(Ytc-Y1)/(T-1) | (11)
960 | FOR Ts=1 TO T | (11), t
970 | Y(Ts)=Y=Y1+DelY*(Ts-1) | (11)
980 | Yb(Ts)=(Y-Muy)/Sigy | (12)
990 | IF M>2 THEN 1020
1000 | R=EXP(-Y) | (57)
1010 | GOTO 1030
1020 | R=(1+Y/M)^(-M) | (13) & (66)
1030 | P(Ts)=P=(1-R)^K | (66)
1040 | Q=1-P^Rl | (84)
1050 | IF N>2 THEN 1080
1060 | Z(Ts)=Z=-LOG(Q) | (21) & (60)
1070 | GOTO 1090
1080 | Z(Ts)=Z=N*(Q^(-1/N)-1) | (21) & (84)
1090 | Zb(Ts)=(Z-Muz)/Sigz | (22)
1100 | NEXT Ts
1110 | Dbz=10*LGT(Z(T)/Z(1)) | (23)
1120 | PRINT " Y THRESHOLDS NORMALIZED PROBABILITIES"
1130 | FOR Ts=1 TO T
1140 | PRINT Y(Ts),Yb(Ts),P(Ts)
1150 | NEXT Ts
1160 | PRINT
1170 | PRINT " Z THRESHOLDS NORMALIZED PROBABILITIES"
1180 | FOR Ts=1 TO T
1190 | PRINT Z(Ts),Zb(Ts),P(Ts)
1200 | NEXT Ts
1210 | PRINT
1220 | PRINT "Dbz =" ; Dbz

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1230 PRINT
1240 PRINT " Pd1 =";Pd1;"          Pd2 =";Pd2;"          Pd3 =";Pd3
1250 PRINT
1260 PRINT "SIGNAL-TO-NOISE RATIOS (DB) FOR INITIAL TRANSFORMATION:"
1270 IF M>2 THEN 1400
1280 FOR Ts=1 TO T          | (94), t
1290 Y=Y(Ts)
1300 Q=(1-EXP(-Y))^(K-1)    | (95)
1310 D1=-LOG(1-(1-Pd1)/Q)   | (94)
1320 Rt1=Y/D1-1            | (94)
1330 D2=-LOG(1-(1-Pd2)/Q)
1340 Rt2=Y/D2-1
1350 D3=-LOG(1-(1-Pd3)/Q)
1360 Rt3=Y/D3-1
1370 PRINT 10*LGT(Rt1),10*LGT(Rt2),10*LGT(Rt3)
1380 NEXT Ts
1390 GOTO 1530
1400 Rm=-1/M                | (99)
1410 FOR Ts=1 TO T          | (99), t
1420 Ym=Y(Ts)/M             | (99) & (100)
1430 Q=(1+Ym)^(-M)          | (100)
1440 Q=(1-Q)^(K-1)          | (100)
1450 D1=(1-(1-Pd1)/Q)^Rm-1  | (99)
1460 Rt1=Ym/D1-1            | (99)
1470 D2=(1-(1-Pd2)/Q)^Rm-1
1480 Rt2=Ym/D2-1
1490 D3=(1-(1-Pd3)/Q)^Rm-1
1500 Rt3=Ym/D3-1
1510 PRINT 10*LGT(Rt1),10*LGT(Rt2),10*LGT(Rt3)
1520 NEXT Ts
1530 PRINT
1540 PRINT "SIGNAL-TO-NOISE RATIOS (DB) FOR NEW TRANSFORMATION:"
1550 IF N>2 THEN 1680
1560 FOR Ts=1 TO T          | SIMILAR
1570 Z=Z(Ts)                 | TO
1580 Q=(1-EXP(-Z))^(L-1)     | (94)
1590 D1=-LOG(1-(1-Pd1)/Q)    | &
1600 Rt1=Z/D1-1              | (95)
1610 D2=-LOG(1-(1-Pd2)/Q)
1620 Rt2=Z/D2-1
1630 D3=-LOG(1-(1-Pd3)/Q)
1640 Rt3=Z/D3-1
1650 PRINT 10*LGT(Rt1),10*LGT(Rt2),10*LGT(Rt3)
1660 NEXT Ts
1670 GOTO 1810
1680 Rn=-1/N                | SIMILAR
1690 FOR Ts=1 TO T          | TO
1700 Zn=Z(Ts)/N             | (99)
1710 Q=(1+Zn)^(-N)          | &
1720 Q=(1-Q)^(L-1)          | (100)
1730 D1=(1-(1-Pd1)/Q)^Rn-1
1740 Rt1=Zn/D1-1
1750 D2=(1-(1-Pd2)/Q)^Rn-1
1760 Rt2=Zn/D2-1
1770 D3=(1-(1-Pd3)/Q)^Rn-1
1780 Rt3=Zn/D3-1
1790 PRINT 10*LGT(Rt1),10*LGT(Rt2),10*LGT(Rt3)
1800 NEXT Ts
1810 PRINT
1820 END

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P1 = .85 Dby = 10 T = 7
 M = 8 K = 12 N = 16 L = 24
 Muy = 3.94631506068 Sigy = 2.09538979024
 Muz = 4.32438211761 Sigz = 1.69328544323

Y THRESHOLDS	NORMALIZED	PROBABILITIES
5.7085601983	.841010653879	.85
14.2714004957	4.92752493268	.996679075641
22.8342407932	9.01403921147	.999753631023
31.3970810906	13.1005534903	.999965311792
39.9599213881	17.1870677691	.99999280763
48.5227616855	21.2735820479	.999998067526
57.085601983	25.3600963267	.999999374796

Z THRESHOLDS	NORMALIZED	PROBABILITIES
5.86721821227	.911149446675	.85
11.8779764405	4.46091021044	.996679075641
16.8024176865	7.36912705339	.999753631023
21.0784326281	9.89440414638	.999965311792
24.909916314	12.1571553566	.99999280763
28.4120734509	14.2254168838	.999998067526
31.6575625522	16.1420985125	.999999374796

Dbz = 7.32045232969

Pd1 = .5 Pd2 = .9 Pd3 = .99

SIGNAL-TO-NOISE RATIOS (DB) FOR INITIAL TRANSFORMATION:

7.18029480646	16.5239686654	26.8808804974
12.6997498763	21.2426978346	31.503743108
14.8466399857	23.3091924052	33.5584316626
16.269595645	24.6986419313	34.9428284925
17.3390310658	25.7492651765	35.9905902914
18.1963923488	26.5945353622	36.8340131363
18.9120679996	27.3017783991	37.5399641554

SIGNAL-TO-NOISE RATIOS (DB) FOR NEW TRANSFORMATION:

7.40601673703	16.6306689407	26.9721139815
11.9559811212	20.4527339701	30.7066466017
13.5624054665	21.9841310487	32.2269170012
14.587035072	22.975309045	33.2129962117
15.3357177535	23.704135924	33.9387696157
15.922698397	24.2777465869	34.5103143719
16.4038217386	24.749160188	34.9802228898

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10 ! NORMALIZER & ORING, EQUISPACED IN POWER, DIFF. NO. OF THRESHOLDS
20 P1=.85 ! SPECIFIED PROBABILITY, (9)
30 Dby=10 ! DECIBEL RATIO OF THRESHOLDS, (10)
40 T=7 ! INITIAL NUMBER OF THRESHOLDS, (10)
50 U=15 ! NEW NUMBER OF THRESHOLDS, (24)
60 M=8 ! INITIAL NORMALIZER SIZE > 2, (62)
70 ! FOR NO INITIAL NORMALIZATION, THAT IS, M INFINITE, SET M <= 2
80 K=12 ! INITIAL OR-ING SIZE > 0, (63)
90 N=16 ! NEW NORMALIZER SIZE > 2, (81)
100 ! FOR NO NEW NORMALIZATION, THAT IS, N INFINITE, SET N <= 2
110 L=24 ! NEW OR-ING SIZE > 0, (82)
120 Pd1=.5 ! SPECIFIED
130 Pd2=.9 ! DETECTION
140 Pd3=.99 ! PROBABILITIES, (30)
150 PRINT "P1 =";P1;" Dby =";Dby;" T =";T;" U =";U
160 PRINT "M =";M;" K =";K;" N =";N;" L =";L
170 REDIM Y(1:T),Yb(1:T),P(1:T),Zp(1:U),Zbp(1:U),Pp(1:U)
180 DIM Y(99),Yb(99),P(99),Zp(99),Zbp(99),Pp(99)
190 Rk=1/K ! (67)
200 R1=1/L ! (84)
210 IF M>2 THEN 310
220 F1=F2=0 ! M INFINITE
230 FOR Ms=1 TO K ! (56), m
240 R1=1/Ms
250 F1=F1+R1
260 F2=F2+R1*R1
270 NEXT Ms
280 Muy=F1 ! (56)
290 Sigy=SQR(F2) ! (56)
300 GOTO 400
310 R1=1/M ! M > 2
320 R2=2/M
330 F1=F2=1
340 FOR Ks=1 TO K ! (79), k
350 F1=F1*Ks/(Ks-R1) ! (79)
360 F2=F2*Ks/(Ks-R2) ! (79)
370 NEXT Ks
380 Muy=M*(F1-1) ! (78)
390 Sigy=M*SQR(F2-F1*F1) ! (78)
400 IF N>2 THEN 500
410 G1=G2=0 ! N INFINITE
420 FOR Ns=1 TO L ! (61), n
430 R1=1/Ns
440 G1=G1+R1
450 G2=G2+R1*R1
460 NEXT Ns
470 Muz=G1 ! (61)
480 Sigz=SQR(G2) ! (61)
490 GOTO 590
500 R1=1/N ! N > 2
510 R2=2/N
520 G1=G2=1
530 FOR Ks=1 TO L ! (86), k
540 G1=G1*Ks/(Ks-R1) ! (86)
550 G2=G2*Ks/(Ks-R2) ! (86)
560 NEXT Ks
570 Muz=N*(G1-1) ! (85)
580 Sigz=N*SQR(G2-G1*G1) ! (85)
590 PRINT "Muy =";Muy;" Sigy =";Sigy
600 PRINT "Muz =";Muz;" Sigz =";Sigz
610 PRINT

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620 ! EQUISPACED IN DECIBELS: REMOVE 890-980 AND INSERT 630-720
630 !  $R = 1 - P1^{rk}$  ! (9) & (67)
640 ! IF M>2 THEN 670
650 !  $Y1 = -\text{LOG}(R)$  ! (58)
660 ! GOTO 680
670 !  $Y1 = M * (R^{(-1/M)} - 1)$  ! (67)
680 !  $D1 = 10 * \text{LGT}(Y1)$  ! (14)
690 !  $De1d = Dby / (T - 1)$  ! (15) & (16)
700 ! FOR Ts=1 TO T ! (16), t
710 !  $Dt = D1 + De1d * (Ts - 1)$  ! (16)
720 !  $Y(Ts) = Y = 10^{(Dt/10)}$  ! (17)
730 ! PRESET CONSTANTS: REMOVE 890-990 AND INSERT 740-770
740 ! DATA 1,3,5,7,9,10,11 ! USER MUST INPUT T NUMBERS
750 ! READ Yb(*) ! NORMALIZED Y THRESHOLDS
760 ! FOR Ts=1 TO T ! (18), t
770 !  $Y(Ts) = Y = \text{Muy} + \text{Sigy} * Yb(Ts)$  ! (18)
780 ! PRESET PROBABILITIES: REMOVE 890-1040 AND INSERT 790-880
790 ! DATA .9,.99,.999,.9999,.99999,.999999,.9999999
800 ! READ P(*) ! (19)
810 ! FOR Ts=1 TO T ! (19), t
820 !  $P = P(Ts)$ 
830 !  $R = 1 - P^{rk}$  ! (58)
840 ! IF M>2 THEN 870
850 !  $Y(Ts) = Y = -\text{LOG}(R)$  ! (58)
860 ! GOTO 880
870 !  $Y(Ts) = Y = M * (R^{(-1/M)} - 1)$  ! (9) & (67)
880 !  $Yb(Ts) = (Y - \text{Muy}) / \text{Sigy}$  ! (12)
890 !  $R = 1 - P1^{rk}$  ! (9) & (67)
900 ! IF M>2 THEN 930
910 !  $Y1 = -\text{LOG}(R)$  ! (58)
920 ! GOTO 940
930 !  $Y1 = M * (R^{(-1/M)} - 1)$  ! (67)
940 !  $Ay = 10^{(Dby/10)}$  ! (10)
950 !  $Ytc = Y1 * Ay$  ! (10)
960 !  $Dely = (Ytc - Y1) / (T - 1)$  ! (11)
970 ! FOR Ts=1 TO T ! (11), t
980 !  $Y(Ts) = Y = Y1 + Dely * (Ts - 1)$  ! (11)
990 !  $Yb(Ts) = (Y - \text{Muy}) / \text{Sigy}$  ! (12)
1000 ! IF M>2 THEN 1030
1010 !  $R = \text{EXP}(-Y)$  ! (57)
1020 ! GOTO 1040
1030 !  $R = (1 + Y/M)^{(-M)}$  ! (13) & (66)
1040 !  $P(Ts) = (1 - R)^K$  ! (66)
1050 ! NEXT Ts

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```

1060 Q1=1-P(1)^R1      ! (84)
1070 Qtc=1-P(T)^R1     ! (84)
1080 IF N>2 THEN 1120
1090 Z1=-LOG(Q1)        ! (60)
1100 Ztc=-LOG(Qtc)      ! (60)
1110 GOTO 1150
1120 Rn=-1/N            ! (84)
1130 Z1=N*(Q1^Rn-1)     ! (21) & (84)
1140 Ztc=N*(Qtc^Rn-1)   ! (21) & (84)
1150 Delz=(Ztc-Z1)/(U-1) ! (24)
1160 FOR Us=1 TO U      ! (24), u
1170 Zp(Us)=Z=Z1+Delz*(Us-1) ! (24)
1180 Zbp(Us)=(Z-Muz)/Sigz ! (26)
1190 IF N>2 THEN 1220
1200 Pp(Us)=(1-EXP(-Z))^L ! (25) & (59)
1210 GOTO 1230
1220 Pp(Us)=(1-(1+Z/N)^(-N))^L ! (25) & (83)
1230 NEXT Us
1240 Dbz=10*LGT(Zp(U)/Zp(1)) ! (23)
1250 PRINT " Y THRESHOLDS          NORMALIZED          PROBABILITIES"
1260 FOR Ts=1 TO T
1270 PRINT Y(Ts),Yb(Ts),P(Ts)
1280 NEXT Ts
1290 PRINT
1300 PRINT " Z THRESHOLDS          NORMALIZED          PROBABILITIES"
1310 FOR Us=1 TO U
1320 PRINT Zp(Us),Zbp(Us),Pp(Us)
1330 NEXT Us
1340 PRINT
1350 PRINT "Dbz =";Dbz
1360 PRINT
1361 PAUSE
1370 PRINT Pd1 =";Pd1;" Pd2 =";Pd2;" Pd3 =";Pd3
1380 PRINT
1390 PRINT "SIGNAL-TO-NOISE RATIOS (DB) FOR INITIAL TRANSFORMATION:"
1400 IF M>2 THEN 1530
1410 FOR Ts=1 TO T      ! (94), t
1420 Y=Y(Ts)
1430 Q=(1-EXP(-Y))^(K-1) ! (95)
1440 D1=-LOG(1-(1-Pd1)/Q) ! (94)
1450 Rt1=Y/D1-1          ! (94)
1460 D2=-LOG(1-(1-Pd2)/Q)
1470 Rt2=Y/D2-1
1480 D3=-LOG(1-(1-Pd3)/Q)
1490 Rt3=Y/D3-1
1500 PRINT 10*LGT(Rt1),10*LGT(Rt2),10*LGT(Rt3)
1510 NEXT Ts
1520 GOTO 1660

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1530 Rm=-1/M                                ! (99)
1540 FOR Ts=1 TO T                          ! (99), t
1550 Ym=Y(Ts)/M                            ! (99) & (100)
1560 Q=(1+Ym)^(-M)                        ! (100)
1570 Q=(1-Q)^(K-1)                        ! (100)
1580 D1=(1-(1-Pd1)/Q)^Rm-1                ! (99)
1590 Rt1=Ym/D1-1                          ! (99)
1600 D2=(1-(1-Pd2)/Q)^Rm-1
1610 Rt2=Ym/D2-1
1620 D3=(1-(1-Pd3)/Q)^Rm-1
1630 Rt3=Ym/D3-1
1640 PRINT 10*LGT(Rt1),10*LGT(Rt2),10*LGT(Rt3)
1650 NEXT Ts
1660 PRINT
1670 PRINT "SIGNAL-TO-NOISE RATIOS (DB) FOR NEW TRANSFORMATION:"
1680 IF N>2 THEN 1810
1690 FOR Us=1 TO U                          ! SIMILAR
1700 Z=Zp(Us)                              ! TO
1710 Q=(1-EXP(-Z))^(L-1)                  ! (94)
1720 D1=-LOG(1-(1-Pd1)/Q)                 ! &
1730 R1=Z/D1-1                            ! (95)
1740 D2=-LOG(1-(1-Pd2)/Q)
1750 R2=Z/D2-1
1760 D3=-LOG(1-(1-Pd3)/Q)
1770 R3=Z/D3-1
1780 PRINT 10*LGT(R1),10*LGT(R2),10*LGT(R3)
1790 NEXT Us
1800 GOTO 1940
1810 Rn=-1/N                                ! SIMILAR
1820 FOR Us=1 TO U                          ! TO
1830 Zn=Zp(Us)/N                          ! (99)
1840 Q=(1+Zn)^(-N)                        ! &
1850 Q=(1-Q)^(L-1)                        ! (100)
1860 D1=(1-(1-Pd1)/Q)^Rn-1
1870 R1=Zn/D1-1
1880 D2=(1-(1-Pd2)/Q)^Rn-1
1890 R2=Zn/D2-1
1900 D3=(1-(1-Pd3)/Q)^Rn-1
1910 R3=Zn/D3-1
1920 PRINT 10*LGT(R1),10*LGT(R2),10*LGT(R3)
1930 NEXT Us
1940 PRINT
1950 END

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P1 = .85 Dby = 10 T = 7 U = 15
 M = 8 K = 12 N = 16 L = 24
 Muy = 3.94631506068 Sigy = 2.09538979024
 Muz = 4.32438211761 Sigz = 1.69328544323

Y THRESHOLDS	NORMALIZED	PROBABILITIES
5.7085601983	.841010653879	.85
14.2714004957	4.92752493268	.996679075641
22.8342407932	9.01403921147	.999753631023
31.3970810906	13.1005534903	.999965311792
39.9599213881	17.1870677691	.99999280763
48.5227616855	21.2735820479	.999998067526
57.085601983	25.3600963267	.999999374796

Z THRESHOLDS	NORMALIZED	PROBABILITIES
5.86721821227	.911149446675	.85
7.70938566512	1.99907437995	.956529647638
9.55155311797	3.08699931323	.98667504332
11.3937205708	4.1749242465	.99560662072
13.2358880237	5.26284917978	.998447087595
15.0780554765	6.35077411306	.999415549265
16.9202229294	7.43869904633	.999767363279
18.7623903822	8.52662397961	.999902645376
20.6045578351	9.61454891288	.999957385298
22.4467252879	10.7024738462	.999980574274
24.2888927408	11.7903987794	.999990813211
26.1310601936	12.8783237127	.999995507461
27.9732276465	13.966248646	.999997734746
29.8153950993	15.0541735793	.999998825244
31.6575625522	16.1420985175	.999999374796

Dbz = 7.32045232969

Pd1 = .5 Pd2 = .9 Pd3 = .99

SIGNAL-TO-NOISE RATIOS (DB) FOR INITIAL TRANSFORMATION:

7.18029480646	16.5239686654	26.8808804974
12.6997498763	21.2426978346	31.503743108
14.8466399857	23.3091924052	33.5584316626
16.269595645	24.6986419313	34.9428284925
17.3390310658	25.7492651765	35.9905902914
18.1963923488	26.5945353622	36.8340131363
18.9120679996	27.3017783991	37.5399641554

SIGNAL-TO-NOISE RATIOS (DB) FOR NEW TRANSFORMATION:

7.40601673703	16.6306689407	26.9721139815
9.64374630239	18.3703967446	28.6551266507
10.8741086655	19.4515609092	29.7168048237
11.7563696979	20.2655149832	30.5212118605
12.4659011251	20.9347216111	31.1845531717
13.0680388962	21.5092203112	31.7549390039
13.5941676551	22.0147264365	32.2573356581
14.0624498591	22.4668270369	32.7069765459
14.4847595558	22.8760203338	33.1141652155
14.8694755601	23.2498575248	33.4863332091
15.2228081965	23.5940083337	33.8290709181
15.5495188303	23.9128603228	34.1467104786
15.8533503185	24.2098871812	34.4426853709
16.1373031597	24.4878896233	34.7197662984
16.4038217386	24.749160188	34.9802228898

APPENDIX C. CORROBORATION OF PAGE 28

The CF corresponding to PDF $p_Y(u)$ in (68) is

$$\begin{aligned} f_Y(\xi) &= K \int_0^{\infty} du \exp(i\xi u) \left[1 - \left(1 + \frac{u}{M} \right)^{-M} \right]^{K-1} \left(1 + \frac{u}{M} \right)^{-M-1} = \\ &= K \exp(-i\xi M) \int_0^1 dx (1-x)^{K-1} \exp(i\xi M x^{-1/M}) , \end{aligned} \quad (C-1)$$

where we let $x = (1 + u/M)^{-M}$. Therefore

$$f_Y(\xi) \exp(i\xi M) \sim K \sum_{j=0}^{M-1} \frac{(i\xi M)^j}{j!} \int_0^1 dx (1-x)^{K-1} x^{-j/M} \text{ as } \xi \rightarrow 0, \quad (C-2)$$

where we expanded the second exponential in (C-1) in a power series, and have recognized that the integrals in (C-2) only converge for $j < M$. That is, (C-2) is an asymptotic expansion about $\xi = 0$, up through order $M-1$.

We now employ [1; 3.191 3 and 8.384 1] plus an extended version of the products in (79), namely

$$F_j = \prod_{k=1}^K \left\{ \frac{k}{k - \frac{j}{M}} \right\} \text{ for } 0 \leq j < M , \quad (C-3)$$

to get, from (C-2),

$$f_Y(\xi) \exp(i\xi M) \sim \sum_{j=0}^{M-1} \frac{(i\xi M)^j}{j!} F_j \text{ as } \xi \rightarrow 0 . \quad (C-4)$$

Now let new RV q be defined as

$$q = \frac{Y}{M} + 1 . \quad (C-5)$$

Then its CF is, upon use of (C-4),

$$f_q(\xi) = \overline{\exp(i\xi q)} = \overline{\exp(i\xi Y/M + i\xi)} = f_Y(\xi/M) \exp(i\xi) =$$

$$\sim \sum_{j=0}^{M-1} \frac{(i\xi)^j}{j!} F_j \quad \text{as } \xi \rightarrow 0 . \quad (C-6)$$

Therefore, the moments of RV q , up through order $M-1$, are directly just the products $\{F_j\}$ defined in (C-3), namely

$$\mu_q(j) = F_j \quad \text{for } 0 \leq j < M . \quad (C-7)$$

The usual rules [4; page 70, (3.41)] for proceeding from moments $\{\mu_q(j)\}$ to cumulants $\{\chi_q(j)\}$ apply, at least for $j < M$.

From (C-5), since $y = M(q-1)$, the moments of RV y are

$$\mu_y(j) = \overline{y^j} = M^j \overline{(q-1)^j} = M^j \sum_{n=0}^j (-1)^n \binom{j}{n} \mu_q(j-n) \quad \text{for } j < M . \quad (C-8)$$

This result analytically confirms the conjecture on page 28. The cumulants for y follow readily from (C-5) as

$$\chi_y(1) = M[\chi_q(1) - 1] ,$$

$$\chi_y(j) = M^j \chi_q(j) \quad \text{for } 1 < j < M . \quad (C-9)$$

APPENDIX D. CHARACTERISTIC FUNCTION FOR PDF (68)

An asymptotic expansion for CF $f_y(\xi)$ was derived in (C-4). Here, we will derive a closed form for this CF in terms of exponential integrals.

AVERAGING A NONLINEAR TRANSFORMATION OF RV y

Suppose a positive RV y is subject to a nonlinear transformation, giving output $g(y)$. The average of this latter quantity is

$$\overline{g(y)} = \int_0^{\infty} du \, g(u) \, p_y(u) , \quad (D-1)$$

where $p_y(u)$ is the PDF of RV y . If we integrate by parts on (D-1), there follows

$$\overline{g(y)} = g(0) + \int_0^{\infty} du \, g'(u) [1 - C_y(u)] , \quad (D-2)$$

assuming $C_y(0) = 0$ and

$$\lim_{u \rightarrow +\infty} g(u) [1 - C_y(u)] = 0 . \quad (D-3)$$

Result (D-2) is an alternative to original definition (D-1); an example was utilized in (70).

An example of nonlinear transformation $g(u)$ is

$$g(u) = \exp(i\xi u) , \quad g'(u) = i\xi \exp(i\xi u) , \quad (D-4)$$

for which (D-1) and (D-2) become

$$f_Y(\xi) = \overline{g(y)} = \overline{\exp(i\xi y)} = \int_0^{\infty} du \exp(i\xi u) p_Y(u) = \quad (D-5)$$

$$= 1 + i\xi \int_0^{\infty} du \exp(i\xi u) [1 - C_Y(u)] . \quad (D-6)$$

Requirement (D-3) is satisfied, since $C_Y(u) \rightarrow 1$ as $u \rightarrow +\infty$.

DERIVATION OF CHARACTERISTIC FUNCTION

The CDF of the RV y of interest is given in (66).

Substitution of (D-4) and (66) into (D-6) yields

$$\begin{aligned} f_Y(\xi) &= 1 + i\xi \int_0^{\infty} du \exp(i\xi u) \left\{ 1 - \left[1 - \left(1 + \frac{u}{M} \right)^{-M} \right]^K \right\} = \\ &= 1 + i\xi \int_0^{\infty} du \exp(i\xi u) \sum_{k=1}^K (-1)^{k-1} \binom{K}{k} \left(1 + \frac{u}{M} \right)^{-Mk} = \\ &= 1 + i\xi \sum_{k=1}^K (-1)^{k-1} \binom{K}{k} \int_0^{\infty} du \frac{\exp(i\xi u)}{(1 + u/M)^{Mk}} , \end{aligned} \quad (D-7)$$

where we used (74).

Now define

$$\underline{E}_n(z) \equiv \exp(z) E_n(z) \quad \text{for } n \geq 1 , \quad (D-8)$$

where $E_n(z)$ is the exponential integral of order n [2; 5.1.4].

Also, let $x = 1 + u/M$ in (D-7), thereby leading to

$$\begin{aligned} f_Y(\xi) &= 1 + i\xi M \sum_{k=1}^K (-1)^{k-1} \binom{K}{k} \int_1^{\infty} dx \frac{\exp(i\xi Mx - i\xi M)}{x^{Mk}} = \\ &= 1 + i\xi M \sum_{k=1}^K (-1)^{k-1} \binom{K}{k} \underline{E}_{Mk}(-i\xi M) . \end{aligned} \quad (D-9)$$

This closed form for the CF could utilize the recurrence relations [2; 5.1.14] for the exponential integrals, namely

$$\underline{E}_1(z) = \exp(z) \underline{E}_1(z) ,$$

$$\underline{E}_n(z) = \frac{1}{n-1} \left[1 - z \underline{E}_{n-1}(z) \right] \quad \text{for } n \geq 2 . \quad (D-10)$$

However, this recurrence looks troublesome for large z , due to its alternating character. Also, there appears to be a similar problem with alternating series (D-9) for large K . These latter results were not evaluated numerically.

If we start instead with PDF (68), the CF for y is given by

$$\begin{aligned} f_Y(\xi) &= K \int_0^{\infty} du \exp(i\xi u) \sum_{k=0}^{K-1} (-1)^k \binom{K-1}{k} \left(1 + \frac{u}{M} \right)^{-Mk-M-1} = \\ &= KM \sum_{k=0}^{K-1} (-1)^k \binom{K-1}{k} \underline{E}_{Mk+M+1}(-i\xi M) . \end{aligned} \quad (D-11)$$

This is an alternative equivalent closed form to (D-9) for the CF of RV y .

A couple of special cases of the above results are listed below. For $K = 1$,

$$f_y(\xi) = M \underline{E}_{M+1}(-i\xi M) = 1 + i\xi M \underline{E}_M(-i\xi M) , \quad (D-12)$$

while for $K = 2$,

$$f_y(\xi) = 2M \left[\underline{E}_{M+1}(-i\xi M) - \underline{E}_{2M+1}(-i\xi M) \right] . \quad (D-13)$$

Finally, we have, directly from the PDF in (68), the CF of RV y in the form

$$f_y(\xi) = K \int_0^{\infty} du \exp(i\xi u) \left[1 - \left(1 + \frac{u}{M} \right)^{-M} \right]^{K-1} \left(1 + \frac{u}{M} \right)^{-M-1} . \quad (D-14)$$

Since the integrand decays as u^{-M-1} for large u , this is an attractive numerical form for $M > 3$. And since decent normalization is only attained for large M , form (D-14) is a very promising candidate for an FFT approach in these cases.

POST-AVERAGING OF RANDOM VARIABLE y

Suppose that independent RVs $\{y_j\}$ are subjected to further averaging according to

$$v = \sum_{j=1}^J y_j . \quad (D-15)$$

Then the CF of RV v is a power of the CF of RV y :

$$f_v(\xi) = [f_y(\xi)]^J . \quad (D-16)$$

This is the reason for the emphasis, above, placed on obtaining the CF of RV y .

However, suppose that the numerical evaluation of CF $f_y(\xi)$ is subject to some error; for example, the FFT evaluation of (D-14) would suffer aliasing in the ξ domain due to the necessity of sampling the integral on u . Therefore, let an estimate of $f_y(\xi)$ be available as

$$\tilde{f}_y(\xi) = f_y(\xi) + a(\xi) , \quad (D-17)$$

where $a(\xi)$ is an aliasing component. Then, the accompanying estimate of the CF of RV v is available according to

$$\tilde{f}_v(\xi) \equiv [\tilde{f}_y(\xi)]^J = [f_y(\xi) + a(\xi)]^J \approx f_v(\xi) + J f_y(\xi)^{J-1} a(\xi) , \quad (D-18)$$

where aliasing error $a(\xi)$ is presumed small. In particular, the estimate of the CF of RV v at the origin is

$$\tilde{f}_v(0) \approx 1 + J a(0) . \quad (D-19)$$

If the error in this estimate is desired less than $1E-15$, for example, then the aliasing component at the origin must be bounded according to

$$|a(0)| < \frac{1}{J} 10^{-15} . \quad (D-20)$$

The larger J is in the average (D-15), the tighter is this bound in (D-20). Satisfaction of bound (D-20) will set the increment Δ_u to be used in approximating integral (D-14).

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Spectra and Covariances for "Classical" Nonlinear Signal
Processing Problems Involving Class A Non-Gaussian Noise

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ABSTRACT

Because of the critical rôle of non-Gaussian noise processes in modern signal processing, which usually involves nonlinear operations, it is important to examine the effects of the latter on such noise and the extension to added signal inputs. Here, only non-Gaussian (specifically Class A) noise inputs, with an additive Gaussian component, are considered.

The "classical" problems of zero-memory nonlinear (ZMNL) devices serve to illustrate the approach and to provide a variety of useful output statistical quantities, e.g., mean or "dc" values, mean intensities, covariances, and their associated spectra. Here, Gaussian and non-Gaussian noise fields are introduced, and their respective temporal and spatial outputs are described and numerically evaluated for representative parameters of the noise and the ZMNL devices. Similar statistics for carriers which are phase or frequency modulated by Class A (and Gaussian) noise are also presented, numerically evaluated, and illustrated in the figures. A series of appendices and programs provide the technical support for the numerical analysis.

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LIST OF PRINCIPAL SYMBOLS

A	overlap index (A_A) of Class A noise
$\alpha(R,t)$	input (noise) field
B_v	output covariance function, cf. (2.7a)
b	dimensionless factor, (2.44) et seq.
D_P, D_F	phase- and frequency-modulation scaling factors
δ	(Dirac) delta function
F_1, F_2	characteristic functions
FM	frequency modulation
$f(i\xi)$	Fourier transform of rectifier transfer function
g	ZMNL input-output transfer function
Γ'	$\sigma_G^2 / \Omega_{2A} =$ "Gaussian" factor
$H_1^{(v)}$	scaling parameter for v -th law detector outputs, (2.18a,b,c)
J_0	Bessel function of first kind
$K, K_{L,G}$	covariances
k_L, k_G	normalized covariances
ξ, ξ_1, ξ_2	characteristic function variables
$M_Y, \hat{M}_Y, \hat{M}'_Y$	second-moment functions, (2.7), (2.8), (2.11)
$\mu_F, \hat{\mu}_F$	frequency-modulation indexes
$\mu_P, \hat{\mu}_P$	phase-modulation indexes
v	power law of half-wave rectifier
$\hat{v}, \hat{v}$	normalized wavenumber, cf. (2.33a)
$\Omega_{G,A}$	covariance functions (angle-modulation), section 2.2-6

σ_{2A}	intensity of Class A noise
σ_o	spectral parameter in FM, PM, (2.42c)
PM	phase modulation
$\Delta R, \hat{\Delta R}$	spatial correlation distances
$\hat{R}$	array operator
Re z	real part of z
ρ	"overlap" correlation function, (2.3)
σ_{m+n}^2	noise variance
σ_G^2	intensity of Gauss noise component
T	observation time
T_s	waveform duration of elementary noise source
τ	$t_2 - t_1$, delay (correlation) time
$\hat{\tau}, \hat{\tau}'$	normalized delay parameters
$w(\hat{\omega})$	normalized intensity spectrum, (2.52)
$w_2, \hat{w}_2$	(wavenumber) intensity spectra
$w_{()}, \hat{w}_{()}$	(frequency) intensity spectra
$x, x(t)$	array outputs = inputs to ZMNL devices
$y, y(t)$	outputs of ZMNL devices
Y_a	correlation parameter, cf. (2.7b)
ZMNL	zero-memory nonlinear

LIST OF NORMALIZING AND NORMALIZED PARAMETERS

1. $\hat{t} = \beta\tau$; $\beta = 1/\bar{T}_s$; $\tau = t_2 - t_1$: correlation delay
 $\bar{T}_s$ = mean duration of typical interfering signal
2. $\rho = 1 - |\hat{t}|$ for $|\hat{t}| < 1$, zero otherwise; (2.3): "overlap" correlation function
3. $\hat{\Delta R} = \Delta R / \Delta_L$ (≥ 0); $\Delta R = |R_2 - R_1|$, correlation distance
4. $\hat{\omega} = \omega / \beta = \omega \bar{T}_s$; $\omega = 2\pi f$: normalized (angular) frequency
5. $\hat{\Delta\omega}_L = \Delta\omega_L / \beta$; normalized (frequency) spectrum bandwidth, for Class A noise model
6. $\hat{\Delta\omega}_G = \Delta\omega_G / \beta$; normalized (frequency) spectrum bandwidth, for Gauss noise component
7. Δ_L = rms spread of spatial covariance of non-Gaussian noise field component
8. Δ_G = rms spread of spatial covariance of Gaussian noise field component
9. $\hat{k} = k\Delta_L$: normalized wavenumber
10. $\Delta\omega_N$ = spectrum bandwidth of modulating (RC-Gauss) noise; cf. (2.43) and [1; section 14.1-3]; cf. (11) ff.
11. $\Delta\omega_N \rightarrow \Delta\omega_{N(L)}$ = bandwidth of non-Gaussian component of modulating noise in PM and FM;
 $\Delta\omega_N \rightarrow \Delta\omega_{N(G)}$ = bandwidth of Gaussian component of modulating noise in PM and FM
12. $\zeta = \tau \Delta\omega_N$, correlation variable, cf. (2.43) and (A.6-9) and figures 3.9a through 3.10b; see (A.6-9) for ζ_c
13. $\hat{t}' = \hat{t} - \beta(\Delta R / c_0)$, with $\hat{t} = \tau - \Delta R / c_0$, cf. (2.3a)
14. $\hat{\mu}_F = \mu_F(1 + \Gamma')^{\frac{1}{2}}$: normalized FM index; (2.53)
15. $\hat{\mu}_P = \mu_P(1 + \Gamma')^{\frac{1}{2}}$: normalized PM index; (2.53)
16. $\hat{\omega} = (\omega - \omega_0) / \Delta\omega_N$ = normalized displaced angular frequency, cf. (2.49)

SPECTRA AND COVARIANCES FOR "CLASSICAL" NONLINEAR SIGNAL PROCESSING PROBLEMS INVOLVING CLASS A NON-GAUSSIAN NOISE

PART I. ANALYTIC RESULTS AND NUMERICAL EXAMPLES

1. INTRODUCTION

Non-Gaussian noise fields play a critical rôle in modern signal processing because of the frequently dominant effects of such noise and interference in a wide variety of applications. Communication theory generally, and specifically telecommunications, electromagnetic and acoustic scattering, man-made and natural ambient noise, optics, and underwater acoustics, are common areas of interest in this respect. In the present report we are concerned primarily with underwater acoustic noise phenomena, but the models and results are canonical, that is, they take forms invariant to the particular physical application in question.

Specifically, we are concerned with various second-order statistics of non-Gaussian noise processes and fields after they have been subjected to different types of nonlinear operations, such as rectification and modulation. A generic problem here is the passage of non-Gaussian noise through a zero-memory nonlinear (ZMNL) device. The desired output statistics are typically the mean (dc), mean intensity (power), the covariance or correlation function, and the associated spectra. These last include wavenumber spectra in the case of noise fields, as well as the more general frequency-wavenumber spectra obtained by joint temporal and spatial Fourier transformations. Typical "classical" problems include: (i) rectification, (ii) determination of output spectra and covariances, (iii) calculation of (output) signal-to-noise ratios, (iv) modulation, (v) demodulation, and (vi) special systems, as for example, the spectrum analyzer. These and other problems involving ZMNL devices are described in detail in [1; chapters 5 and 12 - 17]. What is new here is the

use of the approximate second-order probability density functions and characteristic functions in the above applications when the noise processes are non-Gaussian.

A full treatment is given in a current study by Middleton, [2], which is an expanded version of his recent paper [3], which employs some of the results of the present report, namely, the calculated covariances and spectra. Here, we are content to summarize the pertinent analytic results, the corresponding examples of calculated covariances and spectra, and the various computational procedures associated with their evaluation. The details of the derivations are provided in [2] and [3]. Included here, also, is a selection of illustrations of the analytic results.

2. ANALYTICAL RESULTS: A SUMMARY

In the present study, we address three classical problems where the goals are the calculation of the covariance and associated intensity spectrum. Specifically, we consider:

Problem I. The half-wave v -th law rectification of Class A noise fields and processes;

Problem II. Phase modulation of a carrier by a Class A noise process; and

Problem III. Frequency modulation of a carrier by a Class A noise process.

Class A noise, as noted in section 3 of [2], [3], is a canonical form of interference characterized by a coherent structure vis-à-vis the (linear) front-end stages of a typical receiver: negligible transients are produced at the output of these stages. Class B noise, on the other hand, is incoherent and highly impulsive, such that the front-end stages of the receiver generate an output which consists solely of (overlapping) transients. Here, the Class A models are tractable in the required second-order distribution and characteristic functions, whereas the Class B models are not and must

consequently be appropriately approximated in second-order; see [4] and [5] for additional information. In the present report, we shall consider examples of Class A noise only.

2.1 THE SECOND-ORDER CLASS A CHARACTERISTIC FUNCTION

In applications [1] - [3], the second-order characteristic function, $F_2(i\xi_1, i\xi_2)$, plays a key rôle: from it, we may obtain the aforementioned statistics of the outputs of ZMNL devices, spectra of angle-modulated carriers, and other usually second-order statistics of various nonlinear operations arising in a variety of communication and measurement operations.

(See [2], [3] for further discussion.)

Here, we specifically use the approximate Class A noise characteristic function, F_2 , including an additive Gaussian component, given by

$$F_2(i\xi_1, i\xi_2)_{A+G} = \exp[-A(2-\rho)] \sum_{m_1, m_2=0}^{\infty} \frac{[A(1-\rho)]^{m_1+m_2}}{m_1! m_2!} \times \sum_{n=0}^{\infty} \frac{(A\rho)^n}{n!} \exp\left[-\frac{1}{2} Q_{m_1+n, m_2+n}^{(2)}(\xi_1, \xi_2)\right], \quad (2.1)$$

where $A (=A_A)$ is the "overlap" index, and where

$$Q_{m_1+n, m_2+n}^{(2)}(\xi_1, \xi_2) \equiv \xi_1^2 \sigma_{m_1+n}^2 + \xi_2^2 \sigma_{m_2+n}^2 + 2\xi_1 \xi_2 K_{L+G}^{(n)}, \quad (2.2a)$$

$$\sigma_{m+n}^2 \equiv \left(\frac{m+n}{A} + \Gamma'_A\right) \Omega_{2A}; \quad \Omega_{2A} \equiv \frac{1}{2} A \langle B_O^2 \rangle = A \langle L^2 \rangle; \quad \Gamma'_A = \sigma_G^2 / \Omega_{2A}; \quad (2.2b)$$

$$K_{L+G}^{(n)} = \left(n k_L / A + k_G \Gamma'_A\right) \Omega_{2A}; \quad (2.2c)$$

and k_L, k_G are the normalized covariances of the non-Gauss and Gauss components, respectively. Thus, $|k_{L,G}| \leq 1$.

Here, ρ ($=\rho_A$) is the "overlap" correlation function

$$\rho(\tau') \equiv \begin{cases} 1 - \beta|\tau'| & \text{for } \beta|\tau'| \leq 1 \\ 0 & \text{for } \beta|\tau'| > 1 \end{cases}, \quad \beta = 1/\bar{T}_s, \quad (2.3)$$

in which $\bar{T}_s$ is the mean duration of a typical noise-signal of intensity $\langle B_0^2 \rangle / 2 = \langle L^2 \rangle$. The time delay τ' is given by

$$\tau' = \tau - \frac{\Delta R}{c_0} \quad \text{or} \quad \tau' = \tau (= t_2 - t_1), \quad (2.3a)$$

respectively, for space-time fields and received temporal processes. The path delay $\Delta R/c_0$ ($= |R_2 - R_1|/c_0$) accounts for the time differential between propagation paths to the points at which processing occurs, cf. figure 2.1 ff, Case A. The quantities σ_{2A} and σ_G^2 are, respectively, the intensity of the non-Gaussian and Gaussian components which constitute the general Class A model used here. (However, we note that the present Class A model belongs either to the strictly canonical Class A cases, where all interfering sources are equidistant from the observer, or more generally, to the much broader class of situations in which the effective source distribution is concentrated in an annulus whose inner-to-outer radii have a ratio $O(1/2)$ or less. The former is exactly represented by (2.1) to second-order, while the latter is approximately so represented, albeit a good approximation as long as the aforementioned source annulus is not too large. See [5; section V, C], for example. For an exact treatment, see also [6], in an important class of physical models. Finally, differentiation of F_2 , (2.1), in the usual way, gives us the (exact) covariance of the composite Class A and Gauss field, namely,

$$K_{A+G} = - \frac{\partial^2}{\partial \xi_1 \partial \xi_2} F_2 \Big|_{\xi_1 = \xi_2 = 0} = K_A + K_G, \quad (2.4)$$

which in normalized form is

$$k_{A+G}(\Delta R, \tau) = \frac{k_L + \Gamma' k_G}{1 + \Gamma'} . \quad (2.4a)$$

In practice, A is usually less than unity, say $O(0.1 - 0.3)$ typically, so that only a comparatively few terms in $A^{m_1+m_2+n}$ are needed for numerical evaluation of (2.1) and the statistical quantities derived from it, cf. section 2.2 ff. Note that when $\beta|\tau| \geq 1$, $\rho = 0$, and $\Gamma' = 0$, we get

$$\begin{aligned} F_{2-A} &= \left(e^{-A} \sum_{m=0}^{\infty} \frac{A^m}{m!} \exp\left(-\frac{1}{2}\xi_1^2 \sigma_m^2\right) \right) \left(e^{-A} \sum_{n=0}^{\infty} \frac{A^n}{n!} \exp\left(-\frac{1}{2}\xi_2^2 \sigma_n^2\right) \right) \\ &= F_1(i\xi_1)_A F_1(i\xi_2)_A , \end{aligned} \quad (2.4b)$$

as expected: there is now no correlation between process samples. With a Gaussian component, these will be correlated, of course, unless $|\tau| \rightarrow \infty$, so that $k_G \rightarrow 0$, cf. (2.2c).

2.2 PROBLEM I: HALF-WAVE v -TH LAW RECTIFICATION (STATIONARY AND HOMOGENEOUS FIELDS)

Here we consider the problem of obtaining the second-order (second-moment) statistics, M_y , of a sampled noise field, $\alpha(R, t)$, after passage through a ZMNL device, g , when the noise is generally non-Gaussian. Various processing configurations are possible. We show two in figure 2.1, below. Analytically, we have, for stationary and homogeneous inputs [1; section 2.3-2]

$$\begin{aligned} M_y(\Delta R, \tau) &= \overline{g(x_1)g(x_2)} = \frac{1}{(2\pi)^2} \int_{C_1} \int_{C_2} f(i\xi_1) f(i\xi_2) \\ &\times F_2(i\xi_1, i\xi_2; \Delta R, \tau)_x d\xi_1 d\xi_2 = \overline{y_1 y_2} , \end{aligned} \quad (2.5)$$

where $\Delta R = R_2 - R_1$, $\tau = t_2 - t_1$, and $f(i\xi)$ is the Fourier transform of the ZMNL device with $y_1 = y(R_1, t_1)$, etc. In the present cases, we have specifically

$$f(i\xi) = \beta \Gamma(v+1)/(i\xi)^{v+1}, \quad v > -1, \quad (2.6)$$

for these half-wave v -th law rectifiers [1; (2.101a,b)].

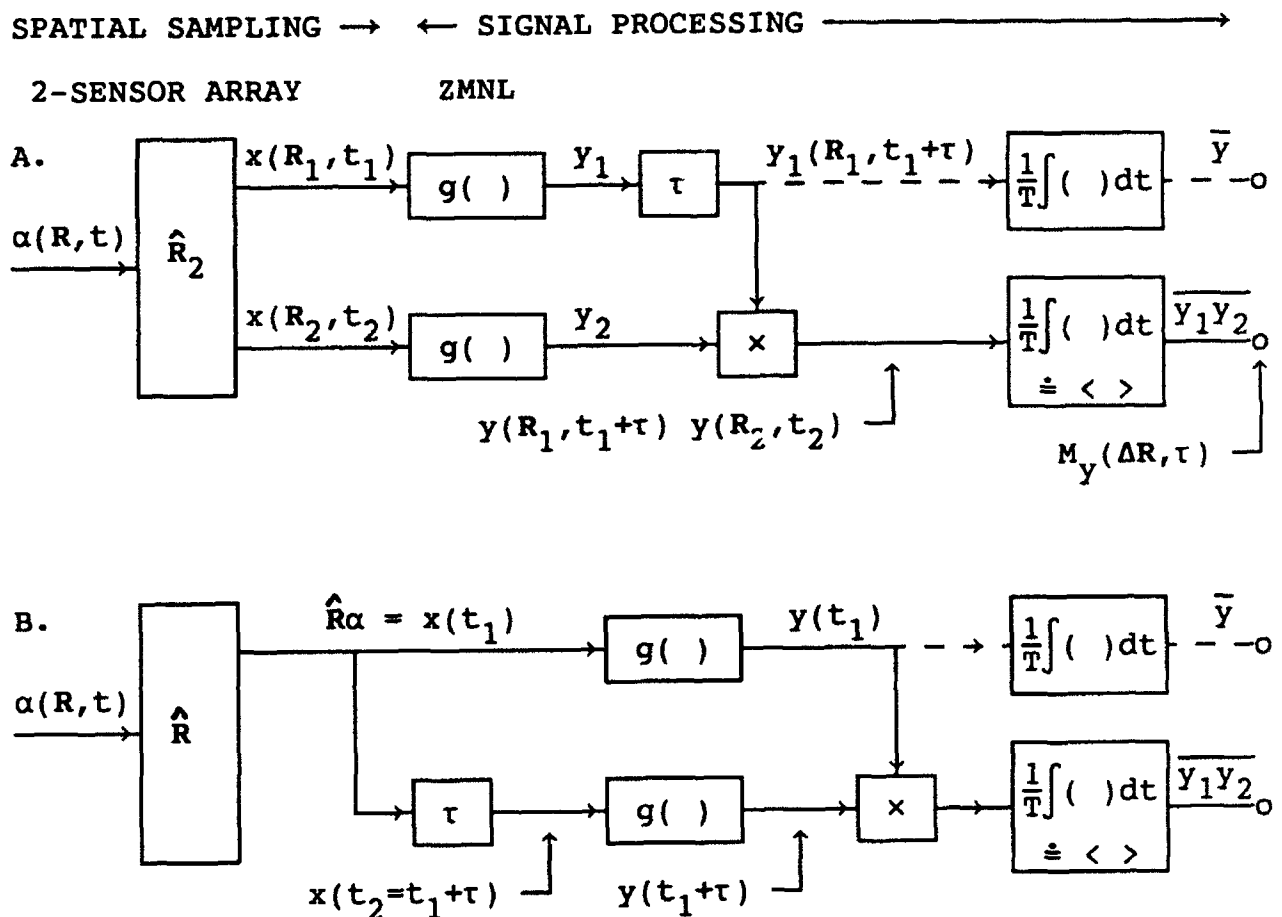


Figure 2.1 A. Two-point sensor array ($\hat{R}_2$) giving sampled field at two space-time points. B. A general array ($\hat{R}$) (preformed beam), converting the field $\alpha(R, t)$ into a single (time) process $x(t_1)$. Both are followed by ZMNL devices, delays, and averaging, as indicated schematically.

For the Class A non-Gaussian noise inputs of section 2.1 above, we find that the (normalized) second-moment M_y for the resulting rectified field is now

$$M_y(\hat{\Delta R}, \tau) = \exp[-A(2-\rho)] \sum_{m_1, m_2=0}^{\infty} \frac{[A(1-\rho)]^{m_1+m_2}}{m_1! m_2!} \sum_{n=0}^{\infty} \frac{(A\rho)^n}{n!} \\ \times \left(\frac{n+m_1}{A} + \Gamma' \right)^{v/2} \left(\frac{n+m_2}{A} + \Gamma' \right)^{v/2} B_v \Big|_{m_1, m_2, n} , \quad (2.7)$$

where we have further postulated the noise field to be isotropic, $\Delta R \rightarrow |\Delta R|$, and where specifically,

$$B_v \Big|_a \equiv B_v(Y|m_1, m_2, n) = \Gamma^2 \left(\frac{v+1}{2} \right) {}_2F_1 \left(-\frac{v}{2}, -\frac{v}{2}; \frac{1}{2}; Y_a^2 \right) \\ + 2Y_a \Gamma^2 \left(\frac{v}{2} + 1 \right) {}_2F_1 \left(\frac{1-v}{2}, \frac{1-v}{2}; \frac{3}{2}; Y_a^2 \right) , \quad (2.7a)$$

$$Y_a \equiv \frac{\frac{n}{A} k_L + \Gamma' k_G}{\left(\frac{m_1+n}{A} + \Gamma' \right)^{1/2} \left(\frac{m_2+n}{A} + \Gamma' \right)^{1/2}} ; \quad a = (m_1, m_2, n) , \quad |Y_a| \leq 1 . \quad (2.7b)$$

Specifically, also, we have the following normalized forms

$$\hat{M}_y \equiv M_y / \Omega_{2A}^v 2^v / 4\pi ; \quad \hat{\tau}' \equiv \beta \tau' , \quad \beta = 1/\bar{T}_s , \quad \text{cf. (2.3)} , \\ \hat{\Delta R} \equiv \Delta R / \Delta_L , \quad \Delta_L = \text{correlation distance,} \quad \Delta R = |R_2 - R_1| . \quad (2.8)$$

For numerical results, we select the following models for the space-time covariance functions of the isotropic and stationary non-Gaussian and Gaussian components of the input noise field:

$$k_L \equiv \exp \left(-\Delta R^2 / \Delta_L^2 - \frac{1}{2} (\Delta \omega_L \tau' / \beta)^2 \right) = \exp \left(-\hat{\Delta R}^2 - \frac{1}{2} (\Delta \hat{\omega}_L \hat{\tau}')^2 \right) ; \quad (2.9a)$$

$$k_G \equiv \exp \left(-\Delta R^2 / \Delta_G^2 - \frac{1}{2} (\Delta \omega_G \tau' / \beta)^2 \right) = \exp \left(-\hat{\Delta R}^2 (\Delta_L / \Delta_G)^2 - \frac{1}{2} (\Delta \hat{\omega}_G \hat{\tau}')^2 \right) ;$$

$$\hat{\Delta\omega}_L \equiv \Delta\omega_L/\beta, \quad \hat{\Delta\omega}_G \equiv \Delta\omega_G/\beta. \quad (2.9b)$$

Here, Δ_G is a correlation distance, and $\Delta\omega_L$, $\Delta\omega_G$ are angular frequency spreads associated with the respective non-Gaussian and Gaussian components of the input field. Note that if we define the correlation distance Δ_L as that where $k_L = 1/e$ ($\hat{r}' = 0$), then $\Delta_L = \Delta R_L$, etc.

For the special cases of v considered here, we also observe (from [1; (A.1-39)]) that B_v may be expressed in closed form:

$$B_0(Y) = \pi + 2 \arcsin(Y), \quad (2.10a)$$

$$B_1(Y) = Y \arcsin(Y) + \left(1 - Y^2\right)^{\frac{1}{2}} + \frac{\pi}{2}Y, \quad (2.10b)$$

$$B_2(Y) = \left(\frac{1}{2} + Y^2\right) \left(\frac{\pi}{2} + \arcsin(Y)\right) + \frac{3}{2}Y \left(1 - Y^2\right)^{\frac{1}{2}}. \quad (2.10c)$$

2.2-1 GAUSS PROCESSES ALONE ($A=0$)

When only a Gauss noise field is originally present, that is, $A = 0$, for example, $\varrho_{2A} = 0$, (2.7) reduces to the classical result [1; page 541, (13.4a)]:

$$\hat{M}'_Y \equiv \hat{M}_Y|_{A=0} = B_v|_{a=0}; \quad M_Y = \frac{(2\psi)^v}{4\pi} \hat{M}_Y; \quad Y_a \rightarrow Y_0 = k_G. \quad (2.11)$$

For comparison with the non-Gaussian cases ($A>0$), we choose to have equal input noise intensities. This means that

$$\psi_{A=0} = \sigma_G^2 + \varrho_{2A} = \varrho_{2A}(1 + \Gamma'),$$

so that

$$\hat{M}_Y|_{A=0} = (1 + \Gamma')^v B_v|_{a=0}, \quad Y_0 = k_G, \quad (2.12)$$

and $\hat{M}'_Y$ is then to be compared with $\hat{M}_Y$, $A > 0$. When Γ' is small, as is usually the case, we can often replace $(1 + \Gamma')^v$ by unity.

At this point, following figure 2.1, we distinguish two classes of operation: (A), where a pair of point sensors is used to sample the noise field and we wish to consider both the space and temporal correlations of the sampled field at the two points (R_1, t_1) , (R_2, t_2) ; and (B), where the space-time field is converted into a random process, $x(t)$, by the beamforming array $(\hat{R})$, with an associated directionality embodied in the resultant beam (vide [7; sections IV B and VI A]).

2.2-2 CASE B, FIGURE 2.1

Let us consider the simpler case (Case B) of the time process first, cf. (B). For this, we set $\Delta R = 0$ formally in (2.7) et seq. above, since $x = \hat{R} \alpha(R, t)$ here and $\tau' = \tau = t_2 - t_1$, cf. (2.3a). See also [3; (3.2) et seq. and (3.11a)]. Then our ad hoc illustrative models of the process covariances k_L , k_G , are, from (2.9a,b), at once*

$$k_L = k_L(\tau) = \exp\left[-\frac{1}{2}(\Delta\omega_L \hat{r}/\beta)^2\right] = \exp\left[-\frac{1}{2}(\hat{\Delta\omega}_L \hat{r})^2\right], \quad (2.13a)$$

$$k_G = k_G(\tau) = \exp\left[-\frac{1}{2}(\Delta\omega_G \hat{r}/\beta)^2\right] = \exp\left[-\frac{1}{2}(\hat{\Delta\omega}_G \hat{r})^2\right]. \quad (2.13b)$$

Accordingly, (2.7) reduces to

$$\begin{aligned} \text{Case B: } M_Y(0, \hat{r}) &\equiv M_Y(\hat{r})_B = (2.7), \text{ with } Y_a = (2.7b), \\ &\text{and } (2.13a,b) \text{ and } \hat{\Delta R} = 0 \text{ therein.} \end{aligned} \quad (2.14)$$

We note that when $|\hat{r}| \geq 1$, $\rho = 0$, and $\hat{M}_Y(0, |\hat{r}| \geq 1)$ reduces to a simpler relation [vis-à-vis (2.7)], viz.:

* A physically derived model of k_G and k_L may be made from [3; (3.11a)] with $L = \hat{R} L$, $\hat{R} = (2.9)$ etc., where L is typically given by [3; (3.3)], for example.

$$\hat{M}_Y(\hat{r})_B = e^{-2A} \sum_{m_1, m_2=0}^{\infty} \frac{A^{m_1+m_2}}{m_1! m_2!} \left(\frac{m_1}{A} + \Gamma' \right)^{v/2} \left(\frac{m_2}{A} + \Gamma' \right)^{v/2} B_v|_a ,$$

(2.14a)

where (2.7b) becomes

$$Y_a = \frac{\Gamma' k_G}{\left(\frac{m_1}{A} + \Gamma' \right)^{1/2} \left(\frac{m_2}{A} + \Gamma' \right)^{1/2}} , \quad \rho = 0 , \quad (2.14b)$$

in $B_v|_a$.

Special cases of interest are:

I. THE INTENSITY $E(y^2)$: $\hat{r} = 0$, $\rho = 1$, $m_1 = m_2 = 0$, and (2.7), (2.14) reduce to

$$\overline{y^2}|_{\text{norm}} = \hat{M}_Y(0)_B = \hat{M}_Y(0,0) = B_v|_{a=n} = e^{-A} \sum_{n=0}^{\infty} \frac{A^n}{n!} \left(\frac{n}{A} + \Gamma' \right)^v ,$$

(2.15)

where now $Y_{a=n} = 1$, e.g., $k_L(0) = 1$ etc., and B_v is independent of n , for example, for $Y_a = 1$,

$$B_v|_{a=n} = \begin{cases} 2\pi & \text{for } v = 0 \\ \pi & \text{for } v = 1 \\ 3\pi/2 & \text{for } v = 2 \end{cases} , \text{ cf. (2.10) .} \quad (2.16a)$$

For general v , $Y_a = 1$, we have (from [1; (A.1-34)])

$$B_v|_{a=n} = 2\pi^{1/2} \Gamma(v+1/2) , \quad v \geq 0 . \quad (2.16b)$$

Thus, (2.15) becomes

$$\overline{y^2}|_{\text{norm}} = \hat{M}_Y(0)_B = \hat{M}_Y(0,0) = 2\pi^{1/2} \Gamma(v+1/2) e^{-A} \sum_{n=0}^{\infty} \frac{A^n}{n!} \left(\frac{n}{A} + \Gamma' \right)^v . \quad (2.17)$$

The unnormalized form is, from (2.8),

$$\bar{y}^2 = M_Y(0)_B = M_Y(0,0) = \frac{2^{v-1}}{\pi^{\frac{1}{2}}} \Omega_{2A}^v \Gamma(v+\frac{1}{2}) H_1^{(v)}(A, \Gamma') \quad (2.18)$$

with

$$H_1^{(v)}(A, \Gamma') \equiv e^{-A} \sum_{n=0}^{\infty} \frac{A^n}{n!} \left(\frac{n}{A} + \Gamma' \right)^v$$

$$= \begin{cases} 1 & \text{for } v = 0, \\ 1 + \Gamma' & \text{for } v = 1, \\ 1/A + (1+\Gamma')^2 & \text{for } v = 2. \end{cases} \quad (2.18a)$$

$$= \begin{cases} 1 & \text{for } v = 0, \\ 1 + \Gamma' & \text{for } v = 1, \\ 1/A + (1+\Gamma')^2 & \text{for } v = 2. \end{cases} \quad (2.18b)$$

$$= \begin{cases} 1 & \text{for } v = 0, \\ 1 + \Gamma' & \text{for } v = 1, \\ 1/A + (1+\Gamma')^2 & \text{for } v = 2. \end{cases} \quad (2.18c)$$

For other values of v (>0), we must evaluate $H_1^{(v)}$ numerically.

II. THE MEAN VALUE, $\bar{y}$; $|\hat{t}| \rightarrow \infty$

Now $\rho = 0$, $n = 0$, $Y_a = 0$, and (2.7) reduces directly, upon use of (2.18), to

$$\begin{aligned} \bar{y}^2 \Big|_{\text{norm}} &= \hat{M}_Y(\infty)_B = \hat{M}_Y(0, \infty) = \Gamma^2 \left(\frac{v+1}{2} \right) \left\{ e^{-A} \sum_{m=0}^{\infty} \frac{A^m}{m!} \left(\frac{m}{A} + \Gamma' \right)^{v/2} \right\}^2 \\ &= \Gamma^2 \left(\frac{v+1}{2} \right) H_1^{(v/2)}(A, \Gamma')^2. \end{aligned} \quad (2.19)$$

The unnormalized form of (2.19) is, from (2.8),

$$\bar{y}^2 = M_Y(\infty)_B = M_Y(0, \infty) = \frac{2^v \Omega_{2A}^v}{4\pi} \Gamma^2 \left(\frac{v+1}{2} \right) H_1^{(v/2)}(A, \Gamma')^2, \quad (2.20)$$

and for v even, we find, from (2.18a,b,c)

$$H_1^{(0)} = 1, \quad H_1^{(1)} = 1 + \Gamma', \quad H_1^{(2)} = \frac{1}{A} + (1 + \Gamma')^2. \quad (2.21)$$

III. THE CONTINUUM INTENSITY: $\overline{y^2} - \bar{y}^2$

From (2.18) and (2.20) we get at once the general result for $\nu \geq 0$,

$$P_c \equiv \overline{y^2} - \bar{y}^2 = 2^\nu \Omega_{2A}^\nu \left\{ \frac{\Gamma(\nu + \frac{1}{2})}{2\pi^{\frac{1}{2}}} H_1^{(\nu)} - \left[\frac{\Gamma(\frac{\nu+1}{2})}{2\pi^{\frac{1}{2}}} H_1^{(\nu/2)} \right]^2 \right\}, \quad (2.22)$$

which is the generalization of [1; (13.7)], in the classical purely Gaussian cases, to the present, dominant non-Gaussian noise component $\Omega_{2A} (>> \sigma_G^2)$. In these classical cases, we can show at once that

$$\lim_{\Omega_{2A} \rightarrow \infty} H_1^{(\nu)} \Omega_{2A}^\nu \rightarrow \lim_{\Omega \rightarrow 0} e^{-A} \sum_{n=0}^{\infty} \frac{A^n}{n!} \left(\frac{n\Omega}{A} + \sigma_G^2 \right)^\nu \rightarrow \sigma_G^{2\nu} (= \psi^\nu), \quad (2.23)$$

where $\Omega_{2A} \rightarrow \infty$ implies $A \rightarrow 0$ and $B_0^2 \rightarrow 0$, cf. (2.2b), so that (2.22) becomes, as expected,

$$P_c|_{\text{Gauss}} = 2^\nu \sigma_G^{2\nu} \left\{ \frac{\Gamma(\nu + \frac{1}{2})}{2\pi^{\frac{1}{2}}} - \frac{\Gamma^2(\frac{\nu+1}{2})}{4\pi} \right\} (> 0), \quad \nu \geq 0. \quad (2.24)$$

Figure 13.5 of [1] shows (2.24) as a function of rectifier law (ν), as well as (2.18), (2.20) in these Gaussian cases. In the present, more general, situation of Class A noise, the results are more complex, as expected, with now two additional parameters (A, Γ'), descriptive of this much broader class of interference.

2.2-3: CASE A, FIGURE 2.1

We turn now to the more general problem of the covariance of the Class A non-Gaussian random field, sampled according to procedure (A), shown schematically in figure 2.1 earlier. Here, $x = \alpha(R, t)$, sensed at (R_1, t_1) , (R_2, t_2) , where $L = L$, cf. (3.3) in [3; (3.2)]. Equation (2.7) applies here, with $\Delta R \neq 0$ (as well as for $\Delta R = 0$), and we use (2.9a,b) for our illustrative examples, which are discussed in section 3 following. At this point, we recall from (2.3a) that the proper time delay to use is $\tau' = \tau - \Delta R/c_0$ in $\rho = \rho(\tau')$, and in some of the structural elements of the noise field covariances, cf. [3; (3.11b,c)].

CASE I: $\hat{t}' = 0$

From (2.7), we have $\rho = 1$, $m_1 = m_2 = 1$, giving

$$\hat{M}_Y(\hat{\Delta R}, 0) = e^{-A} \sum_{n=0}^{\infty} \frac{A^n}{n!} \left(\frac{n}{A} + \Gamma' \right)^v B_v \Big|_{a=n}, \quad \rho = 1, \quad \therefore \tau = \frac{\Delta R}{c_0}, \quad (2.25)$$

where (2.7b) is specifically

$$Y_{a=n} = \frac{\frac{n}{A} k_L(\hat{\Delta R}, 0) + \Gamma' k_G(\hat{\Delta R}, 0)}{\frac{n}{A} + \Gamma'}. \quad (2.25a)$$

For calculations, (2.9a,b) are used, with B_v given by (2.7a), where (2.25a) provides Y_a . When $\hat{\Delta R} = 0$, (2.25) reduces to (2.15) et seq. for the total intensity of the field observed at $R_1 = R_2$.

CASE II: $\hat{\Delta R} \rightarrow \infty, |\tau'| > 1$

When $\hat{\Delta R} \rightarrow \infty$, we obtain different results, depending on τ' . Here $\rho = 0$, $Y_a \rightarrow 0$, cf. (2.9a,b) in (2.7b), and therefore $n = 0$. Accordingly, (2.7) becomes

$$\hat{M}_Y(\infty, |\tau'| > 1) = \hat{M}_Y(\infty, \infty) = \hat{M}_Y(0, \infty) = \bar{y}_{\text{norm}}^2, \quad (2.19) \quad (2.26)$$

The fact that $\hat{\Delta R} \rightarrow \infty$ ensures that $Y_a \rightarrow 0$, a behavior similar to that for Case (B) above, when we consider the purely Gaussian noise process, section 2.2-1.

CASE III: $\hat{\Delta R} \rightarrow \infty, 0 < |\tau'| < 1$

Here, $\rho > 0$ while $Y_a \rightarrow 0$, so that B_v , (2.7b), becomes $\Gamma^2(v + \frac{1}{2})$ once more. The second moment function (2.7) is now

$$\begin{aligned} M_Y(\hat{\Delta R}, \tau') &= \Gamma^2\left(\frac{v+1}{2}\right) \exp[-A(2-\rho)] \sum_{m_1, m_2=0}^{\infty} \frac{[A(1-\rho)]^{m_1+m_2}}{m_1! m_2!} \\ &\times \sum_{n=0}^{\infty} \frac{(Ap)^n}{n!} \left(\frac{n+m_1}{A} + \Gamma'\right)^{v/2} \left(\frac{n+m_2}{A} + \Gamma'\right)^{v/2}, \end{aligned} \quad (2.27)$$

which is a minor simplification of (2.7).

CASE IV: $\hat{\Delta R} \rightarrow \infty, |\tau'| = 0$

In this special situation, where $\tau = \Delta R/c_0 \rightarrow \infty$ in such a way that $\tau' = 0$ and therefore $\rho = 1$, $Y_a = 0$, we obtain directly from (2.27) the comparatively simple result,

$$\hat{M}_Y(\infty, 0) = \Gamma^2\left(\frac{v+1}{2}\right) H_1^{(v)}(> 0). \quad (2.28)$$

2.2-4: REMARKS

At first glance, as $\Delta R \rightarrow \infty$, we might expect M_Y always to reduce to $\bar{y}^2$, e.g., $K_Y \equiv M_Y - \bar{y}^2 = 0$ for the covariance of the rectified space-time field. This is expectedly the case for the covariance (and second-moment) function of the input Class A and Gauss noise field components $\alpha(R,t)$, as we can see directly from (2.9a,b), or from [3; (3.11b,c)] for example, in the physically derived cases. However, the process or field $y = g(x)$ here is the result of a nonlinear operation, cf. (2.5), (2.6), which severely distorts the input waveform and generates all kinds of modulation products, associated with the spatial as well as the temporal variations of the input field. This accounts for the departures in Cases III, IV of $M_Y(\infty, \tau')$ from $\bar{y}^2$, while certainly $M_X(\infty, \tau') \rightarrow \bar{x}^2 = 0$, (since $\bar{x} = 0$ initially here).

From the various limiting results above, we see that

$$M_Y(0,0) > M_Y(\infty,0) \quad \text{and} \quad M_Y(0,0) > M_Y(0,\infty) , \quad (2.29a)$$

and

$$M_Y(\infty,0) \gtrless M_Y(0,\infty) \quad \text{depending on } A, \Gamma', \text{ and } v , \quad (2.29b)$$

with

$$M_Y(0,0) - M_Y(0,\infty) > 0 , \quad \text{cf. (2.19) and (2.20) ,} \quad (2.29c)$$

$$M_Y(\infty,\infty) = M_Y(0,\infty) = \bar{y}^2 , \quad \text{cf. (2.20) and (2.26) ,} \quad (2.29d)$$

whereas

$$M_X(0,0) > M_X(0,\infty) = M_X(\infty,0) = \bar{x}^2 = 0 . \quad (2.29e)$$

Finally, we note that (2.11), (2.12) apply here, also, for the Gauss-alone cases, where now

$$\Omega_{2A}(1 + \Gamma') \rightarrow \sigma_G^2 \quad \text{and} \quad B_v \Big|_{a=0} \rightarrow (1 + \Gamma')^v B_v \Big|_a = 0 .$$

2.2-5: SPECTRA

The various intensity spectra associated with the output of the processor (cf. figure 2.1) are important also, as they show how the energy in this output is distributed. Here, we consider two types of spectra, respectively, for the rectified spatial field (A) and for the process (B), namely the wavenumber and the frequency spectrum of $y(R,0)$ and $y(0,t)$. In particular, wavenumber spectra are useful in the analysis of spatially distributed phenomena, paralleling the analysis of time-dependent phenomena.

I: WAVENUMBER SPECTRUM

The wavenumber intensity spectrum is defined here by

$$W_2(k,0)_Y = W_2(k,\tau)_Y \Big|_{\tau=0} \equiv \iint_{\Delta R} M_Y(\Delta R,0) \exp(ik \cdot \Delta R) d(\Delta R) \quad (2.30a)$$

$$= 2\pi \int_0^\infty M_Y(\Delta R,0) J_0(k\Delta R) \Delta R d(\Delta R) = W_2(k,0)_Y, \quad (2.30b)$$

with

$$k = (k_x, k_y), \quad \Delta R = |\Delta R|, \quad k = |k| \quad (2.30c)$$

for these isotropic fields, where k is an (angular) vector wavenumber. Using the normalization of (2.8), we get, with $\hat{k} \equiv k\Delta_L$,

$$\hat{W}_2(\hat{k},0)_Y \equiv \frac{W_2(k,0)_Y}{\Delta_L^2 2^\nu \Omega^{2\nu}/4\pi} = 2\pi \int_0^\infty \hat{M}_Y(x,0) J_0(\hat{k}x) x dx \quad (2.31)$$

for the normalized wavenumber intensity spectrum.

Since $\hat{M}_y(\infty, 0)$ is nonvanishing, cf. (2.28), there is a dc component, or δ -function, in the general wavenumber spectrum. We will use the relations

$$\int_0^{\infty} x J_0(\hat{k}x) dx = \frac{1}{\hat{k}} \delta(\hat{k} - 0) , \quad \hat{k} = \left(\hat{k}_x^2 + \hat{k}_y^2 \right)^{\frac{1}{2}} = |\hat{k}| , \quad \hat{k} = (\hat{k}, \phi) , \quad (2.32)$$

where we must remember that $\hat{k}$ is two-dimensional. With $\hat{v}$ a vector wavenumber defined by

$$\hat{k} = 2\pi\hat{v} [= (\hat{v}, \phi)] , \quad \hat{k} = 2\pi\hat{v} = 2\pi|\hat{v}| , \quad (2.33a)$$

and using the relation

$$\delta(ax - b) = \frac{1}{a} \delta\left(x - \frac{b}{a}\right) \quad \text{for } a > 0 , \quad (2.33b)$$

we also show that

$$\begin{aligned} \delta(\hat{k}_x - 0) \delta(\hat{k}_y - 0) &= \frac{1}{2\pi\hat{k}} \delta(\hat{k} - 0) = \frac{1}{(2\pi)^3 \hat{v}} \delta(\hat{v} - 0) \\ &= \frac{1}{(2\pi)^2} \delta(\hat{v}_x - 0) \delta(\hat{v}_y - 0) , \quad \hat{v} = |\hat{v}| , \quad \hat{v} = \left(\hat{v}_x^2 + \hat{v}_y^2 \right)^{\frac{1}{2}} . \end{aligned} \quad (2.34)$$

Applying (2.32) - (2.34), with

$$\begin{aligned} \hat{W}_2(\hat{k}, \phi)_y &= 2\pi \int_0^{\infty} x J_0(\hat{k}x) \left[\hat{M}_y(x, 0) - \hat{M}_y(\infty, 0) \right] dx \\ &\quad + 2\pi \hat{M}_y(\infty, 0) \int_0^{\infty} x J_0(\hat{k}x) dx \end{aligned} \quad (2.35a)$$

$$\begin{aligned} &= \hat{W}_2(\hat{k}, 0)_{y\text{-cont}} + (2\pi)^2 \hat{M}_y(\infty, 0) \delta(\hat{k}_x - 0) \delta(\hat{k}_y - 0) \\ &= \hat{W}_2(\hat{k}, 0)_{y\text{-cont}} + \hat{M}_y(\infty, 0) \delta(\hat{v}_x - 0) \delta(\hat{v}_y - 0) , \end{aligned} \quad (2.35b)$$

which defines $\hat{W}_2(\hat{k}, 0)_{y\text{-cont}}$, the continuous portion of the spectrum and shows the dc term in $\hat{k}$ - or $\hat{v}$ -space, as convenient. It is $\hat{W}_{2\text{-cont}}$ with which we are concerned in the specific numerical examples of section 3 ff.

II. FREQUENCY SPECTRUM

Here we employ the Wiener-Khintchine theorem [1; (3.42)] to write for the frequency spectrum of y

$$W_Y(f) = 2 \int_{-\infty}^{\infty} M_Y(0, \tau) \exp(-i\omega\tau) d\tau = B_O \int_0^{\infty} \hat{M}_Y(0, \hat{t}) \cos(\hat{\omega}\hat{t}) d\hat{t}, \quad (2.36)$$

where

$$B_O \equiv \Omega_{2A}^v 2^v / \pi\beta; \quad \hat{t} = \beta\tau; \quad \omega = 2\pi f; \quad \hat{\omega} \equiv \omega/\beta; \quad \therefore \hat{f} = f/\beta. \quad (2.36a)$$

Accordingly, we define the normalized frequency intensity spectrum of y as

$$\hat{W}_Y(\hat{f}) \equiv W_Y(f)/B_O = \int_0^{\infty} \hat{M}_Y(0, \hat{t}) \cos(\hat{\omega}\hat{t}) d\hat{t}. \quad (2.37)$$

Again, there is a dc component, since $\hat{M}_Y(0, \infty) = \bar{y}^2 (> 0)$, cf. (2.20). We have

$$\hat{W}_Y(f) = \int_0^{\infty} [\hat{M}_Y(0, \hat{t}) - \hat{M}_Y(0, \infty)] \cos(\hat{\omega}\hat{t}) d\hat{t} + \frac{1}{2} \hat{M}_Y(0, \infty) \delta(\hat{f}-0), \quad (2.38)$$

since

$$\int_0^{\infty} \cos(\omega x) dx = \pi \delta(\omega-0) = \frac{1}{2} \delta(f-0).$$

As in the wavenumber cases above (Case I), we are concerned with the continuous part of the spectrum, viz.

$$\hat{W}_Y(f)_{\text{cont}} = \int_0^{\infty} \left[\hat{M}_Y(0, \hat{t}) - \bar{Y}^2 \right] \cos(\omega \hat{t}) d\hat{t} , \quad (2.39)$$

which is also illustrated numerically in section 3 ff.

III. WAVENUMBER FREQUENCY SPECTRUM

The wavenumber frequency spectrum is defined by

$$W_2(k, \omega)_Y = \iint_{-\infty}^{\infty} M_Y(\Delta R, \tau) \exp(ik \cdot \Delta R - i\omega \tau) d(\Delta R) d\tau , \quad (2.40)$$

with $\omega = 2\pi f$. The associated wavenumber spectrum $W_2(k, 0)$ used in (2.30) is obtained from $W_2(k, \tau) \big|_{\tau=0}$. In normaliz. form, we have for (2.40), in these isotropic cases,

$$\begin{aligned} \hat{W}_2(\hat{k}, \hat{\omega})_Y &= \left(2^V \Omega_{2A}^V \Delta_L^2 / (4\pi\beta) \right)^{-1} W_2(k, \omega)_Y \\ &= \iiint_{-\infty}^{\infty} \hat{M}_Y(\hat{\Delta R}, \hat{t}) \exp(i\hat{k} \cdot \hat{\Delta R} - i\hat{\omega} \hat{t}) d(\hat{\Delta R}) d\hat{t} ; \end{aligned}$$

$$\hat{W}_2(\hat{k}, \hat{\omega})_Y = 2\pi \int_0^{\infty} \int_0^{\infty} \hat{M}_Y(x, \hat{t}) J_0(\hat{k}x) \exp(-i\hat{\omega} \hat{t}) x dx d\hat{t} . \quad (2.41)$$

The various dc components are readily extracted, as in Cases I and II above. Numerical examples of this joint intensity spectrum are reserved to a possible subsequent study. The results of section 3 show the marginal spectra (Cases I, II) of this more general situation.

2.2-6: FREQUENCY AND PHASE MODULATION BY CLASS A AND GAUSSIAN NOISE

This is a Case (B) situation, cf. figure 2.1, where $\Delta R = 0$ and we are concerned only with the received (non-Gaussian) noise process which is used to angle-modulate a (high frequency) carrier f_0 . For the analysis, see [3; section II].

The general result for the covariance of the carrier modulated by Class A and Gauss noise is found to be

$$K_Y(\tau)_{A+G} = \frac{1}{2} A_0^2 \operatorname{Re} \left[\exp \left(i \omega_0 \tau - D_0^2 \varrho(\tau)_G - A(2-\rho) \right. \right. \\ \left. \left. + 2A(1-\rho) \exp \left[-D_0^2 \varrho_0 / A \right] \right) \right], \quad (2.42)$$

where now, cf. [1; (4.2), (14.14c)],

$$\varrho(\tau)_G \Big|_{A \text{ FM}} = \left(\sigma_G^2 \text{ or } \frac{1}{A} \varrho_{2A} \right) \int_0^{|\tau|} (|\tau| - \lambda) k(\lambda) d\lambda \quad (D_0 = D_F) \\ = \left(1 \text{ or } \frac{1}{A} \right) \int_0^\infty W_{(G \text{ or } L)}(f) \frac{1 - \cos(\omega \tau)}{\omega^2} df. \quad (2.42a)$$

Also, cf. [1; (14.2), (14.14c)],

$$\varrho(\tau)_G \Big|_{A \text{ PM}} = \left(\sigma_G^2 \text{ or } \frac{1}{A} \varrho_{2A} \right) \{k(0) - k(\tau)\}_{(G \text{ or } L)} \quad (D_0 = D_P) \quad (2.42b)$$

and

$$\varrho_0 \rightarrow \varrho_0 \Big|_{A \text{ FM}} = \int_0^\infty W_A(f) df / \omega^2 \quad \text{or} \quad \varrho_0 \Big|_{A \text{ PM}} = \varrho_{2A}. \quad (2.42c)$$

For our numerical examples, we use the RC-spectrum of [1; section 14.1-3], where now

$$k_L(\zeta) = \exp(-b|\zeta|) , \quad k_G(\zeta) = \exp(-|\zeta|) , \quad \zeta = \tau \Delta\omega_N , \quad (2.43)$$

and therefore

$$\begin{aligned} \text{FM: } D_F^2 \varrho(\tau)_A &= \frac{1}{Ab^2} \left(\mu_F^2 \right)_A [\exp(-b|\zeta|) + b|\zeta| - 1] ; \\ D_F^2 \varrho(\tau)_G &= \Gamma' \left(\mu_F^2 \right)_A [\exp(-|\zeta|) + |\zeta| - 1] ; \\ \left(\mu_F^2 \right)_A &= \varrho_{2A} D_F^2 / \Delta\omega_N^2 ; \end{aligned} \quad (2.44a)$$

$$\begin{aligned} \text{PM: } D_P^2 \varrho(\tau)_A &= \frac{1}{A} \left(\mu_P^2 \right)_A [1 - \exp(-b|\zeta|)] ; \\ D_P^2 \varrho(\tau)_G &= \Gamma' \left(\mu_P^2 \right)_A [1 - \exp(-|\zeta|)] ; \\ \left(\mu_P^2 \right)_A &= D_P^2 \varrho_{2A} , \end{aligned} \quad (2.44b)$$

with $b (> 0)$ a dimensionless quantity, as is ζ . The quantity $\Delta\omega_N$ is the bandwidth of the modulating (Gauss) noise, cf. (2.43). Note, also, that

$$\Gamma' \left(\mu_F^2 \right)_A = \sigma_G^2 D_F^2 / \Delta\omega_N^2 = \left(\mu_F^2 \right)_G ; \quad \Gamma' \left(\mu_P^2 \right)_A = \sigma_G^2 D_P^2 = \left(\mu_P^2 \right)_G . \quad (2.45)$$

The quantities $(\mu_{F,P}^2)_{()}$ are the respective modulation indexes for FM and PM, cf. [1; chapter 14].

Finally, we have for ρ in (2.3), now with $\Delta R = 0$,

$$\rho(\tau) \rightarrow \rho(\zeta) = \begin{cases} 1 - \frac{\beta|\zeta|}{\Delta\omega_N} & \text{for } \frac{\beta|\zeta|}{\Delta\omega_N} \leq 1 \\ 0 & \text{for } \frac{\beta|\zeta|}{\Delta\omega_N} > 1 \end{cases} . \quad (2.46)$$

Putting the above (2.43), (2.44) in (2.42), we now specialize our results,

$$K_Y(\tau)_{A+G} = \frac{1}{2} A_0^2 k_0(\tau) \cos(\omega_0 \tau) , \quad \text{with } k_0(0) = 1 , \quad (2.47)$$

to the normalized covariance $k_0(\tau)$, respectively, for FM and PM, and their associated spectra. We have for these carriers modulated by a sum of Gaussian and Class A noise:

I. FREQUENCY MODULATION

$$k_0(\zeta)_{FM} = \exp \left[-\Gamma' \left(\mu_F^2 \right)_A [\exp(-|\zeta|) + |\zeta| - 1] - A(2-\rho) \right. \\ \left. + A\rho \exp \left(-\frac{1}{Ab^2} \left(\mu_F^2 \right)_A [\exp(-b|\zeta|) + b|\zeta| - 1] \right) \right] , \quad (2.48)$$

with $\rho(\tau)$ given by (2.46). Here, $\Omega_0|_{FM} \rightarrow \infty$ in (2.42). Since

$$\lim_{\zeta \rightarrow \infty} k_0(\zeta)_{FM} = 0 ,$$

there is no dc in k_{0-FM} , and hence all the original carrier power ($\sim A_0^2/2$) is distributed into the sideband continuum for this highly nonlinear modulation, as expected [1; section 14.1-2].

The associated intensity spectrum for $k_0|_{FM}$ is defined by

$$W(\hat{\omega})_{A+G}|_{FM} = \int_0^\infty k_0(\zeta)_{FM} \cos(\hat{\omega}\zeta) d\zeta , \quad \hat{\omega} \equiv \frac{\omega - \omega_0}{\Delta\omega_N} , \quad (2.49)$$

which is determined by a direct cosine transform of $k_0(\zeta)_{FM}$. See appendix A.6 ff.

II. PHASE MODULATION

$$k_O(\zeta)_{PM} = \exp \left[-\Gamma' \left(\mu_P^2 \right)_A [1 - \exp(-|\zeta|)] + 2A(1-\rho) \exp \left(-\frac{1}{A} \left(\mu_P^2 \right)_A \right) \right. \\ \left. - A(2-\rho) + A\rho \exp \left(-\frac{1}{A} \left(\mu_P^2 \right)_A [1 - \exp(-b|\zeta|)] \right) \right] ,$$

(2.50)

with $\rho(\zeta)$ again given by (2.46). We note that

$$k_O(0)_{PM} = 1 , \quad (2.51a)$$

as before; that is, the total (normalized) intensity is unity. Also

$$k_O(\infty)_{PM} = \exp \left[-\Gamma' \left(\mu_P^2 \right)_A - 2A \left(1 - \exp \left(-\frac{1}{A} \left(\mu_P^2 \right)_A \right) \right) \right] : \quad (2.51b)$$

this is the fraction of the power remaining in the carrier, so that

$$k_O(0)_{PM} - k_O(\infty)_{PM} = 1 - (2.51b) , \quad (2.51c)$$

which is the fraction of the power distributed in the sideband continuum.

The associated intensity spectrum of the sideband continuum is determined from

$$w(\hat{\omega})_{A+G} \Big|_{PM-cont} = \int_0^\infty \left[k_O(\zeta)_{PM} - k_O(\infty)_{PM} \right] \cos(\hat{\omega}\zeta) d\zeta . \quad (2.52)$$

See section 3 ff. for examples and appendix A.5 for the evaluation methods.

Finally, in the equivalent Gaussian cases (Gauss noise modulation of equal intensity and basic spectrum, e.g.,

$$\Gamma' \left(\mu_F^2 \right)_A \rightarrow \Gamma' \left(\hat{\mu}_F^2 \right)_A = \Gamma' \mu_{FA}^2 (1 + \Gamma') \quad \text{and} \quad k_G \rightarrow k_L ,$$

we see that (2.48), (2.50) reduce to

$$k_O(\zeta)_{\text{FM-Gauss}} = \exp \left[- \left(\hat{\mu}_F^2 \right)_A \Gamma' \left[\exp(-b|\zeta|) + b|\zeta| - 1 \right] / b^2 \right] ,$$

$$\left(\hat{\mu}_F^2 \right)_A = (1 + \Gamma') \left(\mu_F^2 \right)_A ; \quad (2.53a)$$

$$k_O(\zeta)_{\text{PM-Gauss}} = \exp \left[- \left(\hat{\mu}_P^2 \right)_A \Gamma' \left[1 - \exp(-b|\zeta|) \right] \right] ,$$

$$\left(\hat{\mu}_P^2 \right)_A = (1 + \Gamma') \left(\mu_P^2 \right)_A , \quad (2.53b)$$

with spectra obtained as before, from (2.49) and (2.52).

3. NUMERICAL ILLUSTRATIONS AND DISCUSSION

It is convenient to discuss the general results, namely the effects of (ZMNL) nonlinear rectifiers on, and modulation by, a mixture of Gaussian and non-Gaussian noise processes and fields, from the specific numerical calculations presented here in figures 3.1 - 3.10. These constitute a representative selection from the universe of possible parameter states [cf. "Summary of Normalized Parameters" and section 2, preceding]. This is done here on a per-figure basis, as noted below. In each case, the dc component is removed: only the covariance or continuous spectrum is calculated. We recall that there are two cases to distinguish: Case A, $\tau' = \tau - \Delta R/c_0$, a 2-element array; and Case B, $\tau' = \tau = t_2 - t_1$, a preformed beam. See figure 2.1 and (2.3a).

All spectra shown here are normalized to have area (under the spectrum level) of unity, i.e., the spectral normalization is obtained by dividing the spectrum by the value of the associated covariance at its origin. The normalization of the covariances themselves is obtained by dividing by the value at $\hat{f}=0$ or $\hat{\Delta R}=0$.

I. GAUSS NOISE ALONE

FIGURE 3.1

This figure shows the normalized temporal covariance ($\hat{\Delta R}=0$) for both the input and output of a ZMNL half-wave v -th law ($v \geq 0$) detector, when $v = 0, 1, 2$ and when only Gaussian noise ($A=0$) is applied to these nonlinear devices. These curves are based on (2.11) with (2.7a), where $Y_a = k_G$, (2.96), with $\Delta \hat{\omega}_G \equiv \Delta \omega_G / \beta = 5$ here. The normalization is with respect to the covariance maximum; e.g., the normalized covariance shown in figure 3.1 is obtained from $[(2.11)/(2.11)_{\hat{f}=0}]$, $\hat{\Delta R}=0$. These results apply for both cases A,B of figure 2.1, where now $\hat{f}' = \hat{f}$, since $\hat{\Delta R}=0$, cf. (2.3a) and remarks.

As expected (cf. [1; chapter 13]), the general nonlinearity (2.6), $v \geq 0$, contracts the covariance, which is equivalent to spreading the spectrum vis-à-vis the input, cf. figure 3.2, below. Moreover, the greater the distortion ($v=0,2$), usually the greater are these effects. [See appendix A.1.]

FIGURE 3.2

This is the same situation as shown in figure 3.1, except that the normalized intensity (frequency) spectrum is calculated now [cf. section 2.2-5, Case II, (2.39)] with $\hat{M}_y(0, \hat{f})$, (2.11), used in (2.39). Observe the greatly broadened spectra, particularly at the low spectral levels, where the greater spread occurs for the "super-clipper", $v=0$, cf. remarks, figure 3.1; also, appendix A.3.

FIGURE 3.3

For the same purely Gaussian field above, cf. (2.11) and (2.96), with $\hat{f}, \hat{f}'=0$, the spatial covariance is calculated, with parameters $\Delta_L/\Delta_G = 5^{1/2}$, using (2.11) as before. The normalization is with respect to the covariance at $\hat{R}=0$. Again, one observes the same kind of contraction in the covariance as noted in figure 3.1. [See appendix A.2.]

FIGURE 3.4

This is the wavenumber analogue of the frequency spectrum of figure 3.2, now with $\hat{f}', \hat{f}=0$, and is obtained from (2.35a,b) with $\Delta_L/\Delta_G = 5^{1/2}$. The rectification operation similarly spreads the wavenumber spectrum, with the greatest distortion ($v=0$) yielding the greatest wavenumber spread, as expected from the corresponding contraction of the associated covariance, cf. figure 3.3 above. [See appendix A.4.]

CLASS A PLUS GAUSS NOISE

FIGURE 3.5

The temporal covariance here is given by the general result (2.7), with the associated relations (2.7), (2.8), (2.9), wherein $\hat{\Delta R}=0$, so that $\hat{t}=\hat{t}'=t_2-t_1$, as before, and where $B_v|_a$, (2.7a), is now given analytically by (2.10) for $v = 0,1,2$. Here, the parameter values are $\Delta\hat{\omega}_G \equiv \Delta\omega_G/\beta = 5^{1/2}$, as before, now with $A=0.2$, $\Gamma'_A=10^{-3}$, $\Delta\omega_L/\beta \equiv \Delta\hat{\omega}_L = 1$ for the Class A non-Gaussian noise component, typically.

Again, for the super-clipper ($v=0$), the contraction in the normalized covariance is greatest, cf. figure 3.1. But the contribution of the comparatively strong non-Gaussian component exaggerates this effect. [See appendix A.1.]

FIGURE 3.6

The corresponding intensity (frequency) spectrum ($\hat{\Delta R}=0$), obtained from (2.7) in (2.39), however, shows a fine-structure not exhibited when Gauss noise alone ($A=0$) is applied to these ZMNL devices. The spectral levels for the case $v=0$, ($A=0$) and ($A>0$), cf. figure 3.2 with figure 3.6, are approximately the same, whereas the other inputs, cases $v=1,2$, are much elevated as f becomes larger, again due to the presence of the structured Class A noise, when $\beta\bar{T}_s \leq 1$, cf. (2.3): on the average, the original Class A "signals" are of comparatively short duration, or spectrally wide to begin with, so that clipping further spreads the spectrum. [See appendix A.3.]

FIGURE 3.7

The spatial covariance when Class A noise is added to the Gaussian input shows analogous behavior, cf. figures 3.3 and 3.5: the covariance is compressed vis-à-vis the input, but more so than in the Gauss-alone situations. Again, (2.7) - (2.10) are employed. [See appendix A.2.]

FIGURE 3.8

The corresponding wavenumber (intensity) spectrum with Class A noise and the Gaussian component, obtained from (2.7) - (2.10) in (2.39), is shown here. Comparison with figure 3.4 indicates a broader spectral input, due to the non-Gaussian component, but a relatively narrower output, although the latter is still noticeably spread vis-à-vis the original input. [See appendix A.4.]

FIGURE 3.9

Finally, we consider the angle-modulation cases described in section 2.2-6 above, where weak to strong angle modulations ($\mu \sim 1$ to 50) by Class A noise, with a weak ($\Gamma' = 10^{-3}$) Gaussian modulation component, is employed.

For phase modulation by non-Gaussian noise, based on (2.50) with (2.44b), (2.45), (2.46), the resulting normalized intensity (frequency) spectra are obtained by applying (2.50) to (2.52), where $f = \hat{\omega}/2\pi$; $\hat{\omega} = (\omega - \omega_0)/\Delta\omega_N$, cf. (2.49). Note the "spike" at $f \sim 0.1$, followed by a variety of sidelobes which rise as the phase modulation index μ_p increases. The spike is now bounded at $f \approx 0.8$, at the -10 dB level, when $\mu_p = 50$. As expected, the larger indexes (μ_p) produce broader spectra. [See appendix A.5.]

FIGURE 3.10

For frequency modulation by non-Gaussian noise, from (2.48) with (2.49) and (2.44a), the corresponding intensity (frequency) spectra again exhibit a continuous spike ($f < 0.1$). With small modulation indexes (μ_F), the spectra are less broad than for the larger indexes, as expected. The non-Gaussian noise component dominates the spectrum here. [See appendix A.6.]

EXTENSIONS

Other situations where the second-order Class A probability density functions may be applied are noted in [2] and [3]. We list some of the extensions of the analysis to the following "classical" problems:

- 1) The inclusion of representative signals, with Gauss and non-Gauss (Class A) noise, in the problems already treated here (section 2);
- 2) The case of the full-wave square-law rectifier, with both Class A and B noise, as well as Gauss noise;
- 3) The extension of 2) to include general broadband and narrowband signals;
- 4) The calculation of signal-to-noise ratios and deflection criteria, cf. [1; section 5.3-4].
- 5) Covariances and spectra for ZMNL system outputs, with signals as well as non-Gaussian noise inputs;
- 6) The rôle of the electromagnetic (or acoustic) interference (EMI or AcI) scenario, cf. [5; section 2B,5];
- 7) Evaluation of the large (FM,PM) indexes, or asymptotically Gaussian cases, cf. [12].

Further opportunities to extend the classical theory [2],[3], now with non-Gaussian noise inputs, are evident from the examples and methods described in [1; chapters 5, 12 - 16], for instance.

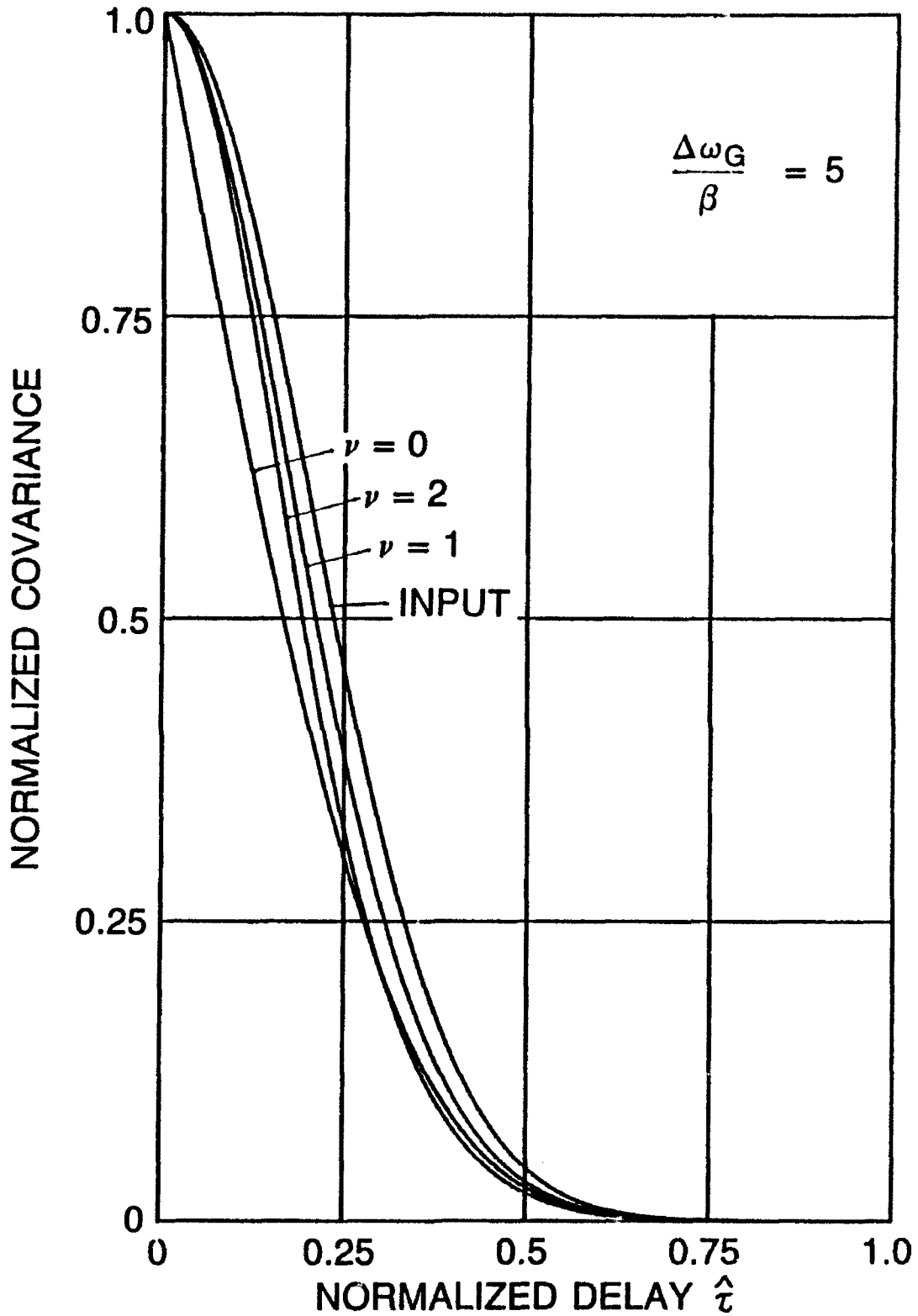


FIGURE 3.1 TEMPORAL COVARIANCE (FOR $\hat{\Delta R}=0$); GAUSS NOISE ONLY;
CF. (2.11) WITH (2.7a), (2.9b), AND APPENDIX A.1

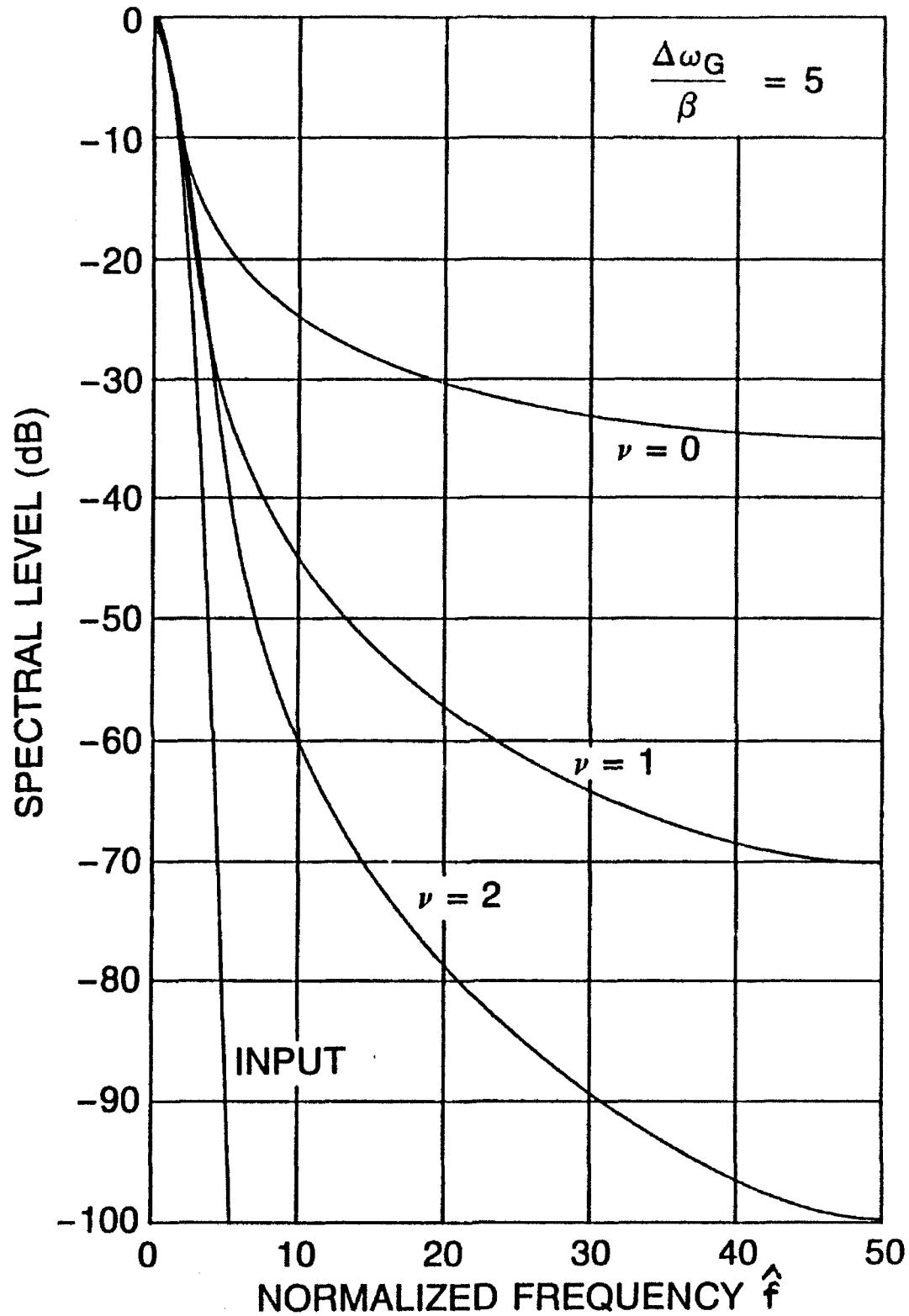


FIGURE 3.2 FREQUENCY (INTENSITY) SPECTRUM (FOR $\hat{\Delta R}=0$); GAUSS NOISE ONLY; CF. (2.39), USED WITH (2.11), AND APPENDIX A.3

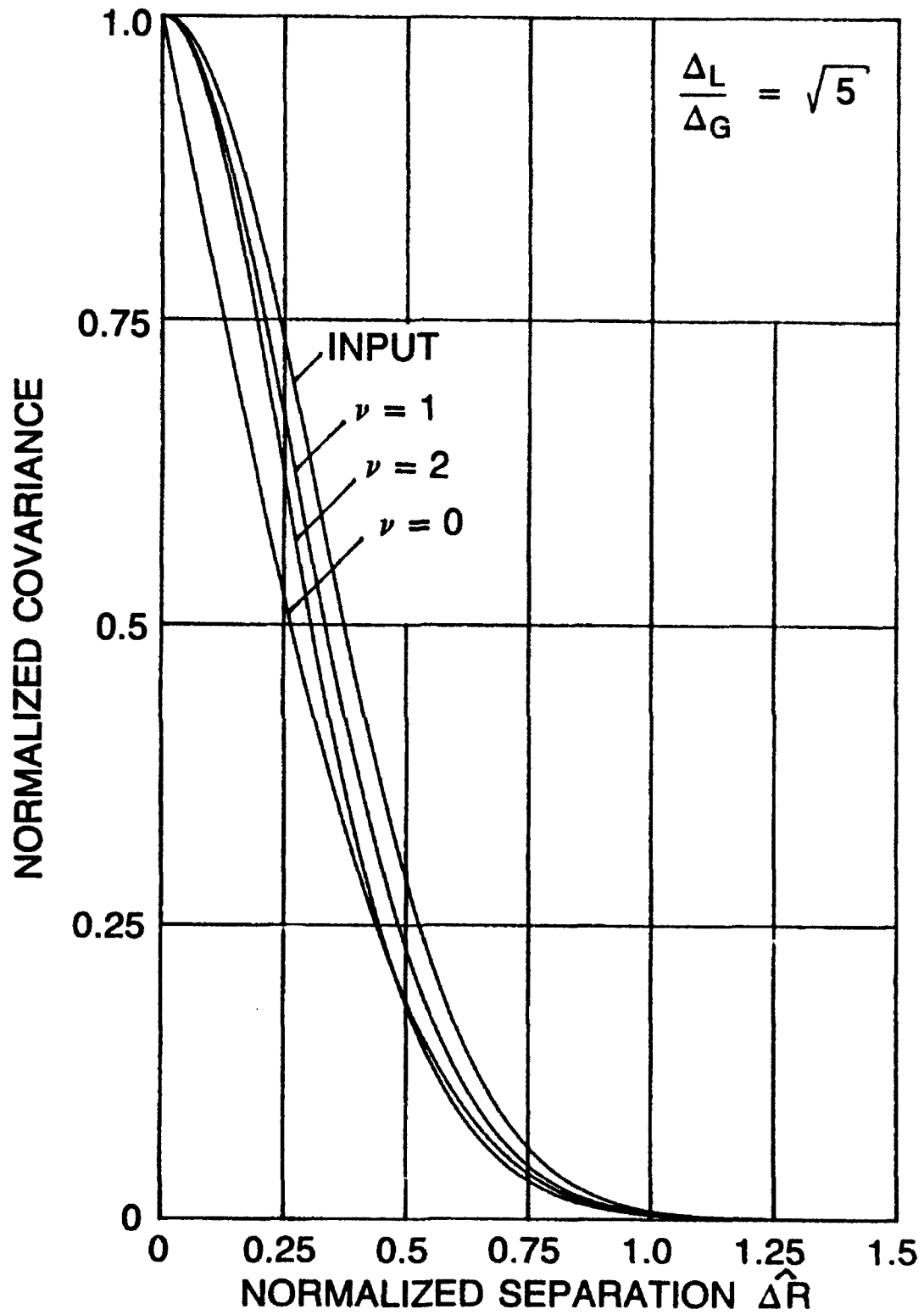


FIGURE 3.3 SPATIAL COVARIANCE (FOR $\hat{t}', \hat{t}=0$); GAUSS NOISE ONLY; CF. (2.11), (2.7b), AND APPENDIX A.2

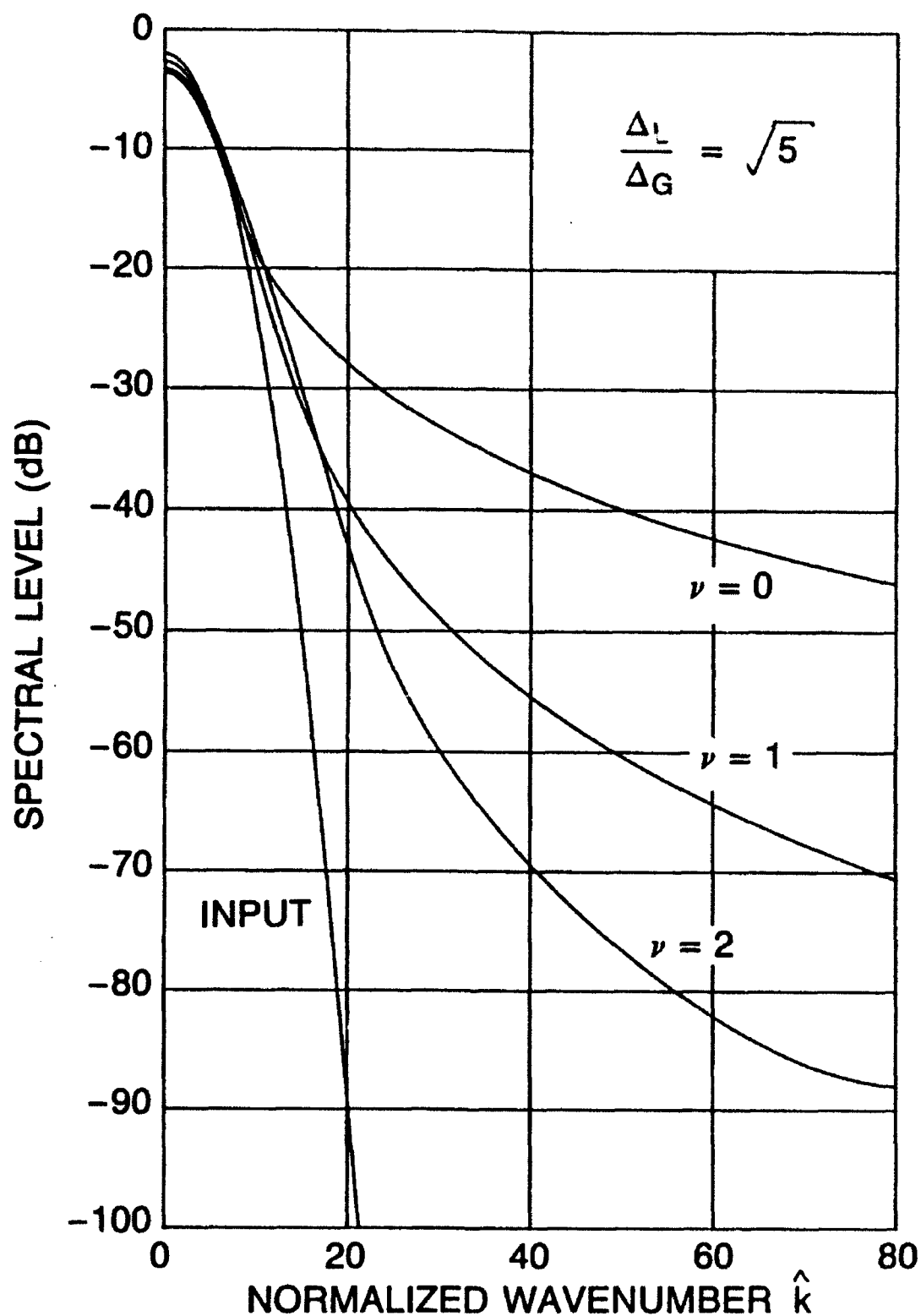


FIGURE 3.4 WAVENUMBER (INTENSITY) SPECTRUM (FOR $\hat{r}', \hat{r}=0$); GAUSS NOISE ONLY; CF. (2.35a,b) WITH (2.11), (2.7b), AND APPENDIX A.4

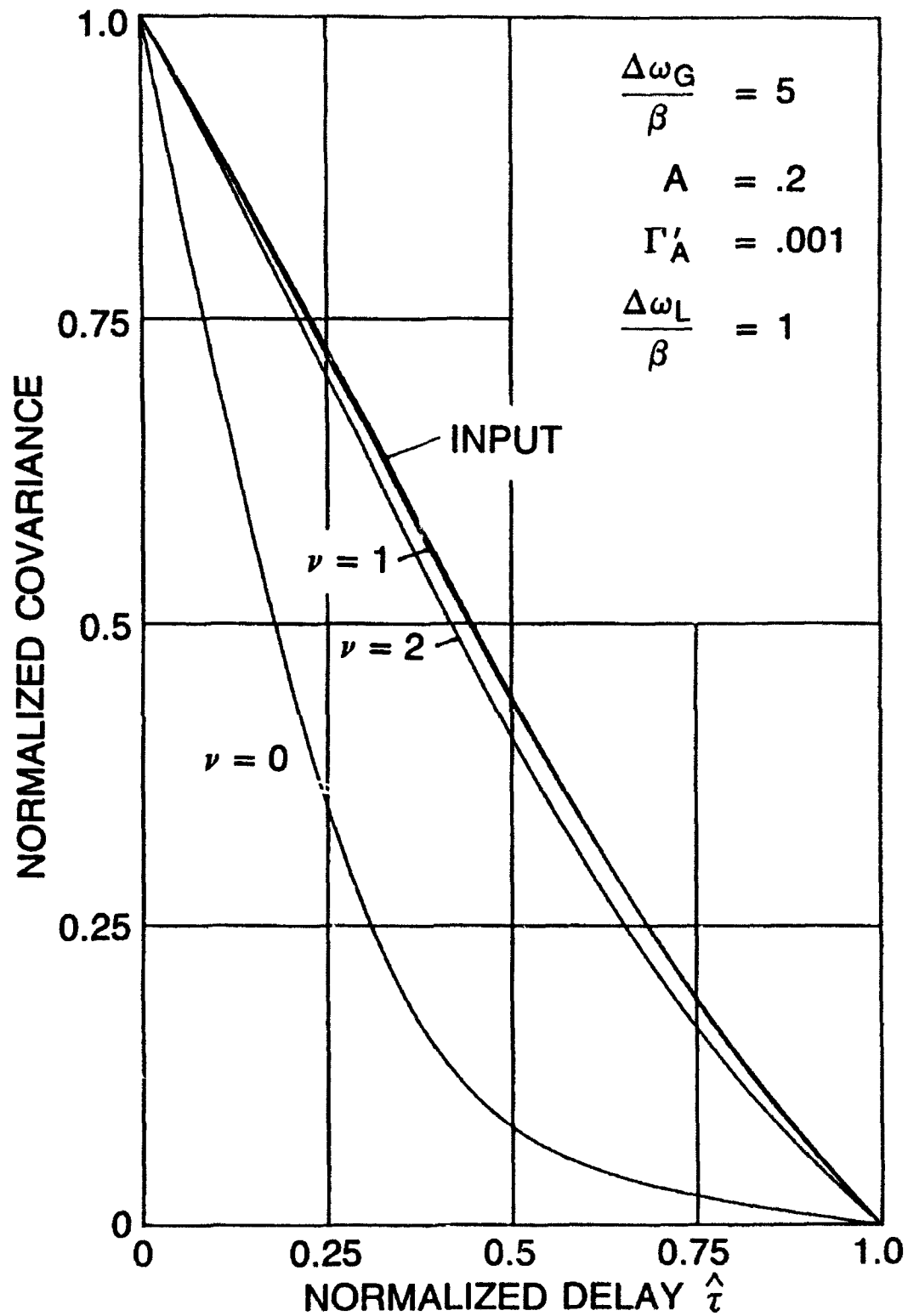


FIGURE 3.5 TEMPORAL COVARIANCE (FOR $\hat{\Delta R}=0$); CLASS A AND GAUSS NOISE; CF. (2.7)-(2.9) AND APPENDIX A.1

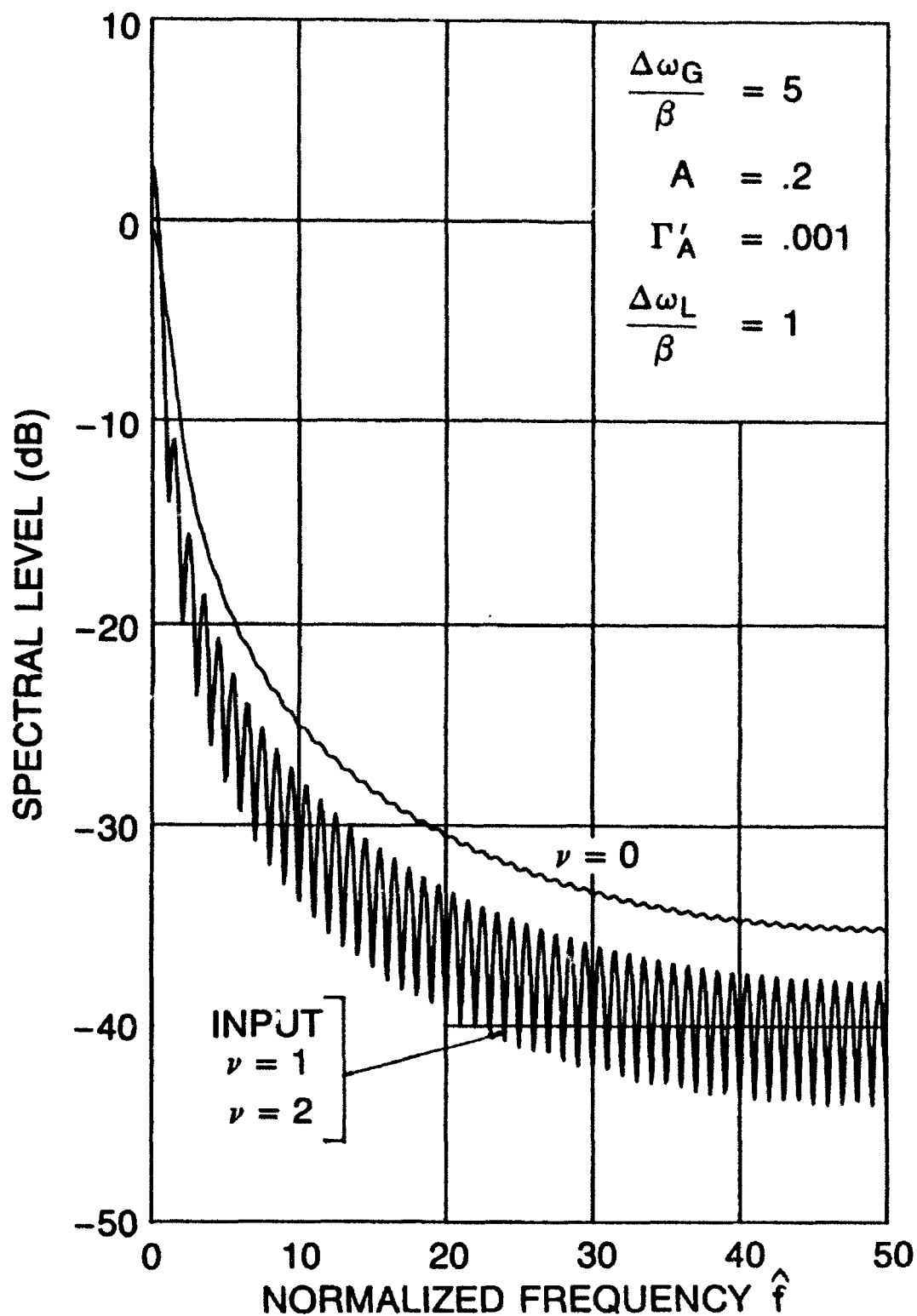


FIGURE 3.6 FREQUENCY (INTENSITY) SPECTRUM (FOR $\hat{\Delta R}=0$); CLASS A AND GAUSS NOISE; CF. (2.7) IN (2.39) WITH APPENDIX A.3

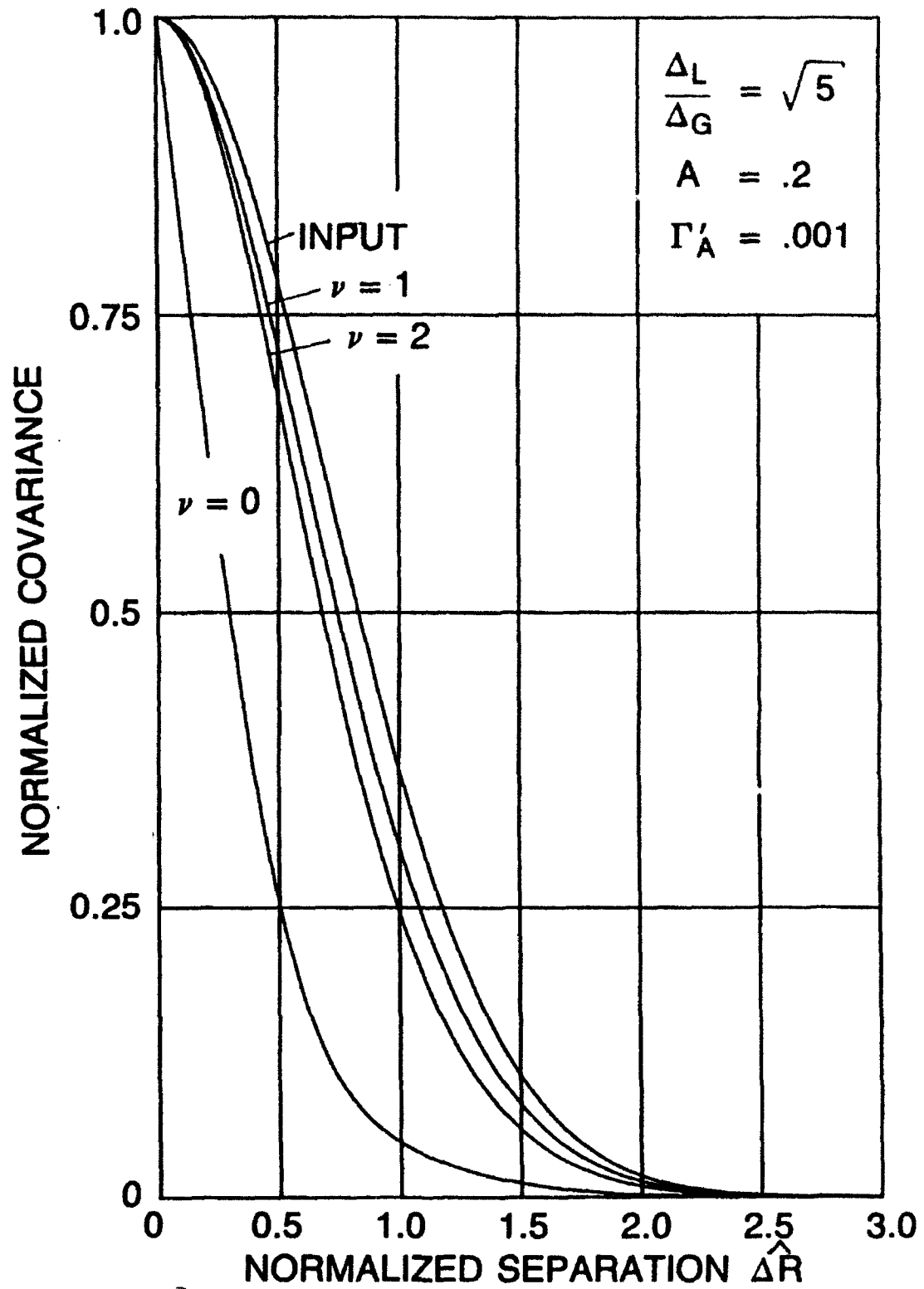


FIGURE 3.7 SPATIAL COVARIANCE (FOR $\hat{r}', \hat{r}=0$); CLASS A AND GAUSS NOISE; CF. (2.7)-(2.10) WITH APPENDIX A.2

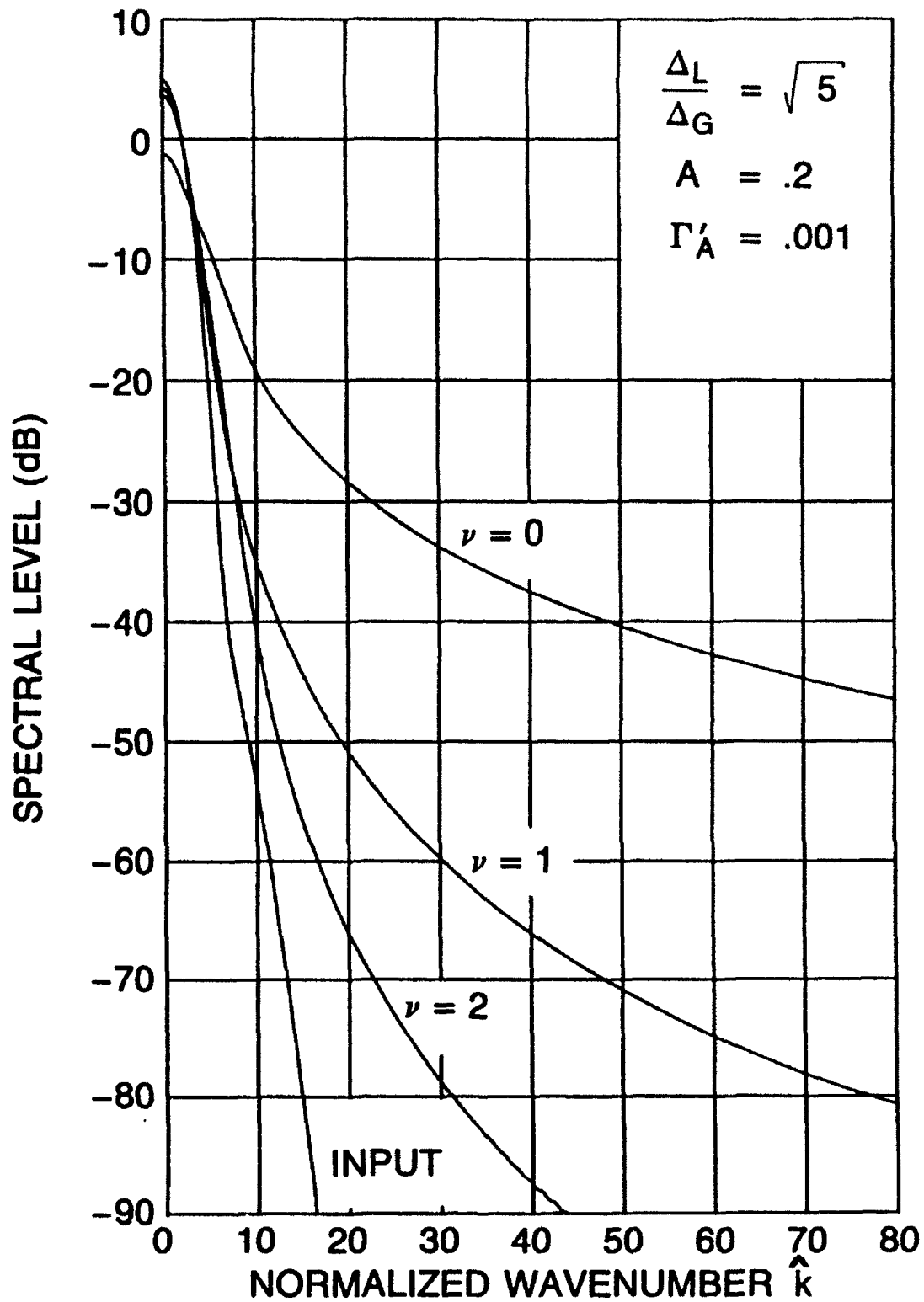


FIGURE 3.8 WAVENUMBER SPECTRUM (FOR $\hat{t}', \hat{t}=0$); CLASS A AND GAUSS NOISE; CF. (2.7)-(2.10) IN (2.38a,b) AND APPENDIX A.4

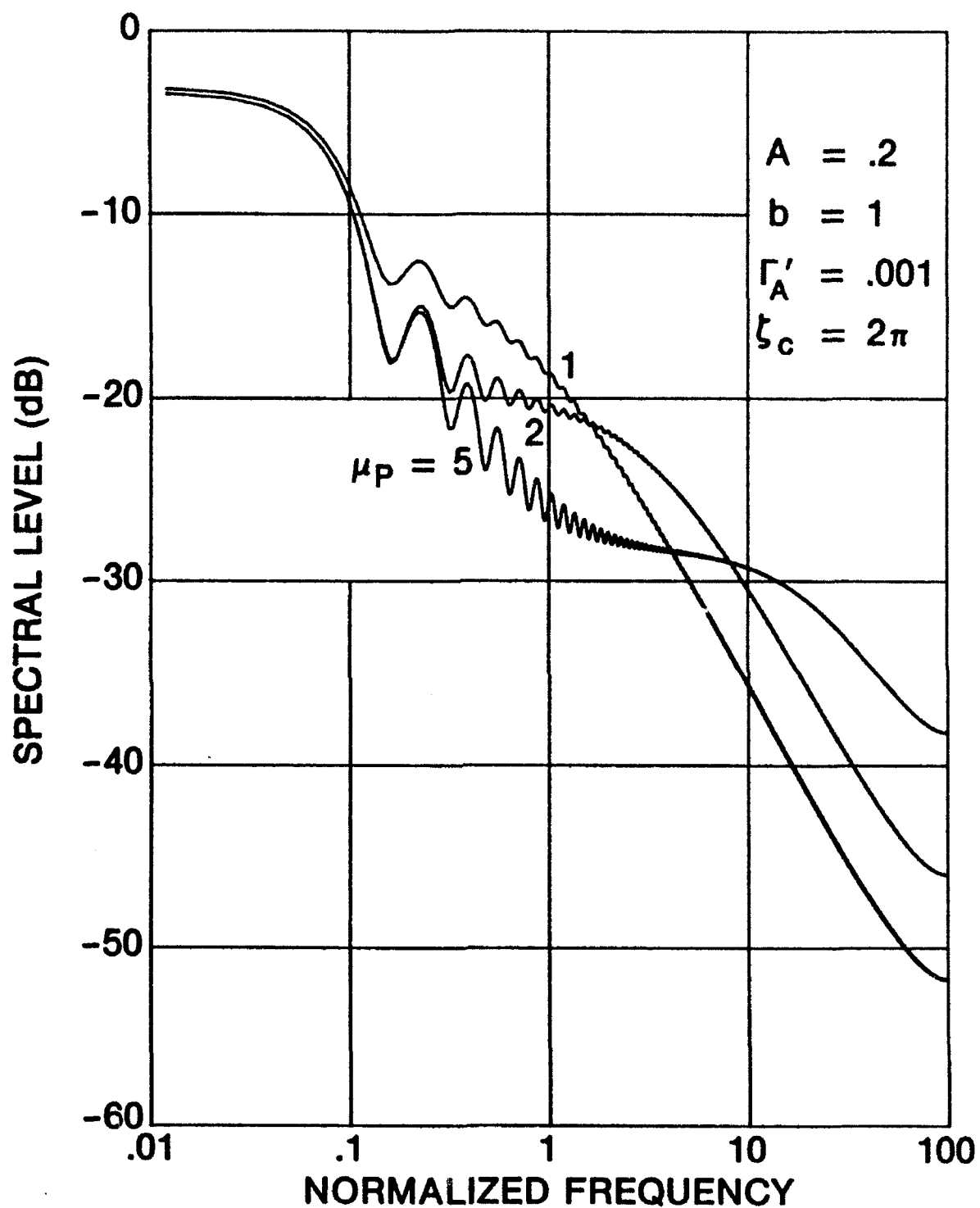


FIGURE 3.9a PHASE MODULATION (INTENSITY) SPECTRUM FOR INDEX $\mu_p=1,2,5$, CLASS A AND GAUSS NOISE; CF. (2.50) WITH (2.44b), (2.45), (2.46) IN (2.52), AND APPENDIX A.5

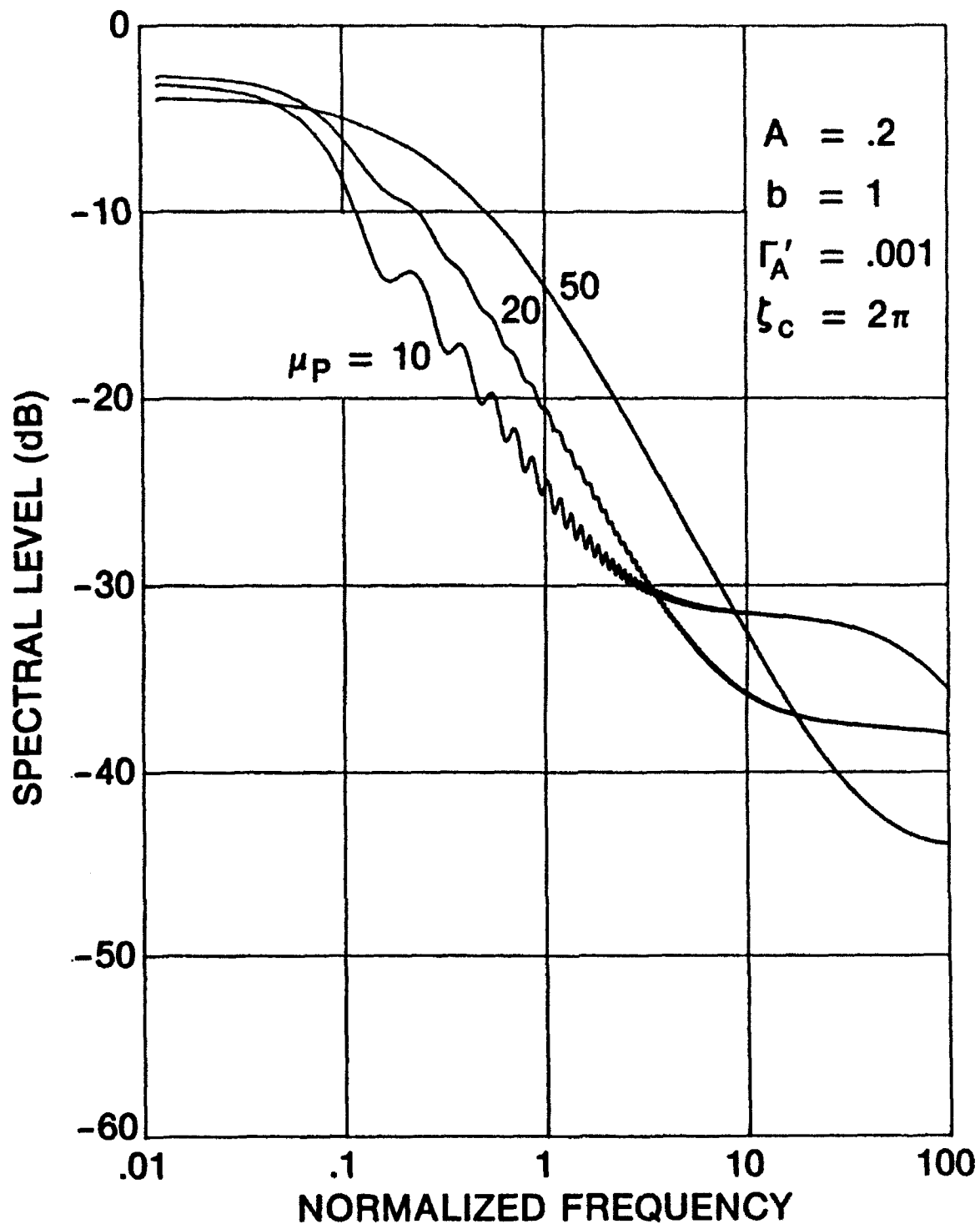


FIGURE 3.9b PHASE MODULATION (INTENSITY) SPECTRUM FOR INDEX $\mu_P=10,20,50$, CLASS A AND GAUSS NOISE; CF. (2.50) WITH (2.44b), (2.45), (2.46) IN (2.52), AND APPENDIX A.5

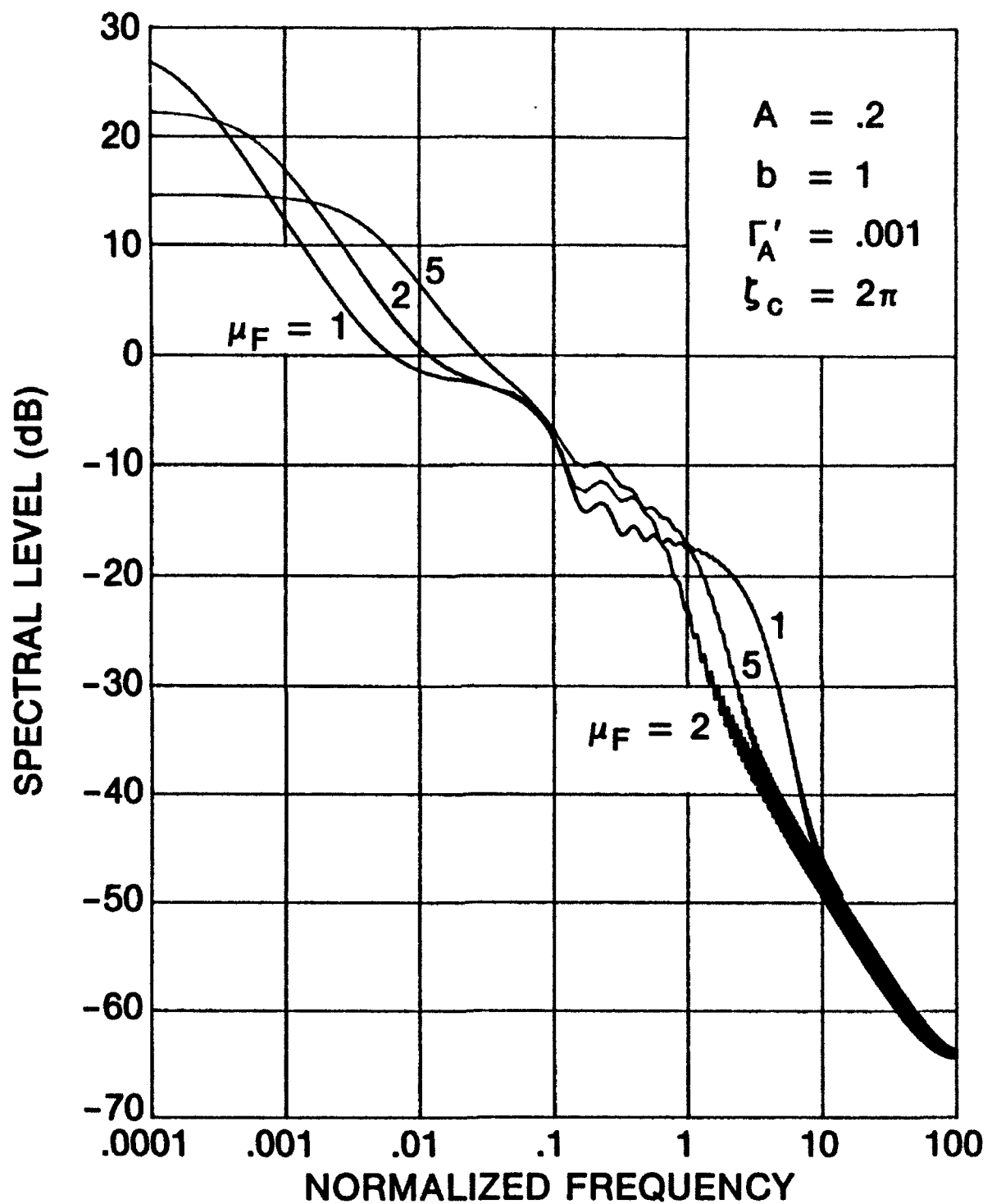


FIGURE 3.10a FREQUENCY MODULATION (INTENSITY) SPECTRUM
 FOR INDEX $\mu_F=1,2,5$, CLASS A AND GAUSS NOISE;
 CF. (2.48) WITH (2.49), (2.44a), AND APPENDIX A.6

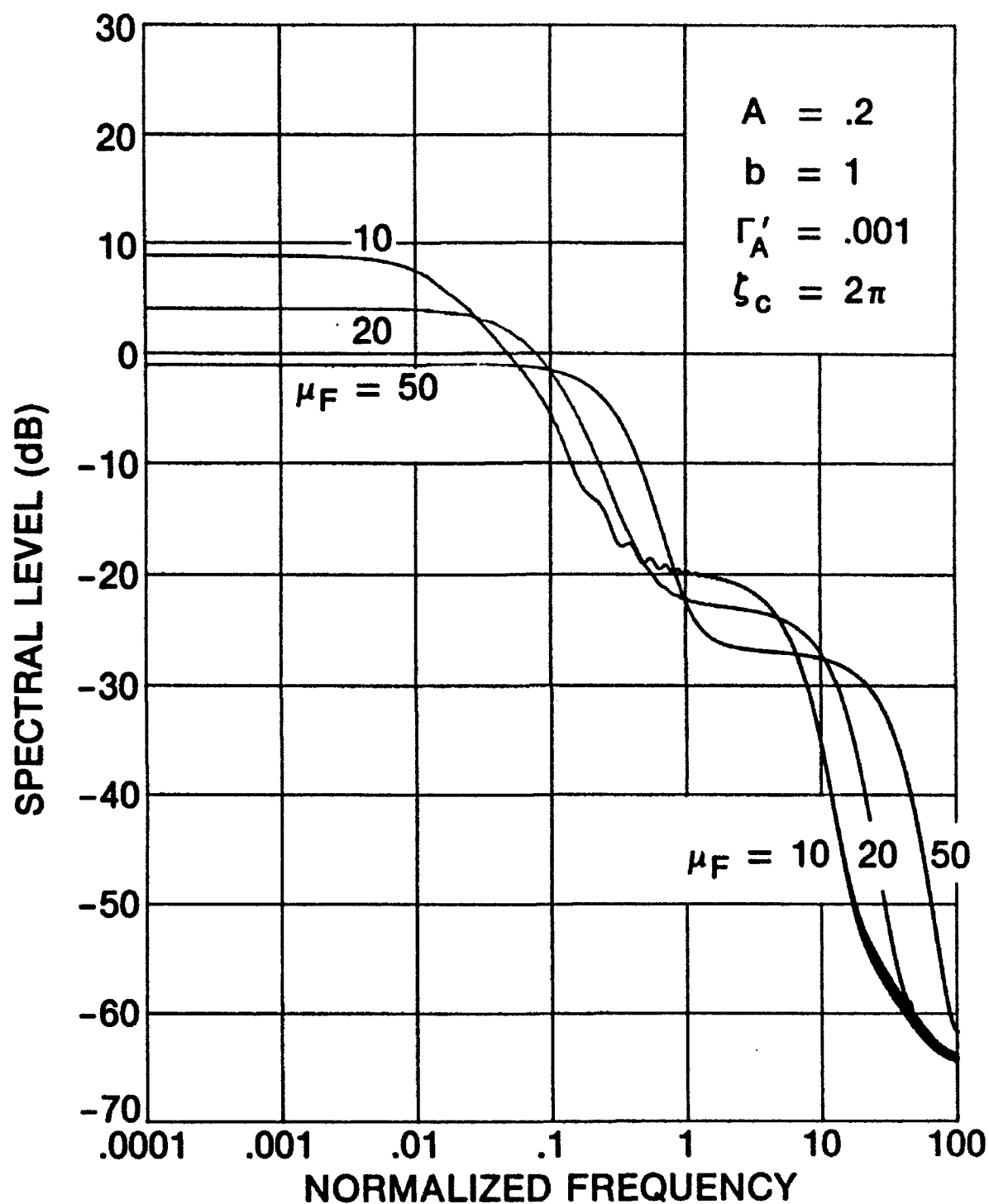


FIGURE 3.10b FREQUENCY MODULATION (INTENSITY) SPECTRUM
 FOR INDEX $\mu_F=10,20,50$, CLASS A AND GAUSS NOISE;
 CF. (2.48) WITH (2.49), (2.44a), AND APPENDIX A.6

PART II. MATHEMATICAL AND COMPUTATIONAL PROCEDURES

4. SOME PROPERTIES OF THE COVARIANCE FUNCTION

In this section, we collect some useful relations for the covariance and auxiliary functions encountered in the numerical evaluation. These are necessary for rapid computation of the multiple series involved here and also serve as checks on the numerical procedures employed.

4.1 SIMPLIFICATION AND EVALUATION OF $B_v(Y)$

The function $B_v(Y)$ is defined by the following combination of hypergeometric functions:

$$B_v(Y) = \Gamma^2\left(\frac{v+1}{2}\right) F\left(-\frac{v}{2}, -\frac{v}{2}; \frac{1}{2}; Y^2\right) + \\ + 2 \Gamma^2\left(\frac{v}{2} + 1\right) Y F\left(\frac{1-v}{2}, \frac{1-v}{2}; \frac{3}{2}; Y^2\right) \quad \text{for } Y^2 \leq 1. \quad (4.1)$$

For the upper F function in (4.1), we have [1; (A.1.39b)]

$$v = 0, \quad F\left(0, 0; \frac{1}{2}; Y^2\right) = 1; \\ v = 1, \quad F\left(-\frac{1}{2}, -\frac{1}{2}; \frac{1}{2}; Y^2\right) = Y \arcsin(Y) + \left(1 - Y^2\right)^{\frac{1}{2}}; \\ v = 2, \quad F\left(-1, -1; \frac{1}{2}; Y^2\right) = 1 + 2Y^2; \quad (4.2)$$

where arcsin is the principal value inverse sine function. On the other hand, for the latter F function in (4.1), we have [1; (A.1.39a) and (A.1.39c)]

$$v = 0, \quad F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; Y^2\right) = \frac{\arcsin(Y)}{Y} ;$$

$$v = 1, \quad F\left(0, 0; \frac{3}{2}; Y^2\right) = 1 ;$$

$$v = 2, \quad F\left(-\frac{1}{2}, -\frac{1}{2}; \frac{3}{2}; Y^2\right) = \frac{3}{4}\left(1 - Y^2\right)^{\frac{1}{2}} + \frac{1 + 2Y^2}{4Y} \arcsin(Y). \quad (4.3)$$

When these quantities are substituted in the above expression for $B_v(Y)$, we find the following relatively simple relations:

$$B_0(Y) = \pi + 2 \arcsin(Y) ,$$

$$B_1(Y) = Y \arcsin(Y) + \left(1 - Y^2\right)^{\frac{1}{2}} + \frac{\pi}{2}Y ,$$

$$B_2(Y) = \left(\frac{1}{2} + Y^2\right) \left(\frac{\pi}{2} + \arcsin(Y)\right) + \frac{3}{2}Y \left(1 - Y^2\right)^{\frac{1}{2}} . \quad (4.4)$$

These three quantities can be computed simultaneously by the following very compact computer coding in BASIC:

```

Y2=Y*Y
Sq=SQR(1.-Y2)
T=ASN(Y)+1.5707963267948966
B0=T+T
B1=Y*T+Sq
B2=(.5+Y2)*T+1.5*Y*Sq

```

(4.5)

Thus, the rather formidable expression, above, for $B_v(Y)$ can be evaluated by the use of just one square root and one arcsin when $v = 0, 1, 2$.

The following limiting values, which are obvious, are needed for various special cases:

$$\begin{aligned}
B_0(0) &= \pi, & B_0(1) &= 2\pi, \\
B_1(0) &= 1, & B_1(1) &= \pi, \\
B_2(0) &= \pi/4, & B_2(1) &= 3\pi/2, \\
B_3(0) &= 1, & B_3(1) &= 15\pi/4, \\
B_4(0) &= 9\pi/16, & B_4(1) &= 105\pi/8.
\end{aligned} \tag{4.6}$$

These are special cases of

$$B_\nu(0) = r^2 \left(\frac{\nu + 1}{2} \right), \tag{4.7}$$

$$B_\nu(1) = 2 \pi^{\frac{1}{2}} \Gamma\left(\nu + \frac{1}{2}\right), \tag{4.8}$$

the latter following from [10; (15.1.20)].

4.2 LIMITING VALUES OF THE COVARIANCE FUNCTION

The covariance function at normalized separation $\hat{\Delta R}$ and delay $\hat{t}$ is given by (2.7) as

$$\begin{aligned}
\hat{M}_Y(\hat{\Delta R}, \hat{t}) &= \exp[-A(2-\rho)] \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{[A(1-\rho)]^{m_1+m_2}}{m_1! m_2!} \\
&\times \sum_{n=0}^{\infty} \frac{(A\rho)^n}{n!} \left(\frac{n+m_1}{A} + \Gamma'_A \right)^{\nu/2} \left(\frac{n+m_2}{A} + \Gamma'_A \right)^{\nu/2} B_\nu(Y), \tag{4.9}
\end{aligned}$$

where

$$\rho = \rho(\hat{t}) = \max\{0, 1 - |\hat{t}|\}, \tag{4.10}$$

$$Y = Y(m_1, m_2, n) = \frac{\frac{n}{A} k_L + \Gamma'_A k_G}{\left(\frac{n+m_1}{A} + \Gamma'_A \right)^{\frac{1}{2}} \left(\frac{n+m_2}{A} + \Gamma'_A \right)^{\frac{1}{2}}}, \tag{4.11}$$

$$k_L = k_L(\hat{\Delta R}, \hat{t}) = \exp \left[-\hat{\Delta R}^2 - \frac{1}{2} \left(\frac{\Delta \omega_L}{\beta} \right)^2 \hat{t}^2 \right], \quad (4.12)$$

$$k_G = k_G(\hat{\Delta R}, \hat{t}) = \exp \left[- \left(\frac{\Delta_L}{\Delta_G} \right)^2 \hat{\Delta R}^2 - \frac{1}{2} \left(\frac{\Delta \omega_G}{\beta} \right)^2 \hat{t}^2 \right]. \quad (4.13)$$

The functions ρ , k_L , k_G can be replaced by other functional dependencies, if desired. The function $B_v(Y)$ has been considered earlier and considerably simplified for $v = 0, 1, 2$.

4.3 VALUE AT INFINITY

As $\hat{\Delta R}$ or $\hat{t} \rightarrow \pm\infty$, then

$$\rho \rightarrow 0, \quad k_L \rightarrow 0, \quad k_G \rightarrow 0, \quad Y \rightarrow 0. \quad (4.14)$$

(If $|\hat{t}|$ remains less than 1 as $\hat{\Delta R}$ tends to infinity, then ρ does not approach zero; this nuance has been discussed elsewhere in this report.) Then, it follows that

$$\begin{aligned} \hat{M}_Y &\rightarrow \exp(-2A) \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{A^{m_1+m_2}}{m_1! m_2!} \left(\frac{m_1}{A} + \Gamma'_A \right)^{v/2} \left(\frac{m_2}{A} + \Gamma'_A \right)^{v/2} B_v(0) \\ &= B_v(0) \left(\exp(-A) \sum_{m=0}^{\infty} \frac{A^m}{m!} \left(\frac{m}{A} + \Gamma'_A \right)^{v/2} \right)^2, \end{aligned} \quad (4.15)$$

because the sum on n can be terminated with the $n = 0$ term.

The sum on m can be effected in closed form, for $v = 0, 2, 4$, etc., by using the following results:

$$\sum_{m=0}^{\infty} \frac{A^m}{m!} = \exp(A), \quad (4.16)$$

$$\sum_{m=0}^{\infty} \frac{A^m}{m!} m = \sum_{m=1}^{\infty} \frac{A^m}{(m-1)!} = A \exp(A) , \quad (4.17)$$

$$\begin{aligned} \sum_{m=0}^{\infty} \frac{A^m}{m!} m^2 &= \sum_{m=1}^{\infty} \frac{A^m}{(m-1)!} (m-1+1) \\ &= \sum_{m=2}^{\infty} \frac{A^m}{(m-2)!} + \sum_{m=1}^{\infty} \frac{A^m}{(m-1)!} = (A^2 + A) \exp(A) . \end{aligned} \quad (4.18)$$

There follows

$$\hat{M}_Y(\infty) = \begin{cases} \pi & \text{for } v = 0 \\ \frac{\pi}{4} (1 + \Gamma'_A)^2 & \text{for } v = 2 \\ \frac{9\pi}{16} \left[\frac{1}{A} + (1 + \Gamma'_A)^2 \right]^2 & \text{for } v = 4 \end{cases} . \quad (4.19)$$

The case for $v = 1$ requires a numerical summation, once A and Γ'_A are specified. When these limiting values are subtracted from the correlation function, we obtain the covariance function.

4.4 VALUE AT THE ORIGIN

For $\hat{\Delta R} = 0$, $\hat{t} = 0$, then

$$\rho = 1, \quad k_L = 1, \quad k_G = 1, \quad (4.20)$$

and

$$\hat{M}_Y(0,0) = \exp(-A) \sum_{n=0}^{\infty} \frac{A^n}{n!} \left(\frac{n}{A} + \Gamma'_A \right)^v B_v(1), \quad (4.21)$$

because the sums on m_1 and m_2 can be terminated with the zero terms, thereby also leading to $Y = 1$.

The sum on n can be accomplished in closed form, for $v = 0, 1, 2$, etc., by using results given earlier. There follows

$$\hat{M}_Y(0,0) = \begin{cases} 2\pi & \text{for } v = 0 \\ \pi(1 + \Gamma'_A) & \text{for } v = 1 \\ \frac{3}{2}\pi \left[\frac{1}{A} + (1 + \Gamma'_A)^2 \right] & \text{for } v = 2 \end{cases}. \quad (4.22)$$

PART III. APPENDICES AND PROGRAMS

APPENDIX A.1 - EVALUATION OF COVARIANCE FUNCTION
FOR ZERO SEPARATION ($\hat{\Delta R} = 0$)

A program for the numerical evaluation of covariance $\hat{M}_Y(\hat{\Delta R}, \hat{r})$ for $\hat{\Delta R} = 0$ is contained in this appendix. Inputs required of the user are A , Γ'_A , $(\Delta\omega_L/\beta)^2$, $(\Delta\omega_G/\beta)^2$, $\delta(\hat{r})$, $N(\hat{r})$, in lines 20 - 70. Since we are generally interested in values of A less than 1, the series for $\hat{M}_Y$ in (4.9) will not have to be taken to very large values of m_1 , m_2 , n ; accordingly, the values of $\{A^k/k!\}$ are tabulated once in lines 260 - 300 with a tolerance of $1E-10$ set in line 80.

The values of the covariance at infinity, as given by (4.19), are computed and subtracted in lines 220 - 240 and 400 - 420; this is in anticipation of taking a Fourier transform of a covariance function which decays to zero for large arguments $\hat{\Delta R}$.

The functions $B_Y(Y)$ and $\hat{M}_Y(\hat{\Delta R}, \hat{r})$ are available in the two subroutines starting at lines 1010 and 1120, respectively. The latter subroutine actually calculates the covariance at general nonzero values of both $\hat{\Delta R}$ and $\hat{r}$, although we only employ it for $\hat{\Delta R} = 0$ in this appendix; see lines 10 and 380. Also, for $\hat{\Delta R} = 0$, the parameter $Lg2 = (\Delta_L/\Delta_G)^2$ is not relevant and, hence, is entered as zero in line 380.

The exponential Gaussian forms for k_L and k_G are used in lines 1200 and 1210, while the triangular form for ρ is entered in line 1240. Any of these can be replaced, if desired, by forms more appropriate to the user.

The program is written in BASIC for the Hewlett Packard 9000 Computer Model 520. The designation DOUBLE denotes integer variables, not double precision. The output from the program is stored in data files AOT0, AOT1, AOT2, for $v = 0, 1, 2$, respectively.

```

10    Rc=0.                                ! DelR^
20    A=.2                                ! A(subA)
30    Gp=.001                             ! GAMMA'(subA)
40    Wlb2=1.                             ! (DelW(subL)/Beta)^2
50    Wgb2=25.                             ! (DelW(subG)/Beta)^2
60    Dtc=.01                             ! INCREMENT IN Tau^
70    Ntc=200                             ! NUMBER OF Tau^ VALUES
80    Tolerance=1.E-10
90    COM Af(0:40),C(0:80),Sq(0:80)
100   COM DOUBLE J                        ! INTEGER
110   DIM Kag(200),Tc(0:200),F0(0:200),F1(0:200),F2(0:200)
120   DOUBLE Ntc,K                        ! INTEGERS
130   FOR K=0 TO Ntc
140     Tc=K*Dtc                          ! Tau^
150     Rho=MAX(0.,1.-ABS(Tc))             ! Rho
160     T2=.5*Tc*Tc
170     K1=EXP(-Wlb2*T2)
180     Kg=EXP(-Wgb2*T2)
190     Kag(K)=(Rho*K1+Gp*Kg)/(1.+Gp)      ! INPUT COVARIANCE
200   NEXT K
210   A1=1./A                             ! A>0 REQUIRED
220   F0inf=PI
230   F1inf=FNFIinf(A,Gp)
240   F2inf=.25*PI*(1.+Gp)*(1.+Gp)
250   Af(0)=1.
260   FOR K=1 TO 40
270     J=K
280     Af(K)=T=Af(K-1)*A/K                ! A^K/K!
290     IF T<Tolerance THEN 320
300   NEXT K
310   PRINT "40 TERMS IN Af(*)"
320   FOR K=0 TO J*2
330     C(K)=T=K*A1+Gp
340     Sq(K)=1./SQR(T)
350   NEXT K
360   FOR K=0 TO Ntc
370     Tc(K)=Tc=K*Dtc                    ! Tau^
380     CALL Myc(Rc,Tc,A,Gp,Wlb2,Wgb2,0.,F0(K),F1(K),F2(K))
390   NEXT K
400   MAT F0=F0-(F0inf)
410   MAT F1=F1-(F1inf)
420   MAT F2=F2-(F2inf)
430   MAT F0=F0/(F0(0))
440   MAT F1=F1/(F1(0))
450   MAT F2=F2/(F2(0))
460   PRINT "INFINITY: ";F0inf;F1inf;F2inf
470   PRINT "MINIMA: ";MIN(F0(*));MIN(F1(*));MIN(F2(*))
480   PRINT "AT Ntc: ";F0(Ntc);F1(Ntc);F2(Ntc)
490   CREATE DATA "AIT1",8
500   ASSIGN #1 TO "AIT1"
510   PRINT #1;Kag(*)
520   CREATE DATA "AOT0",8
530   ASSIGN #1 TO "AOT0"
540   PRINT #1;F0(*)
550   CREATE DATA "AOT1",8

```

```

560  ASSIGN #1 TO "AOT1"
570  PRINT #1;F1(*)
580  CREATE DATA "AOT2",8
590  ASSIGN #1 TO "AOT2"
600  PRINT #1;F2(*)
610  ASSIGN #1 TO *
620  Tcmax=Dtc*Ntc
630  GINIT 200/260
640  PLOTTER IS 505,"HPGL"
650  PRINTER IS 505
660  LIMIT PLOTTER 505,0,200,0,260
670  VIEWPORT 22,85,19,122
680  WINDOW 0.,1.,0.,1.
690  PRINT "VS5"
700  GRID .25,.25
710  PRINT "VS36"
720  PLOT Tc(*),Kag(*)
730  PENUP
740  PLOT Tc(*),F0(*)
750  PENUP
760  PLOT Tc(*),F1(*)
770  PENUP
780  PLOT Tc(*),F2(*)
790  PENUP
800  PAUSE
810  PRINTER IS CRT
820  PLOTTER 505 IS TERMINATED
830  END
840  !
850  DEF FNF1inf(A,Gp)                ! for v(=nu) = 1
860  Tol=1.E-18
870  Ag=A*Gp
880  T=1.
890  S=SQR(1.+Ag)
900  FOR M=2 TO 100
910  T=T*A/M
920  P=T*SQR(M+Ag)
930  S=S+P
940  IF P<S*Tol THEN 970
950  NEXT M
960  PRINT "100 TERMS IN FNF1inf"
970  T=Gp+A*S*S+2.*SQR(Ag)*S
980  RETURN EXP(-2.*A)*T
990  FNEED
1000  !
1010  SUB Bnu(Y,B0,B1,B2)              ! Bv(Y) for v=0,1,2
1020  IF Y>1. THEN PRINT "Y = 1 +";Y-1.
1030  IF Y>1. THEN Y=1.
1040  Y2=Y*Y
1050  Sq=SQR(1.-Y2)
1060  T=ASN(Y)+1.5707963267948966
1070  B0=T+T
1080  B1=Y*T+Sq
1090  B2=(.5+Y2)*T+1.5*Y*Sq
1100  SUBEND
1110  !

```

```

1120 SUB Myc(Rc,Tc,A,Gp,W1b2,Wgb2,Lg2,S0,S1,S2)
1130 COM Af(*),C(*),Sq(*)
1140 COM DOUBLE J ! INTEGER
1150 ALLOCATE Ap(0:J),Ap1(0:J)
1160 DOUBLE K,M1,M2,N,K1,K2 ! INTEGERS
1170 A1=1./A ! A>0 REQUIRED
1180 T2=.5*Tc*Tc
1190 R2=Rc*Rc
1200 K1=EXP(-R2-W1b2*T2)
1210 Kg=EXP(-Lg2*R2-Wgb2*T2)
1220 Ak=A1*K1
1230 Gk=Gp*Kg
1240 Rho=MAX(0.,1.-ABS(Tc)) ! Rho
1250 Rho1=1.-Rho
1260 Ap(0)=Ap1(0)=Pk=Pk1=1.
1270 FOR K=1 TO J
1280 Pk=Pk*Rho
1290 Pk1=Pk1*Rho1
1300 T=Af(K)
1310 Ap(K)=T*Pk
1320 Ap1(K)=T*Pk1
1330 NEXT K
1340 S0m1=S1m1=S2m1=0.
1350 FOR M1=0 TO J
1360 S0m2=S1m2=S2m2=0.
1370 FOR M2=0 TO J
1380 S0n=S1n=S2n=0.
1390 FOR N=0 TO J
1400 K1=N+M1
1410 K2=N+M2
1420 T=Ap(N)
1430 P=C(K1)*C(K2)
1440 Y=(N*Ak+Gk)*Sq(K1)*Sq(K2)
1450 CALL Bru(Y,B0,B1,B2)
1460 S0n=S0n+T*B0
1470 S1n=S1n+T*SQR(P)*B1
1480 S2n=S2n+T*P*B2
1490 NEXT N
1500 T2=Ap1(M2)
1510 S0m2=S0m2+T2*S0n
1520 S1m2=S1m2+T2*S1n
1530 S2m2=S2m2+T2*S2n
1540 NEXT M2
1550 T1=Ap1(M1)
1560 S0m1=S0m1+T1*S0m2
1570 S1m1=S1m1+T1*S1m2
1580 S2m1=S2m1+T1*S2m2
1590 NEXT M1
1600 T=EXP(-A*(2.-Rho))
1610 S0=T*S0m1
1620 S1=T*S1m1
1630 S2=T*S2m1
1640 SUBEND

```


APPENDIX A.2 - EVALUATION OF COVARIANCE FUNCTION FOR ZERO DELAY ($\hat{r}, \hat{r}' = 0$)

A program for the numerical evaluation of covariance $\hat{M}_y(\hat{\Delta R}, \hat{r})$ for $\hat{r}, \hat{r}' = 0$ is contained in this appendix. Inputs required of the user are $A, \Gamma'_A, (\Delta_L/\Delta_G)^2, \delta(\hat{\Delta R}), N(\hat{\Delta R})$, in lines 20 - 60. The tolerance for terminating the triple infinite sums is set at $1E-15$ in line 70. The output from the program is stored in data files AOR0, AOR1, AOR2, for $v = 0, 1, 2$, respectively. Other relevant comments are made in appendix A.1.

The limit of $\hat{M}_y$ at $\hat{\Delta R} = \infty$ (when $\hat{r} = 0$) is given by the closed form results

$$\hat{M}_y(\infty, 0) = \begin{cases} \pi & \text{for } v = 0 \\ 1 + \Gamma'_A & \text{for } v = 1 \\ \frac{\pi}{4} \left[\frac{1}{A} + (1 + \Gamma'_A)^2 \right] & \text{for } v = 2 \end{cases} \quad (\text{A.2-1})$$

These values have been subtracted from $\hat{M}_y$ so that we can Bessel transform a function which tends to zero as $\hat{\Delta R} \rightarrow \infty$.

```

10   Tc=0.                ! Tau^
20   A=.2                 ! A(subA)
30   Gp=.001              ! GAMMA'(subA)
40   Lg2=5.              ! (De1L/De1G)^2
50   Drc=.005             ! INCREMENT IN De1R^
60   Nrc=900              ! NUMBER OF De1R^ VALUES
70   Tolerance=1.E-15
80   COM Af(0:40),C(0:80),Sq(0:80)
90   COM DOUBLE J         ! INTEGER
100  DIM Rc(0:900),Kag(0:900),F0(0:900),F1(0:900),F2(0:900)
110  DOUBLE Nrc,K         ! INTEGERS
120  A1=1./A              ! A>0 REQUIRED
130  F0inf=PI             ! LIMITS FOR
140  F1inf=1.+Gp          ! Rc=infinity
150  F2inf=.25*PI*((1.+Gp)*(1.+Gp)+A1) ! AND Tc=0
160  Af(0)=1.
170  FOR K=1 TO 40
180    J=K
190    Af(K)=T=Af(K-1)*A/K ! A^K/K!
200    IF T<Tolerance THEN 230
210  NEXT K
220  PRINT "40 TERMS IN Af(*)"
230  FOR K=0 TO J*2
240    C(K)=T=K*A1+Gp
250    Sq(K)=1./SQR(T)
260  NEXT K

```

```

270   FOR K=0 TO Nrc
280   Rc(K)=Rc=K*Drc                      ! DelR^
290   R2=Rc*Rc
300   K1=EXP(-R2)
310   Kg=EXP(-Lg2*R2)
320   Rho=MAX(0.,1.-ABS(Tc))              ! Rho
330   Kag(K)=(Rho*K1+Gp*Kg)/(1.+Gp)      ! INPUT COVARIANCE
340   CALL Myc(Rc,Tc,A,Gp,0.,0.,Lg2,F0(K),F1(K),F2(K))
350   NEXT K
360   MAT F0=F0-(F0inf)
370   MAT F1=F1-(F1inf)
380   MAT F2=F2-(F2inf)
390   MAT F0=F0/(F0(0))
400   MAT F1=F1/(F1(0))
410   MAT F2=F2/(F2(0))
420   PRINT "INFINITY: ";F0inf;F1inf;F2inf
430   PRINT "MINIMA:  ";MIN(F0(*));MIN(F1(*));MIN(F2(*))
440   PRINT "AT Nrc:  ";F0(Nrc);F1(Nrc);F2(Nrc)
450   CREATE DATA "AIR1",33
460   ASSIGN #1 TO "AIR1"
470   PRINT #1;Kag(*)
480   CREATE DATA "AOR0",33
490   ASSIGN #1 TO "AOR0"
500   PRINT #1;F0(*)
510   CREATE DATA "AOR1",33
520   ASSIGN #1 TO "AOR1"
530   PRINT #1;F1(*)
540   CREATE DATA "AOR2"
550   ASSIGN #1 TO "AOR2"
560   PRINT #1;F2(*)
570   ASSIGN #1 TO *
580   Rcmax=Drc*Nrc
590   GINIT 200/260
600   PLOTTER IS 505,"HPGL"
610   PRINTER IS 505
620   LIMIT PLOTTER 505,0,200,0,260
630   VIEWPORT 22,85,19,122
640   WINDOW 0.,3.,0.,1.
650   PRINT "VS5"
660   GRID .5,.25
670   PRINT "VS36"
680   PLOT Rc(*),Kag(*)
690   PENUP
700   PLOT Rc(*),F0(*)
710   PENUP
720   PLOT Rc(*),F1(*)
730   PENUP
740   PLOT Rc(*),F2(*)
750   PENUP
760   PAUSE
770   PRINTER IS CRT
780   PLOTTER 505 IS TERMINATED
790   END
800   !
810   SUB Bnu(Y,B0,B1,B2)                  ! Bv(Y) for v=0,1,2
820   ! SEE APPENDIX A.1
900   SUBEND
910   !
920   SUB Myc(Rc,Tc,A,Gp,W1b2,Wgb2,Lg2,S0,S1,S2)
930   ! SEE APPENDIX A.1
1440  SUBEND

```

APPENDIX A.3 - EVALUATION OF TEMPORAL INTENSITY SPECTRUM FOR ZERO SEPARATION ($\hat{\Delta R} = 0$)

A program for the numerical evaluation of the Fourier transform of covariance $\hat{M}_y(0, f) - \hat{M}_y(\infty)$ is contained in this appendix. Inputs required of the user are listed in lines 10 - 30. The data input, AOT0 or AOT1 or AOT2, as generated by means of the program in appendix A.1, is injected by means of lines 410, 600, and 790.

In order to keep the FFT (fast Fourier transform) size, N in lines 30 and 320, at reasonable values, the data sequence is collapsed, without any loss of accuracy, according to the method given in [8; pages 7 - 8] and [9; pages 13 - 16]. The integration rule documented here is the trapezoidal rule; this procedure is very accurate and efficient and is recommended for numerical Fourier transforms.

```

10  Ntc=200                ! NUMBER OF Tau^ VALUES
20  Dtc=.01                ! INCREMENT IN Tau^
30  N=1024                 ! SIZE OF FFT; N > Ntc REQUIRED
40  DOUBLE Ntc,N,N4,N2,Ns  ! INTEGERS
50  N4=N/4
60  N2=N/2
70  REDIM Cos(0:N4),X(0:N-1),Y(0:N-1)
80  DIM Cos(256),X(1023),Y(1023),A(200)
90  T=2.*PI/N
100 FOR Ns=0 TO N4
110  Cos(Ns)=COS(T*Ns)      ! QUARTER-COSINE TABLE IN Cos(*)
120 NEXT Ns
130 GINIT 200/260
140 PLOTTER IS 505,"HPGL"
150 PRINTER IS 505
160 LIMIT PLOTTER 505,0,200,0,260
170 VIEWPORT 22,85,19,122
180 WINDOW 0,N2,-5,1
190 PRINT "VS5"
200 GRID N/10,1
210 PRINT "VS36"
220 ASSIGN #1 TO "AIT1"
230 READ #1;A(*)
240 MAT X=(0.)
250 MAT Y=(0.)
260 X(0)=.5*A(0)
270 FOR Ns=1 TO Ntc-1
280  X(Ns)=A(Ns)
290 NEXT Ns
300 X(Ntc)=.5*A(Ntc)
```

```

310   MAT X=X*(Dtc*4.)
320   CALL Fft14(N,Cos(*),X(*),Y(*))
330   FOR Ns=0 TO N2
340   Ar=X(Ns)
350   IF Ar>0. THEN 380
360   PENUP
370   GOTO 390
380   PLOT Ns,LGT(Ar)
390   NEXT Ns
400   PENUP
410   ASSIGN #1 TO "AOT0"
420   READ #1;A(*)
430   MAT X=(0.)
440   MAT Y=(0.)
450   X(0)=.5*A(0)
460   FOR Ns=1 TO Ntc-1
470   X(Ns)=A(Ns)
480   NEXT Ns
490   X(Ntc)=.5*A(Ntc)
500   MAT X=X*(Dtc*4.)
510   CALL Fft14(N,Cos(*),X(*),Y(*))
520   FOR Ns=0 TO N2
530   Ar=X(Ns)
540   IF Ar>0. THEN 570
550   PENUP
560   GOTO 580
570   PLOT Ns,LGT(Ar)
580   NEXT Ns
590   PENUP
600   ASSIGN #1 TO "AOT1"
610   READ #1;A(*)
620   MAT X=(0.)
630   MAT Y=(0.)
640   X(0)=.5*A(0)
650   FOR Ns=1 TO Ntc-1
660   X(Ns)=A(Ns)
670   NEXT Ns
680   X(Ntc)=.5*A(Ntc)
690   MAT X=X*(Dtc*4.)
700   CALL Fft14(N,Cos(*),X(*),Y(*))
710   FOR Ns=0 TO N2
720   Ar=X(Ns)
730   IF Ar>0. THEN 760
740   PENUP
750   GOTO 770
760   PLOT Ns,LGT(Ar)
770   NEXT Ns
780   PENUP
790   ASSIGN #1 TO "AOT2"
800   READ #1;A(*)
810   MAT A=A/(A(0))
820   MAT X=(0.)
830   MAT Y=(0.)

```

```

840   X(0)=.5*A(0)
850   FOR Ns=1 TO Ntc-1
860   X(Ns)=A(Ns)
870   NEXT Ns
880   X(Ntc)=.5*A(Ntc)
890   MAT X=X*(Dtc*4.)
900   CALL Fft14(N,Cos(*),X(*),Y(*))
910   FOR Ns=0 TO N2
920   Ar=X(Ns)
930   IF Ar>0. THEN 960
940   PENUP
950   GOTO 970
960   PLOT Ns,LGT(Ar)
970   NEXT Ns
980   PENUP
990   PAUSE
1000  END
1010  !
1020  SUB Fft14(DOUBLE N,REAL Cos(*),X(*),Y(*)) ! N<=2^14=16384; 0 SUBS
1030  DOUBLE Log2n,N1,N2,N3,N4,J,K ! INTEGERS < 2^31 = 2,147,483,648
1040  DOUBLE I1,I2,I3,I4,I5,I6,I7,I8,I9,I10,I11,I12,I13,I14,L(0:13)
1050  IF N=1 THEN SUBEXIT
1060  IF N>2 THEN 1140
1070  A=X(0)+X(1)
1080  X(1)=X(0)-X(1)
1090  X(0)=A
1100  A=Y(0)+Y(1)
1110  Y(1)=Y(0)-Y(1)
1120  Y(0)=A
1130  SUBEXIT
1140  A=LOG(N)/LOG(2.)
1150  Log2n=A
1160  IF ABS(A-Log2n)<1.E-8 THEN 1190
1170  PRINT "N =";N;"IS NOT A POWER OF 2; DISALLOWED."
1180  PAUSE
1190  N1=N/4
1200  N2=N1+1
1210  N3=N2+1
1220  N4=N3+N1
1230  FOR I1=1 TO Log2n
1240  I2=2^(Log2n-I1)
1250  I3=2*I2
1260  I4=N/I3
1270  FOR I5=1 TO I2
1280  I6=(I5-1)*I4+1
1290  IF I6<=N2 THEN 1330
1300  A1=-Cos(N4-I6-1)
1310  A2=-Cos(I6-N1-1)
1320  GOTO 1350
1330  A1=Cos(I6-1)
1340  A2=-Cos(N3-I6-1)
1350  FOR I7=0 TO N-I3 STEP I3
1360  I8=I7+I5-1
1370  I9=I8+I2
1380  T1=X(I8)
1390  T2=X(I9)

```

```

1400    T3=Y(I8)
1410    T4=Y(I9)
1420    A3=T1-T2
1430    A4=T3-T4
1440    X(I8)=T1+T2
1450    Y(I8)=T3+T4
1460    X(I9)=A1*A3-A2*A4
1470    Y(I9)=A1*A4+A2*A3
1480    NEXT I7
1490    NEXT I5
1500    NEXT I1
1510    I1=Log2n+1
1520    FOR I2=1 TO 14
1530    L(I2-1)=1
1540    IF I2>Log2n THEN 1560
1550    L(I2-1)=2^(I1-I2)
1560    NEXT I2
1570    K=0
1580    FOR I1=1 TO L(13)
1590    FOR I2=I1 TO L(12) STEP L(13)
1600    FOR I3=I2 TO L(11) STEP L(12)
1610    FOR I4=I3 TO L(10) STEP L(11)
1620    FOR I5=I4 TO L(9) STEP L(10)
1630    FOR I6=I5 TO L(8) STEP L(9)
1640    FOR I7=I6 TO L(7) STEP L(8)
1650    FOR I8=I7 TO L(6) STEP L(7)
1660    FOR I9=I8 TO L(5) STEP L(6)
1670    FOR I10=I9 TO L(4) STEP L(5)
1680    FOR I11=I10 TO L(3) STEP L(4)
1690    FOR I12=I11 TO L(2) STEP L(3)
1700    FOR I13=I12 TO L(1) STEP L(2)
1710    FOR I14=I13 TO L(0) STEP L(1)
1720    J=I14-1
1730    IF K>J THEN 1800
1740    A=X(K)
1750    X(K)=X(J)
1760    X(J)=A
1770    A=Y(K)
1780    Y(K)=Y(J)
1790    Y(J)=A
1800    K=K+1
1810    NEXT I14
1820    NEXT I13
1830    NEXT I12
1840    NEXT I11
1850    NEXT I10
1860    NEXT I9
1870    NEXT I8
1880    NEXT I7
1890    NEXT I6
1900    NEXT I5
1910    NEXT I4
1920    NEXT I3
1930    NEXT I2
1940    NEXT I1
1950    SUBEND

```

APPENDIX A.4 - EVALUATION OF WAVENUMBER INTENSITY SPECTRUM FOR ZERO DELAY ($\hat{r}, \hat{r}' = 0$)

A program for the numerical evaluation of the zeroth-order Bessel transform of covariance $\hat{M}_y(\hat{\Delta R}, 0) - \hat{M}_y(\infty)$ is contained in this appendix. Inputs required of the user are listed in lines 10 - 40 and are coupled to appendix A.2, where the data input, AOR0 or AOR1 or AOR2, was generated. The numerical Bessel transform is accomplished by means of Simpson's rule with end correction [11; pages 414 - 418], and is exceedingly accurate for the small increment, .005, in $\hat{\Delta R}$ employed in line 30.

```

10   Dkc=.4                ! INCREMENT IN k^
20   Nkc=200              ! NUMBER OF k^ VALUES
30   Drc=.005            ! INCREMENT IN DeIR^
40   Nrc=900             ! NUMBER OF DeIR^ VALUES
50   DOUBLE Nrc,Nkc,I,Ns ! INTEGERS
60   REDIM C(0:Nrc)
70   REDIM Wi(0:Nkc),W0(0:Nkc),W1(0:Nkc),W2(0:Nkc)
80   DIM C(900),Wi(200),W0(200),W1(200),W2(200)
90   ASSIGN #1 TO "AIR1"
100  READ #1;C(*)
110  FOR I=0 TO Nkc
120  Kc=I*Dkc              ! k^
130  T=Kc*Drc
140  Se=So=0.
150  FOR Ns=1 TO Nrc-1 STEP 2
160  So=So+Ns*FNJo(T*Ns)*C(Ns)
170  NEXT Ns
180  FOR Ns=2 TO Nrc-2 STEP 2
190  Se=Se+Ns*FNJo(T*Ns)*C(Ns)
200  NEXT Ns
210  Wi(I)=C(0)+16.*So+14.*Se
220  NEXT I
230  MAT Wi=Wi*(Drc*Drc*2.*PI/15.)
240  ASSIGN #1 TO "AOR0"
250  READ #1;C(*)
260  FOR I=0 TO Nkc
270  Kc=I*Dkc
280  T=Kc*Drc
290  Se=So=0.
300  FOR Ns=1 TO Nrc-1 STEP 2
310  So=So+Ns*FNJo(T*Ns)*C(Ns)
320  NEXT Ns
330  FOR Ns=2 TO Nrc-2 STEP 2
340  Se=Se+Ns*FNJo(T*Ns)*C(Ns)
350  NEXT Ns
360  W0(I)=C(0)+16.*So+14.*Se
370  NEXT I
380  MAT W0=W0*(Drc*Drc*2.*PI/15.)

```

```

390  ASSIGN #1 TO "AOR1"
400  READ #1;C(*)
410  FOR I=0 TO Nkc
420  Kc=I*Dkc
430  T=Kc*Drc
440  Se=So=0.
450  FOR Ns=1 TO Nrc-1 STEP 2
460  So=So+Ns*FNJo(T*Ns)*C(Ns)
470  NEXT Ns
480  FOR Ns=2 TO Nrc-2 STEP 2
490  Se=Se+Ns*FNJo(T*Ns)*C(Ns)
500  NEXT Ns
510  W1(I)=C(0)+16.*So+14.*Se
520  NEXT I
530  MAT W1=W1*(Drc*Drc*2.*PI/15.)
540  ASSIGN #1 TO "AOR2"
550  READ #1;C(*)
560  ASSIGN #1 TO *
570  FOR I=0 TO Nkc
580  Kc=I*Dkc
590  T=Kc*Drc
600  Se=So=0.
610  FOR Ns=1 TO Nrc-1 STEP 2
620  So=So+Ns*FNJo(T*Ns)*C(Ns)
630  NEXT Ns
640  FOR Ns=2 TO Nrc-2 STEP 2
650  Se=Se+Ns*FNJo(T*Ns)*C(Ns)
660  NEXT Ns
670  W2(I)=C(0)+16.*So+14.*Se
680  NEXT I
690  MAT W2=W2*(Drc*Drc*2.*PI/15.)
700  GINIT 200/260
710  PLOTTER IS 505,"HPGL"
720  PRINTER IS 505
730  LIMIT PLOTTER 505,0,200,0,260
740  VIEWPORT 22,85,19,122
750  WINDOW 0,Nkc,-9,1
760  PRINT "VS5"
770  GRID 25,1
780  PRINT "VS36"
790  FOR I=0 TO Nkc
800  W=Wi(I)
810  IF W>0. THEN 840
820  PENUP
830  GOTO 850
840  PLOT I,LGT(W)
850  NEXT I
860  PENUP

```



```

870   FOR I=0 TO Nkc
880   W=W0(I)
890   IF W>0. THEN 920
900   PENUP
910   GOTO 930
920   PLOT I,LGT(W)
930   NEXT I
940   PENUP
950   FOR I=0 TO Nkc
960   W=W1(I)
970   IF W>0. THEN 1000
980   PENUP
990   GOTO 1010
1000  PLOT I,LGT(W)
1010  NEXT I
1020  PENUP
1030  FOR I=0 TO Nkc
1040  W=W2(I)
1050  IF W>0. THEN 1080
1060  PENUP
1070  GOTO 1090
1080  PLOT I,LGT(W)
1090  NEXT I
1100  PENUP
1110  PAUSE
1120  PRINTER IS CRT
1130  PLOTTER 505 IS TERMINATED
1140  END
1150  !
1160  DEF FNJo(X)          ! Jo(X) FOR ALL X
1170  Y=ABS(X)
1180  IF Y>8. THEN 1280
1190  T=Y*Y                ! HART, #5845
1200  P=2271490439.5536033-T*(5513584.5647707522-T*5292.6171303845574)
1210  P=2334489171877869.7-T*(47765559442673.588-T*(462172225031.71803-T*P))
1220  P=185962317621897804.-T*(44145829391815982.-I*P)
1230  Q=204251483.52134357+T*(494030.79491813972+T*(884.72036756175504+T))
1240  Q=2344750013658996.8+T*(15015462449769.752+T*(64398674535.133256+T*Q))
1250  Q=185962317621897733.+T*Q
1260  Jo=P/Q
1270  RETURN Jo
1280  Z=8./Y              ! HART, #6546 & 6946
1290  T=Z*Z
1300  Pn=2204.5010439651804+T*(128.67758574871419+T*.90047934748028803)
1310  Pn=8554.8225415066617+T*(8894.4375329606194+T*Pn)
1320  Pd=2214.0488519147104+T*(130.88490049992388+T)
1330  Pd=8554.8225415066628+T*(8903.8361417095954+T*Pd)
1340  Qn=13.990976865960680+T*(1.0497327982345548+T*.00935259532940319)
1350  Qn=-37.510534954957112-T*(46.093826814625175+T*Qn)
1360  Qd=921.56697552653090+T*(74.428389741411179+T)
1370  Qd=2400.6742371172675+T*(2971.9837452084920+T*Qd)
1380  T=Y-.78539816339744828
1390  Jo=.28209479177387820*SQR(Z)*(COS(T)*Pn/Pd-SIN(T)*Z*Qn/Qd)
1400  RETURN Jo
1410  FNEED

```

APPENDIX A.5 - EVALUATION OF PHASE MODULATION INTENSITY SPECTRUM

The normalized covariance function for phase modulation is given by (2.50) in the main text, namely

$$k_o(\zeta) = \exp\left[-\Gamma'_A \mu_P^2 [1 - \exp(-\zeta)] - A[2 - \rho(\zeta)] + 2A [1 - \rho(\zeta)] \exp\left(-\mu_P^2/A\right) + A \rho(\zeta) \exp\left(-\frac{\mu_P^2}{A} [1 - \exp(-b\zeta)]\right)\right] \quad (\text{A.5-1})$$

for $\zeta \geq 0$, where ζ is the time delay and $\rho(\zeta)$ is the temporal normalized covariance of the field process. Also $\mu_P^2 = \mu_{PG}^2$. Since (A.5-1) involves an exponential of an exponential of an exponential, and because a wide range of parameter values are of interest, care must be taken in numerical evaluation of this covariance and its transform.

Observe first that

$$k_o(0) = 1 \quad \text{since } \rho(0) = 1. \quad (\text{A.5-2})$$

Also, as delay $\zeta \rightarrow +\infty$, then $\rho \rightarrow 0$, giving

$$k_o(\infty) = \exp\left[-\Gamma'_A \mu_P^2 - 2A + 2A \exp\left(-\mu_P^2/A\right)\right] \neq 0. \quad (\text{A.5-3})$$

The spectrum of interest is given by

$$W_o(\omega) = 4 \int_0^{\infty} d\zeta \cos(\omega\zeta) k_o(\zeta) \quad \text{for } \omega \geq 0; \quad \omega = 2\pi f. \quad (\text{A.5-4})$$

The nonzero value of (A.5-3) at $\zeta = \infty$ leads to an impulse in spectrum $W_o(\omega)$ at $\omega = 0$. This limiting value, $k_o(\infty)$, must be subtracted from covariance (A.5-1) prior to the numerical Fourier transform indicated by (A.5-4).

For $\Gamma'_A \mu_P^2 \ll 1$, the term

$$\exp\left(-\Gamma'_A \mu_P^2 [1 - \exp(-\zeta)]\right) \quad (\text{A.5-5})$$

approaches its limiting value at $\zeta = +\infty$ as follows:

$$\begin{aligned} & \exp\left(-\Gamma'_A \mu_P^2 [1 - \exp(-\zeta)]\right) - \exp\left(-\Gamma'_A \mu_P^2\right) = \\ & = \exp\left(-\Gamma'_A \mu_P^2\right) \left[\exp\left(\Gamma'_A \mu_P^2 \exp(-\zeta)\right) - 1\right] = \\ & \sim \exp\left(-\Gamma'_A \mu_P^2\right) \Gamma'_A \mu_P^2 \exp(-\zeta) . \end{aligned} \quad (\text{A.5-6})$$

This is a fairly rapid decay with ζ and will not lead to numerical difficulty when $\Gamma'_A \mu_P^2 \ll 1$.

For large $b\mu_P^2/A$, the term

$$\exp\left(-\frac{\mu_P^2}{A} [1 - \exp(-b\zeta)]\right) \quad (\text{A.5-7})$$

is very sharp near $\zeta = 0$; in fact, it is given approximately by

$$\exp\left(-\frac{\mu_P^2}{A} b\zeta\right) \quad \text{for } \zeta \text{ near } 0 . \quad (\text{A.5-8})$$

Therefore, we define the sharp component of covariance $k_o(\zeta)$ as

$$k_s(\zeta) = \exp\left[-A + A \exp\left(-\frac{\mu_P^2}{A} b\zeta\right)\right] - \exp(-A) \quad \text{for all } \zeta . \quad (\text{A.5-9})$$

Then

$$k_s(0) = 1 - \exp(-A) , \quad k_s(\infty) = 0 . \quad (\text{A.5-10})$$

Now we let

$$\begin{aligned} k_o(\zeta) &= [k_o(\zeta) - k_s(\zeta)] + k_s(\zeta) = \\ &\equiv k_f(\zeta) + k_s(\zeta) , \end{aligned} \quad (\text{A.5-11})$$

where $k_f(\zeta)$ is a flat function near $\zeta = 0$. Then we can express the desired difference as

$$\begin{aligned} k_o(\zeta) - k_o(\infty) &= [k_f(\zeta) - k_o(\infty)] + k_s(\zeta) = \\ &\equiv k_1(\zeta) + k_s(\zeta) , \end{aligned} \quad (\text{A.5-12})$$

where functions $k_1(\zeta)$ and $k_s(\zeta)$ both decay to 0 at $\zeta = \infty$. We now employ two separate FFTs on each of the functions in (A.5-12). The sharp component, $k_s(\zeta)$, must be sampled with a very small increment, $\Delta\zeta$, when $b\mu_p^2/A$ is large. On the other hand, the flat component

$$k_1(\zeta) = k_f(\zeta) - k_o(\infty) \quad (\text{A.5-13})$$

can be sampled in a coarser fashion. Finally, if $b\mu_p^2/A$ is moderate, we work directly with $k_o(\zeta) - k_o(\infty)$ without breaking it into any components.

Two programs are furnished in this appendix, one for moderate $b\mu_p^2/A$, and the other for the flat component (A.5-13) when $b\mu_p^2/A$ is large. For sake of brevity, the Fourier transform of the sharp component (A.5-9) is straightforward and is not presented. The particular covariance $\rho(\zeta)$ adopted is triangular,

$$\rho(\zeta) = 1 - \frac{|\zeta|}{\zeta_c} \quad \text{for } |\zeta| < \zeta_c , \quad 0 \text{ otherwise} , \quad (\text{A.5-14})$$

but can easily be replaced. The parameter ζ_c is the cutoff value of covariance $\rho(\zeta)$.

The number of samples, N , taken of the covariance, in order to perform the FFT of (A.5-4), is rather large, so as to guarantee a very small value of truncation error at the upper end of the integral, despite the small increment $\Delta\zeta$. In order to keep the FFT size, M_f , at reasonable values, the data sequence is collapsed without any loss of accuracy according to the method given in [8; pages 7 - 8] and [9; pages 13 - 16]. The trapezoidal rule is used to approximate the integral in (A.5-4), for reasons given in [8; appendix A].

```

10  ! SPECTRUM FOR PHASE MODULATION - MODERATE
20  Mup=1.                ! MUsupP
30  Gp=.001               ! Gamma'
40  Bs=1.                 ! b
50  A=.2                  ! A
60  Zc=2.*PI              ! Rho(Z) = 0 for |Z|>Zc; Z=zeta
70  Delz=.005             ! Zeta increment
80  N=60000               ! Maximum number of samples of ko(zeta)
90  Mf=16384              ! Size of FFT
100 DOUBLE N,Mf,Ms,Ns     ! INTEGERS
110 DIM X(16384),Y(16384),Cos(4096)
120 REDIM X(0:Mf-1),Y(0:Mf-1),Cos(0:Mf/4)
130 MAT X=(0.)
140 MAT Y=(0.)
150 T=2.*PI/Mf
160 FOR Ms=0 TO Mf/4
170 Cos(Ms)=COS(T*Ms)      ! QUARTER-COSINE TABLE
180 NEXT Ms
190 Ta=Gp*Mup*Mup
200 IF A=0. THEN 220
210 Tb=Mup*Mup/A
220 Tc=2.*A*FNExp(Tb)
230 Kinf=FNExp(Ta+2.*A-Tc) ! CORRELATION AT INFINITY
240 COM A,Bs,Zc,Ta,Tb,Tc,Kinf
250 T=1.-Kinf
260 PRINT 0,T
270 X(0)=T*.5              ! TRAPEZOIDAL RULE
280 FOR Ns=1 TO N
290 Corr=FNKo(Ns*Delz)     ! CORRELATION ko(zeta)
300 IF Ns<6 THEN PRINT Ns,Corr
310 IF ABS(Corr)<1.E-30 THEN 350
320 Ms=Ns MODULO Mf       ! COLLAPSING
330 X(Ms)=X(Ms)+Corr
340 NEXT Ns
350 PRINT "Final value of Corr =";Corr;" Ns =";Ns
360 MAT X=X*(Delz*4.)
370 CALL Fft14(Mf,Cos(*),X(*),Y(*))

```

```

380  GINIT
390  PLOTTER IS "GRAPHICS"
400  GRAPHICS ON
410  WINDOW -2,2,-60,0
420  LINE TYPE 3
430  GRID 1,10
440  LINE TYPE 1
450  Delf=1./<Mf*Delz>
460  FOR Ms=1 TO Mf/2
470  F=Ms*Delf          ! FREQUENCY
480  PLOT LGT(F),10.*LGT(X(Ms))
490  NEXT Ms
500  PENUP
510  PAUSE
520  END
530  !
540  DEF FNExp(Xminus)    ! EXP(-X) WITHOUT UNDERFLOW
550  IF Xminus>708.3 THEN RETURN 0.
560  RETURN EXP(-Xminus)
570  FNEND
580  !
590  DEF FNKo(Zeta)       ! CORRELATION ko(zeta)
600  COM A,Bs,Zc,Ta,Tb,Tc,Kinf
610  Rho=MAX(0.,1.-Zeta/Zc) ! TRIANGULAR RHO
620  E1=Ta*(1.-FNExp(Zeta))
630  E2=Tb*(1.-FNExp(Bs*Zeta))
640  E3=A*Rho*FNExp(E2)
650  RETURN FNExp(E1+A*(2.-Rho)-Tc*(1.-Rho)-E3)-Kinf
660  FNEND
670  !
680  SUB Fft14(DOUBLE N,REAL Cos(*),X(*),Y(*)) ! N<=2^14=16384; 0 SUBS
690  ! SEE APPENDIX A.3

```

```

10  ! SPECTRUM FOR PHASE MODULATION - FLAT COMPONENT
20  Mup=1.          ! MUsupP
30  Gp=.001        ! Gamma'
40  Bs=1.          ! b
50  A=0.           ! A
60  Zc=2.*PI       ! Rho(Z) = 0 for |Z|>Zc; Z=zeta
70  Delz=.005      ! Zeta increment
80  N=60000        ! Maximum number of samples of k1(zeta)
90  Mf=16384       ! Size of FFT
100  DOUBLE N,Mf,Ms,Ns ! INTEGERS
110  DIM X(16384),Y(16384),Cos(4096)
120  REDIM X(0:Mf-1),Y(0:Mf-1),Cos(0:Mf/4)
130  MAT X=(0.)
140  MAT Y=(0.)
150  T=2.*PI/Mf
160  FOR Ms=0 TO Mf/4
170  Cos(Ms)=COS(T*Ms) ! QUARTER-COSINE TABLE
180  NEXT Ms
190  Ta=Gp*Mup*Mup
200  IF A=0. THEN 220

```

```

210 Tb=Mup*Mup/A
220 Tc=2.*A*FNExp(Tb)
230 Tb=5.E55
240 Kinf=FNExp(Ta+2.*A-Tc) ! CORRELATION AT INFINITY
250 Ea=FNExp(A)
260 Tbb=Tb*Bb
270 COM A,Bs,Zc,Ta,Tb,Tc,Kinf,Ea,Tbb
280 T=1.-Kinf-(1.-Ea) ! SUBTRACT SHARP COMPONENT
290 PRINT 0,T
300 X(0)=T*.5 ! TRAPEZOIDAL RULE
310 FOR Ns=1 TO N
320 Corr=FNK1(Ns*Delz) ! CORRELATION k1(zeta)
330 IF Ns<6 THEN PRINT Ns,Corr
340 IF ABS(Corr)<1.E-30 THEN 380
350 Ms=Ns MODULO Mf ! COLLAPSING
360 X(Ms)=X(Ms)+Corr
370 NEXT Ns
380 PRINT "Final value of Corr =";Corr;" Ns =";Ns
390 MAT X=X*(Delz*4.)
400 CALL Fft14(Mf,Cos(*),X(*),Y(*))
410 GINIT
420 PLOTTER IS "GRAPHICS"
430 GRAPHICS ON
440 WINDOW -2,2,-60,0
450 LINE TYPE 3
460 GRID 1,10
470 LINE TYPE 1
480 Delf=1./(Mf*Delz)
490 FOR Ms=1 TO Mf/2
500 F=Ms*Delf ! FREQUENCY
510 T=X(Ms)
520 IF T>0. THEN 550
530 PENUP
540 GOTO 560
550 PLOT LGT(F),10.*LGT(T)
560 NEXT Ms
570 PENUP
580 PAUSE
590 END
600 !
610 DEF FNExp(Xminus) ! EXP(-X) WITHOUT UNDERFLOW
620 IF Xminus>708.3 THEN RETURN 0.
630 RETURN EXP(-Xminus)
640 FNEND
650 !
660 DEF FNK1(Zeta) ! CORRELATION k'(zeta)
670 COM A,Bs,Zc,Ta,Tb,Tc,Kinf,Ea,Tbb
680 Rho=MAX(0.,1.-Zeta/Zc) ! TRIANGULAR RHO
690 E1=Ta*(1.-FNExp(Zeta))
700 E2=Tb*(1.-FNExp(Bs*Zeta))
710 E3=A*Rho*FNExp(E2)
720 E4=FNExp(Tbb*Zeta)
730 Sharp=FNExp(A*(1.-E4))-Ea ! ks(zeta)
740 RETURN FNExp(E1+A*(2.-Rho)-Tc*(1.-Rho)-E3)-Kinf-Sharp
750 FNEND
760 !
770 SUB Fft14(DOUBLE N,REAL Cos(*),X(*),Y(*)) ! N<=2^14=16384; 0 SUBS
780 ! SEE APPENDIX A.3

```

APPENDIX A.6 - EVALUATION OF FREQUENCY MODULATION INTENSITY SPECTRUM

The normalized covariance function for frequency modulation is given by (2.48) in the main text, namely

$$k_0(\zeta) = \exp\left[-\Gamma'_A \mu_F^2 [\exp(-\zeta) + \zeta - 1] - A[2 - \rho(\zeta)] + A \rho(\zeta) \exp\left[-\frac{\mu_F^2}{Ab^2} [\exp(-b\zeta) + b\zeta - 1]\right]\right] \text{ for } \zeta \geq 0, \quad (\text{A.6-1})$$

where ζ is the time delay and $\rho(\zeta)$ is the temporal normalized covariance of the field process. Also, $\mu_F^2 = \mu_{FG}^2$ and $b = \Delta\omega_A/\Delta\omega_N$. Since (A.6-1) involves an exponential of an exponential of an exponential, and because a wide range of parameter values are of interest, care must be taken in numerical evaluation of this covariance and its transform.

Observe that

$$k_0(0) = 1, \quad \text{because } \rho(0) = 1. \quad (\text{A.6-2})$$

Also, as delay $\zeta \rightarrow +\infty$, then $\rho \rightarrow 0$, giving

$$k_0(\zeta) \sim \exp\left[-\Gamma'_A \mu_F^2 (\zeta - 1) - 2A\right] \equiv k_1(\zeta) \text{ for } \zeta > 0. \quad (\text{A.6-3})$$

This term, $k_1(\zeta)$, decays slowly with ζ if $\Gamma'_A \mu_F^2 \ll 1$.

The spectrum of interest is given by

$$W_0(\omega) = 4 \int_0^{\infty} d\zeta \cos(\omega\zeta) k_0(\zeta) \text{ for } \omega \geq 0; \quad \omega = 2\pi f. \quad (\text{A.6-4})$$

The spectrum corresponding to the limiting component, $k_1(\zeta)$ in (A.6-3), is directly available in closed form as

$$\begin{aligned}
 W_1(\omega) &= 4 \int_0^{\infty} d\zeta \cos(\omega\zeta) k_1(\zeta) = \\
 &= \exp\left(\Gamma'_A \mu_F^2 - 2A\right) \frac{4 \Gamma'_A \mu_F^2}{\left(\Gamma'_A \mu_F^2\right)^2 + \omega^2} .
 \end{aligned} \tag{A.6-5}$$

If $\Gamma'_A \mu_F^2 \ll 1$, this latter quantity is large and very sharply peaked at $\omega = 0$; hence, this term has been subtracted out and handled separately when $\Gamma'_A \mu_F^2 \ll 1$. The residual covariance, $k_0(\zeta) - k_1(\zeta)$, then decays very rapidly with ζ and is easily handled directly by means of an FFT. This breakdown is not necessary when $\Gamma'_A \mu_F^2 \sim 1$ and is avoided, then, by handling $k_0(\zeta)$ directly in one FFT.

For $\mu_F^2/A \gg 1$, the term

$$\exp\left(-\frac{\mu_F^2}{Ab^2} [\exp(-b\zeta) + b\zeta - 1]\right) \tag{A.6-6}$$

inside the exponential in (A.6-1) behaves like

$$\exp\left(-\frac{\mu_F^2}{A} \frac{1}{2} \zeta^2\right) \quad \text{near } \zeta = 0 , \tag{A.6-7}$$

where its major sharp contribution arises. For example, if $\mu_F = 50$, $A = 1$, then increment $\Delta\zeta = .005$ leads to values for (A.6-7) of

$$\exp(-0.156 n^2) \text{ at } \zeta = n \Delta\zeta , \tag{A.6-8}$$

which is adequately sampled in order to track its dominant contribution; the actual sequence of values is 1, .856, .536, .247, .083. For smaller μ_F^2/A , this sampling interval is more than adequate.

Two programs are furnished in this appendix, one each for the cases of large and small $\Gamma'_A \mu_F^2$. The particular covariance $\rho(\zeta)$ adopted is triangular,

$$\rho(\zeta) = 1 - \frac{|\zeta|}{\zeta_c} \quad \text{for } |\zeta| < \zeta_c, \quad 0 \text{ otherwise,} \quad (\text{A.6-9})$$

but can easily be replaced. The parameter ζ_c is the cutoff covariance value and is specified numerically in the examples in figures 3.9a through 3.10b.

```

10  ! SPECTRUM FOR FREQUENCY MODULATION - LARGE  $G_p \mu_F^2$ 
20  Muf=50.                ! MUsubF
30  Gp=.001                ! Gamma'
40  Bs=1.                  ! b
50  A=.2                   ! A
60  Zc=2.*PI              ! Rho(Z) = 0 for |Z|>Zc; Z=zeta
70  Delz=.005              ! Zeta increment
80  N=60000                ! Maximum number of samples of ko(zeta)
90  Mf=16384               ! Size of FFT
100  DOUBLE N,Mf,Ms,Ns     ! INTEGERS
110  DIM X(16384),Y(16384),Cos(4096)
120  REDIM X(0:Mf-1),Y(0:Mf-1),Cos(0:Mf/4)
130  MAT X=(0.)
140  MAT Y=(0.)
150  T=2.*PI/Mf
160  FOR Ms=0 TO Mf/4
170  Cos(Ms)=COS(T*Ms)      ! QUARTER-COSINE TABLE
180  NEXT Ms
190  Ta=Gp*Muf*Muf
200  IF A=0. THEN 220
210  Tb=Muf*Muf/(A*Bs*Bs)
220  Tc=FNExp(2.*A-Ta)*Ta
230  Td=Ta*Ta
240  COM A,Bs,Zc,Ta,Tb
250  X(0)=.5                ! TRAPEZOIDAL RULE
260  FOR Ns=1 TO N
270  Corr=FNKo(Ns*Delz)     ! CORRELATION ko(zeta)
280  IF Corr<1.E-20 THEN 320
290  Ms=Ns MODULO Mf        ! COLLAPSING
300  X(Ms)=X(Ms)+Corr
310  NEXT Ns
320  PRINT "Final value of Corr =";Corr;" Ns =";Ns
330  MAT X=X*(Delz)
340  CALL Fft14(Mf,Cos(*),X(*),Y(*))

```

```

350  GINIT
360  PLOTTER IS "GRAPHICS"
370  GRAPHICS ON
380  WINDOW -4,2,-70,30
390  LINE TYPE 3
400  GRID 1,10
410  LINE TYPE 1
420  Delf=1./(Mf*Delz)
430  FOR Ms=1 TO Mf/2
440  F=Ms*Delf          ! FREQUENCY
450  T=X(Ms)
460  IF T>0. THEN 490
470  PENUP
480  GOTO 500
490  PLOT LGT(F),10.*LGT(T)
500  NEXT Ms
510  PENUP
520  Add=X(0)-Tc/Td      ! ORIGIN CORRECTION
530  F=1.E-4
540  FOR Ns=1 TO 100
550  W=2.*PI*F
560  W1=Tc/(Td+W*W)
570  T=W1+Add
580  IF T>0. THEN 610
590  PENUP
600  GOTO 620
610  PLOT LGT(F),10.*LGT(W1+Add)
620  F=F*1.1             ! FREQUENCY
630  IF F>Delf THEN 650
640  NEXT Ns
650  PENUP
660  PAUSE
670  END
680  !
690  DEF FNExp(Xminus)    ! EXP(-X) WITHOUT UNDERFLOW
700  IF Xminus>708.3 THEN RETURN 0.
710  RETURN EXP(-Xminus)
720  FNEND
730  !
740  DEF FNKo(Zeta)       ! CORRELATION ko(zeta)
750  COM A,Bs,Zc,Ta,Tb
760  E1=FNExp(Zeta)+Zeta-1.
770  T=Bs*Zeta
780  E2=FNExp(T)+T-1.
790  Rho=MAX(0.,1-Zeta/Zc) ! TRIANGULAR RHO
800  T=Ta*E1+A*(2.-Rho)-A*Rho*FNExp(Tb*E2)
810  RETURN FNExp(T)
820  FNEND
830  !
840  SUB Fft14(DOUBLE N,REAL Cos(*),X(*),Y(*)) ! N<=2^14=16384; 0 SUBS
850  ! SEE APPENDIX A.3

```

```

10  ! SPECTRUM FOR FREQUENCY MODULATION - SMALL Gp*Muf^2
20  Muf=1.                ! MUsubF
30  Gp=.001              ! Gamma'
40  Bs=1.                ! b
50  A=.2                 ! A
60  Zc=2.*PI             ! Rho(Z) = 0 for |Z|>Zc; Z=zeta
70  Delz=.005            ! Zeta increment
80  N=10000              ! Maximum number of samples of ko(zeta)
90  Mf=8192              ! Size of FFT
100 DOUBLE N,Mf,Ms,Ns    ! INTEGERS
110 DIM X(8192),Y(8192),Cos(2048)
120 REDIM X(0:Mf-1),Y(0:Mf-1),Cos(0:Mf/4)
130 MAT X=(0.)
140 MAT Y=(0.)
150 T=2.*PI/Mf
160 FOR Ms=0 TO Mf/4
170   Cos(Ms)=COS(T*Ms)   ! QUARTER-COSINE TABLE
180 NEXT Ms
190 Ta=Gp*Muf*Muf
200 IF A=0. THEN 220
210 Tb=Muf*Muf/(A*Bs*Bs)
220 T=FNExp(2.*A-Ta)
230 Tc=T*Ta
240 Td=Ta*Ta
250 Delf=.1*Ta/(2.*PI)   ! INCREMENT IN FREQUENCY
260 COM A,Bs,Zc,Ta,Tb
270 X(0)=.5*(1.-T)       ! TRAPEZOIDAL RULE
280 FOR Ns=1 TO N
290   Corr=FNK01(Ns*Delz) ! CORRELATION ko(zeta)-k1(zeta)
300   IF ABS(Corr)<1.E-30 THEN 340
310   Ms=Ns MODULO Mf     ! COLLAPSING
320   X(Ms)=X(Ms)+Corr
330 NEXT Ns
340 PRINT "Final value of Corr =";Corr;"   Ns =";Ns
350 MAT X=X*(Delz)
360 CALL Fft14(Mf,Cos(*),X(*),Y(*))

```

```

370  GINIT
380  PLOTTER IS "GRAPHICS"
390  GRAPHICS ON
400  WINDOW -4,2,-70,30
410  LINE TYPE 3
420  GRID 1,10
430  LINE TYPE 1
440  FOR Ms=1 TO 2000
450  F=Ms*De1f          ! FREQUENCY
460  W=2.*PI*F
470  W1=Tc/(Td+W*W)     ! SHARP SPECTRAL COMPONENT
480  T=Mf*De1z*F
490  Ns=INT(T)
500  Fr=T-Ns
510  W2=Fr*X(Ns+1)+(1.-Fr)*X(Ns) ! BROAD SPECTRAL COMPONENT
520  PLOT LGT(F),10.*LGT(W1+W2)
530  NEXT Ms
540  Ns=MAX(Ns,1)
550  FOR Ms=Ns TO Mf/2
560  F=Ms/(Mf*De1z)     ! FREQUENCY
570  W=2.*PI*F
580  W1=Tc/(Td+W*W)
590  W2=X(Ms)
600  T=W1+W2
610  IF T>0. THEN 640
620  PENUP
630  GOTO 650
640  PLOT LGT(F),10.*LGT(T)
650  NEXT Ms
660  PENUP
670  PAUSE
680  END
690  !
700  DEF FNExp(Xminus)   ! EXP(-X) WITHOUT UNDERFLOW
710  IF Xminus>708.3 THEN RETURN 0.
720  RETURN EXP(-Xminus)
730  FNEND
740  !
750  DEF FNKoi(Zeta)     ! CORRELATION ko(zeta)-k1(zeta)
760  COM A,Bs,Zc,Ta,Tb
770  E1=FNExp(Zeta)+Zeta-1.
780  T=Bs*Zeta
790  E2=FNExp(T)+T-1.
800  Rho=MAX(0.,1-Zeta/Zc) ! TRIANGULAR RHO
810  T=Ta*E1+A*(2.-Rho)-A*Rho*FNExp(Tb*E2)
820  RETURN FNExp(T)-FNExp(Ta*(Zeta-1.))+2.*A)
830  FNEND
840  !
850  SUB Fft14(DOUBLE N,REAL Cos(*),X(*),Y(*)) ! N<=2^14=16384; 0 SUBS
860  ! SEE APPENDIX A.3

```

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Exact Performance of Filtered and Weighted Energy Detector with
Mismatched Frequency and Time Locations and Characteristics

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ABSTRACT

The characteristic function of the output of a filtered and weighted energy detector, subjected to a nonstationary Gaussian input signal in the presence of colored Gaussian noise, is derived in closed form. This result allows and accounts for mismatch in the receiving filter frequency characteristics relative to the received signal and noise spectra. It also allows for uncertainty in the time location and/or duration of the received signal, by way of arbitrary time weighting of the squared filter output.

Programs for evaluation of the receiver operating characteristics, for white noise inputs as well as colored noise inputs, are furnished in BASIC and exercised for a number of examples. This approach allows for precise quantitative investigation of the degradation associated with filtered energy detection in the presence of uncertainties.

Approved for public release; distribution is unlimited.

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LIST OF SYMBOLS

Δ	time sampling increment, figure 1
k, m	integers denoting discrete time, figure 1 and (1)
$x(k\Delta)$	digital input sequence, figure 1 and (1)
$h(m\Delta)$	digital filter, figure 1 and (5)
$y(k\Delta)$	filter output sequence, figure 1 and (5)
$w(k\Delta)$	summer weights, figure 1, (6) and (7)
z	system output, figure 1 and (6)
$n(k\Delta)$	input noise sequence, (1)
$s(k\Delta)$	input signal sequence, (1)
$g(k\Delta)$	signal gating sequence, (1) and (4)
overbar	ensemble average, (2)
$R_n(m\Delta)$	input noise covariance, (2)
$R_s(m\Delta)$	input signal covariance, (3)
k_a	input signal start time integer, (4)
k_b	input signal stop time integer, (4)
$y_s(k\Delta)$	filter output signal, (5) and (13)
$y_n(k\Delta)$	filter output noise, (5) and (8)
k_c	observation start time integer, (7)
k_d	observation stop time integer, (7)
$k_1\Delta, k_2\Delta$	general covariance times, (9)
C_n	filter output noise covariance, (9)
$\phi_h(j\Delta)$	filter correlation function, (10)
σ_n^2	input noise power, (11) and (45)
C_s	filter output signal covariance, (15)
K_1, K_2	auxiliary integers, (18)

K_0	auxiliary integer, (19)
C_y	covariance of total filter output, (23)
a_k	auxiliary random variables, (24)
C_a	covariance of $\{a_k\}$, (26)
C	matrix with elements (26)
Q	normalized modal matrix of eigenvectors of C , (27)
Λ	diagonal matrix of eigenvalues $\{\lambda_k\}$ of C , (27)
T	transpose of matrix, (27)
A	column vector of $\{a_k\}$, (28)
B	linear transformation of A , (29)
$\{b_k\}$	elements of B , (30)
$f_z(\xi)$	characteristic function of output z , (32)
τ	argument (delay) of correlation functions, (33)
J	number of input signal covariance components, (33)
α_j	input signal scaling, (33)
τ_{sj}	input signal time constant, (33)
σ_s^2	input signal power, (34)
f	frequency, (35)
$G_s(f)$	input signal spectrum, (35)
ω	$2\pi f$, (35)
a_j	input signal exponential parameter, (36)
M	number of input noise covariance components, (44)
β_m	input noise scaling, (44)
τ_{nm}	input noise time constant, (44)
b_m	input noise exponential parameter, (46)

N	number of filter components, (47)
ψ_n	filter scaling, (47)
τ_{hn}	filter time constant, (47)
c_n	filter exponential parameter, (49)
v_n	auxiliary filter constant, (51)
$\underline{H}(z)$	filter z transform, (52)
$\otimes$	convolution, (55) and (A-2)
$C(k\Delta)$	auxiliary function, (56)
μ_m, γ_n	auxiliary constants, (60)
S	auxiliary function, (65) and (69)
g	auxiliary function, (68)
$f_0(\xi)$	characteristic function for noise only, (73)
$p_0(u)$	probability density function for noise only, (73)
Δ_0	sampling increment in ξ applied to $f_0(\xi)$, (73)
$\tilde{p}_0(u)$	approximation to $p_0(u)$, (73) and (74)
N_0	size of FFT for $\tilde{p}_0(u)$, (74)
δ_0	increment in u for $\tilde{p}_0(u)$, (74) and (75)
u_0	maximum useful u for $\tilde{p}_0(u)$, (75)
$f_1(\xi)$	characteristic function for signal present, (76)
$p_1(u)$	probability density function for signal present, (76)
Δ_1	sampling increment in ξ applied to $f_1(\xi)$, (76)
$\tilde{p}_1(u)$	approximation to $p_1(u)$, (76) and (77)
N_1	size of FFT for $\tilde{p}_1(u)$, (77)
δ_1	increment in u for $\tilde{p}_1(u)$, (77) and (78)
u_1	maximum useful u for $\tilde{p}_1(u)$, (78)
$E_k(u)$	exceedance distribution corresponding to $f_k(\xi)$, (80)
Im	imaginary part, (80)

P_D	detection probability
P_F	false alarm probability
CDF	cumulative distribution function
EDF	exceedance distribution function
R	input signal to noise ratio σ_s^2/σ_n^2 , (82)
$\delta_b(\tau)$	unit impulse train with spacing b , (A-2)

EXACT PERFORMANCE OF FILTERED AND WEIGHTED ENERGY DETECTOR WITH
MISMATCHED FREQUENCY AND TIME LOCATIONS AND CHARACTERISTICS

INTRODUCTION

The detection of weak random signals of unknown duration, bandwidth, time location, and frequency shift in the presence of colored noise is often accomplished in practice by filtering the received waveform to the frequency band of interest, squaring the filter output, and time-weighting this quantity prior to accumulation and threshold comparison. The performance of this processor obviously depends on the spectra of the received signal and noise processes as well as on the transfer characteristics of the filter employed. Lack of detailed knowledge of the signal spectral characteristics or frequency shift will often dictate that a fairly broad (mismatched) filter passband be utilized in order that the signal be passed when present. Similarly, uncertainty about the signal location or duration will mandate that the receiver observation time be lengthened in order to ensure capture of any impinging signal energy.

Additionally, the received signal process is often characterized as a finite-duration burst of a stationary process. Since the signal time location and duration are often unknown, the receiving filter must be kept open at all times, thereby allowing the filter output noise to reach its full-strength steady state value. By contrast, a gated input signal leads to a filter output component which gradually builds up in time and

then decays slowly after the signal is turned off. These nonstationary effects, coupled with the particular time weighting employed during the observation interval, influence the performance of the detector in a complicated nonobvious fashion.

There is a need to be able to calculate exactly the performance of this type of signal processor in the presence of such deleterious factors, so that the degradations associated with lack of knowledge can be ascertained quantitatively. Furthermore, it will not suffice to resort to Gaussian approximations for the processor output, because the number of samples employed are not large enough to utilize the central limit theorem, especially on the tails of the distribution (small false alarm probabilities) where we are interested. This need is addressed in this report for the case of Gaussian input signals in Gaussian noise, with the result that programs are furnished for exactly assessing the performance of the mismatched energy detector for a very wide variety of characteristics and selections of parameters. In particular, no presumptions are made about the input signal time-bandwidth product or about the size of the product of observation time and filter bandwidth.

SYSTEM DESCRIPTION AND ASSUMPTIONS

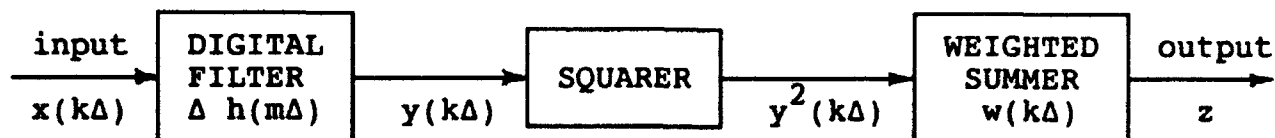


Figure 1. Block Diagram of Detector

The detector of interest is the digital processor depicted in figure 1; time sampling increment Δ is arbitrary but should be approximately matched to the coherence times of the input signal and filter. The digital input sequence $x(k\Delta)$, filter impulse response $\Delta h(m\Delta)$, and summer weights $w(k\Delta)$ are all real. Input $x(k\Delta)$ consists of either noise-alone or gated signal plus noise, according to model

$$x(k\Delta) = \left\{ \begin{array}{c} n(k\Delta) \\ \text{or} \\ s(k\Delta) g(k\Delta) + n(k\Delta) \end{array} \right\} \quad \text{for all } k. \quad (1)$$

Input noise $n(k\Delta)$ is present for all time and is a stationary zero-mean discrete Gaussian sequence with covariance

$$R_n(m\Delta) = \overline{n(k\Delta) n(k\Delta - m\Delta)} \quad \text{for all } m, k, \quad (2)$$

where an overbar denotes an ensemble average. The numerical evaluation of the values of the noise covariance, directly from a specified noise spectrum $G_n(f)$, is considered in appendix A.

Underlying input signal $s(k\Delta)$ in (1) (if present) is also a

stationary zero-mean discrete Gaussian sequence with covariance

$$R_s(m\Delta) = \overline{s(k\Delta) s(k\Delta - m\Delta)} \quad \text{for all } m, k. \quad (3)$$

However, input signal samples $s(k\Delta)$ are gated by function $g(k\Delta)$ which is nonzero only in a limited time region:

$$g(k\Delta) = 1 \quad \text{for } k_a \leq k \leq k_b, \text{ zero otherwise.} \quad (4)$$

This results in a gated burst of stationary signal sequence $s(k\Delta)$ being inputted to digital filter $h(m\Delta)$; the input signal starting and ending times $k_a\Delta$ and $k_b\Delta$, respectively, are generally unknown, except in an approximate way. This generality allows for consideration of input signals of unknown arrival time and duration at the detector input.

The filter output $y(k\Delta)$ can be conveniently broken into signal and noise components, in accordance with (1), and is available by means of discrete convolution

$$y(k\Delta) = \sum_m \Delta h(m\Delta) x(k\Delta - m\Delta) = y_s(k\Delta) + y_n(k\Delta), \quad (5)$$

when an input signal is present. (Summations without limits are over $(-\infty, +\infty)$.) However, due to the gating in (1), filter output signal $y_s(k\Delta)$ will have transient phases, including a buildup just after time $k_a\Delta$ and a decay just after time $k_b\Delta$. This nonstationary behavior is included and exactly accounted for in the following analysis.

The filter output (5) is then squared, scaled by weights $w(k\Delta)$, and summed to give system output

$$z = \sum_k w(k\Delta) y^2(k\Delta) , \quad (6)$$

where the weights are nonzero only in a limited observation time specified by

$$w(k\Delta) \neq 0 \quad \text{for } k_c \leq k \leq k_d . \quad (7)$$

These weights need not be uniform; for example, they could be exponential. Output z is compared with a threshold for a declaration of signal absent versus signal present.

Observation time limits $k_c\Delta$ and $k_d\Delta$ are under the control of the receiver processor, and would hopefully match fairly well the time interval over which the gated input signal is received; see (4). But, in any event, the degree of generality in (7) allows for investigation of observation times that might be too short, thereby losing some signal energy, or too long, thereby picking up additional unwanted noise components; both situations lead to deterioration of the detectability of weak signals and should be avoided if possible. An illustration of the parameters is given in figure 2, for signal-only present.

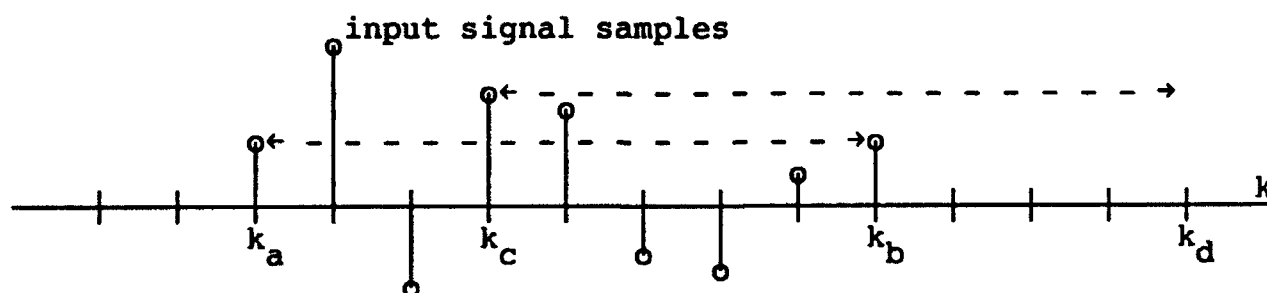


Figure 2. Gated and Observation Intervals; No Noise

FILTER OUTPUT NOISE CHARACTERIZATION

The filter output noise sequence is obtained from discrete convolution (5) and (1) as

$$y_n(k\Delta) = \Delta \sum_m h(m\Delta) n(k\Delta - m\Delta) \quad \text{for all } k, \quad (8)$$

where it is presumed that the filter $h(m\Delta)$ has been open to noise input $n(k\Delta)$ for all time; this results in noise output $y_n(k\Delta)$ being a zero-mean stationary Gaussian sequence. The filter output noise covariance at general times $k_1\Delta$ and $k_2\Delta$ is given by

$$\begin{aligned} C_n(k_1\Delta, k_2\Delta) &= \overline{y_n(k_1\Delta) y_n(k_2\Delta)} = \\ &= \Delta^2 \sum_{mp} h(m\Delta) h(p\Delta) \overline{n(k_1\Delta - m\Delta) n(k_2\Delta - p\Delta)} = \\ &= \Delta^2 \sum_{mp} h(m\Delta) h(p\Delta) R_n(k_1\Delta - k_2\Delta + p\Delta - m\Delta) = \\ &= \sum_j \phi_h(j\Delta) R_n(k_1\Delta - k_2\Delta - j\Delta), \end{aligned} \quad (9)$$

where the filter correlation function is defined as

$$\phi_h(j\Delta) = \Delta^2 \sum_m h(m\Delta) h(m\Delta - j\Delta) \quad \text{for all } j. \quad (10)$$

The right-hand side of (9) is a function only of the time difference $k_1\Delta - k_2\Delta$, in keeping with the stationarity of filter output noise $y_n(k\Delta)$.

If the input noise in figure 1 is white, its covariance in (2) becomes

$$R_n(m\Delta) = \sigma_n^2 \delta_{m0} , \quad (11)$$

where σ_n^2 is the input noise power. In this case, the filter output noise covariance in (9) simplifies to

$$C_n(k_1\Delta, k_2\Delta) = \sigma_n^2 \phi_h(k_1\Delta - k_2\Delta) \text{ for white input noise .} \quad (12)$$

The numerical evaluation of filter correlation function $\phi_h(j\Delta)$ in (10), directly from a specified transfer function $H(f)$, is considered in appendix B. A possible problem with discontinuous functions is treated in appendix C.

FILTER OUTPUT SIGNAL CHARACTERIZATION

The filter output signal sequence (when present) follows from (5) and (1) as

$$y_s(k\Delta) = \Delta \sum_m h(k\Delta - m\Delta) s(m\Delta) g(m\Delta) \quad \text{for all } k. \quad (13)$$

The finite duration of gating sequence $g(k\Delta)$ in (4), as well as the realizability of impulse response $\Delta h(m\Delta)$, will serve to terminate the summation in (13) at finite limits.

We presume that filter $h(m\Delta)$ in figure 1 is realizable; that is,

$$h(m\Delta) = 0 \quad \text{for } m < 0. \quad (14)$$

This makes filter output signal $y_s(k\Delta)$ in (13) equal to zero for $k < k_a$. On the other hand, if the filter has an infinite-duration impulse response $\Delta h(m\Delta)$, then $y_s(k\Delta)$ is nonzero for $k \geq k_a$; see figure 3. However, the filter output noise $y_n(k\Delta)$ will dominate the signal output for $k \gg k_b$, meaning that performance of the detector will be poor in the case when $k_d \gg k_b$. That is, too long an observation time in (7), relative to the signal duration, is detrimental to signal detectability.

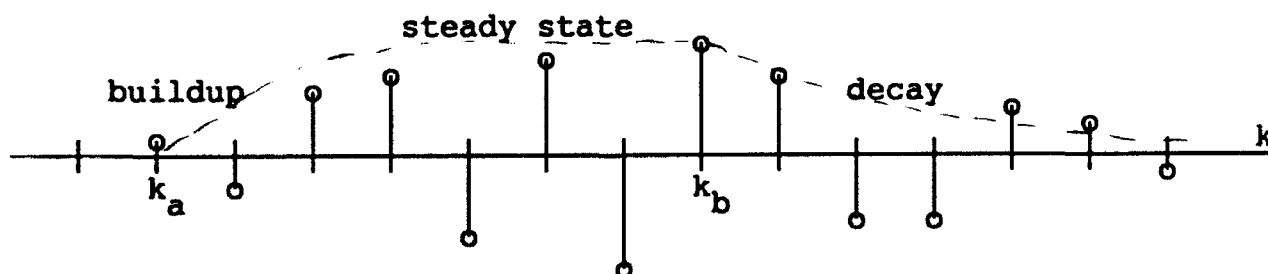


Figure 3. Filter Output Signal Buildup and Decay

The signal output, $y_s(k\Delta)$ in (13), from filter $h(m\Delta)$ is not stationary. The filter output signal covariance is defined as

$$C_s(k_1\Delta, k_2\Delta) = \overline{y_s(k_1\Delta) y_s(k_2\Delta)} = C_s(k_2\Delta, k_1\Delta) \quad \text{for all } k_1, k_2. \quad (15)$$

This function is zero for $k_1 < k_a$ or $k_2 < k_a$; therefore, in the following, we can confine calculation of (15) to $k_1 \geq k_a$ and $k_2 \geq k_a$. The particular filter output signal sample $y_s(k_a\Delta)$ is nonzero only if $h(0) \neq 0$.

Substitution of (13) in (15) and the use of (3) yields

$$\begin{aligned} C_s(k_1\Delta, k_2\Delta) &= \Delta^2 \sum_{mp} h(k_1\Delta - m\Delta) h(k_2\Delta - p\Delta) \overline{s(m\Delta) s(p\Delta)} g(m\Delta) g(p\Delta) = \\ &= \Delta^2 \sum_{mp} h(k_1\Delta - m\Delta) h(k_2\Delta - p\Delta) R_s(m\Delta - p\Delta) g(m\Delta) g(p\Delta). \end{aligned} \quad (16)$$

When we take explicit account of realizability condition (14) and the finite duration of unity gating function $g(k\Delta)$ in (4), the filter output signal covariance in (16) can be efficiently computed according to

$$C_s(k_1\Delta, k_2\Delta) = \Delta^2 \sum_{m=k_a}^{K_1} h(k_1\Delta - m\Delta) \sum_{p=k_a}^{K_2} h(k_2\Delta - p\Delta) R_s(m\Delta - p\Delta), \quad (17)$$

where

$$K_1 = \min(k_1, k_b), \quad K_2 = \min(k_2, k_b). \quad (18)$$

Since $k_b \geq k_a$, $k_1 \geq k_a$, $k_2 \geq k_a$, it follows that $K_1 \geq k_a$ and $K_2 \geq k_a$, thereby making the summations in (17) treat only nonzero entries.

SUMMER CONSIDERATIONS

The weighted summer in figure 1 and (6) - (7) takes its samples at times $k\Delta$, where $k_c \leq k \leq k_d$. If $k_d < k_a$, then no signal gets into the summer, and the signal at the input cannot be detected at all, whether present or not. Since this is not a useful application of the detector in figure 1, the only situation we will address is that where $k_d \geq k_a$, so that some signal (when present) contributes to the summer output.

There are then two possibilities for observation start time $k_c\Delta$, as indicated in figure 4 below. If $k_c < k_a$ (case 1), the summer is taking in noise-only samples for $k_c \leq k < k_a$; this will degrade performance of the detector but is sometimes unavoidable when the signal onset time, $k_a\Delta$, is unknown. In this case, we only need to compute signal output covariance $C_s(k_1\Delta, k_2\Delta)$ for $k_a \leq k_1, k_2 \leq k_d$, since $C_s = 0$ for $k_1 < k_a$ or $k_2 < k_a$.

On the other hand, if $k_c \geq k_a$ (case 2), signal is taken into the summer on all samples available to it. However, if k_c is considerably larger than k_a , a significant portion of the signal contribution can be lost; this degradation of performance is

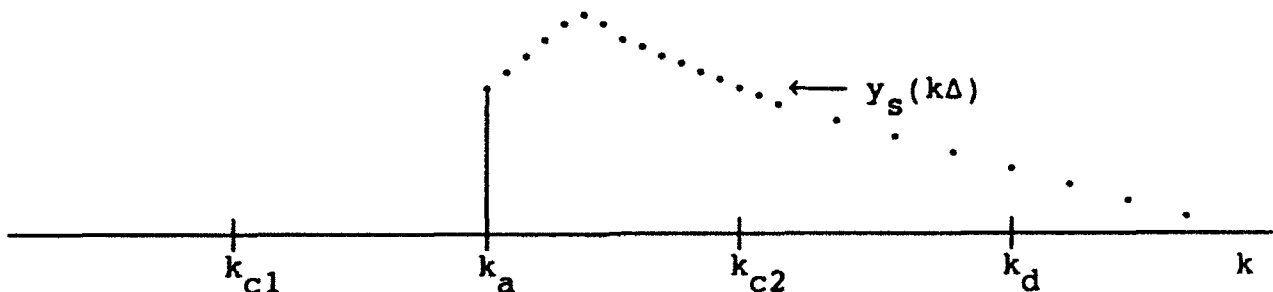


Figure 4. Filter Output Signal $y_s(k\Delta)$

sometimes unavoidable when k_a is unknown. In this latter case, we only need to compute $C_s(k_1\Delta, k_2\Delta)$ for $k_c \leq k_1, k_2 \leq k_d$.

The general rule is that we need to compute, from (17), the filter output signal covariance $C_s(k_1\Delta, k_2\Delta)$ for

$$K_0 \equiv \max(k_a, k_c) \leq k_1, k_2 \leq k_d. \quad (19)$$

Then, if $k_c < k_a$, use 0 for $C_s(k_1\Delta, k_2\Delta)$ for those values where $k_1 < k_a$ or $k_2 < k_a$. Of course, noise covariance $C_n(k_1\Delta, k_2\Delta)$ in (9) or (12) must be computed for $k_c \leq k_1, k_2 \leq k_d$ in all cases.

Since, from (19),

$$k_1 \geq \max(k_a, k_c) \geq k_a, \text{ then } K_1 = \min(k_1, k_b) \geq k_a; \quad (20)$$

also, since

$$k_2 \geq \max(k_a, k_c) \geq k_a, \text{ then } K_2 = \min(k_2, k_b) \geq k_a. \quad (21)$$

Therefore, the sums in (17) always have some entries; that is, the upper limit is never less than the lower limit. The only restriction is

$$k_a \leq k_d. \quad (22)$$

Of course, we must always have $k_a \leq k_b$ and $k_c \leq k_d$.

In summary, the filter output signal covariance follows from (17) and (18), while the filter output noise covariance is available in (9) or (12). The region where the filter output signal covariance $C_s(k_1\Delta, k_2\Delta)$ calculation is nonzero is crosshatched in figure 5 for the case where $k_c < k_a$. If $k_c \geq k_a$, then $C_s(k_1\Delta, k_2\Delta)$ is nonzero in the larger square indicated in figure 5.

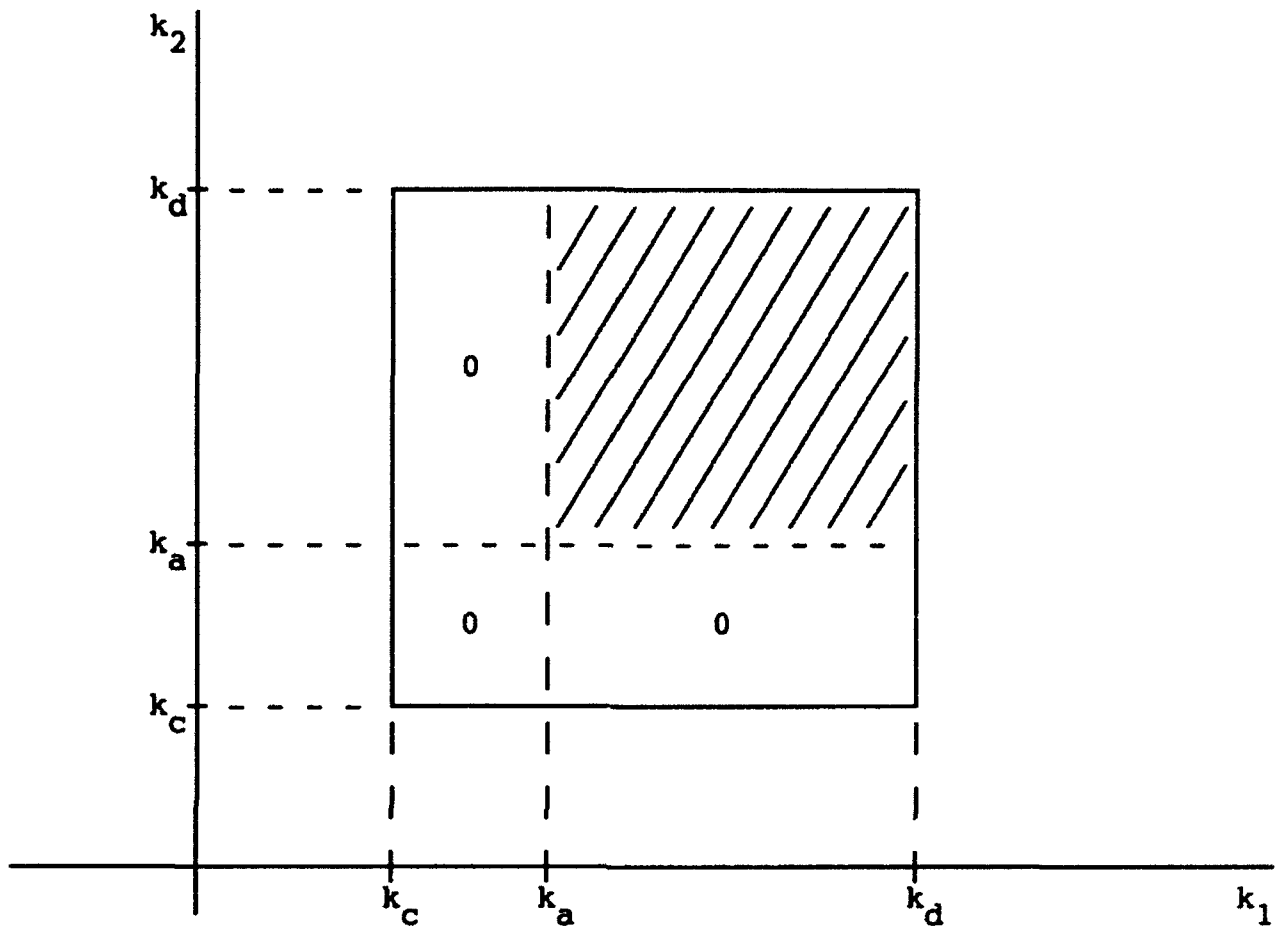


Figure 5. Calculation Region for $C_s(k_1\Delta, k_2\Delta)$

CHARACTERISTIC FUNCTION OF SYSTEM OUTPUT z

The system output z is given in (6) as a sum of weighted and squared correlated zero-mean Gaussian random variables $y(k\Delta)$. Also, the covariance of the noise component $y_n(k\Delta)$ is given by (9) or (12), while the covariance of the signal component $y_s(k\Delta)$ (if present) is given by (17) - (18) and figure 5. Therefore, the covariance of $y(k\Delta)$ is given by

$$C_y(k_1\Delta, k_2\Delta) = C_s(k_1\Delta, k_2\Delta) + C_n(k_1\Delta, k_2\Delta) . \quad (23)$$

The system input signal-to-noise ratio is $R_s(0)/R_n(0)$ in terms of input covariances (2) and (3).

For nonnegative weights $\{w(k\Delta)\}$, we define random variables

$$a_k = \sqrt{w(k\Delta)} y(k\Delta) \quad \text{for } k_c \leq k \leq k_d . \quad (24)$$

Then, (6) - (7) yields the output in the form

$$z = \sum_{k=k_c}^{k_d} a_k^2 , \quad (25)$$

where zero-mean Gaussian random variables $\{a_k\}$ have covariance

$$C_a(k_1, k_2) = \overline{a_{k_1} a_{k_2}} = \sqrt{w(k_1\Delta) w(k_2\Delta)} C_y(k_1\Delta, k_2\Delta) . \quad (26)$$

In order to find the characteristic function of random variable z in (25), we consider the square symmetric covariance matrix C with elements (26) for $k_c \leq k_1, k_2 \leq k_d$. Let Q be the

normalized modal matrix of the orthonormal eigenvectors of C and let Λ be the diagonal matrix of the eigenvalues $\{\lambda_k\}$ of C ; see [1; pages 36 - 39]. Then we have

$$Q^T Q = Q Q^T = I, \quad Q^T C Q = \Lambda. \quad (27)$$

Now let column vector A be made up of elements $\{a_k\}$ for $k_c \leq k \leq k_d$, as defined in (24), in which case (25) and (26) can be expressed as

$$z = A^T A, \quad C = \overline{A A^T}. \quad (28)$$

Also, let column vector B be defined by linear transformation

$$B = Q^T A; \quad \text{then} \quad A = Q B. \quad (29)$$

Substitution of (29) into (28) yields

$$z = A^T A = B^T Q^T Q B = B^T B = \sum_{k=k_c}^{k_d} b_k^2, \quad (30)$$

where $\{b_k\}$ are the elements of B . At the same time, the covariance matrix of vector B is

$$\overline{B B^T} = Q^T \overline{A A^T} Q = Q^T C Q = \Lambda, \quad (31)$$

where we used (29), (28), and (27). Thus, the sum for z in (30) is composed of uncorrelated (and therefore independent) zero-mean Gaussian random variables $\{b_k\}$, where the variance of element b_k is eigenvalue λ_k .

The characteristic function of random variable z in (30) is,

using the independence and Gaussian character of $\{b_k\}$,

$$\begin{aligned}
 f_z(\xi) &= \overline{\exp(i\xi z)} = \prod_{k=k_c}^{k_d} \overline{\exp(i\xi b_k^2)} = \\
 &= \prod_{k=k_c}^{k_d} \int du \exp(i\xi u^2) (2\pi\lambda_k)^{-1/2} \exp\left(\frac{-u^2}{2\lambda_k}\right) = \\
 &= \prod_{k=k_c}^{k_d} \left(1 - i2\xi\lambda_k\right)^{-1/2}.
 \end{aligned} \tag{32}$$

Although this final result is given as a finite product of $k_d - k_c + 1$ principal-value square roots, it can be computed as a single principal-value square root of a finite product of complex first-order polynomials, provided that the location of the product in the complex plane is tracked from $\xi = 0$; for example, see [2; pages B-1 and B-2].

COVARIANCE AND FILTER EXAMPLES

INPUT SIGNAL CHARACTERIZATION

The input signal covariance will be taken to be a sum of damped exponentials:

$$R_s(\tau) = \sum_{j=1}^J \alpha_j \exp(-|\tau|/\tau_{sj}) \quad \text{for all } \tau, \quad (33)$$

where the J exponential parameters $\{\tau_{sj}\}$ are all distinct. The scalings $\{\alpha_j\}$ and time constants $\{\tau_{sj}\}$ can be complex, provided that they occur in complex conjugate pairs; that is, input signal covariance $R_s(\tau)$ must be real for all τ . The origin value,

$$R_s(0) = \sum_{j=1}^J \alpha_j = \sigma_s^2, \quad (34)$$

is the input power of stationary signal sequence $s(k\Delta)$ in (1) and (3), prior to gating. The input signal spectrum corresponding to (33) is, with $\omega = 2\pi f$,

$$G_s(f) = \int d\tau \exp(-i2\pi f\tau) R_s(\tau) = \sum_{j=1}^J \frac{2 \alpha_j \tau_{sj}}{1 + (\omega \tau_{sj})^2}, \quad (35)$$

which is seen to contain $2J$ distinct poles in the complex f -plane. $G_s(f)$ can contain up to $2J-2$ zeros.

For time sampling increment Δ , the sampled input signal covariance is, from (33),

$$R_s(k\Delta) = \sum_{j=1}^J \alpha_j \exp(-a_j |k|) ; \quad a_j = \frac{\Delta}{\tau_{sj}} . \quad (36)$$

The J exponential parameters $\{a_j\}$ are distinct, but they can be complex; in fact, $\{\alpha_j\}$ and $\{a_j\}$ can both be complex provided that they occur in complex conjugate pairs such that $R_s(k\Delta)$ is real for all k . In fact, some of the scalings $\{\alpha_j\}$ can be negative, provided that total spectrum $G_s(f)$ in (35) is nonnegative for all f . The exponential parameters $\{a_j\}$ in (36) must also satisfy $\text{Re } a_j > 0$ in order that the covariance tend to zero at $k = \pm\infty$.

For a particular pair of complex conjugate terms, say

$$\alpha_1 = \alpha_0 , \quad \alpha_2 = \alpha_0^* , \quad \tau_{s1} = \tau_0 , \quad \tau_{s2} = \tau_0^* , \quad a_1 = a_0 , \quad a_2 = a_0^* , \quad (37)$$

the corresponding part of the (continuous) input signal covariance (33) can be expressed in the alternative form

$$\begin{aligned} R_s^0(\tau) &= \alpha_0 \exp(-|\tau|/\tau_0) + \alpha_0^* \exp(-|\tau|/\tau_0^*) = \\ &= 2 \exp(-\omega_r |\tau|) [\alpha_r \cos(\omega_i |\tau|) + \alpha_i \sin(\omega_i |\tau|)] , \end{aligned} \quad (38)$$

where we have defined the real and imaginary parts

$$\alpha_0 = \alpha_r + i\alpha_i , \quad \frac{1}{\tau_0} = \omega_r + i\omega_i . \quad (39)$$

The sampled signal covariance for component (38) is

$$R_s^0(k\Delta) = 2 \exp(-a_r |k|) [\alpha_r \cos(a_i |k|) + \alpha_i \sin(a_i |k|)] , \quad (40)$$

where $a_r = \omega_r \Delta$ and $a_i = \omega_i \Delta$.

The spectral component associated with component covariance (38) is, by reference to (35) and (39),

$$\begin{aligned}
 G_S^O(f) &= \frac{2 \alpha_o \tau_o}{1 + (\omega \tau_o)^2} + \frac{2 \alpha_o^* \tau_o^*}{1 + (\omega \tau_o^*)^2} = \\
 &= 2 \operatorname{Re} \frac{2 \alpha_o / \tau_o}{\omega^2 + 1/\tau_o^2} = 4 \operatorname{Re} \frac{(\alpha_r + i\alpha_i)(\omega_r + i\omega_i)}{\omega^2 + (\omega_r + i\omega_i)^2} = \\
 &= 4 \frac{\alpha_r \omega_r (\omega_r^2 + \omega_i^2 + \omega^2) + \alpha_i \omega_i (\omega_r^2 + \omega_i^2 - \omega^2)}{(\omega_r^2 + (\omega - \omega_i)^2)(\omega_r^2 + (\omega + \omega_i)^2)} = \\
 &= 2 \frac{\alpha_r \omega_r - \alpha_i(\omega - \omega_i)}{\omega_r^2 + (\omega - \omega_i)^2} + 2 \frac{\alpha_r \omega_r + \alpha_i(\omega + \omega_i)}{\omega_r^2 + (\omega + \omega_i)^2} = \\
 &= \frac{\alpha_i + i\alpha_r}{\omega + \omega_i + i\omega_r} + \frac{\alpha_i - i\alpha_r}{\omega + \omega_i - i\omega_r} + \frac{-\alpha_i + i\alpha_r}{\omega - \omega_i + i\omega_r} + \frac{-\alpha_i - i\alpha_r}{\omega - \omega_i - i\omega_r}. \quad (41)
 \end{aligned}$$

If this spectral component $G_S^O(f)$ goes negative for some ranges of f , it must be accompanied by additional terms in summation (35) to keep the total spectrum $G_S(f)$ nonnegative for all frequencies f . Spectral component (41) has four poles in the complex ω -plane at locations $\omega = \omega_i \pm i\omega_r$ and $\omega = -\omega_i \pm i\omega_r$.

Even if coefficients $\{\alpha_j\}$ and $\{\tau_{sj}\}$ in (35) are purely real and the poles are distinct, oscillatory behavior of spectrum $G_S(f)$ is still attainable. For example, with

$$J = 3, \quad \{a_j\} = 1, 8, 15, \quad \{\alpha_j\} = 11, -48, 75, \quad (42)$$

the spectrum (35) is displayed versus $\omega\Delta$ in figure 6. Actually plotted is the scaled spectrum

$$\frac{G_s(f)}{2\Delta} = \sum_{j=1}^J \frac{\alpha_j a_j}{(\omega\Delta)^2 + a_j^2}, \quad (43)$$

where we have used (36) to eliminate $\{\tau_{sj}\}$. The negative coefficient for α_2 causes the dip near $\omega\Delta = 4$.

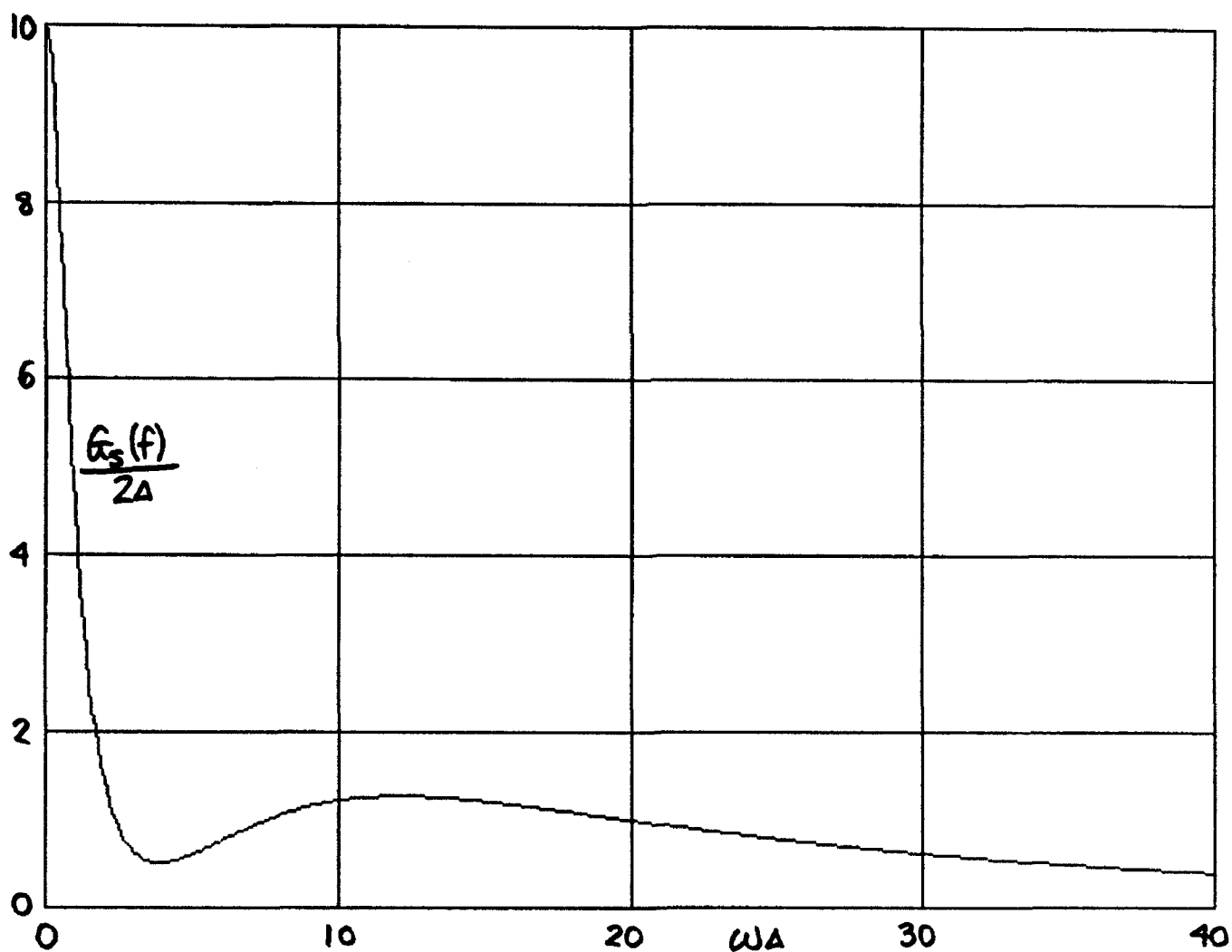


Figure 6. Spectrum (43) for Parameters (42)

INPUT NOISE CHARACTERIZATION

The presentation in this subsection exactly parallels that above for the input signal. The input noise covariance is again taken to be a sum of damped exponentials of the form

$$R_n(\tau) = \sum_{m=1}^M \beta_m \exp(-|\tau|/\tau_{nm}) \quad \text{for all } \tau, \quad (44)$$

where the M (complex) noise time constants $\{\tau_{nm}\}$ are all distinct from one another. (Some of them may equal some of the signal time constants $\{\tau_{sj}\}$ in (33), if desired.) The origin value,

$$R_n(0) = \sum_{m=1}^M \beta_m = \sigma_n^2, \quad (45)$$

is the input power of noise sequence $n(k\Delta)$ in (1).

For time sampling increment Δ , the sampled input noise covariance is, from (44),

$$R_n(k\Delta) = \sum_{m=1}^M \beta_m \exp(-b_m |k|) ; \quad b_m = \frac{\Delta}{\tau_{nm}}. \quad (46)$$

The noise exponential parameters $\{b_m\}$ are distinct, but they can be complex; however, $\text{Re } b_m > 0$. (Some of the $\{b_m\}$ can equal some of the $\{a_j\}$ in (36).) The special case of white noise,

$R_n(k\Delta) = \sigma_n^2 \delta_{k0}$, is handled by choosing $M = 1$, $\beta_1 = \sigma_n^2$, $\tau_{n1} \rightarrow 0$, $b_1 \rightarrow +\infty$.

FILTER CHARACTERIZATION

We first consider an analog filter with a rational transfer function with N distinct poles, namely

$$H(f) = \sum_{n=1}^N \frac{\psi_n}{1 + i2\pi f \tau_{hn}} , \quad (47)$$

where $\{\tau_{hn}\}$ are distinct (complex) time constants. $H(f)$ can have up to $N-1$ zeros. The corresponding impulse response is

$$h(\tau) = \begin{cases} \sum_{n=1}^N \frac{\psi_n}{\tau_{hn}} \exp\left(\frac{-\tau}{\tau_{hn}}\right) & \text{for } \tau \geq 0 \\ 0 & \text{for } \tau < 0 \end{cases} , \quad (48)$$

which is required to be real for all τ . The sampled impulse response is then of the form

$$\Delta h(m\Delta) = \sum_{n=1}^N \psi_n c_n \exp(-c_n m) \quad \text{for } m \geq 0 , \quad c_n = \frac{\Delta}{\tau_{hn}} , \quad (49)$$

where the N filter exponential parameters $\{c_n\}$ are distinct from one another; also, $\text{Re } c_n > 0$.

The filter correlation function is available by substitution of (49) in (10), with the result

$$\phi_h(j\Delta) = \sum_{n=1}^N \psi_n c_n v_n \exp(-c_n |j|) \quad \text{for all } j , \quad (50)$$

where

$$v_n = \sum_{k=1}^N \frac{\psi_k c_k}{1 - \exp(-c_n - c_k)} \quad \text{for } 1 \leq n \leq N . \quad (51)$$

The form of filter correlation $\phi_h(j\Delta)$ in (50) is identical to the sampled input signal covariance (36), in that (50) is also composed of N distinct exponentials; hence, the comments and examples given there are relevant here also.

A useful alternative exists to the calculation of constants $\{v_n\}$ via (51). Namely, define z transform filter

$$\underline{H}(z) = \sum_{m=0}^{\infty} z^{-m} \Delta h(m\Delta) . \quad (52)$$

Replace n by k in (49) and substitute the result into (52); then an interchange of summations yields

$$\underline{H}(z) = \sum_{k=1}^N \frac{\psi_k c_k}{1 - \frac{1}{z} \exp(-c_k)} . \quad (53)$$

It then follows immediately by comparison with (51) that

$$v_n = \underline{H}(\exp(c_n)) \quad \text{for } 1 \leq n \leq N . \quad (54)$$

When $\underline{H}(z)$ is available as a rational function, (54) can be used directly, instead of summation (51).

The exponential parameters $\{c_n\}$, in impulse response (49) and corresponding N -pole filter (53), can be complex, provided that they occur in complex conjugate pairs and that their corresponding coefficients $\{\psi_n\}$ are also conjugate pairs. That is, the sum of two conjugate terms of the form of (49) must be real for all $m \geq 0$. This generality allows for oscillatory impulse responses such as yielded by narrowband filters.

FILTER OUTPUT NOISE COVARIANCE

In the sequel, we will find use for the following closed form for the discrete convolution of two exponentials:

$$\begin{aligned} \exp[-c|k|] \otimes \exp[-b|k|] &\equiv \sum_j \exp[-c|j| - b|k-j|] = \\ &= \left\{ \begin{array}{ll} \frac{\sinh(c) \exp(-b|k|) - \sinh(b) \exp(-c|k|)}{\cosh(c) - \cosh(b)} & \text{for } b \neq c \\ \left[|k| + \frac{\cosh(c)}{\sinh(c)} \right] \exp(-c|k|) & \text{for } b = c \end{array} \right\}. \quad (55) \end{aligned}$$

The filter output noise covariance is given by (9). If we define

$$C(k\Delta) = \sum_j \phi_h(j\Delta) R_n(k\Delta - j\Delta), \quad (56)$$

then the stationarity of filter output noise $y_n(k\Delta)$ in (8) allows us to express (9) as

$$C_n(k_1\Delta, k_2\Delta) = C(k_1\Delta - k_2\Delta). \quad (57)$$

Therefore, we can concentrate on evaluation of $C(k\Delta)$ in (56) for representative filter correlations ϕ_h and input noise covariances R_n .

In particular, we will use the general filter correlation (50) and input noise covariance (46), where we presume that none of the filter parameters $\{c_n\}$ are equal to any of the input noise parameters $\{b_m\}$. That is,

$$c_n \neq b_m \quad \text{for all } n, m. \quad (58)$$

Substitution of (50) and (46) in (56) and use of (55) results in

$$\begin{aligned} C(k\Delta) &= \sum_j \sum_{n=1}^N \psi_n c_n v_n \exp(-c_n |j|) \sum_{m=1}^M \beta_m \exp(-b_m |k-j|) = \\ &= \sum_{n=1}^N \sum_{m=1}^M \psi_n c_n v_n \beta_m \sum_j \exp[-c_n |j| - b_m |k-j|] = \\ &= \sum_{m=1}^M \mu_m \exp(-b_m |k|) + \sum_{n=1}^N \gamma_n \exp(-c_n |k|), \end{aligned} \quad (59)$$

where we defined auxiliary constants

$$\begin{aligned} \mu_m &= \beta_m \sum_{n=1}^N \frac{\psi_n c_n v_n \sinh(c_n)}{\cosh(c_n) - \cosh(b_m)} \quad \text{for } 1 \leq m \leq M, \\ \gamma_n &= \psi_n c_n v_n \sum_{m=1}^M \frac{\beta_m \sinh(b_m)}{\cosh(b_m) - \cosh(c_n)} \quad \text{for } 1 \leq n \leq N. \end{aligned} \quad (60)$$

Condition (58) keeps all the denominators in (60) from becoming zero. On the other hand, if one of the filter parameters $\{c_n\}$ is equal to one of the noise parameters $\{b_m\}$, then it is merely necessary to utilize instead the second line of (55) for that particular n, m pair in the j sum in the middle line of (59).

End result (59) for the filter output noise covariance is a compact expression that is capable of being quickly computed, once constants $\{\mu_m\}$ and $\{\gamma_n\}$ have been evaluated by means of (60) and stored. Again, we encounter a sum of decaying exponentials, albeit of $M + N$ terms now.

FILTER OUTPUT NOISE COVARIANCE FOR WHITE NOISE INPUT

When the input noise is white, then as noted under (46), we have for the input noise covariance,

$$R_n(k\Delta) = \sigma_n^2 \delta_{k0}, \quad \text{that is, } M = 1, \beta_1 = \sigma_n^2, b_1 \rightarrow +\infty. \quad (61)$$

Use of this result in (60) yields $\mu_1 \rightarrow 0$, $\gamma_n \rightarrow \sigma_n^2 \psi_n c_n v_n$, in which case the filter output noise covariance (59) becomes

$$C(k\Delta) = \sigma_n^2 \sum_{n=1}^N \psi_n c_n v_n \exp(-c_n |k|), \quad (62)$$

where $\{v_n\}$ are given by (51).

Actually, (62) is a special case of the general white noise result obtained by substitution of (61) into (56); namely, for arbitrary filter $\Delta h(m\Delta)$, the filter output noise covariance is

$$C(k\Delta) = \sigma_n^2 \phi_h(k\Delta) \quad \text{for white noise input,} \quad (63)$$

where filter correlation ϕ_h is given by (10).

FILTER OUTPUT SIGNAL COVARIANCE

The nonstationary filter output signal covariance is given by double summation (17) in conjunction with (18). If we substitute input signal covariance $R_s(k\Delta)$ given by (36), along with filter impulse response $\Delta h(m\Delta)$ given by (49), into (17), there follows

$$\begin{aligned}
 C_s(k_1\Delta, k_2\Delta) &= \sum_{m=k_a}^{K_1} \sum_{n=1}^N \psi_n c_n \exp(-c_n(k_1-m)) \sum_{p=k_a}^{K_2} \sum_{q=1}^N \psi_q c_q \times \\
 &\times \exp(-c_q(k_2-p)) \sum_{j=1}^J \alpha_j \exp(-a_j|m-p|) = \\
 &= \sum_{j=1}^J \alpha_j \sum_{n=1}^N \psi_n c_n \exp(-c_n k_1) \sum_{q=1}^N \psi_q c_q \exp(-c_q k_2) \times \\
 &\times \sum_{m=k_a}^{K_1} \exp(c_n m) \sum_{p=k_a}^{K_2} \exp(c_q p - a_j|m-p|) . \quad (64)
 \end{aligned}$$

In order to evaluate the innermost double summation in (64), we let $A = \exp(a_j)$, $B = \exp(c_q)$, $C = \exp(c_n)$ and we presume that $k_1 \leq k_2$. It then follows from (18) that $K_1 \leq K_2$, and the double sum in the last line of (64) can be developed thusly:

$$\begin{aligned}
 S(A, B, C, k_a, K_1, K_2) &\equiv \sum_{m=k_a}^{K_1} C^m \sum_{p=k_a}^{K_2} B^p A^{-|m-p|} = \quad (65) \\
 &= \sum_{m=k_a}^{K_1} C^m \left[\sum_{p=k_a}^m B^p A^{p-m} + \sum_{p=m}^{K_2} B^p A^{m-p} - B^m \right] =
 \end{aligned}$$

$$= \sum_{m=k_a}^{K_1} (C/A)^m \sum_{p=k_a}^m (BA)^p + \sum_{m=k_a}^{K_1} (CA)^m \sum_{p=m}^{K_2} (B/A)^p - \sum_{m=k_a}^{K_1} (CB)^m . \quad (66)$$

Now we use the fact that

$$\sum_{m=L}^M z^m = \frac{z^L - z^{M+1}}{1 - z} \quad \text{for } z \neq 1, \quad L \leq M, \quad (67)$$

and we define auxiliary function

$$g(z, k) \equiv \frac{z^k}{1 - z} \quad \text{for } z \neq 1. \quad (68)$$

Then, after some amount of manipulation, the sums in (66) can be expressed in the closed form

$$\begin{aligned} S(A, B, C, k_a, K_1, K_2) = & g(CB, k_a) \frac{A^2 - CB}{(A-B)(A-C)} + g(CB, K_1+1) \frac{B(A^2 - 1)}{(A-B)(1-BA)} \\ & + g(B/A, K_2+1) [g(CA, K_1+1) - g(CA, k_a)] - g(BA, k_a) g(C/A, K_1+1), \end{aligned} \quad (69)$$

provided that $B \neq A$ and $C \neq A$. (B cannot equal $1/A$, nor can C equal $1/A$ or $1/B$, because the real parts of c_n and a_j are always positive.)

We now employ definition (65) in order to express the filter output signal covariance in (64) in the final form

$$\begin{aligned} C_S(k_1\Delta, k_2\Delta) = & \sum_{j=1}^J \alpha_j \sum_{n=1}^N \psi_n c_n \exp(-c_n k_1) \sum_{q=1}^N \psi_q c_q \exp(-c_q k_2) \times \\ & \times S(\exp(a_j), \exp(c_q), \exp(c_n), k_a, K_1, K_2), \end{aligned} \quad (70)$$

provided that none of the filter parameters $\{c_n\}$ are equal to any of the input signal parameters $\{a_j\}$. That is,

$$c_n \neq a_j \quad \text{for all } n, j. \quad (71)$$

This result, (70), cannot be further reduced without the various parameters being specified. The triple summation will not be overly time consuming unless the input signal or filter are characterized by a large number of poles, that is, large J or N .

The result in (70) holds for $k_1 \leq k_2$; the range for $k_1 > k_2$ is most easily handled by use of symmetry relation (15). Also, a significant computational aid is available by using the recurrence relation for the $g(z, k)$ function in (68), namely $g(z, k) = z g(z, k-1)$.

In summary, k_a, k_b, k_c, k_d are given integers satisfying $k_a \leq k_b, k_c \leq k_d, k_a \leq k_d$. Also, $K_0 = \max(k_a, k_c)$, $K_1 = \min(k_1, k_b)$, $K_2 = \min(k_2, k_d)$. Since we keep $k_1 \leq k_2$, then $K_1 \leq K_2$. Integers k_1 and k_2 must vary over the range

$$K_0 \leq k_1 \leq k_2 \leq k_d, \quad (72)$$

meaning that K_1 varies in the range $\min(K_0, k_b)$ to $\min(k_d, k_b)$.

From (69), we see that we have to compute function values $g(CB, k_a), g(CA, k_a), g(BA, k_a)$, as well as the arrays of function values of $g(CB, k), g(CA, k), g(B/A, k), g(C/A, k)$ for k in the range $\min(K_0, k_b)+1$ to $\min(k_d, k_b)+1$.

PROGRAMS

A program listing for the white noise input case is presented in appendix D; it consists of two separate parts. The first part computes the filter output signal and noise covariance matrices for unity input signal and noise powers, the total filter output covariance matrix of elements C_y in (23) for the various signal-to-noise ratios of interest, the weighted covariance matrix C of elements C_a in (26) and its corresponding eigenvalue matrix Λ in (27). The output of these eigenvalues to storage completes the first part.

The second part lists the program that takes in these eigenvalues and computes the exceedance distribution functions, for noise-alone as well as for signal present, from characteristic function (32); these are the false alarm and detection probabilities, respectively. The precise method of handling all the widely different signal-to-noise ratios of interest is presented in the next section. At the end of appendix D, there is also a listing of the simulation program utilized to verify these theoretical derivations and results.

The corresponding program for the colored noise case is listed in appendix E. A similar breakdown into two parts has been adopted; however, once the eigenvalues have been computed at the end of the first part, the identical program in part 2 of appendix D is used for distribution calculations and is therefore not listed again. Also, the simulation program use to check the colored noise case is similar to that in part 3 of appendix D.

ON CALCULATION OF RECEIVER OPERATING CHARACTERISTICS

The evaluation of the receiver operating characteristics for a number of different input signal-to-noise ratios typically involves the Fourier transforms of a set of characteristic functions of the decision variable, which frequently have widely different ranges and rates of variation. In order to make these calculations tractable and easily plotted, it is necessary to resort to FFTs [2] with carefully chosen sizes and sampling increments. We now present the reasoning behind our choices of these parameters and their interrelationships.

For ease of presentation, we will consider the numerical evaluation of a probability density function $p_0(u)$ from a given (noise-only) characteristic function $f_0(\xi)$; the extension to an exceedance distribution function [2] is immediate. We have

$$p_0(u) = \frac{1}{2\pi} \int d\xi \exp(-i\xi u) f_0(\xi) =$$

$$\approx \frac{1}{2\pi} \Delta_0 \sum_k \exp(-ik\Delta_0 u) f_0(k\Delta_0) \equiv \tilde{p}_0(u) , \quad (73)$$

where the trapezoidal approximation with sampling increment Δ_0 in ξ has been employed. The approximation $\tilde{p}_0(u)$ defined by (73) is an aliased version of $p_0(u)$ and has period $u_0 = 2\pi/\Delta_0$ in u . In order to keep aliasing effects negligible in $\tilde{p}_0(u)$, it will be necessary to choose ξ increment Δ_0 small enough that the aliasing lobes at separation u_0 don't overlap. Therefore, we can restrict the evaluation of $\tilde{p}_0(u)$ to N_0 equally spaced points over that interval u_0 , according to

$$\tilde{p}_0\left(\frac{2\pi n}{N_0\Delta_0}\right) = \frac{1}{2\pi} \Delta_0 \sum_k \exp(-i2\pi kn/N_0) f_0(k\Delta_0) \quad \text{for } 0 \leq n \leq N_0-1. \quad (74)$$

Furthermore, by collapsing the sequence $\{f_0(k\Delta_0)\}$ modulo N_0 , (74) can be accomplished exactly and efficiently as an N_0 -point FFT when N_0 is a power of 2; see [2]. The increment in argument u of $\tilde{p}_0(u)$ in (74) and the maximum useful value of u are

$$\delta_0 = \frac{2\pi}{N_0\Delta_0}, \quad u_0 = \frac{2\pi}{\Delta_0}. \quad (75)$$

Now suppose that we also want to evaluate the probability density function $p_1(u)$ corresponding to a different (signal present) characteristic function $f_1(\xi)$ according to

$$\begin{aligned} p_1(u) &= \frac{1}{2\pi} \int d\xi \exp(-i\xi u) f_1(\xi) = \\ &\approx \frac{1}{2\pi} \Delta_1 \sum_k \exp(-ik\Delta_1 u) f_1(k\Delta_1) \equiv \tilde{p}_1(u), \end{aligned} \quad (76)$$

where the sampling increment in ξ is now Δ_1 . By identical reasoning to that above, we can get samples of periodic (aliased) approximation $\tilde{p}_1(u)$ according to

$$\tilde{p}_1\left(\frac{2\pi n}{N_1\Delta_1}\right) = \frac{1}{2\pi} \Delta_1 \sum_k \exp(-i2\pi kn/N_1) f_1(k\Delta_1) \quad \text{for } 0 \leq n \leq N_1-1, \quad (77)$$

which can be realized as an N_1 -point FFT. The increment in u and the maximum useful value of u for this latter case are

$$\delta_1 = \frac{2\pi}{N_1\Delta_1}, \quad u_1 = \frac{2\pi}{\Delta_1}. \quad (78)$$

If we now want to eliminate the arguments of functions $\tilde{p}_0(u)$ and $\tilde{p}_1(u)$ and be able to directly plot $\tilde{p}_1(u)$ versus $\tilde{p}_0(u)$ (that is, y versus x), this can be easily accomplished if we force the u increments in (74) and (77) to be equal, namely, $\delta_1 = \delta_0$. But, then, according to (75) and (78), this means choosing

$$N_1 \Delta_1 = N_0 \Delta_0 . \quad (79)$$

Since integers N_0 and N_1 will be limited to powers of 2, this will mean repeated halving of the ξ increments Δ_0 and Δ_1 in (73) and (76), respectively. The values of $\tilde{p}_1$ yielded by (77) can then be directly plotted versus those values of $\tilde{p}_0$ yielded by (74), up to $n = \min(N_0, N_1) = N_0$.

There are several alternative procedures that could have been adopted. For example, one option is to halve the ξ increment, that is, take $\Delta_1 = \Delta_0/2$, but keep $N_1 = N_0$. This will require discarding every other value put out by FFT (74), in order that the remaining values occur at the same u arguments yielded by (77). This is wasteful of FFT (74) and has not been adopted here. Nor have we employed the possibility of interpolation of (77) to determine approximately what the values of $\tilde{p}_1$ would be at the arguments of $\tilde{p}_0$ in (74).

The need to decrease the size of the ξ increment from Δ_0 in (74) to Δ_1 in (77) is fundamental. It arises from the fact that as the input signal-to-noise ratio is increased, the contribution of the corresponding probability density function $p_k(u)$, $k = 0, 1, 2, \dots$, moves to higher values on the u scale (thereby

leading to the desired higher detection probabilities). In order that approximation $\tilde{p}_k(u)$ not be severely aliased, the ξ increment Δ_k must therefore be decreased; for example, see upper limits u_0 and u_1 in (75) and (78), respectively.

With these points in mind, the following procedure has been adopted and utilized in the programs in this report. First, for the noise-only probability density function $p_0(u)$, a satisfactory value for ξ -increment Δ_0 in (73) is found such that aliasing is insignificant in $\tilde{p}_0(u)$. This selection of Δ_0 is arrived at by looking at a plot of (74) and modifying Δ_0 appropriately by trial and error. An FFT size of $N_0 = 128$ is utilized to evaluate (74), which is then stored; this size for N_0 has generally proven to be sufficient to keep track of the variations of $p_0(u)$.

When we encounter the characteristic function $f_1(\xi)$ and probability density function $p_1(u)$ for the first (lowest) nonzero signal-to-noise ratio of interest, it is usually necessary to decrease the ξ increment to $\Delta_1 = \Delta_0/2$ or $\Delta_0/4$ in order to control the aliasing inherent in approximation $\tilde{p}_1(u)$. At the same time, N_1 is scaled up by 2 or 4, according to (79), thereby maintaining the same u arguments for (74) and (77), as desired. Again, a plot of aliased version $\tilde{p}_1(u)$, obtained by means of (77), serves as the decision point for making an acceptable choice of value for Δ_1 .

For the successively larger signal-to-noise ratios and their corresponding probability density functions $p_k(u)$, $k=1,2,\dots$, occasional repeated halvings of the ξ increment according to

$\Delta_k = \Delta_{k-1}/2$ are made, but only when needed to keep aliasing insignificant in $\tilde{p}_k(u)$. At those times that such a halving is necessary and made, the FFT size is doubled according to $N_k = 2 N_{k-1}$. Otherwise, Δ_k and N_k are kept at their same values as used for the $k-1$ run. By the time the largest signal-to-noise ratio case of interest is reached, the FFT size N_k can get rather large. We are limited in our application to a maximum FFT size of 16384; if aliasing is significant at this size, we are unable to accurately numerically handle these larger signal-to-noise ratios by the adopted procedure. Interpolation would then be the recommended alternative, as alluded to above.

The procedure above has been explained in terms of probability density functions rather than the exceedance distribution functions which are of actual interest. That is, the relevant exceedance distribution functions are plots of the detection probabilities for various signal-to-noise ratios versus the false alarm probability. The only basic change is to replace the top lines of (73) and (76) by the exceedance distributions, which are also obtained by means of Fourier transforms, namely [2; (5) - (6)]

$$E_k(u) = \frac{1}{2} + \int_{0+}^{+\infty} d\xi \operatorname{Im} \left\{ \exp(-i\xi u) \frac{f_k(\xi)}{\pi\xi} \right\}, \quad (80)$$

for $k = 0, 1, 2, \dots$

RESULTS

COMPARISON WITH SIMULATION

In figure 7, the cumulative (CDF) and exceedance distribution functions (EDF) for a white noise input excitation are presented. The input signal and filter are both characterized by one pole, that is, $J = 1$ in (33) - (36) and $N = 1$ in (47) - (50). The simulation employed $1E6$ trials and verifies the theoretical calculations down to the probability level of $1E-5$ plotted. A listing of the simulation program, including all the parameter values that were utilized here, is given at the end of appendix D. The gating and observation parameters are $k_a = 4$, $k_b = 11$ and $k_c = 2$, $k_d = 16$, respectively; thus, it can be seen that the transient buildup and decay are a prominent part of the filter output signal for this particular example.

The results in figure 8 pertain to a colored noise input with one pole, that is, $M = 1$ in (44) - (46). Again, $1E6$ trials were used in the simulation and they confirm the theoretical result down to the $1E-5$ level of probabilities. A listing of the simulation program is given at the end of appendix D.

Absolute time is unimportant to the performance of the energy detector in figure 1. That is, any common constant can be added to or subtracted from k_a , k_b , k_c , k_d , without changing the receiver operating characteristics. Thus, k_a could always be selected as 0 without loss of generality.

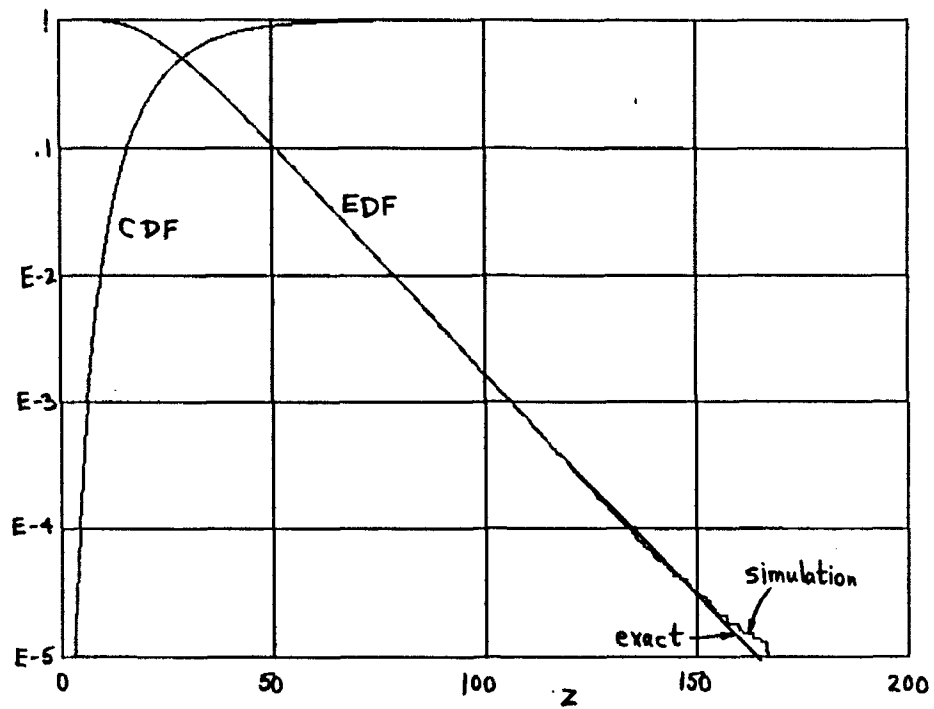


Figure 7. Cumulative and Exceedance Distributions for White Noise

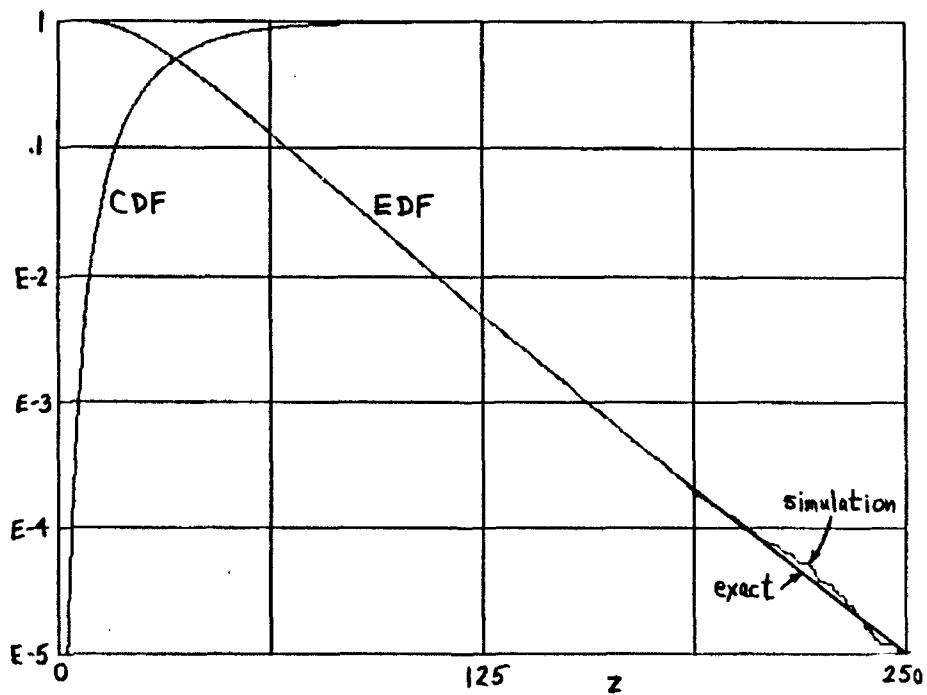


Figure 8. Cumulative & Exceedance Distributions for Colored Noise

RECEIVER OPERATING CHARACTERISTICS

The receiver operating characteristics, that is, detection probability P_D versus false alarm probability P_F for a set of signal-to-noise ratios, for the system in figure 1 will be considered in this section. The first example is evaluated for the following set of parameters, where all times are in seconds. Also, all parameters are real.

$$\begin{aligned} k_a &= 4, k_b = 11, k_c = 2, k_d = 16, \Delta = .2, \\ J &= 3, \{\alpha_j\} = \{11 \quad -48 \quad 75\}, \{\tau_{sj}\} = \{1 \quad 1/8 \quad 1/15\}, \\ M &= 2, \{\beta_m\} = \{39 \quad 60\}, \{\tau_{nm}\} = \{.2 \quad .4\}, \\ N &= 4, \{\psi_n\} = \{1 \quad 2 \quad 3 \quad 4\}, \{\tau_{hn}\} = \{.3 \quad .5 \quad .7 \quad .9\}. \end{aligned} \quad (81)$$

Twenty nonzero values of the system input signal-to-noise ratio

$$R = \frac{\sigma_s^2}{\sigma_n^2} \quad (82)$$

were utilized, namely $R = 5(1.2)27.8$ dB. For computation of the exceedance distribution function P_F directly from the characteristic function, the initial ξ increment Δ_0 in (73) and (74) was taken as .00025; this yielded round-off errors less than $1E-15$ in the false alarm probability calculation. Repeated occasional halving of the ξ increment, as explained in the previous section, was utilized, eventually requiring an FFT size of 16384 for the largest signal-to-noise ratios cases for P_D .

The receiver operating characteristics are plotted on normal probability paper in figure 9 and cover a wide range of values,

ranging from $1E-10$ for P_F to $.999$ for P_D . Since Gaussian random variables would plot on such paper as a set of parallel straight lines, the curvatures of these results illustrate that the Gaussian assumption for decision variable z in figure 1 is not warranted, at least for this example. The major reason for this behavior is the relatively small number of samples used in the detector; see the top line of (81). This is also a partial explanation for the rather large values of the per-sample signal-to-noise ratio R required at the high quality region at the top left of figure 9. Another factor to notice is that the input signal duration is only $k_b\Delta - k_a\Delta = 1.4$ seconds, meaning that the filter output signal never reaches steady state before the input signal is turned off; all these transient signal effects and their limitations on performance have been included exactly in the analysis and numerical results presented here.

The effect of halving the sampling increment Δ in figure 1 is considered in the next example but done in such a manner as to keep the total input signal duration the same. That is, the parameter values in (81) are kept the same except for the following changes:

$$k_a = 8, k_b = 22, k_c = 4, k_d = 32, \Delta = .1. \quad (83)$$

Notice that all absolute times, such as $k_a\Delta$, are kept fixed. The corresponding receiver operating characteristics are plotted in figure 10. Performance has been degraded by a couple of dB. For example, to realize $P_F = 1E-3$ and $P_D = .5$, the per-sample input

signal-to-noise ratio R must now be 16 dB, whereas it was 13.8 dB in the previous figure. This is a degradation of 2.2 dB. The main reason for this behavior is again the inability of the filter output signal to reach steady state and thereby contribute significantly to the summer output. By contrast, the filter output noise is in steady state (by assumption) for both examples.

We now return to the original sampling increment $\Delta = .2$ seconds employed in (81). When the observation interval coincides with the input signal nonzero-excitation duration, that is, $k_a = k_c = 4$, $k_b = k_d = 11$, the receiver operating characteristics in figure 11 illustrate additional improvement. For example, probabilities $P_F = 1E-3$ and $P_D = .5$ can now be realized with $R = 13.1$ dB, which is an improvement of .7 dB relative to the signal-to-noise ratio required for the broader observation interval, $k_c = 2$, $k_d = 16$, used in (81).

Finally, an example with a wide observation interval, namely $k_c = 0$, $k_d = 25$, is displayed in figure 12. The desired probabilities can now be achieved only if the input signal-to-noise ratio is increased to 14.7 dB, a degradation of 1.6 dB relative to the best case above.

These examples illustrate the utility of being able to investigate quantitatively and accurately the effects of nonstationarity in a system, without having to make questionable assumptions about, for example, how closely steady state was or was not realized. They also afford a dependable verification or

rejection of the Gaussian approximation for the decision variable; in a related work [3], the latter approximation was found to be too optimistic for most working ranges of this detection system.

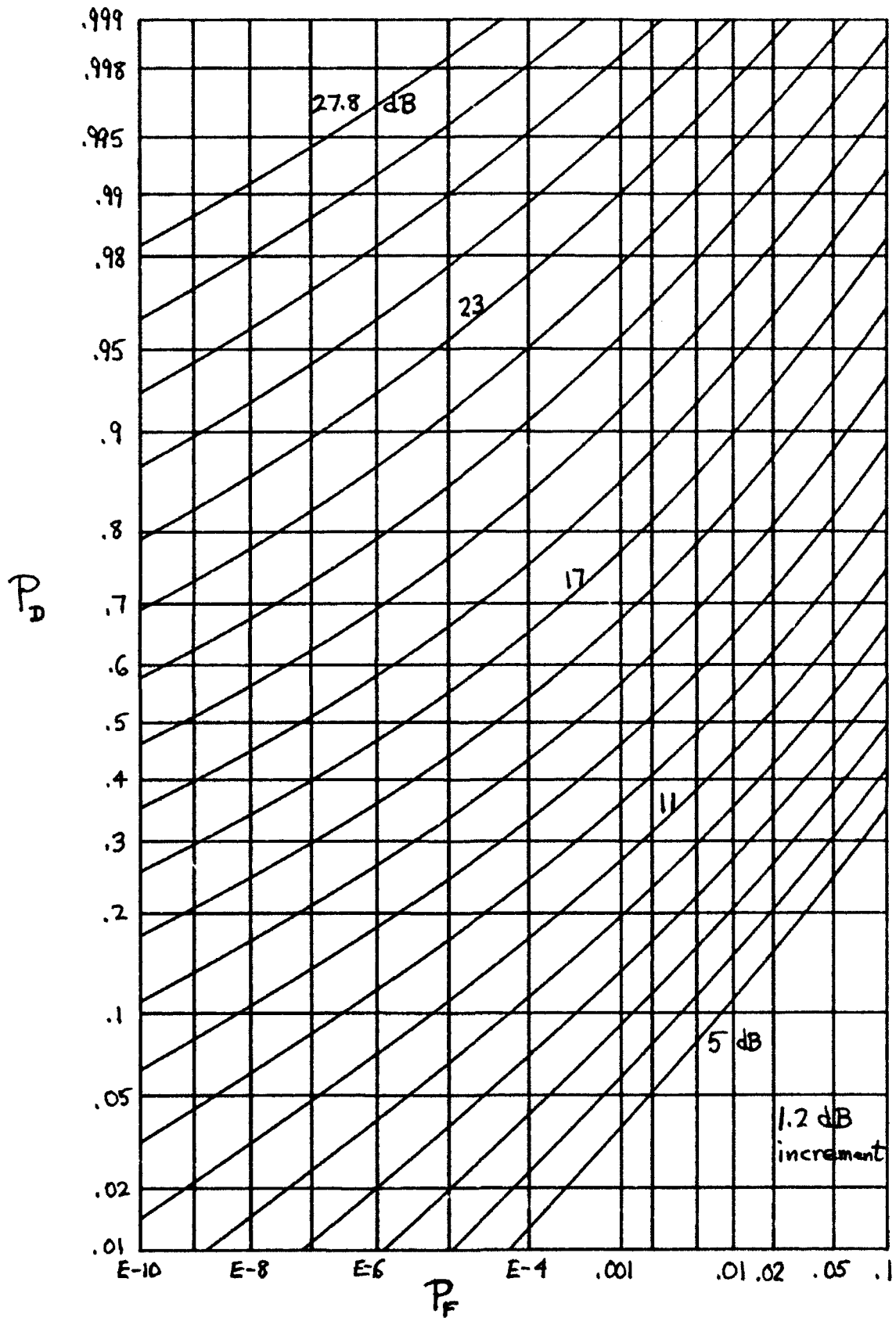


Figure 9. Receiver Operating Characteristics; Example A

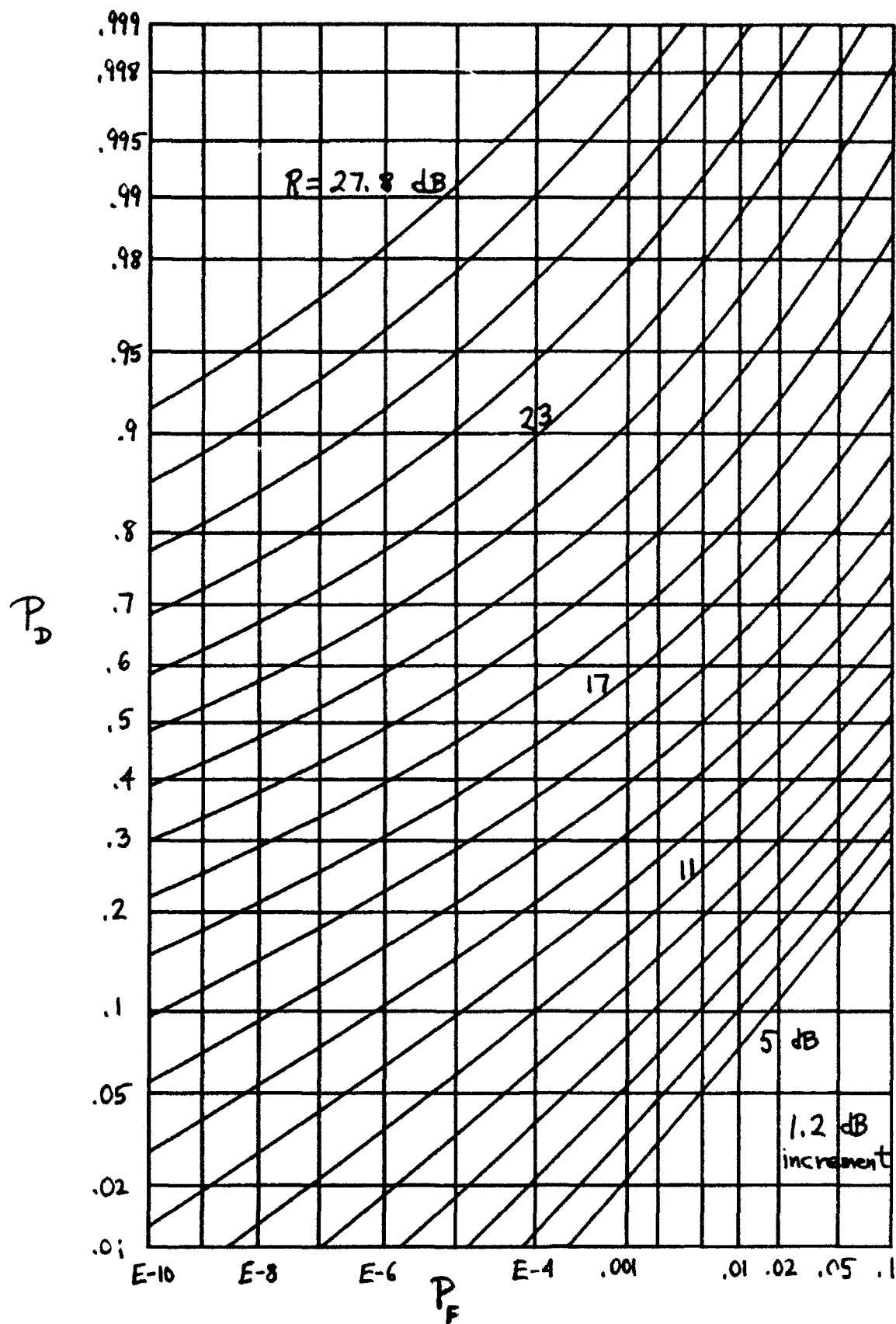


Figure 10. Receiver Operating Characteristics; Example B

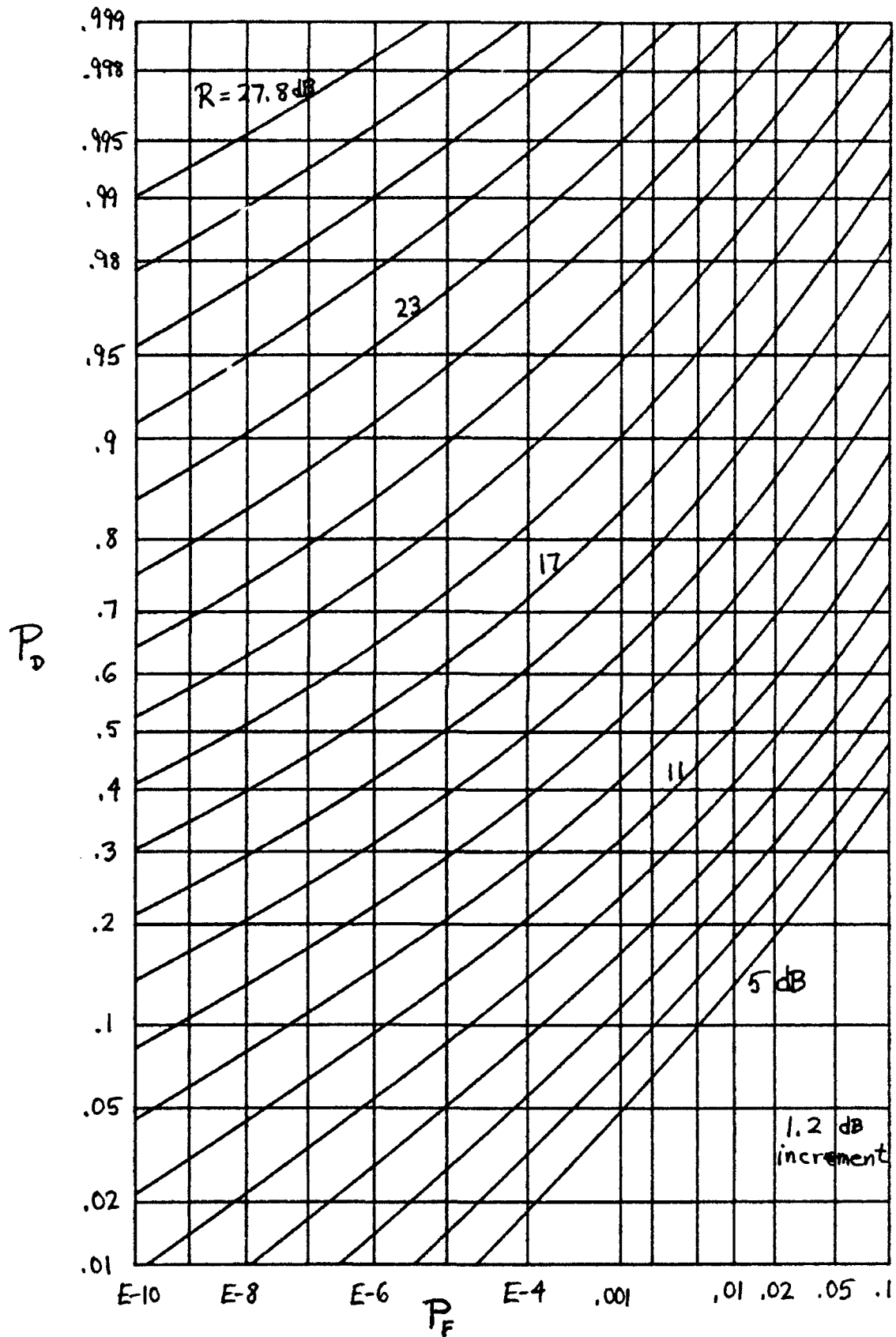


Figure 11. Receiver Operating Characteristics; Example C

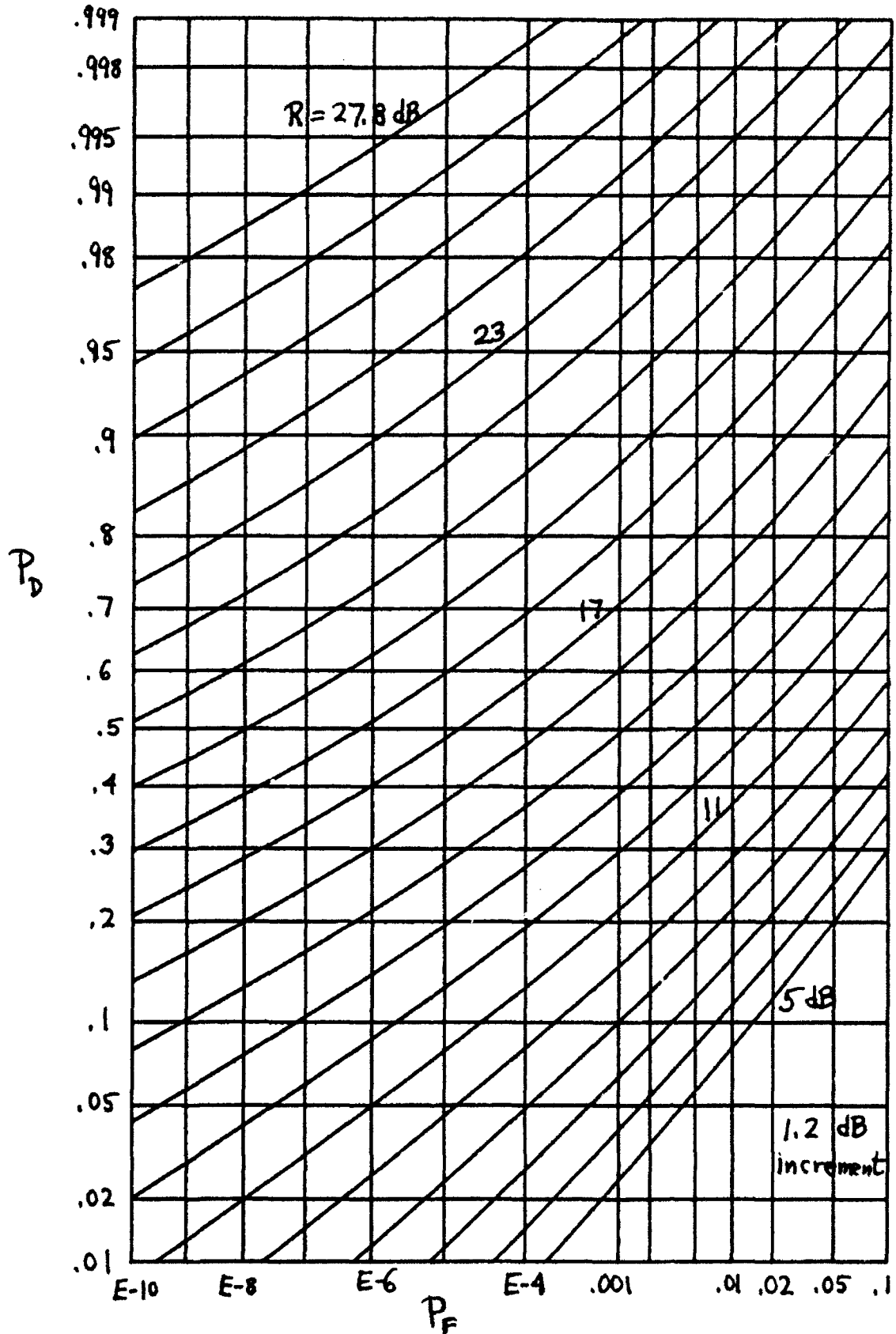


Figure 12. Receiver Operating Characteristics; Example D

SUMMARY

Programs have been written which enable exact analysis of performance of the mismatched bandpass energy detector of figure 1, both for the white noise input case as well as the colored noise input case. These results allow for arbitrary sampling time increment Δ and for arbitrary input signal spectra, input noise spectra, and filter transfer functions. By these means, exact quantitative evaluations and degradations can be determined for various combinations of uncertainty regarding the input signal time location and duration as well as its center frequency, bandwidth, and spectrum. Included in the analysis and programs are the buildup and decay (or any portion thereof) of the nonstationary output signal from the filter, when excited by a burst-like input signal, regardless of the lengths or locations of the excitation and observation intervals. Both programs have been compared with simulation results and confirmed down to as low a probability level as possible consistent with the number of trials used.

APPENDIX A. NUMERICAL EVALUATION OF NOISE COVARIANCE R_n

If the noise covariance $R_n(\tau)$ is not available in analytic form, but the noise spectrum $G_n(f)$ is specified, the following numerical procedure can be employed to evaluate the required samples $R_n(k\Delta)$. We begin with

$$R_n(\tau) = \int df \exp(i2\pi f\tau) G_n(f) =$$

$$\approx \Delta_f \sum_m \exp(i2\pi m\Delta_f \tau) G_n(m\Delta_f) \equiv \tilde{R}_n(\tau) , \quad (\text{A-1})$$

where the trapezoidal rule with frequency increment Δ_f was used. But $\tilde{R}_n(\tau)$ in (A-1) can be developed as

$$\tilde{R}_n(\tau) = \int df \exp(i2\pi f\tau) G_n(f) \Delta_f \delta_{\Delta_f}(f) =$$

$$= R_n(\tau) \otimes \delta_{1/\Delta_f}(\tau) = \sum_m R_n\left(\tau - \frac{m}{\Delta_f}\right) . \quad (\text{A-2})$$

Now suppose that the noise covariance

$$R_n(\tau) \approx 0 \quad \text{for } |\tau| > \tau_0 . \quad (\text{A-3})$$

Then, in order to avoid significant aliasing in (A-2), we must take Δ_f small enough that

$$\frac{1}{2\Delta_f} > \tau_0 , \quad (\text{A-4})$$

in which case (A-2) yields

$$\tilde{R}_n(\tau) \approx R_n(\tau) \quad \text{for } |\tau| < \tau_0 < \frac{1}{2\Delta_f} . \quad (\text{A-5})$$

The required samples of $\tilde{R}_n(\tau)$ follow from (A-1) according to

$$\tilde{R}_n(k\Delta) = \Delta_f \sum_m \exp(i2\pi m \Delta_f k \Delta) G_n(m \Delta_f) . \quad (\text{A-6})$$

Now take frequency increment

$$\Delta_f = \frac{1}{N\Delta} , \quad (\text{A-7})$$

where, according to (A-4), integer N must be taken large enough that $N\Delta/2 > \tau_0$, that is,

$$N > \frac{2\tau_0}{\Delta} . \quad (\text{A-8})$$

Then, from (A-5) - (A-7),

$$R_n(k\Delta) \approx \tilde{R}_n(k\Delta) = \frac{1}{N\Delta} \sum_m \exp(i2\pi mk/N) G_n\left(\frac{m}{N\Delta}\right) \quad \text{for } |k| < \frac{N}{2} . \quad (\text{A-9})$$

Alternatively, this can be expressed as

$$R_n(k\Delta) \approx \tilde{R}_n(k\Delta) = \frac{1}{N\Delta} \sum_{m=0}^{N-1} \exp(i2\pi mk/N) \tilde{G}_n\left(\frac{m}{N\Delta}\right) \quad \text{for } |k| < \frac{N}{2} , \quad (\text{A-10})$$

where collapsed (prealiased) noise spectrum and sequence

$$\tilde{G}_n(f) \equiv \sum_j G_n\left(f - \frac{j}{\Delta}\right) , \quad \tilde{G}_n\left(\frac{m}{N\Delta}\right) = \sum_j G_n\left(\frac{m}{N\Delta} - \frac{j}{\Delta}\right) \quad \text{for } 0 \leq m \leq N-1 . \quad (\text{A-11})$$

The procedure in (A-10) can be efficiently realized as an N-point FFT of N nonzero numbers. The major conditions that must be met are (A-8) in conjunction with (A-3).

APPENDIX B. NUMERICAL EVALUATION OF FILTER CORRELATION ϕ_h

Suppose that we want digital filter correlation

$$\phi_h(j\Delta) = \Delta^2 \sum_m h(m\Delta) h(m\Delta - j\Delta) , \quad (B-1)$$

defined in (10), but all that we have is the filter transfer function $H(f)$, where

$$h(\tau) = \int df \exp(i2\pi f\tau) H(f) \quad \text{for all } \tau . \quad (B-2)$$

Impulse response $h(\tau)$ is presumed real. The direct relationship between $\phi_h(j\Delta)$ and $H(f)$ is the subject of this appendix.

First, define periodic function

$$\tilde{H}(f) \equiv \sum_m H\left(f - \frac{m}{\Delta}\right) \quad \text{for all } f . \quad (B-3)$$

Then, there follows

$$\begin{aligned} \tilde{H}(f) &= H(f) \otimes \delta_{1/\Delta}(f) = \\ &= \int d\tau \exp(-i2\pi f\tau) h(\tau) \Delta \delta_{\Delta}(\tau) = \sum_m \exp(-i2\pi f\Delta m) \Delta h(m\Delta) , \end{aligned} \quad (B-4)$$

where we used the fact that convolution in the frequency domain is equivalent to Fourier transformation of a product in the time domain. That is, $\tilde{H}(f)$ is the Fourier transform of the samples of impulse response $h(\tau)$ taken at time increment Δ . However, care must be taken at points of discontinuity of $h(\tau)$; for example, if $h(\tau)$ is discontinuous at $k\Delta$, the contribution to the right-hand

side of (B-4), at $m = k$, is $\frac{1}{2}[h(k\Delta+) + h(k\Delta-)]$. This point and an example are discussed more fully in appendix C.

Now, there follows from (B-1) and (B-2),

$$\begin{aligned}
 \phi_h(j\Delta) &= \Delta^2 \sum_m \int df \exp(i2\pi f\Delta m) H(f) \int du \exp[-i2\pi u\Delta(m-j)] H^*(u) = \\
 &= \Delta \iint df du H(f) H^*(u) \exp(i2\pi u\Delta j) \Delta \sum_m \exp[i2\pi(f - u)\Delta m] = \\
 &= \Delta \iint df du H(f) H^*(u) \exp(i2\pi u\Delta j) \sum_m \delta\left(f - u - \frac{m}{\Delta}\right) = \\
 &= \Delta \sum_m \int df H(f) H^*\left(f - \frac{m}{\Delta}\right) \exp\left[i2\pi\left(f - \frac{m}{\Delta}\right)\Delta j\right] = \\
 &= \Delta \int df \exp(i2\pi f\Delta j) H(f) \sum_m H^*\left(f - \frac{m}{\Delta}\right) = \\
 &= \Delta \int df \exp(i2\pi f\Delta j) H(f) \tilde{H}^*(f) , \tag{B-5}
 \end{aligned}$$

where we used (B-3).

Now, let I_n denote the following interval of length $1/\Delta$ on the f axis:

$$I_n \equiv \left(\frac{n-\frac{1}{2}}{\Delta}, \frac{n+\frac{1}{2}}{\Delta}\right) . \tag{B-6}$$

Then, the filter correlation ϕ_h follows from (B-5) according to

$$\phi_h(j\Delta) = \Delta \sum_n \int_{I_n} df \exp(i2\pi f\Delta j) H(f) \tilde{H}^*(f) =$$

$$\begin{aligned}
&= \Delta \sum_n \int_{I_0} du \exp[i2\pi(u + \frac{n}{\Delta})\Delta j] H(u + \frac{n}{\Delta}) \tilde{H}^*(u) = \\
&= \Delta \int_{I_0} du \exp(i2\pi u \Delta j) \tilde{H}^*(u) \sum_n H(u + \frac{n}{\Delta}) = \\
&= \Delta \int_{-.5/\Delta}^{.5/\Delta} du \exp(i2\pi u \Delta j) |\tilde{H}(u)|^2, \quad (B-7)
\end{aligned}$$

where we let $u = f - n/\Delta$ and used (B-3) and the periodicity of $\tilde{H}(f)$. This result indicates that we must first alias the given transfer function $H(f)$ according to (B-3) and then Fourier transform its magnitude-square over an interval of length $1/\Delta$.

An alternative approach that leads to the same result (B-7) is to use (B-2) and (B-3) in the form

$$h(m\Delta) = \int df \exp(i2\pi f \Delta m) H(f) = \int_{I_0} df \exp(i2\pi f \Delta m) \tilde{H}(f). \quad (B-8)$$

Use of this latter result in (B-1) leads to filter correlation

$$\begin{aligned}
\phi_h(j\Delta) &= \Delta^2 \sum_m \int_{I_0} df \exp(i2\pi f \Delta m) \tilde{H}(f) \int_{I_0} du \exp[-i2\pi u \Delta(m-j)] \tilde{H}^*(u) = \\
&= \Delta \iint_{I_0} df du \exp(i2\pi u \Delta j) \tilde{H}(f) \tilde{H}^*(u) \Delta \sum_m \exp[i2\pi(f - u)\Delta m] = \\
&= \Delta \iint_{I_0} df du \exp(i2\pi u \Delta j) \tilde{H}(f) \tilde{H}^*(u) \sum_m \delta\left(f - u - \frac{m}{\Delta}\right) =
\end{aligned}$$

$$= \Delta \int_{I_0} df \exp(i2\pi f \Delta j) |\tilde{H}(f)|^2, \quad (B-9)$$

since only one impulse, at $u = f$, lies in interval I_0 . Result (B-9) is identical to (B-7). Notice that we do not get

$$\Delta \int df \exp(i2\pi f \Delta j) |H(f)|^2 = \Delta \int_{I_0} df \exp(i2\pi f \Delta j) \sum_n \left| H\left(f + \frac{n}{\Delta}\right) \right|^2. \quad (B-10)$$

As for the actual numerical evaluation of (B-9), suppose we sample the integral on f with increment $\Delta_f = 1/(K\Delta)$ and use the trapezoidal approximation; then there follows

$$\phi_h(j\Delta) \approx \frac{1}{K} \sum_{k=-K/2}^{K/2} w_k \exp\left(i2\pi \frac{k}{K} j\right) \left| \tilde{H}\left(\frac{k}{K\Delta}\right) \right|^2, \quad (B-11)$$

where weights $\{w_k\}$ are given by

$$w_k = \begin{cases} \frac{1}{2} & \text{for } |k| = K/2 \\ 1 & \text{for } |k| < K/2 \end{cases}. \quad (B-12)$$

By using the periodicity K of the $\exp$ and $\tilde{H}$ terms in (B-11), that sum may be written exactly in the standard FFT form

$$\phi_h(j\Delta) \approx \frac{1}{K} \sum_{k=0}^{K-1} \exp(i2\pi jk/K) \left| \tilde{H}\left(\frac{k}{K\Delta}\right) \right|^2 \quad \text{for } |j| < \frac{K}{2}. \quad (B-13)$$

This is a good approximation, provided that the FFT size K satisfies $K > 2\tau_h/\Delta$, where τ_h is the effective duration of filter correlation $\phi_h(\tau)$.

APPENDIX C. DISPLACED SAMPLING AND FOURIER TRANSFORM OF DISCONTINUOUS FUNCTION

GENERAL RELATIONS

Let $a(t)$ and $A(f)$ be a Fourier transform pair:

$$A(f) = \int dt \exp(-i2\pi ft) a(t) . \quad (C-1)$$

Also, let $b(t)$ and $B(f)$ be a Fourier transform pair. Then, Parseval's theorem states that

$$\int dt a(t) b^*(t) = \int df A(f) B^*(f) . \quad (C-2)$$

We now apply this relation to the case where

$$b(t) = \Delta \delta_{\Delta}(t - \alpha) , \quad B(f) = \delta_{1/\Delta}(f) \exp(-i2\pi f\alpha) . \quad (C-3)$$

There follows

$$L \equiv \Delta \sum_n a(n\Delta + \alpha) = \int dt a(t) \Delta \delta_{\Delta}(t - \alpha) = \quad (C-4)$$

$$= \int df A(f) \delta_{1/\Delta}(f) \exp(i2\pi f\alpha) = \sum_n A\left(\frac{n}{\Delta}\right) \exp(i2\pi n\alpha/\Delta) \equiv R . \quad (C-5)$$

If the samples required in the sums in (C-4) and (C-5) encounter a discontinuity of $a(t)$ or $A(f)$, the value used in (C-4) or (C-5) must be taken as the average value approached from both sides of the discontinuity. Also, the sums in (C-4) and (C-5) may have to be interpreted as principal value, if necessary for convergence.

EXAMPLE

$$a(t) = \begin{cases} \exp[-(b+ic)t] & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}, \quad b > 0. \quad (C-6)$$

$$A(f) = \frac{1}{b + i(c + 2\pi f)} \quad \text{for all } f. \quad (C-7)$$

Function $a(t)$ is discontinuous at $t = 0$, while $A(f)$ is continuous for all f .

Let $0 < \alpha < \Delta$; then the sum in (C-4) is

$$L = \Delta \sum_{n=0}^{\infty} \exp[-(b+ic)(n\Delta+\alpha)] = \frac{\Delta \exp[-\alpha(b+ic)]}{1 - \exp[-\Delta(b+ic)]} \quad \text{for } 0 < \alpha < \Delta. \quad (C-8)$$

At the same time, the sum in (C-5) is

$$R = \sum_n \frac{\exp(i2\pi n\alpha/\Delta)}{b + i(c + 2\pi n/\Delta)} \quad \text{for all } \alpha. \quad (C-9)$$

When $\alpha \neq 0$ (or an integer multiple of Δ), the phase factor in the numerator of (C-9) yields a convergent sum for the real and imaginary parts, without the need for a principal value interpretation. It has also been verified numerically that (C-8) and (C-9) are equal, as predicted by (C-4) and (C-5); that is,

$$\sum_n \frac{\exp(i2\pi n\alpha/\Delta)}{b + i(c + 2\pi n/\Delta)} = \frac{\Delta \exp[-\alpha(b+ic)]}{1 - \exp[-\Delta(b+ic)]} \quad \text{for } 0 < \alpha < \Delta. \quad (C-10)$$

On the other hand, if $\alpha = 0$, then the sum in (C-4) is, accounting for the discontinuity of $a(t)$,

$$L = \frac{\Delta}{2} + \Delta \sum_{n=1}^{\infty} \exp[-(b+ic)n\Delta] = \frac{\Delta}{2} \frac{1 + \exp[-\Delta(b+ic)]}{1 - \exp[-\Delta(b+ic)]} . \quad (C-11)$$

Also, then, the sum in (C-5) is

$$R = \sum_n \frac{1}{b + i(c + 2\pi n/\Delta)} = \sum_n \frac{b - i(c + 2\pi n/\Delta)}{b^2 + (c + 2\pi n/\Delta)^2} , \quad (C-12)$$

the imaginary part of which must be interpreted as a principal value sum. Again, it has been numerically verified (next page) that (C-11) and (C-12) are equal; that is,

$$\sum_n \frac{1}{b + i(c + 2\pi n/\Delta)} = \frac{\Delta}{2} \frac{1 + \exp[-\Delta(b+ic)]}{1 - \exp[-\Delta(b+ic)]} . \quad (C-13)$$

Notice that the limit of (C-10) as $\alpha \rightarrow 0+$ does not yield (C-13).

If we apply result (C-11) to $\tilde{H}(f)$ in the last entry in (B-4), we have, for example (304),

$$\tilde{H}(f) = \frac{a}{2} \frac{1 + \exp(-a-i2\pi f\Delta)}{1 - \exp(-a-i2\pi f\Delta)} . \quad (C-14)$$

By contrast, the limit of (C-10) as $\alpha \rightarrow 0+$ yields

$$\frac{a}{1 - \exp(-a-i2\pi f\Delta)} . \quad (C-15)$$

BASIC PROGRAM FOR NUMERICAL VERIFICATION

```

10  Bs=.57                      | b
20  Cs=.71                      | c
30  De=.93                      | delta
40  Al=.1                       | alpha
50  PRINT Bs,Cs,De,Al
60  CALL Exp(-Al*Bs,-Al*Cs,Nr,Ni)
70  CALL Exp(-De*Bs,-De*Cs,Er,Ei)
80  CALL Div(De*Nr,De*Ni,1-Er,-Ei,Lr,Li)
90  IF Al=0. THEN Lr=Lr-.5*De
100 PRINT Lr,Li                 | LEFT-HAND SIDE L
110 A=2*PI/De
120 B=A*Al
130 B2=Bs*Bs
140 Rr=Ri=0.
150 DOUBLE Ns                   | INTEGER
160 FOR Ns=-1E5 TO 1E5         | PRINCIPAL VALUE SUM
170 T=B*Ns
180 C=COS(T)
190 S=SIN(T)
200 T=Cs+A*Ns
210 D=B2+T*T
220 Dr=(C*Bs+S*T)/D
230 Di=(S*Bs-C*T)/D
240 Rr=Rr+Dr
250 Ri=Ri+Di
260 NEXT Ns
270 PRINT Rr,Ri                 | RIGHT-HAND SIDE R
280 END
290 !
300 SUB Div(X1,Y1,X2,Y2,U,V)   | Z1/Z2
310 T=X2*X2+Y2*Y2
320 U=(X1*X2+Y1*Y2)/T
330 V=(Y1*X2-X1*Y2)/T
340 SUBEND
350 !
360 SUB Exp(X,Y,U,V)           | EXP(Z)
370 E=EXP(X)
380 U=E*COS(Y)
390 V=E*SIN(Y)
400 SUBEND

.57      .71      .93      0
.729357897734  -.80564044434
.729357647983  -.805640755434

.57      .71      .93      .001
1.19310532814  -.806028668009
1.19353711305  -.806028668087

.57      .71      .93      .1
1.07135515262  -.839119622089
1.07135537888  -.839119622093

```

APPENDIX D. PROGRAMS FOR WHITE INPUT NOISE

Three BASIC programs are listed in this appendix. All the signal, noise, and filter parameters are restricted to be real here. The first program computes the filter output signal and noise covariance matrices for a white noise input, the sums of these matrices scaled by the various signal-to-noise ratios and then weighted, and the eigenvalues which are then stored. The particular subroutine listed here, SUB Svd, actually computes the eigenvectors in addition; a faster alternative would be to replace this subroutine by one which calculates only the eigenvalues. Twenty different nonzero values of the input signal-to-noise ratio are allowed in the program and must be chosen and input by the user.

These eigenvalues serve as the input to the second program which computes the cumulative and exceedance distribution functions according to the method given in [2] and then plots the receiver operating characteristics by elimination of the threshold variable according to the method described in (73) - (80) in the main text.

The third program is the simulation program used to check the two programs above. It is written for single-pole processes and allows for a colored input noise process but could be easily modified to handle more general processes. A run of $1E6$ trials took 7 hours on the Hewlett-Packard 9000 computer.

```

10  ! COMPUTE COVARIANCE MATRICES AND EIGENVALUES FOR WHITE
20  ! NOISE INPUT; STORE EIGENVALUES IN "EIG" IN LINE 1720
30  Ka=-10          ! INPUT SIGNAL START
40  Kb=5            ! INPUT SIGNAL END; Kb >= Ka
50  Kc=0            ! ACCUMULATOR START
60  Kd=5            ! ACCUMULATOR END; Kd >= Kc,Ka
70  Delta=.2        ! TIME SAMPLING INCREMENT (SECONDS)
80  J=3             ! NUMBER OF SIGNAL COMPONENTS
90  DATA 11.,-48.,75. ! SIGNAL SCALINGS (WE SET SUM = 1)
100 DATA 1.,.125,.066667 ! SIGNAL TIME CONSTANTS (SECONDS)
110 N=4             ! NUMBER OF FILTER COMPONENTS
120 DATA 1.,2.,3.,4. ! FILTER SCALINGS (ARBITRARY)
130 DATA .3,.5,.7,.9 ! FILTER TIME CONSTANTS (SECONDS)
140 Nr=20           ! NUMBER OF SIGNAL-TO-NOISE RATIOS
150 DATA 0,1,2,3,4,5,6,7,8,9,10 ! INPUT SNRS IN DB
160 DATA 11,12,13,14,15,16,17,18,19
170 IF (Ka<=Kb) AND (Kc<=Kd) AND (Ka<=Kd) THEN 200
180 PRINT "PROBLEM WITH PARAMETERS"
190 STOP
200 DIM Alpha(10),A(10),Psi(10),C(10),R(20),W(100)
210 DIM Pc(10),Ec(10),Pcv(10),Cn(200),Ea(10),E(0,1000)
220 DIM Gca(100),Gcda(100),Gcb(100),Gbda(100),Cs(5000)
230 DIM Cw(0,10000),Ca(0,10000),D(100),Eig(0,2000)
240 DOUBLE Ka,Kb,Kc,Kd,J,N,Nr,Ns,Ks,Kdc,Ko ! INTEGERS
250 DOUBLE L1,L2,L,L11,Js,Qs,Ki,Ks1,K1,Ks2,K2,I
260 REDIM Alpha(1:J),A(1:J)
270 READ Alpha(*),A(*)
280 MAT Alpha=Alpha/(SUM(Alpha))
290 MAT A=(Delta)/A
300 REDIM Psi(1:N),C(1:N)
310 READ Psi(*),C(*)
320 MAT C=(Delta)/C
330 REDIM R(1:Nr)
340 READ R(*)
350 REDIM W(Kc:Kd)
360 CALL Weights(Kc,Kd,W(*))
370 MAT W=SQR(W)
380 REDIM Pc(1:N),Ec(1:N) ! FILTER OUTPUT NOISE COVARIANCE
390 FOR Ns=1 TO N
400 C=C(Ns)
410 Pc(Ns)=Psi(Ns)*C
420 Ec(Ns)=EXP(-C)
430 NEXT Ns
440 REDIM Pcv(1:N)
450 FOR Ns=1 TO N
460 E=Ec(Ns)
470 S=0.
480 FOR Ks=1 TO N
490 S=S+Pc(Ks)/(1.-E*Ec(Ks))
500 NEXT Ks
510 Pcv(Ns)=Pc(Ns)*S
520 NEXT Ns
530 Kdc=Kd-Kc
540 REDIM Cn(0:Kdc)
550 MAT Cn=(0.)
560 FOR Ns=1 TO N
570 Pcv=Pcv(Ns)
580 E=Ec(Ns)

```

```

590 P=1.
600 Cn(0)=Cn(0)+Pcv
610 FOR Ks=1 TO Kdc
620 P=P*E
630 Cn(Ks)=Cn(Ks)+Pcv*P
640 NEXT Ks
650 NEXT Ns
660 FOR Js=1 TO J          ! FILTER OUTPUT SIGNAL COVARIANCE
670 A(Js)=EXP(A(Js))
680 NEXT Js
690 Ko=MAX(Ka,Kc)
700 REDIM Ea(1:N),E(1:N,Ko:Kd)
710 FOR Ns=1 TO N
720 E=Ec(Ns)
730 Ea(Ns)=1./E
740 E(Ns,Ko)=EXP(-C(Ns)*Ko)
750 FOR Ks=Ko+1 TO Kd
760 E(Ns,Ks)=E(Ns,Ks-1)*E
770 NEXT Ks
780 NEXT Ns
790 L1=MIN(Ko,Kb)+1
800 L2=MIN(Kd,Kb)+1
810 REDIM Gca(L1:L2),Gcda(L1:L2),Gcb(L1:L2),Gbda(L1:L2)
820 L=Kd-Ko+1
830 REDIM Cs(1:L*(L+1)/2)
840 L11=L1+1
850 FOR Js=1 TO J
860 A1=Alpha(Js)
870 A=A(Js)
880 A1=A*A-1.
890 FOR Ns=1 TO N
900 C=Ea(Ns)
910 Ca=C*A
920 Cda=C/A
930 Cal=1.-Ca
940 Ac=A-C
950 Ae=A1*Pc(Ns)
960 Gca=Ca^Ka/Cal
970 Gca(L1)=Ca^L1/Cal
980 Gcda(L1)=Cda^L1/(1.-Cda)
990 FOR Ks=L11 TO L2
1000 Gca(Ks)=Ca*Gca(Ks-1)
1010 Gcda(Ks)=Cda*Gcda(Ks-1)
1020 NEXT Ks
1030 FOR Qs=1 TO N
1040 B=Ea(Qs)
1050 Aee=Ae*Pc(Qs)
1060 Ba=B*A
1070 Bda=B/A
1080 Cb=C*B
1090 Bal=1.-Ba
1100 Cb1=1.-Cb
1110 Gba=Ba^Ka/Bal
1120 Ab=A-B
1130 F=B*A1/(Ab*Bal)
1140 S1=Cb^Ka/Cb1*(A1+Cb1)/(Ab*Ac)
1150 Gcb(L1)=Cb^L1/Cb1
1160 Gbda(L1)=Bda^L1/(1.-Bda)

```



```

1170 FOR Ks=L11 TO L2
1180 Gcb(Ks)=Cb*Gcb(Ks-1)
1190 Gbda(Ks)=Bda*Gbda(Ks-1)
1200 NEXT Ks
1210 Ki=0
1220 FOR Ks1=Ko TO Kd
1230 K1=MIN(Ks1,Kb)+1
1240 S2=S1+Gcb(K1)*F-Gba*Gcda(K1)
1250 S3=Gca(K1)-Gca
1260 E=Aee*E(Ns,Ks1)
1270 FOR Ks2=Ks1 TO Kd
1280 Ki=Ki+1
1290 K2=MIN(Ks2,Kb)+1
1300 S=S2+Gbda(K2)*S3      ! SUM S
1310 Cs(Ki)=Cs(Ki)+E*E(Qs,Ks2)*S
1320 NEXT Ks2
1330 NEXT Ks1
1340 NEXT Qs
1350 NEXT Ns
1360 NEXT Js
1370 L=Kd-Kc+1      ! TOTAL WEIGHTED COVARIANCE MATRIX
1380 REDIM Cw(1:L,1:L),Ca(1:L,1:L),D(1:L),Eig(0:Nr,1:L)
1390 L11=Kc-1
1400 R=0.
1410 FOR I=0 TO Nr
1420 Ki=0
1430 IF I=0 THEN 1450
1440 R=10.^(R(I)*.1)      ! INPUT POWER SIGNAL-TO-NOISE RATIO
1450 FOR Ks1=Kc TO Kd
1460 W=W(Ks1)      ! WEIGHTS (w(k))
1470 FOR Ks2=Ks1 TO Kd
1480 IF Ks1>=Ko THEN 1510
1490 Cs=0.
1500 GOTO 1530
1510 Ki=Ki+1
1520 Cs=Cs(Ki)
1530 Pr=W*W(Ks2)*(Cn(Ks2-Ks1)+R*Cs)
1540 L1=Ks1-L11
1550 L2=Ks2-L11
1560 Cw(L1,L2)=Cw(L2,L1)=Pr
1570 NEXT Ks2
1580 NEXT Ks1
1590 CALL Svd(L,L,Cw(*),Ca(*),D(*)) ! EIGENVALUES
1600 MAT SORT D(*) DES
1610 IF D(L)>0. THEN 1640
1620 PRINT "PROBLEM: SOME NON-POSITIVE EIGENVALUES"
1630 PAUSE
1640 PRINT "I =",I,"    CONDITION NUMBER =",D(1)/D(L)
1650 PRINT D(*)
1660 PRINT
1670 FOR Ks=1 TO L
1680 Eig(I,Ks)=D(Ks)      ! STORE EIGENVALUES
1690 NEXT Ks
1700 NEXT I
1710 MASS STORAGE IS ":CS80,7"
1720 ASSIGN #1 TO "EIG"
1730 PRINT #1;Nr,Kc,Kd,Eig(*)
1740 ASSIGN #1 TO *
1750 END
1760 !

```

```

1770 SUB Weights(DOUBLE Kc,Kd,REAL W(*))
1780 DOUBLE Ks          ! INTEGER
1790 F=1.                ! DECAY FACTOR (DIMENSIONLESS)
1800 W(Kd)=1.
1810 FOR Ks=Kd-1 TO Kc STEP -1
1820 W(Ks)=W(Ks+1)*F      ! EXPONENTIAL WEIGHTS
1830 NEXT Ks
1840 SUBEND
1850 !
1860 SUB Svd(DOUBLE M,N,REAL A(*),V(*),W(*))
1870 ALLOCATE Rv1(1:N)    ! NUMERICAL RECIPES, PAGES 60-64
1880 IF M>=N THEN 1910    ! A(*) IS OVER-WRITTEN
1890 PRINT "M<N IS DISALLOWED"
1900 PAUSE
1910 DOUBLE I,J,K,L,Ils,Nm,Jj ! INTEGERS (NOT DOUBLE PRECISION)
1920 G=Scale=Anorm=0.
1930 FOR I=1 TO N
1940 L=I+1
1950 Rv1(I)=Scale*G
1960 G=S=Scale=0.
1970 IF I>M THEN 2250
1980 FOR K=I TO M
1990 Scale=Scale+ABS(A(K,I))
2000 NEXT K
2010 IF Scale=0. THEN 2250
2020 FOR K=I TO M
2030 Ra=A(K,I)=A(K,I)/Scale
2040 S=S+Ra*Ra
2050 NEXT K
2060 F=A(I,I)
2070 G=-SQR(S)
2080 IF F<0. THEN G=-G
2090 H=F*G-S
2100 A(I,I)=F-G
2110 IF I=N THEN 2220
2120 FOR J=L TO N
2130 S=0.
2140 FOR K=I TO M
2150 S=S+A(K,I)*A(K,J)
2160 NEXT K
2170 F=S/H
2180 FOR K=I TO M
2190 A(K,J)=A(K,J)+F*A(K,I)
2200 NEXT K
2210 NEXT J
2220 FOR K=I TO M
2230 A(K,I)=A(K,I)*Scale
2240 NEXT K
2250 W(I)=Scale*G
2260 G=S=Scale=0.
2270 IF (I>M) OR (I=N) THEN 2570
2280 FOR K=L TO N
2290 Scale=Scale+ABS(A(I,K))
2300 NEXT K
2310 IF Scale=0. THEN 2570

```

```

2320   FOR K=L TO N
2330   Ra=A(I,K)=A(I,K)/Scale
2340   S=S+Ra*Ra
2350   NEXT K
2360   F=A(I,L)
2370   G=-SQR(S)
2380   IF F<0. THEN G=-G
2390   H=F*G-S
2400   A(I,L)=F-G
2410   FOR K=L TO N
2420   Rv1(K)=A(I,K)/H
2430   NEXT K
2440   IF I=M THEN 2540
2450   FOR J=L TO M
2460   S=0.
2470   FOR K=L TO N
2480   S=S+A(J,K)*A(I,K)
2490   NEXT K
2500   FOR K=L TO N
2510   A(J,K)=A(J,K)+S*Rv1(K)
2520   NEXT K
2530   NEXT J
2540   FOR K=L TO N
2550   A(I,K)=A(I,K)*Scale
2560   NEXT K
2570   Anorm=MAX(Anorm,ABS(W(I))+ABS(Rv1(I)))
2580   NEXT I
2590   FOR I=N TO 1 STEP -1
2600   IF I>=N THEN 2770
2610   IF G=0. THEN 2740
2620   FOR J=L TO N
2630   V(J,I)=A(I,J)/A(I,L)/G
2640   NEXT J
2650   FOR J=L TO N
2660   S=0.
2670   FOR K=L TO N
2680   S=S+A(I,K)*V(K,J)
2690   NEXT K
2700   FOR K=L TO N
2710   V(K,J)=V(K,J)+S*V(K,I)
2720   NEXT K
2730   NEXT J
2740   FOR J=L TO N
2750   V(I,J)=V(J,I)=0.
2760   NEXT J
2770   V(I,I)=1.
2780   G=Rv1(I)
2790   L=I
2800   NEXT I
2810   FOR I=N TO 1 STEP -1
2820   L=I+1
2830   G=W(I)
2840   IF I>=N THEN 2880
2850   FOR J=L TO N
2860   A(I,J)=0.
2870   NEXT J

```

```

2880 IF G=0. THEN 3050
2890 G=1./G
2900 IF I=N THEN 3010
2910 FOR J=L TO N
2920 S=0.
2930 FOR K=L TO M
2940 S=S+A(K,I)*A(K,J)
2950 NEXT K
2960 F=S/A(I,I)*G
2970 FOR K=I TO M
2980 A(K,J)=A(K,J)+F*A(K,I)
2990 NEXT K
3000 NEXT J
3010 FOR J=I TO M
3020 A(J,I)=A(J,I)*G
3030 NEXT J
3040 GOTO 3080
3050 FOR J=I TO M
3060 A(J,I)=0.
3070 NEXT J
3080 A(I,I)=A(I,I)+1.
3090 NEXT I
3100 FOR K=N TO 1 STEP -1
3110 FOR Its=1 TO 30
3120 FOR L=K TO 1 STEP -1
3130 Nm=L-1
3140 IF (ABS(Rv1(L))+Anorm)=Anorm THEN 3360
3150 IF (ABS(W(Nm))+Anorm)=Anorm THEN 3170
3160 NEXT L
3170 C=0.
3180 S=1.
3190 FOR I=L TO K
3200 F=S*Rv1(I)
3210 Rv1(I)=C*Rv1(I)
3220 IF (ABS(F)+Anorm)=Anorm THEN 3360
3230 G=W(I)
3240 H=SQR(F*F+G*G)
3250 W(I)=H
3260 H=1./H
3270 C=G*H
3280 S=-F*H
3290 FOR J=1 TO M
3300 Y=A(J,Nm)
3310 Z=A(J,I)
3320 A(J,Nm)=Y*C+Z*S
3330 A(J,I)=-Y*S+Z*C
3340 NEXT J
3350 NEXT I
3360 Z=W(K)
3370 IF L<>K THEN 3440
3380 IF Z>=0. THEN 3430
3390 W(K)=-Z
3400 FOR J=1 TO N
3410 V(J,K)=-V(J,K)
3420 NEXT J
3430 GOTO 3970

```

```

3440 IF Its<30 THEN 3470
3450 PRINT "NO CONVERGENCE IN 30 ITERATIONS"
3460 PAUSE
3470 X=W(L)
3480 Nm=K-1
3490 Y=W(Nm)
3500 G=Rv1(Nm)
3510 H=Rv1(K)
3520 F=((Y-Z)*(Y+Z)+(G-H)*(G+H))/(2.*H*Y)
3530 G=SQR(F*F+1.)
3540 Aa=ABS(G)
3550 IF F<0. THEN Aa=-Aa
3560 F=((X-Z)*(X+Z)+H*((Y/(F+Aa))-H))/X
3570 C=S=1.
3580 FOR J=L TO Nm
3590 I=J+1
3600 G=Rv1(I)
3610 Y=W(I)
3620 H=S*G
3630 G=C*G
3640 Z=SQR(F*F+H*H)
3650 Rv1(J)=Z
3660 C=F/Z
3670 S=H/Z
3680 F=X*C+G*S
3690 G=-X*S+G*C
3700 H=Y*S
3710 Y=Y*C
3720 FOR Jj=1 TO N
3730 X=V(Jj,J)
3740 Z=V(Jj,I)
3750 V(Jj,J)=X*C+Z*S
3760 V(Jj,I)=-X*S+Z*C
3770 NEXT Jj
3780 Z=SQR(F*F+H*H)
3790 W(J)=Z
3800 IF Z=0. THEN 3840
3810 Z=1./Z
3820 C=F*Z
3830 S=H*Z
3840 F=C*G+S*Y
3850 X=-S*G+C*Y
3860 FOR Jj=1 TO M
3870 Y=A(Jj,J)
3880 Z=A(Jj,I)
3890 A(Jj,J)=Y*C+Z*S
3900 A(Jj,I)=-Y*S+Z*C
3910 NEXT Jj
3920 NEXT J
3930 Rv1(L)=0.
3940 Rv1(K)=F
3950 W(K)=X
3960 NEXT Its
3970 NEXT K
3980 SUBEND

```

```

10 ! PLOT ROC FROM EIGENVALUES "EIG", LINES 120 & 1060
20 Delxi=.08 ! INITIAL INCREMENT ON CHAR. FUNCTION
30 M=128 ! INITIAL SIZE OF FFT
40 Bs=0. ! SHIFT b
50 Tol0=1.E-36 ! TOLERANCE FOR FALSE ALARM PROBABILITY
60 Tol1=1.E-32 ! TOLERANCE FOR DETECTION PROBABILITIES
70 DOUBLE M,Nr,Kc,Kd,Kdc,Ms,I,Ks,Ns,Im,Mm ! INTEGERS
80 DIM E(0:20,1:100),D(1:100),Cos(0:4096) ! 20 NONZERO SNRs
90 DIM X(0:16383),Y(0:16383),Pr(0:20,0:127)
100 PRINT "Delxi =";Delxi,"M =";M,"b =";Bs
110 MASS STORAGE IS ";CS80,7"
120 ASSIGN #1 TO "EIG"
130 READ #1;Nr,Kc,Kd ! NO. OF SNRs, START & END OF SUMMER
140 Kdc=Kd-Kc+1
150 REDIM E(0:Nr,1:Kdc),D(1:Kdc)
160 READ #1;E(*) ! EIGENVALUES
170 GINIT
180 PLOTTER IS "GRAPHICS"
190 GRAPHICS ON
200 REDIM Cos(0:M/4),X(0:M-1),Y(0:M-1)
210 A=2.*PI/M
220 FOR Ms=0 TO M/4
230 Cos(Ms)=COS(A*Ms) ! INITIAL QUARTER-COSINE TABLE
240 NEXT Ms
250 Tol=Tol0
260 FOR I=0 TO Nr ! Nr NONZERO SIGNAL-TO-NOISE RATIOS
270 IF I>0 THEN Tol=Tol1
280 FOR Ks=1 TO Kdc
290 D(Ks)=E(I,Ks) ! COPY EIGENVALUES
300 NEXT Ks
310 Mux=SUM(D) ! MEAN OF RANDOM VARIABLE x
320 R=0. ! ARGUMENT OF SQUARE ROOT
330 P=1. ! POLARITY INDICATOR
340 Muy=Mux+Bs ! MEAN OF y = x + b
350 MAT X=(0.)
360 MAT Y=(0.)
370 Y(0)=.5*Delxi*Muy
380 Ns=0
390 Ns=Ns+1 ! LOOP ON xi
400 Xi=Delxi*Ns ! ARGUMENT xi OF CHAR. FN.
410 Txi=Xi*2. ! CALCULATION OF
420 Pr=1. ! CHARACTERISTIC
430 Pi=0. ! FUNCTION fy(xi)
440 FOR Ks=1 TO Kdc
450 Te=Txi*D(Ks)
460 Pe=Pr+Te*Pi
470 Pi=Pi-Te*Pr
480 Pr=Pe
490 NEXT Ks
500 CALL Sqr(Pr,Pi,Sr,Si)
510 De=Sr*Sr+Si*Si
520 Fyr=Sr/De
530 Fyi=-Si/De
540 Ro=R
550 R=ATN(Fyi/Fyr)
560 IF ABS(R-Ro)>1.6 THEN P=-P
570 IF P>0. THEN 600

```

```

580   Fyr=-Fyr
590   Fyi=-Fyi
600   Ms=Ns MODULO M           ! COLLAPSING
610   A=Fyr/Ns
620   B=Fyi/Ns
630   X(Ms)=X(Ms)+A
640   Y(Ms)=Y(Ms)+B
650   IF A*A+B*B<Tol THEN 670
660   GOTO 390
670   CALL Fft14(M,Cos(*),X(*),Y(*))
680   GCLEAR
690   WINDOW 0,M,-18,0
700   LINE TYPE 3
710   GRID M/8,1
720   LINE TYPE 1
730   FOR Ks=0 TO M-1
740   T=Y(Ks)/PI-Ks/M
750   Y(Ks)=Pr=.5+T           ! EXCEEDANCE PROBABILITY IN Y(*)
760   IF Pr>=1.E-16 THEN Y=LGT(Pr)!Y(Ks)=PROB(x>2 PI Ks/(M Delxi)-Bs)
770   IF Pr<=-1.E-16 THEN Y=-32.-LGT(-Pr)
780   IF ABS(Pr)<1.E-16 THEN Y=-16.
790   PLOT Ks,Y
800   NEXT Ks
810   PENUP
820   IF I=Nr THEN 940
830   PRINT I;
840   INPUT "SCALE FFT SIZE BY 1,2,4,8,...:",Im
850   Mm=MIN(M*Im,16384)
860   IF Mm=M THEN 940
870   Delxi=Delxi*M/Mm
880   M=Mm
890   REDEFIN Cos(0:M/4),X(0:M-1),Y(0:M-1)
900   A=2.*PI/M
910   FOR Ms=0 TO M/4
920   Cos(Ms)=COS(A*Ms)       ! QUARTER-COSINE TABLE; <= 4096
930   NEXT Ms
940   FOR Ks=0 TO 127
950   Pr(I,Ks)=Y(Ks)         ! STORE FOR ROC PLOT
960   NEXT Ks
970   NEXT I
980   FOR Ks=0 TO 127
990   IF Pr(0,Ks)<.1 THEN 1010
1000  NEXT Ks
1010  Ms=Ks-1
1020  FOR Ks=Ms TO 127
1030  IF Pr(0,Ks)<1E-10 THEN 1050
1040  NEXT Ks
1050  Ns=Ks
1060  ASSIGN #1 TO "ROC"
1070  PRINT #1;Ms,Ns,Nr
1080  PRINT #1;Pr(*)
1090  ASSIGN #1 TO *
1100  DIM Pf(1:14),Pd(1:18)
1110  DATA 1E-10,1E-9,1E-8,1E-7,1E-6,1E-5,1E-4
1120  DATA .001,.002,.005,.01,.02,.05,.1
1130  READ Pf(*)

```

```

1140 DATA .01,.02,.05,.1,.2,.3,.4,.5,.6,.7
1150 DATA .8,.9,.95,.98,.99,.995,.998,.999
1160 READ Pd(*)
1170 FOR I=1 TO 14
1180 Pf(I)=FNInvphi(Pf(I))
1190 NEXT I
1200 FOR I=1 TO 18
1210 Pd(I)=FNInvphi(Pd(I))
1220 NEXT I
1230 GCLEAR
1240 X1=Pf(1)
1250 X2=Pf(14)
1260 Y1=Pd(1)
1270 Y2=Pd(18)
1280 WINDOW X1,X2,Y1,Y2
1290 LINE TYPE 3
1300 FOR I=1 TO 14
1310 MOVE Pf(I),Y1
1320 DRAW Pf(I),Y2
1330 NEXT I
1340 FOR I=1 TO 18
1350 MOVE X1,Pd(I)
1360 DRAW X2,Pd(I)
1370 NEXT I
1380 PENUP
1390 LINE TYPE 1
1400 FOR Ks=Ms TO Ns
1410 Pr(0,Ks)=FNInvphi(Pr(0,Ks)) ! FALSE ALARM PROBABILITY
1420 NEXT Ks
1430 FOR I=1 TO Nr
1440 FOR Ks=Ms TO Ns
1450 X=Pr(0,Ks)
1460 Pr=Pr(I,Ks)
1470 IF Pr>.9999 THEN 1510
1480 Pr(I,Ks)=Y=FNInvphi(Pr)
1490 PLOT X,Y
1500 IF Pr<.01 THEN 1520
1510 NEXT Ks
1520 PENUP
1530 NEXT I
1540 PAUSE
1550 END
1560 !
1570 DEF FNArg(X,Y) ! PRINCIPAL ARG(Z)
1580 IF X=0. THEN RETURN .5*PI*SGN(Y)
1590 A=ATN(Y/X)
1600 IF X>0. THEN RETURN A
1610 IF Y<0. THEN RETURN A-PI
1620 RETURN A+PI
1630 FNEND
1640 !
1650 SUB Sqr(X,Y,U,V) ! PRINCIPAL SQR(Z)
1660 F=SQR(SQR(X*X+Y*Y))
1670 T=.5*FNArg(X,Y)
1680 U=F*COS(T)
1690 V=F*SIN(T)
1700 SUBEND
1710 !

```



```

1720 DEF FNInvphi(X)      ! AMS 55, 26.2.23
1730 IF X=.5 THEN RETURN 0.
1740 P=MIN(X,1.-X)
1750 P=MAX(P,1.E-20)
1760 T=-LOG(P)
1770 T=SQR(T+T)
1780 P=1.+T*(1.432788+T*(.189269+T*.001308))
1790 P=T-(2.515517+T*(.802853+T*.010328))/P
1800 IF X<.5 THEN P=-P
1810 RETURN P
1820 FNEND
1830 !
1840 SUB Fft14(DOUBLE N,REAL Cos(*),X(*),Y(*)) ! N<=2^14=16384; 0 SUPS
1850 DOUBLE Log2n,N1,N2,N3,N4,J,K ! INTEGERS < 2^31 = 2,147,483,648
1860 DOUBLE I1,I2,I3,I4,I5,I6,I7,I8,I9,I10,I11,I12,I13,I14,L(0:13)
1870 IF N=1 THEN SUBEXIT
1880 IF N>2 THEN 1960
1890 A=X(0)+X(1)
1900 X(1)=X(0)-X(1)
1910 X(0)=A
1920 A=Y(0)+Y(1)
1930 Y(1)=Y(0)-Y(1)
1940 Y(0)=A
1950 SUBEXIT
1960 A=LOG(N)/LOG(2.)
1970 Log2n=A
1980 IF ABS(A-Log2n)<1.E-8 THEN 2010
1990 PRINT "N =";N;"IS NOT A POWER OF 2; DISALLOWED."
2000 PAUSE
2010 N1=N/4
2020 N2=N1+1
2030 N3=N2+1
2040 N4=N3+N1
2050 FOR I1=1 TO Log2n
2060 I2=2^(Log2n-I1)
2070 I3=2*I2
2080 I4=N/I3
2090 FOR I5=1 TO I2
2100 I6=(I5-1)*I4+1
2110 IF I6<=N2 THEN 2150
2120 A1=-Cos(N4-I6-1)
2130 A2=-Cos(I6-N1-1)
2140 GOTO 2170
2150 A1=Cos(I6-1)
2160 A2=-Cos(N3-I6-1)
2170 FOR I7=0 TO N-I3 STEP I3
2180 I8=I7+I5-1
2190 I9=I8+I2
2200 T1=X(I8)
2210 T2=X(I9)
2220 T3=Y(I8)
2230 T4=Y(I9)

```

```

2240      A3=T1-T2
2250      A4=T3-T4
2260      X(I8)=T1+T2
2270      Y(I8)=T3+T4
2280      X(I9)=A1*A3-A2*A4
2290      Y(I9)=A1*A4+A2*A3
2300      NEXT I7
2310      NEXT I5
2320      NEXT I1
2330      I1=Log2n+1
2340      FOR I2=1 TO 14
2350      L(I2-1)=1
2360      IF I2>Log2n THEN 2380
2370      L(I2-1)=2^(I1-I2)
2380      NEXT I2
2390      K=0
2400      FOR I1=1 TO L(13)
2410      FOR I2=I1 TO L(12) STEP L(13)
2420      FOR I3=I2 TO L(11) STEP L(12)
2430      FOR I4=I3 TO L(10) STEP L(11)
2440      FOR I5=I4 TO L(9) STEP L(10)
2450      FOR I6=I5 TO L(8) STEP L(9)
2460      FOR I7=I6 TO L(7) STEP L(8)
2470      FOR I8=I7 TO L(6) STEP L(7)
2480      FOR I9=I8 TO L(5) STEP L(6)
2490      FOR I10=I9 TO L(4) STEP L(5)
2500      FOR I11=I10 TO L(3) STEP L(4)
2510      FOR I12=I11 TO L(2) STEP L(3)
2520      FOR I13=I12 TO L(1) STEP L(2)
2530      FOR I14=I13 TO L(0) STEP L(1)
2540      J=I14-1
2550      IF K>J THEN 2620
2560      A=X(K)
2570      X(K)=X(J)
2580      X(J)=A
2590      A=Y(K)
2600      Y(K)=Y(J)
2610      Y(J)=A
2620      K=K+1
2630      NEXT I14
2640      NEXT I13
2650      NEXT I12
2660      NEXT I11
2670      NEXT I10
2680      NEXT I9
2690      NEXT I8
2700      NEXT I7
2710      NEXT I6
2720      NEXT I5
2730      NEXT I4
2740      NEXT I3
2750      NEXT I2
2760      NEXT I1
2770      SUBEND

```

```

10  ! SIMULATION FOR ONE-POLE PROCESSES
20  As=.85                      ! INPUT SIGNAL PARAMETER
30  Bs=.77                      ! INPUT NOISE PARAMETER; LINE 160
40  Cs=1.13                    ! FILTER PARAMETER
50  R=.7                        ! INPUT POWER SNR
60  Ka=4
70  Kb=11
80  Kc=2
90  Kd=16
100 Bins=1000
110 Num=1E6                    ! NUMBER OF TRIALS
120 Deltaz=.25                 ! BIN SIZE FOR OUTPUT Z
130 DOUBLE Ka,Kb,Kc,Kd,Bins,Num,It,K,M
140 DIM E(100),S(100),Yn(100),Ys(100),T(1000)
150 Ea=EXP(-As)
160 Eb=EXP(-Bs)                ! WHITE NOISE: FOR Bs=inf, SET Eb=0
170 Ec=EXP(-Cs)
180 Fa=SQR(R*(1.-Ea*Ea))
190 Fb=SQR(1.-Eb*Eb)
200 REDIM E(0:Kd-Ka)
210 E(0)=Cs
220 FOR K=1 TO Kd-Ka
230 E(K)=E(K-1)*Ec
240 NEXT K
250 REDIM S(-30:Kd),Yn(Kc-1:Kd),Ys(Ka:Kd),T(1:Bins)
260 FOR It=1 TO Num
270 S(-30)=Nk=Yk=0.
280 FOR K=-29 TO Kc-1          ! ALLOW STEADY STATE
290 R1=2.*RND-1.              ! BOX-MULLER
300 R2=2.*RND-1.
310 R3=R1*R1+R2*R2
320 IF R3>1. THEN 290
330 R3=SQR(-2.*LOG(R3)/R3)
340 R1=R1*R3
350 R2=R2*R3
360 S(K)=Fa*R1+Ea*S(K-1)      ! FILTER INPUT SIGNAL
370 Nk=Fb*R2+Eb*Nk            ! FILTER INPUT NOISE
380 Yk=Cs*Nk+Ec*Yk           ! FILTER OUTPUT NOISE
390 NEXT K
400 Yn(Kc-1)=Yk
410 FOR K=Kc TO Kd
420 R1=2.*RND-1.
430 R2=2.*RND-1.
440 R3=R1*R1+R2*R2
450 IF R3>1. THEN 420
460 R3=SQR(-2.*LOG(R3)/R3)
470 R1=R1*R3
480 R2=R2*R3
490 M=K-1

```

```

500 S(K)=Fa*R1+Ea*S(M) | FILTER INPUT SIGNAL
510 Nk=Fb*R2+Eb*Nk | FILTER INPUT NOISE
520 Yn(K)=Cs*Nk+Ec*Yn(M) | FILTER OUTPUT NOISE
530 NEXT K
540 FOR K=Ka TO Kd
550 S=0.
560 FOR M=Ka TO MIN(K,Kb)
570 S=S+E(K-M)*S(M) | FILTER OUTPUT SIGNAL
580 NEXT M
590 Ys(K)=S
600 NEXT K
610 Z=0.
620 FOR K=Kc TO Ka-1
630 T=Yn(K)
640 Z=Z+T*T
650 NEXT K
660 FOR K=Ka TO Kd
670 T=Ys(K)+Yn(K)
680 Z=Z+T*T
690 NEXT K
700 K=INT(Z/Deltaz)+1
710 K=MIN(K,Bins)
720 T(K)=T(K)+1.
730 NEXT K
740 FOR K=Bins-1 TO 1 STEP -1
750 T(K)=T(K)+T(K+1)
760 NEXT K
770 MAT T=T/(Num)
780 GINIT
790 PLOTTER IS "GRAPHICS"
800 GRAPHICS ON
810 WINDOW 0,Bins,-5,0
820 GRID Bins/4,1
830 FOR K=1 TO Bins
840 T=T(K)
850 IF T>0. THEN 870
860 GOTO 890
870 PLOT K,LGT(T)
880 NEXT K
890 PENUP
900 FOR K=Bins TO 1 STEP -1
910 T=1.-T(K)
920 IF T>0. THEN 940
930 GOTO 960
940 PLOT K,LGT(T)
950 NEXT K
960 PENUP
970 PAUSE
980 DUMP GRAPHICS
990 END

```

APPENDIX E. PROGRAM FOR COLORED INPUT NOISE

One program is listed in this appendix. It computes the covariance matrices and eigenvalues as described in appendix D, but now for a colored noise input. The program for computing the receiver operating characteristics from the eigenvalues is identical to that listed above and therefore has not been repeated here.

```

10  | COMPUTE COVARIANCE MATRICES AND EIGENVALUES
20  | STORE EIGENVALUES IN "EIG" IN LINE 2260
30  Ka=-10          | INPUT SIGNAL START
40  Kb=5            | INPUT SIGNAL END; Kb >= Ka
50  Kc=0            | ACCUMULATOR START
60  Kd=5            | ACCUMULATOR END; Kd >= Kc,Ka
70  Delta=.2        | TIME SAMPLING INCREMENT (SECONDS)
80  J=3             | NUMBER OF SIGNAL COMPONENTS
90  DATA 11.,-48.,75. | SIGNAL SCALINGS (WE SET SUM = 1)
100 DATA 1.,.125,.066667 | SIGNAL TIME CONSTANTS (SECONDS)
110 M=2             | NUMBER OF NOISE COMPONENTS
120 DATA 39.,60.    | NOISE SCALINGS (WE SET SUM = 1)
130 DATA .2,.4      | NOISE TIME CONSTANTS (SECONDS)
140 N=4             | NUMBER OF FILTER COMPONENTS
150 DATA 1.,2.,3.,4. | FILTER SCALINGS (ARBITRARY)
160 DATA .3,.5,.7,.9 | FILTER TIME CONSTANTS (SECONDS)
170 Nr=20           | NUMBER OF SIGNAL-TO-NOISE RATIOS
180 DATA 0,1,2,3,4,5,6,7,8,9,10 | INPUT SNRS IN DB
190 DATA 11,12,13,14,15,16,17,18,19
200 IF (Ka<=Kb) AND (Kc<=Kd) AND (Ka<=Kd) THEN 230
210 PRINT "PROBLEM WITH PARAMETERS"
220 STOP
230 DIM Alpha(10),A(10),Beta(10),B(10),Psi(10),C(10),R(20)
240 DIM W(100),Pc(10),Ec(10),Pcv(10),Eb(10),Chb(10),Shb(10)
250 DIM Chc(10),Shc(10),Mu(10),Gamma(10),Cn(200),Ea(10)
260 DIM E(0,1000),Gca(100),Gcda(100),Gcb(100),Gbda(100)
270 DIM Cs(5000),Cw(0,10000),Ca(0,10000),D(100),Eig(0,2000)
280 DOUBLE Ka,Kb,Kc,Kd,J,M,N,Nr,Ns,Ks,Ms,Kdc,Ko
290 DOUBLE L1,L2,L,L11,Js,Qs,Ki,Ks1,K1,Ks2,K2,I
300 REDIM Alpha(1:J),A(1:J)
310 READ Alpha(*),A(*)
320 MAT Alpha=Alpha/(SUM(Alpha))
330 MAT A=(Delta)/A
340 REDIM Beta(1:M),B(1:M)
350 READ Beta(*),B(*)
360 MAT Beta=Beta/(SUM(Beta))
370 MAT B=(Delta)/B
380 REDIM Psi(1:N),C(1:N)
390 READ Psi(*),C(*)
400 MAT C=(Delta)/C
410 REDIM R(1:Nr)
420 READ R(*)
430 REDIM W(Kc:Kd)
440 CALL Weights(Kc,Kd,W(*))
450 MAT W=SQR(W)

```

```

1070 Cn(Ks)=Cn(Ks)+Mu*P
1080 NEXT Ks
1090 NEXT Ms
1100 FOR Ns=1 TO N
1110 Ga=Gamma(Ns)
1120 E=Ec(Ns)
1130 P=1.
1140 Cn(0)=Cn(0)+Ga
1150 FOR Ks=1 TO Kdc
1160 P=P*E
1170 Cn(Ks)=Cn(Ks)+Ga*P
1180 NEXT Ks
1190 NEXT Ns
1200 FOR Js=1 TO J          ! FILTER OUTPUT SIGNAL COVARIANCE
1210 A(Js)=EXP(A(Js))
1220 NEXT Js
1230 Ko=MAX(Ka,Kc)
1240 REDIM Ea(1:N),E(1:N,Ko:Kd)
1250 FOR Ns=1 TO N
1260 E=Ec(Ns)
1270 Ea(Ns)=1./E
1280 E(Ns,Ko)=EXP(-C(Ns)*Ko)
1290 FOR Ks=Ko+1 TO Kd
1300 E(Ns,Ks)=E(Ns,Ks-1)*E
1310 NEXT Ks
1320 NEXT Ns
1330 L1=MIN(Ko,Kb)+1
1340 L2=MIN(Kd,Kb)+1
1350 REDIM Gca(L1:L2),Gcda(L1:L2),Gcb(L1:L2),Gbda(L1:L2)
1360 L=Kd-Ko+1
1370 REDIM Cs(1:L*(L+1)/2)
1380 L11=L1+1
1390 FOR Js=1 TO J
1400 A1=Alpha(Js)
1410 A=A(Js)
1420 A1=A*A-1.
1430 FOR Ns=1 TO N
1440 C=Ea(Ns)
1450 Ca=C*A
1460 Cda=C/A
1470 Cal=1.-Ca
1480 Ac=A-C
1490 Ae=A1*Pc(Ns)
1500 Gca=Ca^Ka/Cal
1510 Gca(L1)=Ca^L1/Cal
1520 Gcda(L1)=Cda^L1/(1.-Cda)
1530 FOR Ks=L11 TO L2
1540 Gca(Ks)=Ca*Gca(Ks-1)
1550 Gcda(Ks)=Cda*Gcda(Ks-1)
1560 NEXT Ks
1570 FOR Qs=1 TO N
1580 B=Ea(Qs)
1590 Aee=Ae*Pc(Qs)
1600 Ba=B*A
1610 Bda=B/A
1620 Cb=C*B
1630 Bal=1.-Ba
1640 Cb1=1.-Cb
1650 Gba=Ba^Ka/Bal
1660 Ab=A-B
1670 F=B*A1/(Ab*Bal)

```

```

1680 S1=Cb^Ka/Cb1*(A1+Cb1)/(Ab*Ac)
1690 Gcb(L1)=Cb^L1/Cb1
1700 Gbda(L1)=Bda^L1/(1.-Bda)
1710 FOR Ks=L11 TO L2
1720 Gcb(Ks)=Cb*Gcb(Ks-1)
1730 Gbda(Ks)=Bda*Gbda(Ks-1)
1740 NEXT Ks
1750 Ki=0
1760 FOR Ks1=Ko TO Kd
1770 K1=MIN(Ks1,Kb)+1
1780 S2=S1+Gcb(K1)*F-Gba*Gcda(K1)
1790 S3=Gca(K1)-Gca
1800 E=Aee*E(Ns,Ks1)
1810 FOR Ks2=Ks1 TO Kd
1820 Ki=Ki+1
1830 K2=MIN(Ks2,Kb)+1
1840 S=S2+Gbda(K2)*S3      I  SUM S
1850 Cs(K1)=Cs(K1)+E*E(Qs,Ks2)*S
1860 NEXT Ks2
1870 NEXT Ks1
1880 NEXT Qs
1890 NEXT Ns
1900 NEXT Js
1910 L=Kd-Kc+1      I  TOTAL WEIGHTED COVARIANCE MATRIX
1920 REDIM Cw(1:L,1:L),Ca(1:L,1:L),D(1:L),Eig(0:Nr,1:L)
1930 L11=Kc-1
1940 R=0.
1950 FOR I=0 TO Nr
1960 Ki=0
1970 IF I=0 THEN 1990
1980 R=10.^(R(I)*.1)      I  INPUT POWER SIGNAL-TO-NOISE RATIO
1990 FOR Ks1=Kc TO Kd
2000 W=W(Ks1)      I  WEIGHTS (w(k))
2010 FOR Ks2=Ks1 TO Kd
2020 IF Ks1>=Ko THEN 2050
2030 Cs=0.
2040 GOTO 2070
2050 Ki=Ki+1
2060 Cs=Cs(Ki)
2070 Pr=W*W(Ks2)*(Cn(Ks2-Ks1)+R*Cs)
2080 L1=Ks1-L11
2090 L2=Ks2-L11
2100 Cw(L1,L2)=Cw(L2,L1)=Pr
2110 NEXT Ks2
2120 NEXT Ks1
2130 CALL Sud(L,L,Cw(*),Ca(*),D(*)) I  EIGENVALUES
2140 MAT SORT D(*) DES
2150 IF D(L)>0. THEN 2180
2160 PRINT "PROBLEM: SOME NON-POSITIVE EIGENVALUES"
2170 PAUSE
2180 PRINT "I =",I;"  CONDITION NUMBER =",D(1)/D(L)
2190 PRINT D(*)
2200 PRINT
2210 FOR Ks=1 TO L
2220 Eig(I,Ks)=D(Ks)      I  STORE EIGENVALUES
2230 NEXT Ks
2240 NEXT I
2250 MASS STORAGE IS ";CS80,7"
2260 ASSIGN #1 TO "EIG"
2270 PRINT #1;Nr,Kc,Kd,Eig(*)
2280 ASSIGN #1 TO *
2290 END

```

REFERENCES

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- [3] A. H. Nuttall, **Operating Characteristics for Weighted Energy Detector with Gaussian Signals**, NUSC Technical Report 8753, Naval Underwater Systems Center, New London, CT, 16 July 1990.

NUSC Technical Document 8921
12 December 1991

Comparison of Two Kernels for the
Modified Wigner Distribution Function

Chintana Griffin and Albert H. Nuttall

ABSTRACT

This document contains the lecture presentation of the paper entitled "Comparison of Two Kernels for the Modified Wigner Distribution Function," given at the Society of Photo-Optical Instrumentation Engineers International Symposium on Optical Applied Science and Engineering, Conference 1566 on Advanced Signal Processing Algorithms, Architectures, and Implementations, 21 - 26 July 1991, San Diego, California.

We compare the modified Wigner distribution functions obtained via the Choi-Williams kernel and its rotation, as well as by the tilted Gaussian kernel. Based on several commonly used examples, we demonstrate that the modified Wigner distribution obtained via the Gaussian kernel can minimize the artifacts more effectively and had the capability of selectively filtering out undesired components.

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COMPARISON OF TWO KERNELS FOR THE MODIFIED WIGNER DISTRIBUTION FUNCTION

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OBJECTIVE

- COMPARE TILTED GAUSSIAN AND CHOI-WILLIAMS KERNELS IN TERMS OF MODIFIED WIGNER DISTRIBUTIONS FOR MULTI-COMPONENT SIGNALS.

CRITERIA FOR THE COMPARISON ARE

- CROSS-TERM SUPPRESSION
- DESIRED COMPONENT EXTRACTION

OUTLINE

- **WIGNER DISTRIBUTION & COMPLEX AMBIGUITY FUNCTION**
- **MODIFIED WIGNER DISTRIBUTION**

TILTED GAUSSIAN KERNEL

CHOI-WILLIAMS KERNEL

- **ROTATION OF KERNEL**
- **EXAMPLES**
- **CONCLUSION**

WIGNER DISTRIBUTION & COMPLEX AMBIGUITY FUNCTION

$$\text{TCF:} \quad R(t,\tau) = s\left(t+\frac{\tau}{2}\right) s^*\left(t-\frac{\tau}{2}\right)$$

$$\text{SCF:} \quad \Phi(v,f) = S\left(f+\frac{v}{2}\right) S^*\left(f-\frac{v}{2}\right)$$

$$\text{WDF:} \quad W(t,f) = \int R(t,\tau) \exp(-j2\pi f\tau) d\tau$$

$$= \int \Phi(v,f) \exp(j2\pi v t) dv$$

$$\text{CAF:} \quad \chi(v,\tau) = \int R(t,\tau) \exp(-j2\pi v t) dt$$

$$= \int \Phi(v,f) \exp(j2\pi f\tau) df$$

$$W(t,f) = \iint \chi(v,\tau) \exp(j2\pi v t) \exp(-j2\pi f\tau) d\tau dv$$

WIGNER DISTRIBUTION & COMPLEX AMBIGUITY FUNCTION

WDF

AUTO-TERMS



CONCENTRATED NEAR ORIGIN

CROSS-TERMS



SHIFTED AWAY FROM ORIGIN

CAF

THE AUTO-TERMS AND CROSS-TERMS IN WDF, (t, f) DOMAIN, ARE
ESSENTIALLY SEPARATED IN CAF, (ν, τ) DOMAIN. THE SEPARATION
OF THE AUTO-TERMS AND CROSS-TERMS IN CAF DOMAIN MAKES
 (ν, τ) DOMAIN IDEAL FOR KERNEL DESIGN AND FILTERING.

MODIFIED TIME-FREQUENCY REPRESENTATIONS

modified CAF:

$$\hat{\chi}(v,\tau) = \chi(v,\tau) \tilde{V}(v,\tau)$$

modified SCF:

$$\hat{\Phi}(v,f) = \Phi(v,f) \overset{\dagger}{\oplus} \tilde{V}(v,f)$$

modified TCF:

$$\hat{R}(t,\tau) = R(t,\tau) \overset{\dagger}{\oplus} v(t,\tau)$$

modified WDF:

$$\hat{W}(t,f) = W(t,f) \overset{\ddagger}{\oplus} V(t,f)$$

KERNEL IN EACH DOMAIN

SCF: $\tilde{V}(v,f) = \int \tilde{v}(v,\tau) \exp(-j2\pi f\tau) d\tau$

TCF: $v(t,\tau) = \int \tilde{v}(v,\tau) \exp(j2\pi v t) dv$

WDF: $V(t,f) = \int v(t,\tau) \exp(-j2\pi f\tau) d\tau$

$$= \int \tilde{V}(v,f) \exp(j2\pi v t) dv$$

$$= \iint \tilde{v}(v,\tau) \exp(j2\pi v t - j2\pi f\tau) dv d\tau$$

x

⊕ DENOTES CONVOLUTION ON X

TILTED GAUSSIAN KERNEL

$$\begin{aligned} \text{CAF:} \quad \tilde{v}(v,\tau) &= \exp \left[-\pi \left(\frac{v^2}{B^2} + \frac{\tau^2}{D^2} + 2r \frac{v\tau}{BD} \right) \right] \\ \text{WDF:} \quad V(t,f) &= \frac{BD}{\sqrt{1-r^2}} \exp \left[-\frac{\pi}{1-r^2} (B^2 f^2 + D^2 t^2 + 2rBtDf) \right] \end{aligned}$$

ARBITRARY B,D , AND r
DIMENSIONLESS TILT PARAMETER |r|<1

CHOI-WILLIAMS KERNEL

CAF: $\tilde{v}(v,\tau) = \exp(-v^2\tau^2/\sigma^2), \quad \sigma > 0$

WDF:
$$\begin{aligned} V(t,f) &= 2\pi^{1/2} \sigma \int_0^\infty \cos(2\pi v t) \exp(-\pi^2 \sigma^2 f^2 v^2) \frac{dv}{v} \\ &= 2\pi^{1/2} \sigma \int_0^\infty \cos(2\pi f \tau) \exp(-\pi^2 \sigma^2 \tau^2) \frac{d\tau}{\tau}, \\ t \neq 0, f \neq 0 \end{aligned}$$

ROTATION OF KERNEL

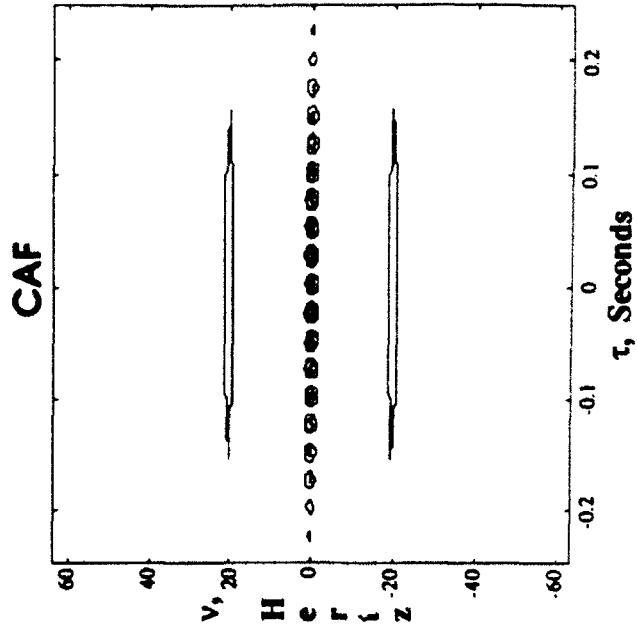
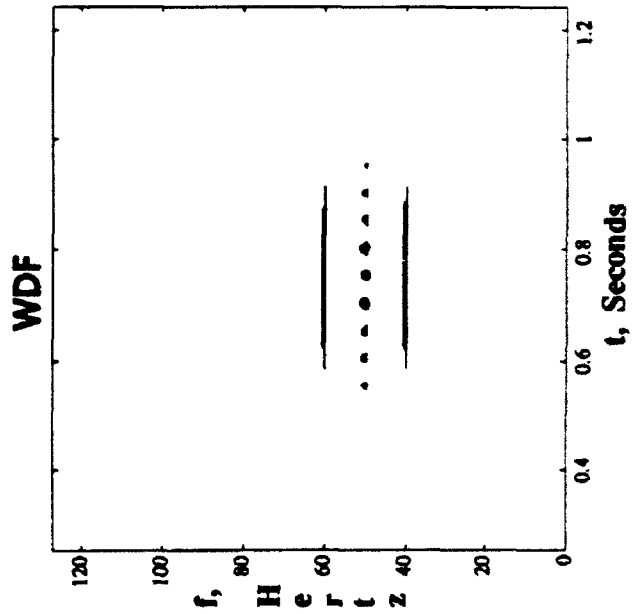
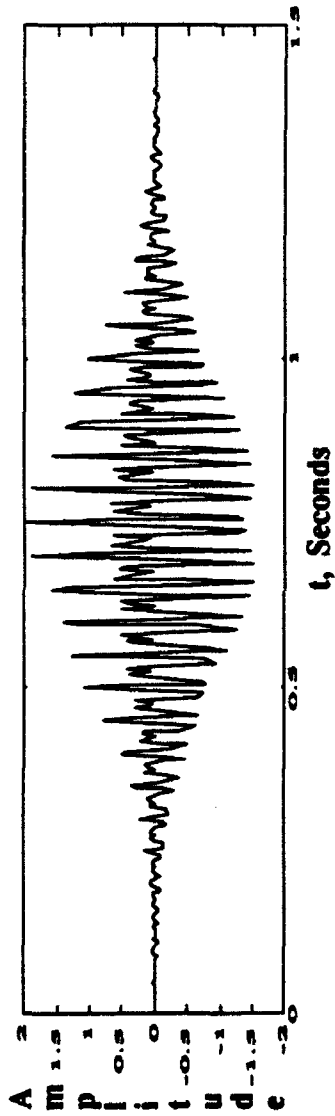
EXPRESS $\tilde{v}$ IN TERMS OF A NORMALIZED FUNCTION $\tilde{u}$:

$$\tilde{v}(v,\tau) = \tilde{u}(v/B, \tau/D)$$

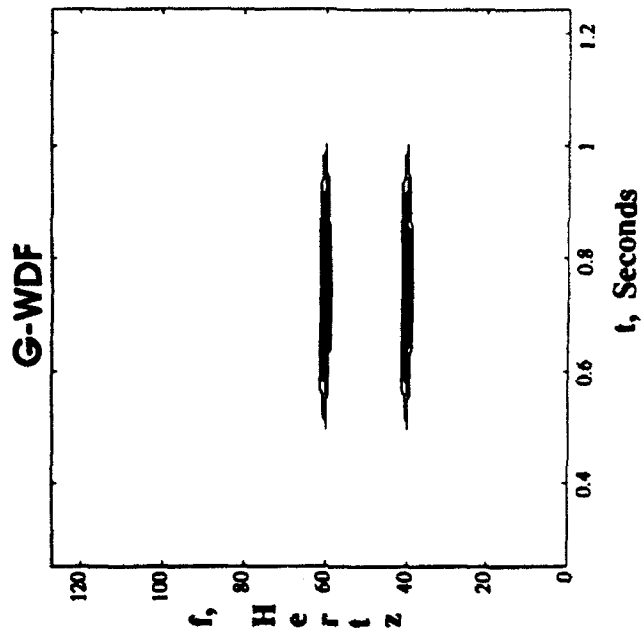
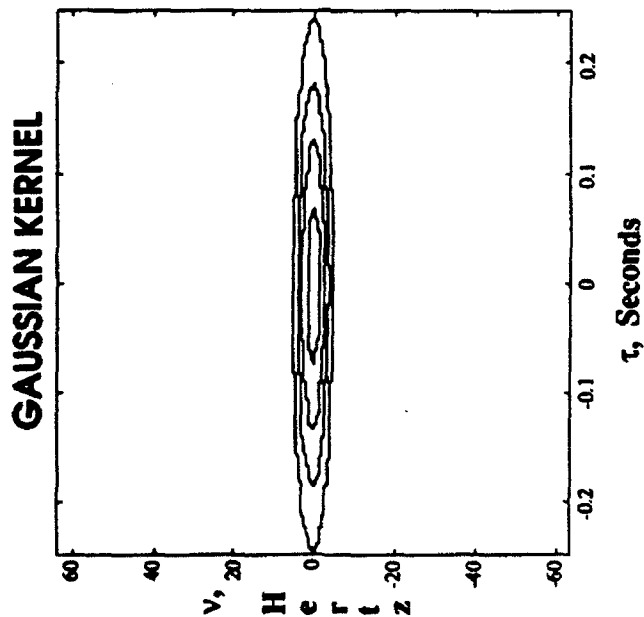
LET θ BE THE ANGLE OF ROTATION IN THE $v/B, \tau/D$ PLANE,
 $C=\cos(\theta)$, $S=\sin(\theta)$. THEN THE ROTATED WEIGHTING IS

$$\begin{aligned} \tilde{r}(v,\tau) &= \tilde{u}\left(C\frac{v}{B} + S\frac{\tau}{D}, C\frac{\tau}{D} - S\frac{v}{B}\right) \\ &= \tilde{v}\left(Cv + S\frac{B}{D}\tau, C\tau - S\frac{D}{B}v\right) \end{aligned}$$

EXAMPLE 1: TWO AM SINEWAVES

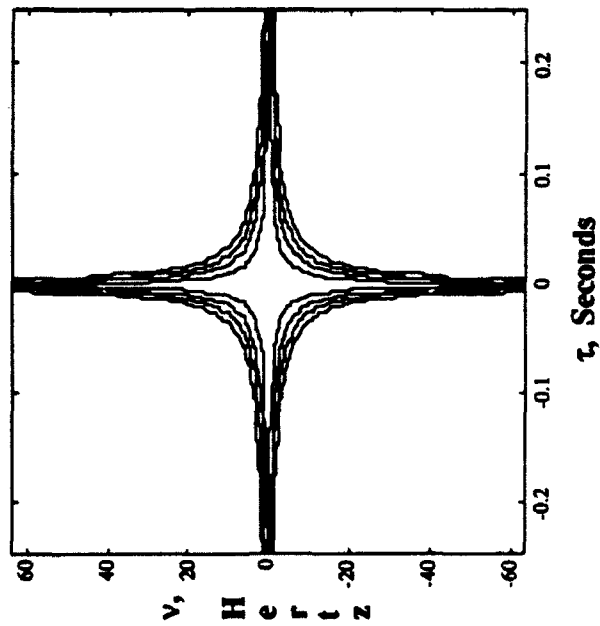


EXAMPLE 1: TWO AM SINEWAVES

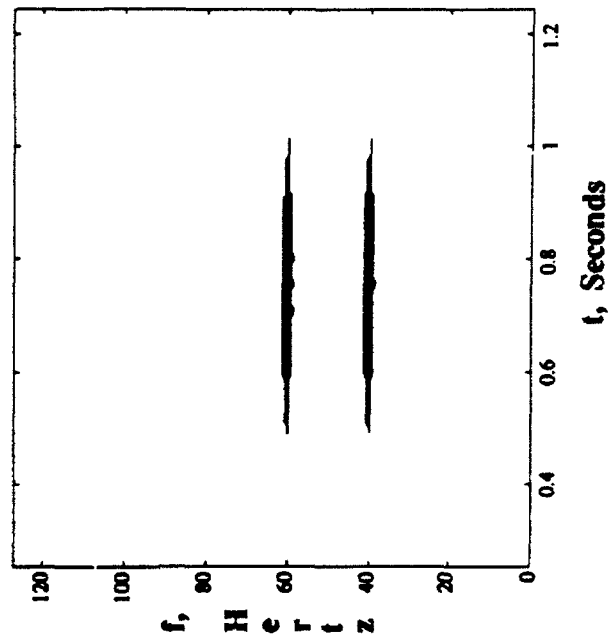


EXAMPLE 1: TWO AM SINEWAVES

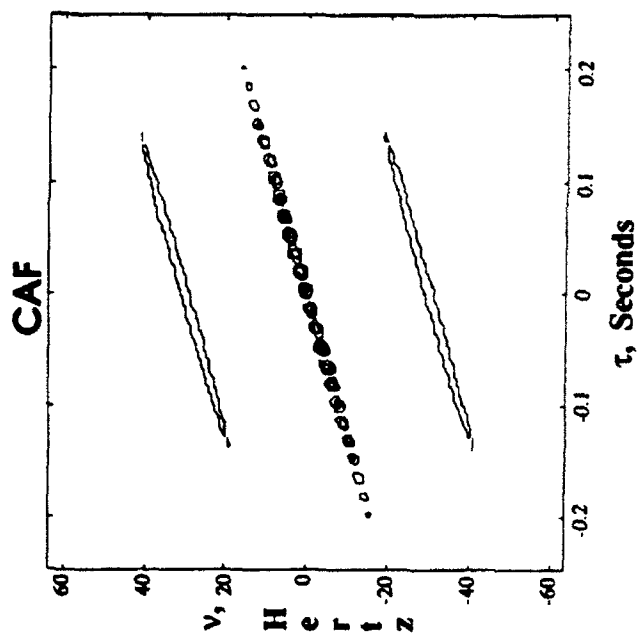
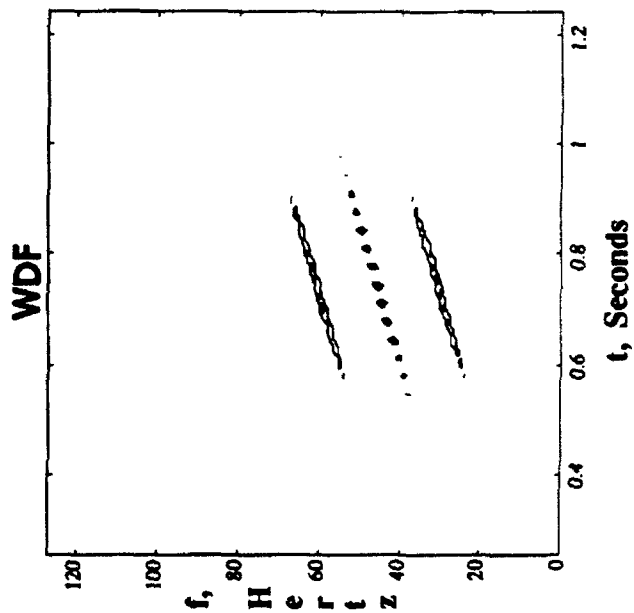
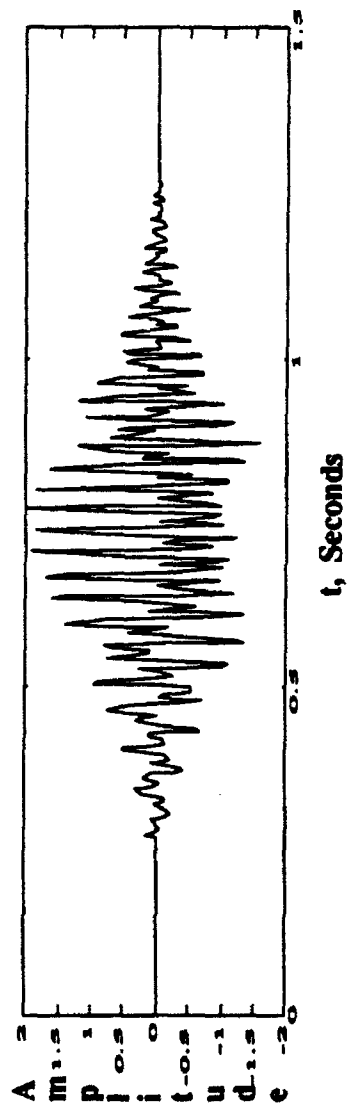
CHOI-WILLIAMS KERNEL



C-WDF

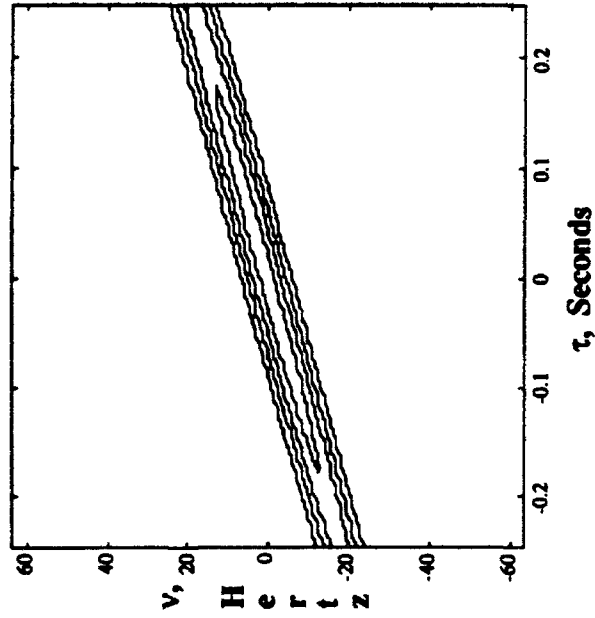


EXAMPLE 2: TWO PARALLEL LFMS

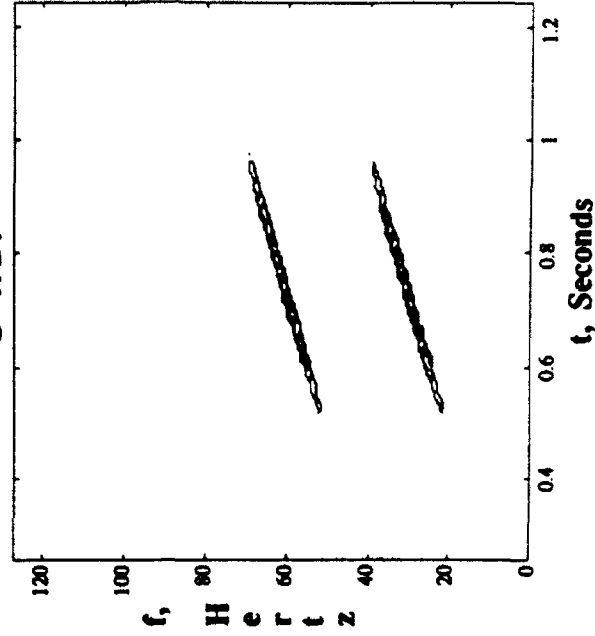


EXAMPLE 2: TWO PARALLEL LFMS

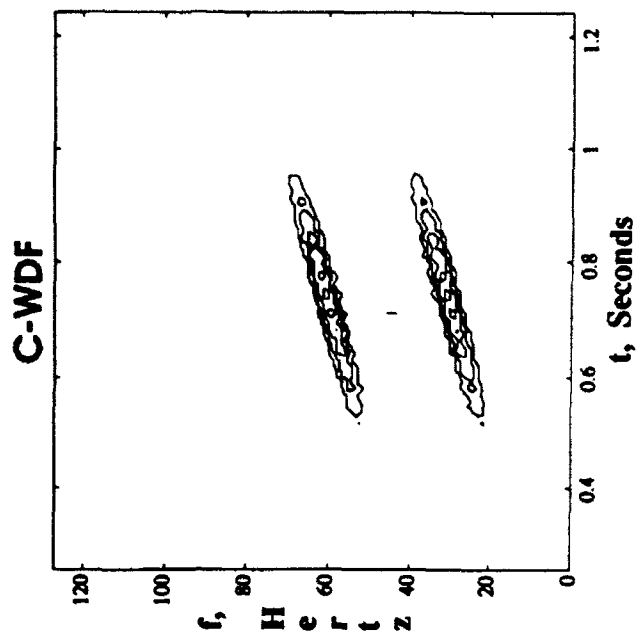
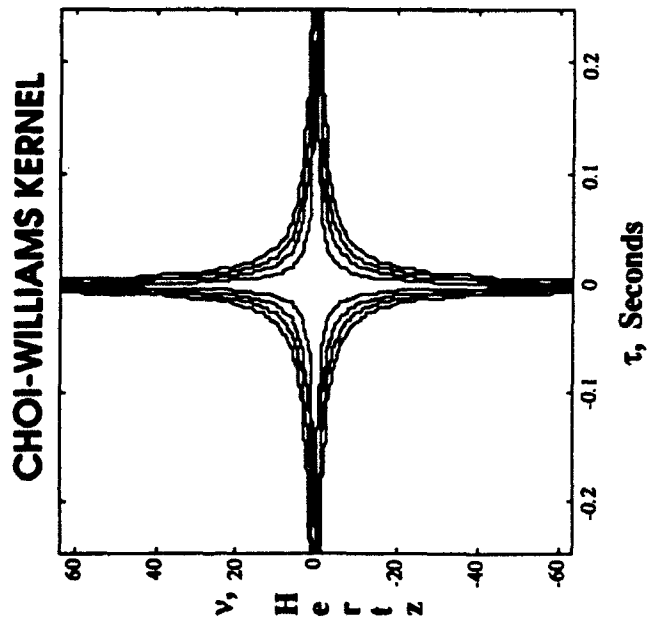
TILTED GAUSSIAN KERNEL



G-WDF

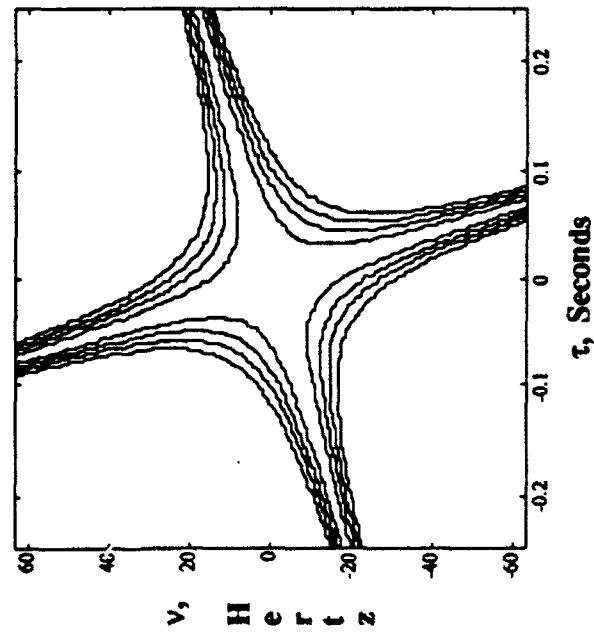


EXAMPLE 2: TWO PARALLEL LFMS

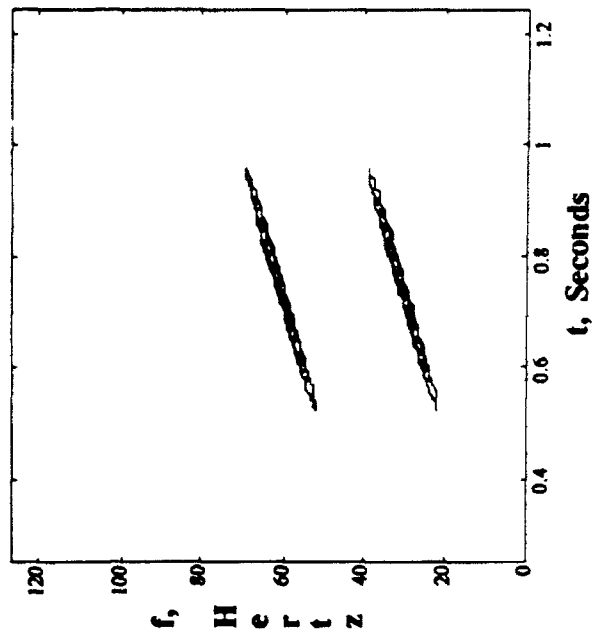


EXAMPLE 2: TWO PARALLEL LFMS

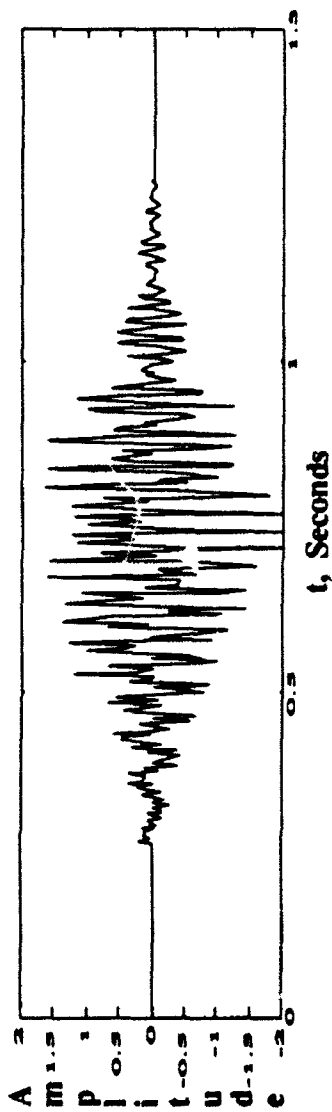
ROTATED CHOI-WILLIAMS KERNEL



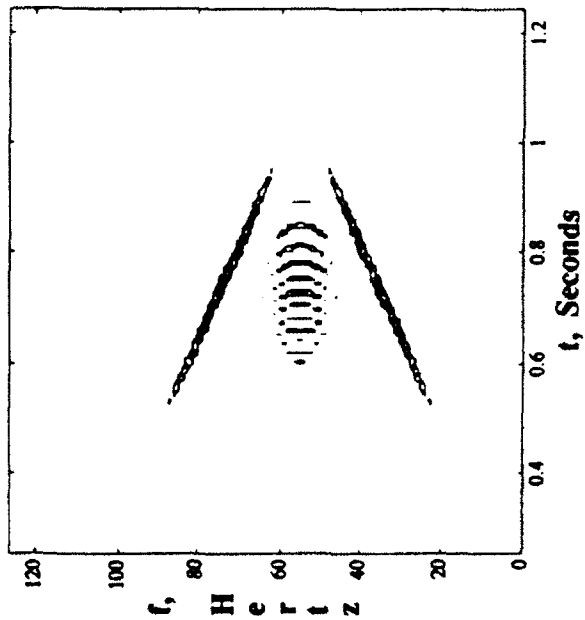
C-WDF



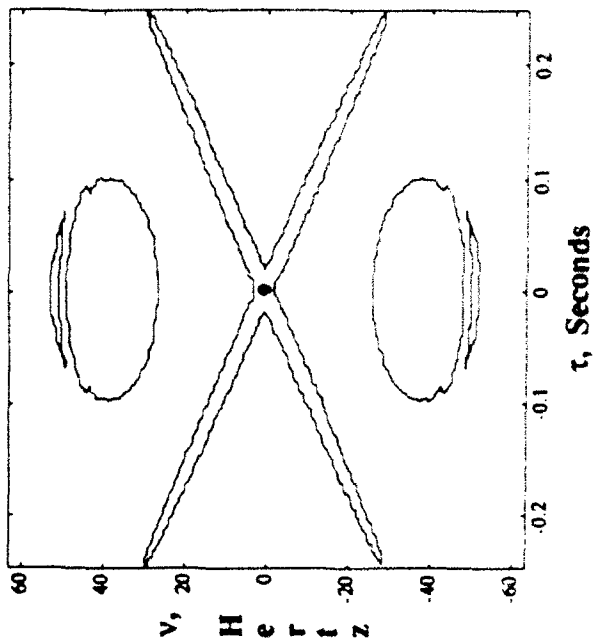
EXAMPLE 3: UPSLIDE, DOWNSLIDE LFMS



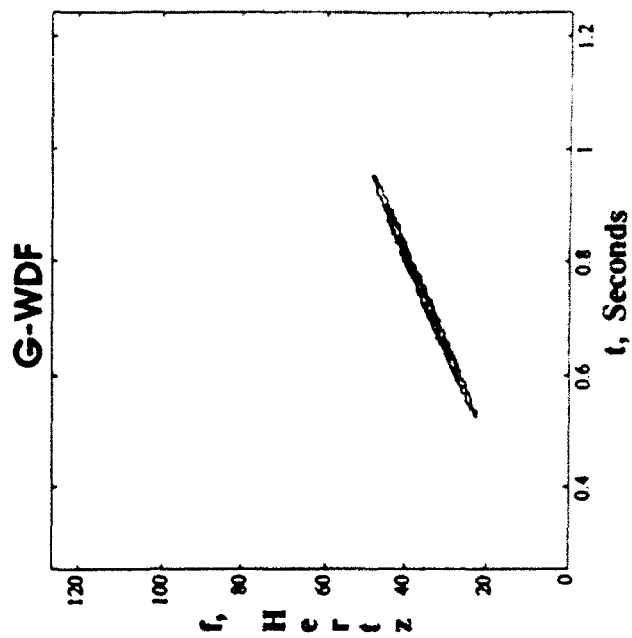
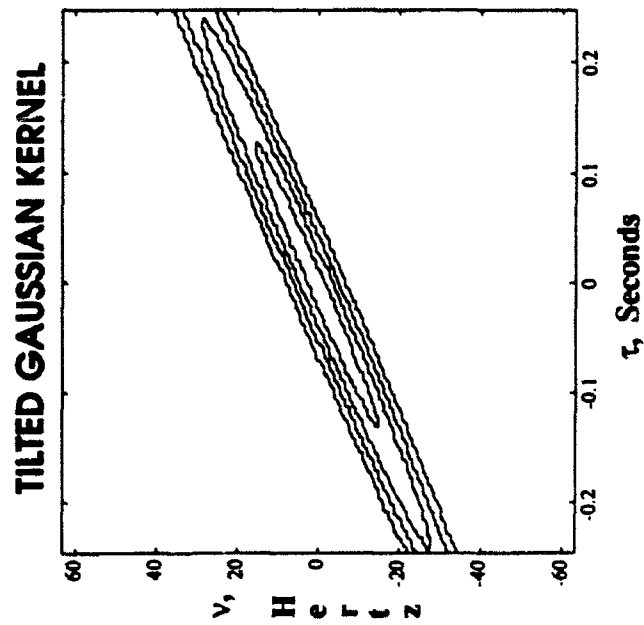
WDF



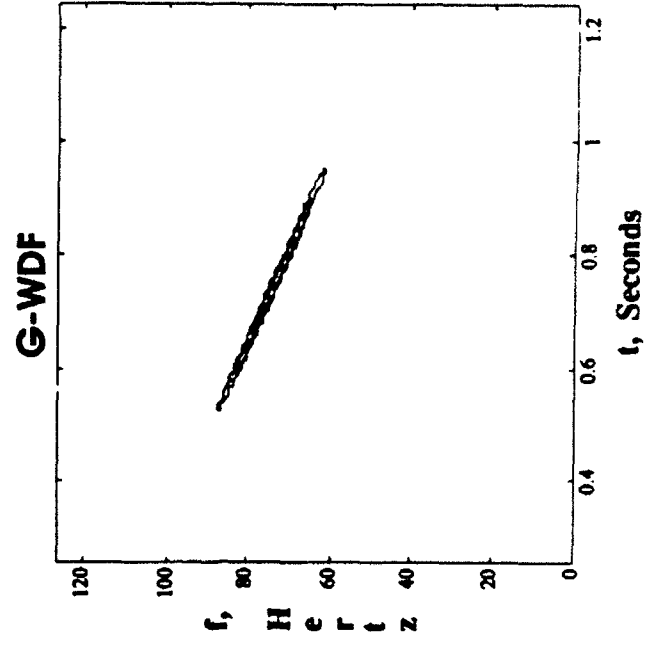
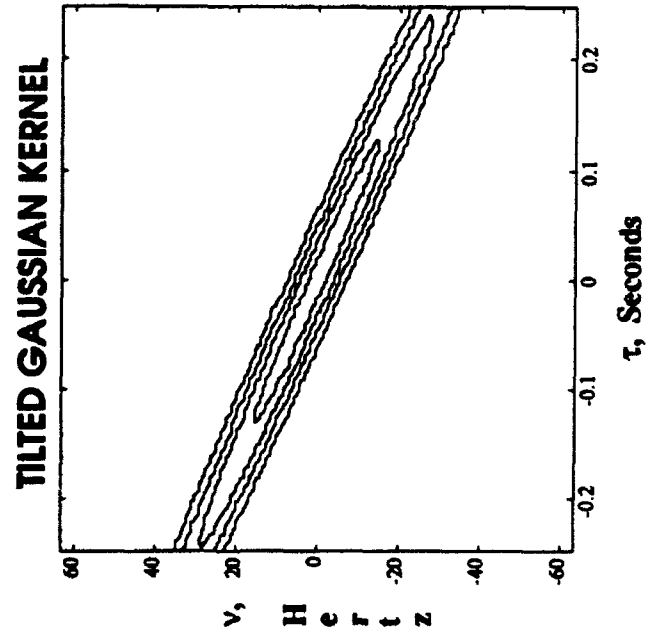
CAF



EXAMPLE 3: UPSLIDE, DOWNSLIDE LFMS

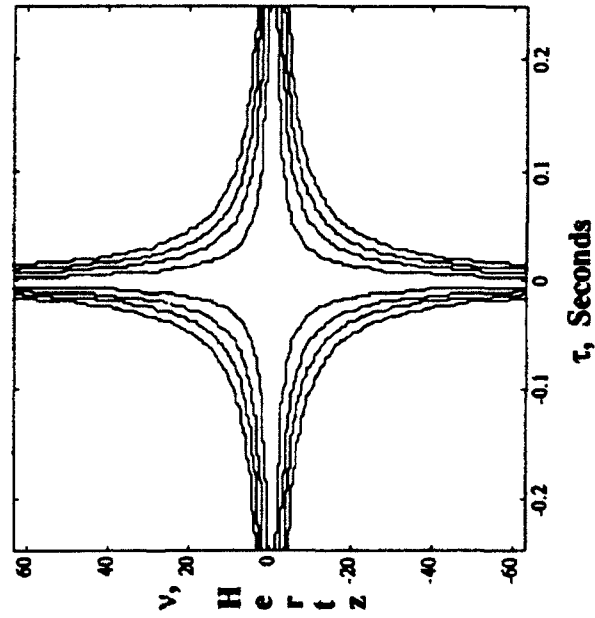


EXAMPLE 3: UPSLIDE, DOWNSLIDE LFMS

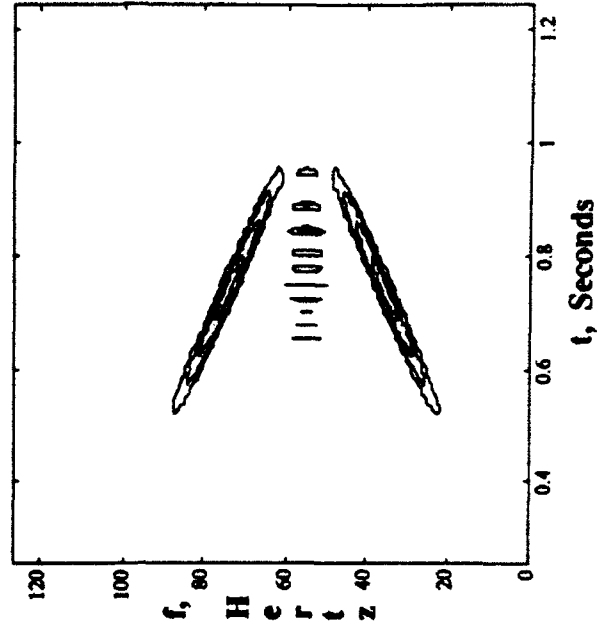


EXAMPLE 3: UPSLIDE, DOWNSLIDE LFMS

CHOI-WILLIAMS KERNEL

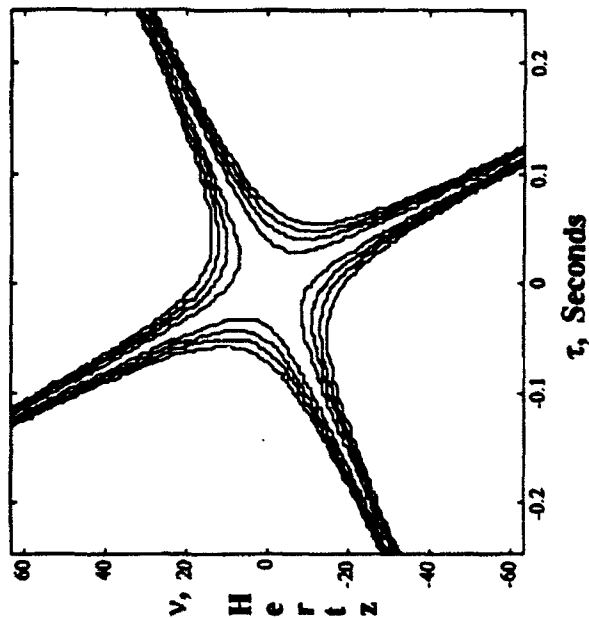


C-WDF

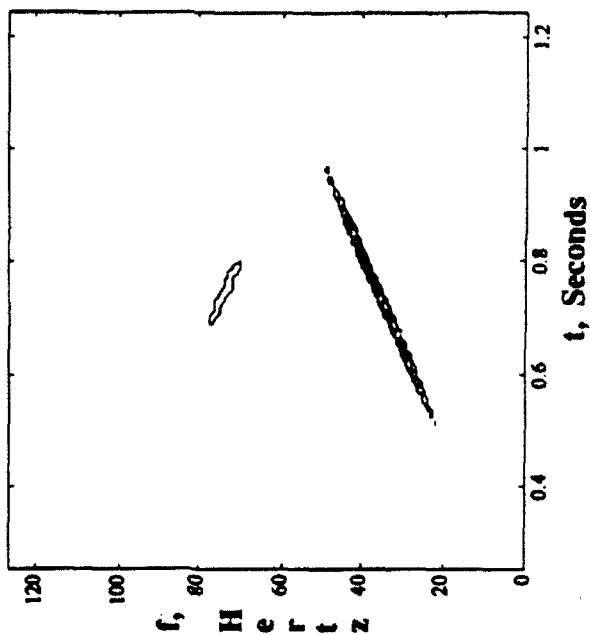


EXAMPLE 3: UPSLIDE, DOWNSLIDE LFMS

ROTATED CHOI-WILLIAMS KERNEL



C-WDF



CONCLUSIONS

TILTED GAUSSIAN KERNEL

- EFFECTIVELY SUPPRESSES THE CROSS-TERMS AND RECOVERS THE AUTO-TERMS IN ALL CASES CONSIDERED HERE
- EFFECTIVELY EXTRACTS SELECTED COMPONENTS, PROVIDED THE ENERGY DISTRIBUTIONS DO NOT SIGNIFICANTLY OVERLAP IN THE (t, f) DOMAIN

CHOI-WILLIAMS KERNEL

- EFFECTIVELY SUPPRESSES THE CROSS-TERMS AND RECOVERS THE AUTO-TERMS ONLY FOR THE CASE WHERE THE SLOPES OF THE LINES OF INSTANTANEOUS FREQUENCY ARE ZERO; I.E. NO FM
- HAS NO CAPABILITY TO EXTRACT SELECTED COMPONENTS

CONCLUSIONS

ROTATED CHOI-WILLIAMS KERNEL

- SUBSTANTIALLY IMPROVES THE CROSS-TERM SUPPRESSION AND THE AUTO-TERM RECOVERY FOR THE CASE WHERE THE SLOPES OF THE LINES OF INSTANTANEOUS FREQUENCY ARE NON-ZERO
- EXTRACTION OF SELECTED COMPONENTS WITH DISTINCT SLOPES OF THE LINES OF INSTANTANEOUS FREQUENCY CAN BE DONE WITH LIMITED SUCCESS

NUWC-NL Technical Report 10015
19 February 1992

Explicit Solution of Difference Equation for the
Wavenumber Response of Fluid-Loaded Stiffened Plate

Albert H. Nuttall

ABSTRACT

A method for solving a difference equation encountered for excitation of a line-driven, fluid-loaded, infinite flat plate with nonperiodic attached rib stiffeners is presented. For the case of excitation by Q complex exponentials, a linear set of Q simultaneous equations must be solved for each wavenumber k . For the most general excitation, a linear integral equation must be solved at each k . A possible shortcut for obtaining solutions at certain shifted values of wavenumber k is pointed out.

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LIST OF SYMBOLS

L	periodic inter rib spacing
Δ	offset of one set of rib stiffeners to another set
Q	number of sets of attached rib stiffeners, (1)
D	rigidity of plate, (1)
$w(x)$	transverse plate displacement, (1)
m	mass per unit area of plate, (1)
ω	applied excitation frequency (radians/second), (1)
$P_e(x)$	external pressure due to line-force, (1)
$P_a(x,0)$	acoustic pressure on upper surface of plate, (1)
$P_q(x)$	total pressure exerted by q-th set of rib stiffeners
k	wavenumber, (2)
$W(k)$	wavenumber response, (2)
$F(k)$	auxiliary function, (2)
$Y(k)$	auxiliary function, (2)
k_L	wavenumber defined by periodic spacing, (2)
B_q	dynamic structural mass of q-th rib stiffener set, (2)
Δ_q	q-th offset distance, (2)
F_0	magnitude of applied line force, (3)
x_0	point of application of line force, (3)
$S(k)$	auxiliary function, (4)
k_b	free in-vacuum plate wavenumber, (4)
ρ_0	mass density of acoustic fluid, (4)
k_0	acoustic wavenumber ω/c_0 , (4)
m'_q	mass per unit length of q-th set of rib stiffeners

m, n, p, q integers
 ε real nonzero variable, (5)
 $\{A_n\}$ given sequence, (5)
 α_q q -th exponential factor, (6)
 $\underline{W}_q(k)$ auxiliary summation function, (8)
 $\underline{F}_q(k)$ auxiliary summation function, (13)
 $\underline{Y}_{pq}(k)$ auxiliary summation function, (14)
 $N(k)$ numerator term for $Q = 2$, (20)
 $D(k)$ denominator term for $Q = 2$, (21)
 $B(u)$ Fourier sum, (22)
 $\underline{W}(u, k)$ auxiliary summation function, (24)
 $\underline{F}(u, k)$ auxiliary summation function, (29)
 $\underline{Y}(v-u, k)$ auxiliary summation function, (30)
 $\underline{W}_p(u, k)$ p -th order approximation to $\underline{W}(u, k)$, (33)

EXPLICIT SOLUTION OF DIFFERENCE EQUATION FOR THE WAVENUMBER RESPONSE OF FLUID-LOADED STIFFENED PLATE

INTRODUCTION*

Acoustic radiation from the upper surface of a fluid-loaded stiffened plate has been examined by Cray [1]. The formulation developed there produces an implicit algebraic equation for the Fourier transformed plate wavenumber response. This response must be manipulated into an explicit form in order to obtain a solution for the plate's structural response and to obtain the near- and far-field generated acoustic response. The following is a brief synopsis of the formulation for the acoustic radiation from a stiffened plate.

The infinite plate investigated in [1] was configured to have two infinite sets of attached rib stiffeners. The stiffeners composing a given set were identical and were spaced periodically with distance L . One set of stiffeners, however, was shifted by an amount Δ from the other set. In this manner, portions of the plate were configured with repeating sections having nonperiodic rib spacing.

The governing equation of motion for the surface displacement of the fluid-loaded isotropic plate for an applied line-force, with Q sets of attached rib stiffeners, is given by

$$D \frac{d^4 w(x)}{dx^4} - m \omega^2 w(x) = P_e(x) - P_a(x, 0) - \sum_{q=1}^Q P_q(x) , \quad (1)$$

*The introduction was contributed by Dr. Benjamin A. Cray.

where D is the rigidity of the plate, $w(x)$ is the transverse plate displacement, m is the mass per unit area of the plate, ω is the applied excitation frequency, $P_e(x)$ is the external pressure due to the applied line-force, $P_a(x,0)$ is the acoustic pressure acting on the upper surface of the plate, and $\{P_q(x)\}$ are the total pressures exerted by each set of attached rib stiffeners.

Rotary inertia and shear deformation effects within the plate and rib stiffeners were neglected. The stiffeners exerted reactive forces upon the plate but applied no angular moments.

Equation (1) was transformed, term by term, using exponential Fourier transforms. The spatial transform variable, k , has the physical significance of wavenumber. It was assumed that the transforms are well defined and exist over the entire domain of integration.

Upon transforming (1) into the wavenumber domain, the implicit form of the wavenumber response, for the fluid-loaded stiffened plate, can be written as

$$W(k) = F(k) - Y(k) \sum_{n=-\infty}^{\infty} W(k + nk_L) \times \\ \times \left[B_1 + B_2 \exp(ink_L \Delta_1) + \dots + B_Q \exp(ink_L \Delta_{Q-1}) \right], \quad (2)$$

where $W(k)$ is the (transformed) wavenumber response. The quantities on the right-hand side of (2) are defined as

$$F(k) = \frac{F_0 \exp(-ikx_0)}{S(k)}, \quad Y(k) = \frac{1}{S(k)}, \quad (3)$$

where

$$S(k) = \left\{ \begin{array}{ll} D(k^4 - k_b^4) - \frac{i \rho_o \omega^2}{\sqrt{k_o^2 - k^2}} & \text{for } |k| < k_o \\ D(k^4 - k_b^4) - \frac{\rho_o \omega^2}{\sqrt{k^2 - k_o^2}} & \text{for } |k| > k_o \end{array} \right\}, \quad (4)$$

where $k_b = (m \omega^2 / D)^{1/4}$ is the free in-vacuum plate wavenumber, $k_L = 2\pi/L$ is the wavenumber defined by the periodic spacing, and $B_q = m'_q \omega^2 / L$ is the dynamic structural mass of the q-th rib stiffener set. Also, Δ_q is the q-th offset distance.

Notice in (2) that the wavenumber or spectral response $W(k)$ appears implicitly. The difficulty now lies in determining an explicit expression for the plate's wavenumber response $W(k)$. That is, it is necessary to manipulate (2) such that the summation which contains the shifted wavenumber responses $\{W(k + nk_L)\}$ may be rewritten in terms of known quantities.

In [1], Cray obtained an explicit expression for $W(k)$ for the case of a single offset, that is, $Q = 2$ in (2). This corresponds to two sets of rib stiffeners, where one set is offset from the other. The following development is a significant extension of the configuration of two sets of rib stiffeners to the case of an arbitrary number of rib stiffener sets, each with different offsets $\{\Delta_q\}$, and to arbitrary excitation functions.

DESCRIPTION AND SOLUTION OF DIFFERENCE EQUATION

In the following mathematical development, k and ε are arbitrary real variables, while m, n, p, q are integers. Also, summations without limits are from $-\infty$ to $+\infty$. We are interested in finding the solution $W(k)$ to the difference equation

$$W(k) = F(k) - Y(k) \sum_n A_n W(k + \varepsilon n), \quad (5)$$

where $F(k)$ and $Y(k)$ are given known functions of k , and sequence $\{A_n\}$ is also known. For $\varepsilon = 0$, the solution to (5) is immediate; hence, ε is nonzero and fixed in the following.

In fact, we are interested in the particular case where A_n is a finite sum of Q complex exponentials in n (compare with the second line of (2)):

$$A_n = \sum_{q=1}^Q B_q \exp(in\alpha_q) \quad \text{for all } n, \quad (6)$$

where $\{\alpha_q\}$, $1 \leq q \leq Q$, are arbitrary (complex) distinct constants, and $\{B_q\}$ are arbitrary complex constants. The $\{B_q\}$ and $\{\alpha_q\}$ are given. Furthermore, since

$$\exp(in\alpha_q) = \exp(in[\alpha_q - p2\pi]) \quad \text{for all } p, \quad (7)$$

there is no loss of generality in assuming that $|\operatorname{Re}(\alpha_q)| \leq \pi$ for all $1 \leq q \leq Q$. If $\operatorname{Im}(\alpha_q)$ is nonzero for any q , then $|A_n|$ cannot remain bounded at both limits of $n = \pm\infty$. The earlier case solved in [1] corresponded to $Q = 2$ and the special value of $\alpha_1 = 0$.

DEFINITION AND PROPERTIES OF AUXILIARY FUNCTIONS

Define functions

$$\underline{W}_q(k) = \sum_n W(k + \epsilon n) \exp(in\alpha_q) \quad \text{for } 1 \leq q \leq Q. \quad (8)$$

(The additional dependence of $\underline{W}_q(k)$ on ϵ is suppressed notationally.) It then follows that

$$\begin{aligned} \underline{W}_q(k + \epsilon m) &= \sum_n W(k + \epsilon m + \epsilon n) \exp(in\alpha_q) = \\ &= \sum_p W(k + \epsilon p) \exp(i(p-m)\alpha_q) = \exp(-im\alpha_q) \underline{W}_q(k), \end{aligned} \quad (9)$$

where we let $p = m + n$ and used (8). This is the key relation regarding the functions $\underline{W}_q(k)$ defined in (8), namely

$$\underline{W}_q(k + \epsilon m) = \exp(-im\alpha_q) \underline{W}_q(k) \quad \text{for } 1 \leq q \leq Q. \quad (10)$$

With the aid of (6) and (8), the summation on n in (5) can now be manipulated into the form

$$\sum_n A_n W(k + \epsilon n) = \sum_n W(k + \epsilon n) \sum_{q=1}^Q B_q \exp(in\alpha_q) = \sum_{q=1}^Q B_q \underline{W}_q(k). \quad (11)$$

This enables us to express (5) in the alternative form

$$W(k) = F(k) - Y(k) \sum_{q=1}^Q B_q \underline{W}_q(k). \quad (12)$$

SOLUTION OF EQUATION (12)

At this point, in analogy to (8), it is convenient to define two additional functions, namely,

$$\underline{F}_q(k) = \sum_n F(k + \epsilon n) \exp(in\alpha_q) \quad \text{for } 1 \leq q \leq Q, \quad (13)$$

$$\underline{Y}_{pq}(k) = \sum_n Y(k + \epsilon n) \exp[in(\alpha_p - \alpha_q)] \quad \text{for } 1 \leq p, q \leq Q. \quad (14)$$

Now, replace k by $k + \epsilon m$ in (12) and use (10), obtaining

$$W(k + \epsilon m) = F(k + \epsilon m) - Y(k + \epsilon m) \sum_{q=1}^Q B_q \exp(-im\alpha_q) \underline{W}_q(k). \quad (15)$$

When this equation is multiplied by $\exp(im\alpha_p)$ and summed over all m , there follows, by use of (8), (13), and (14),

$$\underline{W}_p(k) = \underline{F}_p(k) - \sum_{q=1}^Q \underline{Y}_{pq}(k) B_q \underline{W}_q(k) \quad \text{for } 1 \leq p \leq Q. \quad (16)$$

This relation constitutes Q linear equations (at each k) in the Q unknown functions $\underline{W}_q(k)$, $1 \leq q \leq Q$. The remaining quantities in (16) can be obtained from (6), (13), and (14).

When (16) is solved for all the $\{\underline{W}_q(k)\}$, then (12) directly yields the original quantity of interest, namely $W(k)$ in (5), for that particular value of k . Although this procedure appears to require the solution of a new set of Q simultaneous linear equations for each k of interest, there is a shortcut that might be useful in some cases. Namely, reference to (15) reveals that

$W(k + \epsilon m)$ can now be calculated easily for nonzero m , for that k , once $F(k + \epsilon m)$ and $Y(k + \epsilon m)$ have been computed, without the need for another solution set. The same solutions $\{W_q(k)\}$ are used on the right-hand side of (15), regardless of the value of m under consideration; only the finite summation in (15) need be redone. Thus, the solution W to (5) at arguments $k, k \pm \epsilon, k \pm 2\epsilon, \dots$ can be found from the solution of one set of Q simultaneous equations, (16).

For the special case of $Q = 1$, (16) can be immediately solved to yield [1]

$$\underline{W}_1(k) = \frac{\underline{F}_1(k)}{1 + B_1 \underline{Y}_{11}(k)} = \frac{\sum_n F(k + \epsilon n) \exp(in\alpha_1)}{1 + B_1 \sum_n Y(k + \epsilon n)} . \quad (17)$$

Then (12) yields the explicit result (for $Q = 1$)

$$W(k) = F(k) - B_1 Y(k) \frac{\sum_n F(k + \epsilon n) \exp(in\alpha_1)}{1 + B_1 \sum_n Y(k + \epsilon n)} . \quad (18)$$

In this form, it is possible to directly evaluate $W(k)$ at any k of interest merely by computing the terms encountered on the right-hand side of (18). However, shortcut (15) is still better for the particular arguments $\{k + \epsilon m\}$.

For $Q = 2$, when the solutions of (16) are substituted into (12), the function $W(k)$ is given by the explicit expression

$$W(k) = F(k) - Y(k) \frac{N(k)}{D(k)}, \quad (19)$$

where

$$\begin{aligned} N(k) = & B_1 \underline{F}_1(k) + B_2 \underline{F}_2(k) + B_1 B_2 \times \\ & \times \left[\underline{F}_1(k) \underline{Y}_{22}(k) + \underline{F}_2(k) \underline{Y}_{11}(k) - \underline{F}_1(k) \underline{Y}_{21}(k) - \underline{F}_2(k) \underline{Y}_{12}(k) \right] \end{aligned} \quad (20)$$

and

$$\begin{aligned} D(k) = & 1 + B_1 \underline{Y}_{11}(k) + B_2 \underline{Y}_{22}(k) + \\ & + B_1 B_2 \left[\underline{Y}_{11}(k) \underline{Y}_{22}(k) - \underline{Y}_{12}(k) \underline{Y}_{21}(k) \right]. \end{aligned} \quad (21)$$

A program for the evaluation of (19) - (21), for the case where $Y(k)$ is real, is presented in the appendix; also, B_1 , B_2 , α_1 , and α_2 are real. Then $D(k)$ is real, but $N(k)$, $F(k)$, and $W(k)$ are complex. Also, $\underline{Y}_{22}(k) = \underline{Y}_{11}(k)$ is real, while $\underline{F}_1(k)$, $\underline{F}_2(k)$, and $\underline{Y}_{21}(k)^* = \underline{Y}_{12}(k)$ are complex. The program uses these properties.

In general, for any Q , if (16) is solved analytically for the set $\{\underline{W}_q(k)\}$ in terms of $\{B_q\}$, $\{\underline{F}_q(k)\}$, and $\{\underline{Y}_{pq}(k)\}$, these results can be substituted or utilized in (12) to get an explicit expression for $W(k)$ that is valid for all k . For large Q , this will be impractical; instead, numerical solution of the Q simultaneous equations, (16), will be required at each k of interest. The only exception is to use (15) for the particular arguments $\{k + \epsilon m\}$. (All results above actually hold for k and ϵ complex.)

SOLUTION FOR GENERAL SEQUENCE $\{A_n\}$

In this section, we no longer restrict sequence $\{A_n\}$ to have exponential form (6). Rather, for bounded given sequence $\{A_n\}$, define function

$$B(u) = \frac{1}{2\pi} \sum_n A_n \exp(-inu) \quad \text{for } |u| < \pi, \quad (22)$$

where u is real. Then, the sequence values can be found from

$$A_n = \int_{-\pi}^{\pi} du B(u) \exp(inu) \quad \text{for all } n. \quad (23)$$

The fastest variation with n that (23) allows for A_n is $\exp(\pm in\pi)$; however, this is no loss of generality, as seen by reference to (7). The case considered earlier in (6) corresponds to $B(u)$ being composed of a set of Q impulses of area B_q located at $u = \alpha_q$, with $|\alpha_q| \leq \pi$, when α_q is real.

Now, define function

$$\underline{W}(u, k) = \sum_n W(k + \epsilon n) \exp(inu) \quad \text{for } |u| \leq \pi. \quad (24)$$

(Again, the notational dependence on ϵ is suppressed.) The key relation that $\underline{W}(u, k)$ satisfies is

$$\begin{aligned} \underline{W}(u, k + \epsilon m) &= \sum_n W(k + \epsilon m + \epsilon n) \exp(inu) = \\ &= \sum_p W(k + \epsilon p) \exp(i(p-m)u) = \exp(-imu) \underline{W}(u, k), \end{aligned} \quad (25)$$

where we let $p = m + n$ and used (24). This relation,

$$\underline{W}(u, k + \epsilon m) = \exp(-imu) \underline{W}(u, k) , \quad (26)$$

is the analogue of (10) earlier.

The summation in (5) can now be expressed as

$$\begin{aligned} \sum_n A_n W(k + \epsilon n) &= \sum_n W(k + \epsilon n) \int_{-\pi}^{\pi} du B(u) \exp(inu) = \\ &= \int_{-\pi}^{\pi} du B(u) \underline{W}(u, k) , \end{aligned} \quad (27)$$

where we used (23) and (24). Therefore, the equation of interest, (5), now takes the form

$$W(k) = F(k) - Y(k) \int_{-\pi}^{\pi} du B(u) \underline{W}(u, k) . \quad (28)$$

We now define, in analogy to (13) and (14), the two functions

$$\underline{F}(u, k) = \sum_n F(k + \epsilon n) \exp(inu) \quad \text{for } |u| \leq \pi , \quad (29)$$

$$\underline{Y}(v-u, k) = \sum_n Y(k + \epsilon n) \exp(in(v-u)) \quad \text{for } |u|, |v| \leq \pi . \quad (30)$$

When we replace k by $k + \epsilon m$ in (28), and use (26), there follows

$$W(k + \epsilon m) = F(k + \epsilon m) - Y(k + \epsilon m) \int_{-\pi}^{\pi} du B(u) \underline{W}(u, k) \exp(-imu) . \quad (31)$$

We now multiply (31) by $\exp(imv)$, v real, and sum over all m , to

obtain

$$\underline{W}(v,k) = \underline{F}(v,k) - \int_{-\pi}^{\pi} du \underline{Y}(v-u,k) B(u) \underline{W}(u,k) \quad \text{for } |v| \leq \pi, \quad (32)$$

where we used (24), (29), and (30). This is a linear integral equation, with kernel $\underline{Y}(v-u,k) B(u)$, for unknown $\underline{W}(u,k)$; that is, for each k of interest, a new linear integral equation must be solved. Then, $\underline{W}(u,k)$ will be known for $|u| \leq \pi$ at that particular k .

Once $\underline{W}(u,k)$ is known, (28) gives desired solution $W(k)$ at that particular k value, since $F(k)$, $Y(k)$, and $B(u)$ are known functions. If a large number of k values are of interest, (32) can involve a great deal of computational effort; however, arbitrary bounded sequence $\{A_n\}$ is now allowed in (5). Furthermore, (31) then yields the solution for W at arguments $k, k \pm \varepsilon, k \pm 2\varepsilon, \dots$ without the need to solve another integral equation.

The linear integral equation in (32) could be solved by recursion; for example, the p -th order approximation is given by

$$\underline{W}_p(v,k) \equiv \underline{F}(v,k) - \int_{-\pi}^{\pi} du \underline{Y}(v-u,k) B(u) \underline{W}_{p-1}(u,k) \quad \text{for } |v| \leq \pi. \quad (33)$$

A possible starting value for this recursion is $\underline{W}_0(u,k) = \underline{F}(u,k)$.

SUMMARY

When the governing differential equation of motion for the surface displacement of the fluid-loaded isotropic plate for an applied line-force, with Q sets of attached rib stiffeners, is transformed into the wavenumber domain, a difference equation is encountered. Solution of this latter equation for exponential excitation, for a particular value of wavenumber k , is accomplished through the definition of auxiliary functions involving sums of displaced wavenumber responses with exponential factors. Finally, solution of a simultaneous set of Q linear equations completes the required calculations. For the most general excitation, the set of equations becomes infinite, that is, a linear integral equation must be solved.

APPENDIX. PROGRAM FOR (19) - (21)

In the following program for solution $W(k)$ given by (19) - (21), specified function $Y(k)$ is presumed real; however, $F(k)$ is allowed to be complex. The following example corresponds to

$$F(k) = \exp(-.71 k^2) + i \exp(-.91 k^2 - k) ,$$

$$Y(k) = \exp(-.63 k^2 + k) .$$

```

10    K=.7                ! k, wavenumber
20    E=.4                ! epsilon
30    L1=.1               ! alpha1
40    L2=.2               ! alpha2
50    B1=1.1
60    B2=1.3
70    CALL Fqk(K,E,L1,L2,F1r,F1i,F2r,F2i) ! (13)
80    CALL Ypqk(K,E,L1,L2,Y11,Y12r,Y12i)  ! (14)
90    B=B1*B2
100   Y=Y11-Y12r
110   D=Y11*Y11-Y12r*Y12r-Y12i*Y12i
120   D=1.+(B1+B2)*Y11+B*D                ! (21)
130   Nr=(F1r+F2r)*Y-(F1i-F2i)*Y12i
140   Nr=B1*F1r+B2*F2r+B*Nr                ! (20)
150   Ni=(F1i+F2i)*Y+(F1r-F2r)*Y12i
160   Ni=B1*F1i+B2*F2i+B*Ni                ! (20)
170   PRINT
180   PRINT "DENOMINATOR = ";D
190   Y=FNK(K)/D
200   CALL F(K,Fr,Fi)
210   Wr=Fr-Y*Nr
220   Wi=Fi-Y*Ni                            ! (19)
230   PRINT Wr,Wi
240   END
250   !

```

```

260 SUB Fqk(K,E,L1,L2,F1r,F1i,F2r,F2i) ! (13)
270 DOUBLE N ! INTEGER
280 F1r=F1i=F2r=F2i=0.
290 FOR N=-40 TO 40
300 CALL F(K+E*N,Fr,Fi)
310 IF ABS(N)=40 THEN PRINT ABS(Fr)+ABS(Fi);
320 C=COS(N*L1)
330 S=SIN(N*L1)
340 F1r=F1r+Fr*C-Fi*S
350 F1i=F1i+Fr*S+Fi*C
360 C=COS(N*L2)
370 S=SIN(N*L2)
380 F2r=F2r+Fr*C-Fi*S
390 F2i=F2i+Fr*S+Fi*C
400 NEXT N
410 SUBEND
420 !

430 SUB Ypqk(K,E,L1,L2,Y11,Y12r,Y12i) ! (14)
440 DOUBLE N ! INTEGER
450 Y11=Y12r=Y12i=0.
460 FOR N=-40 TO 40
470 T=FNY(K+E*N)
480 IF ABS(N)=40 THEN PRINT T;
490 A=N*(L1-L2)
500 Y11=Y11+T
510 Y12r=Y12r+T*COS(A)
520 Y12i=Y12i+T*SIN(A)
530 NEXT N
540 SUBEND
550 !

560 SUB F(K,Fr,Fi) ! F(k)
570 Fr=Fi=0.
580 A=.71*K*K
590 IF A>100. THEN 610
600 Fr=EXP(-A)
610 A=.91*K*K+K
620 IF A>100. THEN 640
630 Fi=EXP(-A)
640 SUBEND
650 !

660 DEF FNY(K) ! real Y(k)
670 A=.63*K*K-K
680 IF A>100. THEN RETURN 0.
690 RETURN EXP(-A)
700 FNEND

```

REFERENCES

- [1] B. A. Cray, Near-Field and Far-Field Sound Radiation from a Line-Driven Fluid-Loaded Infinite Flat Plate Having Periodic and Nonperiodic Attached Rib Stiffeners, NUWC-NL Technical Report TR 10013, Naval Undersea Warfare Center, New London, CT, March 1992.

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Exact Detection Performance of Multiple-Pulse Frequency-Shift
Signals in a Partially-Correlated Fading Medium with
Generalized Noncentral Chi-Squared Statistics

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ABSTRACT

The transmitted signal consists of K pulses separated in time, frequency space so as to be nonoverlapping. In passing through the medium to the receiver, each signal pulse is subjected to fading. In particular, pulse pairs which are closely spaced in time, frequency space can fade in a highly dependent fashion, while those more widely separated can have relatively independent fading behavior; that is, the transmitted frequency-shift-keyed signal pulses undergo partially-correlated fading of a very general character that contains both deterministic components as well as random components. The amplitude-fading statistics are not limited to be Rayleigh.

Additive zero-mean Gaussian noise, which is stationary over the total signal transmission time and which has a flat spectrum over the total signal bandwidth, is present at the input to the receiver, in addition to the fluctuating signal pulses (when present).

Receiver processing consists of matched filtering of each of the K time-delayed, frequency-shifted pulse locations, followed by squared-envelope detection, sampling, summation, and comparison of this decision variable with a fixed threshold for a statement on signal presence or absence.

The characteristic function of the decision variable is derived in closed form for a very general model of partially-correlated fading that subsumes Rayleigh, Rician, and noncentral

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Chi probability densities for the amplitude variate as special cases. This characteristic function depends on the number K of signal pulses, the number M of fading components, the deterministic received signal energy in each fading component, the average random received signal energy in each fading component, as well as the received noise spectral density level. Numerous special cases are pointed out and specific results are given in detail.

An efficient expansion for the exceedance distribution, for one of the cases, is listed and exercised for a representative numerical example. Comparisons with earlier approximations reveal them to have been pessimistic by several dB in ranges considered typical for practical applications. Also, the effects of correlated fading of the signal pulses are found to be not overly detrimental until the normalized covariance coefficient of adjacent pulses gets larger than approximately .5.

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LIST OF SYMBOLS

FSK	frequency shift keying
t	time, (1)
$s(t)$	transmitted real signal waveform, (1)
$\underline{s}(t)$	transmitted signal complex envelope, (1)
f_o	carrier frequency, (1)
Re	real part, (1)
$\tilde{E}$	transmitted signal energy, (2)
r	amplitude scaling in transmission, (3)
θ	phase shift in transmission, (3)
t_d	time delay of received signal, (3)
f_d	frequency shift of received signal, (3)
overbar	ensemble average, (4)
$n(t)$	received noise waveform, (5)
$\underline{n}(t)$	received noise complex envelope, (5)
N_o	received noise one-sided spectral level, (5), (10)
A	local reference amplitude scaling, (6)
ϕ	local reference phase shift, (6)
$\alpha(t)$	complex envelope of received waveform, (7)
$\beta(t)$	complex envelope of local reference, (7)
γ	output of receiver processor, (8), figure 1
μ	parameter, $2 r A \tilde{E}$, (8), (17)
n_r, n_i	real and imaginary noise components, (9)
$\delta(t)$	delta function, (10)
σ^2	parameter, $2 N_o A^2 \tilde{E}$, (12), (17)
Prob	exceedance probability, (14)

$f_c(\xi)$	conditional characteristic function of γ , (16)
$\chi(j)$	j-th conditional cumulant of γ , (19)
d	conditional deflection criterion, (21)
K	number of transmitted signal pulses, (22)
r_k	amplitude scaling of k-th signal pulse, (22)
θ_k	phase shift of k-th signal pulse, (22)
A_k	local reference scaling for k-th pulse, (22)
ϕ_k	local reference phase shift for k-th pulse, (22)
$\alpha_k(t)$	complex envelope of k-th received waveform, (22)
$\beta_k(t)$	complex envelope of k-th local reference, (22)
$\underline{s}_k(t)$	transmitted k-th signal pulse complex envelope, (22)
$\tilde{E}_k$	transmitted signal energy in k-th pulse, (23)
μ_k	k-th parameter, $2 r_k A_k \tilde{E}_k$, (24)
σ_k^2	k-th parameter, $2 N_0 A_k^2 \tilde{E}_k$, (24)
$f_c(\xi)$	conditional characteristic function of γ , (26), (27)
$\chi(j)$	j-th conditional cumulant of γ , (28)
d_K	conditional deflection of γ , (30)
$\overline{d_K}$	average deflection of output γ in figure 1, (32), (33)
A	common receiver weighting for K pulses, (35), (38)
$\tilde{E}$	common transmitted energy of K signal pulses, (36), (38)
S	sum of K power fading variates $\{r_k^2\}$, (37)
$f_S(\xi)$	characteristic function of random variable S, (41)
$f_Y(\xi)$	unconditional characteristic function of output γ , (42)
q_k	r_k^2 , power scaling of k-th signal pulse, (43), (44)
t_k	time location of k-th FSK signal pulse, (44)
f_k	frequency location of k-th FSK signal pulse, (44)
$q(t, f)$	continuous fading process, time t and frequency f, (45)

M	number of fading components, (45)
$c_m(t,f)$	deterministic component of fading model, (45)
$g_m(t,f)$	random component of fading model, (45)
R_{mn}	covariance of g_m with g_n , (47)
τ, ν	delay and shift parameters in covariance R_{mn} , (47)
D_{km}	deterministic received signal energy in m-th component of k-th pulse, (50)
D_k	deterministic received signal energy in k-th pulse, (51)
E_{km}	average random received signal energy in m-th component of k-th pulse, (52)
E_k	average random received signal energy in k-th pulse, (53)
$\tilde{q}_k$	zero-mean component of q_k , (54)
$\tilde{R}_{kj}$	covariance between q_k and q_j , (55)
$\tilde{R}_{kk}$	variance of q_k , (56), (58)
ρ_{kj}	covariance coefficient between q_k and q_j , (56)
δ_{mn}	1 for $m = n$; 0 otherwise, (61)
$r^{(m)}$	relative power measure of m-th random fading component, (67)
X	$M \times 1$ Gaussian random column vector, (71)
E	mean vector, (72)
$C, \text{Cov}(X)$	covariance matrix of vector X , (72), (B-1)
Λ	eigenvalue matrix of C , (73)
Q	normalized modal matrix of C , (73)
diag	diagonal matrix, (74)
λ_m	m-th eigenvalue of matrix Λ , $1 \leq m \leq M$, (74)
V_m	m-th eigenvector of matrix Q , (74)
v_{mn}	n-th component of vector V_m , (74)
ϵ_m	deterministic parameter, (75)

$f_q(\xi)$	characteristic function of power-scaling $q(t,f)$, (76)
$\chi_q(p)$	p -th cumulant of $q(t,f)$, (77)
$\underline{E}$	transformed mean vector, (91)
ϵ_n	components of $\underline{E}$, (91)
$S^{(m)}$	power sum over signal pulse numbers, $1 \leq m \leq M$, (95)
$f_S(\xi)$	characteristic function of sum S , (96)
$f^{(m)}(\xi)$	characteristic function of $S^{(m)}$, (97)
$Y^{(m)}$	$K \times 1$ random column vector, (98)
$C^{(m)}$	covariance matrix of vector $Y^{(m)}$, (100)
$\Lambda^{(m)}$	eigenvalue matrix of $C^{(m)}$, (101)
$Q^{(m)}$	normalized modal matrix of $C^{(m)}$, (101)
$\lambda_k^{(m)}$	k -th eigenvalue of matrix $\Lambda^{(m)}$, (102)
$V_k^{(m)}$	k -th eigenvector of matrix $Q^{(m)}$, (102)
$v_{kj}^{(m)}$	j -th component of vector $V_k^{(m)}$, (102)
$\epsilon_k^{(m)}$	transformed mean vector, (103)
h_k	auxiliary constants, $1 \leq k \leq K$, (111)
$\chi_S(p)$	p -th cumulant of sum S , (122)
$\chi_Y(p)$	p -th cumulant of output Y , (125)
$\underline{C}^{(m)}$	normalized covariance matrix, (127)
$\underline{\Lambda}^{(m)}$	eigenvalue matrix of $\underline{C}^{(m)}$, (128)
$\underline{\lambda}_k^{(m)}$	k -th eigenvalue of matrix $\underline{\Lambda}^{(m)}$, (128)
$\underline{\epsilon}_k^{(m)}$	auxiliary constants, (137)
$\underline{C}$	common normalized covariance matrix, (140)
$\underline{\Lambda}$	eigenvalue matrix of $\underline{C}$, (141)
$\underline{\lambda}_k$	k -th eigenvalue of matrix $\underline{\Lambda}$, (141)
$\underline{h}_k$	auxiliary constants, (145)
C	covariance matrix of $g_1(t,f)$, (152)

ϕ	normalized power-scaling, (161)
σ_m^2	variance of $g_m(t,f)$, (162)
f_ϕ	characteristic function of ϕ , (163)
η_m	normalized constants, (164)
PDF	probability density function, (166), figure 2
p_ϕ	probability density function of ϕ , (166)
θ	normalized amplitude-scaling, (167)
p_θ	probability density function of θ , (168)
C, S	auxiliary functions, (172)
η	constant, (174), (179)
I_N	modified Bessel function of order N , (180)
p_Y	probability density function of processor output y
T_{mk}	auxiliary parameters, (182)
e_{mk}	auxiliary parameters, (182)
Prob	probability, (183)
u	threshold, (183)
g_p	expansion coefficient, (183), (186)
F	scale factor, (184)
α_p	auxiliary parameters, (185)
$H_n(x)$	auxiliary function, (187)
ψ_m	fractional strengths of random components, (190)
SNR	signal-to-noise ratio
Cov_1	normalized covariance $\frac{R_{11}(t_{k+1}-t_k, 0)}{R_{11}(0, 0)}$, below (193)
$Cov(\tau, v)$	normalized covariance $R_{11}(\tau, v)/R_{11}(0, 0)$, (194)
det	determinant of matrix, (B-4)

Exact Detection Performance of Multiple-Pulse Frequency-Shift
Signals in a Partially-Correlated Fading Medium with
Generalized Noncentral Chi-Squared Statistics

INTRODUCTION

The transmission of a succession of time-delayed and/or frequency-shifted signal pulses through a fading medium leads to received waveforms which vary in amplitude and phase in a random fashion and with possibly complicated statistical dependencies. The evaluation of the detection capability of matched filter processing and incoherent combination in such fading situations and noise has been the subject of many studies over the years [1 - 15]. Some of these efforts have been aimed at obtaining approximations to the performance of the systems of interest, while others have yielded exact results in selected special cases.

Results for partially-correlated fading have been obtained in [3;4;6;7;9;10;11;12;14;15]. In particular, in [4;6;10;11;12;14], the characteristic functions of the decision variables have been obtained in closed form for the case of partially correlated fading of the received signal pulses with Rayleigh amplitude statistics and additive Gaussian noise. The results in [12] are approximate; however, the signal amplitude fading is not limited to Rayleigh statistics there, but rather can have a Chi distribution of an arbitrary number of degrees of freedom.

The case of independent signal fading in noise, but allowing correlated clutter, was addressed in [13]. An extension of [12]

to partially correlated signal fading in a system with normalization was solved in an approximate fashion in [14]. Finally, a case of partially correlated signal fading with Rayleigh amplitude statistics in the presence of noise and K-distributed clutter was solved exactly in [15].

Here, we will give exact results for a fading medium which has both deterministic components as well as random components with arbitrary covariance coefficients and additive Gaussian noise. Also, frequency shift keyed (FSK) signals will be allowed, with arbitrary (non-overlapping) occupancy in time, frequency space. In addition, the signal amplitude statistics will not be limited to Rayleigh, but will include the range of possibilities inherent in the noncentral Chi distribution of an arbitrary number of degrees of freedom. The end result is a closed form for the characteristic function of the decision variable that, although complicated in appearance, is amenable to efficient computer evaluation of the detection characteristics by means of a single fast Fourier transform (FFT). No matrix inverses are required. However, a novel expansion technique for the detection probability is derived which is efficient and accurate; a program incorporating this expansion is listed and exercised for several examples.

Extensions to a more general form of processor will also be solved but not evaluated. In particular, the characteristic function of the most general complex second-order form of arbitrarily-correlated complex Gaussian random variables with arbitrary means will be obtained in closed form.

RECEIVER PROCESSING FOR A SINGLE PULSE

In this section, we will describe the basic model of the single-pulse transmitted signal, the received signal with fading, the received noise, and the receiver processing. The extension to multiple signal pulses will be undertaken in the next section.

We presume that the transmitted real signal $s(t)$ is narrowband with low-frequency complex envelope $\underline{s}(t)$ superposed on carrier frequency f_0 :

$$s(t) = \text{Re}\{\underline{s}(t) \exp(i2\pi f_0 t)\} . \quad (1)$$

The transmitted signal energy is then

$$\tilde{E} \equiv \int dt [s(t)]^2 = \frac{1}{2} \int dt |\underline{s}(t)|^2 , \quad (2)$$

where integrals without limits are over $(-\infty, +\infty)$. The time-bandwidth product of the single pulse complex envelope $\underline{s}(t)$ is arbitrary; thus, for example, $\underline{s}(t)$ could contain linear frequency modulation along with rectangular or Gaussian amplitude modulation.

The received signal waveform is

$$\text{Re}\{r \underline{s}(t-t_d) \exp[i2\pi(f_0+f_d)t + i\theta]\} , \quad (3)$$

where r is a (dimensionless) amplitude scaling, t_d is a time delay, f_d is a frequency shift, and θ is a phase shift. Real random unknowns r and θ do not vary with time over the duration of pulsed signal $\underline{s}(t)$. Delay t_d and shift f_d are presumed known at the receiver. The average received signal energy is, using

(3) and (2),

$$\overline{r^2} \frac{1}{2} \int dt |\underline{s}(t)|^2 = \overline{r^2} \tilde{E} . \quad (4)$$

The received zero-mean additive noise waveform $n(t)$ is

$$n(t) = \text{Re}\{\underline{n}(t) \exp(i2\pi f_0 t)\} , \quad (5)$$

which is presumed stationary over the duration of the signal.

The spectrum of received noise $n(t)$ in the neighborhood of frequency f_0 is flat, with one-sided spectral level N_0 watts/Hz.

The reference waveform employed at the receiver corresponds to the matched filter to the transmitted signal, namely

$$A \text{Re}\{\underline{s}(t-t_d) \exp[i2\pi(f_0+f_d)t + i\phi]\} , \quad (6)$$

which utilizes knowledge of t_d and f_d . The local reference level A and local phase shift ϕ in (6) are irrelevant to the processing employed here; that is, the performance in terms of the receiver operating characteristics (detection probability P_D versus false alarm probability P_F) is independent of A and ϕ , at least for this case of one signal pulse.

We now define the two analytic functions

$$\begin{aligned} \alpha(t) &= r \underline{s}(t-t_d) \exp[i2\pi(f_0+f_d)t+i\theta] + \underline{n}(t) \exp(i2\pi f_0 t) , \\ \beta(t) &= A \underline{s}(t-t_d) \exp[i2\pi(f_0+f_d)t+i\phi] , \end{aligned} \quad (7)$$

which are recognized as corresponding to the received waveform and the local reference, respectively. The total output of the matched-filter squared-envelope detector to the received signal and noise waveform is then given by

$$\begin{aligned}
\gamma &= \left| \int dt \alpha(t) \beta^*(t) \right|^2 = \\
&= \left| r \exp(i\theta) A \int dt |\underline{s}(t-t_d)|^2 + A \int dt \underline{n}(t) \underline{s}^*(t-t_d) \exp(-i2\pi f_d t) \right|^2 \\
&= \left| 2 r A \tilde{E} + n_r + i n_i \right|^2 \equiv \left| \mu + n_r + i n_i \right|^2, \quad (8)
\end{aligned}$$

where we used (2) and defined the zero-mean processor output complex noise variate

$$n_r + i n_i = \exp(-i\theta) A \int dt \underline{n}(t) \underline{s}^*(t-t_d) \exp(-i2\pi f_d t). \quad (9)$$

The two covariances of the received noise complex envelope $\underline{n}(t)$ are derived in appendix A; they are given by (A-8) and (A-11) as

$$\overline{\underline{n}(t_1) \underline{n}(t_2)} = 0, \quad \overline{\underline{n}(t_1) \underline{n}^*(t_2)} = 2 N_0 \delta(t_1 - t_2). \quad (10)$$

Use of these relations with (9) enables us to determine the two averages

$$\overline{(n_r + i n_i)^2} = \overline{n_r^2} - \overline{n_i^2} + i 2 \overline{n_r n_i} = 0, \quad (11)$$

$$\overline{|n_r + i n_i|^2} = \overline{n_r^2} + \overline{n_i^2} = 2 N_0 A^2 \int dt |\underline{s}(t-t_d)|^2 = 4 N_0 A^2 \tilde{E},$$

where we also used (2). Combining the information in the two lines of (11), there follows, for the two zero-mean real variates n_r and n_i ,

$$\overline{n_r^2} = \overline{n_i^2} = 2 N_0 A^2 \tilde{E} \equiv \sigma^2, \quad \overline{n_r n_i} = 0. \quad (12)$$

These results are independent of the parameters θ , t_d , and f_d .

Since noise input $\underline{n}(t)$ is Gaussian and operation (9) is linear, real variates n_r and n_i are Gaussian with joint probability density function

$$p(n_r, n_i) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{n_r^2 + n_i^2}{2\sigma^2}\right). \quad (13)$$

From this result, we can derive the conditional exceedance distribution function of matched filter output γ in (8), for a given value of random variable r . In terms of the two auxiliary parameters μ and σ^2 defined in (8) and (12) respectively, we have, for $u > 0$, exceedance probability

$$\begin{aligned} \text{Prob}(\gamma > u) &= \text{Prob}\left((\mu + n_r)^2 + n_i^2 > u\right) = \\ &= \iint_C dx \, dy \, \frac{1}{2\pi\sigma^2} \exp\left(-\frac{(x - \mu)^2 + y^2}{2\sigma^2}\right) = \\ &= \frac{1}{2\pi\sigma^2} \int_{\sqrt{u}}^{\infty} dr' \, r' \int_{-\pi}^{\pi} d\theta' \exp\left(-\frac{r'^2 - 2\mu r' \cos\theta' + \mu^2}{2\sigma^2}\right) = \\ &= \frac{1}{\sigma^2} \int_{\sqrt{u}}^{\infty} dr' \, r' \exp\left(-\frac{r'^2 + \mu^2}{2\sigma^2}\right) I_0\left(\frac{\mu r'}{\sigma^2}\right) = Q\left(\frac{\mu}{\sigma}, \frac{\sqrt{u}}{\sigma}\right), \end{aligned} \quad (14)$$

where C is the region exterior to a circle of radius $\sqrt{u}$ centered at the origin of the x, y plane, and Q is Marcum's function [1].

The conditional probability density function of γ , for the same given value of amplitude scaling r , is

$$p(u) = -\frac{d}{du} \text{Prob}(\gamma > u) = \frac{1}{2\sigma^2} \exp\left(-\frac{u + \mu^2}{2\sigma^2}\right) I_0\left(\frac{\mu \sqrt{u}}{\sigma^2}\right), \quad (15)$$

for $u > 0$. Therefore, the conditional characteristic function of the squared-envelope detector output γ , for given r , is

$$f_c(\xi) = \int du \exp(i\xi u) p(u) = \frac{1}{2\sigma^2} \int_0^\infty du \exp\left(i\xi u - \frac{u + \mu^2}{2\sigma^2}\right) I_0\left(\frac{\mu \sqrt{u}}{\sigma^2}\right) \\ = \frac{1}{1 - i\xi 2\sigma^2} \exp\left(\frac{i\xi \mu^2}{1 - i\xi 2\sigma^2}\right), \quad (16)$$

where we used [16; 6.631 4]. The two important parameters here were defined in (8) and (12) according to

$$\mu = 2 r A \tilde{E}, \quad \sigma^2 = 2 N_0 A^2 \tilde{E}. \quad (17)$$

Notice that, in this case of matched filtering, the detailed behavior of complex envelope $\underline{s}(t)$ is not relevant; only the total transmitted signal energy $\tilde{E}$ is of consequence to the conditional characteristic function (16) of output γ in (8). The results in (16) and (17) are independent of the particular value of the medium phase shift θ encountered in (3) or (7); this is due to the processing method adopted at the receiver, namely envelope detection of the matched filter output.

The cumulants of random variable γ , for given fixed amplitude scaling r , are available by developing the logarithm of (16) in a power series in $i\xi$:

$$\ln f_c(\xi) = -\ln(1 - i\xi 2\sigma^2) + i\xi \mu^2 (1 - i\xi 2\sigma^2)^{-1} = \\ = \sum_{j=1}^{\infty} (i\xi)^j \left(\frac{(2\sigma^2)^j}{j} + \mu^2 (2\sigma^2)^{j-1} \right). \quad (18)$$

There follows immediately the j -th (conditional) cumulant of γ as

$$\chi(j) = (j-1)! (2\sigma^2)^j \left(1 + j \frac{\mu^2}{2\sigma^2}\right) \quad \text{for } j \geq 1. \quad (19)$$

In particular, the first two cumulants are

$$\chi(1) = 2\sigma^2 + \mu^2, \quad \chi(2) = 4\sigma^4 \left(1 + \frac{\mu^2}{\sigma^2}\right). \quad (20)$$

It is convenient to define a conditional deflection criterion at the squared-envelope detector output γ as the ratio of the difference of means, with and without signal, to the noise-alone standard deviation. That is,

$$d \equiv \frac{\chi_{S+N}(1) - \chi_N(1)}{\chi_N^{1/2}(2)} = \frac{\mu^2}{2\sigma^2} = r^2 \frac{\tilde{E}}{N_0}, \quad (21)$$

where we used (20) and (17). This conditional deflection depends on amplitude scaling value r , of course. However, it is independent of local reference values A and ϕ in (6), as expected, since performance measures should not depend on receiver processor absolute levels or phase shifts, at least for the case of a single pulse.

RECEIVER PROCESSING FOR MULTIPLE SIGNAL PULSES

In this section we will generalize to the case where K signal pulses are transmitted, all of which are nonoverlapping in time, frequency space; the results derived in the previous section will be used freely in the sequel. The transmitted energy in the k -th pulse is $\tilde{E}_k$, where $1 \leq k \leq K$. The k -th signal pulse undergoes (dimensionless) amplitude scaling r_k and phase shift θ_k in transmission through the medium to the receiver. For each k , random variables r_k and θ_k do not vary with time over the duration of the k -th individual signal pulse; this is a generalization of (3). The k -th local reference waveform employed at the receiver utilizes constant amplitude scaling A_k and phase shift ϕ_k , which can be chosen for best performance; compare (6).

It is now necessary to generalize the definitions in (7) to K pairs of analytic functions, namely

$$\begin{aligned}\alpha_k(t) &= r_k \underline{s}_k(t-t_d) \exp[i2\pi(f_o+f_d)t+i\theta_k] + \underline{n}(t) \exp(i2\pi f_o t) , \\ \beta_k(t) &= A_k \underline{s}_k(t-t_d) \exp[i2\pi(f_o+f_d)t+i\phi_k] ,\end{aligned}\tag{22}$$

which correspond to the k -th received waveform and local reference, respectively. A block diagram of the signal processing employed at the receiver is depicted in figure 1. The k -th signal complex envelope $\underline{s}_k(t)$ in (22) includes the time delay t_k and the frequency shift f_k associated with the k -th pulse, in accordance with the FSK pattern employed at the transmitter and known to the receiver. The K matched filter outputs are envelope detected, squared, and then summed to yield

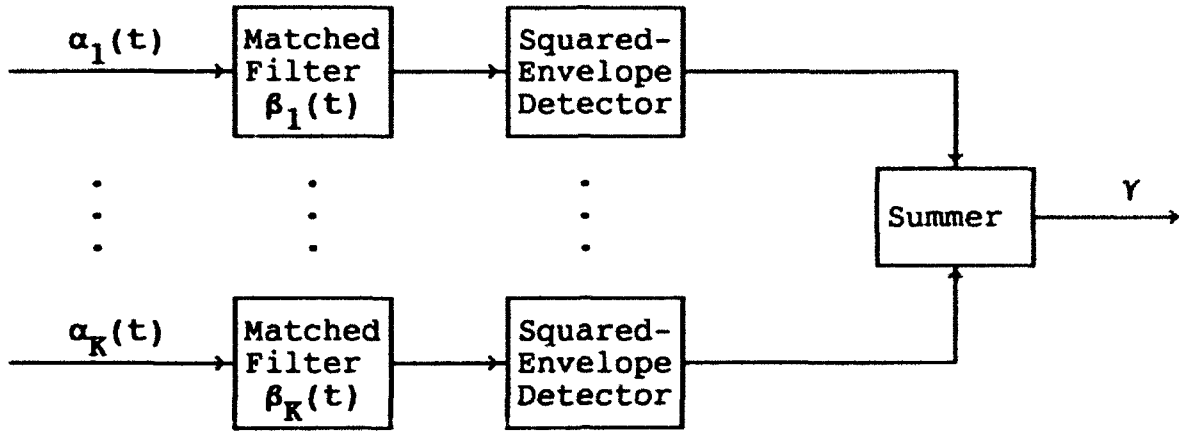


Figure 1. Block Diagram of Receiver Processing

output γ , which is compared with a threshold for purposes of declaring signal presence or absence. Observe that the matched filter $\beta_k(t)$ incorporates the local reference scaling A_k and phase shift ϕ_k and therefore accounts for (complex) weighting prior to the summation indicated in figure 1. The absolute level of weights $\{A_k\}$ does not matter and can be chosen freely; however, their values relative to each other will affect the detection capability of the processor in figure 1.

The energy in the k-th (real) reference waveform is, from (22) and (2),

$$\frac{1}{2} \int dt |\beta_k(t)|^2 = A_k^2 \tilde{E}_k, \quad (23)$$

where $\tilde{E}_k$ is the transmitted energy in the k-th signal pulse. In analogy to (17), we define the K parameters

$$\mu_k = 2 r_k A_k \tilde{E}_k, \quad \sigma_k^2 = 2 N_O A_k^2 \tilde{E}_k \quad \text{for } 1 \leq k \leq K. \quad (24)$$

The noise density level N_0 is a common constant applicable to all K pulses because the received noise is stationary over the total time extent, and it is white over the entire frequency band of the K received signals.

The K signals $\{\underline{s}_k(t)\}$ are nonoverlapping in time, frequency space, (t, f) . Also, the received noise is Gaussian. Therefore, the K output noise variables from the matched filters in figure 1 are statistically independent of each other; mathematically, we are using orthogonality relation

$$\int dt \underline{s}_k(t) \underline{s}_j^*(t) = 0 \quad \text{for } k \neq j. \quad (25)$$

This is due to nonoverlapping, in time or frequency or both, of all the component FSK pulses $\{\underline{s}_k(t)\}$, each with their individual delay, shift t_k, f_k . These observations, plus relations (8) and (16), allow us to determine the conditional characteristic function of output γ from figure 1 in the compact form

$$f_c(\xi) = \prod_{k=1}^K \left(1 - i\xi 2\sigma_k^2 \right)^{-1} \exp \left[i\xi \sum_{k=1}^K \frac{\mu_k^2}{1 - i\xi 2\sigma_k^2} \right]. \quad (26)$$

It should be noticed that this result is independent of the set of phase shifts $\{\theta_k\}$ encountered in transmission; see (22) and (24). The reason for this development is the method of receiver processing adopted in figure 1, namely envelope detection of each of the matched filter outputs prior to summation. However, (26) depends significantly on the amplitude scalings $\{r_k\}$ through parameters $\{\mu_k\}$ in (24); explicitly, the alternative form of (26) is

$$f_c(\xi) = \prod_{k=1}^K \left(1 - i\xi \frac{4 N_0 A_k^2 \tilde{E}_k}{r_k^2 A_k^2 \tilde{E}_k} \right)^{-1} \exp \left[i\xi \sum_{k=1}^K \frac{4 r_k^2 A_k^2 \tilde{E}_k}{1 - i\xi \frac{4 N_0 A_k^2 \tilde{E}_k}{r_k^2 A_k^2 \tilde{E}_k}} \right] . \quad (27)$$

Also, characteristic function (27) is independent of the receiver set of phase shifts $\{\phi_k\}$ used in local references (22), again due to the method of processing employed in figure 1. However, (27) depends upon the receiver weights $\{A_k\}$ in a complicated fashion. In order to determine a reasonable method of selecting this set of weights, we consider the following development.

The (conditional) cumulants of system output random variable γ , for a given fixed set of amplitude scalings $\{r_k\}$, are available by developing the logarithm of (26) or (27) in a power series in $i\xi$. Reference to (18) and (19) immediately yields the j -th (conditional) cumulant of γ as

$$\chi(j) = (j-1)! \sum_{k=1}^K \left(2\sigma_k^2 \right)^j \left(1 + j \frac{\mu_k^2}{2\sigma_k^2} \right) \quad \text{for } j \geq 1 . \quad (28)$$

In particular, the first two cumulants of γ are

$$\chi(1) = \sum_{k=1}^K \left(2\sigma_k^2 + \mu_k^2 \right) , \quad \chi(2) = 4 \sum_{k=1}^K \left(\sigma_k^4 + \sigma_k^2 \mu_k^2 \right) . \quad (29)$$

We can now define a (conditional) deflection criterion d_K for the output of the processor in figure 1, exactly as was done in (21) earlier for the single pulse case. Substitution of (29) and (24) then yields

$$d_K = \frac{\sum_{k=1}^K \mu_k^2}{2 \left(\sum_{k=1}^K \sigma_k^4 \right)^{1/2}} = \frac{\sum_{k=1}^K r_k^2 A_k^2 \tilde{E}_k^2}{N_0 \left(\sum_{k=1}^K A_k^4 \tilde{E}_k^2 \right)^{1/2}} . \quad (30)$$

The absolute level of receiver weights $\{A_k\}$ cancels out of (30); however, the relative sizes of these weights does affect the value of deflection d_K attainable.

If the average power scaling of the k -th pulse, $\overline{r_k^2}$, is the same for all k , that is,

$$\overline{r_k^2} = \overline{r_1^2} \quad \text{for } 2 \leq k \leq K , \quad (31)$$

then the average deflection of output γ in figure 1 is, from (30),

$$\overline{d_K} = \frac{\overline{r_1^2}}{N_0} \frac{\sum_{k=1}^K A_k^2 \tilde{E}_k^2}{\left(\sum_{k=1}^K A_k^4 \tilde{E}_k^2 \right)^{1/2}} \leq \frac{\overline{r_1^2}}{N_0} \left(\sum_{k=1}^K \tilde{E}_k^2 \right)^{1/2} , \quad (32)$$

with equality only if all the weights A_k are equal. This result is not surprising, since there is already a built-in dependence of matched filter $\beta_k(t)$ on the transmitted signal energy $\tilde{E}_k$; namely, (23) gives this energy as $A_k^2 \tilde{E}_k$. Thus, we have the conclusion that the "best" set of weights $\{A_k\}$, for the case of equal power scalings (31) in the medium, is uniform when local reference $\beta_k(t)$ is chosen according to (22).

More generally, the average deflection follows from (30) as

$$\bar{d}_K = \frac{\sum_{k=1}^K \bar{r}_k^2 A_k^2 \tilde{E}_k^2}{N_0 \left(\sum_{k=1}^K A_k^4 \tilde{E}_k^2 \right)^{1/2}} \leq \frac{1}{N_0} \left(\sum_{k=1}^K \bar{r}_k^2 \tilde{E}_k^2 \right)^{1/2}, \quad (33)$$

with equality if and only if the receiver weights satisfy

$$A_k = A \bar{r}_k^2{}^{1/2} \quad \text{for } 1 \leq k \leq K. \quad (34)$$

Scale factor A is arbitrary, reflecting the fact that the absolute level of set $\{A_k\}$ is irrelevant. It should be noted that the "best" receiver weights in (34) are independent of the transmitted signal energies $\{\tilde{E}_k\}$. This is again due to the fact that matched filter $\beta_k(t)$ in figure 1 and (22) already has an energy proportional to $\tilde{E}_k$; see (23).

Result (34) for the weights $\{A_k\}$ requires knowledge of the average power scaling $\bar{r}_k^2$ applied to the k -th pulse in transmission through the medium. But these average power scalings may be unknown or they may be independent of k , the particular pulse number. In this case, a reasonable choice of receiver weights is simply to take $A_k = A$ for $1 \leq k \leq K$. Then the conditional characteristic function in (27) of processor output γ reduces to

$$f_c(\xi) = \prod_{k=1}^K \left(1 - i\xi 4 N_0 A^2 \tilde{E}_k \right)^{-1} \exp \left[i\xi 4 A^2 \sum_{k=1}^K \frac{\bar{r}_k^2 \tilde{E}_k^2}{1 - i\xi 4 N_0 A^2 \tilde{E}_k} \right] \quad (35)$$

In the (usual) case where all the transmitted signal energies in the K pulses are equal, that is, $\tilde{E}_k = \tilde{E}$ for $1 \leq k \leq K$, then

the conditional characteristic function in (35) reduces to

$$f_c(\xi) = \left(1 - i\xi 4 N_0 A^2 \tilde{E}\right)^{-K} \exp\left[\frac{i\xi 4 A^2 \tilde{E}^2}{1 - i\xi 4 N_0 A^2 \tilde{E}} \sum_{k=1}^K r_k^2\right] . \quad (36)$$

It is now very important to notice that the only way that the amplitude scaling factors $\{r_k\}$ enter this characteristic function is through the single quantity (sufficient statistic)

$$S = \sum_{k=1}^K r_k^2 , \quad (37)$$

which is the sum of all the power-fading variates r_k^2 on the K transmitted pulses. This simplified result in (36) holds when the following two reasonable conditions are satisfied:

$$\tilde{E}_k = \tilde{E} , \quad A_k = A \quad \text{for } 1 \leq k \leq K . \quad (38)$$

That is, the transmitted signal pulse energies are all equal and the receiver scalings $\{A_k\}$ in (22) are all taken equal. This particular case will receive most of our attention.

Since the receiver absolute scaling A in (38) is under our control and does not affect detectability performance, we take, for notational convenience and without loss of generality,

$$A^2 = \frac{1}{2 N_0 \tilde{E}} . \quad (39)$$

Then the conditional characteristic function in (36) of processor output y in figure 1 simplifies to

$$f_c(\xi) = (1 - i2\xi)^{-K} \exp\left[\frac{i\xi}{1 - i2\xi} \frac{2\tilde{E}}{N_0} S\right] , \quad (40)$$

where we used (37).

Another important observation must be made at this point. Suppose the characteristic function of random variable S , defined by (37), is known; that is,

$$f_S(\xi) = \overline{\exp(i\xi S)} \quad (41)$$

is known. Then the unconditional characteristic function $f_Y(\xi)$ of processor output γ in figure 1 is obtained by averaging (40) over S and using (41); there follows

$$f_Y(\xi) = (1 - i2\xi)^{-K} f_S\left(\frac{\xi}{1 - i2\xi} \frac{2\tilde{E}}{N_0}\right). \quad (42)$$

This compact form for the unconditional characteristic function of processor output γ depends critically upon being able to obtain the characteristic function $f_S(\xi)$ in (41) of the power-fading summation S defined in (37). Therefore, effort can be concentrated on attempting to determine (41), either exactly or by use of several low-order moments of S ; see [17] for example. Of course, in the process, the amount of partial correlation between individual pairs of pulse power-fading variates $\{r_k^2\}$ in sum S will come into consideration. In any event, the characteristic function of sum S in (37) is the major item of interest; given this quantity, the processor output γ in figure 1 is completely characterized in terms of its characteristic function (42), at least when conditions (38) are satisfied.

The more general conditional characteristic function in (27), with arbitrary $\{\tilde{E}_k\}$ and $\{A_k\}$, will be treated later, after we have introduced a detailed model of the amplitude fadings $\{r_k\}$.

CHARACTERIZATION OF FADING

We are interested in obtaining characteristic function $f_S(\xi)$ of sum S of power-fading variates $\{r_k^2\}$ in (37); its use in (42) will then yield the desired characteristic function $f_Y(\xi)$ of receiver output y in figure 1. However, we must first concentrate on characterizing the fading which gives rise to sum S .

COVARIANCE COEFFICIENTS OF POWER-FADING VARIATES $\{q_k\}$

Define the k -th power-fading variate q_k as the square of the amplitude-fading variate r_k in (22):

$$q_k = r_k^2 \quad \text{for } 1 \leq k \leq K. \quad (43)$$

The model of fading that we consider here is that power scaling q_k is a sample of a continuous fading process $q(t, f)$, namely

$$q_k = q(t_k, f_k) \quad \text{for } 1 \leq k \leq K, \quad (44)$$

where two-dimensional function of time t and frequency f ,

$$q(t, f) = \sum_{m=1}^M \left[c_m(t, f) + g_m(t, f) \right]^2. \quad (45)$$

Sampling time t_k and frequency f_k correspond to the time-delay location and frequency-shift location, respectively, of the k -th FSK complex envelope signal $\underline{s}_k(t)$ employed in (22).

The number of fading components in model (45) is M . Each component contains a deterministic part $c_m(t, f)$ and a stationary zero-mean Gaussian part $g_m(t, f)$. Without loss of generality, the

nonrandom part satisfies

$$c_m(t, f) \geq 0 \quad \text{for all } m, t, f. \quad (46)$$

Random parts $\{g_m(t, f)\}$ of the joint Gaussian processes are stationary in both t and f , with covariances

$$\overline{g_m(t, f) g_n(t-\tau, f-\nu)} = R_{mn}(\tau, \nu) \quad \text{for } 1 \leq m, n \leq M. \quad (47)$$

(It is possible to generalize to nonstationary Gaussian processes $\{g_m(t, f)\}$; however, each covariance R_{mn} would then be a function of the four variables τ, ν, t, f instead of just differences τ, ν .) From (47), we have property

$$R_{nm}(-\tau, -\nu) = R_{mn}(\tau, \nu). \quad (48)$$

Fading model (44) - (45) does not constitute a multipath medium but does mimic its net effect. Every transmitted pulse $\underline{s}_k(t)$ undergoes just one common time delay t_d and frequency shift f_d , as indicated in (22), which are known and utilized by the receiver. Rather, each signal pulse simply undergoes a different phase shift θ_k and amplitude scaling r_k in (22), the latter of which is characterized through power scaling q_k in (43) - (45).

The latter random variable, q_k , is more general than non-central Chi-squared because the random components $\{g_m(t, f)\}$ in (45) can have unequal variances $\{R_{mm}(0, 0)\}$ and can be correlated with each other; that is, we allow $R_{mn}(\tau, \nu) \neq 0$ for $m \neq n$.

The mean of power-fading variate q_k is given by

$$\overline{q_k} = \sum_{m=1}^M \left[c_m^2(t_k, f_k) + R_{mm}(0, 0) \right] \quad \text{for } 1 \leq k \leq K, \quad (49)$$

where we used (44), (45), and (47). This quantity is independent of pulse number k if all M deterministic components are zero, or if they do not depend on t or f ; that is, if $c_m(t, f)$ is independent of t and f for $1 \leq m \leq M$. Since $\tilde{E}_k$ is the transmitted signal energy in the k -th pulse, then

$$\tilde{E}_k c_m^2(t_k, f_k) \equiv D_{km} = \text{deterministic } \underline{\text{received}} \text{ signal energy} \\ \text{in } m\text{-th component of } k\text{-th pulse,} \quad (50)$$

$$\tilde{E}_k \sum_{m=1}^M c_m^2(t_k, f_k) = \sum_{m=1}^M D_{km} \equiv D_k = \text{deterministic received signal} \\ \text{energy in } k\text{-th pulse,} \quad (51)$$

$$\tilde{E}_k R_{mm}(0, 0) \equiv E_{km} = \text{average random } \underline{\text{received}} \text{ signal energy} \\ \text{in } m\text{-th component of } k\text{-th pulse,} \quad (52)$$

$$\tilde{E}_k \sum_{m=1}^M R_{mm}(0, 0) = \sum_{m=1}^M E_{km} \equiv E_k = \text{average random received signal} \\ \text{energy in } k\text{-th pulse.} \quad (53)$$

The alternating (zero-mean) component of power scaling q_k is then given by (44), (45), and (49) as

$$\tilde{q}_k = q_k - \overline{q_k} = \\ = \sum_{m=1}^M \left[2 c_m(t_k, f_k) g_m(t_k, f_k) + g_m^2(t_k, f_k) - R_{mm}(0, 0) \right]. \quad (54)$$

The covariance between a pair of power-fading variates is

$$\tilde{R}_{kj} = \overline{\tilde{q}_k \tilde{q}_j}, \quad (55)$$

while the covariance coefficient between q_k and q_j is

$$\rho_{kj} = \frac{\tilde{R}_{kj}}{(\tilde{R}_{kk} \tilde{R}_{jj})^{1/2}} . \quad (56)$$

Thus, the fundamental calculation required for determination of covariance coefficient ρ_{kj} is $\tilde{R}_{kj}$ as defined by (55) in conjunction with (54).

When we substitute (54) in (55) and use the fact that $\{g_m(t, f)\}$ are zero-mean joint Gaussian processes, the first-order and third-order moments involving $\{g_m(t, f)\}$ are zero, while the fourth-order moment can be broken down into a sum of products of second-order moments, leading to covariance

$$\begin{aligned} \tilde{R}_{kj} = & 2 \sum_{m,n=1}^M R_{mn}^2(t_k - t_j, f_k - f_j) + \\ & + 4 \sum_{m,n=1}^M R_{mn}(t_k - t_j, f_k - f_j) c_m(t_k, f_k) c_n(t_j, f_j) . \end{aligned} \quad (57)$$

In particular, the variance of power-fading variate q_k is

$$\begin{aligned} \tilde{R}_{kk} = \overline{q_k^2} = & 2 \sum_{m,n=1}^M R_{mn}^2(0, 0) + \\ & + 4 \sum_{m,n=1}^M R_{mn}(0, 0) c_m(t_k, f_k) c_n(t_k, f_k) . \end{aligned} \quad (58)$$

The covariance coefficient ρ_{kj} between q_k and q_j is obtained upon substitution of general results (57) and (58) into (56).

A large number of special cases can be obtained from the general formulation of fading in (45) and (47). Below, we list seven special cases that will occupy most of our attention in the remainder of this report, and which will consistently be referred to according to their number in the sequel.

SPECIAL CASE 1: Zero means $\{c_m(t, f)\}$

$$c_m(t, f) = 0 \quad \text{for } 1 \leq m \leq M. \quad (59)$$

$$\tilde{R}_{kj} = 2 \sum_{m,n=1}^M R_{mn}^2(t_k - t_j, f_k - f_j),$$

$$\tilde{R}_{kk} = 2 \sum_{m,n=1}^M R_{mn}^2(0, 0). \quad (60)$$

SPECIAL CASE 2: Uncorrelated components $\{g_m(t, f)\}$

$$R_{mn}(\tau, \nu) = R_{mm}(\tau, \nu) \delta_{mn} \quad \text{for } 1 \leq m, n \leq M. \quad (61)$$

$$\begin{aligned} \tilde{R}_{kj} &= 2 \sum_{m=1}^M R_{mm}^2(t_k - t_j, f_k - f_j) + \\ &+ 4 \sum_{m=1}^M R_{mm}(t_k - t_j, f_k - f_j) c_m(t_k, f_k) c_m(t_j, f_j), \end{aligned}$$

$$\tilde{R}_{kk} = 2 \sum_{m=1}^M R_{mm}^2(0, 0) + 4 \sum_{m=1}^M R_{mm}(0, 0) c_m^2(t_k, f_k). \quad (62)$$

It is very important to note that correlated fading still exists between the received signal pulses; that is, covariance $\tilde{R}_{kj}$ in (62) is not zero, despite uncorrelated property (61) in case 2.

SPECIAL CASE 3: Zero means and uncorrelated components

$$\tilde{R}_{kj} = 2 \sum_{m=1}^M R_{mm}^2(t_k - t_j, f_k - f_j) ,$$

$$\tilde{R}_{kk} = 2 \sum_{m=1}^M R_{mm}^2(0, 0) ,$$

$$\rho_{kj} = \frac{\sum_{m=1}^M R_{mm}^2(t_k - t_j, f_k - f_j)}{\sum_{m=1}^M R_{mm}^2(0, 0)} . \quad (63)$$

SPECIAL CASE 4: Uncorrelated components with identical covariances $\{R_{mn}(\tau, \nu)\}$

$$R_{mn}(\tau, \nu) = R_{11}(\tau, \nu) \delta_{mn} \quad \text{for all } m, n . \quad (64)$$

$$\begin{aligned} \tilde{R}_{kj} &= 2M R_{11}^2(t_k - t_j, f_k - f_j) + \\ &+ 4 R_{11}(t_k - t_j, f_k - f_j) \sum_{m=1}^M c_m(t_k, f_k) c_m(t_j, f_j) , \end{aligned}$$

$$\tilde{R}_{kk} = 2M R_{11}^2(0, 0) + 4 R_{11}(0, 0) \sum_{m=1}^M c_m^2(t_k, f_k) . \quad (65)$$

SPECIAL CASE 5: Zero means and uncorrelated components with identical covariances

$$\tilde{R}_{kj} = 2M R_{11}^2(t_k - t_j, f_k - f_j) ,$$

$$\tilde{R}_{kk} = 2M R_{11}^2(0, 0) ,$$

$$\rho_{kj} = \left(\frac{R_{11}(t_k - t_j, f_k - f_j)}{R_{11}(0, 0)} \right)^2 . \quad (66)$$

In this special case, the covariance coefficient between power-fading variates $q_k = q(t_k, f_k)$ and $q_j = q(t_j, f_j)$ is the square of the covariance coefficient between the amplitude-fading variates $g_m(t_k, f_k)$ and $g_m(t_j, f_j)$; see (47). Due to the identical covariances, (64), this latter covariance coefficient is the same for every m in the range $1 \leq m \leq M$.

SPECIAL CASE 6: Uncorrelated components with proportional covariances

This is a generalization of special case 4; it allows the random components $\{g_m(t, f)\}$ in (45) to have different strengths, as might be encountered in a fading medium. (Note: In the following, the constant $r^{(m)}$ is the average relative power measure of the m -th random fading component; it must not be confused with the random variable r_k which is the amplitude-scaling on the k -th signal pulse in (22).)

$$R_{mn}(\tau, \nu) = R_{11}(\tau, \nu) r^{(m)} \delta_{mn}, \quad r^{(1)} = 1. \quad (67)$$

$$\begin{aligned} \tilde{R}_{kj} &= 2 R_{11}^2(t_k - t_j, f_k - f_j) \sum_{m=1}^M r^{(m)2} + \\ &+ 4 R_{11}(t_k - t_j, f_k - f_j) \sum_{m=1}^M r^{(m)} c_m(t_k, f_k) c_m(t_j, f_j), \\ \tilde{R}_{kk} &= 2 R_{11}^2(0, 0) \sum_{m=1}^M r^{(m)2} + 4 R_{11}(0, 0) \sum_{m=1}^M r^{(m)} c_m^2(t_k, f_k). \end{aligned} \quad (68)$$

SPECIAL CASE 7: Zero means and uncorrelated components with proportional covariances

Setting the means $\{c_m(t, f)\}$ in (68) to zero,

$$\begin{aligned} \tilde{R}_{kj} &= 2 R_{11}^2(t_k - t_j, f_k - f_j) \sum_{m=1}^M r^{(m)2}, \\ \tilde{R}_{kk} &= 2 R_{11}^2(0, 0) \sum_{m=1}^M r^{(m)2}, \\ \rho_{kj} &= \left(\frac{R_{11}(t_k - t_j, f_k - f_j)}{R_{11}(0, 0)} \right)^2. \end{aligned} \quad (69)$$

Notice, in this special case, that $\{\rho_{kj}\}$ are independent of the values of $\{r^{(m)}\}$, the relative power measures of the M random fading components.

CHARACTERISTIC FUNCTION OF POWER-FADING VARIATE $q(t, f)$

The instantaneous power-fading process is, from (45),

$$q(t, f) = \sum_{m=1}^M \left[c_m(t, f) + g_m(t, f) \right]^2 = X^T X, \quad (70)$$

where $M \times 1$ Gaussian random column vector

$$X = \left[c_1(t, f) + g_1(t, f) \quad \dots \quad c_M(t, f) + g_M(t, f) \right]^T. \quad (71)$$

We now appeal directly to the very general result derived in appendix B, for the characteristic function of a quadratic form and linear form, and identify the quantities there according to

$$N = M, \quad B = I, \quad A = 0, \quad E = \left[c_1(t, f) \quad \dots \quad c_M(t, f) \right]^T, \\ C = \text{Cov}(X) = \left[\overline{g_m(t, f) g_n(t, f)} \right]_1^M = \left[R_{mn}(0, 0) \right]_1^M. \quad (72)$$

It is important to note that $M \times M$ symmetric covariance matrix C is not a function of t, f , under the stationarity assumption (47) for all M random processes $\{g_m(t, f)\}$.

According to (B-10), we must solve the standard characteristic-value matrix equation

$$C Q = Q \Lambda, \quad (73)$$

for $M \times M$ eigenvalue matrix Λ and normalized modal matrix Q , where

$$\Lambda = \text{diag}[\lambda_1 \quad \dots \quad \lambda_M], \quad Q = [V_1 \quad \dots \quad V_M], \quad V_m = \left[v_{m1} \quad \dots \quad v_{mM} \right]^T, \quad (74)$$

and $M \times 1$ column vector V_m is the m -th eigenvector with components

$\{v_{mn}\}$, $1 \leq n \leq M$. Then (B-18) with (74) and (72) yields deterministic parameters

$$\varepsilon_m = V_m^T E = \sum_{n=1}^M v_{mn} c_n(t, f) \equiv \varepsilon_m(t, f), \quad \alpha_m = 0, \quad (75)$$

for $1 \leq m \leq M$. We can now use (B-20) to obtain the characteristic function of power-fading variate $q(t, f)$ as

$$f_q(\xi) = \left[\prod_{m=1}^M (1 - i2\xi\lambda_m) \right]^{-\frac{1}{2}} \exp \left[i\xi \sum_{m=1}^M \frac{\varepsilon_m^2(t, f)}{1 - i2\xi\lambda_m} \right]. \quad (76)$$

The eigenvectors $\{V_m\}$ and nonzero means $\{c_m(t, f)\}$ enter this result through the terms $\{\varepsilon_m(t, f)\}$ defined in (75). This general result will be simplified, below, to the seven special cases that were presented earlier in (59) - (69).

By expanding the logarithm of general result (76) in a power series in $i\xi$, the cumulants of $q(t, f)$ are found to be

$$\chi_q(p) = 2^{p-1} (p-1)! \sum_{m=1}^M \left(\lambda_m + p \varepsilon_m^2(t, f) \right) \lambda_m^{p-1} \quad \text{for } p \geq 1. \quad (77)$$

For $p = 1$ and $p = 2$, these may be verified to equal the results in (49) and (58); the pertinent matrix manipulations are indicated in appendix B for the more general case where linear form $A \neq 0$. The quantities $\{\lambda_m\}$ are the eigenvalues of covariance matrix C in (72), while parameters $\{\varepsilon_m(t, f)\}$ are given by (75) in terms of eigenvectors $\{V_m\}$ of matrix C and mean vector $E = E(t, f)$ in (72).

SPECIAL CASE 1: Zero means; see (59)

Since $c_m(t, f) = 0$, then $\epsilon_m(t, f) = 0$ from (75), and the characteristic function in (76) reduces to

$$f_Q(\xi) = \left[\prod_{m=1}^M (1 - i2\xi\lambda_m) \right]^{-1/2}. \quad (78)$$

SPECIAL CASE 2: Uncorrelated components; see (61)

Here, from (72), we have

$$C = \text{diag}[R_{11}(0,0) \ . \ . \ . \ R_{MM}(0,0)] \ . \quad (79)$$

Then, the eigenvalue matrix Λ in (73) must be identical to C , in which case (73) becomes $C Q = Q C$, where C is diagonal. This forces Q to be diagonal also; finally, normalization of Q yields $Q = I$. There follows, from (74) - (76),

$$v_{mn} = \delta_{mn} \ , \quad \epsilon_m = c_m(t, f) \ , \quad (80)$$

and characteristic function

$$f_Q(\xi) = \left[\prod_{m=1}^M (1 - i2\xi R_{mm}(0,0)) \right]^{-1/2} \exp \left[i\xi \sum_{m=1}^M \frac{c_m^2(t, f)}{1 - i2\xi R_{mm}(0,0)} \right] \ . \quad (81)$$

SPECIAL CASE 3: Zero means and uncorrelated components

Here, we simply set the constants in (81) equal to zero:

$$f_Q(\xi) = \left[\prod_{m=1}^M (1 - i2\xi R_{mm}(0,0)) \right]^{-1/2} \ . \quad (82)$$

SPECIAL CASE 4: Uncorrelated components with identical covariances; see (64)

Now we use (64) on result (81) to get

$$f_q(\xi) = \left(1 - i2\xi R_{11}(0,0)\right)^{-M/2} \exp\left[\frac{i\xi}{1 - i2\xi R_{11}(0,0)} \sum_{m=1}^M c_m^2(t,f)\right]. \quad (83)$$

It is important to observe in this case that only the sum of the squares of the means, $\{c_m(t,f)\}$, matters in so far as the characteristic function of power scaling $q(t,f)$ is concerned.

SPECIAL CASE 5: Zero means and uncorrelated components with identical covariances

Upon setting the means in (83) equal to zero, there follows

$$f_q(\xi) = \left(1 - i2\xi R_{11}(0,0)\right)^{-M/2}. \quad (84)$$

This special case is equivalent to [12; (A-21)] when we take the parameter m there equal to $M/2$.

SPECIAL CASE 6: Uncorrelated components with proportional covariances; see (67)

From (67), there follows

$$R_{mn}(0,0) = R_{11}(0,0) r^{(m)} \delta_{mn}, \quad r^{(1)} = 1. \quad (85)$$

Use of this relation on (81) yields characteristic function

$$f_q(\xi) = \left[\prod_{m=1}^M \left(1 - i2\xi R_{11}(0,0) r^{(m)} \right) \right]^{-\frac{1}{2}} \exp \left[i\xi \sum_{m=1}^M \frac{c_m^2(t,f)}{1 - i2\xi R_{11}(0,0) r^{(m)}} \right]. \quad (86)$$

The cumulants of power scaling $q(t,f)$ can be obtained by expanding the logarithm of characteristic function (86) in a power series in $i\xi$; there follows

$$\frac{\chi_q(p)}{(p-1)! 2^{p-1}} = R_{11}^p(0,0) \sum_{m=1}^M r^{(m)p} + p R_{11}^{p-1}(0,0) \sum_{m=1}^M r^{(m)p-1} c_m^2(t,f) \quad \text{for } p \geq 1. \quad (87)$$

SPECIAL CASE 7: Zero means and uncorrelated components with proportional covariances

Setting the means in (86) to zero, we have

$$f_q(\xi) = \left[\prod_{m=1}^M \left(1 - i2\xi R_{11}(0,0) r^{(m)} \right) \right]^{-\frac{1}{2}}. \quad (88)$$

Actual numerical examples displaying typical probability density functions of power-scaling random variable q , for a variety of different parameter values, will be presented in a later section.

CHARACTERISTIC FUNCTION OF POWER-SUM VARIATE S

The power-sum variate S is given in (37) and (43) as the sum of the power scalings $q_k = q(t_k, f_k)$ over the total of K pulses transmitted and received:

$$S = \sum_{k=1}^K q(t_k, f_k) = \sum_{k=1}^K \sum_{m=1}^M [c_m(t_k, f_k) + g_m(t_k, f_k)]^2, \quad (89)$$

where we used (44) and (45). This double sum can be recognized as a quadratic form in KM correlated nonzero-mean Gaussian random variables. Therefore, the general approach in appendix B can be used directly to find the characteristic function of S for any statistical dependencies between the M zero-mean Gaussian processes $\{g_m(t, f)\}$ and any layout of the K points $\{t_k, f_k\}$ in the time, frequency plane. In order to realize form (89), we identify variate $q = S$, constant $N = KM$, and matrices $B = I$, $A = 0$ in appendix B. Mean column vector E in (B-1) is $KM \times 1$, while symmetric covariance matrix C is $KM \times KM$. Then (B-10) becomes the standard characteristic-value matrix equation [18; section 1.13]

$$C Q = Q \Lambda, \quad C = [\overline{g_m(t_k, f_k) g_n(t_j, f_j)}] = [R_{mn}(t_k - t_j, f_k - f_j)], \quad (90)$$

where $KM \times KM$ modal matrix $Q = [V_1 \dots V_{KM}]$ from (B-11). Also, there follows from (B-18),

$$\underline{E} = Q^T E, \quad \epsilon_n = V_n^T E \quad \text{for } 1 \leq n \leq KM; \quad \alpha_n = 0. \quad (91)$$

The characteristic function of sum S in (89) then follows from (B-20) as

$$f_S(\xi) = \left[\prod_{n=1}^{KM} (1 - i2\xi\lambda_n) \right]^{-\frac{1}{2}} \exp \left[i\xi \sum_{n=1}^{KM} \frac{\epsilon_n^2}{1 - i2\xi\lambda_n} \right]. \quad (92)$$

This general result will now be specialized to the seven cases of particular interest here.

SPECIAL CASE 1: Zero means; see (59)

When $c_m(t, f) = 0$ for $1 \leq m \leq M$, then mean vector E is zero, meaning that its components $\{\epsilon_n\}$ in (91) are zero. The result in (92) then reduces to

$$f_S(\xi) = \left[\prod_{n=1}^{KM} (1 - i2\xi\lambda_n) \right]^{-\frac{1}{2}}. \quad (93)$$

Only the KM eigenvalues of $KM \times KM$ covariance matrix C of the KM random variables $\{g_m(t_k, f_k)\}$ need to be evaluated in this case.

SPECIAL CASE 2: Uncorrelated components; see (61)

From this point on, we shall be interested in the more restricted case where component Gaussian process $g_m(t, f)$ is uncorrelated with (independent of) process $g_n(t', f')$ for $m \neq n$, regardless of the values of t, f and t', f' ; this case was also considered earlier in (61). This will probably encompass most situations of practical interest; furthermore, it still allows for correlated fading between the K received signal pulses. That is, covariance $\tilde{R}_{jk}$ in (62), between power scalings q_k and q_j ,

need not be zero, despite uncorrelated property (61) between components $\{g_m(t, f)\}$ in this special case 2.

In this case, a simpler and more direct approach is possible and is adopted. We begin by expressing (89) as

$$S = \sum_{m=1}^M S^{(m)} , \quad (94)$$

where we have defined the M random variables

$$S^{(m)} = \sum_{k=1}^K \left[c_m(t_k, f_k) + g_m(t_k, f_k) \right]^2 \quad \text{for } 1 \leq m \leq M . \quad (95)$$

This superscript m notation is adopted in order to readily distinguish the fading component numbers, $1 \leq m \leq M$, from the time, frequency signal pulse numbers, $1 \leq k \leq K$; see (45) versus figure 1.

Now, it is important to observe, for this special case 2, that these latter M random variables $\{S^{(m)}\}$ are statistically independent of each other, allowing us to develop the characteristic function of sum S in (94) in the finite product form

$$f_S(\xi) = \prod_{m=1}^M f^{(m)}(\xi) , \quad (96)$$

where the m -th characteristic function is given by average

$$f^{(m)}(\xi) = \overline{\exp(i\xi S^{(m)})} , \quad (97)$$

in terms of quadratic sum $S^{(m)}$ in (95).

In order to ascertain the characteristic function of sum $S^{(m)}$, we define $K \times 1$ Gaussian column vector $y^{(m)}$ according to

$$Y^{(m)} = \left[c_m(t_1, f_1) + g_m(t_1, f_1) \cdot \cdot \cdot c_m(t_K, f_K) + g_m(t_K, f_K) \right]^T. \quad (98)$$

Then, $S^{(m)}$ in (95) can be written in the quadratic form

$$S^{(m)} = Y^{(m)T} Y^{(m)}. \quad (99)$$

We now appeal to the general results in appendix B and identify the quantities there according to $q = S^{(m)}$, $N = K$, and

$$B = I, \quad A = 0, \quad E = E^{(m)} = \overline{Y^{(m)}} = \left[c_m(t_1, f_1) \cdot \cdot \cdot c_m(t_K, f_K) \right]^T, \\ C = C^{(m)} = \text{Cov}(Y^{(m)}) = \left[\overline{g_m(t_k, f_k) g_m(t_j, f_j)} \right]_1^K = \left[R_{mm}(t_k - t_j, f_k - f_j) \right]_1^K \quad (100)$$

The K diagonal elements of matrix $C^{(m)}$ are all equal to $R_{mm}(0, 0)$.

According to (B-10), we must now solve, for each value of m in the range $1 \leq m \leq M$, the $K \times K$ characteristic-value equation

$$C^{(m)} Q^{(m)} = Q^{(m)} \Lambda^{(m)}, \quad (101)$$

for $K \times K$ eigenvalue matrix $\Lambda^{(m)}$ and corresponding $K \times K$ normalized modal matrix $Q^{(m)}$, where

$$\Lambda^{(m)} = \text{diag}[\lambda_1^{(m)} \cdot \cdot \cdot \lambda_K^{(m)}], \quad Q^{(m)} = [v_1^{(m)} \cdot \cdot \cdot v_K^{(m)}],$$

$$v_k^{(m)} = [v_{k1}^{(m)} \cdot \cdot \cdot v_{kK}^{(m)}]^T, \quad (102)$$

and $K \times 1$ vector $v_k^{(m)}$ is the k -th eigenvector with components $\{v_{kj}^{(m)}\}$, $1 \leq j \leq K$. Then, (B-18) with (102) and (100) yields

$$\epsilon_k^{(m)} = v_k^{(m)T} E^{(m)} = \sum_{j=1}^K v_{kj}^{(m)} c_m(t_j, f_j), \quad \alpha_k^{(m)} = 0. \quad (103)$$

We can now use (B-20) to obtain the characteristic function of variate $S^{(m)}$ in (99) as

$$f^{(m)}(\xi) = \left[\prod_{k=1}^K (1 - i2\xi\lambda_k^{(m)}) \right]^{-\frac{1}{2}} \exp \left[i\xi \sum_{k=1}^K \frac{\epsilon_k^{(m)^2}}{1 - i2\xi\lambda_k^{(m)}} \right]. \quad (104)$$

The desired characteristic function of power-sum variate S in (89) is then given by (96), applied to (104):

$$f_S(\xi) = \left[\prod_{m=1}^M \prod_{k=1}^K (1 - i2\xi\lambda_k^{(m)}) \right]^{-\frac{1}{2}} \exp \left[i\xi \sum_{m=1}^M \sum_{k=1}^K \frac{\epsilon_k^{(m)^2}}{1 - i2\xi\lambda_k^{(m)}} \right]. \quad (105)$$

This characteristic function of sum S is a very general result, applicable to the case of uncorrelated components $\{g_m(t, f)\}$; however, it does require the solution of M matrix equations of the form of (101), each matrix being of size $K \times K$. Nevertheless, this approach is significantly simpler than solving the one large $KM \times KM$ matrix equation (90).

SPECIAL CASE 3: Zero means and uncorrelated components

When the means $\{c_m(t, f)\}$ are zero, (103) yields $\epsilon_k^{(m)} = 0$ and the characteristic function in (105) reduces to just the product

$$f_S(\xi) = \left[\prod_{m=1}^M \prod_{k=1}^K (1 - i2\xi\lambda_k^{(m)}) \right]^{-\frac{1}{2}}. \quad (106)$$

Eigenvalues $\lambda_1^{(m)}, \dots, \lambda_K^{(m)}$ are found from $K \times K$ covariance matrix $C^{(m)}$ in (100); this solution must be repeated for $1 \leq m \leq M$.

SPECIAL CASE 4: Uncorrelated components with identical covariances; see (64)

Here, covariance matrix $C^{(m)}$ in (100) is independent of m ; in particular, we now have the single $K \times K$ covariance matrix

$$C = \left[R_{11}(t_k - t_j, f_k - f_j) \right]_1^K. \quad (107)$$

This leads to solution matrices $Q^{(m)}$ and $\Lambda^{(m)}$ in (101) which are also independent of m ; that is, (101) becomes the single $K \times K$ characteristic-value matrix equation

$$C Q = Q \Lambda. \quad (108)$$

Thus, the eigenvalues and eigenvectors in (102) are independent of m . However, the constants $\{\epsilon_k^{(m)}\}$ in (103) still depend on m through their dependencies on means $\{c_m(t, f)\}$; that is, from (103),

$$\epsilon_k^{(m)} = V_k^T E^{(m)} = \sum_{j=1}^K v_{kj} c_m(t_j, f_j). \quad (109)$$

The collection of all these conclusions enables us to reduce the characteristic function (105) of S to the compact closed form

$$f_S(\xi) = \left[\prod_{k=1}^K \left(1 - i2\xi\lambda_k \right) \right]^{-M/2} \exp \left[i\xi \sum_{k=1}^K \frac{h_k}{1 - i2\xi\lambda_k} \right], \quad (110)$$

where we defined the K constants

$$h_k = \sum_{m=1}^M \epsilon_k^{(m)2} \quad \text{for } 1 \leq k \leq K. \quad (111)$$

If the number, M , of components in fading model (45) happens to be even, the calculation of (110) does not involve a square root, thereby further simplifying its numerical evaluation. The collapsing of the KM constants $\{\epsilon_k^{(m)}\}$ into a smaller set of K constants, by means of the sums of squares in (111), is the analog of the result in (83) for an individual fading variate $q(t, f)$ in special case 4.

As a special subcase here, suppose that the M deterministic components $\{c_m(t, f)\}$ in (45) are independent of t and f ; that is,

$$c_m(t_k, f_k) = c_m \quad \text{for } 1 \leq k \leq K. \quad (112)$$

This means that the constants are independent of the locations $\{t_k, f_k\}$ of the signal pulses in the t, f plane, although they can still depend on the component number m . Then, there follows from (109) and (111),

$$\epsilon_k^{(m)} = c_m \sum_{j=1}^K v_{kj}, \quad h_k = \left(\sum_{m=1}^M c_m^2 \right) \left(\sum_{j=1}^K v_{kj} \right)^2. \quad (113)$$

That is, the characteristic function of sum S in (110) depends on the constants $\{c_m\}$ only through their sum of squares. This latter property holds true only when special subcase (112) is valid.

SPECIAL CASE 5: Zero means and uncorrelated components with identical covariances

Upon setting the means $\{c_m(t, f)\}$ to zero, it is seen from (109) and (111) that characteristic function (110) reduces to

$$f_S(\xi) = \left[\prod_{k=1}^K (1 - i2\xi\lambda_k) \right]^{-M/2}. \quad (114)$$

Now, only the K eigenvalues $\{\lambda_k\}$ of $K \times K$ covariance matrix C in (107) need to be evaluated. This result in (114) is essentially identical with [10; (D-14)]. For the special case of $M = 2$, this same result for the characteristic function of S can be shown to follow from [11; (20)].

SPECIAL CASE 6: Uncorrelated components with proportional covariances; see (67)

From (67) and (100), we find that $K \times K$ covariance matrix

$$C^{(m)} = r^{(m)} C, \quad \text{where } C = \left[R_{11}(t_k - t_j, f_k - f_j) \right]_1^K, \quad (115)$$

and $r^{(1)} = 1$. Then (101) yields

$$Q^{(m)} = Q, \quad \Lambda^{(m)} = r^{(m)} \Lambda, \quad (116)$$

where Q and Λ are the solutions to the single $K \times K$ characteristic-value matrix equation

$$C Q = Q \Lambda. \quad (117)$$

We now refer to (103) to get constants

$$\varepsilon_k^{(m)} = V_k^T E^{(m)} = \sum_{j=1}^K v_{kj} c_m(t_j, f_j) . \quad (118)$$

Then (104) yields the characteristic function of $S^{(m)}$ as

$$f^{(m)}(\xi) = \left[\prod_{k=1}^K \left(1 - i2\xi \lambda_k r^{(m)} \right) \right]^{-\frac{1}{2}} \exp \left[i\xi \sum_{k=1}^K \frac{\varepsilon_k^{(m)2}}{1 - i2\xi \lambda_k r^{(m)}} \right] . \quad (119)$$

Finally, (96) gives the characteristic function of sum S as

$$f_S(\xi) = \left[\prod_{m=1}^M \prod_{k=1}^K \left(1 - i2\xi \lambda_k r^{(m)} \right) \right]^{-\frac{1}{2}} \times \\ \times \exp \left[i\xi \sum_{m=1}^M \sum_{k=1}^K \frac{\varepsilon_k^{(m)2}}{1 - i2\xi \lambda_k r^{(m)}} \right] . \quad (120)$$

Here, even if the M deterministic components $\{c_m(t, f)\}$ were independent of t and f , as previously assumed in (112), the characteristic function of S would not depend simply on the sum of squares $\sum_m c_m^2$. To see this, substitute the left member of (113) in (120).

SPECIAL CASE 7: Zero means and uncorrelated components with proportional covariances

Set the means $\{c_m(t, f)\}$ to zero in (118), at which point the characteristic function for S in (120) simplifies to

$$f_S(\xi) = \left[\prod_{m=1}^M \prod_{k=1}^K \left(1 - i2\xi \lambda_k r^{(m)} \right) \right]^{-1/2}. \quad (121)$$

Eigenvalues $\{\lambda_k\}$ are the diagonal elements of eigenvalue matrix Λ in $K \times K$ matrix equation (117), where covariance matrix C is given by (115).

CUMULANTS OF SUM S

For some purposes, such as approximating the probability density function or exceedance distribution function of S , the cumulants are useful; see, for example, [17]. By expanding the natural logarithm of the general characteristic function in (92) in a power series in $i\xi$, the cumulants of S are readily found:

$$\chi_S(p) = (p-1)! \sum_{n=1}^{KM} \lambda_n^{p-1} \left(\lambda_n + p \epsilon_n^2 \right) \quad \text{for } p \geq 1. \quad (122)$$

The cumulants for the seven special cases considered above could also be derived. However, we will only present the results for special case 6, namely uncorrelated components with proportional covariances; see (67). Again, expanding the logarithm of characteristic function (120) in a series in $i\xi$, there follows a slight simplification of (122) for the cumulants of S :

$$\begin{aligned} \chi_S(p) = & (p-1)! \sum_{m=1}^M r^{(m)p} \sum_{k=1}^K \lambda_k^p + \\ & + p! \sum_{m=1}^M \sum_{k=1}^K \left(\lambda_k r^{(m)} \right)^{p-1} \epsilon_k^{(m)2} \quad \text{for } p \geq 1. \end{aligned} \quad (123)$$

CHARACTERISTIC FUNCTION OF PROCESSOR OUTPUT γ

With the characteristic function of sum S in hand, we can now return to the desired result (42) for the characteristic function $f_Y(\xi)$ of processor output γ , when conditions (38) are satisfied. Substitution of general result (92) into (42) yields

$$f_Y(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{n=1}^{KM} \left(1 - i2\xi \left(1 + \frac{2\tilde{E}}{N_0} \lambda_n \right) \right) \right]^{-\frac{1}{2}} \times \\ \times \exp \left[i\xi \frac{2\tilde{E}}{N_0} \sum_{n=1}^{KM} \frac{\epsilon_n^2}{1 - i2\xi \left(1 + \frac{2\tilde{E}}{N_0} \lambda_n \right)} \right]. \quad (124)$$

By expanding the logarithm of (124) in a power series in $i\xi$, the cumulants $\chi_Y(p)$ of processor output γ are found to be given by

$$\frac{\chi_Y(p)}{(p-1)! 2^{p-1}} = \sum_{n=1}^{KM} \left(1 + \frac{2\tilde{E}}{N_0} \lambda_n \right)^p + \frac{2\tilde{E}}{N_0} p \sum_{n=1}^{KM} \left(1 + \frac{2\tilde{E}}{N_0} \lambda_n \right)^{p-1} \epsilon_n^2 - K(M-2) \\ \text{for } p \geq 1. \quad (125)$$

SPECIAL CASE 1: Zero means; see (59)

Upon setting means $\{c_m(t, f)\}$ to zero in (124), the constants $\{\epsilon_n\}$ defined in (91) become zero and (124) reduces to

$$f_Y(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{n=1}^{KM} \left(1 - i2\xi \left(1 + \frac{2\tilde{E}}{N_0} \lambda_n \right) \right) \right]^{-\frac{1}{2}}. \quad (126)$$

Only the eigenvalues $\{\lambda_n\}$ of $KM \times KM$ covariance matrix C of the KM random variables $\{g_m(t_k, f_k)\}$ need to be evaluated in this case.

SPECIAL CASE 2: Uncorrelated components; see (61)

The comments in the bottom paragraph of page 32 and the sequel are relevant at this point and should be reviewed. For convenience, we will utilize a normalized version of covariance matrix $C^{(m)}$ defined in (100):

$$\underline{C}^{(m)} = \frac{C^{(m)}}{R_{mm}(0,0)} = \left[\frac{R_{mm}(t_k - t_j, f_k - f_j)}{R_{mm}(0,0)} \right]_1^K. \quad (127)$$

The corresponding eigenvalue matrix $\underline{\Lambda}^{(m)}$ of normalized $K \times K$ covariance matrix $\underline{C}^{(m)}$ is then given by

$$\underline{\Lambda}^{(m)} = \frac{\underline{\Lambda}^{(m)}}{R_{mm}(0,0)} \equiv \text{diag} \left[\underline{\lambda}_1^{(m)} \cdot \cdot \cdot \underline{\lambda}_K^{(m)} \right] \quad (128)$$

in terms of eigenvalue matrix $\underline{\Lambda}^{(m)}$ in (101), which now becomes

$$\underline{C}^{(m)} \underline{Q}^{(m)} = \underline{Q}^{(m)} \underline{\Lambda}^{(m)}. \quad (129)$$

Normalized modal matrix $\underline{Q}^{(m)}$ is unchanged from (101). The relationship between the eigenvalues in (102) and (128) is

$$\lambda_k^{(m)} = \underline{\lambda}_k^{(m)} R_{mm}(0,0). \quad (130)$$

When we employ special case 2 result (105) in (42), the characteristic function of processor output γ is given by

$$\begin{aligned} f_Y(\xi) = & (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{m=1}^M \prod_{k=1}^K \left(1 - i2\xi \left(1 + \frac{2\tilde{E}}{N_0} \lambda_k^{(m)} \right) \right) \right]^{-\frac{1}{2}} \times \\ & \times \exp \left[i\xi \frac{2\tilde{E}}{N_0} \sum_{m=1}^M \sum_{k=1}^K \frac{\epsilon_k^{(m)2}}{1 - i2\xi \left(1 + \frac{2\tilde{E}}{N_0} \lambda_k^{(m)} \right)} \right]. \end{aligned} \quad (131)$$

However, a combination of (130) and (52) yields

$$\frac{2\tilde{E}}{N_0} \lambda_k^{(m)} = \frac{2\tilde{E}}{N_0} R_{mm}(0,0) \lambda_k^{(m)} = \frac{2E_{1m}}{N_0} \lambda_k^{(m)}, \quad (132)$$

where we have taken note that the same signal energy $\tilde{E}$ was transmitted on all K pulses; see (38). That is, E_{1m} is the average random received signal energy in the m -th component of any one of the pulses. The quantity $2E_{1m}/N_0$ is a measure of the average random received "signal-to-noise ratio" in the m -th component of one pulse. At the same time, from (103),

$$\frac{2\tilde{E}}{N_0} \epsilon_k^{(m)2} = \left[\sum_{j=1}^K v_{kj}^{(m)} c_m(t_j, f_j) \left(\frac{2\tilde{E}}{N_0} \right)^{\frac{1}{2}} \right]^2. \quad (133)$$

But since $c_m(t, f) \geq 0$ without loss of generality (see (46)), we have

$$c_m(t_j, f_j) \left(\frac{2\tilde{E}}{N_0} \right)^{\frac{1}{2}} = \left(\frac{2\tilde{E}}{N_0} c_m^2(t_j, f_j) \right)^{\frac{1}{2}} = \left(\frac{2D_{jm}}{N_0} \right)^{\frac{1}{2}}, \quad (134)$$

where D_{jm} is the deterministic received signal energy in the m -th component of the j -th pulse, as defined in (50). Then (133) can be expressed as

$$\frac{2\tilde{E}}{N_0} \epsilon_k^{(m)2} = \left[\sum_{j=1}^K v_{kj}^{(m)} \left(\frac{2D_{jm}}{N_0} \right)^{\frac{1}{2}} \right]^2 \equiv \epsilon_k^{(m)2}. \quad (135)$$

The quantity $2D_{jm}/N_0$ is a measure of the received deterministic signal-to-noise ratio in the m -th component of the j -th pulse.

Upon combining these definitions, (131) is modified to

$$f_Y(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{m=1}^M \prod_{k=1}^K \left(1 - i2\xi \left(1 + \frac{2E_{1m}}{N_0} \lambda_k^{(m)} \right) \right) \right]^{-\frac{1}{2}} \times \\ \times \exp \left[i\xi \sum_{m=1}^M \sum_{k=1}^K \frac{\varepsilon_k^{(m)2}}{1 - i2\xi \left(1 + \frac{2E_{1m}}{N_0} \lambda_k^{(m)} \right)} \right]. \quad (136)$$

Here, eigenvalues $\lambda_1^{(m)}, \dots, \lambda_K^{(m)}$ are those of the $K \times K$ normalized covariance matrix $\underline{C}^{(m)}$ defined in (127). The constants $\varepsilon_k^{(m)}$ are given, according to (135), by

$$\varepsilon_k^{(m)} = \sum_{j=1}^K v_{kj}^{(m)} \left(\frac{2D_{jm}}{N_0} \right)^{\frac{1}{2}}. \quad (137)$$

The cumulants of processor output y in this case are given by

$$\frac{\chi_Y(p)}{(p-1)! 2^{p-1}} = \sum_{m=1}^M \sum_{k=1}^K \left(1 + \frac{2E_{1m}}{N_0} \lambda_k^{(m)} \right)^p + \\ + p \sum_{m=1}^M \sum_{k=1}^K \left(1 + \frac{2E_{1m}}{N_0} \lambda_k^{(m)} \right)^{p-1} \varepsilon_k^{(m)2} - K(M-2) \quad \text{for } p \geq 1. \quad (138)$$

SPECIAL CASE 3: Zero means and uncorrelated components

When means $\{c_m(t, f)\}$ are zero, then (50) and (137) yield $D_{km} = 0$ and $\varepsilon_k^{(m)} = 0$, thereby causing (136) to reduce to

$$f_Y(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{m=1}^M \prod_{k=1}^K \left(1 - i2\xi \left(1 + \frac{2E_{1m}}{N_0} \lambda_k^{(m)} \right) \right) \right]^{-\frac{1}{2}}. \quad (139)$$

SPECIAL CASE 4: Uncorrelated components with identical covariances; see (64)

Now, normalized covariance matrix $\underline{C}^{(m)}$ in (127) is independent of m ; in particular, we have the single $K \times K$ covariance matrix

$$\underline{C} = \frac{\underline{C}}{R_{11}(0,0)} = \left[\frac{R_{11}(t_k - t_j, f_k - f_j)}{R_{11}(0,0)} \right]_1^K. \quad (140)$$

This leads to solution matrices $\underline{Q}^{(m)}$ and $\underline{\Lambda}^{(m)}$ in (129) which are also independent of m ; that is, (129) becomes the single $K \times K$ characteristic-value matrix equation

$$\underline{C} \underline{Q} = \underline{Q} \underline{\Lambda}; \quad \underline{\Lambda} = \underline{\Lambda}/R_{11}(0,0) = \text{diag}[\underline{\lambda}_1, \dots, \underline{\lambda}_K]. \quad (141)$$

Thus, the eigenvalues in (128) and the eigenvectors in (102) are independent of m . However, the constants $\underline{\varepsilon}_k^{(m)}$ in (137) still depend on m through their dependence on deterministic received signal energies $\{D_{jm}\}$; that is, from (137), now

$$\underline{\varepsilon}_k^{(m)} = \sum_{j=1}^K v_{kj} \left(\frac{2D_{jm}}{N_0} \right)^{\frac{1}{2}}. \quad (142)$$

Also, from (52), (38), and (64), we now have

$$E_{1m} = \tilde{E}_1 R_{mm}(0,0) = \tilde{E} R_{11}(0,0) = E_{11}, \quad (143)$$

which is the average random received signal energy in one component of one pulse. (In this special case 4, we have the alternative result $E_{11} = E_1/M$ from (53).)

The collection of these conclusions enables us to express the characteristic function (136) of processor output γ in the closed compact form

$$f_Y(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{k=1}^K \left(1 - i2\xi \left(1 + \frac{2E_{11}}{N_0} \lambda_k \right) \right) \right]^{-\frac{1}{2}M} \times \\ \times \exp \left[i\xi \sum_{k=1}^K \frac{h_k}{1 - i2\xi \left(1 + \frac{2E_{11}}{N_0} \lambda_k \right)} \right], \quad (144)$$

where we defined the K constants

$$h_k = \sum_{m=1}^M \varepsilon_k^{(m)2} = \sum_{m=1}^M \left\{ \sum_{j=1}^K v_{kj} \left(\frac{2D_{jm}}{N_0} \right)^{\frac{1}{2}} \right\}^2 \quad \text{for } 1 \leq k \leq K. \quad (145)$$

Use of (142) was made here. The cumulants of output γ are now

$$\frac{\chi_Y(p)}{(p-1)! 2^{p-1}} = M \sum_{k=1}^K \left(1 + \frac{2E_{11}}{N_0} \lambda_k \right)^p + \\ + p \sum_{k=1}^K h_k \left(1 + \frac{2E_{11}}{N_0} \lambda_k \right)^{p-1} - K(M-2) \quad \text{for } p \geq 1. \quad (146)$$

As a special subcase, suppose that the M deterministic components $\{c_m(t, f)\}$ in fading model (45) are independent of t and f ; see (112). Then, from (50) and (38),

$$D_{km} = \tilde{E}_k c_m^2(t_k, f_k) = \tilde{E} c_m^2 = D_{1m}, \quad (147)$$

where D_{1m} is the received deterministic signal energy in the m -th component of any one of the pulses. This result allows the constants in (145) to be simplified to

$$\underline{h}_k = \left(\sum_{m=1}^M \frac{2D_{1m}}{N_o} \right) \left(\sum_{j=1}^K v_{kj} \right)^2. \quad (148)$$

Thus, in this special subcase, the constants $\{\underline{h}_k\}$ depend only on the sum of the received deterministic signal energies on all the components.

SPECIAL CASE 5: Zero means and uncorrelated components with identical covariances

Upon setting the means $\{c_m(t, f)\}$ to zero, it follows from (50), (145), and (144) that the characteristic function of processor output γ is

$$f_Y(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{k=1}^K \left(1 - i2\xi \left(1 + \frac{2E_{11}}{N_o} \underline{\lambda}_k \right) \right) \right]^{-\frac{1}{2}M}. \quad (149)$$

Eigenvalues $\underline{\lambda}_1, \dots, \underline{\lambda}_K$ correspond to $K \times K$ normalized covariance matrix $\underline{C}$ given by (140), while E_{11} is the average random received signal energy in one component of one pulse. This exact result replaces the approximation in [12; (8)].

In the special case of $M = 2$, that is, two components in fading model (45), the result in (149) reduces to

$$f_Y(\xi) = \left[\prod_{k=1}^K \left(1 - i2\xi \left(1 + \frac{2E_{11}}{N_o} \underline{\lambda}_k \right) \right) \right]^{-1}. \quad (150)$$

This expression agrees with [11; (24)], except for a scaling of output γ by a factor of 2. There was no need here to evaluate multiple integrals as encountered in [11; (21) - (22)].

SPECIAL CASE 6: Uncorrelated components with proportional covariances; see (67)

If we combine the general relation in (42), for the characteristic function of processor output γ , with the characteristic function in (120) for the sum S in this special case 6, we obtain the form

$$f_{\gamma}(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{m=1}^M \prod_{k=1}^K \left(1 - i2\xi \left(1 + \frac{2\tilde{E}}{N_0} \lambda_k r^{(m)} \right) \right) \right]^{-\frac{1}{2}} \times \\ \times \exp \left[i\xi \frac{2\tilde{E}}{N_0} \sum_{m=1}^M \sum_{k=1}^K \frac{\epsilon_k^{(m)^2}}{1 - i2\xi \left(1 + \frac{2\tilde{E}}{N_0} \lambda_k r^{(m)} \right)} \right]. \quad (151)$$

Now, from (67) and (100), we find that $K \times K$ covariance matrix

$$C^{(m)} = r^{(m)} C, \quad \text{where } C = \left[R_{11}(t_k - t_j, f_k - f_j) \right]_1^K, \quad (152)$$

and $r^{(1)} = 1$. Then (101) yields

$$Q^{(m)} = Q, \quad \Lambda^{(m)} = r^{(m)} \Lambda, \quad (153)$$

where Q and Λ are the solutions to the single $K \times K$ matrix equation

$$C Q = Q \Lambda. \quad (154)$$

Now, define normalized covariance matrix $\underline{C}$ as in (140), with corresponding matrix equation and eigenvalue matrix $\underline{\Lambda}$ as in (141). Then the eigenvalues are related according to

$$\lambda_k = \underline{\lambda}_k R_{11}(0,0), \quad (155)$$

leading to result

$$\frac{2\tilde{E}}{N_0} \lambda_k r^{(m)} = \frac{2\tilde{E}}{N_0} R_{11}(0,0) \lambda_k r^{(m)} = \frac{2E_{11}}{N_0} \lambda_k r^{(m)} = \frac{2E_{1m}}{N_0} \lambda_k, \quad (156)$$

upon use of (38), (52), and (67). The quantity E_{1m} is the average random received signal energy in the m -th component of any one pulse. Also, from (135) and (137),

$$\frac{2\tilde{E}}{N_0} \varepsilon_k^{(m)2} = \varepsilon_k^{(m)2}, \quad \varepsilon_k^{(m)} = \sum_{j=1}^K v_{kj} \left(\frac{2D_{jm}}{N_0} \right)^{\frac{1}{2}}, \quad (157)$$

since normalized modal matrix Q is independent of component number m ; see (153). Use of (156) and (157) in (151) yields the characteristic function of processor output γ in the desired form

$$f_Y(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{m=1}^M \prod_{k=1}^K \left(1 - i2\xi \left(1 + \frac{2E_{1m}}{N_0} \lambda_k \right) \right) \right]^{-\frac{1}{2}} \times \\ \times \exp \left[i\xi \sum_{m=1}^M \sum_{k=1}^K \frac{\varepsilon_k^{(m)2}}{1 - i2\xi \left(1 + \frac{2E_{1m}}{N_0} \lambda_k \right)} \right]. \quad (158)$$

This result is a slight simplification of (136) which allowed uncorrelated components of arbitrary covariances; thus, eigenvalues $\{\lambda_k\}$ and eigenvectors $\{V_k\}$ are independent of m here. Furthermore, (158) simplifies to the result in (144) when the covariances of the random fading components $\{g_m(t,f)\}$ in (45) are identical; see (64).

The cumulants $\{\chi_Y(p)\}$ of processor output γ are found by expanding the logarithm of (158) in a power series in $i\xi$:

$$\frac{\chi_Y(p)}{(p-1)! 2^{p-1}} = \sum_{m=1}^M \sum_{k=1}^K \left(1 + \frac{2E_{1m}}{N_0} \lambda_k\right)^p +$$

$$+ p \sum_{m=1}^M \sum_{k=1}^K \left(1 + \frac{2E_{1m}}{N_0} \lambda_k\right)^{p-1} \varepsilon_k^{(m)2} - K(M-2) \quad \text{for } p \geq 1. \quad (159)$$

SPECIAL CASE 7: Zero means and uncorrelated components with proportional covariances

When we set means $\{c_m(t, f)\}$ to zero, the characteristic function in (158) simplifies to

$$f_Y(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{m=1}^M \prod_{k=1}^K \left(1 - i2\xi \left(1 + \frac{2E_{1m}}{N_0} \lambda_k\right)\right) \right]^{-\frac{1}{2}}. \quad (160)$$

For purposes of review, the quantity E_{km} is the average random received signal energy in the m -th component of the k -th pulse, while D_{km} is the received deterministic signal energy in the m -th component of the k -th pulse; see (50) and (52) as well as fading model (45) - (47). Eigenvalues $\{\lambda_k\}$ correspond to normalized covariance matrix $\underline{C}$ in (140), while coefficients $\{v_{kj}\}$ used in (157) are the corresponding eigenvectors' components; see the $K \times K$ matrix equation in (141), which yields eigenvalue matrix $\underline{\Lambda}$ and normalized modal matrix $\underline{Q}$.

In the sequel to (42), it was noted that the more general conditional characteristic function of γ in (27) would be treated once the fading model had been described. This latter situation with general receiver weights $\{A_k\}$ and transmitted signal energies $\{\tilde{E}_k\}$ is considered in appendix C. In particular, the unconditional characteristic function of processor output γ is derived for general correlated fading components $\{g_m(t,f)\}$ and then specialized to the case of uncorrelated fading components.

EXAMPLES OF PROBABILITY DENSITY OF POWER-SCALING $q(t, f)$

In an earlier section, the characteristic function of power-scaling variate $q = q(t, f)$ was derived for fading model (45), yielding general result (76). This result was then simplified for the seven special cases noted there. In this section, we will present some numerical examples of typical probability density functions of q and thereby illustrate the generality and variety of fading model (45).

The only case we will consider in this section is where the M random components $\{g_m(t, f)\}$ in (45) are uncorrelated with each other and have proportional covariances. This is special case 6; see (67). Also, for notational convenience, we consider the normalized (power) random variable

$$\phi = \frac{q}{R_{11}(0, 0)} = \frac{q}{\sigma_1^2}, \quad (161)$$

where

$$\sigma_m^2 \equiv \overline{g_m^2(t, f)} = R_{mm}(0, 0) = R_{11}(0, 0) r^{(m)} \quad \text{for } 1 \leq m \leq M. \quad (162)$$

Here, we used (47) and (67). Then, from (86), the characteristic function of random variable ϕ is

$$f_\phi(\xi) = f_q\left(\frac{\xi}{R_{11}(0, 0)}\right) = \left[\prod_{m=1}^M \left(1 - i2\xi r^{(m)}\right) \right]^{-\frac{1}{2}} \exp\left[i\xi \sum_{m=1}^M \frac{\eta_m^2 r^{(m)}}{1 - i2\xi r^{(m)}} \right], \quad (163)$$

where we defined (dimensionless) normalization constants

$$\eta_m = \frac{c_m}{\sigma_m} = \frac{c_m(t, f)}{R_{mm}(0, 0)^{1/2}} \quad \text{for } 1 \leq m \leq M. \quad (164)$$

For $M = 1$, the general result in (163) simplifies to

$$f_\phi(\xi) = (1 - i2\xi)^{-1/2} \exp\left[\frac{i\xi \eta_1^2}{1 - i2\xi}\right], \quad (165)$$

for which the corresponding PDF (probability density function) is

$$p_\phi(u) = (2\pi u)^{-1/2} \exp\left[-\frac{1}{2}\left(u + \eta_1^2\right)\right] \cosh(\eta_1 \sqrt{u}) \quad \text{for } u > 0. \quad (166)$$

The amplitude-scaling variate, $r_k = q_k^{1/2}$ from (43), has the corresponding PDF for normalized version

$$\theta = \frac{r}{\sigma_1} = \left[\frac{q}{R_{11}(0, 0)}\right]^{1/2} = \sqrt{\phi}, \quad (167)$$

namely

$$p_\theta(u) = 2 u p_\phi(u^2) = \left(\frac{2}{\pi}\right)^{1/2} \exp\left[-\frac{1}{2}\left(u^2 + \eta_1^2\right)\right] \cosh(\eta_1 u) \quad \text{for } u > 0. \quad (168)$$

This PDF is displayed in figure 2 for $\eta_1 = 0(.5)3$. At the origin this PDF is finite and nonzero, with value $(2/\pi)^{1/2} \exp(-\eta_1^2/2)$, indicating the possibility of occasional deep fades (unless η_1 is large). As parameter η_1 gets large, this PDF approaches Gaussian; in fact, directly from (168) and figure 2, we obtain

$$p_\theta(u) \sim (2\pi)^{-1/2} \exp\left[-\frac{1}{2}\left(u - \eta_1\right)^2\right] \quad \text{for } \eta_1 > 3. \quad (169)$$

The corresponding PDF for normalized power-scaling variate $\phi = q/R_{11}(0,0) = q/\sigma_1^2$ was given in (166) and is plotted in figure 3. It is seen to possess a substantial cusp at the origin, which behaves as $(2\pi u)^{-1/2} \exp(-\eta_1^2/2)$. This is the most severe case of deep fading that model (45) can yield, namely for $M = 1$.

When $M = 2$, that is, two components in fading model (45), the characteristic function in (163) reduces to

$$f_\phi(\xi) = \left(1 - i2\xi\right)^{-1/2} \left(1 - i2\xi r^{(2)}\right)^{-1/2} \exp\left[\frac{i\xi \eta_1^2}{1 - i2\xi} + \frac{i\xi \eta_2^2 r^{(2)}}{1 - i2\xi r^{(2)}}\right]. \quad (170)$$

However, instead of Fourier transforming this result, the PDF of ϕ is best found by reverting to definition (45), namely $q = (c_1 + g_1)^2 + (c_2 + g_2)^2$, and performing the probability integrals directly in the g_1, g_2 plane. The result is

$$p_\phi(u) = \frac{1}{4\pi (r^{(2)})^{1/2}} \int_{-\pi}^{\pi} dt \exp\left[-\frac{1}{2}(C^2 + S^2)\right] \quad \text{for } u > 0, \quad (171)$$

where auxiliary functions

$$C \equiv u^{1/2} \cos(t) - \eta_1, \quad S \equiv \left(\frac{u}{r^{(2)}}\right)^{1/2} \sin(t) - \eta_2. \quad (172)$$

As a special subcase here, for $M = 2$, let $r^{(2)} = 1$; that is, let both random components $\{g_m(t, f)\}$ have equal power. Then (170) yields characteristic function

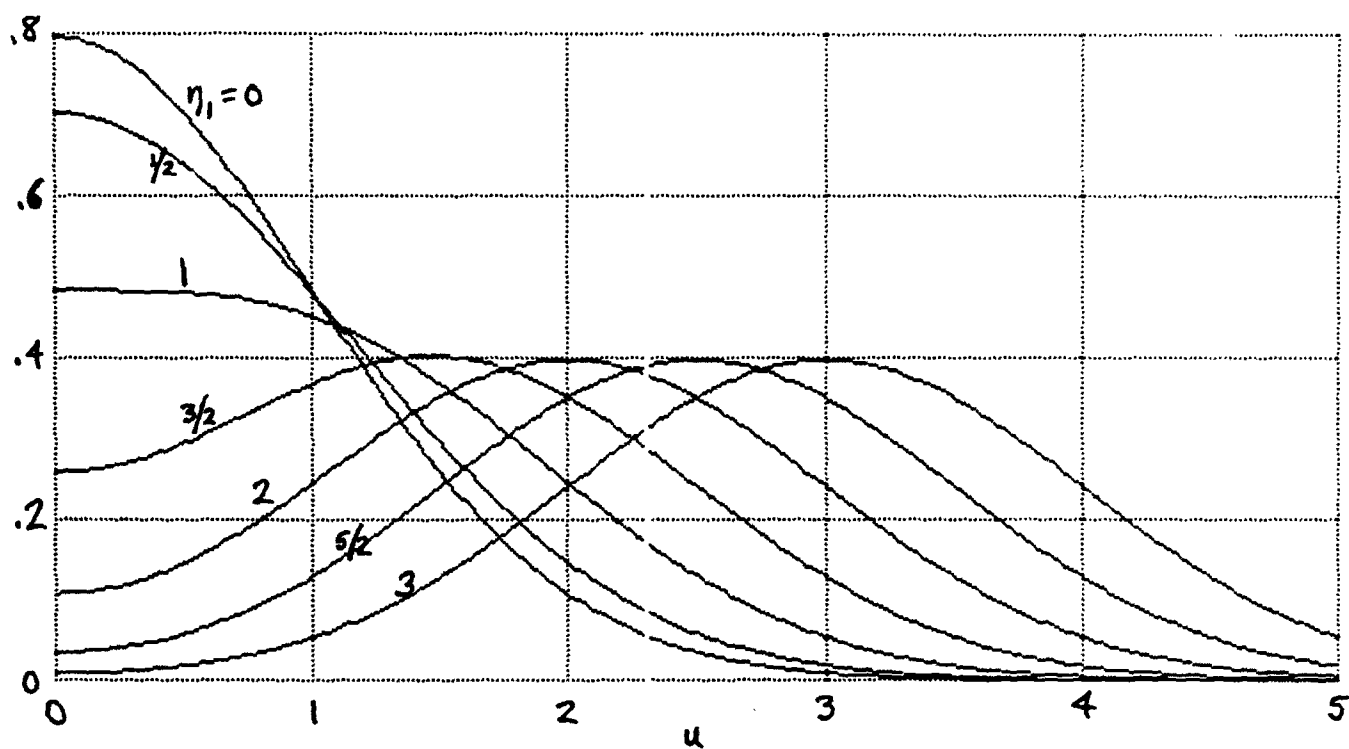


Figure 2. PDF of r/σ_1 for $M = 1$

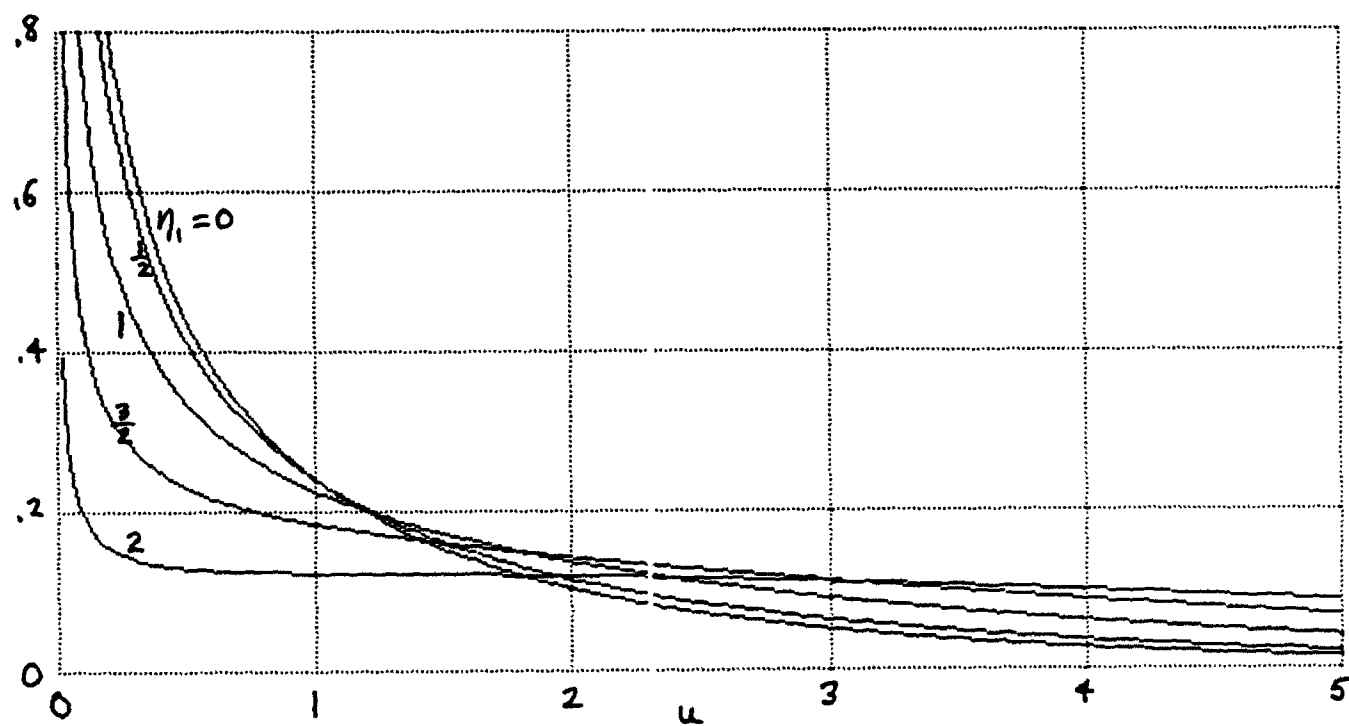


Figure 3. PDF of q/σ_1^2 for $M = 1$

$$f_{\phi}(\xi) = (1 - i2\xi)^{-1} \exp\left[\frac{i\xi \eta^2}{1 - i2\xi}\right], \quad (173)$$

where constant

$$\eta^2 \equiv \eta_1^2 + \eta_2^2 = \frac{c_1^2 + c_2^2}{R_{11}(0,0)} \quad \text{for } M = 2. \quad (174)$$

The corresponding probability density function for ϕ is

$$p_{\phi}(u) = \frac{1}{2} I_0(\eta\sqrt{u}) \exp\left[-\frac{1}{2}(u + \eta^2)\right] \quad \text{for } u > 0. \quad (175)$$

That for $\theta = \sqrt{\phi}$ follows immediately as

$$p_{\theta}(u) = u I_0(\eta u) \exp\left[-\frac{1}{2}(u^2 + \eta^2)\right] \quad \text{for } u > 0. \quad (176)$$

The PDF in (175) is displayed in figure 4. Now, the origin value

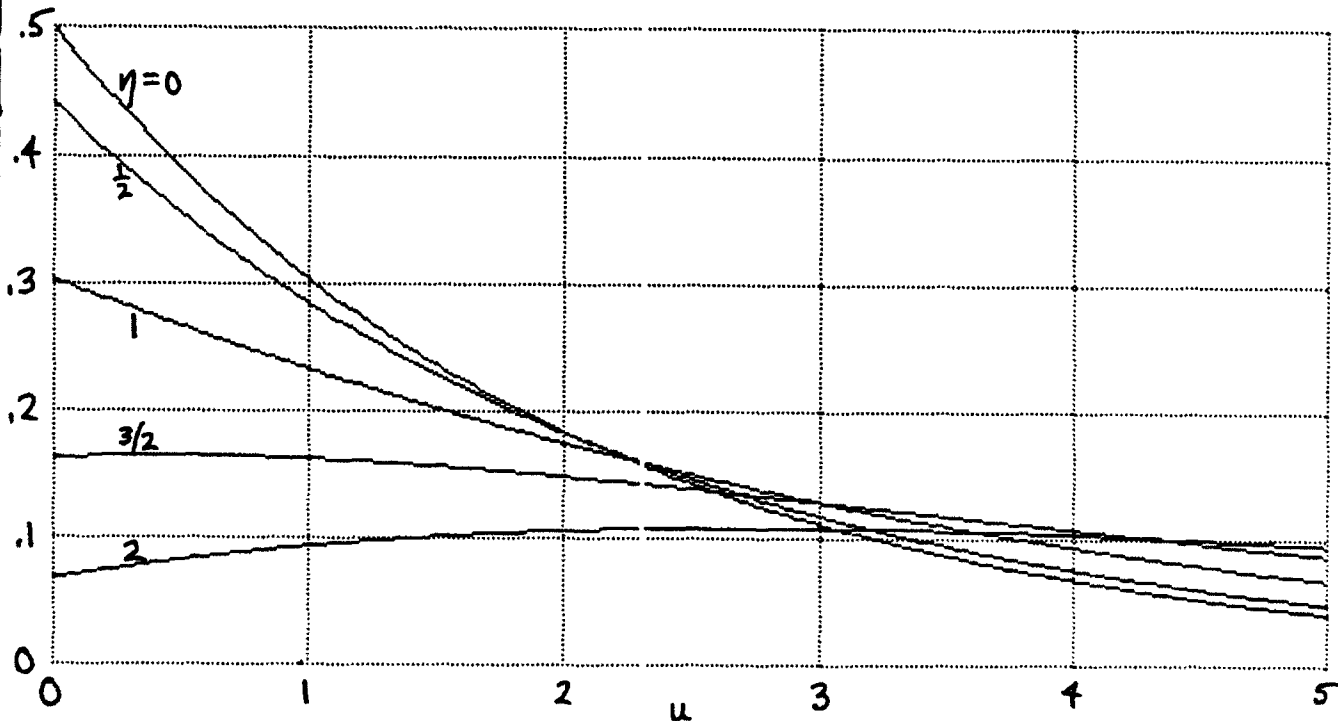


Figure 4. PDF of q/σ_1^2 for $M = 2$, $r^{(2)} = 1$

is finite, namely $.5 \exp(-\eta^2/2)$, in contrast with figure 3 for $M = 1$. The cusps that were in figure 3 are now absent for $M = 2$.

When $M = 2$ but $r^{(2)} \neq 1$, a closed form like (175) is not available for the PDF of ϕ . Instead, we must revert to the general relation in (171) - (172) and perform the numerical integration for each set of parameter values of interest. An example for $r^{(2)} = 1/2$ and $\eta_2 = \eta_1$ is given in figure 5, and a second example for $r^{(2)} = 1/2$ and $\eta_2 = \eta_1/2$ is given in figure 6. A common feature of all these densities is the finite nonzero values at the origin. That is, deep fades are still common for fading model (45) when $M = 2$.

For larger M and completely arbitrary parameters, it is necessary to numerically Fourier transform general characteristic function (163) in order to determine the PDF of ϕ . However, for the special case where

$$M \text{ is even and } r^{(m)} = 1 \text{ for } 1 \leq m \leq M, \quad (177)$$

then (163) reduces to

$$f_{\phi}(\xi) = (1 - i2\xi)^{-M/2} \exp\left[\frac{i\xi \eta^2}{1 - i2\xi}\right], \quad (178)$$

where constant

$$\eta^2 = \sum_{m=1}^M \eta_m^2. \quad (179)$$

The corresponding PDF of ϕ is then

$$p_{\phi}(u) = \frac{1}{2} \left(\frac{\sqrt{u}}{\eta}\right)^N I_N(\eta\sqrt{u}) \exp\left[-\frac{1}{2}(u + \eta^2)\right] \quad \text{for } u > 0, \quad (180)$$

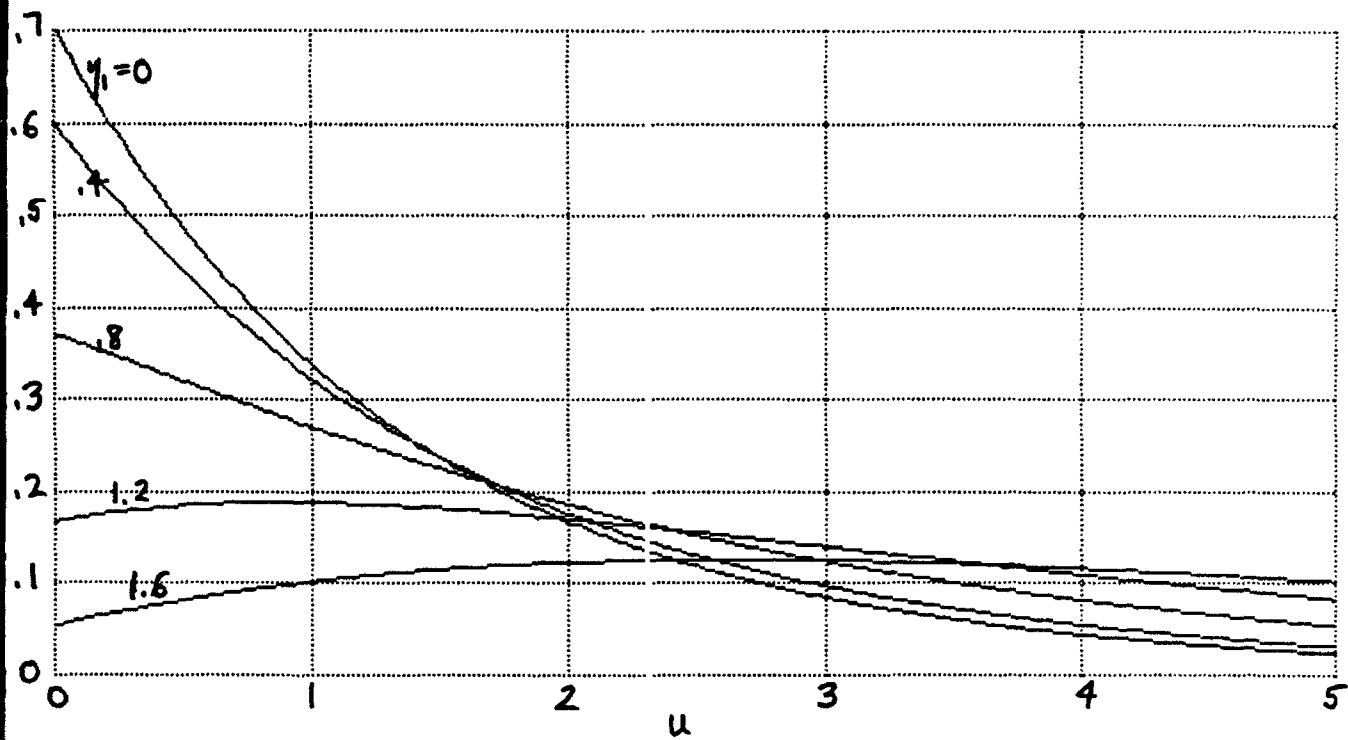


Figure 5. PDF of q/σ_1^2 for $M = 2$, $r^{(2)} = 1/2$, $\eta_2 = \eta_1$

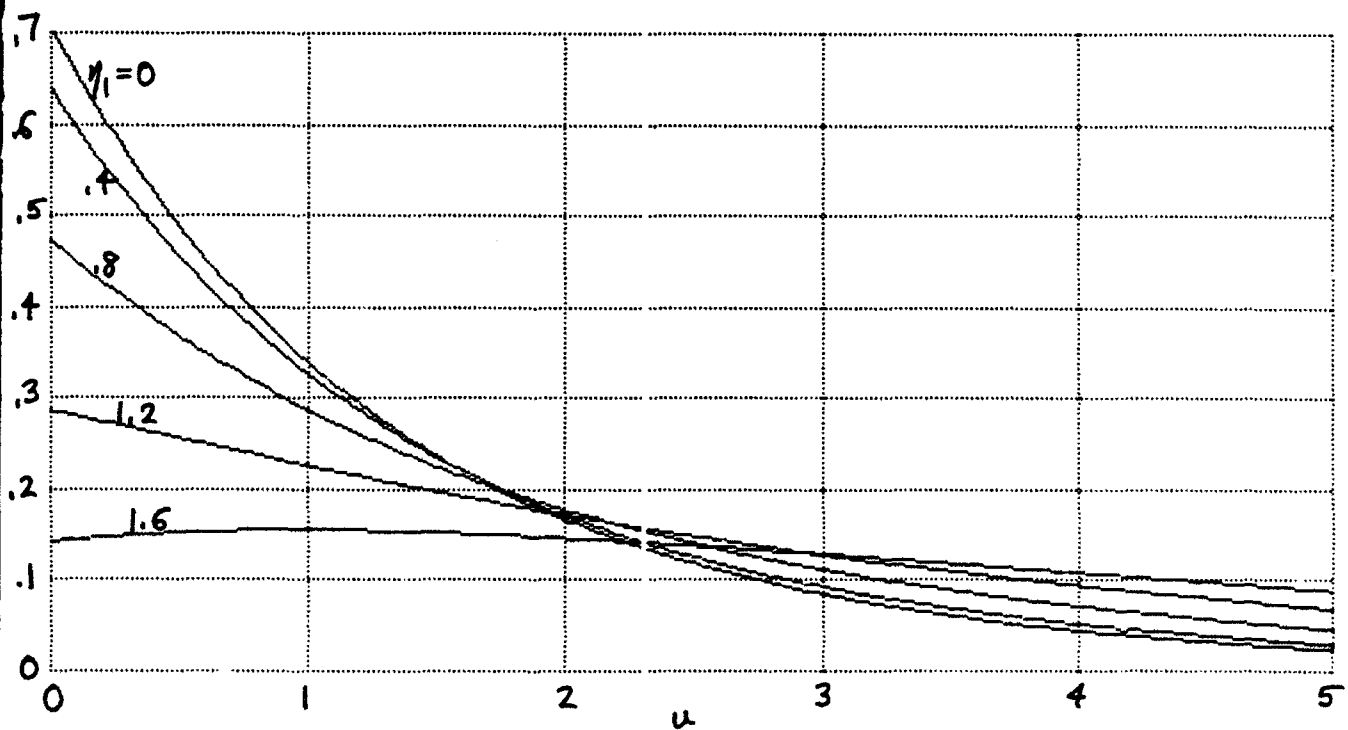


Figure 6. PDF of q/σ_1^2 for $M = 2$, $r^{(2)} = 1/2$, $\eta_2 = \eta_1/2$

where $N = M/2 - 1$. This reduces to (175) for $M = 2$.

The PDF for $\theta = \sqrt{\phi}$ follows as given by the rule given in (168), namely

$$p_{\theta}(u) = u \left(\frac{u}{h} \right)^N I_N(hu) \exp \left[-\frac{1}{2}(u^2 + h^2) \right] \quad \text{for } u > 0. \quad (181)$$

This reduces to (176) for $M = 2$.

Two examples of PDF (180) for random variable ϕ are displayed in figure 7 for $M = 4$ and in figure 8 for $M = 6$, respectively. The former PDF is zero at the origin, while the latter is zero and has zero first derivative at the origin. Thus, deep fades are less likely for the larger values of M in fading model (45).

A general procedure for determining the probability density function $p_{\phi}(u)$ of random variable ϕ from characteristic function $f_{\phi}(\xi)$ in (163), for arbitrary M , $\{r^{(m)}\}$, $\{h_m\}$, is presented at the end of appendix D. It employs some efficient recursions for fast and accurate numerical evaluation of $p_{\phi}(u)$.

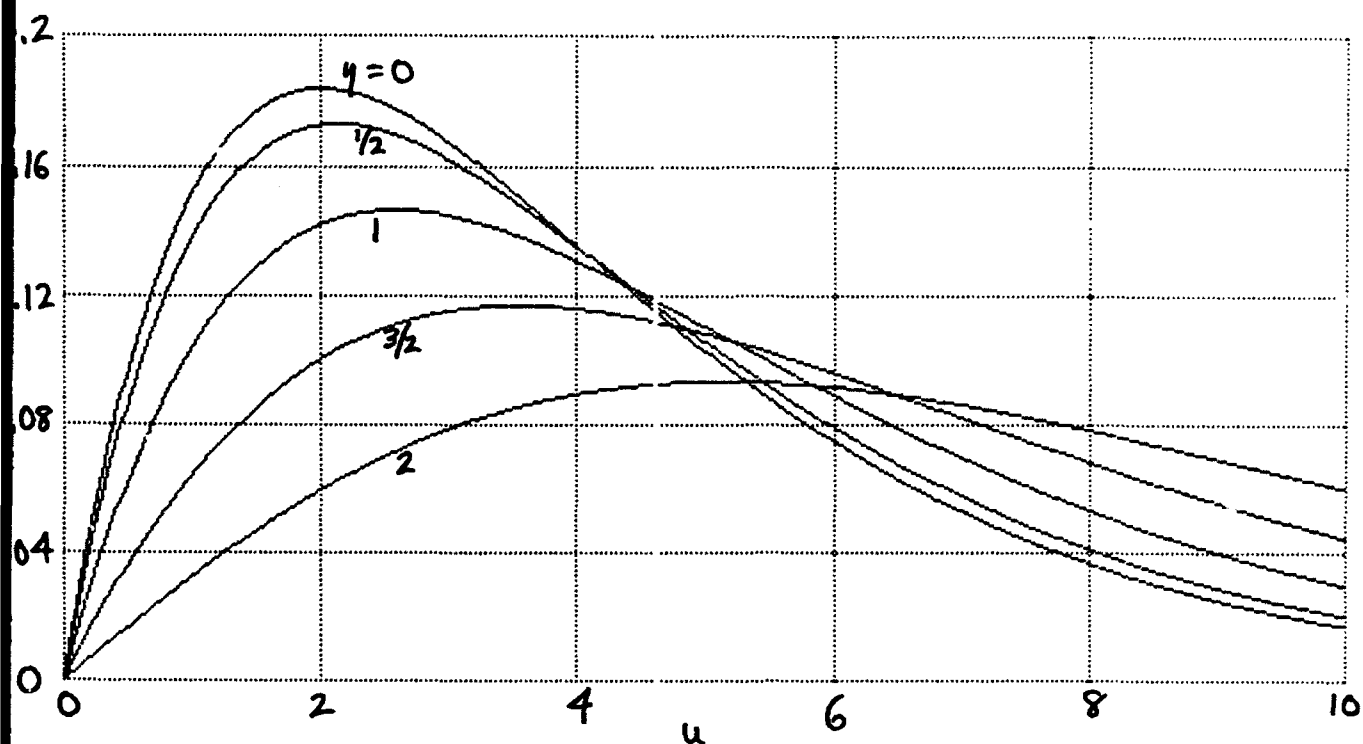


Figure 7. PDF of q/σ_1^2 for $M = 4$, $r^{(m)} = 1$ for all m

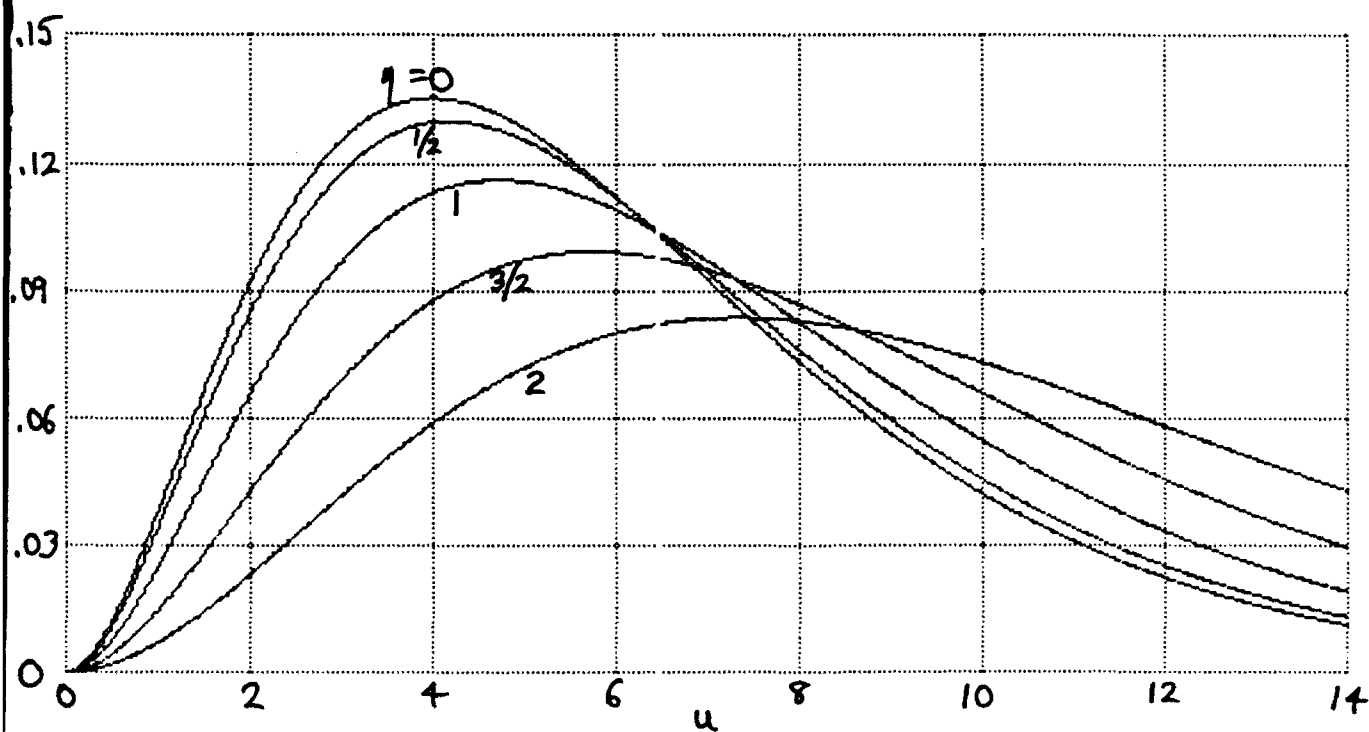


Figure 8. PDF of q/σ_1^2 for $M = 6$, $r^{(m)} = 1$ for all m

PROBABILITY DISTRIBUTION OF PROCESSOR OUTPUT γ

The characteristic functions $f_{\gamma}(\xi)$ of processor output γ , under a variety of special cases, were derived in an earlier section; in particular, see (124) - (160). Although these are compact closed forms, they can encounter computational problems if we attempt to directly utilize Fourier transforms to determine the corresponding probability density functions and exceedance distribution functions. In particular, since the characteristic functions decay only as ξ^{-K} as $\xi \rightarrow \infty$, truncation error can become a significant problem, especially for small K , the number of signal pulses.

EXPANSION FOR DISTRIBUTION FUNCTION

In appendix D, series expansions for $f_{\gamma}(\xi)$ and the corresponding probability density function $p_{\gamma}(u)$ are derived in a form which involves only positive expansion coefficients and terms. Furthermore, efficient and accurate recursions are developed for the evaluation of the coefficients, the probability density function, and the exceedance distribution function.

As an example of the general procedure, the results for general characteristic functions (136) and (158) are summarized here. Define respectively, for $1 \leq m \leq M$, $1 \leq k \leq K$,

$$T_{mk} = \frac{2E_{1m}}{N_0} \lambda_k^{(m)} \quad (136) \quad \text{or} \quad T_{mk} = \frac{2E_{1m}}{N_0} \lambda_k, \quad e_{mk} = \varepsilon_k^{(m)^2} \quad (158) \quad (182)$$

Then, the exceedance distribution function of processor output γ is given by (D-16) as

$$\text{Prob}(\gamma > u) = F \sum_{p=0}^{\infty} g_p H_{K+p-1}\left(\frac{u}{2}\right) \quad \text{for } u > 0, \quad (183)$$

where the quantities required are found in the following fashion.

$$F = \left[\prod_{m=1}^M \prod_{k=1}^K (1 + T_{mk}) \right]^{-1/2}, \quad (184)$$

$$\alpha_0 = -\frac{1}{2} \sum_{m=1}^M \sum_{k=1}^K \frac{e_{mk}}{1 + T_{mk}},$$

$$\alpha_p = \frac{1}{2} \sum_{m=1}^M \sum_{k=1}^K \frac{T_{mk}^{p-1}}{(1 + T_{mk})^p} \left[\frac{T_{mk}}{p} + \frac{e_{mk}}{1 + T_{mk}} \right] \quad \text{for } p \geq 1, \quad (185)$$

$$g_0 = \exp(\alpha_0), \quad g_p = \frac{1}{p} \sum_{n=1}^p n \alpha_n g_{p-n} \quad \text{for } p \geq 1, \quad (186)$$

$$H_n(x) = \exp(-x) \sum_{k=0}^n \frac{x^k}{k!} \quad \text{for } n \geq 0, \quad x \geq 0. \quad (187)$$

Since the eigenvalues of a covariance matrix can never be negative, it is seen from (185) that all the $\{\alpha_p\}$ are nonnegative, with the exception of α_0 . However, α_0 is used only once in (186) to generate a positive g_0 , while the recursion in (186) for $\{g_p\}$ utilizes only nonnegative quantities. Thus, all the coefficients $\{g_p\}$ and scale factor F in (183) are nonnegative. The function $H_n(x)$ in (187) is obviously positive

for $x \geq 0$, thereby guaranteeing that no negative terms will appear in expansion (183). Furthermore, there is a very efficient recursion for the $\{H_n(x)\}$; see (D-18). A sample program for the evaluation of a modified form of (183) is given in appendix D.

In the absence of signal, all the $\{T_{mk}\}$ and $\{e_{mk}\}$ in (182) are zero. Then, all the $\{\alpha_p\}$ and $\{g_p\}$ become zero except for g_0 which is 1. Expansion (183) for the exceedance distribution function of processor output γ then reduces to simply the one term $H_{K-1}(u/2)$, which is consistent with the noise-only characteristic function $(1 - i2\xi)^{-K}$ for processor output γ ; see (42), for example. That is, the false alarm probability is

$$P_F = H_{K-1}\left(\frac{u}{2}\right). \quad (188)$$

An error bound for sum (183), terminated at the $p = N$ term, is given in (D-20). Also, the modifications required to treat the slightly different forms of characteristic functions encountered in (124) and (144) are presented in (D-23) - (D-26). Finally, a refinement of the expansion procedure in (183), which is more rapidly convergent and useful for larger signal-to-noise ratios, is given in (D-27) - (D-36).

FUNDAMENTAL INPUTS AND COMPUTATIONAL PROCEDURE

In the remainder of this section, we will restrict attention to characteristic function (158) which pertains to special case 6, namely uncorrelated components with proportional covariances; see (67). However, before we list the fundamental inputs that are required to conduct the numerical evaluation of the exceedance distribution function of processor output γ in this case, we make an additional modification for later convenience in plotting and comparison.

From (52), (38), and (67), we have

$$E_{km} = E_{1m} = \tilde{E} R_{11}(0,0) r^{(m)} . \quad (189)$$

Then, using (53), the average random received signal energy in one pulse is

$$E_1 = \sum_{m=1}^M E_{1m} = \tilde{E} R_{11}(0,0) \sum_{m=1}^M r^{(m)} . \quad (190)$$

This enables us to express the signal-to-noise ratio measure as

$$\frac{E_{1m}}{N_o} = \frac{E_1}{N_o} \psi_m , \quad \text{where} \quad \psi_m \equiv \frac{r^{(m)}}{\sum_{m=1}^M r^{(m)}} \quad \text{for } 1 \leq m \leq M . \quad (191)$$

The quantities $\{\psi_m\}$ represent the fractional strengths of each of the M random components $\{g_m(t,f)\}$ in fading model (45).

The fundamental inputs required to evaluate the exceedance distribution function of processor output γ are now

K , number of signal pulses in figure 1,
 M , number of fading components in (45),
 E_1/N_0 , signal energy to noise density ratio,
 $\{r^{(m)}\}$ for $1 \leq m \leq M$; see (67),
 $\{D_{km}/N_0\}$ for $1 \leq k \leq K, 1 \leq m \leq M$; see (50),
 $R_{11}(\tau, \nu)/R_{11}(0, 0)$; see (47),
 $\{t_k\}$ and $\{f_k\}$ for $1 \leq k \leq K$; see (44). (192)

The quantity E_1/N_0 is a measure of the receiver input average random signal-to-noise ratio for one signal pulse, while $r^{(m)}$ is the relative strength of the m -th fading component. Ratio D_{km}/N_0 is a measure of the receiver input deterministic signal-to-noise ratio for the k -th signal pulse and m -th component. The function $R_{11}(\tau, \nu)/R_{11}(0, 0)$ is the normalized fading covariance function for the medium at time separation τ and frequency separation ν . Parameters t_k and f_k are the time and frequency locations, respectively, of the k -th signal pulse in time, frequency space.

The first quantity that must be computed is the normalized $K \times K$ covariance matrix $\underline{C}$ given by (140). Then, its eigenvalue matrix $\underline{\Lambda}$ and normalized modal matrix $\underline{Q}$ are found by solving characteristic-value matrix equation (141). This yields eigenvalues $\{\underline{\lambda}_k\}$ and eigenvectors $\{V_k\}$ for $1 \leq k \leq K$. The components of column vector V_k are $v_{k1}, \dots, v_{kK}$; see (102). We then compute $\underline{\varepsilon}_k^{(m)}$ from (142) or (157) for $1 \leq k \leq K, 1 \leq m \leq M$. At this point, the parameters $\{T_{mk}\}$ and $\{e_{mk}\}$ in (182) can be computed for $1 \leq k \leq K, 1 \leq m \leq M$, and the procedure in (183) - (187) can be employed. In particular, (182) and (191) yield

$$T_{mk} = \frac{2E_1}{N_0} \psi_m \lambda_k \quad \text{for } 1 \leq m \leq M, \quad 1 \leq k \leq K. \quad (193)$$

NUMERICAL PERFORMANCE RESULTS

The first set of examples are selected to enable a comparison with the approximate procedure and results in [12]. In figure 9, below, we plot the required per-pulse input SNR (signal-to-noise ratio) measure E_1/N_0 (in dB) versus the number of signal pulses K , for values of the adjacent-pulse normalized (amplitude) covariance, Cov_1 , equal to 0, .5, $\sqrt{.5}$, and 1. In particular, the covariance $R_{11}(\tau, \nu)$ was taken as exponential in τ , and the signal pulses have no frequency shifts $\{f_k\}$ and are equally spaced in time locations $\{t_k\}$; that is, $\text{Cov}_1 = R_{11}(t_{k+1} - t_k, 0)/R_{11}(0, 0)$ for all k . Also, the KM deterministic signal-to-noise ratios $\{D_{km}/N_0\}$ are all zero, and the M strengths $\{r^{(m)}\}$ are all equal to 1; this duplicates the situation in [12]. The particular case in figure 9 pertains to false alarm probability $P_F = 1E-6$, detection probability $P_D = .5$, and $M = 1$ fading component; this last choice corresponds to parameter $m = 1/2$ in [12]. The results here in figure 9 for $\text{Cov}_1 = 0$ and $\text{Cov}_1 = 1$ agree precisely with $\rho = 0$ and $\rho = 1$ in [12; figure 11], as expected, while the results for $\text{Cov}_1 = \sqrt{.5}$ are in rather good agreement with those for $\rho = .5$. This selection of parameters reflects the property that ρ in [12] is Cov_1^2 here; see the section entitled Main Program in appendix D for additional details.

The only change in figure 10 is to require a larger detection probability, namely $P_D = .9$. Although the $\text{Cov}_1 = 0$ and 1 results

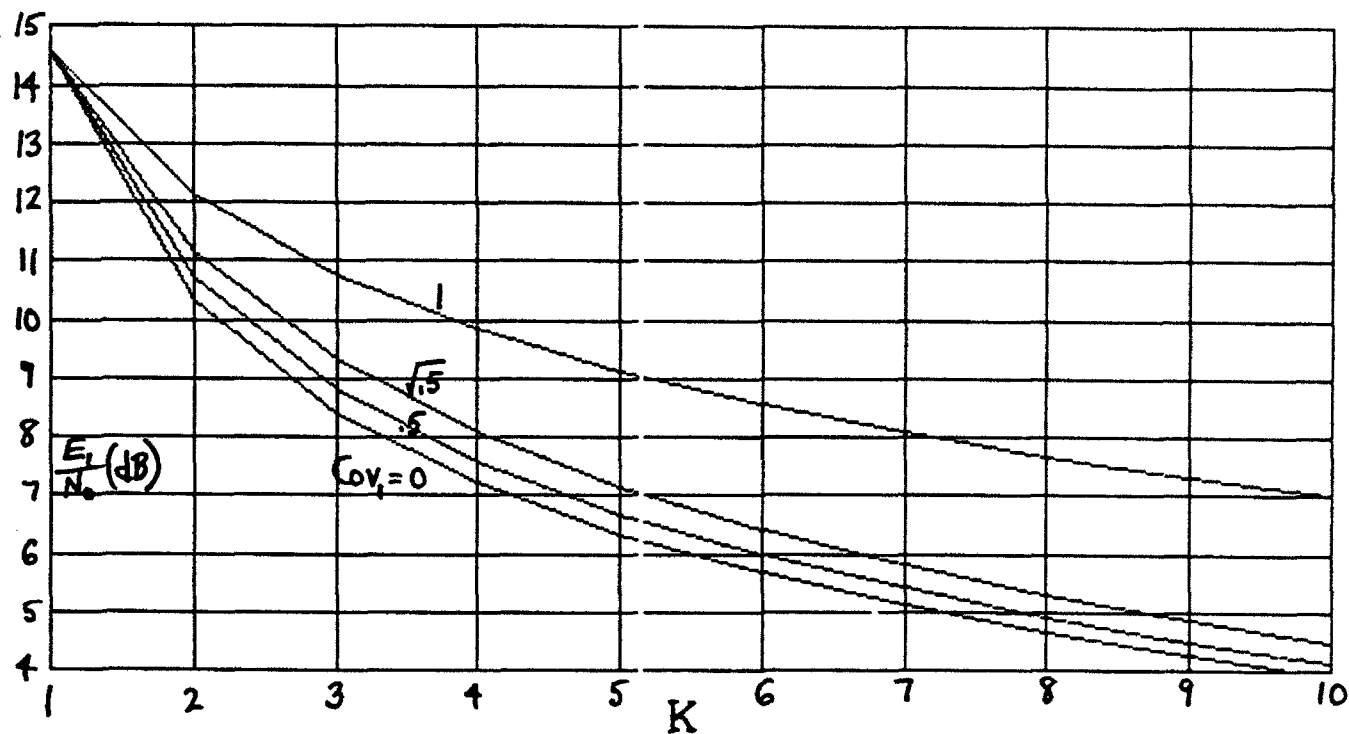


Figure 9. Required SNR for $P_F = 1E-6$, $P_D = .5$, $M = 1$

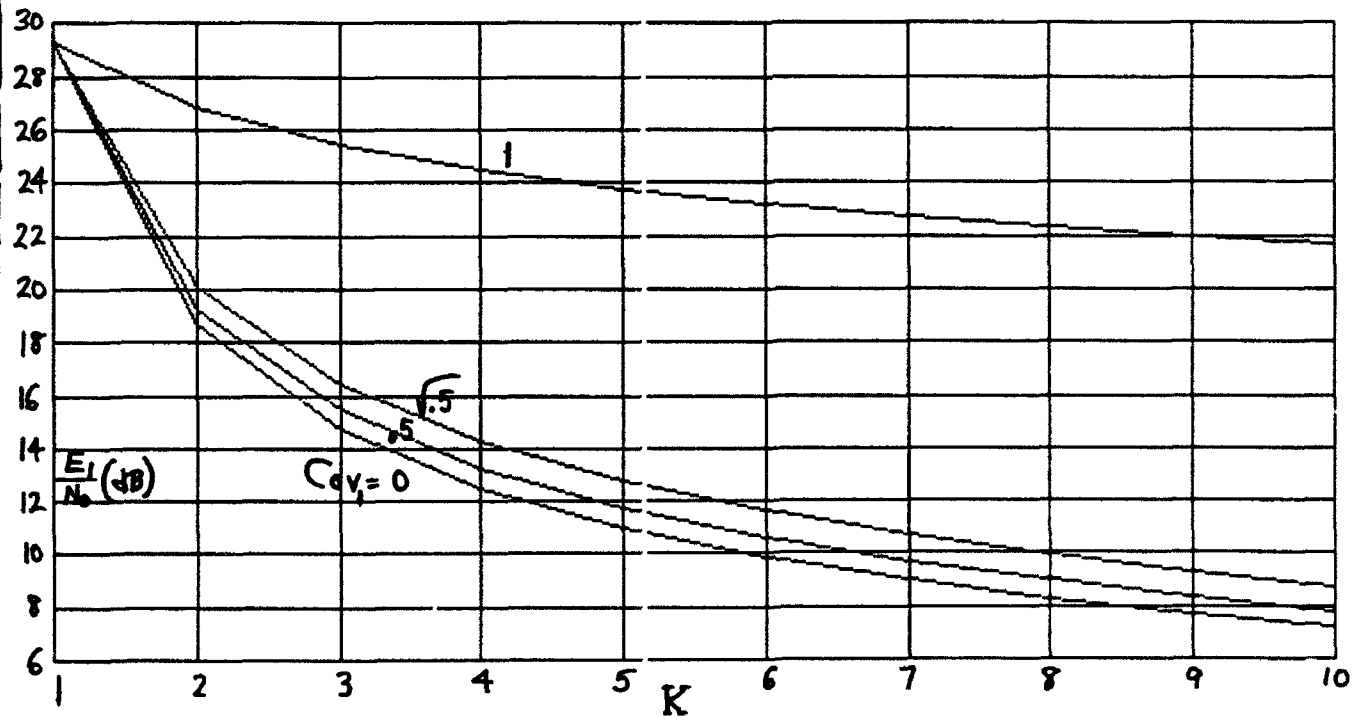


Figure 10. Required SNR for $P_F = 1E-6$, $P_D = .9$, $M = 1$

agree with [12; figure 12], the exact results here for $\text{Cov}_1 = \sqrt{.5}$ are markedly different from the approximate results for $\rho = .5$ in the earlier work. For example, 2.6 dB less is required for $K = 2$ and $K = 3$, while 2.3 dB less is required for $K = 4$. That is, the earlier approximation in [12] was somewhat pessimistic in its performance predictions.

We must observe from figure 10 that the effect of correlated fading is not overly significant until the normalized covariance approaches 1. For example, for $K = 2$, the cost of Cov_1 increasing from 0 to $\sqrt{.5}$ is 1.4 dB, while the cost of Cov_1 going from $\sqrt{.5}$ to 1 is an additional 6.7 dB. The distinction is even greater for $K = 10$, with the same comparison requiring 1.6 dB versus an additional 12.9 dB.

The example in figure 11 pertains to $M = 2$ fading components, (which corresponds to $m = 1$ in [12]); all other parameters are the same as figure 9 above. These results can be compared directly with [12; figure 7]; they reveal identical performance for $\text{Cov}_1 = 0$ and 1, and rather good agreement for $\text{Cov}_1 = \sqrt{.5}$ (versus $\rho = .5$).

When the detection probability P_D is increased to .9 in figure 12, these exact results reveal that lower values of E_1/N_0 are required than the approximation in [12; figure 8] predicted, at least for $\text{Cov}_1 = \sqrt{.5}$ ($\rho = .5$). Also, as was seen in figure 10, the cost of normalized covariance Cov_1 approaching 1 is very significant in terms of increased signal level; that is, the $\text{Cov}_1 = \sqrt{.5}$ curves are well below the $\text{Cov}_1 = 1$ curves in figures

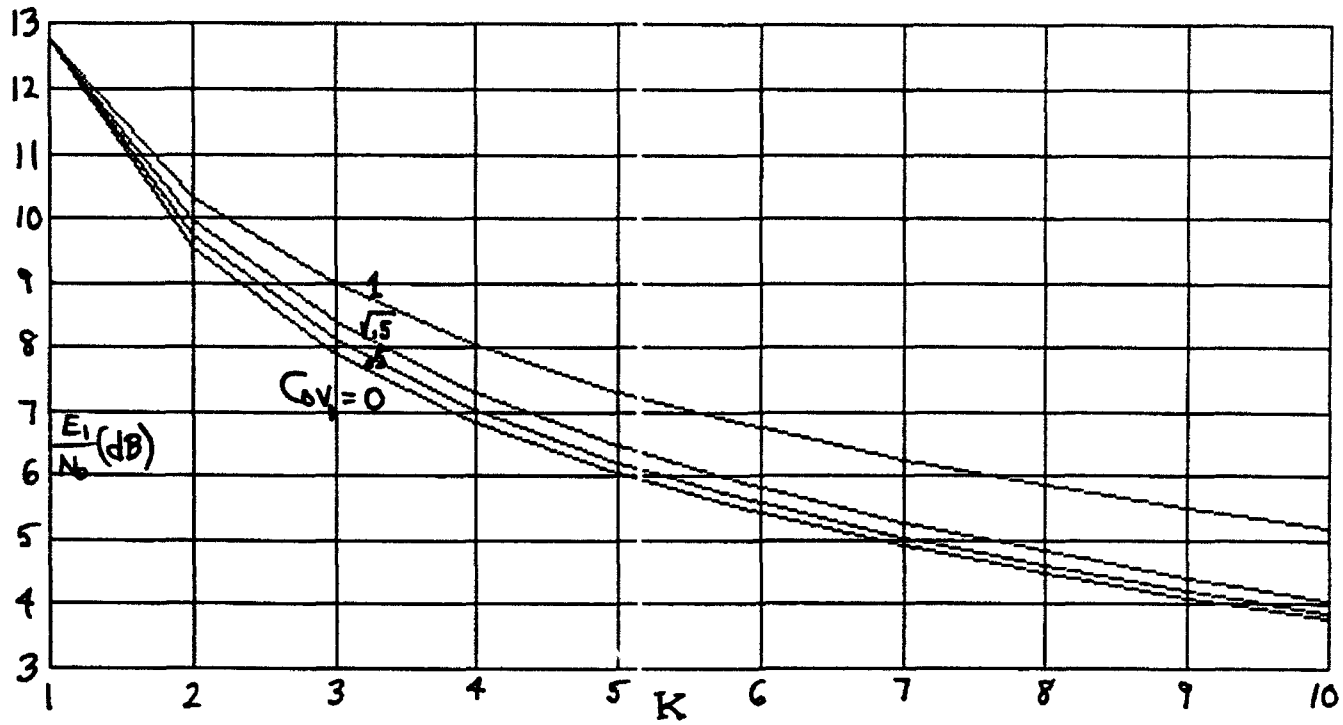


Figure 11. Required SNR for $P_F = 1E-6$, $P_D = .5$, $M = 2$

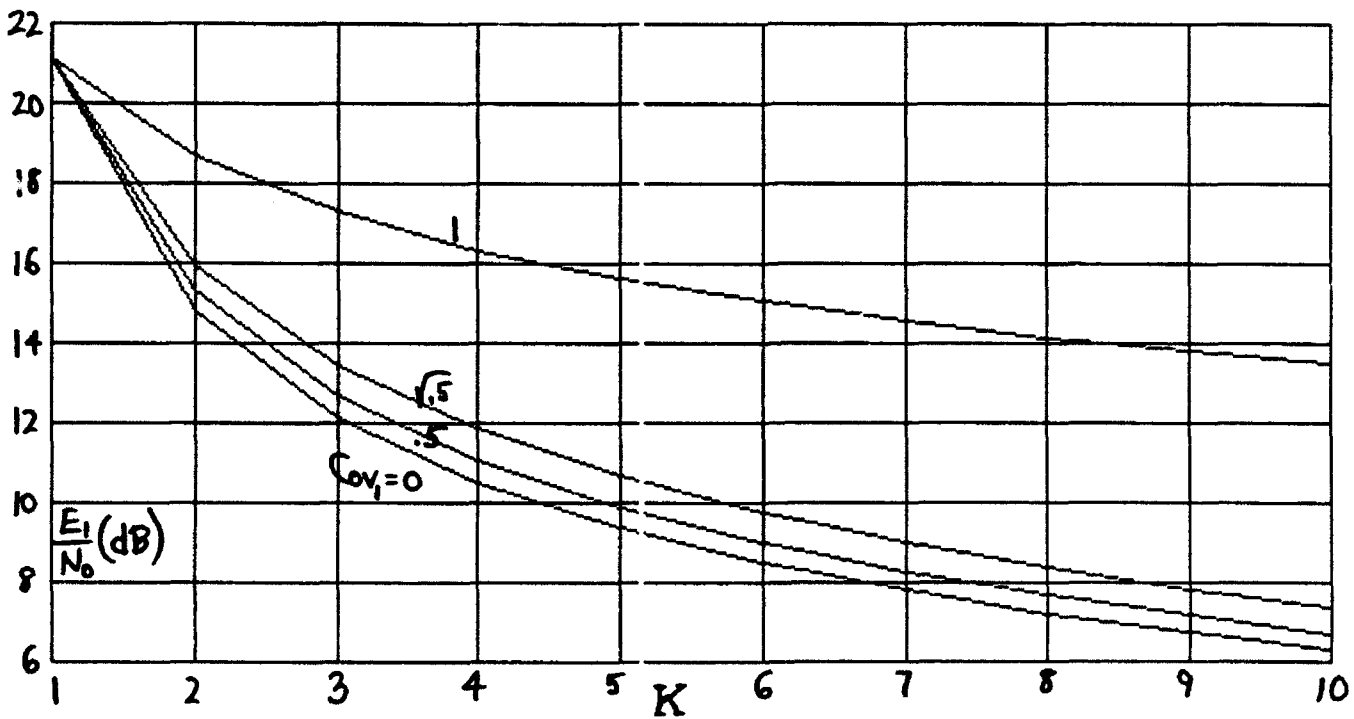


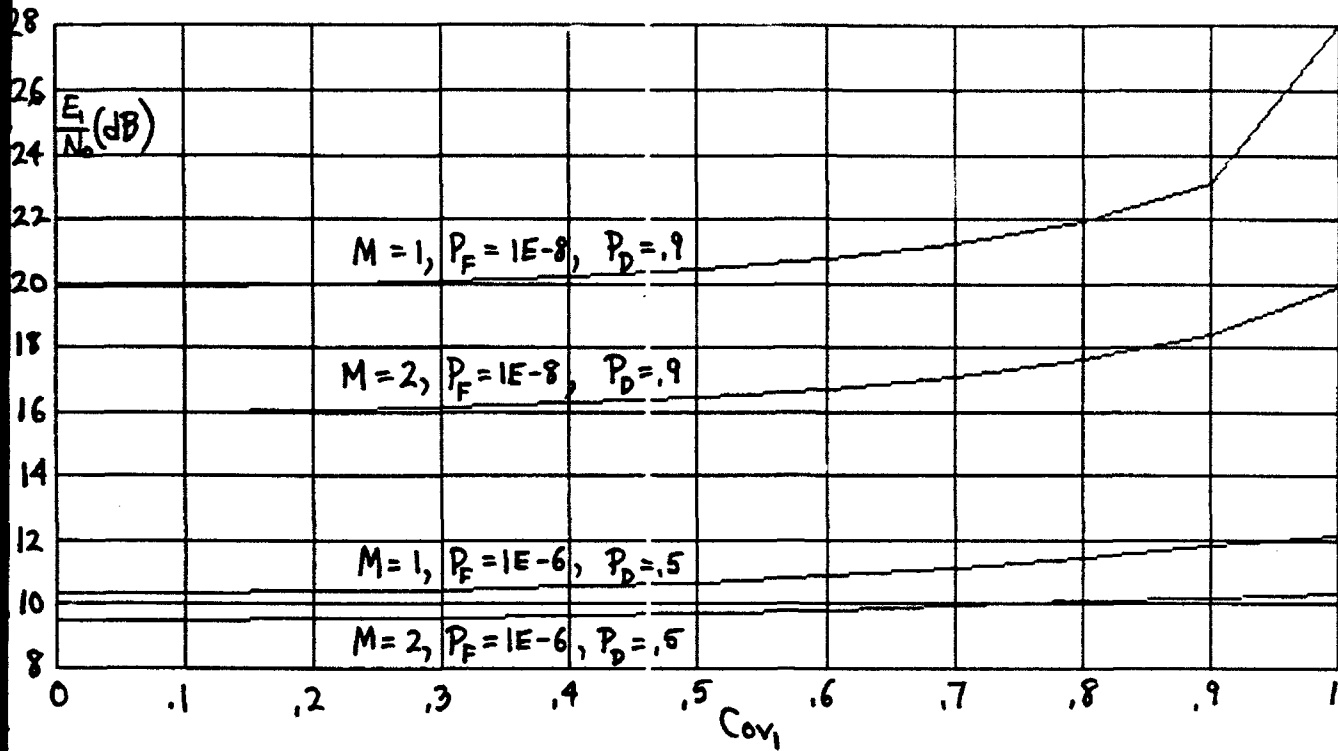
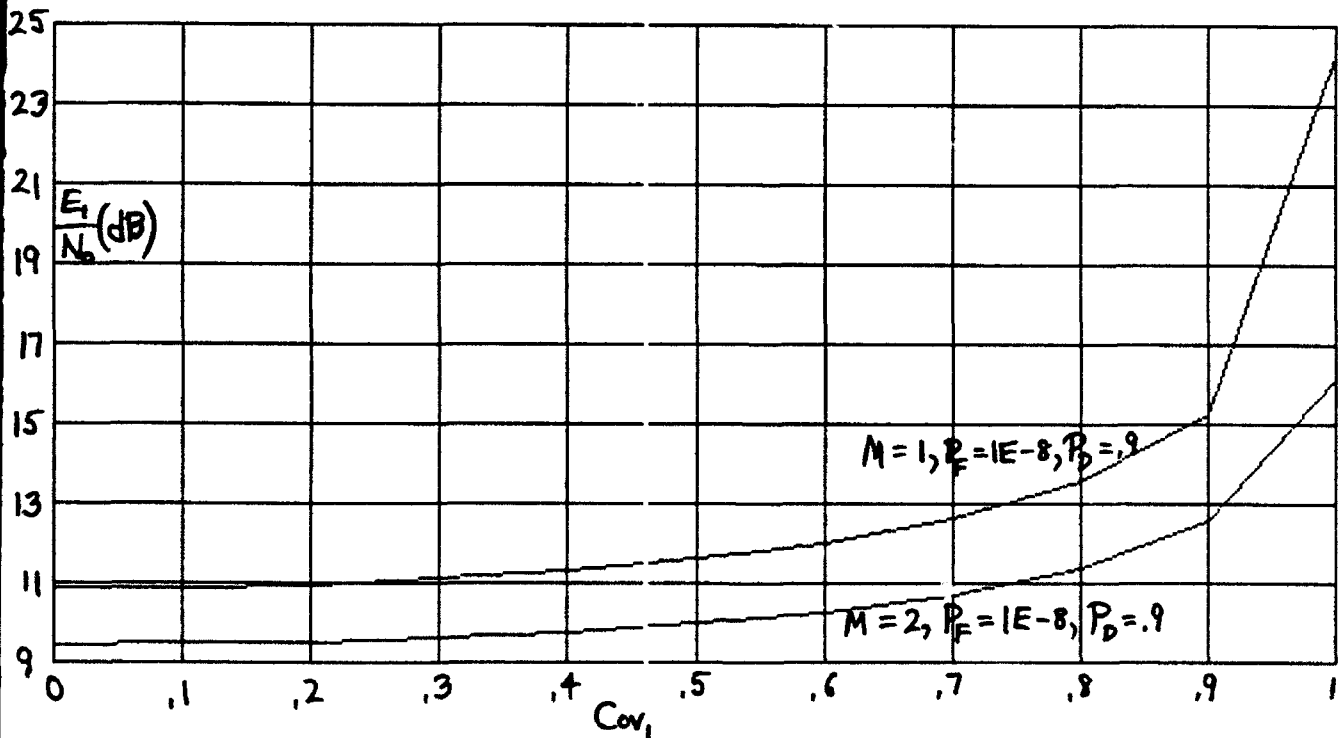
Figure 12. Required SNR for $P_F = 1E-6$, $P_D = .9$, $M = 2$

10 and 12, especially for the larger K values.

The detrimental effect of highly correlated fading pulses is studied quantitatively in figure 13, where normalized covariance Cov_1 is varied from 0 to 1 as the other various parameters are kept fixed. The number K of signal pulses is kept at 2. These plots reveal that for detection probability $P_D = .5$, increased Cov_1 does not lead to a significantly large increase in required input signal level. However, for the better detection probability of $P_D = .9$, higher covariances can be very damaging, requiring additional signal strength to maintain the desired level of performance. For example, as Cov_1 increases from .9 to 1 in the uppermost example in figure 13, the signal must be increased by 4.9 dB. (The kinks in the curves are due to discretization of the abscissa at increment .1 for Cov_1 .)

When K is increased to 6, the results in figure 14 reveal this effect in a more pronounced fashion. An additional 9 dB is now required when Cov_1 is increased from .9 to 1 for the upper curve.

All of the above results have had deterministic signal-to-noise ratio measures $\{D_{km}/N_O\}$ equal to zero. The program listed in appendix D has the capability of incorporating arbitrary values for these parameters as well as others, such as $\{r^{(m)}\}$. An example where all the potential is exercised, and all parameters have nonzero values, is displayed in figure 15. Here, the detection probability P_D is varied from .5 to .999 and the required E_1/N_O is calculated (in dB). It is seen that a sharp

Figure 13. Required SNR versus Cov_1 , $K = 2$ Figure 14. Required SNR versus Cov_1 , $K = 6$

increase is observed as P_D increases above .9, eventually tending to ∞ as $P_D \rightarrow 1$. The particular parameter values are listed below:

$$P_F = 1E-6, \quad K = 4, \quad M = 2,$$

$$r^{(m)} = 1 \quad \text{for } 1 \leq m \leq M,$$

$$D_{km}/N_O = 1 \quad \text{for } 1 \leq k \leq K, \quad 1 \leq m \leq M,$$

$$\{t_k\} = 1, 2, 3, 4, \quad \{f_k\} = 1, 4, 2, 3,$$

$$\text{Cov}(\tau, \nu) = \frac{R_{11}(\tau, \nu)}{R_{11}(0, 0)} = \exp\left(-\frac{\tau^2}{11} - \frac{\nu^2}{10}\right). \quad (194)$$

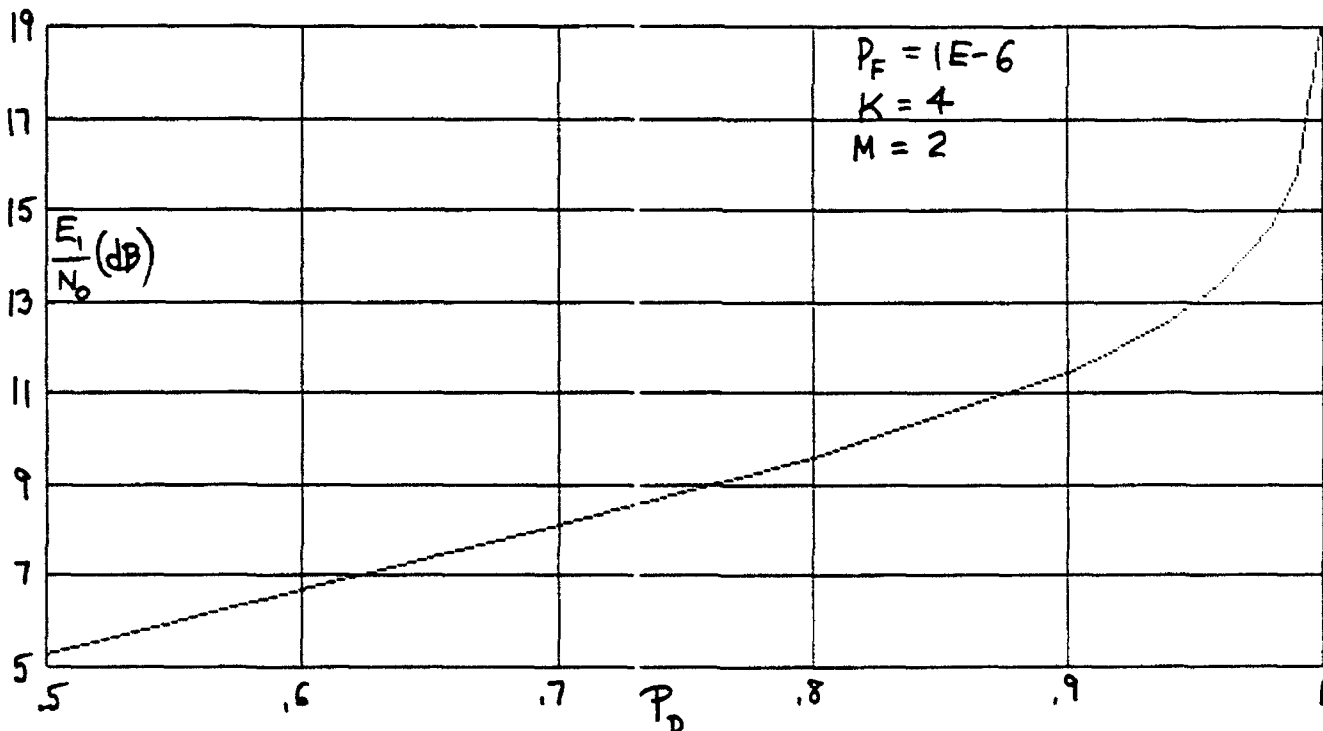


Figure 15. Required SNR versus P_D

SUMMARY

The characteristic function of processor output γ in figure 1 has been derived in closed form for a wide variety of fading conditions and signal formats. The corresponding probability density functions and exceedance distribution functions of γ have then been expanded in convergent series, by means of a novel expansion technique which make efficient use of recursions for rapid and accurate evaluation. These series typically require on the order of 30 to 60 terms for ten decimal accuracy. A program is given which incorporates all these features, including allowance for both deterministic and random components of arbitrary strengths in the fading medium.

One of the most useful results pointed out by this study is that the degree of correlation between the fading signal pulses can be fairly significant without suffering great degradations. That is, when the normalized covariance approaches 1, meaning that all signal pulses tend to fade together, the performance losses are potentially large; however, for coefficients below .5, the losses are not too significant.

Comparison of these exact results with an earlier approximate procedure [12] reveals that the earlier approach generally gives pessimistic predictions of performance when the normalized covariance of the fading is intermediate between 0 and 1. In some cases, the discrepancy can be several dB. This result indicates and emphasizes the need for accurate treatment of systems which must perform well, that is, yield high detection

probabilities in fading media. An extension of this work to a fading medium in which the background noise level is unknown and must be estimated from a finite sample in a noise-only region of time, frequency space, is currently underway by this author and will be reported on shortly.

The major result utilized here is the characteristic function of a quadratic form in correlated nonzero-mean Gaussian random variables; this result is presented in appendix B. It relies on the ability to solve the generalized eigenvalue problem; this latter procedure and solution is presented in appendix E. A related problem involving a slightly more general bilinear form is treated in appendix F. Finally, the characteristic function of the most general complex form with both first-order and second-order terms is solved in appendix G. These results are not utilized in this technical report but are presented for completeness and for possible future use and reference.

APPENDIX A. ANALYTIC AND COMPLEX ENVELOPE PROCESSES

Let $n(t)$ be a stationary zero-mean real random process with covariance $R_n(\tau)$ and double-sided spectrum $G_n(f)$:

$$R_n(\tau) = \overline{n(t) n(t-\tau)} , \quad G_n(f) = \int d\tau \exp(-i2\pi f\tau) R_n(\tau) . \quad (A-1)$$

The analytic process $n_+(t)$ is generated by eliminating the negative frequencies in $n(t)$ and by doubling its positive frequency contributions; that is, $n(t)$ is passed through a filter with transfer function $2 U(f)$, where U is the unit step function.

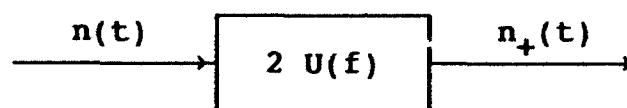


Figure A-1. Generation of Analytic Process

The analytic process can be expressed in terms of its real and imaginary parts according to

$$n_+(t) = n(t) + i n_H(t) , \quad (A-2)$$

where $n_H(t)$ is the Hilbert transform of $n(t)$. The complex envelope process $\underline{n}(t)$ is obtained by frequency down-shifting analytic process $n_+(t)$, in order to center its one-sided spectrum about $f = 0$. Thus, if f_0 is a representative center frequency of $n_+(t)$, then

$$\underline{n}(t) = n_+(t) \exp(-i2\pi f_0 t) . \quad (A-3)$$

The original real process $n(t)$ then follows from (A-2) and (A-3):

$$n(t) = \operatorname{Re}\left[n_+(t)\right] = \operatorname{Re}\left[\underline{n}(t) \exp(i2\pi f_0 t)\right] . \quad (\text{A-4})$$

The spectrum of $n_+(t)$ follows from figure A-1 as

$$G_{n_+}(f) = 4 U(f) G_n(f) , \quad (\text{A-5})$$

while the spectrum of $\underline{n}(t)$ is obtained from (A-3) according to

$$G_{\underline{n}}(f) = G_{n_+}(f+f_0) = 4 U(f+f_0) G_n(f+f_0) . \quad (\text{A-6})$$

The covariances corresponding to (A-5) and (A-6) are

$$\overline{n_+(t) n_+^*(t-\tau)} \quad \text{and} \quad \overline{\underline{n}(t) \underline{n}^*(t-\tau)} , \quad (\text{A-7})$$

respectively. The two complementary covariances are zero; that is,

$$\overline{n_+(t) \underline{n}(t-\tau)} = 0 , \quad \overline{\underline{n}(t) n_+(t-\tau)} = 0 . \quad (\text{A-8})$$

These results follow from the fact that transfer function $2 U(f)$ in figure A-1 is single-sided; that is, $U(f) = 0$ for $f < 0$.

A situation that frequently arises in practice is where $n(t)$ is a noise process with a spectrum $G_n(f)$ that is flat in a broad band of width W in the neighborhood of f_0 , which essentially covers the signals and filters of interest. The spectra of the various processes are illustrated in figure A-2, where N_d is the double-sided noise spectral density level of $n(t)$ in the neighborhood of $\pm f_0$. The spectrum $G_{\underline{n}}(f)$ of the complex envelope $\underline{n}(t)$ is flat in the neighborhood of $f = 0$ and has level $4 N_d$ watts/Hz. Therefore, its covariance

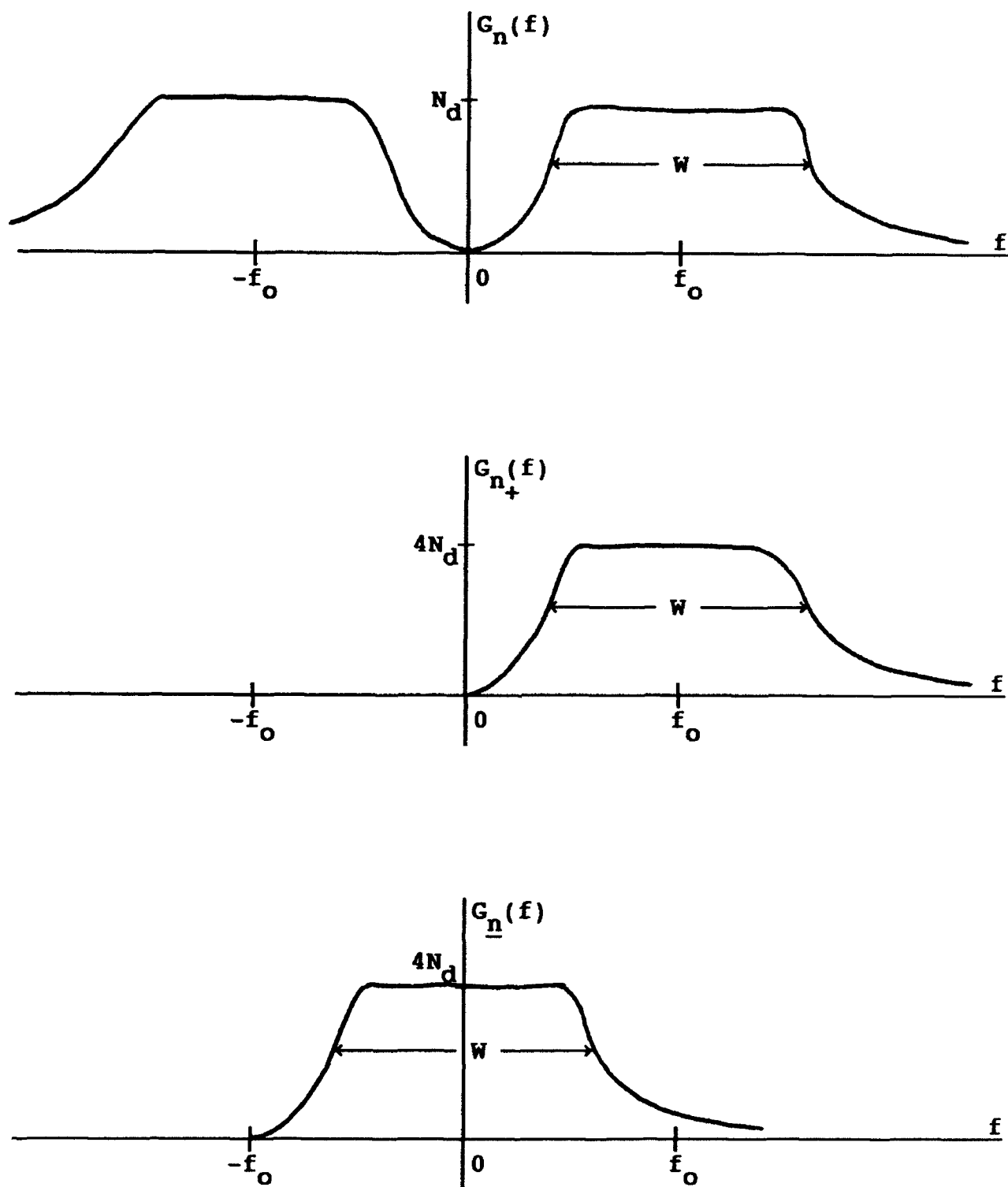


Figure A-2. Spectra of Analytic and Complex Envelope Processes

$$R_{\underline{n}}(\tau) = \int df \exp(i2\pi f\tau) G_{\underline{n}}(f) \quad (A-9)$$

is (relative to the signal and filter time functions) a sharp pulse centered at $\tau = 0$, with area $G_{\underline{n}}(0) = 4 N_d$. Then, a good approximation for many purposes is to say that

$$R_{\underline{n}}(\tau) \approx 4 N_d \delta(\tau) . \quad (A-10)$$

There is no need to let W and f_o tend to infinity in order to utilize this result; it only requires that W be somewhat larger than the bandwidths of the signals and filters of interest.

In terms of the one-sided noise spectral density level of $n(t)$, that is, $N_o = 2 N_d$, approximation (A-10) becomes

$$R_{\underline{n}}(\tau) \approx 2 N_o \delta(\tau) . \quad (A-11)$$

This result may be compared with [19; page 48, (3.10) - (3.1) and page 72, (6.21) - (6.22)], where his N is our N_o .

It should be noted that original covariance $R_n(\tau)$ in (A-1) cannot be recovered from this approximation. That is, via (A-4),

$$R_n(\tau) = \frac{1}{2} \operatorname{Re} \left(R_{\underline{n}}(\tau) \exp(i2\pi f_o \tau) \right) = \quad (A-12)$$

$$\stackrel{?}{=} \frac{1}{2} \operatorname{Re} \left(4 N_d \delta(\tau) \exp(i2\pi f_o \tau) \right) = 2 N_d \delta(\tau) , \quad (A-13)$$

which is incorrect. The flaw is that the use of $\delta(\tau)$, which is tantamount to $W \rightarrow \infty$, must be accompanied by having let $f_o \rightarrow \infty$, in addition. That is, $\exp(i2\pi f_o \tau)$ in (A-12) must vary as fast as $R_{\underline{n}}(\tau)$ in order for the narrowband representation to be valid. The correct end result for $f_o = W/2$ is $R_n(\tau) \rightarrow N_d \delta(\tau)$ as $W \rightarrow \infty$.

APPENDIX B. CHARACTERISTIC FUNCTION OF QUADRATIC AND LINEAR FORM IN CORRELATED NONZERO-MEAN GAUSSIAN RANDOM VARIABLES

The $N \times 1$ real vector X is composed of correlated Gaussian components with mean vector E and covariance matrix C ; that is,

$$\bar{X} = E, \quad \text{Cov}(X) = \overline{(X - E)(X - E)^T} = C. \quad (\text{B-1})$$

Covariance matrix C is $N \times N$, real, symmetric, and nonnegative definite. Real $N \times 1$ vector E is completely arbitrary.

We shall be interested in obtaining the characteristic function of the quadratic and linear real form in X given by

$$q = q(X) = X^T B X + 2 A^T X, \quad (\text{B-2})$$

where $N \times N$ matrix B is real, symmetric, and positive definite, while real $N \times 1$ vector A is completely arbitrary. By completing the square, (B-2) can be written in the alternative form

$$q = (X + B^{-1} A)^T B (X + B^{-1} A) - A^T B^{-1} A. \quad (\text{B-3})$$

Therefore, the minimum possible value of q is $-A^T B^{-1} A$. The case of an indefinite matrix B is undertaken below (B-42).

Random vector X is composed of correlated Gaussian random variables; its probability density function is [20; section 8-3]

$$p(X) = (2\pi)^{-N/2} (\det C)^{-1/2} \exp\left[-\frac{1}{2}(X - E)^T C^{-1} (X - E)\right]. \quad (\text{B-4})$$

The characteristic function of interest is then given by

$$f_q(\xi) = \overline{\exp(i\xi q)} = \int dX p(X) \exp[i\xi q(X)]. \quad (\text{B-5})$$

For convenience, we introduce variable

$$z = i2\xi . \quad (B-6)$$

Substitution of (B-2) and (B-4) in (B-5) yields

$$\begin{aligned} f_q(\xi) &= (2\pi)^{-N/2} (\det C)^{-1/2} \times \\ &\times \int dX \exp \left[-\frac{1}{2} (X - E)^T C^{-1} (X - E) + \frac{1}{2} z X^T B X + z A^T X \right] = \\ &= (2\pi)^{-N/2} (\det C)^{-1/2} \times \\ &\times \int dX \exp \left[-\frac{1}{2} X^T (C^{-1} - zB) X + (C^{-1} E + zA)^T X - \frac{1}{2} E^T C^{-1} E \right] . \quad (B-7) \end{aligned}$$

Now we have the result [20; section 8-3]

$$\int dX \exp \left[-\frac{1}{2} X^T U X + v^T X \right] = (2\pi)^{N/2} (\det U)^{-1/2} \exp \left[\frac{1}{2} v^T U^{-1} v \right] . \quad (B-8)$$

This enables the reduction of the integral in (B-7) to

$$\begin{aligned} f_q(\xi) &= (\det C)^{-1/2} (\det(C^{-1} - zB))^{-1/2} \times \\ &\times \exp \left[\frac{1}{2} (C^{-1} E + zA)^T (C^{-1} - zB)^{-1} (C^{-1} E + zA) - \frac{1}{2} E^T C^{-1} E \right] . \quad (B-9) \end{aligned}$$

Although closed form, (B-9) is not too useful numerically because it requires an inverse of matrix $C^{-1} - zB$ for each new value of z ($= i2\xi$) of interest.

A much more compact and useful form of (B-9) can be obtained by means of the following procedure. For the given covariance matrix C and quadratic-form matrix B , solve the generalized characteristic-value equation

$$C Q = B^{-1} Q \Lambda \quad (B-10)$$

for $N \times N$ eigenvalue matrix Λ and normalized modal matrix Q , where

$$\Lambda = \text{diag}[\lambda_1 \dots \lambda_N] , \quad Q = [V_1 \dots V_N] , \quad V_n = [v_{n1} \dots v_{nN}]^T , \quad (B-11)$$

and $N \times 1$ column vector V_n is the n -th eigenvector with scalar components $\{v_{np}\}$. A procedure for obtaining the solutions Λ and Q to (B-10) is given in appendix E, when matrix B is positive definite.

(It should be noted that (B-10) does not have exactly the same solutions as the equation $B C Q' = Q' \Lambda'$; in fact, $B C$ is not generally symmetric, even if B is diagonal [18; page 79, (253)]. The connection is $\Lambda' = \Lambda$, $Q' = Q D$, with D diagonal.)

Then, we have the two very important properties [18; pages 74 - 77] of solutions Q and Λ :

$$Q^T B^{-1} Q = I , \quad Q^T C Q = \Lambda . \quad (B-12)$$

By means of these two relations, a number of simplifications of (B-9) are possible. We begin by observing that

$$B^{-1} = Q^{-T} Q^{-1} , \quad B = Q Q^T , \quad C = Q^{-T} \Lambda Q^{-1} , \quad C^{-1} = Q \Lambda^{-1} Q^T . \quad (B-13)$$

(Notice that $Q^T Q \neq I$.) We now employ (B-13) in (B-9) to get

$$C^{-1} - zB = Q (\Lambda^{-1} - zI) Q^T , \quad (C^{-1} - zB)^{-1} = Q^{-T} (\Lambda^{-1} - zI)^{-1} Q^{-1} . \quad (B-14)$$

At the same time, using (B-13),

$$\begin{aligned} \det(C) \det(C^{-1} - z B) &= \det(I - z B C) = \det(I - z Q \Lambda Q^{-1}) = \\ &= \det(I - z \Lambda) = \prod_{n=1}^N (1 - z \lambda_n) . \end{aligned} \quad (B-15)$$

Meanwhile, if we denote twice the argument of the exponential in (B-9) by t , we have, by means of (B-14), the alternative expression

$$t = (C^{-1} E + zA)^T Q^{-T} (\Lambda^{-1} - zI)^{-1} Q^{-1} (C^{-1} E + zA) - E^T C^{-1} E . \quad (B-16)$$

Now, by means of (B-13), develop the term

$$\begin{aligned} Q^{-1} (C^{-1} E + zA) &= Q^{-1} C^{-1} E + zQ^{-1} A = \Lambda^{-1} Q^T E + zQ^T B^{-1} A = \\ &= \Lambda^{-1} \underline{E} + z\underline{A} , \end{aligned} \quad (B-17)$$

where we have defined $N \times 1$ vectors

$$\begin{aligned} \underline{E} &\equiv Q^T E = [\varepsilon_1 \dots \varepsilon_N]^T , \quad \varepsilon_n = V_n^T E , \\ \underline{A} &\equiv Q^T B^{-1} A = [\alpha_1 \dots \alpha_N]^T , \quad \alpha_n = V_n^T B^{-1} A . \end{aligned} \quad (B-18)$$

Then t in (B-16) becomes, again using (B-13),

$$\begin{aligned} t &= (\Lambda^{-1} \underline{E} + z\underline{A})^T (\Lambda^{-1} - zI)^{-1} (\Lambda^{-1} \underline{E} + z\underline{A}) - E^T Q \Lambda^{-1} Q^T E = \\ &= (\Lambda^{-1} \underline{E} + z\underline{A})^T (\Lambda^{-1} - zI)^{-1} (\Lambda^{-1} \underline{E} + z\underline{A}) - \underline{E}^T \Lambda^{-1} \underline{E} = \\ &= \sum_{n=1}^N \frac{(\varepsilon_n / \lambda_n + z \alpha_n)^2}{1 / \lambda_n - z} - \sum_{n=1}^N \frac{\varepsilon_n^2}{\lambda_n} = \\ &= z \sum_{n=1}^N \frac{(\varepsilon_n + \alpha_n)^2}{1 - z \lambda_n} - z \sum_{n=1}^N \alpha_n^2 . \end{aligned} \quad (B-19)$$

By collecting all these results together, we can express the characteristic function in (B-9) in the compact form

$$f_Q(\xi) = \left[\prod_{n=1}^N (1 - i2\xi\lambda_n) \right]^{-\frac{1}{2}} \exp \left[i\xi \sum_{n=1}^N \frac{(\epsilon_n + \alpha_n)^2}{1 - i2\xi\lambda_n} - i\xi \sum_{n=1}^N \alpha_n^2 \right]. \quad (B-20)$$

The last constant in (B-20) has the alternative representation

$$- \sum_{n=1}^N \alpha_n^2 = - \underline{A}^T \underline{A} = - \underline{A}^T \underline{B}^{-1} \underline{Q} \underline{Q}^T \underline{B}^{-1} \underline{A} = - \underline{A}^T \underline{B}^{-1} \underline{A}, \quad (B-21)$$

where we used (B-18) and (B-13); this is just the residual constant encountered in (B-3). Thus, the characteristic function of the leading (nonnegative) quadratic form in (B-3) is just (B-20) without the second summation inside the exponential.

Additional relations available from (B-18) and (B-13) are

$$\sum_{n=1}^N \epsilon_n^2 = \underline{E}^T \underline{E} = \underline{E}^T \underline{Q} \underline{Q}^T \underline{E} = \underline{E}^T \underline{B} \underline{E},$$

$$\sum_{n=1}^N \epsilon_n \alpha_n = \underline{E}^T \underline{A} = \underline{E}^T \underline{Q} \underline{Q}^T \underline{B}^{-1} \underline{A} = \underline{E}^T \underline{A} = \underline{A}^T \underline{E}. \quad (B-22)$$

However, we need the constants $\{\alpha_n\}$ and $\{\epsilon_n\}$ individually for characteristic function (B-20). The quantities $\{2\lambda_n\}$ and $\{(\epsilon_n + \alpha_n)^2\}$ should be computed once and stored in arrays prior to the computation of $f_Q(\xi)$ in (B-20) at the numerous ξ values required.

Strictly, the leading product in (B-20) is a product of N principal-value square roots. However, it can be evaluated

numerically as a single square root of a product of the N terms $\{1 - i2\xi\lambda_n\}$, provided that the location of this product is tracked in the complex plane from the point 1 when $\xi = 0$. See, for example, [21; page B-5, lines 220 - 270].

If the linear form in (B-2) is absent, then vector $A = 0$, $\underline{A} = 0$, and $\alpha_n = 0$ for all n . However, we still need quantities

$$\epsilon_n = V_n^T E \quad \text{for } 1 \leq n \leq N, \quad (\text{B-23})$$

for the exponent in (B-20), thereby necessitating calculation of eigenvectors $\{V_n\}$. Only when mean vector E is also zero (in addition to A) do just the eigenvalues $\{\lambda_n\}$ of (B-10) suffice for calculation of characteristic function $f_q(\xi)$ in (B-20).

In this latter case, the matrix $B C$ can be considered instead, since it has the same eigenvalues $\{\lambda_n\}$. This follows from the following manipulations: (B-10) and (B-11) can be written as

$$C [V_1 \dots V_N] = B^{-1} [V_1 \dots V_N] \text{diag}[\lambda_1 \dots \lambda_N], \quad (\text{B-24})$$

or

$$C V_n = B^{-1} V_n \lambda_n \quad \text{for } 1 \leq n \leq N. \quad (\text{B-25})$$

Therefore

$$0 = (C - B^{-1} \lambda_n) V_n = B^{-1} (B C - \lambda_n I) V_n \quad (\text{B-26})$$

or, since B is positive definite,

$$\det(B C - \lambda_n I) = 0 \quad \text{for } 1 \leq n \leq N. \quad (\text{B-27})$$

Thus, $\{\lambda_n\}$ are the eigenvalues of matrix $B C$ as well as (B-10).

CUMULANTS OF q

Before determining the cumulants of q , we present a few relations which will enable easier manipulations of various matrices encountered below. From (B-13), there follows

$$\begin{aligned} B C &= Q \Lambda Q^{-1}, & B C B &= Q \Lambda Q^T, \\ B C B C &= Q \Lambda^2 Q^{-1}, & B C B C B &= Q \Lambda^2 Q^T. \end{aligned} \quad (B-28)$$

Therefore

$$\begin{aligned} \text{tr}(B C) &= \text{tr}(\Lambda Q^{-1} Q) = \text{tr}(\Lambda) = \sum_{n=1}^N \lambda_n, \\ \text{tr}((B C)^2) &= \text{tr}(\Lambda^2 Q^{-1} Q) = \text{tr}(\Lambda^2) = \sum_{n=1}^N \lambda_n^2. \end{aligned} \quad (B-29)$$

Now, if we expand the natural logarithm of the general characteristic function $f_q(\xi)$ in (B-20) in a power series in $i\xi$, we can easily pick off the cumulants as

$$\begin{aligned} \chi_q(1) &= \mu_q = \sum_{n=1}^N (\lambda_n + \epsilon_n^2 + 2\epsilon_n \alpha_n) = \text{tr}(B C) + E^T B E + 2 A^T E, \\ \chi_q(p) &= 2^{p-1} (p-1)! \sum_{n=1}^N \lambda_n^{p-1} (\lambda_n + p(\epsilon_n + \alpha_n)^2) \text{ for } p \geq 2. \end{aligned} \quad (B-30)$$

Here, we used (B-29) and (B-22). In particular, the variance of q is

$$\begin{aligned} \chi_q(2) &= \sigma_q^2 = 2 \sum_{n=1}^N (\lambda_n^2 + 2\lambda_n(\epsilon_n + \alpha_n)^2) = \\ &= 2 \text{tr}((B C)^2) + 4 (B E + A)^T C (B E + A). \end{aligned} \quad (B-31)$$

This latter relation follows from these manipulations:

$$\begin{aligned}
 \sum_{n=1}^N \lambda_n (\varepsilon_n + \alpha_n)^2 &= (\underline{E} + \underline{A})^T \Lambda (\underline{E} + \underline{A}) = \\
 &= (\underline{E} + \underline{B}^{-1} \underline{A})^T \underline{Q} \Lambda \underline{Q}^T (\underline{E} + \underline{B}^{-1} \underline{A}) = (\underline{E} + \underline{B}^{-1} \underline{A})^T \underline{B} \underline{C} \underline{B} (\underline{E} + \underline{B}^{-1} \underline{A}) = \\
 &= (\underline{B} \underline{E} + \underline{A})^T \underline{C} (\underline{B} \underline{E} + \underline{A}) . \tag{B-32}
 \end{aligned}$$

Here, we used (B-18) and (B-28).

ALTERNATIVE DERIVATION OF CHARACTERISTIC FUNCTION (B-20)

We start again with conditions (B-1) and (B-2). Then, solve (B-10) for $\underline{Q}$ and Λ , as before. Now, consider the linear transformation of Gaussian random vector $\underline{X}$ according to

$$\underline{Y} = \underline{Q}^T (\underline{X} - \underline{E}) = [y_1 \dots y_N]^T . \tag{B-33}$$

Then $N \times 1$ Gaussian vector $\underline{Y}$ has mean $\bar{\underline{Y}} = 0$ and covariance matrix

$$\overline{\underline{Y} \underline{Y}^T} = \underline{Q}^T \overline{(\underline{X} - \underline{E})(\underline{X} - \underline{E})^T} \underline{Q} = \underline{Q}^T \underline{C} \underline{Q} = \Lambda , \tag{B-34}$$

where we used (B-1) and (B-12). This diagonal matrix means that

$$\overline{y_n} = 0 , \quad \overline{y_n y_m} = \lambda_n \delta_{nm} ; \tag{B-35}$$

that is, $\{y_n\}$ are zero-mean uncorrelated (and therefore independent) Gaussian random variables. This is the key to this development.

At the same time, solving (B-33) for X , we have, with (B-12),

$$X = E + Q^{-T} Y = E + B^{-1} Q Y . \quad (B-36)$$

Then quadratic form (B-2) can be expressed as

$$\begin{aligned} q &= (E + B^{-1} Q Y)^T B (E + B^{-1} Q Y) + 2 A^T (E + B^{-1} Q Y) = \\ &= Y^T Y + 2 (E^T Q + A^T B^{-1} Q) Y + E^T B E + 2 A^T E = \\ &= Y^T Y + 2 (\underline{E}^T + \underline{A}^T) Y + E^T B E + 2 A^T E = \\ &= \sum_{n=1}^N \left(y_n^2 + 2(\varepsilon_n + \alpha_n) y_n + \varepsilon_n^2 + 2\varepsilon_n \alpha_n \right) , \end{aligned} \quad (B-37)$$

where we used (B-12), (B-18), and (B-22). Therefore, using the independence property derived in (B-34) and (B-35), the characteristic function of q is

$$\begin{aligned} f_q(\xi) &= \overline{\exp(i\xi q)} = \prod_{n=1}^N \left\{ \int \frac{dy_n}{(2\pi\lambda_n)^{1/2}} \exp\left(-\frac{y_n^2}{2\lambda_n}\right) \times \right. \\ &\quad \left. \times \exp\left[i\xi\left(y_n^2 + 2(\varepsilon_n + \alpha_n)y_n + \varepsilon_n^2 + 2\varepsilon_n\alpha_n\right)\right] \right\} = \\ &= \prod_{n=1}^N \left\{ \left(1 - i2\xi\lambda_n\right)^{-1/2} \exp\left[i\xi \frac{(\varepsilon_n + \alpha_n)^2}{1 - i2\xi\lambda_n} - i\xi\alpha_n^2\right] \right\} , \end{aligned} \quad (B-38)$$

which is equal to (B-20).

PROPERTIES OF EIGENVECTORS $\{V_n\}$

Consider the representation of modal matrix Q in terms of eigenvectors $\{V_n\}$ in (B-11). Then the first relation in (B-12) and the second relation in (B-13) yield, respectively,

$$V_n^T B^{-1} V_m = \delta_{nm} , \quad \sum_{n=1}^N V_n V_n^T = B . \quad (B-39)$$

Similarly, the second relation in (B-12) and the fourth relation in (B-13) yield, respectively,

$$V_n^T C V_m = \lambda_n \delta_{nm} , \quad \sum_{n=1}^N \frac{1}{\lambda_n} V_n V_n^T = C^{-1} . \quad (B-40)$$

Thus, there are two orthogonality relations satisfied by the eigenvectors $\{V_n\}$, namely the leading relations in (B-39) and (B-40). If, in addition, matrix B or C is diagonal, then the trailing relations in (B-39) and (B-40) yield an additional orthogonality property.

In terms of the components (B-11), the relations above become

$$\sum_{k,j=1}^N v_{nk} (B^{-1})_{kj} v_{mj} = \delta_{nm} , \quad \sum_{n=1}^N v_{nk} v_{nj} = (B)_{kj} , \quad (B-41)$$

and

$$\sum_{k,j=1}^N v_{nk} (C)_{kj} v_{mj} = \lambda_n \delta_{nm} , \quad \sum_{n=1}^N \frac{1}{\lambda_n} v_{nk} v_{nj} = (C^{-1})_{kj} . \quad (B-42)$$

GENERAL SYMMETRIC MATRIX B

All the results above are based on the premise that matrix B in the quadratic form in (B-2) is positive definite. However, there are examples where this is not the situation, in which case the corresponding procedure for solution of (B-10) in appendix E (that was mentioned under (B-11)) is not applicable. We now give an alternative procedure for obtaining the characteristic function of q in (B-2) for any real matrix B, whether definite or not. Matrix B can be taken symmetric without loss of generality, since only the symmetric part of B is active in (B-2).

We first observe that the N×N covariance matrix C in (B-1) is always nonnegative definite because

$$V^T C V = V^T \overline{(X - E)(X - E)^T} V = \overline{(V^T(X - E))^2} \geq 0 \quad (B-43)$$

for any N×1 real vector V. We shall presume that C is positive definite. Instead of solving (B-10), we solve the alternative generalized real characteristic-value matrix equation

$$B Q = C^{-1} Q \Lambda \quad (B-44)$$

for N×N normalized modal matrix Q and diagonal eigenvalue matrix Λ . A procedure for this solution is given in appendix E; see (E-14) and sequel.

Then, these solutions satisfy [18; pages 74 - 77]

$$Q^T C^{-1} Q = I, \quad (B-45)$$

$$Q^T B Q = \Lambda \equiv \text{diag}[\lambda_1 \ . \ . \ . \ \lambda_N] . \quad (B-46)$$

Now let linearly transformed $N \times 1$ random vector

$$Z = Q^{-1} (X - E) \equiv [z_1 \ . \ . \ . \ z_N]^T . \quad (B-47)$$

Then mean $\bar{Z} = 0$ and covariance matrix

$$\overline{Z Z^T} = Q^{-1} \overline{(X - E)(X - E)^T} Q^{-T} = Q^{-1} C Q^{-T} = I , \quad (B-48)$$

upon use of (B-1) and (B-45). Solve (B-47) for X , obtaining

$$X = Q Z + E . \quad (B-49)$$

Now, substitute (B-49) into quadratic and linear form (B-2):

$$\begin{aligned} q &= (Q Z + E)^T B (Q Z + E) + 2 A^T (Q Z + E) = \\ &= (Z + Q^{-1} E)^T Q^T B Q (Z + Q^{-1} E) + 2 A^T Q (Z + Q^{-1} E) = \\ &= (Z + \underline{E})^T \underline{A} (Z + \underline{E}) + 2 \underline{A}^T (Z + \underline{E}) , \end{aligned} \quad (B-50)$$

where we used (B-46) and defined two auxiliary $N \times 1$ vectors

$$\underline{E} = Q^{-1} E \equiv [\epsilon_1 \ . \ . \ . \ \epsilon_N]^T \quad (B-51)$$

and

$$\underline{A} = Q^T A \equiv [\alpha_1 \ . \ . \ . \ \alpha_N]^T . \quad (B-52)$$

Therefore, (B-50) can now be expressed as

$$q = \sum_{n=1}^N (\lambda_n z_n + \lambda_n \epsilon_n + 2 \alpha_n)(z_n + \epsilon_n) . \quad (B-53)$$

The characteristic function of q can now be readily found, with the assistance of covariance property (B-48), in its most compact form

$$\begin{aligned}
f_q(\xi) &= \overline{\exp(i\xi q)} = \\
&= \prod_{n=1}^N \left(\int dz_n (2\pi)^{-\frac{1}{2}} \exp\left[-\frac{1}{2}z_n^2 + i\xi(\lambda_n z_n + \lambda_n \epsilon_n + 2\alpha_n)(z_n + \epsilon_n)\right] \right) = \\
&= \left[\prod_{n=1}^N (1 - i2\xi\lambda_n) \right]^{-\frac{1}{2}} \exp\left[i\xi \sum_{n=1}^N \frac{\lambda_n \epsilon_n^2 + 2\epsilon_n \alpha_n + i2\xi\alpha_n^2}{1 - i2\xi\lambda_n} \right] . \quad (B-54)
\end{aligned}$$

In summary, the following computations must be performed: solve (B-44) for $N \times N$ matrices Q and A ; compute $N \times 1$ vectors $\underline{E}$ and $\underline{A}$ by means of (B-51) and (B-52), respectively; and evaluate $f_q(\xi)$ at desired ξ values by use of (B-54). The only inverse matrix required is that of modal matrix Q ; the solution of (B-44) does not actually require calculation of C^{-1} , as will be seen in appendix E, (E-14) and sequel.

If mean vector E in (B-1) is zero and/or if linear form vector A in (B-2) is zero, the corresponding calculations in (B-51) and (B-52) can be circumvented. If both E and A are zero, the exp term in (B-54) is absent altogether.

In this latter case, only the eigenvalues $\{\lambda_n\}$ of (B-44) need be determined. If we let V_n be the n -th eigenvector (column) of Q , equation (B-44) takes the form

$$B V_n = C^{-1} V_n \lambda_n \quad \text{for } 1 \leq n \leq N . \quad (B-55)$$

This can be manipulated into

$$(B - \lambda_n C^{-1}) V_n = 0 , \quad (B-56)$$

meaning that

$$\det(B - \lambda_n C^{-1}) = 0 \quad (B-57)$$

or, since $\det(C) \neq 0$, that

$$\det(B C - \lambda_n I) = 0 . \quad (B-58)$$

That is, $\{\lambda_n\}$ are the eigenvalues of matrix $B C$. This is a much simpler numerical task than solving (B-44) for both Q and Λ .

If none of the eigenvalues $\{\lambda_n\}$ are zero or near zero, then (B-46) furnishes inverse matrix $Q^{-1} = \Lambda^{-1} Q^T B$ and an alternative to (B-51), namely

$$\begin{aligned} \underline{E} &= \Lambda^{-1} Q^T B E , \\ \epsilon_n &= \frac{1}{\lambda_n} V_n^T B E \quad \text{for } 1 \leq n \leq N . \end{aligned} \quad (B-59)$$

Also, from (B-52), regardless of the sizes of the eigenvalues,

$$\alpha_n = V_n^T A \quad \text{for } 1 \leq n \leq N . \quad (B-60)$$

The interrelationships between Q and Λ in (B-44) and the corresponding matrices Q and Λ in (B-10) are derived in appendix E, (E-24) and sequel. The identity of characteristic functions (B-20) and (B-54) is also verified there.

PROPERTIES OF EIGENVECTORS $\{V_n\}$

Relations (B-45) and (B-46) yield the following orthogonality properties between the eigenvectors:

$$V_n^T C^{-1} V_m = \delta_{nm} , \quad V_n^T B V_m = \lambda_n \delta_{nm} . \quad (B-61)$$

Also, since (B-45) and (B-46) can be alternatively expressed as $C = Q Q^T$ and $B^{-1} = Q \Lambda^{-1} Q^T$, we have

$$\sum_{n=1}^N V_n V_n^T = C , \quad \sum_{n=1}^N \frac{1}{\lambda_n} V_n V_n^T = B^{-1} . \quad (B-62)$$

As a special case of (B-61), it can be seen that $\lambda_n = V_n^T B V_n$; therefore, if matrix B is nonnegative definite, then $\lambda_n \geq 0$ for all n. The following example demonstrates that indefinite symmetric B matrices can lead to negative eigenvalues, even when matrix C is symmetric and positive definite:

$$B = \begin{bmatrix} 4 & -6 \\ -6 & 7 \end{bmatrix} , \quad C = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} ; \quad (B-63)$$

the eigenvalue and eigenvector solutions to (B-44) are

$$\Lambda = \begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix} , \quad Q = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} . \quad (B-64)$$

CUMULANTS OF q

By taking the logarithm of characteristic function $f_q(\xi)$ in (B-54) and expanding in a power series in $i\xi$, the cumulants of quadratic and linear form q are found to be

$$\chi_q^{(m)} = \left\{ \begin{array}{ll} \sum_{n=1}^N \left[\lambda_n (1 + \epsilon_n^2) + 2\epsilon_n \alpha_n \right] & \text{for } m = 1 \\ (m-1)! \, 2^{m-1} \sum_{n=1}^N \lambda_n^{m-2} \left[\lambda_n^2 + m(\lambda_n \epsilon_n + \alpha_n)^2 \right] & \text{for } m \geq 2 \end{array} \right\}. \quad (\text{B-65})$$

APPENDIX C. CHARACTERISTIC FUNCTION OF OUTPUT γ FOR GENERAL TRANSMITTED SIGNAL ENERGIES $\{\tilde{E}_k\}$ AND RECEIVER WEIGHTS $\{A_k\}$

The conditional characteristic function of processor output γ , for arbitrary transmitted signal energies $\{\tilde{E}_k\}$ and receiver weights $\{A_k\}$, was given in (27) as

$$f_c(\xi) = \prod_{k=1}^K \left(1 - i\xi 2\sigma_k^2\right)^{-1} \exp\left[i\xi \sum_{k=1}^K w_k r_k^2\right], \quad (C-1)$$

where

$$\sigma_k^2 = 2 N_O A_k^2 \tilde{E}_k, \quad w_k = \frac{4 A_k^2 \tilde{E}_k}{1 - i\xi 4 N_O A_k^2 \tilde{E}_k}. \quad (C-2)$$

It is important to notice that the coefficients $\{w_k\}$ in the summation in (C-1) are complex and are functions of argument ξ .

Now, we want to find the unconditional characteristic function of γ , after averaging over the statistics of amplitude scalings $\{r_k\}$. From (43) - (45),

$$r_k^2 = \sum_{m=1}^M \left[c_m(t_k, f_k) + g_m(t_k, f_k) \right]^2 = \sum_{m=1}^M x_{mk}^2, \quad (C-3)$$

where Gaussian random variables

$$x_{mk} = c_m(t_k, f_k) + g_m(t_k, f_k) \quad \text{for } 1 \leq m \leq M, \quad 1 \leq k \leq K. \quad (C-4)$$

Therefore, the exponential term in (C-1) becomes

$$\exp\left[i\xi \sum_{k=1}^K w_k r_k^2\right] = \exp\left[i\xi X^T \tilde{W} X\right], \quad (C-5)$$

where $MK \times 1$ random Gaussian real column vector

$$X = [x_{11} \cdots x_{M1} \cdots x_{1k} \cdots x_{Mk} \cdots x_{1K} \cdots x_{MK}]^T, \quad (C-6)$$

and $MK \times MK$ diagonal matrix of complex elements

$$\tilde{W} = \text{diag}[w_1 \cdots w_1 \cdots \underbrace{w_k \cdots w_k}_{M \text{ terms}} \cdots w_K \cdots w_K] . \quad (C-7)$$

The $MK \times MK$ covariance matrix of vector X is

$$K_X = \text{Cov}\{X\} = \overline{(X - \bar{X})(X - \bar{X})^T} \quad (C-8)$$

which is known.

The unconditional characteristic function of γ is available from (C-1) and (C-5) in the form

$$\begin{aligned} f_Y(\xi) &= \overline{f_C(\xi)} = \prod_{k=1}^K \left(1 - i\xi 2\sigma_k^2\right)^{-1} \overline{\exp[i\xi X^T \tilde{W} X]} = \\ &= \prod_{k=1}^K \left(1 - i\xi 2\sigma_k^2\right)^{-1} \int dX (2\pi)^{-MK/2} \left(\det K_X\right)^{-1/2} \times \\ &\quad \times \exp\left[-\frac{1}{2}(X - \bar{X})^T K_X^{-1} (X - \bar{X}) + i\xi X^T \tilde{W} X\right] = \\ &= \prod_{k=1}^K \left(1 - i\xi 2\sigma_k^2\right)^{-1} \left(\det(I - i\xi 2 \tilde{W} K_X)\right)^{-1/2} \times \\ &\quad \times \exp\left[\frac{1}{2} \bar{X}^T (I - i\xi 2 \tilde{W} K_X)^{-1} K_X^{-1} \bar{X} - \frac{1}{2} \bar{X}^T K_X^{-1} \bar{X}\right] . \end{aligned} \quad (C-9)$$

The matrix $\tilde{W}$ depends on ξ through coefficients $\{w_k\}$ in (C-2).

Therefore, in order to compute (C-9), it is necessary to invert a complex $MK \times MK$ matrix for each ξ of interest. However, if mean $\bar{X}$ is zero, only the determinant in (C-9) need be evaluated.

UNCORRELATED FADING COMPONENTS $\{g_m(t, f)\}$

In this subsection, we presume that the fading components in model (45) are uncorrelated and therefore independent. This means that, in (C-4), random variables $x_{m1}, \dots, x_{mK}$ are independent of $x_{n1}, \dots, x_{nK}$ for $m \neq n$. Reference to (C-1) and (C-3) then yields the characteristic function of output γ in the form

$$f_Y(\xi) = \overline{f_C(\xi)} = \prod_{k=1}^K \left(1 - i\xi 2\sigma_k^2\right)^{-1} E(\xi), \quad (C-10)$$

where

$$\begin{aligned} E(\xi) &\equiv \overline{\exp\left[i\xi \sum_{k=1}^K w_k r_k^2\right]} = \overline{\exp\left[i\xi \sum_{k=1}^K w_k \sum_{m=1}^M x_{mk}^2\right]} = \\ &= \prod_{m=1}^M \overline{\exp\left[i\xi \sum_{k=1}^K w_k x_{mk}^2\right]}. \end{aligned} \quad (C-11)$$

Now, let $K \times 1$ real random vector X_m and $K \times K$ complex diagonal matrix W be defined according to

$$X_m = [x_{m1} \ \dots \ x_{mK}]^T, \quad W = \text{diag}[w_1 \ \dots \ w_K]. \quad (C-12)$$

Also, let the covariance matrix of X_m be denoted by K_m , which is a $K \times K$ real matrix, for $1 \leq m \leq M$. Then, the m -th term in (C-11) becomes

$$\begin{aligned} \overline{\exp\left[i\xi \sum_{k=1}^K w_k x_{mk}^2\right]} &= \overline{\exp\left[i\xi X_m^T W X_m\right]} = \int dX_m (2\pi)^{-K/2} \times \\ &\times (\det K_m)^{-1/2} \exp\left[-\frac{1}{2} (X_m - \bar{X}_m)^T K_m^{-1} (X_m - \bar{X}_m) + i\xi X_m^T W X_m\right] = \end{aligned}$$

$$\left(\det(I - i\xi 2WK_m)\right)^{-\frac{1}{2}} \exp\left[\frac{1}{2} \bar{X}_m^T \left(I - i\xi 2WK_m\right)^{-1} K_m^{-1} \bar{X}_m - \frac{1}{2} \bar{X}_m^T K_m^{-1} \bar{X}_m\right] \quad (C-13)$$

Finally, the desired unconditional characteristic function of processor output γ follows from a combination of (C-10), (C-11), and (C-13) as

$$f_Y(\xi) = \prod_{k=1}^K \left(1 - i\xi 2\sigma_k^2\right)^{-1} \prod_{m=1}^M \left\{ \left(\det(I - i\xi 2WK_m)\right)^{-\frac{1}{2}} \times \right. \\ \left. \times \exp\left[\frac{1}{2} \bar{X}_m^T \left(I - i\xi 2WK_m\right)^{-1} K_m^{-1} \bar{X}_m - \frac{1}{2} \bar{X}_m^T K_m^{-1} \bar{X}_m\right] \right\}. \quad (C-14)$$

The major computational burden here consists of the inverses of M complex $K \times K$ matrices for each ξ value of interest. However, if the M covariance matrices $\{K_m\}$ are all identical, the task is considerably simplified. Alternatively, if all the means $\{\bar{X}_m\}$ are zero, then the evaluation is limited to M determinants of the complex $M \times M$ matrices $I - i\xi 2WK_m$. The $K \times K$ complex matrix W is defined by (C-12) and (C-2).

APPENDIX D. DERIVATION OF PROBABILITY DENSITY AND
EXCEEDANCE DISTRIBUTION FROM CHARACTERISTIC FUNCTION

The general forms in (136) and (158) of the characteristic function of processor output γ can be written as

$$f_{\gamma}(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{m=1}^M \prod_{k=1}^K \left(1 - i2\xi(1 + T_{mk}) \right) \right]^{-\frac{1}{2}} \times \\ \times \exp \left[i\xi \sum_{m=1}^M \sum_{k=1}^K \frac{e_{mk}}{1 - i2\xi(1 + T_{mk})} \right], \quad (D-1)$$

where, respectively

$$T_{mk} = \frac{2E_{1m}}{N_0} \lambda_k^{(m)} \quad \text{or} \quad T_{mk} = \frac{2E_{1m}}{N_0} \lambda_k, \quad e_{mk} = \varepsilon_k^{(m)2}. \quad (D-2)$$

We want to determine the corresponding probability density function and exceedance distribution function. There is a possible numerical problem using Fourier transforms on (D-1) directly, since its asymptotic decay is only proportional to ξ^{-K} as $\xi \rightarrow \infty$, which makes accurate results difficult to achieve for small K .

We will expand (D-1) in a power series in $(1 - i2\xi)^{-1}$ and thereby obtain not only an asymptotic expansion of $f_{\gamma}(\xi)$, but in fact, a convergent series which can be transformed term-by-term to yield the probability density function of γ . Integration then yields the exceedance distribution function. Furthermore, it will turn out that all the series coefficients are positive and related by recursions involving only positive terms. These properties enable very accurate and efficient evaluation.

We begin by making the transformation

$$z = \frac{1}{1 - i2\xi} , \quad i2\xi = - \frac{1 - z}{z} . \quad (D-3)$$

Substitution in (D-1) yields, after manipulation,

$$f_Y(\xi) = F z^K D(z) E(z) , \quad (D-4)$$

where

$$F = \left[\prod_{m=1}^M \prod_{k=1}^K (1 + T_{mk}) \right]^{-\frac{1}{2}} , \quad D(z) = \left[\prod_{m=1}^M \prod_{k=1}^K (1 - z r_{mk}) \right]^{-\frac{1}{2}} ,$$

$$r_{mk} = \frac{T_{mk}}{1 + T_m} , \quad E(z) = \exp \left[- \frac{1 - z}{2} \sum_{m=1}^M \sum_{k=1}^K \frac{e_{mk}/(1+T_{mk})}{1 - z r_{mk}} \right] . \quad (D-5)$$

To develop the series of $D(z)$ in powers of z , expand

$$\ln D(z) = - \frac{1}{2} \sum_{m=1}^M \sum_{k=1}^K \ln(1 - z r_{mk}) = \sum_{p=1}^{\infty} \beta_p z^p , \quad (D-6)$$

where

$$\beta_p = \frac{1}{2p} \sum_{m=1}^M \sum_{k=1}^K r_{mk}^p \quad \text{for } p \geq 1 . \quad (D-7)$$

Notice that $\beta_p \geq 0$ for $p \geq 1$ since $T_{mk} \geq 0$; this latter property follows from (D-2) and the fact that eigenvalues of covariance matrices can never be negative.

At the same time, the expansion of $\ln E(z)$ in (D-5) is

$$\ln E(z) = - \frac{1}{2} (1 - z) \sum_{m=1}^M \sum_{k=1}^K \frac{e_{mk}}{1 + T_{mk}} \sum_{p=0}^{\infty} r_{mk}^p z^p . \quad (D-8)$$

By combining the results in (D-6) - (D-8), there follows, after some manipulations,

$$\ln(D(z) E(z)) = \sum_{p=0}^{\infty} \alpha_p z^p, \quad (D-9)$$

where

$$\alpha_0 = -\frac{1}{2} \sum_{m=1}^M \sum_{k=1}^K \frac{e_{mk}}{1 + T_{mk}},$$

$$\alpha_p = \frac{1}{2} \sum_{m=1}^M \sum_{k=1}^K \frac{T_{mk}^{p-1}}{(1 + T_{mk})^p} \left[\frac{T_{mk}}{p} + \frac{e_{mk}}{1 + T_{mk}} \right] \quad \text{for } p \geq 1. \quad (D-10)$$

Notice that $\alpha_p \geq 0$ for $p \geq 1$.

The desired product for use in (D-4) is then [17; page 93]

$$D(z) E(z) = \exp \left(\sum_{p=0}^{\infty} \alpha_p z^p \right) = \sum_{p=0}^{\infty} g_p z^p, \quad (D-11)$$

where

$$g_0 = \exp(\alpha_0), \quad g_p = \frac{1}{p} \sum_{n=1}^p n \alpha_n g_{p-n} \quad \text{for } p \geq 1. \quad (D-12)$$

It is important to notice that $g_p \geq 0$ for $p \geq 0$; that is, the recursion in (D-12) involves no negative quantities, thereby avoiding cancellation error.

Finally, the characteristic function in (D-4) can be expressed in the desired power series in z ($= (1 - i2\xi)^{-1}$) as

$$f_Y(\xi) = F z^K \sum_{p=0}^{\infty} g_p z^p. \quad (D-13)$$

Now, we know the following Fourier transform pair between a characteristic function and a probability density function:

$$z^n = \frac{1}{(1 - i2\xi)^n} \longleftrightarrow \frac{u^{n-1} \exp(-u/2)}{2^n (n-1)!} \quad \text{for } u > 0. \quad (\text{D-14})$$

This enables us to write the probability density function of γ , directly from (D-13), in the form

$$p_\gamma(u) = F \sum_{p=0}^{\infty} g_p \frac{u^{K+p-1} \exp(-u/2)}{2^{K+p} (K+p-1)!} \quad \text{for } u > 0, \quad (\text{D-15})$$

where scale factor F is given by (D-5). All the terms in this series are nonnegative.

Integration on the positive tail of density p_γ gives the exceedance distribution function of γ :

$$\text{Prob}(\gamma > u) = \int_u^{\infty} dt \, F_\gamma(t) = F \sum_{p=0}^{\infty} g_p H_{K+p-1}\left(\frac{u}{2}\right) \quad \text{for } u > 0, \quad (\text{D-16})$$

where F is given by (D-5) and

$$H_n(x) \equiv \exp(-x) \sum_{k=0}^n \frac{x^k}{k!} \quad \text{for } n \geq 0, \, x \geq 0. \quad (\text{D-17})$$

This latter sequence of functions is easily generated by use of the coupled recurrences

$$H_0(x) = \exp(-x), \quad H_n(x) = T_n(x) + H_{n-1}(x) \quad \text{for } n \geq 1,$$

$$T_0(x) = \exp(-x), \quad T_n(x) = T_{n-1}(x) \frac{x}{n} \quad \text{for } n \geq 1. \quad (\text{D-18})$$

The two recurrences derived above, namely (D-12) and (D-18), utilize only nonnegative quantities. The single negative term, α_0 in (D-10), is immediately converted to positive quantity g_0 in (D-12) and α_0 is not encountered again.

Since processor output γ can never be negative (see figure 1), we have, from (D-16) and (D-17),

$$1 = \text{Prob}(\gamma > 0) = F \sum_{p=0}^{\infty} g_p . \quad (\text{D-19})$$

This relation can be used to furnish an upper bound on the error incurred by using up to the $p = N$ term in series (D-16) for the exceedance distribution function. In particular, the error is

$$F \sum_{p=N+1}^{\infty} g_p H_{K+p-1}\left(\frac{u}{2}\right) \leq F \sum_{p=N+1}^{\infty} g_p = 1 - F \sum_{p=0}^N g_p \quad (\text{D-20})$$

for all $u \geq 0$. The upper bound of 1 on $H_n(x)$ follows immediately from definition (D-17). Since scale factor F and coefficients $\{g_p\}$ for $0 \leq p \leq N$ must be computed anyway, in order to utilize (D-16), error bound (D-20) is simple to compute. It says that the error at any u is never larger than the error at the origin.

However, (D-16) converges too slowly in some cases of large signal-to-noise ratio; only sequence $\{g_p\}$ converges to zero. Result (D-19) allows a modification to sum (D-16) which will converge much more rapidly. From (D-17), since $H_n(x) \rightarrow 1$ as $n \rightarrow \infty$, we express (D-16) as

$$\begin{aligned}
\text{Prob}(\gamma > u) &= F \sum_{p=0}^{\infty} g_p \left[H_{K+p-1} \left(\frac{u}{2} \right) - 1 + 1 \right] = \\
&= 1 - F \sum_{p=0}^{\infty} g_p \left[1 - H_{K+p-1} \left(\frac{u}{2} \right) \right]. \quad (D-21)
\end{aligned}$$

Now, both factors in the latter sum decay to zero as p increases. In fact,

$$1 - H_{n-1}(x) \leq \exp(-x) \frac{x^n}{n!} \frac{n+1}{n+1-x} \quad \text{for } n > x. \quad (D-22)$$

This alternative in (D-21) is utilized in the program listed at the end of this appendix. Recursions for $1 - H_n(x)$, which are obvious modifications of (D-18), are also used.

As noted at the beginning of this appendix, form (D-1) encompasses (136) and (158) in the main text. It therefore also covers (139) and (160), because these are reductions that can be obtained, respectively, by setting all $\{e_{mk}\}$ to zero. If all the $\{e_{mk}\}$ and $\{T_{mk}\}$ are zero, then all $\{\alpha_p\}$ are zero, from (D-10). Then, all $\{g_p\}$ are zero except $g_0 = 1$, from (D-12). In this case of noise-only, series (D-16) has just the one term $H_{K-1}(u/2)$.

The results above, for the expansions of the characteristic function, probability density function, and exceedance distribution function, also apply directly to forms (124) (and (126)) as well as (144) (and (149)) in the main text, but with the following changes. For (124), define instead

$$T_n = \frac{2\tilde{E}}{N_0} \lambda_n, \quad e_n = \frac{2\tilde{E}}{N_0} \epsilon_n^2, \quad F = \left[\prod_{n=1}^{KM} (1 + T_n) \right]^{-\frac{1}{2}}, \quad (D-23)$$

along with

$$\alpha_0 = -\frac{1}{2} \sum_{n=1}^{KM} \frac{e_n}{1 + T_n},$$

$$\alpha_p = \frac{1}{2} \sum_{n=1}^{KM} \frac{T_n^{p-1}}{(1 + T_n)^p} \left[\frac{T_n}{p} + \frac{e_n}{1 + T_n} \right] \quad \text{for } p \geq 1. \quad (D-24)$$

The recurrence (D-12) involving $\{g_p\}$ is unchanged.

For (144), define instead

$$T_k = \frac{2E_{11}}{N_0} \lambda_k, \quad F = \left[\prod_{k=1}^K (1 + T_k) \right]^{-M/2}, \quad (D-25)$$

along with

$$\alpha_0 = -\frac{1}{2} \sum_{k=1}^K \frac{h_k}{1 + T_k},$$

$$\alpha_p = \frac{1}{2} \sum_{k=1}^K \frac{T_k^{p-1}}{(1 + T_k)^p} \left[\frac{M}{p} T_k + \frac{h_k}{1 + T_k} \right] \quad \text{for } p \geq 1. \quad (D-26)$$

Again, the recurrence involving $\{g_p\}$ is unchanged.

REFINEMENT

In some cases, a refinement of the above procedure may be worthwhile, especially for larger signal-to-noise ratios. We illustrate it with reference to (150), which applies to $M = 2$, two components in fading model (45). In this case, we have characteristic function

$$f_Y(\xi) = \left[\prod_{k=1}^K (1 - i2\xi Q_k) \right]^{-1}, \quad (D-27)$$

where

$$Q_k = 1 + \frac{2E_{11}}{N_0} \lambda_k \quad \text{for } 1 \leq k \leq K. \quad (D-28)$$

These quantities $\{Q_k\}$ are always larger than 1 for nonzero signal-to-noise ratio.

Instead of making transformation (D-3), we modify it to the more general form

$$z = \frac{1}{1 - i2\xi Q}, \quad (D-29)$$

where Q can be chosen larger than 1 if desired. Then, defining

$$F = Q^K \left[\prod_{k=1}^K Q_k \right]^{-1}, \quad t_k = \frac{Q_k - Q}{Q_k} \quad \text{for } 1 \leq k \leq K, \quad (D-30)$$

characteristic function (D-27) can be developed as

$$f_Y(\xi) = F z^K \left[\prod_{k=1}^K (1 - t_k z) \right]^{-1} = F z^K \exp \left[\sum_{p=1}^{\infty} \alpha_p z^p \right], \quad (D-31)$$

now with

$$p \alpha_p = \sum_{k=1}^K t_k^p = \sum_{k=1}^K \left(\frac{Q_k - Q}{Q_k} \right)^p \quad \text{for } p \geq 1. \quad (\text{D-32})$$

If we take the constant Q to have value

$$\tilde{Q} = \min_{1 \leq k \leq K} Q_k, \quad (\text{D-33})$$

then all the $\{t_k\}$ and $\{\alpha_p\}$ are nonnegative, and we still have all the desirable properties listed above. The value $\tilde{Q}$ is always larger than 1; it enables (D-31) to furnish a better fit to (D-27) with fewer terms than arbitrarily forcing $Q = 1$. Larger values than $\tilde{Q}$ can yield still more-rapidly convergent series, but at the expense of some negative $\{g_p\}$. There follows, from (D-31)

$$f_Y(\xi) = F z^K \sum_{p=0}^{\infty} g_p z^p = F \sum_{p=0}^{\infty} g_p (1 - i2\xi Q)^{-K-p} \quad (\text{D-34})$$

for any Q ; see (D-11) and (D-12) for the $\{g_p\}$. Expansion (D-34) should be compared with the moment procedure in [17; (102) and (105)], where a different basis set, namely the generalized Laguerre functions, is used for the series expansion.

The corresponding exceedance distribution function is

$$\text{Prob}(\gamma > u) = F \sum_{p=0}^{\infty} g_p H_{K+p-1} \left(\frac{u}{2Q} \right) \quad \text{for } u > 0. \quad (\text{D-35})$$

Alternatively, we have the more rapidly convergent form

$$\text{Prob}(\gamma > u) = 1 - F \sum_{p=0}^{\infty} g_p \left[1 - H_{K+p-1} \left(\frac{u}{2Q} \right) \right] \quad \text{for } u > 0. \quad (\text{D-36})$$

PROBABILITY DENSITY FROM CHARACTERISTIC FUNCTION (163)

The characteristic function of normalized random variable $\phi = q/R_{11}(0,0)$ defined in (161) was presented in (163). The corresponding probability density function $p_\phi(u)$ was then evaluated for a few special cases in (166), (171), (175), and (180). More generally, if we apply the expansion technique above to general result (163), we obtain the following procedure for $p_\phi(u)$: given M , $\{r^{(m)}\}$, $\{\eta_m^2\}$, and threshold u ,

$$F = \left(\prod_{m=1}^M r^{(m)} \right)^{-1/2}, \quad \alpha_0 = -\frac{1}{2} \sum_{m=1}^M \eta_m^2, \quad (D-37)$$

$$p \alpha_p = \frac{1}{2} \sum_{m=1}^M \frac{(r^{(m)} - 1)^{p-1}}{r^{(m)p}} (r^{(m)} - 1 + p \eta_m^2) \quad \text{for } p \geq 1, \quad (D-38)$$

$$g_0 = \exp(\alpha_0), \quad g_p = \frac{1}{p} \sum_{n=1}^p n \alpha_n g_{p-n} \quad \text{for } p \geq 1, \quad (D-39)$$

$$h_0(u) = \frac{u^{1/2 M - 1} \exp(-u/2)}{2^{M/2} \Gamma(M/2)},$$

$$h_p(u) = h_{p-1}(u) \frac{u}{M - 2 + 2p} \quad \text{for } p \geq 1, \quad (D-40)$$

$$p_\phi(u) = F \sum_{p=0}^{\infty} g_p h_p(u) \quad \text{for } u > 0. \quad (D-41)$$

A function subprogram for this particular procedure is listed on the next page.

FUNCTION SUBPROGRAM FOR (D-37) - (D-41)

```

10  DEF FNPdf(DOUBLE M,REAL U,Rs(*),Etsq(*))
20  DOUBLE Ms,Ps      !  INTEGERS
30  ALLOCATE Q(1:M),R(1:M),S(1:M),A(1:100),G(0:100)
40  M2=M/2.
50  M21=M2-1.
60  U2=U/2.
70  F=1.
80  S=0.
90  FOR Ms=1 TO M
100  Rs=Rs(Ms)
110  T=1./Rs
120  E=Etsq(Ms)
130  Q(Ms)=1.
140  R(Ms)=1.-T
150  S(Ms)=E*T
160  F=F*Rs
170  S=S+E
180  NEXT Ms
190  G(0)=EXP(-.5*S)
200  H=.5*U2^M21*EXP(-U2)/(SQR(F)*FNGamma(M2))
210  Pdf=G(0)*H
220  Q=1.
230  FOR Ps=1 TO 100
240  S=0.
250  FOR Ms=1 TO M
260  R=R(Ms)
270  IF Ps=1 THEN 290
280  Q(Ms)=Q(Ms)*R
290  S=S+Q*(R+Ps*S(Ms))
300  NEXT Ms
310  A(Ps)=.5*S
320  S=0.
330  FOR Ms=1 TO Ps
340  S=S+A(Ms)*G(Ps-Ms)
350  NEXT Ms
360  G(Ps)=G=S/Ps
370  H=H*U2/(M21+Ps)
380  T=G*H
390  Pdf=Pdf+T
400  IF T<Pdf*1.E-15 THEN 440
410  NEXT Ps
420  PRINT "100 TERMS ARE INSUFICIENT"
430  PAUSE
440  RETURN Pdf
450  END

```

MAIN PROGRAM

A program for the evaluation of (D-21), by means of (D-10) and recursions (D-12) and (D-18), is presented below. The desired false alarm and detection probabilities, P_f and P_d respectively, of the processor in figure 1 are specified in lines 10 and 20. Parameter K in line 30 is the number of signal pulses, while M in line 40 is the number of components in fading model (45). The M power ratios $\{r^{(m)}\}$ are inputted in line 50, while the MK deterministic input signal-to-noise ratio measures $\{D_{km}/N_0\}$ are entered in line 60. The K time locations $\{t_k\}$ and frequency locations $\{f_k\}$ of the K transmitted signal pulses are entered in lines 70 and 80. Guesses at the required average random input signal-to-noise ratio measure E_1/N_0 are required in lines 90 and 100; the program will search for the required threshold u and required input signal-to-noise ratio E_1/N_0 to meet the specifications.

The normalized covariance function of the medium fading is described in function subprogram DEF FNCov(Tau,Nu); currently, it allows for exponential decay in time separation τ and frequency separation ν , but can be easily modified. It must be noted that covariance $R_{mn}(\tau,\nu)$ defined in (47) involves a product of the amplitude-fading quantities $\{g_m(t,f)\}$, not their squares. This distinction, relative to covariance $\tilde{R}_{kj}$ in (55) and normalized covariance ρ_{kj} in (56) between power-fading variates $\{g_m^2(t,f)\}$, must be carefully observed and maintained; furthermore, it is worthwhile to review (66) and (69) at this point, which pertain

only to special cases 5 and 7.

Quantities that need be computed only once and then used repeatedly for different signal-to-noise ratios, such as the eigenvalues and eigenvectors and some auxiliary arrays, are evaluated in the main program and passed in common to function subprogram DEF FNPd for the detection probability. Similarly, for speed and storage purposes, we have identified, for use in this subprogram, the array variables $A(p) = p \alpha_p$ for $p \geq 1$ and $G(p) = F g_p$ for $p \geq 0$. A separate listing for the false alarm probability P_F in (188) is given in function subprogram DEF FNPf.

The program is listed in BASIC for the Hewlett-Packard 9000 computer. On this particular device, the notation DOUBLE denotes INTEGER variables, not double precision. As a numerical check, the values printed out are threshold $U = 38.2583364$ and signal-to-noise ratio measure $\text{dB} = 10 \log_{10}(E_1/N_0) = 13.4744256$. Since the mean of processor output γ for noise-only is $\bar{\gamma} = 2K = 6$, the normalized threshold is $U/\bar{\gamma} = 6.376$, for this example of $P_F = 1E-6$.

```

10  Pf=1.E-6           ! FALSE ALARM PROBABILITY
20  Pd=.9              ! DETECTION PROBABILITY
30  K=3                ! NUMBER OF SIGNAL PULSES
40  M=2                ! NUMBER OF FADING COMPONENTS
50  DATA 1.,1.        ! POWER RATIOS r(m)   FOR 1 <= m <= M
60  DATA 0,0,0,0,0,0  ! Dkm/No FOR 1 <= k <= K, 1 <= m <= M
70  DATA .1,.3,.5     ! TIMES tk (SEC)       FOR 1 <= k <= K
80  DATA .2,.6,.4     ! FREQUENCIES fk (HZ)  FOR 1 <= k <= K
90  E1no0=10.          ! E1/No STARTING VALUE
100 E1no1=1.           ! E1/No INCREMENT
110 Ef=1.E-15          ! TOLERANCE ON Pf
120 Ed=1.E-10          ! TOLERANCE ON Pd
130 PRINT
140 PRINT "Pf =";Pf;"   K =";K;"   M =";M
150 OPTION BASE 1
160 COM DOUBLE K,M      ! INTEGERS (NOT DOUBLE PRECISION)
170 DOUBLE Ms,Ks,Js     ! INTEGERS
180 DIM Rs(5),Dn(10,5),Ts(10),Fs(10),Psi(5),Cbar(10,10)
190 DIM U(10,10),V(10,10),Eig(10),Sq(10,5)
200 COM U,Prod(5,10),Es(5,10),T(5,10),Q(5,10),S(5,10),B(5,10)
210 COM A(100),G(0:100)
220 REDIM Rs(1:M),Dn(1:K,1:M),Ts(1:K),Fs(1:K),Psi(1:M),Cbar(1:K,1:K)
230 REDIM U(1:K,1:K),V(1:K,1:K),Eig(1:K),Sq(1:K,1:M)
240 REDIM Prod(1:M,1:K),Es(1:M,1:K),T(1:M,1:K)
250 REDIM Q(1:M,1:K),S(1:M,1:K),B(1:M,1:K)
260 READ Rs(*),Dn(*),Ts(*),Fs(*) ! Dn(*) WAS FILLED IN THE ORDER:
270 S=0.                ! Dn(1,1),Dn(1,2),Dn(1,3),...,Dn(k,m),...,Dn(K,M)
280 FOR Ms=1 TO M
290 S=S+Rs(Ms)
300 NEXT Ms
310 FOR Ms=1 TO M
320 Psi(Ms)=Rs(Ms)/S
330 NEXT Ms
340 FOR Ks=1 TO K
350 FOR Js=1 TO K
360 Cov=FNCov(Ts(Ks)-Ts(Js),Fs(Ks)-Fs(Js))
370 Cbar(Ks,Js)=Cov      ! NORMALIZED COVARIANCE MATRIX
380 NEXT Js
390 NEXT Ks
400 MAT U=Cbar
410 CALL Svd(K,K,U(*),V(*),Eig(*)) ! OUTPUTS: U(*),V(*),Eig(*)
420 PRINT "EIGENVALUES:"
430 PRINT Eig(*)

```

```

440   FOR Ms=1 TO M
450   T=2.*Psi(Ms)
460   FOR Ks=1 TO K
470   Prod(Ms,Ks)=T*Eig(Ks) ! 2 Psi Eig
480   NEXT Ks
490   NEXT Ms
500   FOR Ks=1 TO K
510   FOR Ms=1 TO M
520   Sq(Ks,Ms)=SQR(2.*Dn(Ks,Ms))
530   NEXT Ms
540   NEXT Ks
550   FOR Ks=1 TO K
560   FOR Ms=1 TO M
570   S=0.
580   FOR Js=1 TO K
590   S=S+V(Js,Ks)*Sq(Js,Ms)
600   NEXT Js
610   Es(Ms,Ks)=S*S          ! emk
620   NEXT Ms
630   NEXT Ks
640   U1=1.                  ! THRESHOLD INCREMENT
650   U0=-LOG(Pf)-U1         ! THRESHOLD STARTING VALUE
660   CALL Inversfunction1(-Pf,Ef,U0,U1,U)
670   PRINT "THRESHOLD U =" ; U
680   CALL Inversfunction2(Pd,Ed,E1no0,E1no1,E1no)
690   Db=10.*LGT(E1no)
700   PRINT "dB =" ; Db
710   END
720   !
730   DEF FNCov(Tau,Nu)      ! NORMALIZED COVARIANCE
740   A=LOG(2.)*.5           ! R11(Tau,Nu)/R11(0,0)
750   B=1.3
760   Cov=EXP(-A*ABS(Tau)-B*ABS(Nu))
770   RETURN Cov
780   FNEND
790   !
800   DEF FNPf(U)            ! PROBABILITY OF FALSE ALARM
810   COM DOUBLE K           ! INTEGER
820   DOUBLE Ks
830   U2=U*.5
840   Pf=T=EXP(-U2)
850   FOR Ks=1 TO K-1
860   T=T*U2/Ks
870   Pf=Pf+T
880   NEXT Ks
890   RETURN -Pf            ! - TO YIELD INCREASING FUNCTION
900   FNEND
910   !

```

```

920  DEF FNPd(E1no)
930  COM DOUBLE K,M          ! INTEGERS
940  COM U,Prod(*),Es(*),T(*),Q(*),S(*),B(*),A(*),G(*)
950  DOUBLE Ms,Ks,K1,P
960  Tol=1.E-10              ! RELATIVE ERROR OF SUM
970  FOR Ms=1 TO M
980  FOR Ks=1 TO K
990  T(Ms,Ks)=E1no*Prod(Ms,Ks)
1000 NEXT Ks
1010 NEXT Ms
1020 F=1.
1030 S=0.
1040 FOR Ms=1 TO M
1050 FOR Ks=1 TO K
1060 T=T(Ms,Ks)
1070 Q(Ms,Ks)=Q+1./T
1080 S(Ms,Ks)=T/Q
1090 F=F*Q
1100 S=S+Es(Ms,Ks)/Q
1110 NEXT Ks
1120 NEXT Ms
1130 G(0)=EXP(-.5*S)/SQR(F)
1140 K1=K-1
1150 U2=U*.5
1160 T=EXP(-U2)
1170 H1=1.-T
1180 FOR Ks=1 TO K1
1190 T=T*U2/Ks
1200 H1=H1-T
1210 NEXT Ks
1220 Pd=1.-G(0)*H1
1230 FOR P=1 TO 100
1240 T=T*U2/(K1+P)
1250 H1=H1-T
1260 S=0.
1270 FOR Ms=1 TO M
1280 FOR Ks=1 TO K
1290 Q=Q(Ms,Ks)
1300 IF P>1 THEN 1330
1310 B(Ms,Ks)=B+1./Q
1320 GOTO 1340
1330 B(Ms,Ks)=B+B(Ms,Ks)*S(Ms,Ks)
1340 S=S+B*(T(Ms,Ks)/P+Es(Ms,Ks)/Q)
1350 NEXT Ks
1360 NEXT Ms
1370 A(P)=.5*S*P
1380 S=0.
1390 FOR Ms=1 TO P
1400 S=S+A(Ms)*G(P-Ms)
1410 NEXT Ms
1420 G(P)=G+S/P
1430 Del=G*H1
1440 Pd=Pd-Del
1450 IF Del<Pd*Tol THEN 1490
1460 NEXT P
1470 PRINT "100 TERMS ARE INSUFFICIENT"
1480 PAUSE
1490 PRINT "Pd =";Pd,"    P =";P,"    E1/No =";E1no
1500 RETURN Pd
1510 FNEND
1520 !

```

```
1530 SUB Inversfunction1(Desired,Error,X1,Del,X2)
1540 X2=X1+Del
1550 F1=FNPf(X1)
1560 F2=FNPf(X2)
1570 IF F2>=Desired THEN 1620
1580 X1=X2
1590 X2=X2+Del
1600 F1=F2
1610 GOTO 1560
1620 IF F1<=Desired THEN 1680
1630 X2=X1
1640 X1=X1-Del
1650 F2=F1
1660 F1=FNPf(X1)
1670 GOTO 1620
1680 Xa=X1
1690 Xb=X2
1700 IF F2-Desired<Desired-F1 THEN 1770
1710 T=X1
1720 X1=X2
1730 X2=T
1740 T=F1
1750 F1=F2
1760 F2=T
1770 IF ABS(F2-Desired)<Error THEN 1870
1780 IF F2=F1 THEN 1870
1790 T=(X1*(F2-Desired)-X2*(F1-Desired))/(F2-F1)
1800 T=MAX(T,Xa)
1810 T=MIN(T,Xb)
1820 X1=X2
1830 X2=T
1840 F1=F2
1850 F2=FNPf(X2)
1860 GOTO 1770
1870 SUBEND
1880 !
1890 SUB Inversfunction2(Desired,Error,X1,Del,X2)
1900 X2=X1+Del
1910 F1=FNPd(X1)
1920 F2=FNPd(X2)
1930 IF F2>=Desired THEN 1980
1940 X1=X2
1950 X2=X2+Del
1960 F1=F2
1970 GOTO 1920
1980 IF F1<=Desired THEN 2040
1990 X2=X1
2000 X1=X1-Del
```



```

2010 F2=F1
2020 F1=FNPd(X1)
2030 GOTO 1980
2040 Xa=X1
2050 Xb=X2
2060 IF F2-Desired<Desired-F1 THEN 2130
2070 T=X1
2080 X1=X2
2090 X2=T
2100 T=F1
2110 F1=F2
2120 F2=T
2130 IF ABS(F2-Desired)<Error THEN 2230
2140 IF F2=F1 THEN 2230
2150 T=(X1*(F2-Desired)-X2*(F1-Desired))/(F2-F1)
2160 T=MAX(T,Xa)
2170 T=MIN(T,Xb)
2180 X1=X2
2190 X2=T
2200 F1=F2
2210 F2=FNPd(X2)
2220 GOTO 2130
2230 SUBEND
2240 !
2250 SUB Svd(DOUBLE M,N,REAL A(*),V(*),W(*))
2260 ! THIS SUBROUTINE COMPUTES THE SINGULAR VALUE DECOMPOSITION
2270 ! OF AN ARBITRARY REAL MxN MATRIX A:  $A = U W V^t$ ,  $M \geq N$ .
2280 ! U IS MxN, V IS NxN, W IS NxN:  $W = \text{DIAG}(D(n))$ .
2290 ALLOCATE Rv1(1:N) ! NUMERICAL RECIPES, PAGES 60-64
2300 IF M>N THEN 2330 ! A(*) IS OVER-WRITTEN
2310 PRINT "M<N IS DISALLOWED"
2320 PAUSE
2330 DOUBLE I,J,K,L,Its,Nm,Jj ! INTEGERS (NOT DOUBLE PRECISION)
2340 G=Scale=Anorm=0.
2350 FOR I=1 TO N
2360 L=I+1
2370 Rv1(I)=Scale*G
2380 G=S=Scale=0.
2390 IF I>M THEN 2670
2400 FOR K=I TO M
2410 Scale=Scale+ABS(A(K,I))
2420 NEXT K
2430 IF Scale=0. THEN 2670
2440 FOR K=I TO M
2450 Aa=A(K,I)=A(K,I)/Scale
2460 S=S+Aa*Aa
2470 NEXT K
2480 F=A(I,I)
2490 G=-SQR(S)
2500 IF F<0. THEN G=-G

```

```

2510 H=F*G-S
2520 A(I,I)=F-G
2530 IF I=N THEN 2640
2540 FOR J=L TO N
2550 S=0.
2560 FOR K=I TO M
2570 S=S+A(K,I)*A(K,J)
2580 NEXT K
2590 F=S/H
2600 FOR K=I TO M
2610 A(K,J)=A(K,J)+F*A(K,I)
2620 NEXT K
2630 NEXT J
2640 FOR K=I TO M
2650 A(K,I)=A(K,I)*Scale
2660 NEXT K
2670 W(I)=Scale*G
2680 G=S=Scale=0.
2690 IF (I>M) OR (I=N) THEN 2990
2700 FOR K=L TO N
2710 Scale=Scale+ABS(A(I,K))
2720 NEXT K
2730 IF Scale=0. THEN 2990
2740 FOR K=L TO N
2750 Aa=A(I,K)=A(I,K)/Scale
2760 S=S+Aa*Aa
2770 NEXT K
2780 F=A(I,L)
2790 G=-SQRT(S)
2800 IF F<0. THEN G=-G
2810 H=F*G-S
2820 A(I,L)=F-G
2830 FOR K=L TO N
2840 Rv1(K)=A(I,K)/H
2850 NEXT K
2860 IF I=M THEN 2960
2870 FOR J=L TO M
2880 S=0.
2890 FOR K=L TO N
2900 S=S+A(J,K)*A(I,K)
2910 NEXT K
2920 FOR K=L TO N
2930 A(J,K)=A(J,K)+S*Rv1(K)
2940 NEXT K
2950 NEXT J
2960 FOR K=L TO N
2970 A(I,K)=A(I,K)*Scale
2980 NEXT K
2990 Anorm=MAX(Anorm,ABS(W(I))+ABS(Rv1(I)))
3000 NEXT I

```

```

3010  FOR I=N TO 1 STEP -1
3020  IF I>=N THEN 3190
3030  IF G=0. THEN 3160
3040  FOR J=L TO N
3050  V(J,I)=A(I,J)/A(I,L)/G
3060  NEXT J
3070  FOR J=L TO N
3080  S=0.
3090  FOR K=L TO N
3100  S=S+A(I,K)*V(K,J)
3110  NEXT K
3120  FOR K=L TO N
3130  V(K,J)=V(K,J)+S*V(K,I)
3140  NEXT K
3150  NEXT J
3160  FOR J=L TO N
3170  V(I,J)=V(J,I)=0.
3180  NEXT J
3190  V(I,I)=1.
3200  G=R*V(I)
3210  L=I
3220  NEXT I
3230  FOR I=N TO 1 STEP -1
3240  L=I+1
3250  G=W(I)
3260  IF I>=N THEN 3300
3270  FOR J=L TO N
3280  A(I,J)=0.
3290  NEXT J
3300  IF G=0. THEN 3470
3310  G=1./G
3320  IF I=N THEN 3430
3330  FOR J=L TO N
3340  S=0.
3350  FOR K=L TO M
3360  S=S+A(K,I)*A(K,J)
3370  NEXT K
3380  F=S/A(I,I)*G
3390  FOR K=I TO M
3400  A(K,J)=A(K,J)+F*A(K,I)
3410  NEXT K
3420  NEXT J
3430  FOR J=I TO M
3440  A(J,I)=A(J,I)*G
3450  NEXT J
3460  GOTO 3500
3470  FOR J=I TO M
3480  A(J,I)=0.
3490  NEXT J
3500  A(I,I)=A(I,I)+1.

```

```

3510 NEXT I
3520 FOR K=N TO 1 STEP -1
3530 FOR Its=1 TO 30
3540 FOR L=K TO 1 STEP -1
3550 Nm=L-1
3560 IF (ABS(Rv1(L))+Anorm)=Anorm THEN 3780
3570 IF (ABS(W(Nm))+Anorm)=Anorm THEN 3590
3580 NEXT L
3590 C=0.
3600 S=1.
3610 FOR I=L TO K
3620 F=S*Rv1(I)
3630 Rv1(I)=C*Rv1(I)
3640 IF (ABS(F)+Anorm)=Anorm THEN 3780
3650 G=W(I)
3660 H=SQR(F*F+G*G)
3670 W(I)=H
3680 H=1./H
3690 C=G*H
3700 S=-F*H
3710 FOR J=1 TO M
3720 Y=A(J,Nm)
3730 Z=A(J,I)
3740 A(J,Nm)=Y*C+Z*S
3750 A(J,I)=-Y*S+Z*C
3760 NEXT J
3770 NEXT I
3780 Z=W(K)
3790 IF L<>K THEN 3860
3800 IF Z>=0. THEN 3850
3810 W(K)=-Z
3820 FOR J=1 TO N
3830 V(J,K)=-V(J,K)
3840 NEXT J
3850 GOTO 4390
3860 IF Its<30 THEN 3890
3870 PRINT "NO CONVERGENCE IN 30 ITERATIONS"
3880 PAUSE
3890 X=W(L)
3900 Nm=K-1
3910 Y=W(Nm)
3920 G=Rv1(Nm)
3930 H=Rv1(K)
3940 F=((Y-Z)*(Y+Z)+(G-H)*(G+H))/(2.*H*Y)
3950 G=SQR(F*F+1.)
3960 Aa=ABS(G)
3970 IF F<0. THEN Aa=-Aa
3980 F=((X-Z)*(X+Z)+H*((Y/(F+Aa))-H))/X
3990 C=S=1.
4000 FOR J=L TO Nm

```

```
4010 I=J+1
4020 G=Rv1(I)
4030 Y=W(I)
4040 H=S*G
4050 G=C*G
4060 Z=SQR(F*F+H*H)
4070 Rv1(J)=Z
4080 C=F/Z
4090 S=H/Z
4100 F=X*C+G*S
4110 G=-X*S+G*C
4120 H=Y*S
4130 Y=Y*C
4140 FOR Jj=1 TO N
4150 X=V(Jj,J)
4160 Z=V(Jj,I)
4170 V(Jj,J)=X*C+Z*S
4180 V(Jj,I)=-X*S+Z*C
4190 NEXT Jj
4200 Z=SQR(F*F+H*H)
4210 W(J)=Z
4220 IF Z=0. THEN 4260
4230 Z=1./Z
4240 C=F*Z
4250 S=H*Z
4260 F=C*G+S*Y
4270 X=-S*G+C*Y
4280 FOR Jj=1 TO M
4290 Y=A(Jj,J)
4300 Z=A(Jj,I)
4310 A(Jj,J)=Y*C+Z*S
4320 A(Jj,I)=-Y*S+Z*C
4330 NEXT Jj
4340 NEXT J
4350 Rv1(L)=0.
4360 Rv1(K)=F
4370 W(K)=X
4380 NEXT Its
4390 NEXT K
4400 SUBEND
```

APPENDIX E. SOLUTION OF GENERALIZED EIGENVALUE PROBLEM

C and B are $N \times N$ real symmetric matrices, and B is positive definite. The important alternative case where B is not positive definite will be undertaken in (E-14) and sequel. We want the normalized modal matrix Q and the eigenvalue matrix Λ of the generalized characteristic-value problem [18; page 74] encountered in (B-10), namely

$$C Q = B^{-1} Q \Lambda . \quad (E-1)$$

Then, it is known that [18; pages 74 - 77]

$$Q^T B^{-1} Q = I \quad \text{and} \quad Q^T C Q = \Lambda , \quad (E-2)$$

which is a simultaneous reduction of two matrices to diagonal form. Alternatively, when both equations in (E-2) are satisfied, then (E-1) follows.

The first relation in (E-2) sets the scalings on the eigenvectors $\{V_n\}$ of Q; in fact, it reads $V_n^T B^{-1} V_n = 1$ for $1 \leq n \leq N$. An alternative scaling choice would be to make the eigenvectors of unit length, that is, $V_n^T V_n = 1$. However, this would result in $Q^T B^{-1} Q$ becoming a diagonal matrix with non-unity elements; this latter alternative is not adopted here. Therefore, the eigenvectors $\{V_n\}$ in (E-2) do not have unit length, that is, $V_n^T V_n \neq 1$ generally. The eigenvectors are unique within their polarities.

A procedure for determining Q and Λ is now presented. Solve for the square root matrix S in

$$B = S S^T , \quad (E-3)$$

where S need not be symmetric. For example, the lower (or upper) triangular square root matrix will suffice. Then there follows

$$B^{-1} = S^{-T} S^{-1} \quad \text{and} \quad S^T B^{-1} S = I . \quad (E-4)$$

Now compute $N \times N$ matrix

$$A = S^T C S , \quad (E-5)$$

which is symmetric. Solve the standard eigenvalue problem

$$A U = U \gamma \quad (E-6)$$

for normalized modal matrix U and diagonal eigenvalue matrix γ .

Then it is known that these solutions satisfy

$$U^T U = I \quad \text{and} \quad U^T A U = \gamma . \quad (E-7)$$

It will now be shown that

$$Q = S U \quad \text{and} \quad \Lambda = \gamma \quad (E-8)$$

are the desired matrix solutions to (E-1). Substitution of (E-8) in the first relation in (E-2) yields

$$Q^T B^{-1} Q = U^T S^T B^{-1} S U = U^T I U = I , \quad (E-9)$$

where we used (E-4) and (E-7). Also, substitution of (E-8) in the second relation in (E-2) yields

$$Q^T C Q - \Lambda = U^T S^T C S U - \gamma = U^T A U - \gamma = 0 , \quad (E-10)$$

where we used (E-5) and (E-7). Thus, both relations in (E-2) are satisfied by the matrix assignments in (E-8).

In summary, in order to solve (E-1): compute square root matrix S in (E-3); evaluate matrix A via (E-5); solve (E-6) for U and γ ; determine Q and Λ by means of (E-8). Notice there is never any need to compute inverse matrix B^{-1} ; however, matrix B must be positive definite for the operation in (E-3) to succeed.

We now present an example which illustrates many of the quantities considered above; let matrices

$$B = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 \\ 1 & 8 \end{bmatrix}. \quad (\text{E-11})$$

Then the solutions of (E-1) are

$$\Lambda = \begin{bmatrix} 20 & 0 \\ 0 & 12 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}. \quad (\text{E-12})$$

The column vectors of Q are not orthogonal; in fact,

$$Q^T Q = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}. \quad (\text{E-13})$$

Rather, the orthogonality relations of (E-2) may be readily verified to be satisfied in this example, namely $Q^T B^{-1} Q = I$ and $Q^T C Q = \Lambda$.

GENERAL SYMMETRIC MATRIX B

In attempting to solve (E-1) for Q and Λ , the square root of matrix B was required in (E-3). If B is not positive definite, complex solutions would be required there. Here, we will solve the alternative generalized characteristic-value matrix equation encountered in (B-44), namely

$$B Q = C^{-1} Q \Lambda , \quad (E-14)$$

where covariance matrix C is real symmetric and positive definite, while matrix B is real symmetric but can be indefinite. The solutions Q and Λ will then satisfy the relations

$$Q^T C^{-1} Q = I \quad \text{and} \quad Q^T B Q = \Lambda . \quad (E-15)$$

To this aim, first solve for the square root matrix S in

$$C = S S^T . \quad (E-16)$$

Then

$$C^{-1} = S^{-T} S^{-1} \quad \text{and} \quad S^T C^{-1} S = I . \quad (E-17)$$

Next, compute the $N \times N$ matrix

$$A = S^T B S , \quad (E-18)$$

which is symmetric. Solve the standard characteristic-value equation

$$A U = U \gamma \quad (E-19)$$

for U and γ , for which we have orthogonality properties

$$U^T U = I \quad \text{and} \quad U^T A U = \gamma . \quad (E-20)$$

It will now be shown that the solutions to (E-14) are

$$Q = S U \quad \text{and} \quad \Lambda = \gamma . \quad (E-21)$$

Substitution of (E-21) in the first relation in (E-15) yields

$$Q^T C^{-1} Q = U^T S^T C^{-1} S U = U^T I U = U^T U = I , \quad (E-22)$$

where we used (E-17) and (E-20). Similarly, substitution of (E-21) in the second relation in (E-15) yields

$$Q^T B Q - \Lambda = U^T S^T B S U - \gamma = U^T A U - \gamma = 0 , \quad (E-23)$$

where we used (E-18) and (E-20). Thus, both relations in (E-15) are verified by solutions (E-21).

In summary, in order to solve (E-14): compute square root matrix S in (E-16); evaluate matrix A according to (E-18); solve (E-19) for U and γ ; determine Q and Λ by means of (E-21). This procedure requires covariance matrix C to be positive definite. Inverses of matrices C , B , or S are not required.

(If square root matrix S in (E-16) is taken lower triangular, then S^{-1} is also lower triangular and can be very easily computed directly from S . This enables calculation of inverse matrix C^{-1} by means of (E-17), if desired.)

INTERRELATIONSHIPS OF SOLUTIONS

Here we shall relate the solutions Q and Λ of (E-1) to the solutions Q and Λ of (E-14). In particular, we maintain that the specific relations are

$$Q = Q^{-T} \Lambda^{\frac{1}{2}} \quad \text{and} \quad \Lambda = \Lambda. \quad (\text{E-24})$$

To verify this claim, substitute (E-24) into the first relation in (E-2) and obtain

$$Q^T B^{-1} Q - I = \Lambda^{\frac{1}{2}} Q^{-1} B^{-1} Q^{-T} \Lambda^{\frac{1}{2}} - I = \Lambda^{\frac{1}{2}} \Lambda^{-1} \Lambda^{\frac{1}{2}} - I = 0, \quad (\text{E-25})$$

where we used (E-15). Similarly, substitution in the second relation in (E-2) yields

$$Q^T C Q - \Lambda = \Lambda^{\frac{1}{2}} Q^{-1} C Q^{-T} \Lambda^{\frac{1}{2}} - \Lambda = \Lambda^{\frac{1}{2}} I \Lambda^{\frac{1}{2}} - \Lambda = 0, \quad (\text{E-26})$$

where we also used (E-15). Thus, both relations in (E-2) are satisfied by interrelationships (E-24).

The connection between eigenvectors is obtained by utilizing (E-15) to give

$$Q^{-T} = B Q \Lambda^{-1}. \quad (\text{E-27})$$

Substitution in (E-24) then yields

$$Q = B Q \Lambda^{-\frac{1}{2}}; \quad \text{that is,} \quad V_n = B V_n \lambda_n^{-\frac{1}{2}}. \quad (\text{E-28})$$

The eigenvalue relation is, directly from (E-24),

$$\lambda_n = \lambda_n. \quad (\text{E-29})$$

Finally, the connection between vectors $\underline{E}$ and $\underline{A}$ in (B-18) and the corresponding vectors $\underline{E}$ and $\underline{A}$ in (B-51) and (B-52) is as follows: the use of (E-28) in the first relation in (B-18) yields

$$\epsilon_n = V_n^T E = \lambda_n^{-\frac{1}{2}} V_n^T B E = \lambda_n^{\frac{1}{2}} \epsilon_n , \quad (E-30)$$

the last relation based on (B-59). And substitution of (E-28) in the second relation in (B-18) yields

$$\alpha_n = V_n^T B^{-1} A = \lambda_n^{-\frac{1}{2}} V_n^T B B^{-1} A = \lambda_n^{-\frac{1}{2}} \alpha_n , \quad (E-31)$$

using (B-60).

It can now be seen that the use of (E-29) - (E-31) in characteristic function (B-20) immediately converts it to (B-54). This is a direct confirmation of the fact that the characteristic function of q is unique, regardless of how derived. However, the recommended procedure is that given in (B-43) and sequel, because it is more general, allowing symmetric matrix B to be indefinite. In fact, relation (E-28) is valid only when all eigenvalues λ_n are positive. If this is not the case, although Q cannot be found using real arithmetic, the technique summarized under (E-23), for determining solutions Q and Λ of (E-14), is still valid and can always be used with real arithmetic. Eigenvalues $\{\lambda_n\}$ can take on positive and negative values. The only restriction is that symmetric matrix C be positive definite.

APPENDIX F. STATISTICS OF REAL BILINEAR FORM

Suppose we encounter real bilinear form [22; (13.5-1)]

$$b = X^T A Y , \quad (F-1)$$

where random vector X is $M \times 1$, random vector Y is $N \times 1$, and arbitrary matrix A is $M \times N$, all quantities being real. We allow random vectors X and Y to be arbitrarily correlated with each other, but to have zero means here; generalizations to nonzero means can be made along the lines of appendix B.

Let $L = M+N$ and define $L \times 1$ real vector

$$Z = \begin{bmatrix} X \\ Y \end{bmatrix} . \quad (F-2)$$

Let $L \times L$ matrix

$$B = \frac{1}{2} \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix} = B^T . \quad (F-3)$$

Then (F-1) becomes

$$b = Z^T B Z , \quad (F-4)$$

which is a quadratic form in L variables; $L = M+N$. Matrix B is symmetric but not positive definite. Letting $r = \min(M, N)$, numerical investigation has revealed that B has r positive eigenvalues $\mu_1, \dots, \mu_r$; r eigenvalues which are their negatives $-\mu_1, \dots, -\mu_r$; and the rest zero.

The mean of real random vector Z is zero, and its (arbitrary) $L \times L$ covariance matrix is

$$C = \overline{Z Z^T} = \overline{\begin{bmatrix} X \\ Y \end{bmatrix} \begin{bmatrix} X^T & Y^T \end{bmatrix}} = \begin{bmatrix} \overline{X X^T} & \overline{X Y^T} \\ \overline{Y X^T} & \overline{Y Y^T} \end{bmatrix} = \begin{bmatrix} C_{xx} & C_{xy} \\ C_{xy}^T & C_{yy} \end{bmatrix}, \quad (F-5)$$

which is symmetric. C is also nonnegative definite, because

$$V^T C V = V^T \overline{Z Z^T} V = \overline{(V^T Z)^2} \geq 0 \quad (F-6)$$

for any $L \times 1$ vector V . We presume that C is positive definite.

In order to convert quadratic form (F-4) to diagonal form with uncorrelated variates, we first solve the $L \times L$ generalized characteristic-value equation (see (E-14) and sequel)

$$B Q = C^{-1} Q \Lambda \quad (F-7)$$

for $L \times L$ matrices Q and Λ . Then, we have properties [18; pages 74 - 77]

$$Q^T C^{-1} Q = I \quad \text{and} \quad Q^T B Q = \Lambda = \text{diag}[\lambda_1 \dots \lambda_L]. \quad (F-8)$$

Now let linearly transformed random vector

$$W = Q^{-1} Z = [w_1 \dots w_L] ; \quad Z = Q W. \quad (F-9)$$

Vector W also has zero mean. Its covariance matrix is

$$\overline{W W^T} = Q^{-1} \overline{Z Z^T} Q^{-T} = Q^{-1} C Q^{-T} = I, \quad (F-10)$$

upon use of (F-5) and (F-8). Also, bilinear form (F-4) becomes

$$b = Z^T B Z = W^T Q^T B Q W = W^T \Lambda W = \sum_{k=1}^L \lambda_k w_k^2, \quad (F-11)$$

where we used (F-9) and (F-8). Thus, b is a sum of L weighted squares of uncorrelated zero-mean unit-variance random variables. Notice that not all eigenvalues $\{\lambda_k\}$ need be positive; some can be zero and some can be negative.

If random vectors X and Y are joint-Gaussian, then vector W in (F-9) is also Gaussian, with probability density function

$$p(W) = \prod_{k=1}^L \left((2\pi)^{-\frac{1}{2}} \exp(-w_k^2/2) \right), \quad (F-12)$$

based on (F-10). The characteristic function of bilinear form b in (F-11) is then

$$\begin{aligned} f_b(\xi) &= \overline{\exp(i\xi b)} = \prod_{k=1}^L \left(\int dw_k (2\pi)^{-\frac{1}{2}} \exp\left[-w_k^2/2 + i\xi\lambda_k w_k^2\right] \right) = \\ &= \left[\prod_{k=1}^L \left(1 - i2\xi\lambda_k \right) \right]^{-\frac{1}{2}}. \end{aligned} \quad (F-13)$$

Since only the eigenvalues $\{\lambda_k\}$ of A in (F-8) are required to evaluate characteristic function (F-13), it is not actually necessary to determine the eigenvectors Q in matrix equation (F-7). In fact, by a procedure identical to that given earlier in (B-55) - (B-58), it may be shown that $\{\lambda_k\}$ are the eigenvalues of nonsymmetric $L \times L$ matrix $B C$. Reference to (F-3) and (F-5) reveals that this latter matrix is given by

$$B C = \frac{1}{2} \begin{bmatrix} A C_{xy}^T & A C_{yy} \\ A^T C_{xx} & A^T C_{xy} \end{bmatrix}. \quad (F-14)$$

The rank of matrix $B C$ is generally $R = 2 \min(M, N)$; that is, the product for characteristic function $f_b(\xi)$ in (F-13) will contain only R terms, which is less than L except when $M = N$.

The claim for the rank of $B C$ is based upon the following observation. Suppose that $M \leq N$; then (F-1) can be written as

$$b = \sum_{m=1}^M \sum_{n=1}^N x_m a_{mn} y_n = \sum_{m=1}^M x_m v_m, \quad (F-15)$$

where (linearly transformed) Gaussian random variables

$$v_m = \sum_{n=1}^N a_{mn} y_n \quad \text{for } 1 \leq m \leq M. \quad (F-16)$$

Thus, b is a sum of products of just $2M$ correlated random variables. Therefore, the characteristic function in (F-13) can have no more than $2M$ terms. Direct numerical evaluation of the eigenvalues $\{\lambda_k\}$ of nonsymmetric $L \times L$ matrix $B C$ in (F-14) has verified that its rank is generally $R = 2 \min(M, N)$, and that the R nonzero eigenvalues can be positive and negative with no obvious interrelations. (If $N < M$, the summation order in (F-15) can be reversed, thereby giving b as a sum of products of $2N$ correlated random variables, and hence, rank $2N$ for matrix $B C$.)

The generalization of (F-1) to the form

$$b = X^T A_{11} X + X^T A_{12} Y + Y^T A_{21} X + Y^T A_{22} Y \quad (F-17)$$

can be directly fit into the framework of (F-2) and (F-4) if we generalize definition (F-3) to $L \times L$ matrix

$$B = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} . \quad (F-18)$$

We can then replace this B matrix by its symmetric part, since only the symmetric part of B is active in quadratic form (F-4).

APPENDIX G. CHARACTERISTIC FUNCTION OF MOST GENERAL
COMPLEX FORM WITH FIRST-ORDER AND SECOND-ORDER TERMS

Let Z be a complex $N \times 1$ random vector with real and imaginary parts X and Y ; that is, $Z = X + i Y$. Let D_1 and D_2 be complex $N \times 1$ constant vectors. Let C_1, C_2, C_3, C_4 be complex $N \times N$ constant matrices, which need not be Hermitian or symmetric.

The most general first-order complex form is

$$f_1 = D_1^T Z + D_2^T Z^* = (D_1 + D_2)^T X + i(D_1 - D_2)^T Y = H^T W, \quad (G-1)$$

where

$$H = \begin{bmatrix} D_1 + D_2 \\ i(D_1 - D_2) \end{bmatrix}, \quad W = \begin{bmatrix} X \\ Y \end{bmatrix}. \quad (G-2)$$

H is a complex $2N \times 1$ constant vector and is completely arbitrary; that is, every complex element of H can be independently specified. W is a real $2N \times 1$ random vector.

The most general second-order complex form is

$$f_2 = Z^H C_1 Z + Z^H C_2 Z^* + Z^T C_3 Z + Z^T C_4 Z^* = W^T M W, \quad (G-3)$$

where

$$M = \begin{bmatrix} C_1 + C_2 + C_3 + C_4 & i(C_1 - C_2 + C_3 - C_4) \\ -i(C_1 + C_2 - C_3 - C_4) & C_1 - C_2 - C_3 + C_4 \end{bmatrix}. \quad (G-4)$$

M is a complex $2N \times 2N$ constant matrix, which need not be Hermitian. M is completely arbitrary; that is, every complex element of M can be independently specified.

The pertinent statistics of real vector W are

$$\begin{aligned}\bar{W} &= \begin{bmatrix} \bar{X} \\ \bar{Y} \end{bmatrix}, \quad K = \text{Cov}(W) = \overline{(W - \bar{W})(W - \bar{W})^T} = \\ &= \overline{\begin{bmatrix} X - \bar{X} \\ Y - \bar{Y} \end{bmatrix} \begin{bmatrix} X^T - \bar{X}^T & Y^T - \bar{Y}^T \end{bmatrix}} = \begin{bmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{bmatrix}. \quad (G-5)\end{aligned}$$

Here, $\bar{W}$ is a real $2N \times 1$ constant vector, while K is a real $2N \times 2N$ symmetric constant matrix.

The general complex form of interest is

$$c = f_2 + f_1 = W^T M W + H^T W. \quad (G-6)$$

For X and Y joint-Gaussian, W is Gaussian with probability density function

$$p(W) = (2\pi)^{-N} (\det K)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(W - \bar{W})^T K^{-1} (W - \bar{W})\right). \quad (G-7)$$

The statistical quantity of interest is, for complex scalar α , the characteristic function of complex random variable c in (G-6); in particular, the characteristic function is defined here as the average of the exponential

$$\begin{aligned}f_c(\alpha) &= \overline{\exp(\alpha c)} = \overline{\exp\left(\alpha W^T M W + \alpha H^T W\right)} = (2\pi)^{-N} (\det K)^{-\frac{1}{2}} \times \\ &\times \int dW \exp\left(-\frac{1}{2}(W - \bar{W})^T K^{-1} (W - \bar{W}) + \alpha W^T M W + \alpha H^T W\right) = \\ &= d^{-\frac{1}{2}} \exp(t), \quad (G-8)\end{aligned}$$

where

$$d = \det(I - 2 \alpha M K) = \det(I - 2 \alpha K M) ,$$

$$t = - \frac{1}{2} \bar{W}^T K^{-1} \bar{W} + \frac{1}{2} V^T (I - 2 \alpha K M)^{-1} K V ,$$

$$V = K^{-1} \bar{W} + \alpha H . \quad (G-9)$$

K is a real $2N \times 2N$ matrix, M is a complex $2N \times 2N$ matrix which need not be Hermitian, $\bar{W}$ is a real $2N \times 1$ vector, H and V are complex $2N \times 1$ vectors, and α is a complex scalar.

In general, we must invert $2N \times 2N$ real symmetric matrix K. Also, we must invert $2N \times 2N$ complex matrix $I - 2 \alpha K M$, which need not be Hermitian, and which depends on argument α . The average of the exponential in (G-8) is the type of operation encountered in appendix C.

If $\bar{X} = 0$, $\bar{Y} = 0$, $D_1 = 0$, $D_2 = 0$, then $\bar{W} = 0$, $H = 0$, $V = 0$, $t = 0$, and we need only evaluate complex determinant $d = \det(I - 2 \alpha K M)$, which depends on α .

SPECIAL CASE: $Y = 0$

Then complex forms

$$\begin{aligned} f_1 &= (D_1 + D_2)^T X \equiv D^T X , \\ f_2 &= X^T (C_1 + C_2 + C_3 + C_4) X \equiv X^T C X , \\ c &= X^T C X + D^T X . \end{aligned} \quad (G-10)$$

Matrices C and D are complex and completely arbitrary.

Identify in the subsection above,

$$2N \rightarrow N , \quad W \rightarrow X , \quad M \rightarrow C , \quad H \rightarrow D , \quad K \rightarrow K_{xx} , \quad (G-11)$$

thereby getting characteristic function

$$f_c(\alpha) = \overline{\exp(\alpha c)} = d^{-\frac{1}{2}} \exp(t) , \quad (G-12)$$

where

$$\begin{aligned} d &= \det(I - 2 \alpha K_{xx} C) , \\ t &= -\frac{1}{2} \bar{X}^T K_{xx}^{-1} \bar{X} + \frac{1}{2} V^T (I - 2 \alpha K_{xx} C)^{-1} K_{xx} V , \\ V &= K_{xx}^{-1} \bar{X} + \alpha D . \end{aligned} \quad (G-13)$$

K_{xx} is a real $N \times N$ matrix, C is a complex $N \times N$ matrix which need not be Hermitian, $\bar{X}$ is a real $N \times 1$ vector, D and V are complex $N \times 1$ vectors, and α is a complex scalar.

JOINT CHARACTERISTIC FUNCTION OF REAL AND IMAGINARY PARTS OF c

For the general complex form c in (G-6), the joint characteristic function of c_r and c_i is, for real ξ and ζ ,

$$f(\xi, \zeta) \equiv \overline{\exp(i\xi c_r + i\zeta c_i)} . \quad (G-14)$$

But

$$\begin{aligned} \xi c_r + \zeta c_i &= \xi (W^T M_r W + H_r^T W) + \zeta (W^T M_i W + H_i^T W) = \\ &= W^T \tilde{M} W + \tilde{H}^T W , \end{aligned} \quad (G-15)$$

where real matrices

$$\tilde{M} \equiv \xi M_r + \zeta M_i , \quad \tilde{H} \equiv \xi H_r + \zeta H_i . \quad (G-16)$$

Therefore, identifying $\alpha \rightarrow i$, $M \rightarrow \tilde{M}$, $H \rightarrow \tilde{H}$, in (G-8), there follows the joint characteristic function of c_r and c_i as

$$f(\xi, \zeta) = \overline{\exp(i W^T \tilde{M} W + i \tilde{H}^T W)} = d^{-\frac{1}{2}} \exp(t) , \quad (G-17)$$

where

$$\begin{aligned} d &= \det(I - i 2 \tilde{M} K) = \det(I - i 2 K \tilde{M}) , \\ t &= -\frac{1}{2} \bar{W}^T K^{-1} \bar{W} + \frac{1}{2} V^T (I - i 2 K \tilde{M})^{-1} K V , \end{aligned}$$

$$V = K^{-1} \bar{W} + i \tilde{H} . \quad (G-18)$$

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Statistics of a Whiteness Measure

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ABSTRACT

A whiteness measure of a random number generator is defined as the sum of squares of all the off-zero elements of the sample covariance function of a finite segment of data of length K . The mean and variance of this whiteness measure are evaluated exactly, while its cumulative and exceedance distribution functions are determined by simulations. It is found that the variance of the whiteness measure must be broken into the two separate cases where K is even versus K is odd. This necessitates the analytic evaluation of some high-order moments in order to determine the variance exactly.

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LIST OF SYMBOLS

K	number of data points, (1)
x_k	k-th data value, (1)
IID	independent identically distributed
$E()$	expectation of random variable, (1)
F	fourth moment of random variable x_k , (1)
n	n-th delay, (2)
R_n	sample covariance at delay n , (2)
W_K	whiteness measure, (3)
Var	variance, (10)
V_K	variance of K-th whiteness measure W_K , (10)
A, B, C, D	unknown constants in (10) and (35)
ϕ_n	sum of delayed products of data, (11)
$A, \tilde{B}, \tilde{C}$	constants for K odd, (45)
A, B, C	constants for K even, (48)
$p(x)$	probability density function of random variable x , (51)
D	constant for K even or odd, (55)
M	size of fast Fourier transform, (56)
$\{X_m\}$	fast Fourier transform of data $\{x_k\}$, (56)
u	threshold value, (56)
$CDF(u)$	cumulative distribution function, (57)
$EDF(u)$	exceedance distribution function, (57)
V_n	interval probabilities, (B-1)
C	cumulative distribution function, (B-3)
E	exceedance distribution function, (B-3)
μ_n	n-th moment, (B-8)

STATISTICS OF A WHITENESS MEASURE

INTRODUCTION

When a random number generator is designed to yield zero-mean independent random variables, one useful test of its validity is afforded by its sample covariance function. This quantity would ideally be zero for all delays except the origin value. However, in practice, due to the finite length of data generated and used to test the generator, the sample covariance function is not identically zero but fluctuates about zero. A measure of the whiteness of the generator is afforded by the sum of squares of all the off-zero elements of the sample covariance function, relative to the square of its origin value. This measure was suggested in [1; appendix C].

In this report, we investigate the statistics of this whiteness measure, including its cumulative and exceedance distribution functions and its mean and variance. Since a sample covariance involves products of data values, the squared covariance depends on fourth-order products of the data, and the variance of this sample quantity involves eighth-order products of the data under various delays. It is this latter high-order product which greatly complicates the statistical analysis and which necessitates a roundabout procedure for exact evaluation of the variance of the whiteness measure. The probability distributions of this measure are determined by simulation for two types of random variables, uniform and Gaussian.

MEAN AND VARIANCE OF WHITENESS MEASURE

Consider real data sequence $x_0, x_1, \dots, x_{K-1}$ of K data points which are independent and identically distributed (IID) with a symmetric probability density function about zero. This zero-mean sequence will have all odd-order moments equal to zero. Also, assume that the data are scaled to have unit variance and a fourth moment of value F ; that is

$$E(x_k^2) = 1, \quad E(x_k^4) = F, \quad \text{for } 0 \leq k \leq K-1, \quad (1)$$

where E denotes the expectation. This situation includes the uniform random number generator and the Gaussian random number generator, for example. For the usual uniform random variable distributed over $(-\frac{1}{2}, \frac{1}{2})$, we have scaled its output by $\sqrt{12}$ for present purposes in order to realize variance 1. Thus, $F = 1.8$ for the uniform case, while $F = 3$ for Gaussian numbers.

The sample covariance of the available data is defined as

$$R_n = \frac{1}{K} \sum_k x_k x_{k-n} \quad \text{for all } n. \quad (2)$$

Ideally, we might like to have sequence $\{R_n\}$ equal to zero for $n \neq 0$. However, this is never the case, although the $\{R_n\}$ for $n \neq 0$ are much smaller than R_0 when K is large. The mean value of R_0 is easily seen to be 1, by reference to (1). A measure of the whiteness of data sequence $\{x_k\}$ is afforded by the sum of squares of all the off-zero elements of sequence $\{R_n\}$:

$$W_K \equiv \sum_{n \neq 0} R_n^2 = 2 \sum_{n=1}^{K-1} R_n^2 \quad \text{for } K \geq 2. \quad (3)$$

MEAN OF WHITENESS MEASURE W_K

The mean value of random variable R_n^2 follows from (2) as

$$\begin{aligned} E(R_n^2) &= E\left(\frac{1}{K^2} \sum_k \sum_j x_k x_{k-n} x_j x_{j-n}\right) = \\ &= \frac{1}{K^2} \sum_k \sum_j E(x_k x_{k-n} x_j x_{j-n}) . \end{aligned} \quad (4)$$

Since we are only interested in values of $n > 0$ according to (3), the expectation in (4) is nonzero only when $k = j$; here, we are utilizing both the IID and the zero-mean properties of $\{x_k\}$. Then, (4) becomes, upon use of (1),

$$E(R_n^2) = \frac{1}{K^2} \sum_{k=n}^{K-1} 1 = \frac{K-n}{K^2} \quad \text{for } 1 \leq n \leq K-1 . \quad (5)$$

(For completeness, $E(R_0^2) = (F + K - 1)/K$; $\text{Variance}(R_0) = (F-1)/K$. Thus, R_0 clusters around 1 as $K \rightarrow \infty$, while $R_n \rightarrow 0$ as $K \rightarrow \infty$ for fixed $n \neq 0$.) Use of result (5) in (3) yields the desired mean value of whiteness measure W_K as

$$E(W_K) = \frac{2}{K^2} \sum_{n=1}^{K-1} (K-n) = \frac{K-1}{K} . \quad (6)$$

Notice that this mean value is independent of fourth-moment F and that it approaches 1 as $K \rightarrow \infty$. Recall that $E(R_0) = 1$ for comparison.

VARIANCE OF WHITENESS MEASURE W_K

The direct evaluation of the variance of random variable W_K in (3) would require a very tedious procedure. Whereas the mean evaluation in (4) only encountered fourth-order products of delayed versions of $\{x_k\}$, we would now encounter eighth-order products, requiring a complicated counting procedure to account for all the various types of terms. Specifically, from (2) and (3), we have whiteness measure

$$W_K = \frac{2}{K^2} \sum_{n=1}^{K-1} \sum_{k=n}^{K-1} \sum_{j=n}^{K-1} x_k x_{k-n} x_j x_{j-n} , \quad (7)$$

leading to mean square value

$$E(W_K^2) = \frac{4}{K^4} \sum_{n=1}^{K-1} \sum_{m=1}^{K-1} \sum_{k=n}^{K-1} \sum_{j=n}^{K-1} \sum_{q=m}^{K-1} \sum_{p=m}^{K-1} E(x_k x_{k-n} x_j x_{j-n} x_q x_{q-m} x_p x_{p-m}) . \quad (8)$$

Not only would this eighth-order average have to be evaluated for all possible values of n, m, k, j, q, p , but the sixth-order summation would then have to be conducted. The only reasonable case that can be evaluated from (8) is that for the term proportional to F^2 . It is obtained only for the special choices $n = m$ and $k = j = q = p$; then the right-hand side of (8) reduces to

$$\frac{4}{K^4} \sum_{n=1}^{K-1} \sum_{k=n}^{K-1} F^2 = \frac{4}{K^4} \sum_{n=1}^{K-1} (K - n) F^2 = \frac{2(K - 1)}{K^3} F^2 . \quad (9)$$

Notice that moments of $\{x_k\}$ above the fourth need not be known.

The difficulty of attempting to evaluate (8) directly forces us to attack the problem from a different aspect. Specifically, we adopt a shortcut to obtain, exactly, the variance of whiteness measure W_K . First, observe from (8) that the mean square value of W_K contains a denominator of K^4 . Secondly, it has been observed from simulations that the variance of W_K goes to zero proportional to $1/K$ for large K . Therefore, the form of the variance, V_K , of random variable W_K must be

$$V_K = \text{Var}(W_K) = \frac{A K^3 + B K^2 + C K + D}{K^4}, \quad (10)$$

where A, B, C, D are unknown constants. In order to determine these four constants, we will evaluate, exactly, the variance V_K of W_K for a sufficient number of low-order values of K , and then solve the four simultaneous linear equations yielded by (10).

For convenience, we define the sums

$$\phi_n = \sum_{k=n}^{K-1} x_k x_{k-n} \quad \text{for } 1 \leq n \leq K-1. \quad (11)$$

Then

$$R_n = \frac{1}{K} \phi_n \quad \text{for } 1 \leq n \leq K-1, \quad (12)$$

as seen from (2). The whiteness measure in (3) then takes the form

$$W_K = \frac{2}{K^2} \sum_{n=1}^{K-1} \phi_n^2 \quad \text{for } K \geq 2. \quad (13)$$

For $K = 1$, there are no terms in the sum, yielding $W_1 = 0$.

SPECIAL CASE $K = 2$

We have, from (11) and (13),

$$\phi_1 = x_1 x_0, \quad W_2 = \frac{2}{4} \phi_1^2 = \frac{1}{2} x_1^2 x_0^2. \quad (14)$$

Therefore, upon use of the IID property of the $\{x_k\}$ and (1),

$$E(W_2^2) = \frac{1}{4} E(x_1^4 x_0^4) = \frac{1}{4} F^2. \quad (15)$$

The variance of W_2 then follows as

$$V_2 = \text{Var}(W_2) = E(W_2^2) - E(W_2)^2 = \frac{1}{4}(F^2 - 1), \quad (16)$$

where we used (6).

SPECIAL CASE $K = 3$

The procedure for the remaining cases is similar to that detailed above for $K = 2$; therefore, the following presentation will be abbreviated, and only the main results will be listed. We have

$$\phi_1 = x_1 x_0 + x_2 x_1, \quad \phi_2 = x_2 x_0, \quad (17)$$

$$W_3 = \frac{2}{9} [\phi_1^2 + \phi_2^2] = \frac{2}{9} [x_1^2 (x_0 + x_2)^2 + x_2^2 x_0^2], \quad (18)$$

$$W_3^2 = \frac{4}{81} [x_1^4 (x_0 + x_2)^4 + x_2^4 x_0^4 + 2 x_1^2 (x_0 + x_2)^2 x_2^2 x_0^2]. \quad (19)$$

The mean value of (19) is given by

$$E(W_3^2) = \frac{4}{81} \left(F(F + 6 + F) + F^2 + 2(F + F) \right) = \frac{4}{81} (3F^2 + 10F) . \quad (20)$$

Finally, the variance of W_3 is

$$V_3 = \frac{4}{81} (3F^2 + 10F - 9) . \quad (21)$$

SPECIAL CASE $K = 4$

In this case, we have

$$\phi_1 = x_1 x_0 + x_2 x_1 + x_3 x_2 , \quad \phi_2 = x_2 x_0 + x_3 x_1 , \quad \phi_3 = x_3 x_0 , \quad (22)$$

$$W_4 = \frac{2}{16} [\phi_1^2 + \phi_2^2 + \phi_3^2] , \quad (23)$$

$$64 W_4^2 = \phi_1^4 + \phi_2^4 + \phi_3^4 + 2 \phi_1^2 \phi_2^2 + 2 \phi_1^2 \phi_3^2 + 2 \phi_2^2 \phi_3^2 . \quad (24)$$

The mean value of (24) will be found in stages. The six components of (24) have the following average values:

$$E(\phi_3^4) = E(x_3^4 x_0^4) = F^2 , \quad (25)$$

$$E(\phi_3^2 \phi_2^2) = E(x_3^2 x_0^2 (x_2 x_0 + x_3 x_1)^2) = F + F = 2F , \quad (26)$$

$$E(\phi_3^2 \phi_1^2) = E(x_3^2 x_0^2 (x_1 x_0 + x_2 x_1 + x_3 x_2)^2) = F + 1 + F = 2F + 1 , \quad (27)$$

$$E(\phi_2^4) = E((x_2 x_0 + x_3 x_1)^4) = F^2 + 6 + F^2 = 2F^2 + 6 , \quad (28)$$

$$\begin{aligned}
E(\phi_2^2 \phi_1^2) &= E((x_2 x_0 + x_3 x_1)^2 (x_1 x_0 + x_2 x_1 + x_3 x_2)^2) = \\
&= E\left([x_2^2 x_0^2 + x_3^2 x_1^2 + 2 x_3 x_2 x_1 x_0][x_1^2 x_0^2 + x_2^2 x_1^2 + \right. \\
&\quad \left.+ x_3^2 x_2^2 + 2 x_2 x_1^2 x_0 + 2 x_3 x_2 x_1 x_0 + 2 x_3 x_2^2 x_1]\right) = \\
&= F + F + F + F + F + F + 4 = 6F + 4 , \tag{29}
\end{aligned}$$

$$\phi_1^2 = x_1^2(x_0 + x_2)^2 + x_3^2 x_2^2 + 2 x_3 x_2 x_1(x_0 + x_2) , \tag{30}$$

$$\begin{aligned}
\phi_1^4 &= x_1^4(x_0 + x_2)^4 + x_3^4 x_2^4 + 6 x_3^2 x_2^2 x_1^2(x_0 + x_2)^2 + \\
&\quad + 4 x_3 x_2 x_1^3(x_0 + x_2)^3 + 4 x_3^3 x_2^3 x_1(x_0 + x_2) , \tag{31}
\end{aligned}$$

$$E(\phi_1^4) = F(F + 6 + F) + F^2 + 6(1 + F) = 3F^2 + 12F + 6 . \tag{32}$$

Combining these results into (24), we have mean square value

$$E(W_4^2) = \frac{1}{32}(3F^2 + 16F + 11) \tag{33}$$

and variance

$$V_4 = \frac{1}{32}(3F^2 + 16F - 7) . \tag{34}$$

The analytical derivations of V_5 , V_6 , V_7 , V_8 are deferred to appendix A due to their lengthy calculations and need for a shorthand notation. It will turn out that we also need all of these latter results when we find the constants A, B, C, D in variance expression (10).

GENERAL DETERMINATION OF VARIANCE OF W_K

The general form for the variance V_K of whiteness measure W_K is given by (10) for arbitrary K and is repeated below:

$$V_K = \text{Var}(W_K) = \frac{A K^3 + B K^2 + C K + D}{K^4} . \quad (35)$$

However, analytic determination of V_K for $K = 2, 3, 4, 5, 6, 7, 8$ (see appendix A also) have revealed that separate forms like (35) must be employed for K even versus K odd. That is, two different sets of constants A, B, C, D apply in the even versus odd cases of K . The available analytic results for V_K (above and in appendix A) are summarized below:

$$V_1 = 0 \quad (\text{see the line under (13)}) , \quad (36)$$

$$V_2 = \frac{1}{4}(F^2 - 1) , \quad (37)$$

$$V_3 = \frac{4}{81}(3F^2 + 10F - 9) , \quad (38)$$

$$V_4 = \frac{1}{32}(3F^2 + 16F - 7) , \quad (39)$$

$$V_5 = \frac{8}{625}(5F^2 + 38F - 11) , \quad (40)$$

$$V_6 = \frac{1}{324}(15F^2 + 144F - 23) , \quad (41)$$

$$V_7 = \frac{4}{2401}(21F^2 + 246F - 23) , \quad (42)$$

$$V_8 = \frac{1}{256}(7F^2 + 96F - 3) . \quad (43)$$

If we take K equal to the odd values 1, 3, 5, 7 in (35) and use results (36), (38), (40), (42), we obtain four simultaneous linear equations for the constants A, B, C, D . Their solution leads to the following expression for the variance V_K of W_K :

$$V_K = \frac{K-1}{K^4} (A K^2 + \tilde{B} K + \tilde{C}) \quad \text{for } K \text{ odd} , \quad (44)$$

where

$$A = 4F + \frac{4}{3} , \quad \tilde{B} = 2F^2 - 4F - \frac{38}{3} , \quad \tilde{C} = -4F + 8 . \quad (45)$$

When (45) is substituted into (44), the variance expression can be rearranged in terms of powers of F :

$$V_K = \frac{2(K-1)}{K^4} \left[KF^2 + 2(K^2 - K - 1)F + \frac{1}{3}(2K^2 - 19K + 12) \right] \quad \text{for } K \text{ odd} . \quad (46)$$

The F^2 term here confirms (9), as anticipated.

If we take K equal to the even values 2, 4, 6, 8 in (35) and use results (37), (39), (41), (43), we obtain four different simultaneous linear equations for the constants A, B, C, D . Their solution leads to the following expression for the variance V_K of W_K :

$$V_K = \frac{1}{K^3} (A K^2 + B K + C) \quad \text{for } K \text{ even} , \quad (47)$$

where

$$A = 4F + \frac{4}{3} , \quad B = 2F^2 - 8F - 14 , \quad C = -2F^2 + \frac{62}{3} . \quad (48)$$

When (48) is substituted into (47), the variance expression can be rearranged in terms of powers of F according to

$$V_K = \frac{2}{K^3} \left[(K-1)F^2 + 2K(K-2)F + \frac{1}{3}(2K^2 - 21K + 31) \right] \quad \text{for } K \text{ even} . \quad (49)$$

Again, the F^2 dependence in (9) is confirmed.

The asymptotic behavior of variance V_K for large K is given by

$$V_K \sim \left(4F + \frac{4}{3} \right) \frac{1}{K} \quad \text{as } K \rightarrow \infty \quad (50)$$

for both K odd and K even. This is due to the fact that constant A in (35) is identical for the odd and even cases; compare (45) and (48). Thus, whiteness measure W_K tends to cluster around 1 as $K \rightarrow \infty$. Recall that $R_0 \rightarrow 1$, while $R_n \rightarrow 0$ for fixed n , as $K \rightarrow \infty$.

The end results for variance V_K of whiteness measure W_K are given by (44) and (47), or by (46) and (49). Plots of V_K for the uniform random variable and the Gaussian random variable $\{x_k\}$ are displayed in figures 1 and 2, respectively. A short tabulation of V_K is given in table 1 for the uniform, Gaussian, exponential, and alternating random variables $\{x_k\}$. The probability density functions of $\{x_k\}$ for these four cases are, respectively,

$$p_u(x) = .5/\sqrt{3} \quad \text{for } |x| < \sqrt{3} , \quad F = 1.8 ; \quad (51)$$

$$p_g(x) = (2\pi)^{-1/2} \exp(-x^2/2) , \quad F = 3 ; \quad (52)$$

$$p_e(x) = \frac{1}{\sqrt{2}} \exp(-\sqrt{2}|x|) , \quad F = 6 ; \quad (53)$$

$$p_a(x) = \frac{1}{2} \delta(x-1) + \frac{1}{2} \delta(x+1) , \quad F = 1 . \quad (54)$$

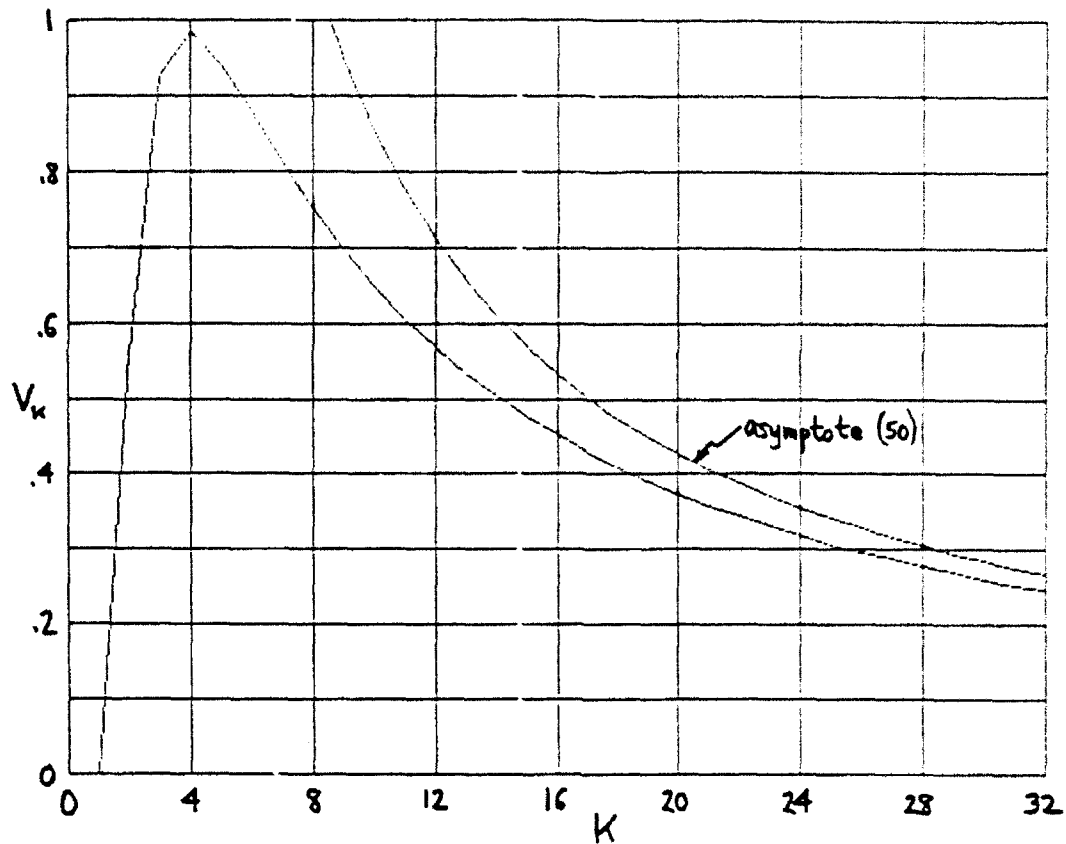


Figure 1. Variance V_K for Uniform Random Variables $\{x_k\}$

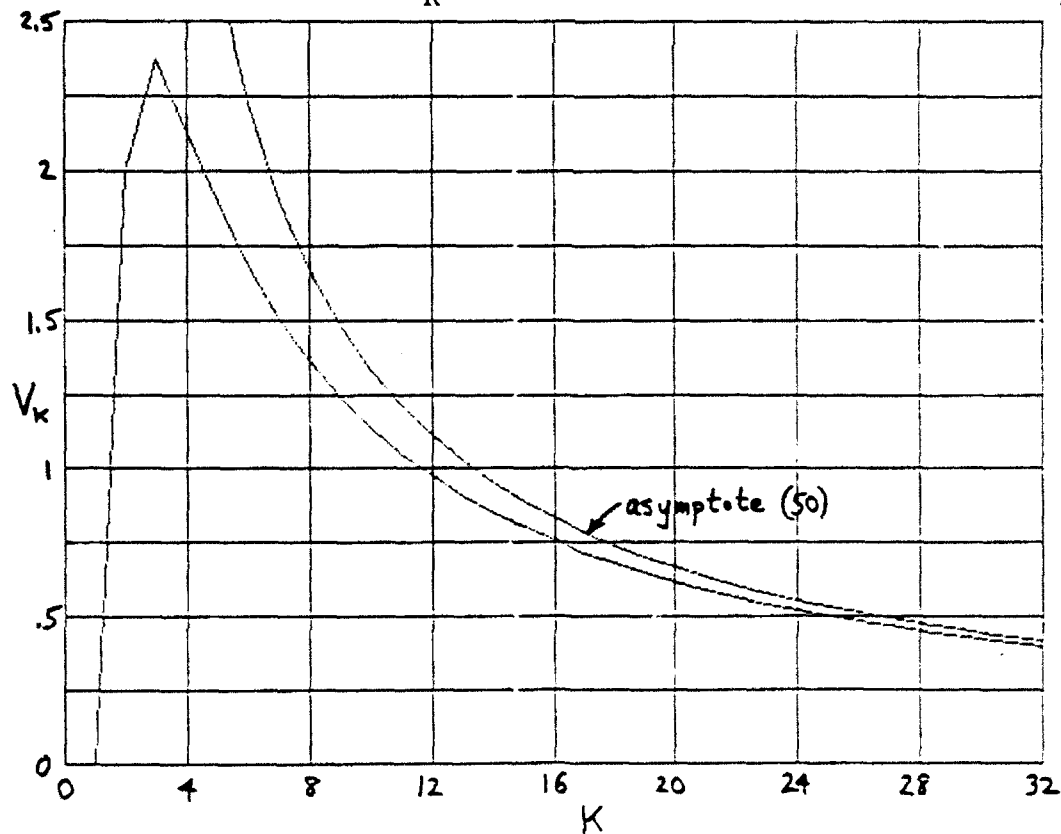


Figure 2. Variance V_K for Gaussian Random Variables $\{x_k\}$

A short table of the variances for these four examples is given below. For the alternating example, $x_k = \pm 1$ and $F = 1$, whiteness measure W_2 for $K = 2$ is always equal to $1/2$, thereby leading to variance $V_2 = 0$. The smallest possible example of F is 1, as realized in the alternating random variable case.

Table 1. Variance V_K of Whiteness Measure W_K

K	Uniform	Gaussian	Exponential	Alternating
2	.56	2.	8.75	.0
3	.92444	2.37037	7.85185	.19753
4	.985	2.125	6.15625	.375
5	.94208	1.8944	5.0816	.4096
6	.87901	1.67901	4.26235	.41975
7	.81273	1.50604	3.68013	.40650
8	.75188	1.35938	3.22266	.39063
16	.45117	.75586	1.60986	.25977
32	.24569	.39722	.79987	.14771
64	.12804	.20346	.39808	.07852
128	.06534	.10295	.19850	.04045

If we combine (47) with the multiplied-out version of (44), the variance V_K can indeed be written in the form (35) for all K , where the constants A , B , C are as given in (48), but constant D must be taken according to the two different values

$$D = \begin{cases} 0 & \text{for } K \text{ even} \\ 4(F - 2) & \text{for } K \text{ odd} \end{cases} . \quad (55)$$

Notice that, despite (8) involving eighth-order products, nothing above fourth-order moment F of $\{x_k\}$ is required in these results.

PROBABILITY DISTRIBUTIONS OF WHITENESS MEASURE

The direct evaluation of whiteness measure W_K , according to its definition (3) in conjunction with (2), is very time consuming for large K , due to all the multiplications required. An attractive alternative, in terms of fast Fourier transforms, was derived in [1; appendix C] and is employed here; the program utilized is listed in appendix B. The key relation relative to (3) is [1; (C-5)]

$$W_K = \frac{1}{K^2 M^2} \left[M \sum_{m=0}^{M-1} |X_m|^4 - \left(\sum_{m=0}^{M-1} |X_m|^2 \right)^2 \right] , \quad (56)$$

where M is the size of the fast Fourier transform $\{X_m\}$ of data $\{x_k\}$. The only restriction on M is that we must use $M \geq 2K - 1$; then, the right-hand side of (56) is independent of M . (For $K = 1$, $X_m = x_0$ for $0 \leq m \leq M-1$, leading to $W_1 = 0$, as noted under (13).) Again, notice that the whiteness measure W_K depends on fourth-order products of the data or its transform.

The cumulative distribution function (CDF) and exceedance distribution function (EDF) of whiteness measure W_K ,

$$\text{CDF}(u) = \text{Prob}(W_K < u) , \quad \text{EDF}(u) = \text{Prob}(W_K > u) , \quad (57)$$

for the case where data $\{x_k\}$ is uniformly distributed over $-\sqrt{3}, \sqrt{3}$ [see (51)], are displayed in figures 3 - 10 for $K = 2, 3, 4, 8, 16, 32, 64, 128$, respectively. These results were determined by using at least one million trials for W_K as defined in (56). The

exceedance distribution function for small K has a cusp near zero argument which disappears for larger K . However, random variable W_K does not approach Gaussian as K increases; rather, as shown in figure 10 for $K = 128$, the right-hand tail appears to approach exponential behavior. For a bounded random variable, $|x_k| < B$, the value of W_K is bounded according to

$$W_K < \frac{(K - 1)(2K - 1)}{3K} B^4 . \quad (58)$$

In the case of the uniform random variable x_k , where $B = \sqrt{3}$, (58) yields 4.5 for $K = 2$, 10 for $K = 3$, and 15.75 for $K = 4$.

Although the mean of W_{128} is $127/128$ and its variance is $V_{128} = .06534$, the standard deviation of W_{128} is 0.256; this leads to the possibility of large values of W_{128} on occasion. For example, figure 10 shows that the whiteness measure can reach a value of 1.8 or larger about 1% of the time. If a candidate uniform random number generator has probability distributions for W_K which differ significantly from figures 3 - 10, it is suspect and should be more thoroughly investigated before further use.

The corresponding cumulative and exceedance distribution functions of the whiteness measure W_K for a Gaussian random number generator [see (52)] are displayed in figures 11 - 18 for $K = 2, 3, 4, 8, 16, 32, 64, 128$, respectively. The first observation to make is that the positive tail of W_K can now reach much larger values when K is small. However, for the larger values of K , the probability distributions of W_K appear to be approaching a common behavior, regardless of the distribution of the underlying data $\{x_k\}$; compare figures 10 and 18 for $K = 128$.

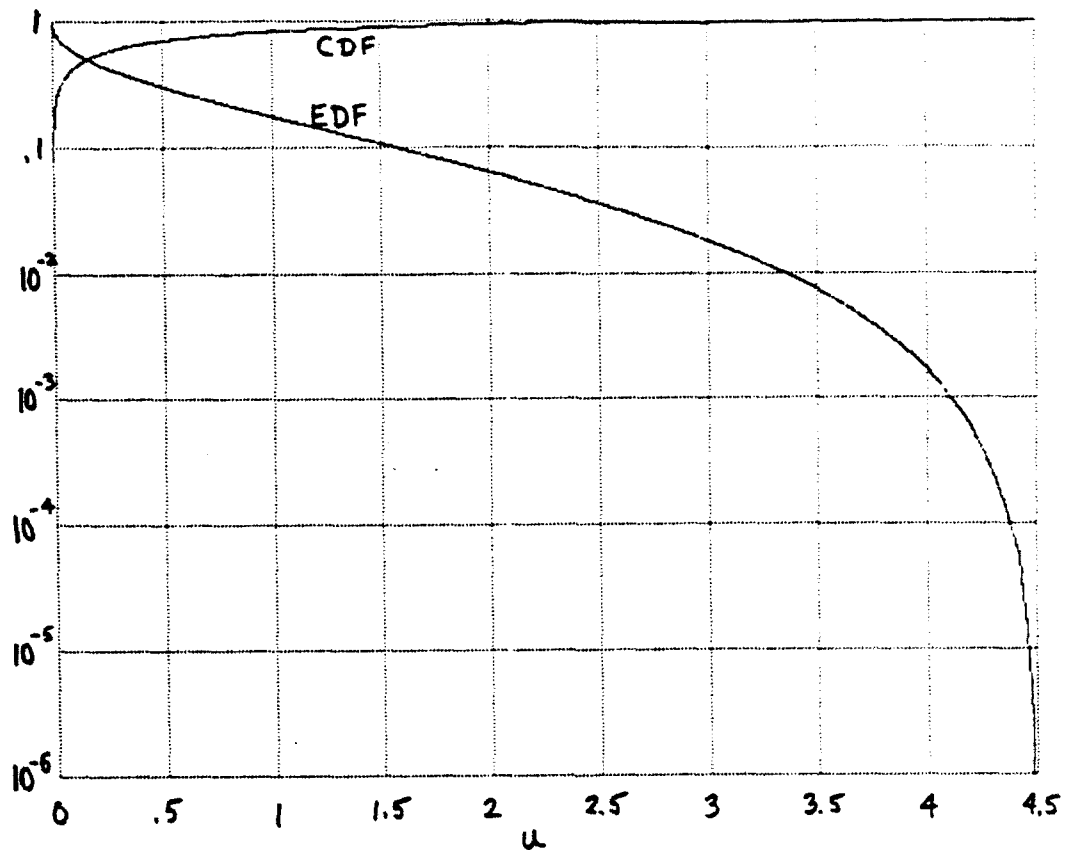


Figure 3. Distributions of W_2 for Uniform Random Variables

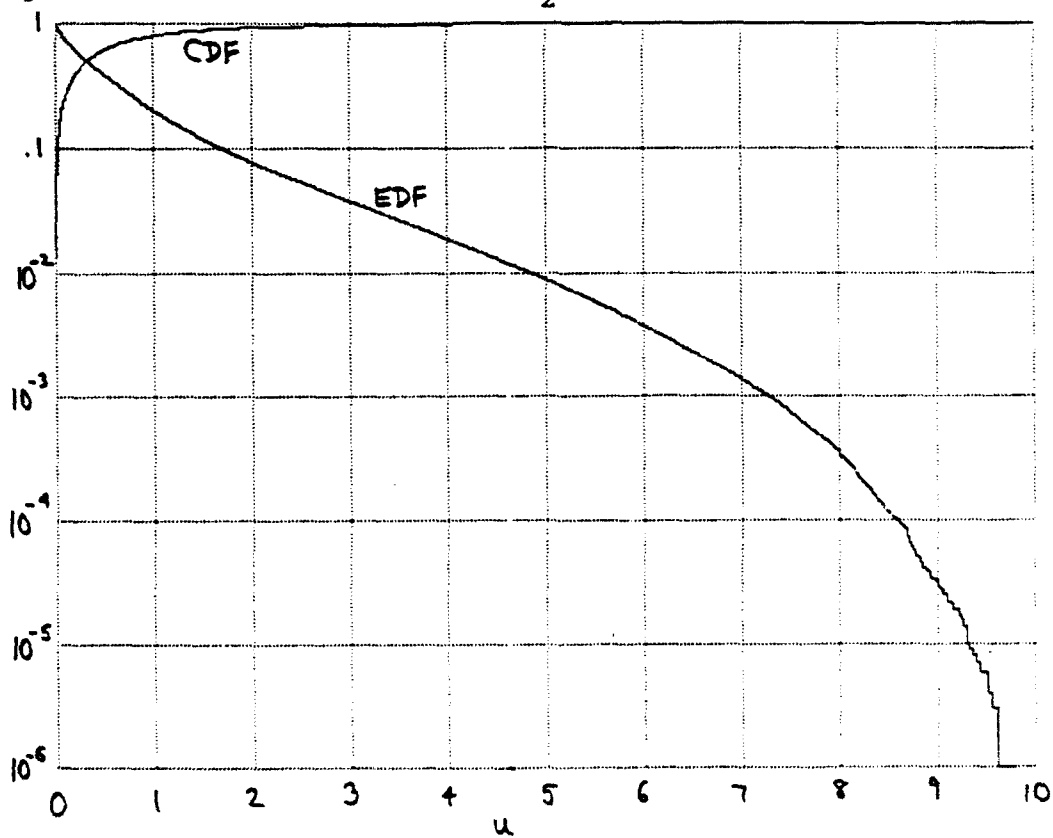
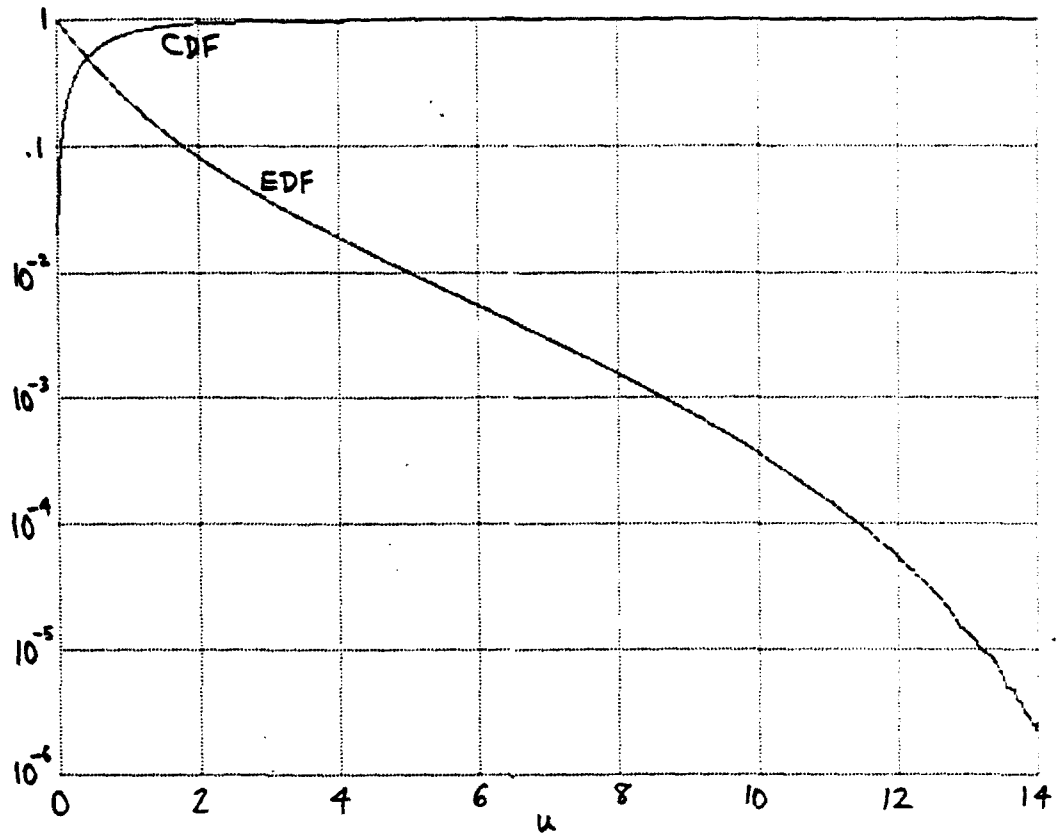
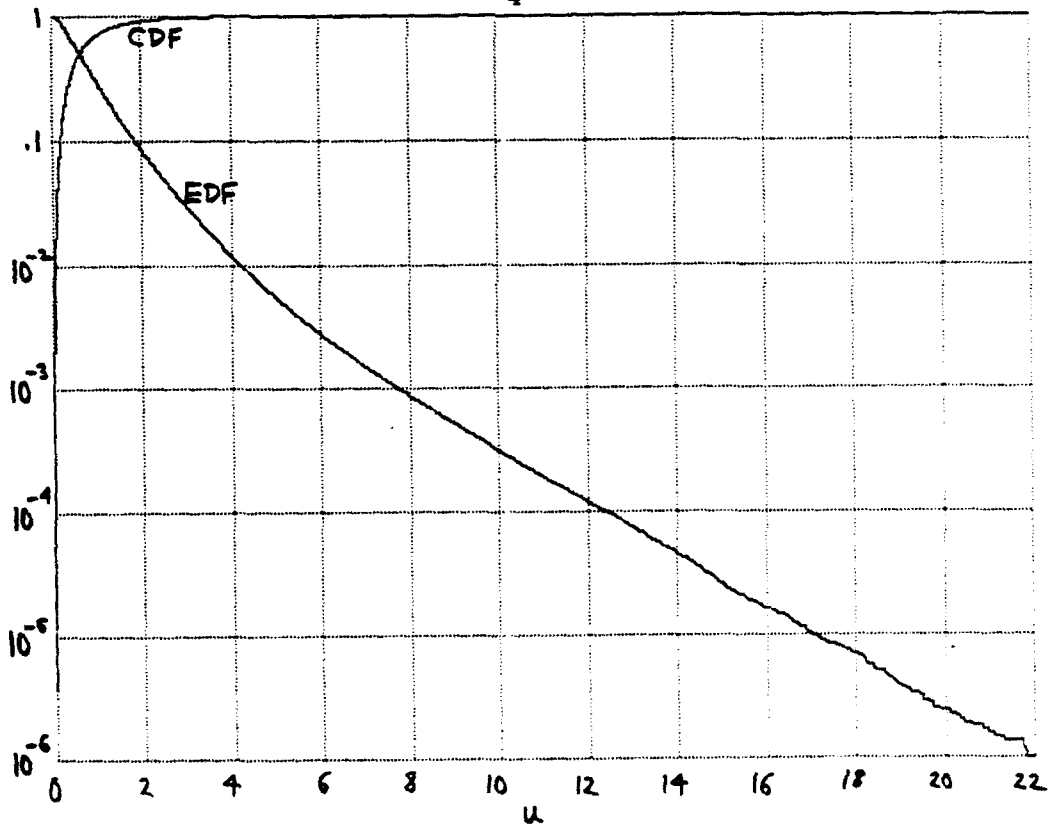


Figure 4. Distributions of W_3 for Uniform Random Variables

Figure 5. Distributions of W_4 for Uniform Random VariablesFigure 6. Distributions of W_8 for Uniform Random Variables

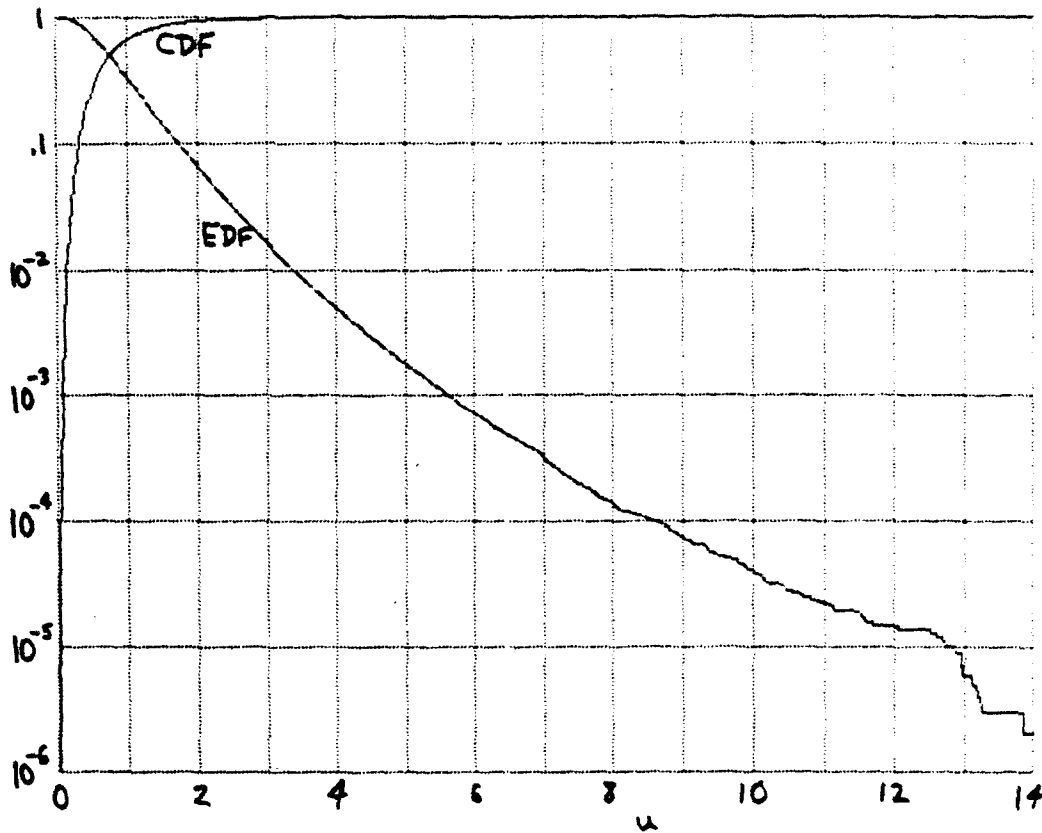


Figure 7. Distributions of W_{16} for Uniform Random Variables

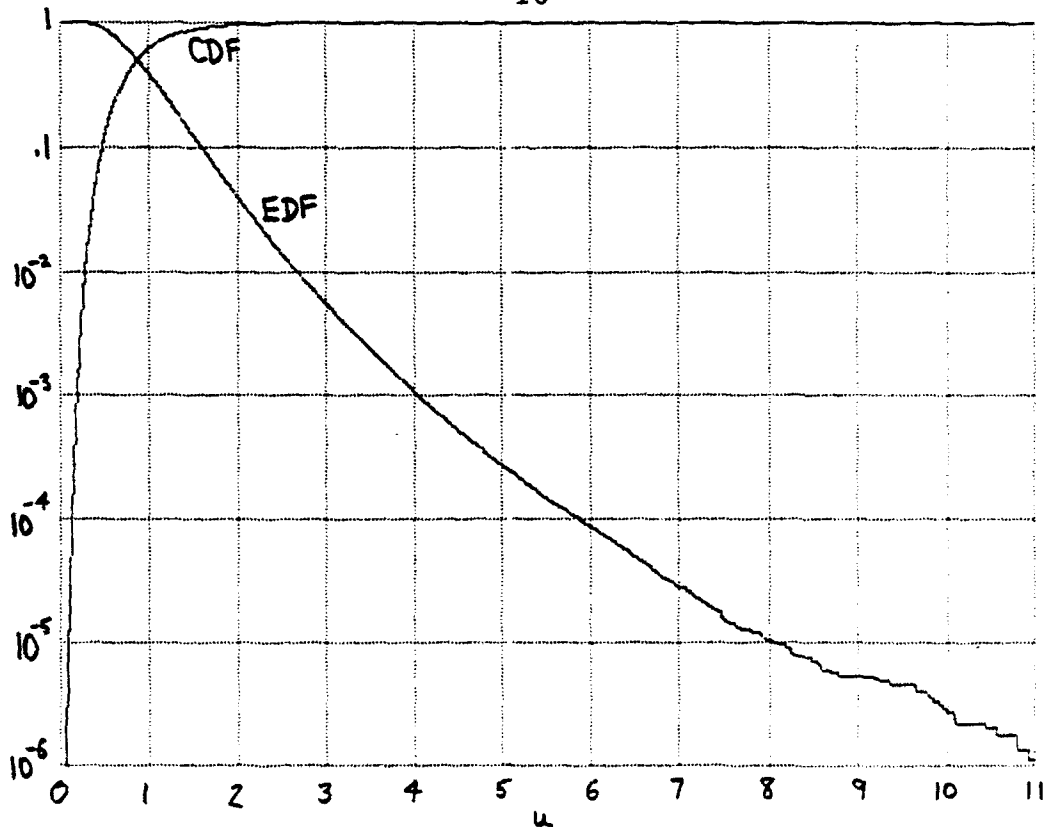
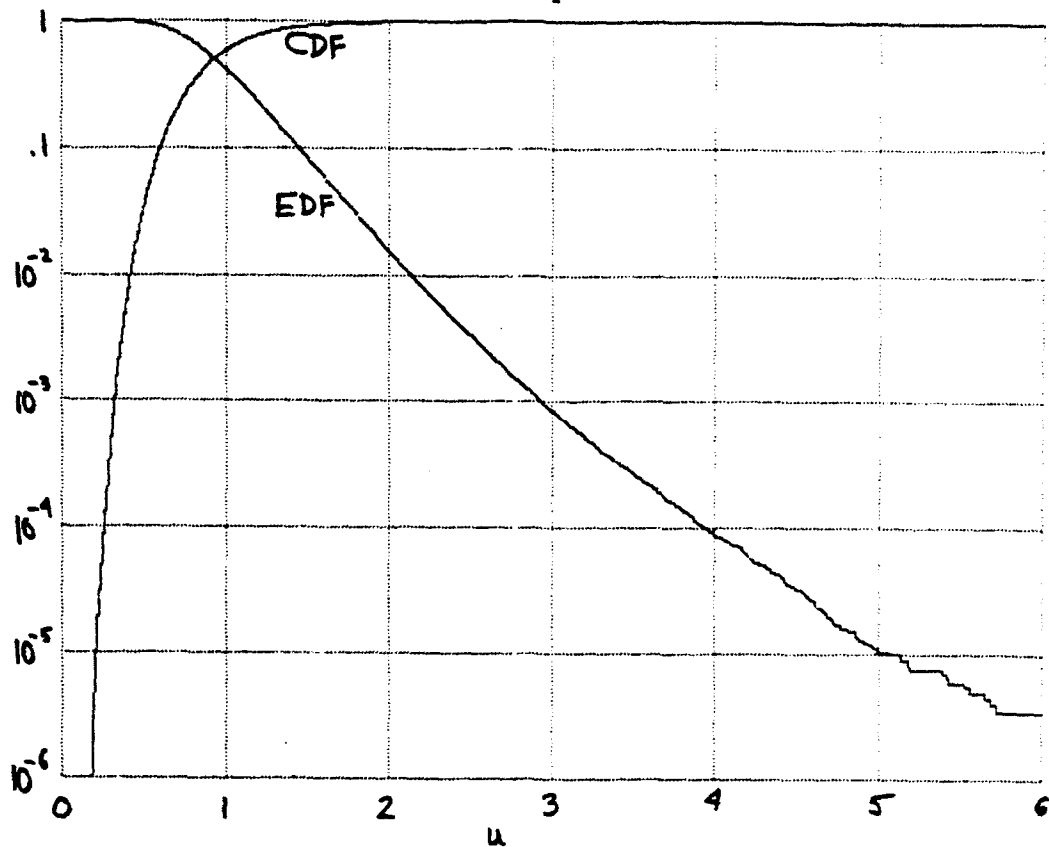
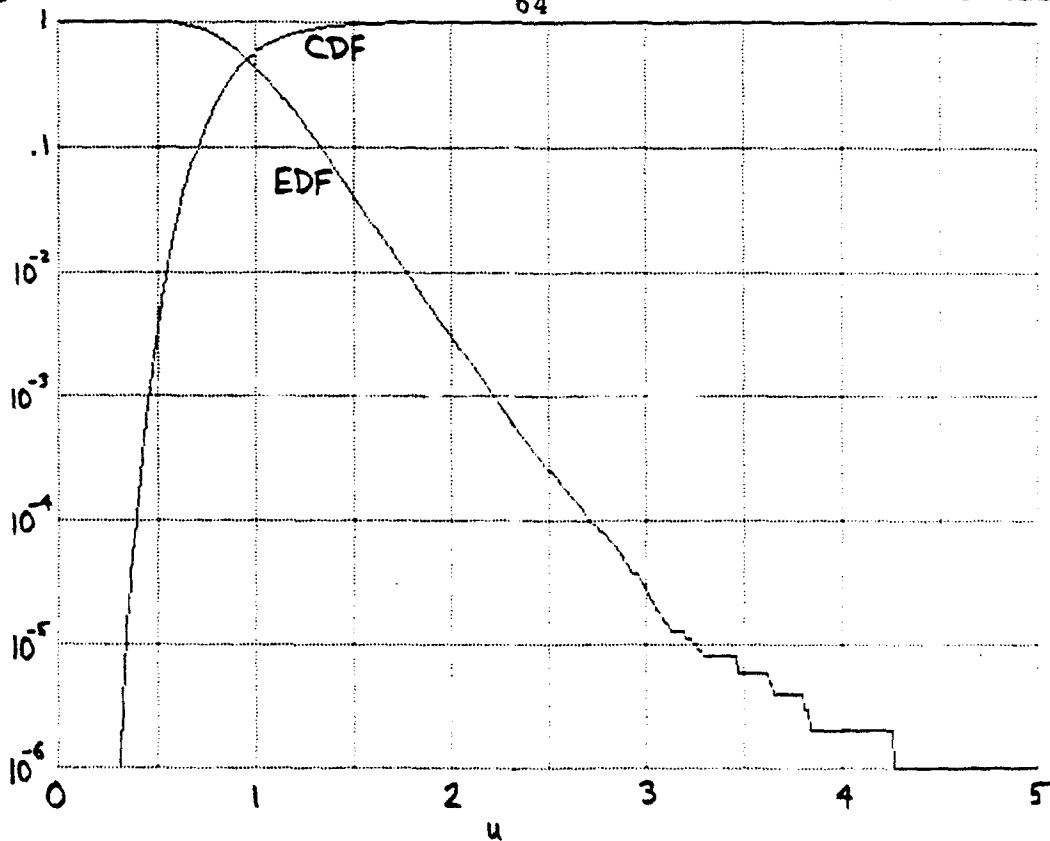
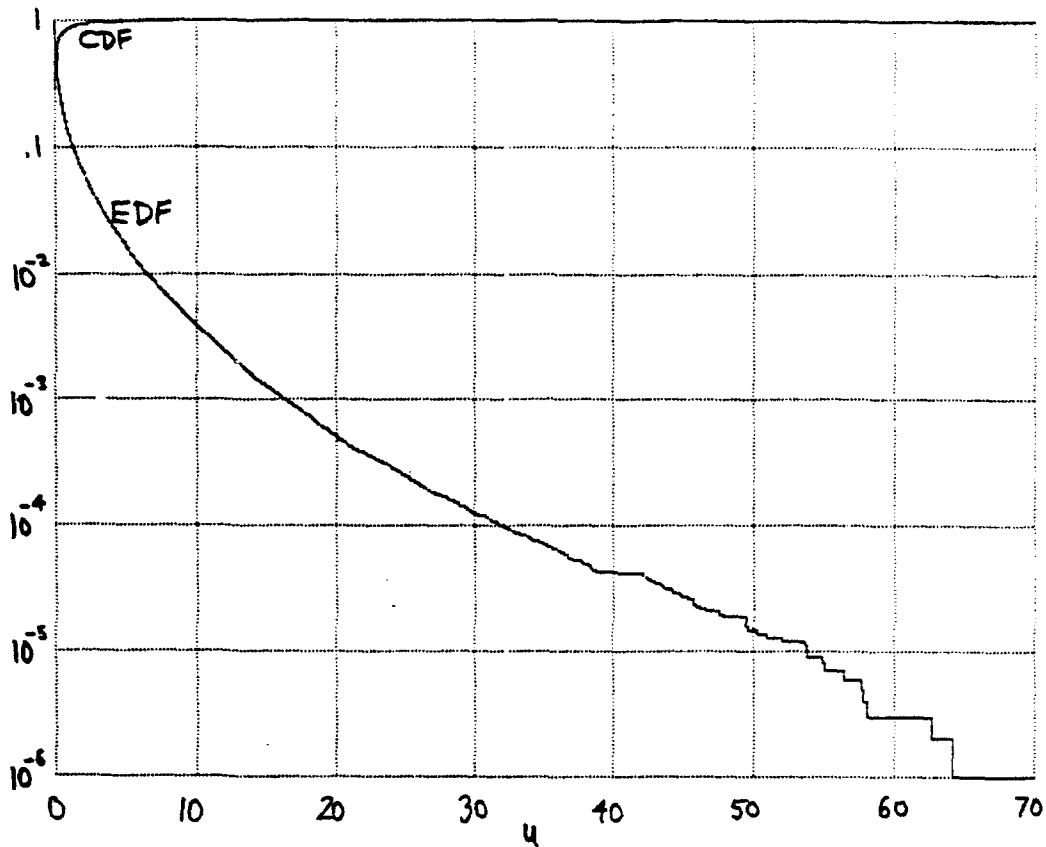
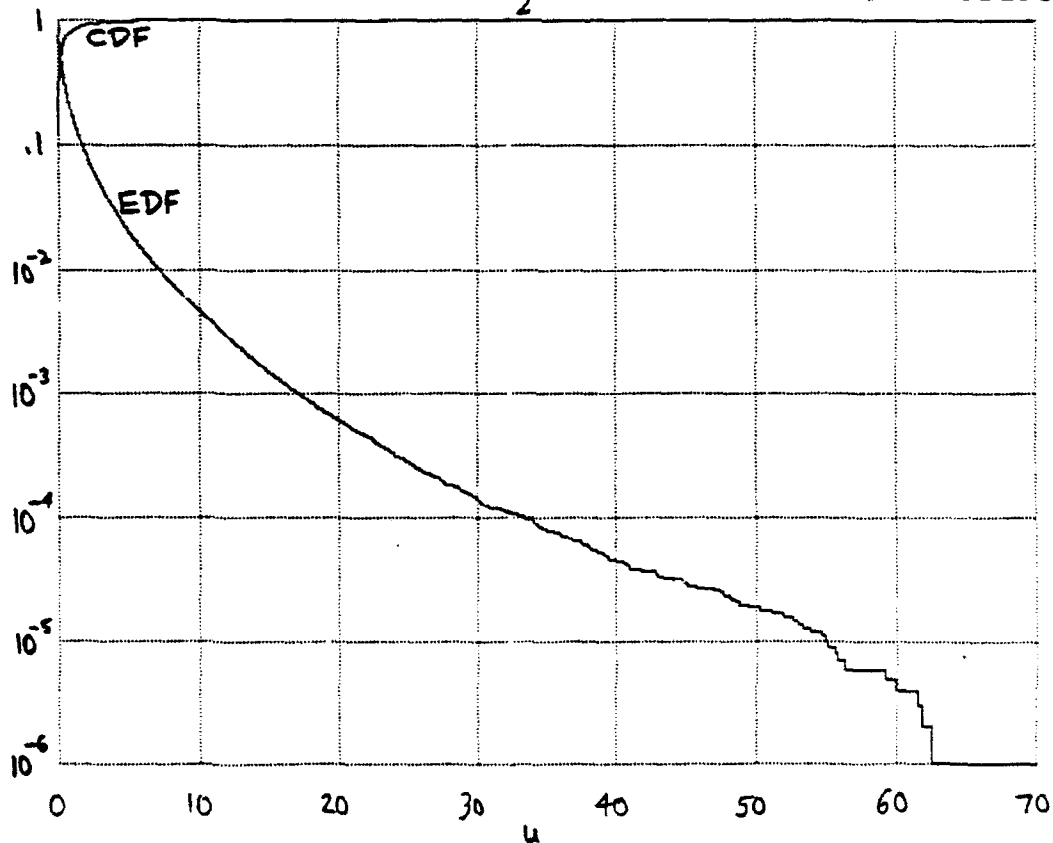


Figure 8. Distributions of W_{32} for Uniform Random Variables

Figure 9. Distributions of W_{64} for Uniform Random VariablesFigure 10. Distributions of W_{128} for Uniform Random Variables

Figure 11. Distributions of W_2 for Gaussian Random VariablesFigure 12. Distributions of W_3 for Gaussian Random Variables

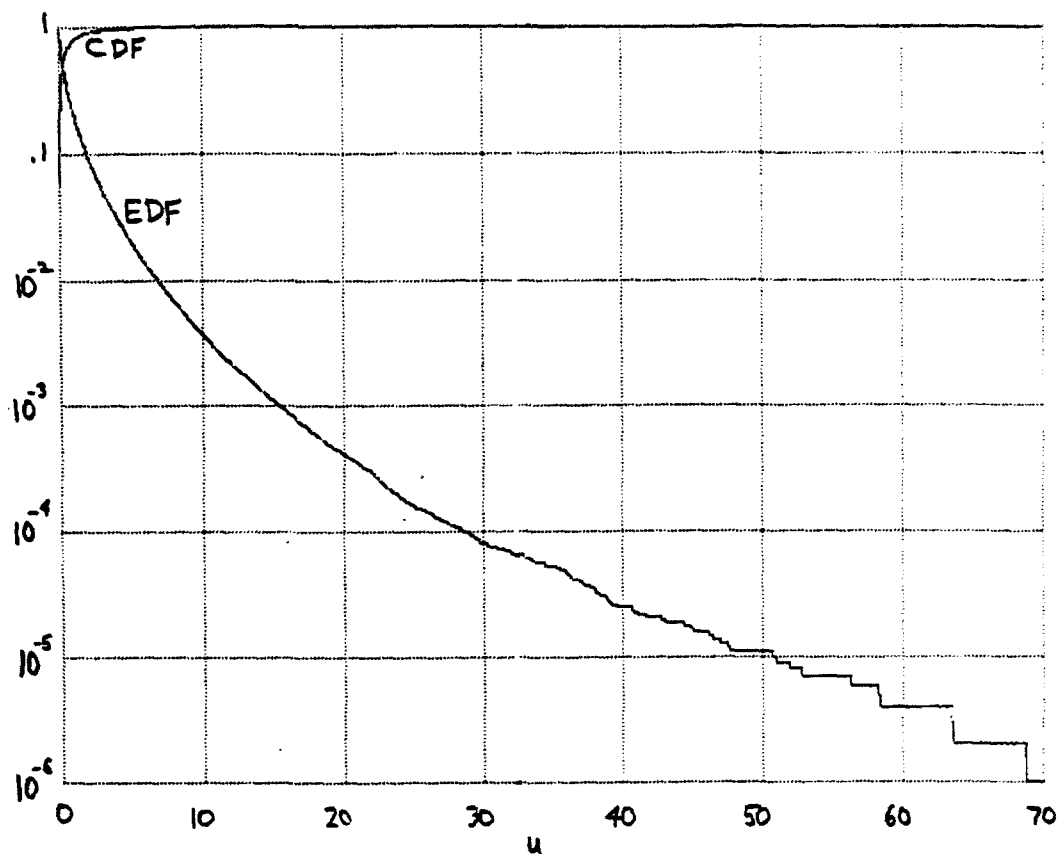


Figure 13. Distributions of W_4 for Gaussian Random Variables

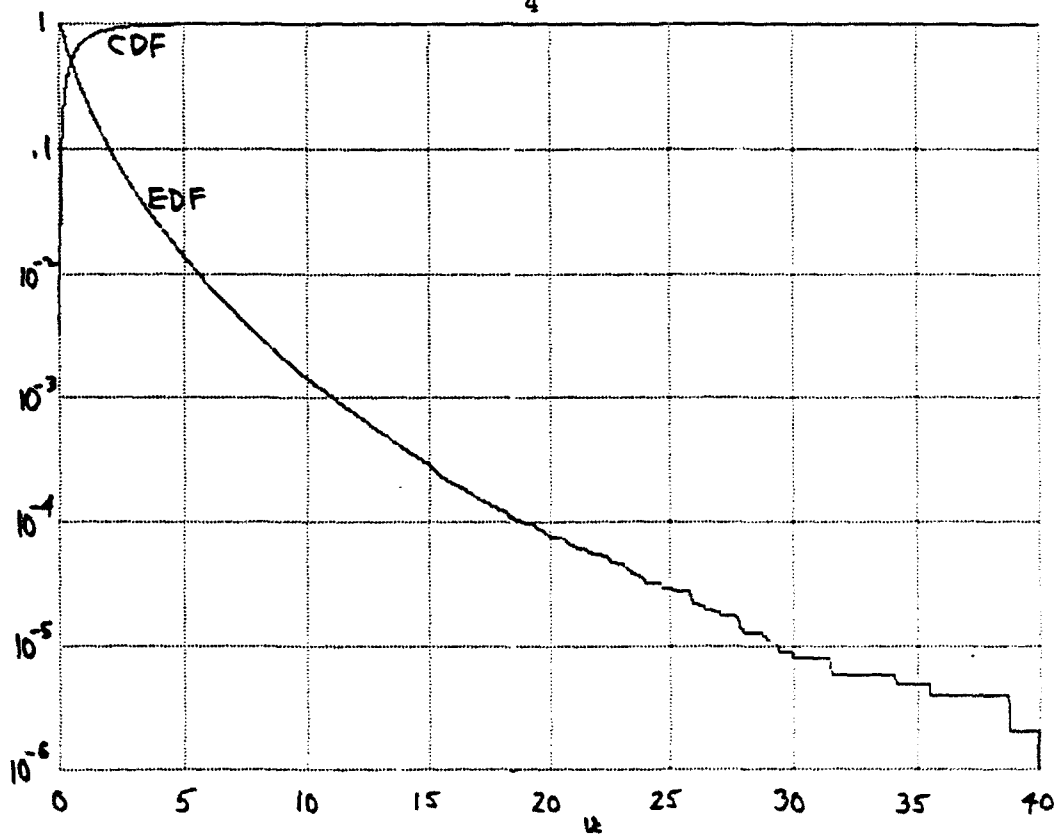


Figure 14. Distributions of W_8 for Gaussian Random Variables

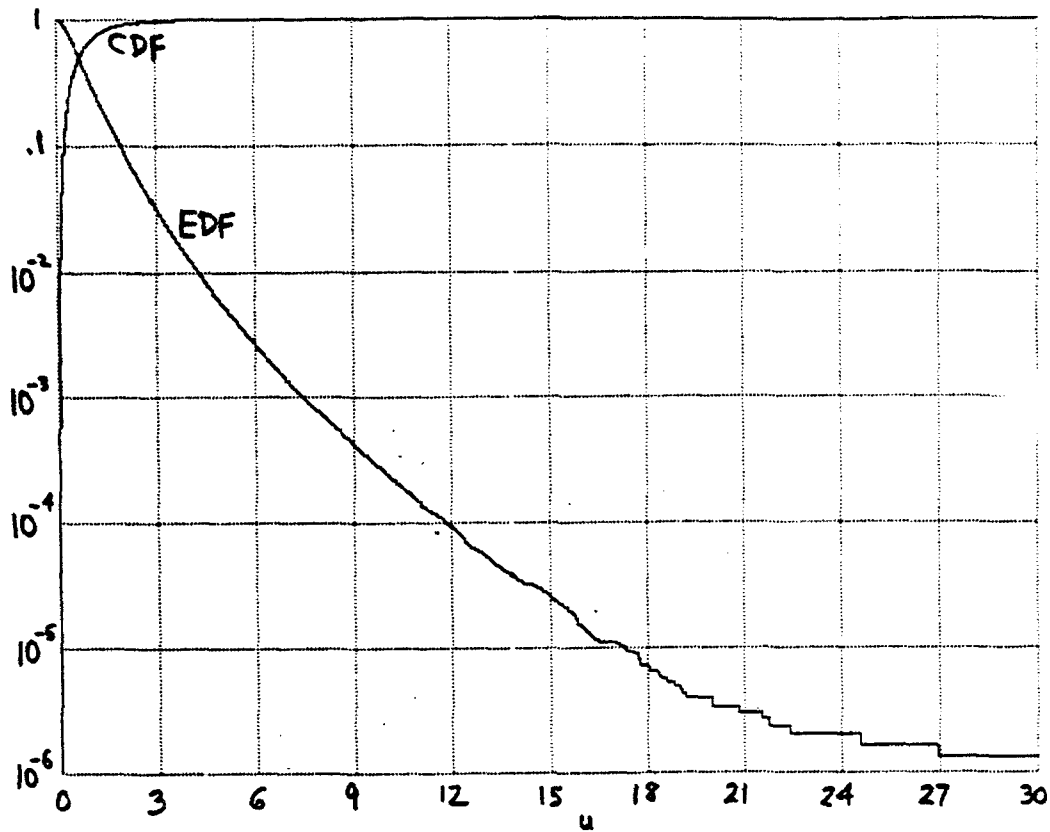


Figure 15. Distributions of W_{16} for Gaussian Random Variables

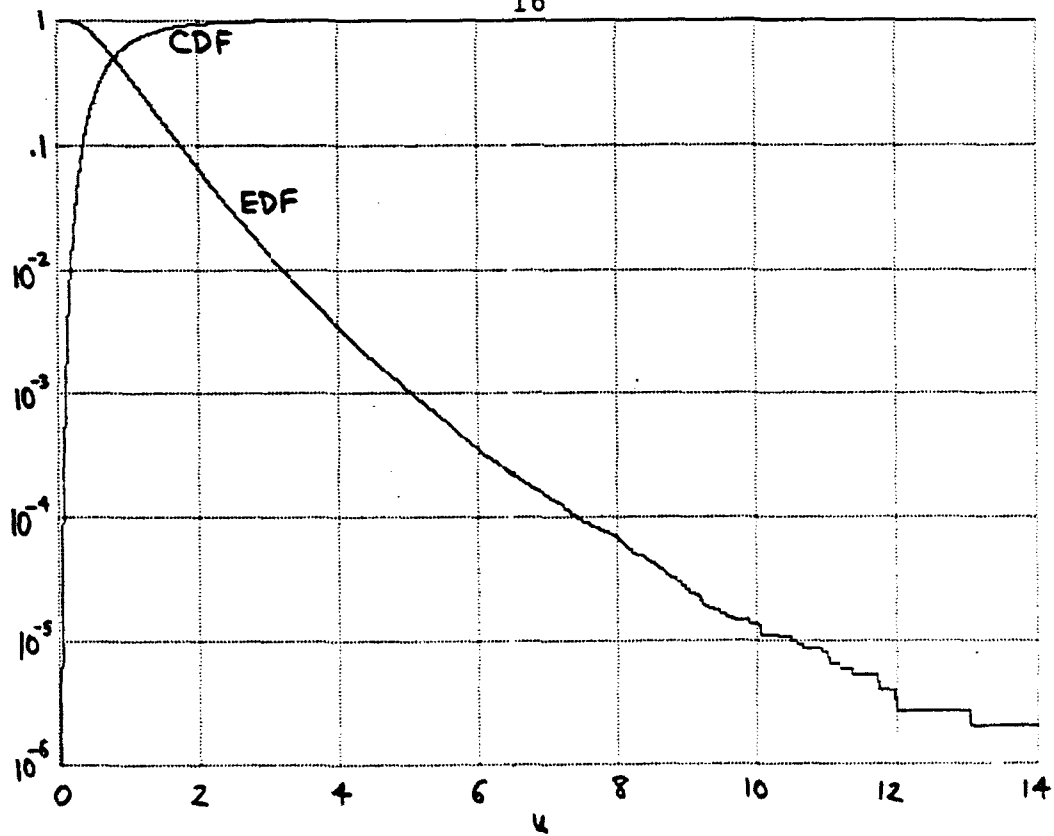
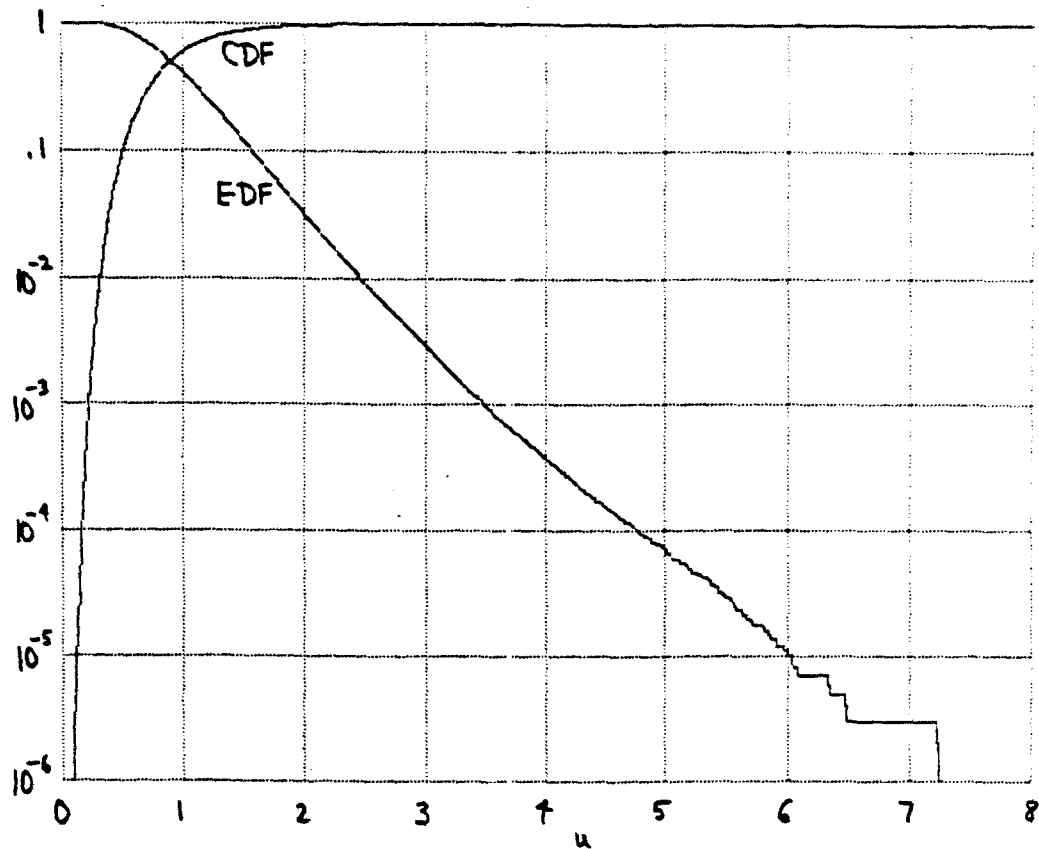
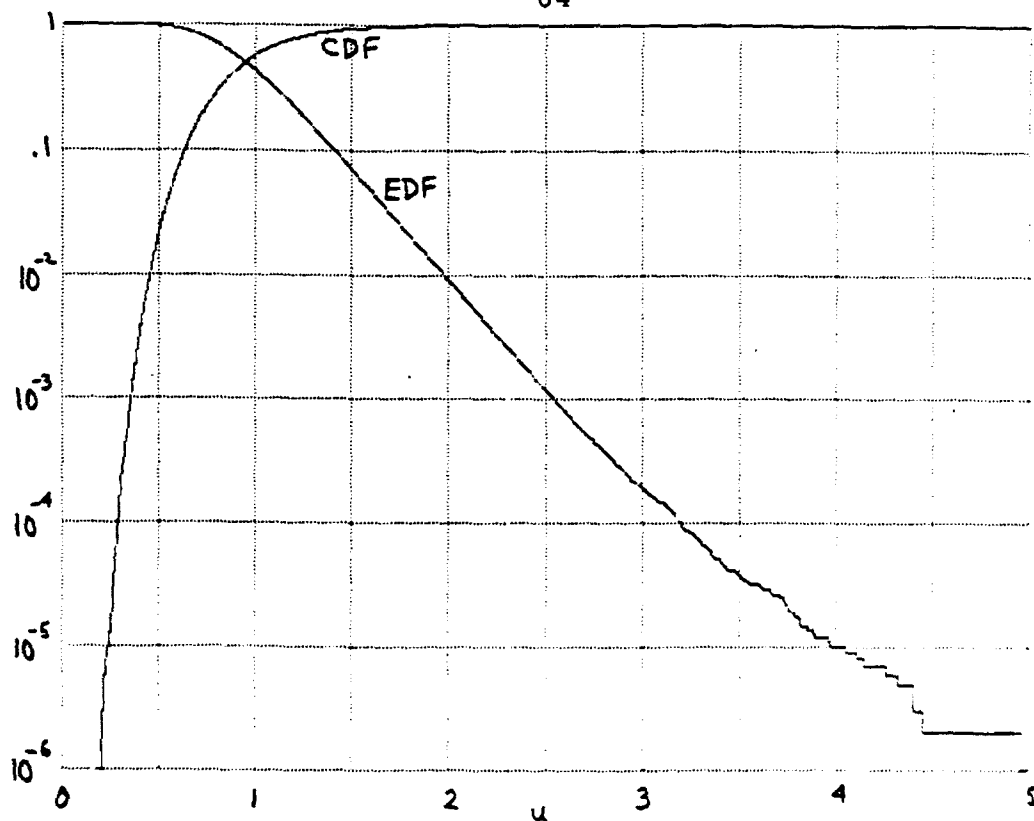


Figure 16. Distributions of W_{32} for Gaussian Random Variables

Figure 17. Distributions of W_{64} for Gaussian Random VariablesFigure 18. Distributions of W_{128} for Gaussian Random Variables

SUMMARY

The statistics of a whiteness measure, for testing a random number generator, have been investigated in terms of the mean, variance, and probability distributions. The mean and variance results are exact and have been borne out by numerous simulations for different noise sources $\{x_k\}$ and data sizes K . These results, for whiteness measure W_K defined in (3), are summarized below:

$$E(W_K) = \frac{K-1}{K}, \quad V_K = \text{Var}(W_K) = \frac{A K^3 + B K^2 + C K + D}{K^4}, \quad (59)$$

where

$$A = 4F + \frac{4}{3}, \quad B = 2F^2 - 8F - 14, \quad C = -2F^2 + \frac{62}{3} \quad \text{for all } K,$$

$$\text{while } D = \begin{cases} 0 & \text{for } K \text{ even} \\ 4(F-2) & \text{for } K \text{ odd} \end{cases}. \quad (60)$$

The mean of whiteness measure W_K is independent of fourth-order moment F , while the variance of W_K depends on F , but not on sixth or eighth-order moments of data $\{x_k\}$. That is, the eighth-order product encountered in the general mean-square expression (8) never requires knowledge higher than fourth-order for its evaluation. This result applies for a symmetric zero-mean probability density function for unit-variance data $\{x_k\}$.

The cumulative and exceedance probability distributions were determined by simulations involving more than one million trials each and therefore have good reliability approximately down to the .0001 probability level.

APPENDIX A. DERIVATION OF VARIANCE OF WHITENESS MEASURE W_K

The variances V_K of whiteness measure W_K for $K = 2, 3, 4$ were derived in (14) - (34) in the main text. We now present the derivations for the remaining cases, $K = 5, 6, 7, 8$, that are necessary in order to determine V_K for all K .

SPECIAL CASE $K = 5$

$$\phi_1 = x_1 x_0 + x_2 x_1 + x_3 x_2 + x_4 x_3, \quad \phi_4 = x_4 x_0,$$

$$\phi_2 = x_2 x_0 + x_3 x_1 + x_4 x_2, \quad \phi_3 = x_3 x_0 + x_4 x_1, \quad (A-1)$$

$$W_5 = \frac{2}{25} [\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2], \quad (A-2)$$

$$\begin{aligned} \frac{625}{4} W_5^2 = & \phi_1^4 + \phi_2^4 + \phi_3^4 + \phi_4^4 + 2 \phi_1^2 \phi_2^2 + 2 \phi_1^2 \phi_3^2 + 2 \phi_1^2 \phi_4^2 + \\ & + 2 \phi_2^2 \phi_3^2 + 2 \phi_2^2 \phi_4^2 + 2 \phi_3^2 \phi_4^2. \end{aligned} \quad (A-3)$$

The component averages required are developed in detail as follows:

$$E(\phi_4^4) = E(x_4^4 x_0^4) = F^2, \quad (A-4)$$

$$E(\phi_4^2 \phi_3^2) = E(x_4^2 x_0^2 (x_3 x_0 + x_4 x_1)^2) = F + F = 2F, \quad (A-5)$$

$$\begin{aligned} E(\phi_4^2 \phi_2^2) &= E(x_4^2 x_0^2 [x_3 x_1 + x_2 (x_0 + x_4)]^2) = \\ &= E(x_4^2 x_0^2 [x_3^2 x_1^2 + x_2^2 (x_0 + x_4)^2 + 2 x_3 x_2 x_1 (x_0 + x_4)]) = \end{aligned}$$

$$= 1 + (F + F) = 2F + 1 , \quad (A-6)$$

$$\begin{aligned} E(\phi_4^2 \phi_1^2) &= E(x_4^2 x_0^2 [x_1(x_0 + x_2) + x_3(x_2 + x_4)]^2) = \\ &= E(x_4^2 x_0^2 [x_1^2(x_0 + x_2)^2 + x_3^2(x_2 + x_4)^2 + 2 x_3 x_1(x_0 + x_2)(x_2 + x_4)]) = \\ &= (F + 1) + (1 + F) = 2F + 2 , \quad (A-7) \end{aligned}$$

$$E(\phi_3^4) = E([x_3 x_0 + x_4 x_1]^4) = F^2 + 6 + F^2 = 2F^2 + 6 , \quad (A-8)$$

$$\begin{aligned} E(\phi_3^2 \phi_2^2) &= E([x_3 x_0 + x_4 x_1]^2 [x_3 x_1 + x_2(x_4 + x_0)]^2) = \\ &= E([x_3^2 x_0^2 + x_4^2 x_1^2 + 2 x_4 x_3 x_1 x_0] [x_3^2 x_1^2 + x_2^2(x_4 + x_0)^2 + \\ &+ 2 x_3 x_2 x_1(x_4 + x_0)]) = F + (1+F) + F + (F+1) = 4F + 2 , \quad (A-9) \end{aligned}$$

$$\begin{aligned} E(\phi_3^2 \phi_1^2) &= E([x_3 x_0 + x_4 x_1]^2 [x_1(x_2 + x_0) + x_3(x_4 + x_2)]^2) = \\ &= E([x_3^2 x_0^2 + x_4^2 x_1^2 + 2 x_4 x_3 x_1 x_0] [x_1^2(x_2 + x_0)^2 + x_3^2(x_4 + x_2)^2 + \\ &+ 2 x_3 x_1(x_2 + x_0)(x_4 + x_2)]) = \\ &= (1 + F) + F(1 + 1) + F(1 + 1) + (F + 1) + 4 = 6F + 6 , \quad (A-10) \end{aligned}$$

$$\begin{aligned} E(\phi_2^4) &= E([x_3 x_1 + x_2(x_4 + x_0)]^4) = \\ &= F^2 + 6(1 + 1) + F(F + 6 + F) = 3F^2 + 6F + 12 , \quad (A-11) \end{aligned}$$

$$\begin{aligned}
E(\phi_2^2 \phi_1^2) &= E\left([x_3 x_1 + x_2(x_4 + x_0)]^2 [x_1(x_2 + x_0) + x_3(x_4 + x_2)]^2\right) \\
&= E\left([x_3^2 x_1^2 + x_2^2(x_4 + x_0)^2 + 2 x_3 x_2 x_1(x_4 + x_0)] \times \right. \\
&\quad \times \left.[x_1^2(x_2 + x_0)^2 + x_3^2(x_4 + x_2)^2 + 2 x_3 x_1(x_2 + x_0)(x_4 + x_2)]\right) = \\
&= F(1 + 1) + F(1 + 1) + \\
&\quad + E\left([x_4^2 + 2 x_4 x_0 + x_0^2][x_2^4 + 2 x_2^3 x_0 + x_2^2 x_0^2]\right) + \\
&\quad + E\left([x_4^2 + 2 x_4 x_0 + x_0^2][x_4^2 x_2^2 + 2 x_4 x_2^3 + x_2^4]\right) + \\
&\quad + 4 E\left(x_2(x_4 + x_0)(x_2 + x_0)(x_4 + x_2)\right) = \\
&= 4F + (F + 1 + F + F) + (F + F + 1 + F) + 4(1 + 1) = 10F + 10, \\
&\hspace{15em} (A-12)
\end{aligned}$$

$$\begin{aligned}
E(\phi_1^4) &= E\left([x_1(x_2 + x_0) + x_3(x_4 + x_2)]^4\right) = \\
&= E\left(x_1^4(x_2 + x_0)^4 + 6 x_1^2(x_2 + x_0)^2 x_3^2(x_4 + x_2)^2 + x_3^4(x_4 + x_2)^4\right) = \\
&= F(F + 6 + F) + 6 E\left([x_2^2 + 2 x_2 x_0 + x_0^2][x_4^2 + 2 x_4 x_2 + x_2^2]\right) + \\
&\quad + F(F + 6 + F) = 4F^2 + 12F + 6(1 + F + 1 + 1) = \\
&= 4F^2 + 18F + 18. \hspace{10em} (A-13)
\end{aligned}$$

Now, we combine all the component averages, above, to obtain mean square value

$$E(W_5^2) = \frac{8}{625}(5F^2 + 38F + 39) \quad (A-14)$$

and variance

$$V_5 = \frac{8}{625}(5F^2 + 38F - 11) . \quad (A-15)$$

SPECIAL CASE K = 6

Now, we adopt a very useful shorthand notation to handle the rest of the cases of interest. For example, here, $\phi_5 = x_5 x_0$ and $\phi_5^2 = x_5^2 x_0^2$, which is denoted by 5500; that is, the superfluous x is ignored when possible. Also, $x_4 x_2^2 x_0$ is denoted by 4220. With this background, we now have

$$\begin{aligned} \phi_5^2 &= 5500 , \quad \phi_4^2 = 4400+5511+2(5410) , \\ \phi_3^2 &= 3300+4411+5522+2(4310+5320+5421) , \\ \phi_2^2 &= 2200+3311+4422+5533+2(3210+4220+5320+4321+5331+5432) , \\ \phi_1^2 &= 1100+2211+3322+4433+5544+ \\ &+2(2110+3210+4310+5410+3221+4321+5421+4332+5432+5443) . \end{aligned} \quad (A-16)$$

From (13), there follows

$$W_6 = \frac{2}{36} \sum_{n=1}^5 \phi_n^2 = \frac{1}{18} (\phi_1^2 + \phi_2^2 + \dots + \phi_5^2) \quad (A-17)$$

and

$$324 W_6^2 = \phi_1^4 + \phi_2^4 + \phi_3^4 + \phi_4^4 + \phi_5^4 + 2(\phi_1^2 \phi_2^2 + \dots + \phi_4^2 \phi_5^2) . \quad (\text{A-18})$$

We also abbreviate the following ensemble averages as follows

$$E(\phi_m^2 \phi_n^2) = T_{mn} . \quad (\text{A-19})$$

Then, there follows, in a straightforward but tedious manner,

$$\begin{aligned} T_{55} &= F^2 , \quad T_{54} = F+F = 2F , \quad T_{53} = F+1+F = 2F+1 , \\ T_{52} &= F+1+1+F = 2F+2 , \quad T_{51} = F+1+1+1+F = 2F+3 , \\ T_{44} &= F^2+6+F^2 = 2F^2+6 , \quad T_{43} = (F+F+1)+(1+F+F) = 4F+2 , \\ T_{42} &= (F+1+F+1)+(1+F+1+F) = 4F+4 , \\ T_{41} &= (F+1+1+F+F)+(F+F+1+1+F)+4 = 6F+8 , \\ T_{33} &= 3F^2+4(1+1+1)+2(1+1+1) = 3F^2+18 , \\ T_{32} &= (F+F+1+F)+(1+F+F+1)+(F+1+F+F)+4(1) = 8F+8 , \\ T_{31} &= (F+1+F+F+1)+(F+F+1+F+F)+(1+F+F+1+F)+4(1+1) = 10F+13 , \\ T_{22} &= 4F^2+4(1+F+1+1+F+1)+2(1+F+1+1+F+1) = 4F^2+12F+24 , \\ T_{21} &= (F+F+F+1+1)+(F+F+F+F+1)+(1+F+F+F+F)+(1+1+F+F+F)+12 = 14F+18 , \\ T_{11} &= 5F^2+(4+2)(F+1+1+1+F+1+1+F+1+F) = 5F^2+24F+36 . \quad (\text{A-20}) \end{aligned}$$

The desired average is, from (A-18) - (A-20),

$$324 E(W_6^2) = 15F^2 + 144F + 202 . \quad (\text{A-21})$$

The variance of W_6 is then

$$V_6 = \frac{1}{324} (15F^2 + 144F - 23) . \quad (\text{A-22})$$

SPECIAL CASE K = 7

Continuing in the fashion established above, we now have

$$\phi_6^2 = 6600 , \quad \phi_5^2 = 5500+6611+2(6510) ,$$

$$\phi_4^2 = 4400+5511+6622+2(5410+6420+6521) ,$$

$$\phi_3^2 = 3300+4411+5522+6633+2(4310+5320+6330+5421+6431+6532) ,$$

$$\phi_2^2 = 2200+3311+4422+5533+6644+$$

$$+2(3210+4220+5320+6420+4321+5331+6431+5432+6442+6543) ,$$

$$\phi_1^2 = 1100+2211+3322+4433+5544+6655+2(2110+3210+4310+5410+6510+3221+4321+5421+6521+4332+5432+6532+5443+6543+6554) . \quad (A-23)$$

From (13),

$$w_7 = \frac{2}{49} (\phi_1^2 + \phi_2^2 + \cdots + \phi_6^2) \quad (A-24)$$

and therefore

$$\frac{2401}{4} w_7^2 = \phi_1^4 + \cdots + \phi_6^4 + 2(\phi_1^2 \phi_2^2 + \cdots \phi_5^2 \phi_6^2) . \quad (A-25)$$

The required averages are as follows:

$$T_{66} = F^2 , \quad T_{65} = F+F = 2F , \quad T_{64} = F+1+F = 2F+1 ,$$

$$T_{63} = F+1+1+F = 2F+2 , \quad T_{62} = F+1+1+1+F = 2F+3 ,$$

$$T_{61} = F+1+1+1+1+F = 2F+4 , \quad T_{55} = F^2+F^2+4+2 = 2F^2+6 ,$$

$$T_{54} = (F+F+1)+(1+F+F) = 4F+2 , \quad T_{53} = (F+1+F+1)+(1+F+1+F) = 4F+4 ,$$

$$T_{52} = (F+1+1+F+1)+(1+F+1+1+F) = 4F+6 ,$$

$$T_{51} = (F+1+1+1+F+F)+(F+F+1+1+1+F)+4 = 6F+10 ,$$

$$T44 = 3F^2 + 4(1+1+1) + 2(1+1+1) = 3F^2 + 18 ,$$

$$T43 = (F+F+1+1) + (1+F+F+1) + (1+1+F+F) = 6F+6 ,$$

$$T42 = (F+1+F+1+F) + (1+F+1+F+1) + (F+1+F+1+F) + 4(1) = 8F+11 ,$$

$$T41 = (3F+3) + (4F+2) + (3F+3) + 4(1+1) = 10F+16 ,$$

$$T33 = 4F^2 + 4(1+1+F+1+1+1) + 2(1+1+F+1+1+1) = 4F^2 + 6F + 30 ,$$

$$T32 = (3F+2) + (3F+2) + (3F+2) + (3F+2) + 4(1+1) = 12F+16 ,$$

$$T31 = (3F+3) + (4F+2) + (4F+2) + (3F+3) + 4(1+1+1) = 14F+22 ,$$

$$T22 = 5F^2 + (4+2)(1+F+1+1+1+F+1+1+F+1) = 5F^2 + 18F + 42 ,$$

$$T21 = 2(3F+3) + 3(4F+2) + 4(1+1+1+1) = 18F+28 ,$$

$$T11 = 6F^2 + (4+2)(F+1+1+1+1+F+1+1+1+F+1+1+F+1+F) = 6F^2 + 30F + 60 .$$

(A-26)

The average of interest is, from (A-25) and (A-26),

$$\frac{2401}{4} E(W_7^2) = 21F^2 + 246F + 418 , \quad (A-27)$$

leading to variance

$$V_7 = \frac{4}{2401} (21F^2 + 246F - 23) . \quad (A-28)$$

SPECIAL CASE K = 8

This is the last case that we need to evaluate. We now have

$$\begin{aligned}
 \phi_7^2 &= 7700, \quad \phi_6^2 = 6600+7711+2(7610), \\
 \phi_5^2 &= 5500+6611+7722+2(6510+7520+7621), \\
 \phi_4^2 &= 4400+5511+6622+7733+2(5410+6420+7430+6521+7531+7632), \\
 \phi_3^2 &= 3300+4411+5522+6633+7744+ \\
 &\quad +2(4310+5320+6330+7430+5421+6431+7441+6532+7542+7643), \\
 \phi_2^2 &= 2200+3311+4422+5533+6644+7755+2(3210+4220+5320+6420+ \\
 &\quad +7520+4321+5331+6431+7531+5432+6442+7542+6543+7553+7654), \\
 \phi_1^2 &= 1100+2211+3322+4433+5544+6655+7766+ \\
 &\quad +2(2110+3210+4310+5410+6510+7610+3221+4321+5421+6521+7621+ \\
 &\quad +4332+5432+6532+7632+5443+6543+7643+6554+7654+7665). \quad (A-29)
 \end{aligned}$$

From (13) again,

$$W_8 = \frac{2}{64} (\phi_1^2 + \phi_2^2 + \dots + \phi_7^2), \quad (A-30)$$

giving

$$1024 W_8^2 = \phi_1^4 + \dots + \phi_7^4 + 2(\phi_1^2 \phi_2^2 + \dots + \phi_6^2 \phi_7^2). \quad (A-31)$$

The averages needed are listed below.

$$\begin{aligned}
 T_{77} &= F^2, \quad T_{76} = 2F, \quad T_{75} = 2F+1, \quad T_{74} = 2F+2, \\
 T_{73} &= 2F+3, \quad T_{72} = 2F+4, \quad T_{71} = 2F+5,
 \end{aligned}$$

$$T66 = F^2 + F^2 + 4 + 2 = 2F^2 + 6, \quad T65 = (F + F + 1) + (1 + F + F) = 4F + 2,$$

$$T64 = (F + 1 + F + 1) + (1 + F + 1 + F) = 4F + 4,$$

$$T63 = (F + 1 + 1 + F + 1) + (1 + F + 1 + 1 + F) = 4F + 6,$$

$$T62 = (F + 1 + 1 + 1 + F + 1) + (1 + F + 1 + 1 + 1 + F) = 4F + 8,$$

$$T61 = (3F + 4) + (3F + 4) + 4 = 6F + 12,$$

$$T55 = 3F^2 + 4(1 + 1 + 1) + 2(1 + 1 + 1) = 3F^2 + 18,$$

$$T54 = (F + F + 1 + 1) + (1 + F + F + 1) + (1 + 1 + F + F) = 6F + 6,$$

$$T53 = 3(2F + 3) = 6F + 9,$$

$$T52 = (3F + 3) + (2F + 4) + (3F + 3) + 4(1) = 8F + 14,$$

$$T51 = (3F + 4) + (4F + 3) + (3F + 4) + 4(1 + 1) = 10F + 19,$$

$$T44 = 4F^2 + 4(6) + 2(6) = 4F^2 + 36,$$

$$T43 = (3F + 2) + (2F + 3) + (2F + 3) + (3F + 2) + 4(1) = 10F + 14,$$

$$T42 = 4(3F + 3) + 4(1 + 1) = 12F + 20,$$

$$T41 = 2(3F + 4) + 2(4F + 3) + 4(1 + 1 + 1) = 14F + 26,$$

$$T33 = 5F^2 + 4(2F + 8) + 2(2F + 8) = 5F^2 + 12F + 48,$$

$$T32 = 4(3F + 3) + (4F + 2) + 4(1 + 1 + 1) = 16F + 26,$$

$$T31 = 2(3F + 4) + 3(4F + 3) + 4(1 + 1 + 1 + 1) = 18F + 33,$$

$$T22 = 6F^2 + 4(4F + 11) + 2(4F + 11) = 6F^2 + 24F + 66,$$

$$T21 = 2(3F + 4) + 4(4F + 3) + 4(1 + 1 + 1 + 1 + 1) = 22F + 40,$$

$$T11 = 7F^2 + 4(6F + 15) + 2(6F + 15) = 7F^2 + 36F + 90. \quad (A-32)$$

The desired average is therefore

$$1024 E \left(W_8^2 \right) = 28F^2 + 384F + 772, \quad (A-33)$$

giving variance

$$V_8 = \frac{1}{256} \left(7F^2 + 96F - 3 \right). \quad (A-34)$$

APPENDIX B. PROGRAM FOR ESTIMATION OF DISTRIBUTIONS OF W_K

```

10  T=1E6                ! NUMBER OF TRIALS  "NUWC TR10237"
20  K=32                 ! NUMBER OF DATA POINTS, ARBITRARY
30  M=64                 ! FFT SIZE, M >= 2K-1, POWER OF 2
40  L=11000              ! NUMBER OF LEVELS FOR DISTRIBUTION.
50  Dw=.001              ! INCREMENT IN W
60  Gr=1000              ! GRID SPACING
70  PRINTER IS PRT
80  PRINT "K =";K;"  T =";T;"  Dw =";Dw;"  UNIFORM"
90  PRINTER IS CRT
100 DOUBLE T,K,M,L,M1,M2,K1,Ts,Ks ! INTEGERS, NOT DP
110 DIM Cos(512),X(2048),Y(2048),V(30000)
120 M1=M-1
130 REDIM Cos(0:M/4),X(0:M1),Y(0:M1),V(0:L)
140 A=2.*PI/M
150 FOR Ms=0 TO M/4
160 Cos(Ms)=COS(A*Ms)    ! QUARTER-COSINE TABLE IN Cos(*)
170 NEXT Ms
180 M2=M/2
190 K1=K-1
200 T1=1./T
210 F=12./(K*M)          ! UNIT-VARIANCE UNIFORM
220 F=F*F                 ! RANDOM VARIABLES {x(subk)}
230 Mu=K1/K              ! EXACT MEAN OF WK
240 Mu1=Var=0.
250 Ta=TIMEDATE
260 FOR Ts=1 TO T
270 FOR Ks=0 TO K1
280 X(Ks)=RND-.5          ! ZERO MEAN
290 Y(Ks)=0.              ! REAL INPUT
300 NEXT Ks
310 FOR Ks=K TO M1
320 X(Ks)=Y(Ks)=0.
330 NEXT Ks
340 CALL Fft14(M,Cos(*),X(*),Y(*))
350 S2=S4=0.
360 FOR Ms=1 TO M2-1      ! ZERO TO NYQUIST
370 X=X(Ms)
380 Y=Y(Ms)
390 A=X*X+Y*Y
400 S2=S2+A
410 S4=S4+A*A
420 NEXT Ms
430 X=X(0)
440 A=X(M2)
450 X=X*X
460 A=A*A
470 S2=X+A+2.*S2
480 S4=X*X+A*A+2.*S4
490 W=F*(M*S4-S2*S2)      ! WHITENESS MEASURE WK

```



```

500 Mu1=Mu1+W
510 Var=Var+(W-Mu)*(W-Mu)! USE KNOWN MEAN Mu
520 Ms=INT(W/Dw)
530 Ms=MIN(Ms,L)
540 V(Ms)=V(Ms)+T1 ! INCREMENTAL PROBABILITIES
550 NEXT Ts
560 Tb=TIMEDATE
570 PRINTER IS PRT
580 PRINT (Tb-Ta)/3600;" HOURS"
590 PRINT
600 Mu1=Mu1/T ! ESTIMATED MEAN OF WK
610 Var=Var/T ! ESTIMATED VARIANCE OF WK
620 PRINT "Mu1 =";Mu1;" Mu =";Mu
630 PRINT "Var =";Var
640 PRINT
650 PLOTTER IS "GRAPHICS"
660 GRAPHICS ON
670 WINDOW 0,L,-6,0
680 LINE TYPE 3
690 GRID Gr,1
700 LINE TYPE 1
710 C=0.
720 FOR Ms=0 TO L-1
730 C=C+V(Ms) ! CDF OF WHITENESS MEASURE WK
740 IF C>0. THEN 760
750 GOTO 770
760 PLOT Ms+1,LGT(C)
770 NEXT Ms
780 PENUP
790 E=S1=S2=0.
800 FOR Ms=L TO 1 STEP -1
810 E=E+V(Ms) ! EDF OF WHITENESS MEASURE WK
820 S1=S1+E
830 S2=S2+S1
840 IF E>0. THEN 860
850 GOTO 870
860 PLOT Ms,LGT(E)
870 NEXT Ms
880 PLOT 0,0
890 PENUP
900 Mu1=Dw*(.5+S1) ! ESTIMATED MEAN OF WK
910 Mu2=2.*Dw*Dw*S2 ! SEE APPENDIX C
920 PRINT "Mu1 =";Mu1;" Mu =";Mu
930 PRINT "Var =";Mu2-Mu*Mu ! ESTIMATED VARIANCE OF WK
940 PRINT
950 PRINTER IS CRT
960 PAUSE
970 END
980 !
990 SUB Fft14(DOUBLE N,REAL Cos(*),X(*),Y(*)) ! N=2^14=16384; 0 SUBS

```

APPENDIX C. EVALUATION OF MOMENTS DIRECTLY
FROM MEASURED EXCEEDANCE DISTRIBUTION

Let x be a positive random variable with probability density function p , cumulative distribution function (CDF) C , and exceedance distribution function (EDF) E . Let the measurements of these distributions be the interval probabilities

$$V_n = \text{Prob}(n\Delta \leq x < (n+1)\Delta) \quad \text{for } 0 \leq n. \quad (\text{C-1})$$

Then

$$1 = \int_0^{\infty} dx \, p(x) = \sum_{n=0}^{\infty} \int_{n\Delta}^{(n+1)\Delta} dx \, p(x) = \sum_{n=0}^{\infty} V_n. \quad (\text{C-2})$$

At the same time, we can express

$$V_n = C((n+1)\Delta) - C(n\Delta) = E(n\Delta) - E((n+1)\Delta), \quad (\text{C-3})$$

which can be inverted, leading respectively to EDF and CDF

$$E(n\Delta) = \text{Prob}(x \geq n\Delta) = \sum_{m=n}^{\infty} V_m \quad \text{for } n \geq 0, \quad (\text{C-4})$$

$$C(n\Delta) = \text{Prob}(x < n\Delta) = \sum_{m=0}^{n-1} V_m \quad \text{for } n \geq 1. \quad (\text{C-5})$$

There also follows

$$E(0) = 1, \quad E((n+1)\Delta) = E(n\Delta) - V_n \quad \text{for } n \geq 0, \quad (\text{C-6})$$

or, as an alternative form to (C-4) if desired,

$$E(\Delta) = 1 - V_0, \quad E(2\Delta) = 1 - V_0 - V_1, \quad E(3\Delta) = 1 - V_0 - V_1 - V_2, \quad \dots \quad (\text{C-7})$$

The first two moments of random variable x can be developed as

$$\mu_1 = \int_0^{\infty} dx \, x \, p(x) = \int_0^{\infty} dx \, E(x) \approx \Delta \left[\frac{1}{2} E(0) + \sum_{n=1}^{\infty} E(n\Delta) \right], \quad (C-8)$$

and

$$\mu_2 = \int_0^{\infty} dx \, x^2 \, p(x) = 2 \int_0^{\infty} dx \, x \, E(x) \approx 2 \Delta^2 \sum_{n=1}^{\infty} n \, E(n\Delta). \quad (C-9)$$

These results can be rapidly evaluated by recursion. For $E((N+1)\Delta) = 0$, use

```

E=S1=S2=0.
FOR Ns=N TO 1 STEP -1
E=E+V(Ns)
S1=S1+E
S2=S2+S1
NEXT Ns
Mu1=Delta*(.5+S1)
Mu2=2.*Delta*Delta*S2

```

(C-10)

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- [1] A. H. Nuttall, On Generation of Random Numbers with Specified Distributions or Densities, NUSC Technical Report 6843, Naval Underwater Systems Center, New London, CT, 1 December 1982

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