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AN INTRODUCTION TO THE METHOD OF AVERAGING

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THE METHOD OF AVERAGING IS A POWERFUL TOOL FOR THE INTEGRATION OF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS WHICH HAVE SOME RAPIDLY OSCILLATING VARIABLES AND OTHER SLOWLY OSCILLATING VARIABLES. THIS REPORT BRIDGES THE GAP FROM CASUAL ACQUAINTANCE WITH THE UNDERLYING IDEA TO TECHNICAL READING AND APPLICATION. THREE EXAMPLES ARE WORKED IN DETAIL.

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AN INTRODUCTION

TO THE

METHOD OF AVERAGING

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November 1992

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AN INTRODUCTION TO THE METHOD OF AVERAGING

The Method of Averaging is a powerful tool for the integration of systems of ordinary differential equations which have some rapidly oscillating variables and other slowly oscillating variables. The literature on the method can be very frustrating to the beginner. The purpose of this report is to bridge the gap from casual acquaintance to technical reading. Three worked examples are included.

The first usage of the main idea of the method of averaging is attributed to van der Pol. The mathematical foundations of the method were established by Krylov and Bogoliubov (see Krylov and Bogoliubov in the references). Further refinements have been provided by Bogoliubov and Mitropolsky.

The method is used in the analysis of nonlinear oscillations, which arise in various specialties. There are several versions of the details of the method. For samples, see the references of Jordan and Smith, Meirovitch, Mickens, and Minorsky. My first encounter with the method occured in the setting of satellite orbits.

Consider the equations of motion for a two-body orbit

$$\dot{a} = 0$$
$$\dot{e} = 0$$
$$\dot{I} = 0$$
$$\dot{\omega} = 0$$
$$\dot{\Omega} = 0$$
$$\dot{M} = n$$

This is easy to integrate numerically. However, if a more realistic model is required for a real orbit around a real planet, then the zeros are replaced by other quantities. Even a modest increase in the fidelity of the model leads to a heavily coupled nonlinear system. Numerical difficulties are significant since M varies much faster than the other variables. The rapid change of M forces a small step size on the whole problem.

We will see that under certain circumstances a clever change of variables is available which leads to a simpler system with very desirable properties. The special circumstances are

a. One of the variables changes much more rapidly than the others.

b. The only dependence of the slowly changing variables on the fast variable, is periodic.

The most desirable property is that the resulting equations are easier to integrate, either numerically or analytically. This comes from the fact that our change of variable will give an average value of the original variables, effectively subtracting out periodic behavior. After integrating the simpler equations, the periodic contribution is effectively added back in as part of the transformation of variables.

To emphasize the idea of the method, suppose that some phenomenon of interest behaves as in Figure 1.

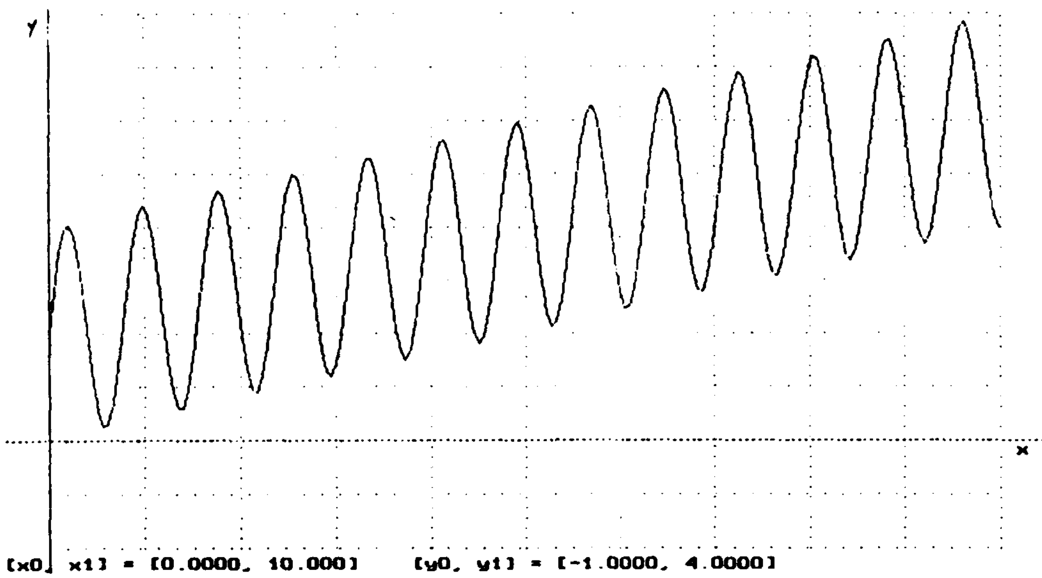

Figure 1

In order to accurately track the variations in this problem, a very small step size is necessary. Notice the general trend of the function to increase linearly. It may be possible to find a linear function and a sinusoidal function whose sum accurately describes the behavior seen in the figure. If so, we could follow the line (which is very easy to integrate accurately with large step size, or even analytically) until we reach the period of interest and then use small steps for evaluation of the sinusoidal contribution.

Sometimes we can find a change of variable which has the same affect by absorbing the periodic behavior into the change of variable. The change of variable to be identified by the method of averaging has this kind of advantage.

AN EXAMPLE

The strategy for finding a clever change of variable will be demonstrated on the system.

$$\dot{x} = \varepsilon \sin y$$

$$\dot{y} = x + \varepsilon \cos y$$

where ε is a small constant.

We seek new variables ξ and χ which lead to a simpler system of differential equations

$$\dot{\xi} = \varepsilon M(\xi) \tag{1}$$

$$\dot{\chi} = \xi + \varepsilon \Omega(\xi) \tag{2}$$

where $M(\xi)$ and $\Omega(\xi)$ are as yet unknown. Notice the absence of χ as an argument of M and Ω. This removal of dependence on the rapidly changing variable is what the method of averaging is designed to do.

We will describe the relationship between x,y and ξ,χ by

$$x = \xi + \varepsilon\eta(\xi,\chi)$$

$$y = \chi + \varepsilon\phi(\xi,\chi)$$

where η and ϕ are as yet unspecified. Notice that since ε is small ξ and χ are closely related to x and y.

Differentiation of these equations yields

$$\dot{x} = \dot{\xi} + \varepsilon \frac{\partial\eta}{\partial\xi}\dot{\xi} + \varepsilon \frac{\partial\eta}{\partial\chi}\dot{\chi}$$

$$\dot{y} = \dot{\chi} + \varepsilon \frac{\partial\phi}{\partial\xi}\dot{\xi} + \varepsilon \frac{\partial\phi}{\partial\chi}\dot{\chi}$$

With substitution this becomes

$$\dot{x} = \varepsilon M + \varepsilon^2 \frac{\partial\eta}{\partial\xi} M + \varepsilon \frac{\partial\eta}{\partial\chi}\xi + \varepsilon^2 \frac{\partial\eta}{\partial\chi}\Omega$$

$$\dot{y} = \xi + \varepsilon\Omega + \varepsilon^2 \frac{\partial\phi}{\partial\xi} M + \varepsilon \frac{\partial\phi}{\partial\chi}\xi + \varepsilon^2 \frac{\partial\phi}{\partial\chi}\Omega$$

This representation has the flavor of a power series in ε. We will compare it to the Taylor expansion of the original equations. Omitting the details we present the terms of the series through $O(\varepsilon)$ when expanded about a point (ξ,χ).

$$\dot{x} = \varepsilon \sin \chi + \varepsilon (y - \chi) \cos \chi$$

$$\dot{y} = \xi + \varepsilon \cos \chi + (x - \xi) - \varepsilon (y - \chi) \sin \chi$$

Comparing the two series and recognizing that $y - \chi$ and $x - \xi$ are each $O(\varepsilon)$, we see that to first order

$$\sin \chi = M + \xi \frac{\partial \eta}{\partial \chi} \tag{3}$$

$$\cos \chi + \eta = \Omega + \xi \frac{\partial \phi}{\partial \chi} \tag{4}$$

NOTE: This method can be used while keeping any desired order of accuracy in ε. But since we have deleted terms, we are now working with an approximation to the original problem.

We now have the problem reformulated in terms of two equations in the four unknown functions M, Ω, η, and ϕ. This means we are free to impose two conditions at our convenience. We use these two conditions to lay the groundwork for the disappearance of χ in our final solution. The conditions are

η is periodic in χ with period 2π,

ϕ is periodic in χ with period 2π.

It is not necessary that the period be 2π, I have simply chosen this value for illustration. Our next step exploits the two conditions just introduced. We will integrate with respect to χ over one period. Several integrals will be zero by periodicity of the anti-derivatives.

$$\int_0^{2\pi} \sin\chi \, d\chi = \int_0^{2\pi} M \, d\chi + \int_0^{2\pi} \xi \frac{\partial \eta}{\partial \chi} \, d\chi \tag{5}$$

$$\int_0^{2\pi} \cos \chi d\chi + \int_0^{2\pi} \eta d\chi = \int_0^{2\pi} \Omega d\chi + \int_0^{2\pi} \xi \frac{\partial \phi}{\partial \chi} \, d\chi \tag{6}$$

Equation (5) immediately reduces to

$$0 = 2\pi M$$

and forces the choice $M = 0$. Hence, equation (3) becomes

$$\sin\chi = \xi \frac{\partial \eta}{\partial \chi}$$

and we can integrate to solve for η.

$$\int \sin \chi d\chi = \int \xi \frac{\partial \eta}{\partial \chi} \, d\chi$$

$$-\cos\chi = \xi\eta + \hat{k}(\xi)$$

$$\eta = -\frac{1}{\xi}\cos\chi + k(\xi)$$

and we are free to choose $k(\xi)$. We will make our choice after we substitute for η and give some attention to equation (6).

$$\int_0^{2\pi}\left(-\frac{1}{\xi}\cos\chi + k(\xi)\right)d\chi = 2\pi\Omega$$

$$k(\xi) = \Omega$$

We now choose $k(\xi) = 0$ so $\Omega = 0$. This means equation (4) becomes

$$\cos\chi - \frac{\cos\chi}{\xi} = \xi\frac{\partial\phi}{\partial\chi}$$

which we can integrate to solve for ϕ

$$\int\cos\chi d\chi - \int\frac{\cos\chi}{\xi}d\chi = \int\xi\frac{\partial\phi}{\partial\chi}d\chi$$

$$\sin\chi - \frac{\sin\chi}{\xi} = \xi\phi + C(\xi)$$

We choose $C(\xi) = 0$ and write

$$\phi = \frac{\sin\chi}{\xi} - \frac{\sin\chi}{\xi^2}$$

Pulling everything together we obtain simplified equations (1) and (2)

$$\dot{\xi} = 0$$

$$\dot{\chi} = \xi$$

These are very easily integrated numerically, or indeed analytically. The transformation back to the original variables is also easy.

$$x = \xi - \varepsilon\frac{\cos\chi}{\xi}$$

$$y = \chi + \varepsilon\frac{\sin\chi}{\xi} - \varepsilon\frac{\sin\chi}{\xi^2}$$

This solves the problem to first order. To make our solution more concrete, let's use it with numbers. Suppose $\varepsilon = 0.01$ and initial conditions $x(0) = 0$, $y(0) = 0$ are specified.

The original equations become

$$\dot{x} = .01 \sin y \quad ; \quad x(0) = 0$$

$$\dot{y} = x + .01 \cos y \quad ; \quad y(0) = 0$$

The transformed equations are

$$\dot{\xi} = 0 \quad ; \quad \xi\ (0) = 0.1$$

$$\dot{\chi} = \xi \quad ; \quad \chi\ (0) = 0$$

where the boundary conditions are determined from

$$0 = \xi(0) - \varepsilon \frac{\cos\chi(0)}{\xi(0)}$$

$$0 = \chi\ (0) + \varepsilon \frac{\sin\chi(0)}{\xi(0)} - \varepsilon \frac{\sin\chi(0)}{\xi(0)^2}$$

By inspection $\chi(0) = 0$ satisfies that second equation and the first equation becomes

$$0 = \xi(0) - \frac{\varepsilon}{\xi(0)}$$

or

$$\xi(0) = \sqrt{\varepsilon}$$

$$= 0.1$$

The solution is

$$\xi = .1$$

$$\chi = t$$

which, after integration and transformation becomes

$$x = .1 - .1 \cos (.1t)$$

$$y = .1t + .1 \sin (.1t) - \sin (.1t)$$

$$= .1t - .9 \sin (.1t)$$

We can graph this to see the exact solution of our approximation to the original problem.

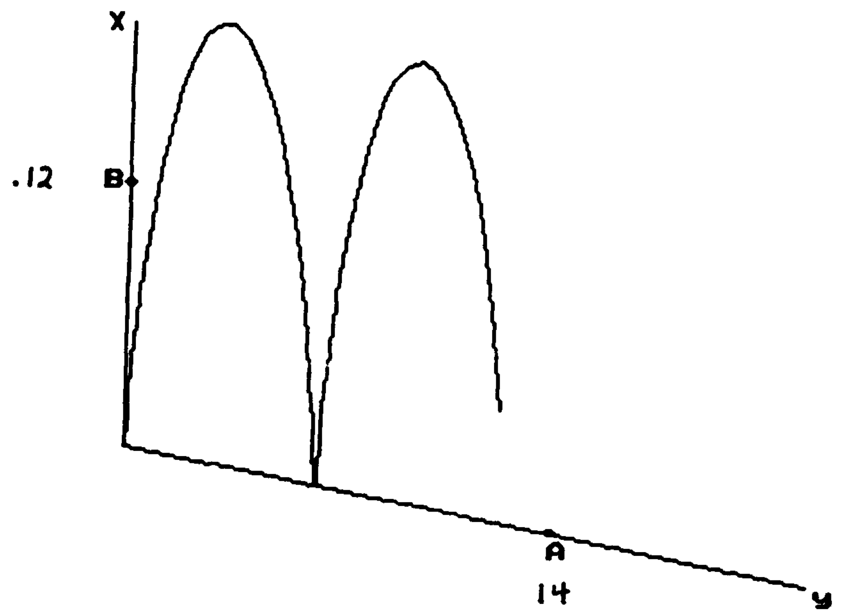

Figure 2

This estimate to the solution of the original problem may be quite rough. But we knew it is was only a first order approximation and ε was not extremely small. Nevertheless, it is useful as an illustrative problem.

The strategy which has been demonstrated is known as the method of averaging. It is due to Soviet Electrical Engineers Krylov and Bogoliubov (see references). The method can be used to any order accuracy and for any size system of slow and fast variables.

A SECOND EXAMPLE

Given: $\dot{x} = \varepsilon \sin^2 y$

$\dot{y} = k + \varepsilon (x^2 + \cos^3 y)$

Solve through second order using the method of averaging. The method is applicable since y changes much more rapidly than x and the dependence of x on y is purely periodic.

To solve the problem through second order we seek a transformation of the form

$$x = \xi + \varepsilon\eta^{(1)} \left(\xi, \chi\right) + \varepsilon^2 \eta^{(2)} \left(\xi, \chi\right) \tag{7}$$

$$y = \chi + \varepsilon \phi^{(1)} \left(\xi, \chi\right) + \varepsilon^2 \phi^{(2)} \left(\xi, \chi\right) \tag{8}$$

The superscripts in parentheses identify distinct functions. They do not indicate derivatives.

When we complete the transformation we will obtain equations of the form

$$\dot{\xi} = \varepsilon M^{(1)} \left(\xi\right) + \varepsilon^2 M^{(2)} \left(\xi\right)$$

$$\dot{\chi} = k + \varepsilon \Omega^{(1)} \left(\xi\right) + \varepsilon^2 \Omega^{(2)} \left(\xi\right)$$

We begin our work toward finding the desired transformation of variables by differentiating the proposed form of the transformation.

$$\dot{x} = \dot{\xi} + \varepsilon \frac{\partial\eta^{(1)}}{\partial\xi} \dot{\xi} + \varepsilon \frac{\partial\eta^{(1)}}{\partial\chi} \dot{\chi} + \varepsilon^2 \frac{\partial\eta^{(2)}}{\partial\xi} \dot{\xi} + \varepsilon^2 \frac{\partial\eta^{(2)}}{\partial\chi} \dot{\chi}$$

$$\dot{y} = \dot{\chi} + \varepsilon \frac{\partial\phi^{(1)}}{\partial\xi} \dot{\xi} + \varepsilon \frac{\partial\phi^{(1)}}{\partial\chi} \dot{\chi} + \varepsilon^2 \frac{\partial\phi^{(2)}}{\partial\xi} \dot{\xi} + \varepsilon^2 \frac{\partial\phi^{(2)}}{\partial\chi} \dot{\chi}$$

Next we substitute for the form we expect $\dot{\xi}$ and $\dot{\chi}$ to have, and write

$$\dot{x} = \varepsilon M^{(1)} + \varepsilon^2 M^{(2)} + \varepsilon^2 \frac{\partial\eta^{(1)}}{\partial\xi} M^{(1)} + \varepsilon \frac{\partial\eta^{(1)}}{\partial\chi} k + \varepsilon^2 \frac{\partial\eta^{(1)}}{\partial\chi} \Omega^{(1)} + \varepsilon^2 \frac{\partial\eta^{(2)}}{\partial\chi} k$$

$$\dot{y} = k + \varepsilon \Omega^{(1)} + \varepsilon^2 \Omega^{(2)} + \varepsilon^2 \frac{\partial\phi^{(1)}}{\partial\xi} M^{(1)} + \varepsilon \frac{\partial\phi^{(1)}}{\partial\chi} k + \varepsilon^2 \frac{\partial\phi^{(1)}}{\partial\chi} \Omega^{(1)} + \varepsilon^2 \frac{\partial\phi^{(2)}}{\partial\chi} k$$

with terms $O(\varepsilon^3)$ omitted.

The last two equations are the beginnings of power series in ε for $\dot{x}$ and $\dot{y}$ in terms of ξ and χ. Since two power series which converge to the same value must be equal term by term, these last two equations must match the corresponding portions of the Taylor series expansions of the original equations. So expand the original equations about $x = \xi$, $y = \chi$.

$$\dot{x} = \varepsilon \sin^2 y \Big|_{\substack{x=\xi \\ y=\chi}} + \varepsilon \frac{\partial}{\partial x} \sin^2 y \Big|_{\substack{x=\xi \\ y=\chi}} (x - \xi) +$$

$$\varepsilon \frac{\partial}{\partial y} \sin^2 y \Big|_{\substack{x=\xi \\ y=\chi}} (y - \chi) + O(\varepsilon^3)$$

$$\dot{y} = k + \varepsilon (x^2 + \cos^3 y) \Big|_{\substack{x=\xi \\ y=\chi}} + \varepsilon \frac{\partial}{\partial x} (x^2 + \cos^3 y) \Big|_{\substack{x=\xi \\ y=\chi}} (x - \xi) +$$

$$\varepsilon \frac{\partial}{\partial y} (x^2 + \cos^3 y) \Big|_{\substack{x=\xi \\ y=\chi}} (y - \chi) + O(\varepsilon^3)$$

$$\dot{x} = \varepsilon \sin^2\chi + 2\varepsilon^2\phi^{(1)} \sin\chi \cos\chi + O(\varepsilon^3)$$

$$\dot{y} = k + \varepsilon (\xi^2 + \cos^3\chi) + 2\varepsilon^2\xi\eta^{(1)} - 3\varepsilon^2\phi^{(1)}\cos^2\chi\sin\chi + O(\varepsilon^3)$$

From setting the terms of the two power series equal we obtain

$$M^{(1)} + \frac{\partial\eta^{(1)}}{\partial\chi} k = \sin^2\chi \tag{9}$$

$$\Omega^{(1)} + \frac{\partial\phi^{(1)}}{\partial\chi} k = \xi^2 + \cos^3\chi \tag{10}$$

$$M^{(2)} + M^{(1)}\frac{\partial\eta^{(1)}}{\partial\xi} + \Omega^{(1)}\frac{\partial\eta^{(1)}}{\partial\chi} + k\,\frac{\partial\eta^{(2)}}{\partial\chi} = 2\phi^{(1)}\sin\chi\cos\chi \tag{11}$$

$$\Omega^{(2)} + M^{(1)}\frac{\partial\phi^{(1)}}{\partial\xi} + \Omega^{(1)}\frac{\partial\phi^{(1)}}{\partial\chi} + k\frac{\partial\phi^{(2)}}{\partial\chi} = 2\xi\eta^{(1)} - 3\phi^{(1)}\cos^2\chi\sin\chi \tag{12}$$

These four equations contain all of the requirements which are to be satisfied by the transformation of variables. It remains for us to find expressions for $M^{(1)}$, $M^{(2)}$, $\Omega^{(1)}$, $\Omega^{(2)}$, $\eta^{(1)}$, $\eta^{(2)}$, $\phi^{(1)}$, and $\phi^{(2)}$ such that the four equations are satisfied. Since we have 4 equations but 8 unknowns, we are free to choose 4 values at random. However, it will be to our advantage to choose cleverly. In particular, it is part of the method of averaging to require that $\eta^{(1)}$, $\eta^{(2)}$, $\phi^{(1)}$, and $\phi^{(2)}$ each be periodic in χ. We will assume period 2π for convenience. (We could have stated this as part of the form of the proposed transformation.)

As a preliminary step to solving the system of 4 equations, we will compare the average value, over one period, of the two sides of each of the equations. Of course, they must be equal. This is the step which gives the method its name. We begin with equation (9).

$$\frac{1}{2\pi}\int_0^{2\pi} M^{(1)}d\chi + \frac{k}{2\pi}\int_0^{2\pi} \frac{\partial\eta^{(1)}}{\partial\chi}\, d\chi = \frac{1}{2\pi}\int_0^{2\pi} \sin^2\chi\, d\chi$$

$$\frac{1}{2\pi} M^{(1)}\chi\Big|_0^{2\pi} + \frac{k}{2\pi}\,\eta^{(1)}\Big|_0^{2\pi} = \frac{1}{2\pi}\left[\frac{\chi}{2} - \frac{\sin(2\chi)}{4}\right]_0^{2\pi}$$

$$M^{(1)} = \frac{1}{2}$$

Now that we know $M^{(1)} = \frac{1}{2}$ lets take another look at equation (9).

$$\frac{1}{2} + \frac{\partial\eta^{(1)}}{\partial\chi}k = \sin^2\chi$$

$$\frac{\partial\eta^{(1)}}{\partial\chi} = \frac{\sin^2\chi - \frac{1}{2}}{k}$$

$$\eta^{(1)} = \frac{-\sin(2\chi)}{4k} + f_1(\xi) \qquad (13)$$

where $f_1(\xi)$ is the constant of integration with respect to χ. Let $f_1(\xi) = 0$. (Actually, I looked ahead before choosing 0). Now consider equation (10).

$$\frac{1}{2\pi}\int_0^{2\pi} \Omega^{(1)}\, d\chi + \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial\phi^{(1)}}{\partial\chi} k\, d\chi = \frac{1}{2\pi}\int_0^{2\pi} \xi^2\, d\chi + \frac{1}{2\pi}\int_0^{2\pi} \cos^3\chi d\chi$$

$$\Omega^{(1)} = \xi^2$$

Remember that many of the integrals we are encountering can be solved by inspection and many are zero. For example, in the present case

$\Omega^{(1)}$ is constant with respect to χ so the first integral was easy.

$\phi^{(1)}$ is periodic (with period 2π) in χ so the second integral was zero.

ξ^2 is constant with respect to χ, so the third integral was easy.

$\cos^3\chi$ is periodic and from our knowledge of the graph of $\cos\chi$, we can conclude the fourth integral is zero.

Now that we know $\Omega^{(1)}$ let's take another look at equation (10).

$$\xi^2 + \frac{\partial\phi^{(1)}}{\partial\chi} k = \xi^2 + \cos^3\chi$$

$$\frac{\partial\phi^{(1)}}{\partial\chi} = \frac{\cos^3\chi}{k}$$

$$\phi^{(1)} = \frac{\sin\chi\ (\cos^2\chi + 2)}{3k} + f_2(\xi) \tag{14}$$

Next we consider equation (11).

$$\frac{1}{2\pi}\int_0^{2\pi} M^{(2)}\, d\chi + \frac{1}{2\pi}\int_0^{2\pi} M^{(1)} \frac{\partial\eta^{(1)}}{\partial\xi}\, d\chi + \frac{1}{2\pi}\int_0^{2\pi} \Omega^{(1)} \frac{\partial\eta^{(1)}}{\partial\chi}\, d\chi$$

$$+ \frac{k}{2\pi}\int_0^{2\pi} \frac{\partial\eta^{(2)}}{\partial\chi}\, d\chi = \frac{1}{2\pi}\int_0^{2\pi} 2\phi^{(1)} \sin\chi\cos\chi\, d\chi$$

$M^{(2)} = 0$

Looking again at equation (11).

$$\xi^2\left(\frac{-\cos(2\chi)}{2k}\right) + k\,\frac{\partial\eta^{(2)}}{\partial\chi} = 2\left(\frac{\sin\chi\ (\cos^2\chi + 2)}{3k} + f_2(\xi)\right)\sin\chi\cos\chi$$

$$\eta^{(2)} = \frac{\xi^2}{2k^2} \int \cos(2\chi)\, d\chi + \frac{2}{3k^2} \int \sin^2\chi \cos^3\chi d\chi + \frac{4}{3k^2} \int \sin^2\chi \cos\chi\, d\chi +$$

$$\frac{2f_2}{k} \int \sin\chi \cos\chi\, d\chi$$

$$= \frac{\xi^2}{2k^2} \frac{\sin(2\chi)}{2} - \frac{2}{3k^2} \frac{\sin^5\chi}{5} + \frac{6}{3k^2} \frac{\sin^3\chi}{3}$$

$$+ \frac{2f_2}{k} \frac{\sin^2\chi}{2} + f_3(\xi) \tag{15}$$

Finally, we compare average values for equation (12). A lot of trigonometric identities are used here. Personal preference has significant impact on form and simplifications. Be aware that verification can be tedious.

$$\frac{1}{2\pi}\int_0^{2\pi} \Omega^{(2)}\, d\chi + \frac{1}{2\pi}\int_0^{2\pi} M^{(1)} \frac{\partial\phi^{(1)}}{\partial\xi}\, d\chi + \frac{1}{2\pi}\int_0^{2\pi} \Omega^{(1)} \frac{\partial\phi^{(1)}}{\partial\chi}\, d\chi +$$

$$\frac{1}{2\pi}\int_0^{2\pi} k \frac{\partial\phi^{(2)}}{\partial\chi}\, d\chi = \frac{1}{2\pi}\int_0^{2\pi} 2\xi\eta^{(1)}\, d\chi - \frac{1}{2\pi}\int_0^{2\pi} 3\phi^{(1)} \cos^2\chi \sin\chi\, d\chi$$

$$\Omega^{(2)} + \frac{1}{2\pi}\frac{1}{2}\int_0^{2\pi} \frac{df_2}{d\xi}\, d\chi = \frac{\xi}{\pi}\int_0^{2\pi} \left(\frac{-\sin(2\chi)}{4k}\right) d\chi$$

$$- \frac{3}{2\pi}\int_0^{2\pi} \left[\frac{\sin\chi(\cos^2\chi + 2)}{3k} + f_2\right] \cos^2\chi \sin\chi\, d\chi$$

$$\Omega^{(2)} + \frac{1}{2}\frac{df_2}{d\xi} = -\frac{1}{2\pi k}\int_0^{2\pi} \sin(\chi\cos^2\chi + 2)\cos^2\chi\sin\chi \, d\chi$$

$$-\frac{3f_2}{2\pi}\int_0^{2\pi} \cos^2\chi\sin\chi \, d\chi$$

$$\Omega^{(2)} + \frac{1}{2}\frac{df_2}{d\xi} = -\frac{1}{2\pi k}\int_0^{2\pi} \sin^2\chi\cos^4\chi \, d\chi - \frac{2}{2\pi k}\int_0^{2\pi} \sin^2\chi\cos^2\chi \, d\chi$$

$$= -\frac{1}{2\pi k}\left[\frac{\sin^3\chi\cos^3\chi}{6} + \frac{\sin^3\chi\cos\chi}{8} - \frac{1}{16}\sin\chi\cos\chi + \frac{\chi}{16}\right]_0^{2\pi}$$

$$-\frac{1}{\pi k}\left[\frac{\sin^3\chi\cos\chi}{4} - \frac{\sin\chi\cos\chi}{8} + \frac{\chi}{8}\right]_0^{2\pi}$$

$$= -\frac{1}{16k} - \frac{1}{4k}$$

$$= -\frac{5}{16k}$$

Now choose $f_2 = \frac{-5\xi}{8k}$ to make $\Omega^{(2)} = 0$.

Note: There is a trade-off here. I could have chosen $f_2 = 0$ long ago and $\Omega^{(2)}$ would have been a constant (still easy to integrate later) and simplified $\phi^{(1)}$ in the transformation. Personal preference was my only motivation.

Now we take another look at equation (12).

$$\frac{-5}{16k} + \xi^2 \frac{1}{3k}\left(\sin\chi\ (-2\cos\chi\ \sin\chi) + \cos\chi\ (\cos^2\chi + 2)\right) + k\frac{\partial\phi^{(2)}}{\partial\chi}$$

$$= \frac{-\xi}{2k}\sin(2\chi) - 3\left(\frac{\sin\chi(\cos^2\chi + 2)}{3k} - \frac{5\xi}{8k}\right)\cos^2\chi\sin\chi$$

$$\frac{\partial\phi^{(2)}}{\partial\chi} = \frac{-\xi}{2k^2}\sin(2\chi) - \frac{1}{k^2}\cos^4\chi\sin^2\chi - \frac{2}{k^2}\cos^2\chi\sin^2\chi +$$

$$\frac{15\xi}{8k}\cos^2\chi\sin\chi + \frac{5}{16k^2} + \frac{2\xi^2}{3k^2}\quad \sin^2\chi\cos\chi - \frac{\xi^2}{3k^2}\cos^3\chi - \frac{2\xi^2}{3k^2}\cos\chi$$

$$\phi^{(2)} = \frac{\xi}{4k^2}\cos(2\chi) - \frac{1}{k^2}\left[\frac{\cos^3\chi\sin^3\chi}{6} - \frac{1}{16}\left(\frac{\sin(4\chi)}{4} - \chi\right)\right]$$

$$- \frac{2}{k^2}\left(\frac{\sin^3\chi\cos\chi}{4} - \frac{\sin\chi\cos\chi}{8} + \frac{\chi}{8}\right) - \frac{15\xi}{8k^2}\frac{\cos^3\chi}{3} + \frac{5\chi}{16k^2} -$$

$$\frac{\xi^2}{3k^2}\sin\chi\ \left(\cos^2\chi + 2\right) + f_4\ (\xi)$$

$$= \frac{1}{k^2}\left[\frac{\xi}{4}\cos(2\chi) - \frac{\cos^3\chi\sin^3\chi}{6} + \frac{5}{64}\sin\ (4\chi)\right.$$

$$\left. - \frac{5\xi}{8}\cos^3\chi - \frac{\xi^2}{3}\sin\chi\ (\cos^2\chi + 2)\right] \tag{16}$$

where we have chosen $f_4(\xi) = 0$. We have not seen any opportunity for a clever choice of f_3. Let $f_3(\xi) = 0$.

The transformed differential equations are

$$\dot{\xi} = \frac{\varepsilon}{2}$$

$$\dot{\chi} = k + \varepsilon\xi^2$$

This can be solved as

$$\xi = \frac{\varepsilon}{2} t + \xi_0$$

$$\dot{\chi} = k + \varepsilon \left(\frac{\varepsilon}{2}t + \xi_0\right)^2$$

$$\dot{\chi} = k + \varepsilon \left(\frac{\varepsilon^2 t^2}{4} + \varepsilon t \xi_0 + \xi_0^2\right)$$

$$\chi = kt + \frac{\varepsilon^3 t^3}{12} + \frac{\varepsilon^2 t^2}{2} \xi_0 + \varepsilon t \xi_0^2 + \chi_0$$

The transformation back to the original variables is obtained by substituting equations (13) - (16) into equations (7) and (8).

$$x = \xi - \frac{\varepsilon \sin(2\chi)}{4k} + \varepsilon^2 \left[\frac{\xi^2 \sin(2\chi)}{4k^2} - \frac{2\sin^5\chi}{15k^2} + \frac{2\sin^3\chi}{3k^2} - \frac{5\xi\sin^2\chi}{8k^2}\right]$$

$$y = \chi + \varepsilon\left[\frac{1}{3k} \sin\chi \,(\cos^2\chi + 2) - \frac{5\xi}{8k}\right] + \varepsilon^2 \frac{1}{k^2}\left[\frac{\xi}{4} \cos(2\chi) - \right.$$

$$\left. \frac{\cos^3\chi\sin^3\chi}{6} + \frac{5}{64} \sin(4\chi) - \frac{5\xi}{8} \cos^3\chi - \frac{\xi^2}{3} \sin\chi \,(\cos^2\chi + 2)\right]$$

This concludes our second example.

Note that different choices for the constants of integration lead to answers which may appear to be different, yet still be correct.

THIRD EXAMPLE

The next example has one fast and two slow variables.

Given:

$$\dot{x}_1 = \varepsilon\ (x_2 + \sin^2 y)$$

$$\dot{x}_2 = \varepsilon \cos^2 y$$

$$\dot{y} = x_1\ x_2 + \varepsilon\ x_1^2 \sin y$$

Solve through 1[st] order using the method of averaging. We seek a transformation of the form

$$x_1 = \xi_1 + \varepsilon\eta_1\ (\xi_1,\ \xi_2,\ \chi) \qquad (17)$$

$$x_2 = \xi_2 + \varepsilon\eta_2\ (\xi_1,\ \xi_2,\chi) \qquad (18)$$

$$y = \chi + \varepsilon\phi\ (\xi_1,\xi_2,\chi) \qquad (19)$$

where we require η_1, η_2, and ϕ to be periodic in χ. Once η_1, η_2, and ϕ are known, we can differentiate and rearrange to get transformed differential equations of the form

$$\dot{\xi}_1 = \varepsilon\ M_1\ (\xi_1,\xi_2)$$

$$\dot{\xi}_2 = \varepsilon\ M_2\ (\xi_1,\ \xi_2)$$

$$\dot{\chi} = \xi_1\ \xi_2 + \varepsilon\ \Omega\ (\xi_1\ ,\ \xi_2\ )$$

where, as indicated in the notation, M_1, M_2, and Ω do not depend on χ.

If we differentiate the proposed transformation, we see that in terms of the new variables, the differential equations are

$$\dot{x}_1 = \dot{\xi}_1 + \varepsilon\ \frac{\partial\eta_1}{\partial\xi_1}\ \dot{\xi}_1 + \varepsilon\ \frac{\partial\eta_1}{\partial\xi_2}\ \dot{\xi}_2 + \varepsilon\ \frac{\partial\eta_1}{\partial\chi}\ \dot{\chi}$$

$$\dot{x}_2 = \dot{\xi}_2 + \varepsilon\ \frac{\partial\eta_2}{\partial\xi_1}\ \dot{\xi}_1 + \varepsilon\ \frac{\partial\eta_2}{\partial\xi_2}\ \dot{\xi}_2 + \varepsilon\ \frac{\partial\eta_2}{\partial\chi}\ \dot{\chi}$$

$$\dot{y} = \dot{\chi} + \varepsilon\ \frac{\partial\phi}{\partial\xi_1}\ \dot{\xi}_1 + \varepsilon\ \frac{\partial\phi}{\partial\xi_2}\ \dot{\xi}_2 + \varepsilon\ \frac{\partial\phi}{\partial\chi}\ \dot{\chi}$$

And if we further substitute, using the form which we desire $\dot{\xi}_1$, $\dot{\xi}_2$, and $\dot{\chi}$ to have, we obtain the first order approximation

$$\dot{x}_1 = \varepsilon M_1 + \varepsilon \frac{\partial \eta_1}{\partial \chi} \xi_1 \xi_2 + 0 (\varepsilon^2)$$

$$\dot{x}_2 = \varepsilon M_2 + \varepsilon \frac{\partial \eta_2}{\partial \chi} \xi_1 \xi_2 + 0(\varepsilon^2)$$

$$\dot{y} = \xi_1 \xi_2 + \varepsilon \Omega + \varepsilon \frac{\partial \phi}{\partial \chi} \xi_1 \xi_2 + 0 (\varepsilon^2)$$

These are power series representations in ε of the differential equations. Another set of power series representations in ε can be obtained by expanding the original equations in Taylor series. After this has been done, we can set the power series representations equal and all of the dot terms will be gone. Further, since both series are power series and they converge to the same function, they are equal term by term (from which we will extract a system of equations to solve). For the original equations, the Taylor series is

$$\dot{x}_1 = \varepsilon \left(x_2 + \sin^2 y\right)\Big|_{\substack{x_1 = \xi_1 \\ x_2 = \xi_2 \\ y = \chi}} + \varepsilon \frac{\partial}{\partial x_1} \left(x_2 + \sin^2 y\right)\Big|_{\substack{x_1 = \xi_1 \\ x_2 = \xi_2 \\ y = \chi}} (x_1 - \xi_1) +$$

$$\frac{\partial}{\partial x_2} \left(x_2 + \sin^2 y\right)\Big|_{\substack{x_1 = \xi_1 \\ x_2 = \xi_2 \\ y = \chi}} (x_2 - \xi_2) + \varepsilon \frac{\partial}{\partial y} \left(x_2 + \sin^2 y\right)\Big|_{\substack{x_1 = \xi_1 \\ x_2 = \xi_2 \\ y = \chi}} (y - \chi) + 0 (\varepsilon^2)$$

but, $(x_1 - \xi_1)$, $(x_2 - \xi_2)$, and $(y - \chi)$ are all $0(\varepsilon)$, which means that through first order

$$\dot{x}_1 = \varepsilon \left(\xi_2 + \sin^2 \chi\right)$$

Similarly,

$$\dot{x}_2 = \varepsilon \cos^2 y \Big|_{\substack{x_1 = \xi_1 \\ x_2 = \xi_2 \\ y = \chi}} + 0 (\varepsilon^2)$$

$= \varepsilon \cos^2\chi$

$$\dot{y} = x_1 x_2 \Big|_{\substack{x_1 = \xi_1 \\ x_2 = \xi_2 \\ y = \chi}} + \varepsilon x_1^2 \sin y \Big|_{\substack{x_1 = \xi_1 \\ x_2 = \xi_2 \\ y = \chi}} +$$

$$\frac{\partial}{\partial x_1} (x_1 x_2) \Big|_{\substack{x_1 = \xi_1 \\ x_2 = \xi_2 \\ y = \chi}} (x_1 - \xi_1) + \frac{\partial}{\partial x_2} (x_1 x_2) \Big|_{\substack{x_1 = \xi_1 \\ x_2 = \xi_2 \\ y = \chi}} + O(\varepsilon^2)$$

$$= \xi_1 \xi_2 + \varepsilon \xi_1^2 \sin\chi + \varepsilon \xi_2 \eta_1 + \varepsilon \xi_1 \eta_2$$

When we match corresponding terms of the power series we get

$$M_1 + \frac{\partial \eta_1}{\partial \chi} \xi_1 \xi_2 = \xi_2 + \sin^2\chi$$

$$M_2 + \frac{\partial \eta_2}{\partial \chi} \xi_1 \xi_2 = \cos^2\chi$$

$$\Omega + \frac{\partial \phi}{\partial \chi} \xi_1 \xi_2 = \xi_1^2 \sin\chi + \xi_2 \eta_1 + \xi_1 \eta_2 \qquad (20)$$

When we compare the average values over one period for each equation, we obtain

$$M_1 = \xi_2 + \frac{1}{2}$$

$$M_2 = \frac{1}{2}$$

which we can substitute back to get

$$\xi_2 + \frac{1}{2} + \frac{\partial \eta_1}{\partial \chi} \xi_1 \xi_2 = \xi_2 + \sin^2\chi$$

$$\frac{\partial \eta_1}{\partial \chi} = \frac{1}{\xi_1 \xi_2} \left(\sin^2\chi - \frac{1}{2}\right)$$

$$\eta_1 = \frac{1}{\xi_1 \xi_2} \left[\frac{\chi}{2} - \frac{1}{4} \sin(2\chi) - \frac{\chi}{2} + f_1 \left(\xi_1 , \xi_2\right)\right]$$

Let $f_1 = 0$. Then

$$\eta_1 = \frac{- \sin (2\chi)}{4 \xi_1 \xi_2} \tag{21}$$

Next,

$$\frac{1}{2} + \frac{\partial \eta_2}{\partial \chi} \xi_1 \xi_2 = \cos^2\chi$$

$$\frac{\partial \eta_2}{\partial \chi} = \frac{1}{\xi_1 \xi_2} \left(\cos^2\chi - \frac{1}{2}\right)$$

$$\eta_2 = \frac{1}{\xi_1 \xi_2} \left[\frac{\chi}{2} + \frac{1}{4} \sin(2\chi) - \frac{\chi}{2} + f_2 \left(\xi_1, \xi_2\right)\right.$$

Let $f_2 = 0$.

$$\eta_2 = \frac{\sin(2\chi)}{4 \xi_1 \xi_2} \tag{22}$$

Now we compare average values for equation (20) and obtain

$\Omega = 0$

Substituting into equation (20).

$$\frac{\partial \phi}{\partial \chi} \xi_1 \xi_2 = \xi_1^2 \sin\chi + \xi_2 \frac{-\sin (2\chi)}{4 \xi_1 \xi_2} + \xi_1 \frac{\sin(2\chi)}{4 \xi_1 \xi_2}$$

$$\frac{\partial \phi}{\partial \chi} = \frac{\xi_1}{\xi_2} \sin\chi - \frac{\sin (2\chi)}{4 \xi_1^2 \xi_2} + \frac{\sin (2\chi)}{4 \xi_1 \xi_2^2}$$

$$\phi = \frac{-\xi_1}{\xi_2} \cos\chi + \left(\frac{\xi_2 - \xi_1}{8 \xi_1^2 \xi_2^2}\right) \cos(2\chi) \tag{23}$$

Now, our new differential equations are

$$\dot{\xi}_1 = \varepsilon\, \xi_2 + \frac{\varepsilon}{2}$$

$$\dot{\xi}_2 = \frac{\varepsilon}{2}$$

$$\dot{\chi} = \xi_1 \xi_2$$

whose solutions are developed by

$$\xi_2 = \frac{\varepsilon}{2}\, t + \xi_{2_0}$$

$$\dot{\xi}_1 = \frac{\varepsilon^2}{2}\, t + \varepsilon\, \xi_{2_0} + \frac{\varepsilon}{2}$$

$$\xi_1 = \frac{\varepsilon^2 t^2}{4} + \varepsilon t\, \xi_{2_0} + \frac{\varepsilon}{2}\, t + \xi_{1_0}$$

$$\dot{\chi} = \left(\frac{\varepsilon^2 t^2}{4} + \varepsilon t \xi_{2_0} + \frac{\varepsilon}{2}\, t + \xi_{1_0}\right)\left(\frac{\varepsilon}{2}\, t + \xi_{2_0}\right)$$

$$\chi = \frac{\varepsilon^3}{8}\,\frac{t^4}{4} + \left(\frac{3\varepsilon^2}{4}\,\xi_{2_0} + \frac{\varepsilon^2}{4}\right)\frac{t^3}{3} +$$

$$\left(\varepsilon\, \xi_{2_0}^2 + \frac{\varepsilon}{2}\xi_{2_0} + \frac{\varepsilon}{2}\xi_{1_0}\right)\frac{t^2}{2} + \xi_{1_0}\xi_{2_0} t + \chi_0$$

which can be transformed back to the original variables by substituting equations (21) - (23) into equations (17) - (19). This concludes the example.

$$x_1 = \xi_1 - \frac{\varepsilon \sin(2\chi)}{4\, \xi_1 \xi_2}$$

$$x_2 = \xi_2 + \frac{\varepsilon \sin(2\chi)}{4\, \xi_1 \xi_2}$$

$$y = \chi + \varepsilon \left[\frac{-\xi_1}{\xi_2} \cos\chi + \frac{\xi_2 - \xi_1}{8\, \xi_1^2\, \xi_2^2} \cos(2\chi)\right]$$

A TRANSITION TO THE LITERATURE

Historically, primes were used to indicate the transformed variables. That is, $x' = \xi$, $y' = \chi$. Because of this, it is not unusual to read of "primed variables" although the primes are almost never actually written. If the method is applied twice within a problem then "doubly primed" variables are obtained. (This would be done to remove the second fastest periodic from a set of equations.)

The generalized method of averaging addresses a system of slow variables

$$\dot{x}_i = \varepsilon f_i^{(1)} (\underline{x}, \underline{y}) + \varepsilon^2 f_i^{(2)} (\underline{x}, \underline{y}) + \ldots.$$

and fast variables

$$\dot{y}_\alpha = w_\alpha (\underline{x}) + \varepsilon u_\alpha^{(1)} (\underline{x}, \underline{y}) + \varepsilon^2 u_\alpha^{(2)} (\underline{x}, \underline{y}) + \ldots$$

The strategy is to seek a transformation of the form

$$x_i = \xi_i + \varepsilon\eta_i^{(1)} (\underline{\xi}, \underline{\chi}) + \varepsilon^2\eta_i^{(2)} (\underline{\xi}, \underline{\chi}) + \ldots$$

$$y_\alpha = \chi_\alpha + \varepsilon \phi_\alpha^{(1)} (\underline{\xi}, \underline{\chi}) + \varepsilon^2 \phi_\alpha^{(2)} (\underline{\xi}, \underline{\chi}) + \ldots$$

where $\xi_i = x_i'$ by notational convention.

The corresponding differential equations will have the form

$$\dot{\xi}_i = \varepsilon M_i^{(1)} (\underline{\xi}) + \varepsilon^2 M_i^{(2)} (\underline{\xi}) + \ldots$$

$$\dot{\chi}_\alpha = w_\alpha (\underline{\xi}) + \varepsilon \Omega_\alpha^{(1)} (\underline{\xi}) + \varepsilon^2 \Omega_\alpha^{(2)} (\underline{\xi}) + \ldots$$

These are solved using the strategy demonstrated in the preceding examples. This concludes my introduction to the method of averaging. I have given a sufficient background for an intelligent reading of TN76-1 (Hoots & Major) which contains details.

For current applications in astrodynamics, see the references under Liu.

References to the Method of Averaging or the KBM method should now be more readable.

References

N.N. Bogoliubov and Y.A. Mitropolsky, Asymptotic Method in the Theory of Non-linear Oscillations, New York, Gorden and Beach, 1961

Feiix R. Hoots and Paul E. Major, The Generalized Method of Averaging, CCZ Technical Note 76-1, 10 Feb 76 (A report from the 14th Aerospace, now Air Force Space Command)

D. W. Jordon and P. Smith, Nonlinear Ordinary Differential Equations (2nd ed), Clarendon Press, 1987 (pp 98-111)

N. Krylov and N. N. Bogoliubov, Introduction to Nonlinear Mechanics (in Russian), Kiev, 1937

Joseph J.F. Liu, Certain Comments on the Method of Averaging and its Application, Northrop Services, Inc., Huntsville, Alabama TN-242-1045, NAS8-21810, Jan 1972

Joseph J.F. Liu, A Second-Order Theory of an Artificial Satellite Under the Influence of the Oblateness of Earth, Northrop Services, Inc., Huntsville, Alabama M-240-1203, NAS8-21810, Jan 1973

Leonard Meirovitch, Methods of Analytical Dynamics, McGraw-Hill, 1970 (pp 302-309, 322-328)

Ronald E. Mickens, An Introduction to Nonlinear Oscillations, Cambridge University Press, 1981 (pp 55-79)

Nicholas Minorsky, Nonlinear Oscillations, Van Nostrand, 1962 (pp 273-281)