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FORCE-FREE TIME-HARMONIC PLASMOIDS

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A heretofore unexplored solution of Maxwell's equations is investigated for time-harmonic waves in a partially ionized gas. The analysis is focussed on the spherically symmetric cases that behave like electromagnetic energy trapped in the form of a "plasmoid". It will be shown that a critical frequency exists, below which the current cannot be carried by electrons and the plasmoid remain stable. Resonant sizes will be shown to exist such that plasmoid will not exchange energy with their external surroundings, and their boundary conditions can be met by vacuum solutions to Maxwell's equations. Virial analysis calculates free-charge density and critical frequency to be consistent with Newtonian mechanics and classical electromagnetics. A stable vortical motion of the plasma will be shown to exactly cancel the dominant component of the electromechanical stresses, with the residual stresses being a strongly decreasing function of frequency.

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PREFACE

Work in this paper is primarily aimed at the elementary development of the theory of time–harmonic electromagnetic fields that have the force–free form for the electric and magnetic fields, as well as the current. It is hoped that enough detail is included for the paper to be used as a tutorial for the majority of people who have never encountered force-free fields, and wish to investigate what, at first blush, seems like an impossible solution of Maxwell's equations.

The research was spurred by attempts to understand experimental results obtained in the late 1960's and early 1970's in areas of fusion weapons and laser fusion in Los Alamos. At the time there were anomalies such as unexplained symmetries in the fragments of disintegrating laser–irradiated pellets, and puzzles about compression and stability of small spherical volumes of plasma in the presence of high–intensity electromagnetic fields. During that time, and since, a large number of people investigated many of the phenomena of interest, without fully being able to resolve all the challenges of plasma inertial confinement. No claims will be made that this work will make any of the inertial confinement problems easier.

In the period of close to two decades after performing the original work, some of the insights gained into solutions of Maxwell's equations in a conducting fluid were seen to have relevance beyond the controlled fusion community. In particular, finite–sized, non–plane–wave, force–free time–harmonic electromagnetic excitations, with electric and magnetic field vectors parallel to each other everywhere, in the presence of plasma current, seemed to be undiscovered solutions that deserved exploration. What makes such solutions interesting is that the Poynting vector is identically zero everywhere inside a finite–sized force–free field, and it cannot dissipate unless either its boundary is radiating or internal stresses make it break up.

Classically, these fields cannot exist in a vacuum. In a quiescient plasma, internal stresses on their accompanying currents can disrupt them as soon as they are created. By a remarkable coincidence, the electromechanical stresses manifest themselves mainly as a pressure field, which can be nearly exactly balanced by vortical fluid motion. This vortical fluid field, otherwise called Beltrami motion, is also self–disruptive, by itself, in a compressible fluid.

The alleged simultaneous occurance of the force–free time–harmonic electromagnetic field and the matching vortical fluid motion, with a well–defined boundary, will be called a "plasmoid". Although the solutions to Maxwell's equations, coupled to fluid–dynamic modes, do not seem to violate any physical principles, it is not immediately obvious how one would go about producing them. Mathematically, it, at first, seemed possible that such excitations could exist at all frequencies. A more physical approach showed that below a critical frequency, determined by its size, a plasmoid represents a plasma state at a local energy maximum, *i.e.* it is unstable. In addition, electrons are unable to carry the current below this critical frequency. Above the critical frequency, the physics becomes more interesting, as the current is carried by the electrons and the plasma is at an energy minimum, even though the total energy is positive.

The author believes that a number of observed phenomena, including ball lightning, "fireballs" and "charge clusters" that occur in strong plasma discharges, may be described, at least in part, by the plasmoids in this theory. Some of the non-linear effects of powerful lasers impinging on small spheres of material might also produce plasmoids of the type to be described.

The work in this paper being only a start, the theory is incomplete. A great deal of work still remains to be done to determine whether real plasmas at elevated temperatures can sustain force-free plasmoids. Open questions of quantum and quantum electrodynamic effects and interactions with the electromagnetic zero-point background remain to be investigated. Puzzles concerning external fields, surface currents and plasmoid interactions also need solving, and may involve quantum effects. The possibility that unexplored and, as yet, unknown principles are involved in explaining the existence of compact, stable, autonomous, highly energetic concentrations of plasma and/or electromagnetic fields cannot be overlooked.

Much of this work can be conducted with quiet thought in libraries and in front of a computer. In general, however, a complementary experimental program is absclutely necessary at this point to focus the theoretical effort.

The author would like to thank Dr. Franklin B. Mead, Chief of the Future Technologies Section in the Astronautical Sciences Division at the Air Force's Phillips Laboratory for the the interest he has shown in this work. Without Dr. Mead's encouragement and kind hospitality the ideas in this work may never have been developed even this far. Research in such areas is very risky and I acknowledge Dr. Mead's bravery, in an era of rapidly diminishing funds, for the support of the work. Thanks are also due to the University of Dayton Research Institute and Dr. Eugene Gerber for providing a pleasant atmosphere for conducting research.

FORCE–FREE TIME–HARMONIC PLASMOIDS

SUMMARY

Maxwell's equations allow time–harmonic solutions wherein the electric and magnetic fields are parallel to each other and both have the force–free form. These fields are accompanied by a time–harmonic current field that also has the force–free form. Such solutions are postulated as a model for long–lived spherical plasmoids. It is shown that a critical frequency exits, which depends on the size of the plasmoid. Below this frequency, the time–harmonic current cannot be carried by electrons in a plasma. Above the critical frequency local perturbations of the current tend to increase the total energy, lending stabilization. The electromechanical stresses, induced by the fields on the current–carrying electrons, can be balanced, to a high degree of accuracy by vortical motion of the supporting plasma. The velocity field of the vortex also has the force–free (Beltrami) form. This vortical fluid motion is stable and exactly compatible with the stresses in the plasmoid. Although it is a theoretical possibility for currents to flow on the outside of a plasmoid to isolate it from external fields, free–space time–harmonic fields can be found that have both their electric and magnetic fields parallel to each other over the surface of a spherical plasmoid having a resonant size, making the fields continuous at the boundary. Surface currents can thereby be eliminated. Only at a resonant size can the outward–going radiation due to the fields at the surface of the plasmoid be nulled and the external fields be joined continuously to the internal fields. An example is given showing that a one–megajoule force–free plasmoid in an STP atmosphere could be described with physically reasonable parameters.

I. INTRODUCTION

Bostick described a plasmoid as a structure made of plasma, "... whose form is determined by the magnetic field it carries along with itself[1] ." His definition attempted to describe the coherent discharges he encountered emanating from a plasma gun. These plasma structures are evidently very stable, have long lifetimes, and can interact with each other[2]. Since the time of Bostick's original work, plasmoid–like structures have been reported by other investigators[3]. Under different conditions they may have names like "anomalous discharges", "charge clusters", and "ball lightning."

The phenomenona that seem to be shared by all the occurences of plasmoids are the presence of strong electromagnetic fields and a current flow near the time and position of plasmoid creation.

There may be a number of distinct species of plasmoids, each deriving its stability from different physical principles. These principles may involve quantum as well as quantum-electrodynamic effects. It is possible that some undiscovered physical laws govern plasmoid generation. What will be done in this work, however, is to assume that at least one family of plasmoids can be described by a classical model that invokes only "standard" physical theory. The physical model to be presented for plasmoids may not be complete, *i.e.*, it may have deficiencies. It may not describe, even approximately, all species of stable plasma excitations lumped under the term "plasmoids." There are, however, compelling reasons to explore a simple theory based on an elementary model, which appears to be relevant to stable plasmoid creation and subsequent evolution. Pushing the theory to its limits, its deficiencies may become evident. At that point, suggestions can be made for improving the model, and where new physics has to be introduced.

A force-free time-harmonic electromagnetic wave trapped in a stabilizing plasma vortical mode is chosen as a model. The functional forms of the time-varying electric and magnetic vector fields are identical, within a constant factor and phase, with this form being identical to that of the fluid velocity field of the plasma. The major electromagnetic stresses, manifested by pressures proportional to the local electric field energy, are balanced by reduced pressures due to fluid motion. The pinch effect due to the time-varying current adds to the stability.

Using a standing-wave model is not new. Kapitza conjectured that ball lightning was explained by such a model[4]. A model having a vortical field identical to a component of the electromagnetic field is not new either. Wells explored such a model a number of years ago[5]. What seems to be new is the combination of the ideas to explain some aspects of plasmoid phenomena.

After some mathematical preliminaries, it will be shown that time-harmonic solutions to Maxwell's equations can have the electric and magnetic fields, as well as the current in the force-free form. It will be shown that the phenomena of interest do not occur below a critical frequency, which is determined by the size of the plasmoid.

II. MATHEMATICAL PRELIMINARIES

A vector field, **F**, will be called "force-free" if it obeys the equation

$$\nabla \times \mathbf{F} = \alpha \mathbf{F}, \tag{1}$$

where α can be a complex-valued function of position. Unless otherwise noted, however, α will be assumed to be constant in time, spatially uniform, and real, inside regions that will be called plasmoids. Equation (1) will not necessarily be assumed to hold outside the boundary of a plasmoid.

At first sight, it may seem that the only solution to eq.(1) is $\mathbf{F} = 0$. The analogy usually brought to mind is that of the magnetic field due to a steady and uniform direct current flowing in a wire. In fig.(1) a steady current, $\mathbf{J}$, with units of amperes per square meter, is assumed flowing in an infinite cylindrical wire, producing a magnetic field $\mathbf{H}$ inside and outside the wire. From Maxwell's equations, $\nabla\times\mathbf{H} = \mathbf{J}$ inside the wire. It is quite obvious that $\mathbf{H}$ and $\mathbf{J}$ are not parallel to each other anywhere.

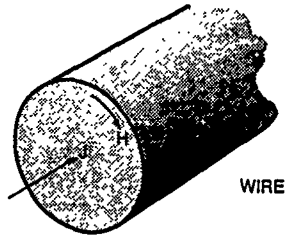

Figure 1. Uniform Current in a Cylindrical Wire. Inside the wire the magnetic field is orthogonal to the current.

In general, however, it is possible to create vector functions for which eq.(1) is true. Taking the curl of eq.(1) inside a plasmoid results in

$$\nabla\times\nabla\times\mathbf{F} = \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F} = \alpha^2\mathbf{F}. \tag{2}$$

From eq.(1), $\mathbf{F}$ arises from the curl of itself so that $\nabla\cdot\mathbf{F} = 0$. Therefore it follows that $\mathbf{F}$ satisfies the vector Helmholtz equation

$$(\nabla^2 + \alpha^2)\mathbf{F} = 0. \tag{3}$$

Suitable solutions to eq.(3) in spherical coordinates can be obtained from solutions of the scalar Helmholtz equation

$$(\nabla^2 + \alpha^2)\psi(\mathbf{r}) = 0, \tag{4}$$

using the formulas[6]

$$\mathbf{L}(\mathbf{r}) = \nabla\psi(\mathbf{r}), \tag{5}$$

$$\mathbf{M}(\mathbf{r}) = \mathbf{L}(\mathbf{r})\times\mathbf{r}, \tag{6}$$

and

$$\mathbf{N}(\mathbf{r}) = \frac{1}{\alpha}\nabla\times\mathbf{M}(\mathbf{r}), \tag{7}$$

where $\mathbf{r}$ is the radius vector from the origin to the field point. The solenoidal vector $\mathbf{F} = \mathbf{M} + \mathbf{N}$ satisfies eq.(1).

Although this prescription will produce a force–free vector field for each ψ satisfying eq.(4), it is beyond the present scope of the present work to investigate the mathematical completeness of this procedure.

A concrete example of a family of force–free vector fields is

$$\mathbf{M}_l(r,\theta,\phi) = (l+1)\cos\theta\sin^{l-1}\theta\frac{j_l(\alpha r)}{\alpha r}\hat{\mathbf{r}} - \left[j_{l-1}(\alpha r) - l\frac{j_l(\alpha r)}{\alpha r}\right]\sin^l\theta\hat{\theta} \pm j_l(\alpha r)\sin^l\theta\hat{\phi}, \tag{8}$$

where l is any integer, $l > 0$, j_l is the familiar spherical Bessel function, and the vectors $\hat{\mathbf{r}}$, $\hat{\theta}$, and $\hat{\phi}$ are the familiar right–handed 3–D polar coordinate unit vectors, with ϕ the polar angle. The vector function $\mathbf{M}_l(r,\theta,\phi)$ is axially symmetric, *i.e.*, there is no dependence on ϕ in any of the vector components.

Figures (2-4) show families of field lines for the solution described by eq.(8) with $l = 1$. All the points lying on a given field line define a family whose locus is the surface of a torus. If any field point approaches the radial distance where $j_1(\alpha r) = 0$, the other points in its family lie on a torus that approaches a spherical overall shape, as in fig.(4). In the limit, the field lines on the spherical part of such surfaces approach the laminar velocity field lines of an incompressible fluid flowing over a rigid sphere, with the poles being the stagnation points[7].

The $l = 1$ case will be very important in the rest of this work, since this case has the highest symmetries and the smallest field–line curvatures for a given radius where the first zero of $j_l(\alpha r)$ occurs. An $l = 1$ spherical plasmoid will henceforth be assumed to have a minimum radius $R_0 = 4.493\ldots/\alpha$. The radius R_0 corresponds to the first zero of $j_1(\alpha r)$. At any radius corresponding to a zero of $j_1(\alpha r)$, the force–free vector field in eq.(8) has no radial component. The importance of this property will become evident below.

Analytic force–free fields in toroidal coordinates have also been investigated.[8]

The name "force–free" came from the observation that a D.C. magnetic field, under vacuum conditions and satisfying eq.(1), would have to arise from a volumetric current field that was parallel to the magnetic field at every point. For such a magnetic field, the stresses on the currents, which come from the Lorentz force term $\mu_0\mathbf{J}\times\mathbf{H}$, where $\mathbf{J}$ is the current density, $\mathbf{H}$ is the magnetic field, and μ_0 is the permeability of the vacuum, would be nil.

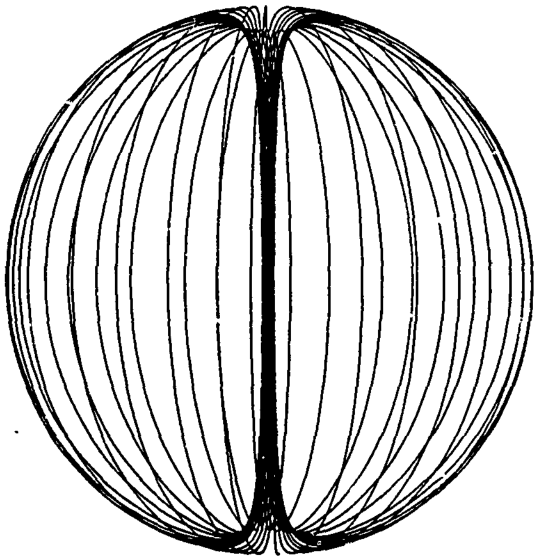

Figure 2. Ergodic flow lines for the vector field in eq.(8), with $\alpha = 1$ and $l = 1$. The starting point for the flow lines is on the equator, at $r = 4.49$. The figure traced out by the lines has the topology of a torus, having a spherical "outer" shape with a "hole" along the symmetry axis. If the radius of the starting point was exactly at the first zero of $j_1(\alpha r)$, *i.e.*, at $r = 4.493\ldots$, the outer shape would exactly be spherical and the hole would have zero volume. This figure is viewed orthogonally to the symmetry axis.

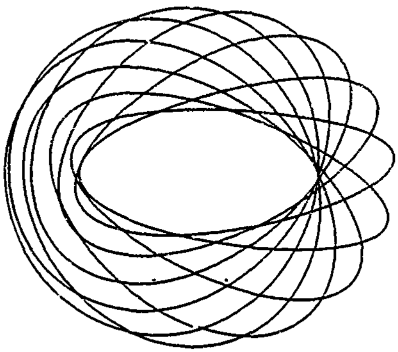

Figure 3. Ergodic flow lines for the vector field in eq.(8), with $\alpha = 1$ and $l = 1$. The starting point for the flow lines is on the equator, at $r = 3.37$, about three-quarters of the distance from the origin to the first zero of $j_1(\alpha r)$. The surface supporting the flow lines is donut-shaped. This figure is viewed at an angle of 45° to the symmetry axis.

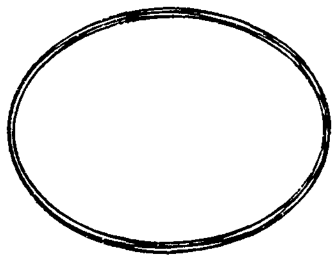

Figure 4. Ergodic flow lines for the vector field in eq.(8), with $\alpha = 1$ and $l = 1$. The starting point for the flow lines is on the equator, at $r = 2.8$, about three-quarters of the distance from the origin to the first zero of $j_1(\alpha r)$. The surface supporting the flow lines resembles a thin donut. This figure is viewed at an angle of 45° to the symmetry axis.

Since the stresses in such a system would be minimized, it follows that the energy directly associated with the magnetic field must be at an extremum. The attempt is made to utilize this principle in the design of compact toroids and spheromaks[9]. In these devices an effort is directed to produce force–free plasma regions that will have long lifetimes and thereby be hospitable for thermonuclear reactions.

Force–free fields were investigated in the nineteenth century by Beltrami[10], and are still under active study[11]. Although the interested reader will find an ample literature on the subject, the non–plane–wave, time–harmonic case, including self–consistent current, has not been published.

III. TIME–HARMONIC FORCE–FREE ELECTROMAGNETIC FIELDS

Assume the electric and magnetic fields in a neutral plasma can be written

$$\mathbf{E}(\mathbf{r},t) \equiv \mathbf{E}(\mathbf{r})\sin(\omega t + \varphi), \tag{9}$$

and

$$\mathbf{H}(\mathbf{r},t) \equiv \mathbf{H}(\mathbf{r})\cos(\omega t + \varphi), \tag{10}$$

where ω is the angular frequency of the time–harmonic electromagnetic field and φ is a phase angle. Note that the fields are described by real numbers in eqs.(9) and (10).

Maxwell's equations in a plasma can be written

$$\nabla \times \mathbf{E}(\mathbf{r}) = \mu\omega\mathbf{H}(\mathbf{r}) = Z_0 k\mathbf{H}(\mathbf{r}), \tag{11}$$

and

$$\nabla \times \mathbf{H}(\mathbf{r}) = \epsilon\omega\mathbf{E}(\mathbf{r}) + \mathbf{J}(\mathbf{r}) = \frac{k}{Z_0}\mathbf{E}(\mathbf{r}) + \mathbf{J}(\mathbf{r}), \tag{12}$$

where μ is the permeability of the plasma, ϵ is the permittivity of the plasma (both assumed spatially invariant), k is the wavenumber, $k = \sqrt{\epsilon\mu}\omega$, and Z_0 is the wave impedance, $Z_0 = \sqrt{\mu/\epsilon} \approx 377.6$ ohms in vacuo.

If the magnetic field, $\mathbf{H}(\mathbf{r})$, and the spatial part of the current, $\mathbf{J}(\mathbf{r})$ are assumed to be force–free, as defined by eq.(1), then it is easy to see, from eq.(11) that the electric field, $\mathbf{E}(\mathbf{r})$ will also have this property. From eqs.(11) and (12)

$$\mathbf{E}(\mathbf{r}) = Z_0 \frac{k}{\alpha} \mathbf{H}(\mathbf{r}), \tag{13}$$

and

$$\mathbf{J}(\mathbf{r}) = \alpha \left(1 - \frac{k^2}{\alpha^2}\right) \mathbf{H}(\mathbf{r}) = \frac{\alpha^2}{\omega\mu} \left(1 - \frac{k^2}{\alpha^2}\right) \mathbf{E}(\mathbf{r}). \tag{14}$$

If no free charge exists associated with the frequency ω, then it can be shown that it is merely sufficient for $\mathbf{H}(\mathbf{r})$ to have the force–free form for all the other electromagnetic quantities to have this form[12]. The reason for this is that under such conditions, $\nabla \cdot \mathbf{E} = 0$, and the electric field must be derived entirely from the curl of a vector field. By introducing the vector potential, $\mathbf{A}$ (see eqs.(17–19)), $\mathbf{E}(\mathbf{r})$ can be shown to be entirely derived from the curl of $\mathbf{H}(\mathbf{r})$, and must therefore have the force–free form if $\mathbf{H}(\mathbf{r})$ is in this form. It then follows from eq.(12) that $\mathbf{J}(\mathbf{r})$ must also have the force–free form.

From eq.(12) it is easy to see that the current and the magnetic field have the same time dependence factor, $\cos(\omega t + \varphi)$. There is a critical frequency, ω_0, for a fixed value of α, where

$$\omega_0 = \frac{\alpha}{\sqrt{\epsilon\mu}} = \alpha c, \tag{15}$$

and c is the speed of light in the plasma. At this critical frequency the phase of the current changes sign with respect to the electric and magnetic fields.

The significance of the last statement can be appreciated by considering the energy, Q, associated with a force–free field. From first principles,

$$Q = \frac{\epsilon}{2} \int \mathbf{E} \cdot \mathbf{E} dv + \frac{\mu}{2} \int \mathbf{H} \cdot \mathbf{H} dv. \tag{16}$$

Both $\mathbf{E}$ and $\mathbf{H}$ can be expressed in terms of the vector potential, $\mathbf{A}$, where

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu}{\alpha} \mathbf{H}(\mathbf{r}, t) \equiv \mathbf{A}(\mathbf{r}) \cos(\omega t + \varphi), \tag{17}$$

$$E(r,t) = -\frac{\partial A(r,t)}{\partial t}, \tag{18}$$

and

$$H(r,t) = \frac{1}{\mu}\nabla\times A(r,t). \tag{19}$$

Equation (16) now becomes

$$Q = \frac{\epsilon}{2}\int E\cdot(-\frac{\partial A}{\partial t})dv + \frac{1}{2}\int H\cdot\nabla\times A dv. \tag{20}$$

From the vector identity

$$\nabla\cdot(A\times H) = H\cdot\nabla\times A - A\cdot\nabla\times H, \tag{21}$$

the fact that $A\times H = 0$, and eq.(12),

$$Q = \frac{\epsilon\omega}{2}\sin^2(\omega t+\varphi)\int E(r)\cdot A(r)dv + \frac{1}{2}\cos^2(\omega t+\varphi)\int A(r)\cdot\left[\epsilon\omega E(r) + J(r)\right]dv. \tag{22}$$

Combining terms,

$$Q = \frac{\epsilon\omega}{2}\int E(r)\cdot A(r)dv + \frac{1}{2}\cos^2(\omega t+\varphi)\int A(r)\cdot J(r)dv. \tag{23}$$

The first term in eq.(23) is independent of time, but the second term is not. Taking the time average, $\overline{Q}$,

$$\overline{Q} = \frac{\epsilon\omega}{2}\int E(r)\cdot A(r)dv + \frac{1}{4}\int A(r)\cdot J(r)dv. \tag{24}$$

The first term in eq.(24) is always non-negative. The second term is positive for $\omega < \omega_0$ and negative for $\omega > \omega_0$. To see this, substituting for $J(r)$ and $A(r)$,

$$\int \mathbf{A}(\mathbf{r})\cdot\mathbf{J}(\mathbf{r})dv = \left(1 - \frac{k^2}{\alpha^2}\right)\mu \int \left|\mathbf{H}(\mathbf{r})\right|^2 dv. \qquad (25)$$

Note that the integral is always positive, but the factor multiplying the integral goes through zero and changes sign at the critical frequency.

The second integral in eq.(24) is the portion of the energy in the electromagnetic field arising from the direct interaction of the current with the magnetic field. Since Lorentz forces on the current are zero, this part of the energy is at an extremum. This extremum represents a minimum. It can be shown that the kinetic energy density of the electrons is equal and opposite in sign to this term. (see Appendix D.)

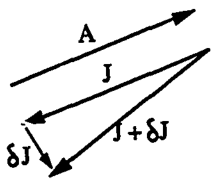

Figure 5. Perturbation of the J field. In a force-free plasmoid above the critical frequency, a small perturbation, $\delta\mathbf{J}$, of the current, $\mathbf{J}$, will add energy to the system.

An interesting consequence of the phase change above a critical frequency is demonstrated using fig.(5). Figure (5) shows the vector potential, **A**, and two currents, **J** and $\mathbf{J} + \delta\mathbf{J}$, all at the same point. The current **J** is the current that would naturally arise in a self-consistent force-free time-harmonic field. Note that **J** is exactly out of phase with **A**. The current $\mathbf{J}+\delta\mathbf{J}$ represents the original current perturbed in such a way that the two magnitudes are equal, i.e.,

$$\left|\mathbf{J}\right| = \left|\mathbf{J} + \delta\mathbf{J}\right|. \qquad (26)$$

A perturbation such as this may arise from a local statistical fluctuation of the plasma, redirecting the current but not changing its magnitude. Since the unperturbed current and the vector potential are 180° out of phase, and since the vector potential is not affected, to lowest order, by a small local current change,

$$(\mathbf{J} + \delta\mathbf{J})\cdot\mathbf{A} > \mathbf{J}\cdot\mathbf{A}, \qquad (27)$$

This implies that a small local deflection of the current would increase the energy of the system, so that it behaves like an elastic body. The same logic predicts the system may be unstable below the critical frequency because a small local perturbation of the current lowers the energy.

As will be seen in the next sections, stable time–harmonic force–free plasmoids may not exist below the critical frequency, from several considerations. The above arguments will be revisited when fluid velocities and stresses are introduced.

IV. RADIATION AND IDEAL PLASMOIDS

Before discussing electronic charge carriers and the conditions for overall stability, the question of radiation can be addressed. The assumption to be made is that plasmoids will form in such a manner that the electric and magnetic fields will be tangent to the surface boundary. The rationale for this will become clearer when fluid velocities are introduced. A self–consistent, stable plasmoid cannot have a component of velocity normal to its surface. Since the velocity field to be introduced will be identical in form to the electric and magnetic fields, the conclusion is reached that the electromagnetic fields must be tangent to the plasmoid surface. Only spherical plasmoids will be discussed, but the results are easily extended to cover possible non–spherical cases.

Instead of going into extensive mathematical proofs, which is done elsewhere[13], and in Appendix A for the $l = 1$ case, analogy will be made to the field inside a perfectly conducting spherical cavity excited at a transverse–electric (TE) resonance frequency. In a TE resonance the electric vector has no radial component and must therefore be zero at the surface of the cavity. The condition for resonance is the vanishing of the total electric field and the normal component of the magnetic field at the cavity surface. These conditions are met by the wavenumber, k, and the cavity radius, a satisfying[6]

$$j_l(ka) = 0, \tag{28}$$

where j_l is the usual l–th order spherical Bessel function.

The source for the cavity field is the surface current, $\hat{\mathbf{n}} \times \mathbf{H}$, at the boundary, where $\hat{\mathbf{n}}$ is the inward–pointing normal and $\mathbf{H}$ is the surface magnetic field. This surface current produces no field outside the cavity.

The analogy to the force–free plasmoid field can now be made. If ψ in eq.(4) is taken to be either of the functions ("o" for odd or "e" for even parity with respect to the polar angle ϕ)

$$\psi_{{}^{e}_{o} l,m} = j_l(kr) P_l^m(\cos\theta) \,{}^{\cos}_{\sin}\, m\phi, \tag{29}$$

then eqs.(5), (6) and (7) give vector fields[6]

$$\mathbf{m}_{{}^{e}_{o}l,m} = \mp \frac{m}{\sin\theta} j_l(kr) P_l^m(\cos\theta) \, {}^{\sin}_{\cos} m\phi \hat{\theta} - j_l(kr) \frac{\partial P_l^m}{\partial \theta} \, {}^{\cos}_{\sin} m\phi \hat{\phi}, \tag{30}$$

and

$$\begin{aligned} \mathbf{n}_{{}^{e}_{o}l,m} = & \frac{l(l+1)}{kr} j_l(kr) P_l^m(\cos\theta) \, {}^{\cos}_{\sin} m\phi \hat{\mathbf{r}} \\ & + \frac{1}{kr} \frac{\partial}{\partial r} \left[r j_l(kr) \right] \frac{\partial}{\partial \theta} P_l^m(\cos\theta) \, {}^{\cos}_{\sin} m\phi \hat{\theta} \\ & \mp \frac{m}{kr \sin\theta} \frac{\partial}{\partial r} \left[r j_l(kr) \right] P_l^m(\cos\theta) \, {}^{\sin}_{\cos} m\phi \hat{\phi}. \end{aligned} \tag{31}$$

For a TE resonance, the electric field can be written ("$\propto$" means "proportional to")

$$\mathbf{E}_{{}^{e}_{o}l,m} \propto \mathbf{m}_{{}^{e}_{o}l,m}, \tag{32}$$

and the magnetic field can be written

$$\mathbf{H}_{{}^{e}_{o}l,m} \propto \mathbf{n}_{{}^{e}_{o}l,m}. \tag{33}$$

The force-free field vector derived from $\psi_{{}^{e}_{o}l,m}$ can be written

$$\mathbf{F}_{{}^{e}_{o}l,m} = \mathbf{m}_{{}^{e}_{o}l,m} + \mathbf{n}_{{}^{e}_{o}l,m}, \tag{34}$$

with α written for k. If the radius a of a force-free plasmoid is chosen to satisfy eq.(28), then both the electric and magnetic fields at the surface will have the form given by eq.(30), because $\mathbf{m}_{{}^{e}_{o}l,m} = 0$ if eq.(28) is true. Since the field of the resonant cavity does not radiate, neither of the fields of the plasmoid can radiate.

The implication is, then, that by choosing a radius, R_0, that simultaneously satisfies

$$j_l(\alpha R_0) = 0 \quad \text{and} \quad j_l(kR_0) = 0 \tag{35}$$

drastically reduces the radiation from the surface of a plasmoid.

The analogy between the cavity and the plasmoid is shown in fig.(6). From now on, it will always be assumed that the plasmoids obey eqs.(35).

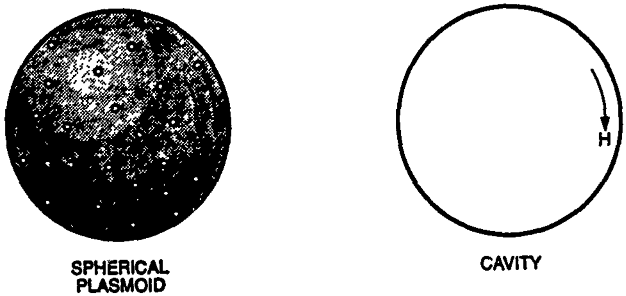

Figure 6. Plasmoid *vs.* cavity fields. At the surface of a perfectly conducting cavity, the electric field of a TM mode is zero, with the corresponding magnetic field spatially identical in form to the plasmoid surface fields.

V. ELECTRONIC CURRENTS

In a real plasmoid high-frequency currents are assumed to be carried by the electrons. In what follows, the electrons will be treated like a gas, neutralized by the backgound of ions, as shown in fig.(7). Difficulties will be ignored that might be encountered when the wavelength of the radiation becomes so small that it approaches the size of the Debye length. Likewise, statistical effects due to temperature will not be considered. In the absence of fluid motion, the overwhelmingly dominant forces assumed acting on the electrons will come from their interaction with the electromagnetic field. Under these conditions the ions will represent a sta-

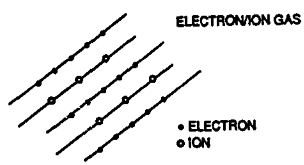

Figure 7. The neutral plasma. The electron density in a plasma will be assumed high enough to be approximated as a gas interacting with the neutralising ions and the neutral atoms. In a plasmoid, above the critical frequency, time-harmonic current is carried by the electrons.

tionary background and will also be ignored in this section. The effects of fluid motion will be investigated in a subsequent section.

Electronic motion is governed by Lorentz forces, demonstrated in fig.(8). In an electromagnetic field the force on an electron is

$$m_e\ddot{\mathbf{r}} = e\mathbf{E} + e\mu\mathbf{v}\times\mathbf{H}, \qquad (36)$$

where m_e is the electronic mass, $\ddot{\mathbf{r}}$ is the acceleration, $\mathbf{v}$ is the electron velocity, and e is the electronic charge. If electrons could exactly follow curved electric field lines the second term in eq.(36) would be zero in force–free fields.

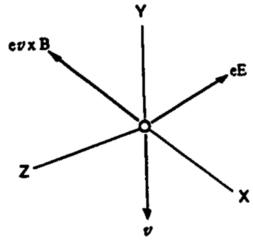

Figure 8. Forces on a moving electron. Electric forces act on the electrons whether or not they are in motion. Magnetic forces act only on moving electrons.

As a first approximation, electrons will be assumed to execute very–small–amplitude simple harmonic motion tangent to the field lines, with the $\mathbf{v}\times\mathbf{H}$ term ignored. In that case, using eq.(9), the first integral of eq.(36) gives the velocity at $\mathbf{r}_0$,

$$\mathbf{v} = \dot{\mathbf{r}} = -\left(\frac{e}{m_e}\right)\frac{\mathbf{E}(\mathbf{r}_0)}{\omega}\cos(\omega t + \varphi). \qquad (37)$$

This velocity can now be used to determine the free–electron charge density, ρ_e. Since $\mathbf{J} = \rho_e\mathbf{v}$, eq.(14) and (37) can be combined to give

$$\rho_e = -\frac{m_e}{e}\left(\frac{\alpha^2 - k^2}{\mu}\right). \qquad (38)$$

Equation (38) says two things about ρ_e. It calculates the magnitude and the *sign* of the electronic charge density. Since the sign of the charge density must be negative, because the sign of the electronic charge, e, is negative, it follows that electrons cannot carry the high–frequency current in force–free plasmoids below the critical frequency. That is, if $k < \alpha$, then the current must be carried by the ions. Since the ionic mass is so much larger than electronic mass, this is clearly an impossibility, at least from a classical viewpoint.

There are several conclusions to be drawn from eq.(38). The first is that self–consistent force–free plasmoids cannot form below the critical frequency. When they can form, the free–electron charge density must be uniform, and increases quadratically with frequency.

Although the above analysis directly gave the charge density by assuming the current was consistent with Maxwell's equations, and electrons obeyed Newton's laws of motion, no mention is made of stability. A full analysis of stability will have to include the effects of the electromagnetic stresses internal to the plasmoid. It will be shown later that these stresses can, for the most part, be replaced by a pressure proportional to the square of the electric field, which can be cancelled by vortical motion, in a non-relativistic approximation. The simplified analysis to be given below does not include electromagnetic stresses, or can be thought of as assuming that they have been cancelled by other mechanisms.

Under this assumption, it will be shown that stability is still not assured unless the frequency and charge density are identical to those that have already been derived from other considerations. It will also be shown that small perturbations in the magnitude of the free-electron charge density will cause the electron gas to contract if the magnitude increases and expand if the magnitude decreases.

VI. VIRIAL CONSIDERATIONS

Consider the moment of inertia, I, of the electrons in a finite plasmoid, and its time derivatives:

$$I = \int_V \rho_m |\mathbf{r}|^2 dV, \tag{39}$$

$$\frac{dI}{dt} = \dot{I} = 2\int_V \rho_m \mathbf{r}\cdot\dot{\mathbf{r}} dV, \tag{40}$$

and

$$\frac{d^2I}{dt^2} = \ddot{I} = 2\int_V \rho_m \left[\mathbf{r}\cdot\ddot{\mathbf{r}} + |\dot{\mathbf{r}}|^2\right] dV, \tag{41}$$

where ρ_m is the (assumed spatially uniform) mass density of the electrons, $\mathbf{r}$ is the position vector associated with an "infinitesimal" blob of electrons, and the integration is over the volume of the plasmoid. Strictly speaking, the integrations in eqs.(39)–(41) should be sums over individual electrons, but their densities are assumed high enough for the integral to be an appropriate approximation. The quantity $\dot{I}$ is will be called the "virial of the electronic motion"[14,15,16,17].

The exclusion of the electromagnetic field densities from the above integrals is assumed to be justified, in the non-relativistic limit, to deal with lowest-order effects arising from variations of charge density and current. Subsequent analysis will deal with the direct effects of field densities on the mechanical stresses in a plasmoid, and how they interact with the fluid dynamics of the plasma. It is assumed that, under non-relativistic conditions, these separate effects can be dealt with independently.

The quantity $\mathbf{r} \equiv \mathbf{r}(x, y, z, t)$ is the instantaneous position of an infinitesimal volume whose mean position is at the fixed point $\mathbf{r}_0 = \mathbf{r}(x, y, z, t = t_0)$, and, to first order, merely oscillates with the same frequency as the applied time-harmonic field. By definition, the velocity, $\mathbf{v}$, is proportional to $\mathbf{J}$:

$$\mathbf{v} = \mathbf{J}/\rho_e, \tag{42}$$

where ρ_e is the (spatially uniform) charge density of free electrons. The charge density, ρ_e, is simply related to the mass density, ρ_m by $\rho_e = (e/m)\rho_m$, where $(e/m) = -1.759\times10^{11}$ $coul/kg$. Since all the quantities are assumed to be harmonic in time, so that $\mathbf{J}(x, y, z, t) \equiv \mathbf{J}(x, y, z)\cos(\omega t + \varphi)$, then $\mathbf{r}$ can be written, for small oscillations about the point $\mathbf{r}_0$,

$$\mathbf{r} = \mathbf{r}_0 + \frac{\mathbf{J}(\mathbf{r}_0)}{\rho_e} \int \cos(\omega t + \varphi)dt = \mathbf{r}_0 + \frac{\mathbf{J}(\mathbf{r}_0)}{\rho_e \omega} \sin(\omega t + \varphi). \tag{43}$$

A necessary and sufficient condition for stability is that the time-integrated average of $\dot{I}$ is zero. This condition implies that the moment of inertia of the electron cloud can assume its original value any number of times, and cannot become uniformly unbounded as time increases. Instead of working directly with $\dot{I}$, however, investigation will be made into conditions under which the time-integrated average of $\ddot{I}$ is zero. This condition is more subtle. A necessary, but not always sufficient, condition for the time-integrated average of $\dot{I}$ to be zero is that the time-integrated average of $\ddot{I}$ be zero. What will be shown is that if the uniform charge density has exactly the right size, and the frequency is above a critical value, the time-integrated average of $\ddot{I}$ is exactly zero if all other stresses are balanced. A larger (smaller) value for the charge density would drive the time-integrated average of $\ddot{I}$ negative (positive), leading to contraction (expansion) and ultimate instabilities. This is because the exact value of uniform charge density derived for a null is the same required from classical mechanics to simultaneously satisfy Maxwell's equations and the laws of inertia, with finite mass electrons in harmonic motion along field lines. Since other factors will enter when considering the electromagnetic field densities, the conditions to be derived will for the charge densities must only be considered necessary, but not sufficient for stability. Gerjouy and Stabler[16] discuss a similar point.

Substituting the time-dependent $\mathbf{r}$ from eq.(43) into eq.(41) and taking the time average automatically gives zero, because of the assumption that the charges oscillate with fixed centers implies confinement. It is far more interesting to substitute for the terms in $\ddot{I}$, writing field quantities in terms of the current, to calculate conditions for stability. Denote the time-integrated average of a quantity by a bar, *i.e.*,

$$\overline{X} = \lim_{T\to\infty}\left[\int_0^T X\,dt\right]. \tag{44}$$

To obtain new information from eqs.(40), (41) note that the terms in the integrand containing $\ddot{\mathbf{r}}$, the acceleration, are related to the volume force on the electrons[18],

$$\rho_m \ddot{\mathbf{r}} = \rho_e \mathbf{E}(\mathbf{r}) \cos(\omega t + \varphi). \tag{45}$$

The term with $|\dot{\mathbf{r}}|^2$ can be written

$$\rho_m |\dot{\mathbf{r}}|^2 = \frac{\rho_m}{\rho_e^2} |\mathbf{J}|^2. \tag{46}$$

From eqs.(14), (40), (41), (43), (44), (45) and (46), with a little algebra,

$$\overline{\frac{d^2 I}{dt^2}} = \int_V \left[\frac{\rho_m}{\rho_e^2} + \frac{\mu_0}{\alpha^2 - k^2}\right] |\mathbf{J}|^2 dV = \left[\frac{\rho_m}{\rho_e^2} + \frac{\mu_0}{\alpha^2 - k^2}\right] \int_V |\mathbf{J}|^2 dV. \tag{47}$$

In order for the right hand side of eq.(47) to be zero the following conditions must hold:

$$k > \alpha, \tag{48}$$

and

$$\rho_e = -\frac{(m/e)}{\mu}\left(\alpha^2 - k^2\right), \tag{49}$$

where $(m/e) = -5.69 \times 10^{-12} kg/coul$ is the mass-to-charge ratio for the electron. Equation (49) can be derived using a Lagrangian formulation. From variational considerations introduced in classical mechanics, the result in eq.(49) can be shown to arise from only assuming that the quantities **E**, **H**, and **J**, have the force-free form, and that the free electrons carrying the current have non-zero mass. Uniform charge density is not an a-priori assumption in the Lagrangian calculation[12].

Besides providing the criterion necessary for stability, eq.(47) predicts the consequences of the magnitude of ρ_e deviating from that in eq.(49). If $|\rho_e|$ increases or decreases in size, the bracketted factor in eq.(47) will decrease or increase, respectively. If this occurs in isolated regions, the bracket, naturally, cannot be taken out of the integral. Physically, the situation is diagrammed in fig.(9). A local increase in $|\rho_e|$ means that the contribution to $\overline{d^2I/dt^2}$ will become negative in the neighborhood of the increase, leading to the contribution to $\overline{dI/dt}$ becoming negative, leading to a local contraction of the electron gas. This is an unstable situation, because it will cause a deviation from force-free motion and could destroy the plasmoid. Similar reasoning leads to the conclusion that a decrease in charge density leads to a net expansion of the plasmoid.

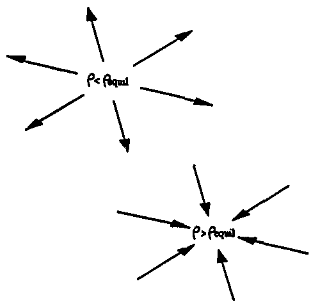

Figure 9. Effects of local charge perturbations. Local fluctuations in electron charge density can cause plasmoids to constrict or expand.

Physically this can be understood by considering the energy of self-interaction of the current in a plasmoid. Two side–by–side line currents, of equal size, in close proximity, attract each other with a force proportional to the square of their current because the magnetic field between them has a smaller energy density than in the region outside. The magnetic "pressure" arising from the electromagetic stress tensor[19] pushes the wires together. This is also the origin of the "pinch effect" in plasma physics. Increasing the free–electron charge in a plasmoid will increase the current, leading, qualitatively, to a volume pinch effect tending to contract the volume of the plasmoid. Similarly, decreasing the charge density decreases the pinch effect and leads to expansion.

This (increase/contraction)–(decrease/expansion) seems to be an inherent instability of a plasmoid where statistical fluctuations of charge density might occur. Any local increase in charge density, causing a contraction, would tend to compress the region, making the free–electron charge density increase even more. Similarly for the decrease/expansion case. This positive feedback is countered by the tendency of an ionized plasma in a strong electromagnetic field to have its radiative recombination rate increase with density and its ionization rate to decrease with density. Statistical fluctuations in charge density would then be nullified to some extent by ionization/recombination processes in real gasses. A full calculation investigating this phenomenon has not been done as yet. The sensitivity of the ionization/recombination rates to gas density, as a function of frequency and electric field amplitude may be a crucial factor in plasmoid stability.

As was already mentioned, the conditions in eqs.(48) and (49) are necessary for stability, if all other stresses are absent. The added condition that makes (48) and (49) sufficient is that $\overline{dI/dt} = 0$, since equations (48) and (49) are the conditions also derived from solving Newton's laws of motion for the plasmoid system when the electromagnetic stresses are ignored. Since the fields are stationary in space and time, except for the harmonically oscillatory part, it follows that at $t = 0$ the plasmoid initially has $\overline{dI/dt} = 0$, where the T in eq.(44) is time for a single cycle of the oscillatory field, $T = 2\pi/\omega$.

Ignoring electromagnetic, thermodynamic and fluid stresses, eqs.(48) and (49), constraining the frequency to be above a critical minimum and the free–electron charge density to be uniform and a specific quadratic function of k and α, represent necessary and sufficient conditions for plasmoid stability.

In the next section, electromagetic stresses will be discussed in more detail. Unless these stresses are balanced, they will tend to make the virial of the electronic motion, $\dot{I}$, positive. In other words, the electromagnetic field trapped in a plasmoid acts like a positive overpressure tending to make the plasmoid expand and dissipate. A mechanism will subsequently be discussed that will balance out all but a very small part of the electromagnetic stresses.

VII. ELECTROMAGNETIC STRESSES

The usual analysis involving the virial theorem[14,15,16,17] includes the electromagnetic energy density in the integrand in eq.(39). The assumptions made here are that the electronic velocities are non–relativistic and the harmonic fields **E** and **H** are non–zero inside the plasmoid and do not have the force–free form outside. It is possible to have an external time–harmonic field that obeys free–space conditions with no discontinuities appearing at the plasmoid surface. The external branch of the electromagnetic field has no currents, does not have the force–free form, and will be assumed to have no stresses associated with it.

It is important now to estimate how close a real physical system can be made to approach the conditions for force–free–field approximations on the vector fields **E**, **H**, and **J**. If electrons were massless, the problem would be far simpler, since they would experience no centripedal accelerations and would conform to electric field lines like currents in wires. In reality, the electrons cannot confine their motion to be strictly along the curved field lines of a force–free field.

The assumption that the electrons undergo simple harmonic motion along electric field lines becomes more accurate as the frequency increases and the electric field decreases.This can be seen from eq.(43), where the amplitude of the harmonic motion is $\mathbf{J}/(\rho_e\omega)$. Substituting for **J** in terms of **E**, a condition for small–amplitude oscillations can be written

$$\frac{|\mathbf{E}|_{max}}{(\frac{m}{e})\omega^2} \ll R_0, \tag{50}$$

where $|\mathbf{E}|_{max}$ is the maximum absolute amplitude of the harmonic electric field within an $l = 1$ plasmoid and R_0 is the radius. Using R.M.K.S. units eq.(50) becomes

$$|\mathbf{E}|_{max} = 2.02 \times 10^7 \zeta \left(\frac{R_0}{\lambda^2}\right) \quad \text{volts/m}, \tag{51}$$

where ζ is an appropriately small factor that satisfies eq.(50), and λ is the free-space wavelength of the harmonic wave.

While eq.(50) gives a criterion for small-amplitude electronic motion, a more exact expression can be written for the forces on individual electrons in a force-free time-harmonic plasmoid. When these forces are known, trajectories may then be calculated, and an assessment can be made of techniques for balancing the forces.

It will be shown below that the condition in eq.(50) is related to the problem of electron "drift" in a plasma. That is, in this context, the tendency for electrons to move orthogonally to electric and magnetic field lines. The approach will be to expand the acceleration field as a perturbation series about the harmonic motion.

For purposes of perturbative expansion, it will be assumed that the frequencies to be encountered will be high enough, and the respective field amplitudes small enough to say that the major part of the electronic velocity will be harmonic motion along electric field lines defined by solutions to eq.(1). Since the field lines have curvature, these oscillations will be accompanied by a drift motion across electric field lines, as in fig.(10), unless the effects of the curvature are balanced by some other mechanism.

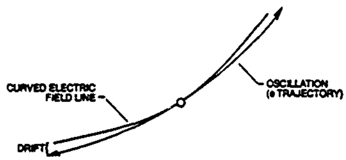

Figure 10. Drift due to field curvature. If the field lines are curved, the electrons will continually drift, to oscillate along different field lines.

There are several reasons why this electronic drift is undesirable: First, as shown above, the current in an ideal force-free plasmoid will not radiate, under the right conditions of size and frequency. Drift currents, not having the force-free form, are capable of radiating, potentially unbalancing the plasmoid mechanically. Next, drift currents may have a divergence that would lead to a change in charge density, if the rest of the plasma were not present. Since the plasma creates large restoring forces to maintain neutrality, drift currents can be responsible for pressure gradients that could disrupt the symmetry of a plasmoid, causing its self-destruction. In any case, drift currents may have non-linear effects that can build in an uncompensated manner and tend to shorten the lifetime of a plasmoid.

It is important, then, to understand the origins and dynamics of drift currents. Toward that end, the forces acting on the free electrons will be examined in detail, assuming non-relativistic conditions.

An electron is acted upon by the Lorentz force

$$\mathbf{F} = m_e\ddot{\mathbf{r}} = e\mathbf{E} + e\dot{\mathbf{r}}\times\mathbf{B}, \tag{52}$$

where $\mathbf{B} = \mu_0\mathbf{H}$ and $\mathbf{r}$ is the instantaneous position of the electron. Instead of an analysis of eq.(52) that will relate the components of $\mathbf{F}$ to elements of the electromagnetic stress tensor[20], the drift motion will be taken as a perturbation determined by the oscillatory motion along electric field lines and the curvature of the components of the field gradients. The connections to the stress tensor also will be discussed below.

It will be assumed, now, that the frequency of oscillations, ω, is so high that in the time needed for one full cycle of the electric field, $T = 2\pi/\omega$, the net motion of the center of mass of an electron oscillation is very small compared to the overall size of the plasmoid. This slow net motion can arise because of field inhomogeneities in the $\mathbf{B}$ and $\mathbf{E}$ vectors as well as the relative local orientation of $\mathbf{B}$ and $\mathbf{E}$. The major electronic motion, then, over many cycles, will be high-frequency oscillation along local electric field lines. This high-frequency motion will be superimposed on another component whose net effect would be to slowly transport electrons onto different electric field lines. The following analysis closely follows the logic on pp. 77–80 of Gekker's book[21].

Assume that the position vector $\mathbf{r}$ may be written as a sum of an oscillatory term, $\mathbf{r}_1$, and and a net transport term, $\mathbf{r}_0$. Let the oscillatory term arise from ignoring the magnetic field's contribution to eq.(52).

It then follows from eqs. (9) and (10) that

$$\mathbf{r}_1 = -\eta\omega^{-2}\mathbf{E}(\mathbf{r}_0, t)\cos(\omega t + \varphi), \tag{53}$$

where $\eta \equiv e/m_e = -1.76\times10^{11}$coul/kg, the ratio of the electronic charge to its mass, and the electric field is calculated at position $\mathbf{r}_0$. The quantity $\mathbf{r}_1$ takes into account only the straight-line motion tangent to a field line. After the contribution from $\mathbf{r}_0$ is added on, the trajectory should contain all the corrections for curvature.

Equation (52) can be rewritten:

$$\ddot{\mathbf{r}}_0 + \ddot{\mathbf{r}}_1 = \eta\mathbf{E}(\mathbf{r}_0 + \mathbf{r}_1, t) + \eta(\dot{\mathbf{r}}_0 + \dot{\mathbf{r}}_1)\times\mathbf{B}(\mathbf{r}_0 + \mathbf{r}_1, t). \tag{54}$$

Expanding the right hand side of eq.(54) as Taylor series in $\mathbf{r}_1$, and rearranging,

$$\begin{aligned}\ddot{\mathbf{r}}_0 + \ddot{\mathbf{r}}_1 =& \eta\left[\mathbf{E}(\mathbf{r}_0,t) + (\mathbf{r}_1\cdot\nabla)\mathbf{E}(\mathbf{r}_0,t) + \frac{1}{2}(\mathbf{r}_1\cdot\nabla)^2\mathbf{E}(\mathbf{r}_0,t) + \dots\right] + \\ &\eta\dot{\mathbf{r}}_0\times\left[\mathbf{B}(\mathbf{r}_0,t) + (\mathbf{r}_1\cdot\nabla)\mathbf{B}(\mathbf{r}_0,t) + \frac{1}{2}(\mathbf{r}_1\cdot\nabla)^2\mathbf{B}(\mathbf{r}_0,t) + \dots\right] + \\ &\eta\dot{\mathbf{r}}_1\times\left[\mathbf{B}(\mathbf{r}_0,t) + (\mathbf{r}_1\cdot\nabla)\mathbf{B}(\mathbf{r}_0,t) + \frac{1}{2}(\mathbf{r}_1\cdot\nabla)^2\mathbf{B}(\mathbf{r}_0,t) + \dots\right].\end{aligned} \tag{55}$$

Subtracting the second time derivative of eq.(53), with more rearranging, the equation for $\ddot{\mathbf{r}}_0$ is

$$\begin{aligned}\ddot{\mathbf{r}}_0 =& \eta\left[(\mathbf{r}_1\cdot\nabla)\mathbf{E}(\mathbf{r}_0,t) + \dot{\mathbf{r}}_1\times\mathbf{B}(\mathbf{r}_0,t)\right] + \\ &\eta\left[\frac{1}{2}(\mathbf{r}_1\cdot\nabla)^2\mathbf{E}(\mathbf{r}_0,t) + \dots\right] + \\ &\eta\dot{\mathbf{r}}_0\times\left[\mathbf{B}(\mathbf{r}_0,t) + (\mathbf{r}_1\cdot\nabla)\mathbf{B}(\mathbf{r}_0,t) + \frac{1}{2}(\mathbf{r}_1\cdot\nabla)^2\mathbf{B}(\mathbf{r}_0,t) + \dots\right] + \\ &\eta\dot{\mathbf{r}}_1\times\left[(\mathbf{r}_1\cdot\nabla)\mathbf{B}(\mathbf{r}_0,t) + \frac{1}{2}(\mathbf{r}_1\cdot\nabla)^2\mathbf{B}(\mathbf{r}_0,t) + \dots\right].\end{aligned} \tag{56}$$

From Maxwell's equation $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$, eq.(53) and the familiar vector identity

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\nabla\mathbf{B} + \mathbf{B}\cdot\nabla\mathbf{A} + \mathbf{A}\times\nabla\mathbf{B} + \mathbf{B}\times\nabla\mathbf{A},$$

eq.(56) becomes

$$
\begin{aligned}
\ddot{\mathbf{r}}_0 = -\frac{1}{2}\nabla\left|\frac{\eta\mathbf{E}(\mathbf{r}_0,t)}{\omega}\right|^2 + \\
\eta\left[\frac{1}{2}(\mathbf{r}_1\cdot\nabla)^2\mathbf{E}(\mathbf{r}_0,t)+\ldots\right]+ \\
\eta\dot{\mathbf{r}}_0\times\left[\mathbf{B}(\mathbf{r}_0,t)+(\mathbf{r}_1\cdot\nabla)\mathbf{B}(\mathbf{r}_0,t)+\frac{1}{2}(\mathbf{r}_1\cdot\nabla)^2\mathbf{B}(\mathbf{r}_0,t)+\ldots\right]+ \\
\eta\dot{\mathbf{r}}_1\times\left[(\mathbf{r}_1\cdot\nabla)\mathbf{B}(\mathbf{r}_0,t)+\frac{1}{2}(\mathbf{r}_1\cdot\nabla)^2\mathbf{B}(\mathbf{r}_0,t)+\ldots\right].
\end{aligned}
\tag{57}
$$

The first term on the right in eq.(57) can obviously lead to gross motion of the center of mass of an electron. It says that the electromagnetic field can produce a volume stress in plasmas that appears to arise from a pressure proportional to the square of the electric field and inversely proportional to the square of the frequency.

Many of the other terms in eq.(57) lead to oscillatory motion, with no time-averaged velocity of the center of mass. To lowest order, assume that the oscillatory part of $\dot{\mathbf{r}}_0$ is negligible. Then, since $\mathbf{r}_1$ and $\mathbf{E}(\mathbf{r}_0,t)$ both have $\sin(\omega t+\varphi)$ dependence, the first higher-order term to contribute a non-zero time-averaged contribution in the first square bracket is

$$(\eta/6)(\mathbf{r}_1\cdot\nabla)^3\mathbf{E}(\mathbf{r}_0,t),$$

with $\sin^4(\omega t+\varphi)$ appearing as the time dependence. Similarly, the first higher-order term to contribute in the last square bracket is

$$(\eta/2)\dot{\mathbf{r}}_1\times(\mathbf{r}_1\cdot\nabla)^2\mathbf{B}(\mathbf{r}_0,t),$$

with $\sin^2(\omega t+\varphi)\cos^2(\omega t+\varphi)$ appearing. To lowest order, none of the terms in the second bracket can contribute to non-oscillatory motion, because $B(\mathbf{r}_0,t)$ has $\cos(\omega t+\varphi)$ dependence.

The electric field must now be restricted in size, not only to make the series in eq.(57) converge, but to prevent the higher-order terms from making a substantial contribution to the motion. A physical restraint will be that the maximum value of $\mathbf{r}_1$ be much smaller than R_0. From eq.(53), define a smallness parameter, ξ, very similar to ζ in eq.(51), where

$$\xi = \frac{\eta|\mathbf{E}|_{max}}{R_0\omega^2} \ll 1, \tag{58}$$

so that

$$|\mathbf{E}|_{max} = \xi \frac{\omega^2 R_0}{\eta}. \tag{59}$$

Note that for a fixed values of ξ and (k/α), ω is inversely proportional to R_0, and the maximum electric field intensity varies inversely with plasmoid radius. Maximum energy densities can therefore be inversely proportional to the square of the radius without violating stability criteria.

It can then be shown that the time–averaged force on an individual electron can be written

$$m_e \overline{\ddot{\mathbf{r}}}_0 = -\frac{1}{4} m_e \nabla \left| \frac{\eta \mathbf{E}(\mathbf{r}_0)}{\omega} \right|^2 \left[1 + O(\xi^2) \right]. \tag{60}$$

In the next section, it will be shown that eq.(60) produces an effective non–uniform fluid pressure proportional to the energy density in the electric field. By introducing vortical motion, variations in the total time–averaged fluid pressure governing the flow within the plasmoid will be cancelled to order ξ^2.

VIII. VORTICAL PLASMA STABILIZATION

The time–averaged volume force on the electron fluid at any point is

$$\mathbf{F}_V = \frac{\rho_e}{\eta} \overline{\ddot{\mathbf{r}}}. \tag{61}$$

From eqs.(38) and (61),

$$\mathbf{F}_V = -\frac{\epsilon}{4} \left(1 - \frac{\alpha^2}{k^2} \right) \nabla |\mathbf{E}(\mathbf{r})|^2 \left[1 + O(\xi^2) \right]. \tag{62}$$

To order ξ^2 this is equivalent to the volume force that would arise from the gradient of the pressure field

$$P(\mathbf{r}) = \frac{\epsilon}{4} \left(1 - \frac{\alpha^2}{k^2} \right) |\mathbf{E}(\mathbf{r})|^2 + P_0, \tag{63}$$

where P_0 is a constant that may not be zero.

A great deal of care must be taken to account for all the stresses due to the electromagnetic field. Equation (57) is correct throughout a region where all the quantities exist and are analytic. It is important to note that the approximation, eq.(63), derived from eq.(57) is *not* what can be derived from a localized interpretation of the meaning of the electromagnetic stress tensor in a plasmoid. This subtle point is worth exploring in a little more depth.

The electromagnetic stress tensor in a homogeneous, isotropic linear medium, ${}^2\mathbf{S}$, is physically defined by its divergence[6,19,20],

$$\nabla\cdot({}^2\mathbf{S}) = \epsilon\Big[(\nabla\times\mathbf{E})\times\mathbf{E} + \mathbf{E}\nabla\cdot\mathbf{E}\Big] + \mu\Big[(\nabla\times\mathbf{H})\times\mathbf{H} + \mathbf{H}\nabla\cdot\mathbf{H}\Big], \tag{64}$$

which can also be expressed in the form

$$\nabla\cdot({}^2\mathbf{S}) = \rho_c\mathbf{E} + \mu\mathbf{J}\times\mathbf{H}, \tag{65}$$

where ρ_c is the net free (unneutralized) charge and $\mathbf{J}$ is the current.

In an ideal force-free plasmoid, there is no free charge, $\rho_c = 0$, and, by definition, $\mathbf{J}\times\mathbf{H} = 0$. The divergence of the electromagnetic stress tensor is therefore identically zero there.

Where trouble can arise is in writing an expression for the stress tensor having the dyadic form

$${}^2\mathbf{S} = \epsilon_0(\mathbf{EE} - \frac{1}{2}|\mathbf{E}|^2\bar{\bar{\mathbf{I}}}) + \mu_0(\mathbf{HH} - \frac{1}{2}|\mathbf{H}|^2\bar{\bar{\mathbf{I}}}), \tag{66}$$

where $\bar{\bar{\mathbf{I}}}$ is the unit dyadic.

It is true that, integrating over the volume and surface area of a plasmoid

$$\int_V(\rho_c\mathbf{E} + \mu_0\mathbf{J}\times\mathbf{H})dv = \oint_A {}^2\mathbf{S}\cdot\hat{\mathbf{n}}da, \tag{67}$$

where $\hat{\mathbf{n}}$ is the outwardly pointing normal. Obviously, the integrals in eq.(67) are equal to zero for a force-free plasmoid.

From eq.(66), the integrand of the second intgral in eq.(67) can be written

$$^{2}\mathbf{S}\cdot\hat{\mathbf{n}} = \epsilon\left[(\mathbf{E}\cdot\hat{\mathbf{n}})\mathbf{E} - \frac{1}{2}|\mathbf{E}|^2\hat{\mathbf{n}}\right] + \mu\left[(\mathbf{H}\cdot\hat{\mathbf{n}})\mathbf{H} - \frac{1}{2}|\mathbf{H}|^2\hat{\mathbf{n}}\right]. \tag{68}$$

The "usual" interpretation of the quantity in eq.(68) is that of a local stress being communicated across the surface of integration, see fig.(11). In this case since both electric and magnetic fields are have no normal components at the surface of the plasmoid, the electromagnetic stress, from eq.(68) amounts to a pressure equal to the total energy density of the electromagnetic field. In reality, there is no local net force in an ideal force-free plasmoid.

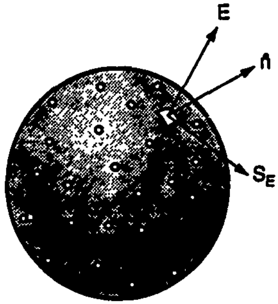

Figure 11. The stress tensor. The "usual" interpretation of the stress tensor translates to a local stress across a surface.

The analysis done to find the pressure due to non-zero-mass electrons finds a quantity, in eq.(63), that differs at the surface of the plasmoid from assuming stresses to arise from eq.(68). Even though, as the frequency increases, with the size of the plasmoid fixed, the differences become smaller, eq.(68) cannot be used and has no relevance to the local pressure field when the fields are assumed to arise from the force-free solutions to Maxwell's equations. Relevance can be re-established by allowing plasmoids to have different fields directly inside and outside their boundaries, forcing currents to flow on their surfaces.

These currents, a single layer needed for the jump in the H field, and a double layer needed for a jump in the E field, will exhibit a reaction to the fields at a plasmoid boundary that will effectively act like an outward local pressure proportional to the local energy density of the electromagnetic field. Within a constant factor, this latter pressure has the same functional form as the first term in eq.(63).

As will be shown below, the pressure represented by the first term in eq.(63) can be balanced by a vortical motion of the fluid in the plasmoid. If extra stresses occur at the surface, they can be balanced by modifying the magnitude of the fluid velocity everywhere. This would, however, unbalance the stresses everywhere else in the plasmoid. In a perfectly non-viscous fluid it would also be possible to modify the velocities found in the outermost "toroids" of the vortical flow to compensate for the stresses surface currents would produce. Since the flows in separate toroids of the vortical motion are independent, the rest of the plasmoid would be unaffected. It is easy to see that even in the absence of viscosity, taking surface currents into account complicates the theory.

In the next section it is shown that some or all plasmoid surface currents can be eliminated if special external fields are present. It will be shown that these external fields can be described by the resonant modes of a perfectly conducting spherical cavity. In view of this, the theory will now be developed ignoring the effects of surface currents.

Even though the gradient of the pressure field in eq.(63) acts only on the electron fluid, it is transmitted to the ions via the electrostatic forces that enfcrce quasineutrality in the plasma. The internal flow of the plasmoid will then be governed by static and dynamic fluid pressure gradients, in addition to the forces in eq.(62). In what follows, it will be assumed that the plasmoid behaves fluid–dynamically like a perfect gas, with constant thermal conductivity and specific heats.

A flow field, with velocity distribution $\mathbf{v}(\mathbf{r})$ will now be introduced that counteracts the pressure in eq.(63). Let

$$\mathbf{v}(\mathbf{r}) = A\mathbf{M}(\mathbf{r}), \tag{69}$$

where A is a constant to be determined and $\mathbf{M}(\mathbf{r})$ obeys eq.(1). In other words, the streamlines of $\mathbf{v}$ are identical with the harmonic electric field lines. This steady flow field is potentially stable, as can be seen by applying the equation for the evolution of vorticity, $\vec{\omega}$[7],

$$\frac{\partial\vec{\omega}}{\partial t} = \nabla\times(\mathbf{v}\times\vec{\omega}), \tag{70}$$

where

$$\vec{\omega} = \nabla\times\mathbf{v} = \alpha\mathbf{v}. \tag{71}$$

In eq.(70), the right–hand side is identically zero because $\mathbf{v}$ has the force–free form, with $\mathbf{v}$ a multiple of $\nabla\times\mathbf{v}$. Force–free vorticies, with vector field $\mathbf{M}(\mathbf{r})$, are not the only candidates fcr stable motion. If the vector function $(\mathbf{v}\times\vec{\omega})$ is the gradient of a scalar function, the vortex formed by the velocity $\mathbf{v}$ is also potentially stable. Hill's spherical vortex is in this latter class. Exactly equivalent to eq.(70) is the expression for the rate of change of $\vec{\omega}$ in an element of fluid that is convected with purely vortical flow[7],

$$\frac{D\vec{\omega}}{Dt} = \vec{\omega}\cdot\nabla\mathbf{v}. \tag{72}$$

From eq.(71) this becomes

$$\frac{D\mathbf{v}}{Dt} = \mathbf{v}\cdot\nabla\mathbf{v}. \tag{73}$$

Since v has the force–free form, the formula for the gradient of a dot product of two vector fields implies

$$\frac{D\mathbf{v}}{Dt} = \frac{1}{2}\nabla|\mathbf{v}|^2. \tag{74}$$

Equation (74) says that the acceleration of an element of fluid moving in a force–free vortex can be derived from the gradient of the square of the velocity. Multiplying both sides of eq.(74) by ρ, the fluid density, and assuming uniform density,

$$\rho\frac{D\mathbf{v}}{Dt} = \frac{1}{2}\nabla\rho|\mathbf{v}|^2, \tag{75}$$

it follows that the volume force on an element of fluid in a force–free vortex may be thought to arise from the pressure field

$$p_V = p_0 - \frac{1}{2}\rho|\mathbf{v}|^2, \tag{76}$$

where p_0 is a constant and ρ is spatially invariant. This pressure can be adjusted to cancel the major part of the electromagnetic pressure.

It is immediately obvious that the value of A in eq.(69) should be

$$A = \pm\sqrt{\frac{1}{2\rho}\epsilon_0 E_0^2\left(1 - \frac{\alpha^2}{k^2}\right)}, \tag{77}$$

where the spatial part of the electric field is $E_0\mathbf{M}(\mathbf{r})$.

The above analysis leading to the result that vortical motion can cancel the major portion of the electromagnetic forces on the electronic fluid, leaving only a uniform residual pressure, p_0, everywhere is so important that it will be derived in another manner.

Some assumptions will be made first:
(1) All processes are adiabatic, so that the specific entropy is spatially uniform.
(2) The gas pressure, p, must equal the surrounding ambient pressure.

(3) The gas density, ρ, must be uniform within the plasmoid.

Any flows that are incompatible with any of the above assumptions will not be considered. Before going ahead with the alternate derivation, these assumptions should be discussed.

Assumption (1) says that all fluid motions were initiated without the addition of heat. Also, no processes occured that generated or transferred heat. If the plasma in the plasmoid was derived from the surrounding plasma then the specific entropy of the plasmoid is equal to the specific entropy of the surrounding plasma.

Assumption (2) says that the only pressure gradients that can remain in the plasmoid, after the vortical motion cancels the electromagnetic pressure, must arise from the thermodynamic gas pressure. Assuming an "atmosphere" surrounding the plasmoid exerts an isotropic pressure p_a, the internal thermodynamic pressure must match this for stability, after surface–current effects are included. In large systems, p_a will probably arise from the surrounding gas pressure. In small systems, there may be a component of pressure from a Casimir force. Casimir forces will arise in situations where, owing to the high conductivity of the plasma due to large numbers of free electrons, components of the electromagnetic zero–point background energy cannot penetrate uniformly into the plasma. This will result in a lower energy density within the plasmoid, and an apparent pressure gradient due to the vacuum. It is beyond the scope of this note to investigate zero–point effects.

Assumption (3) must hold in order to allow the forces that arise from vortical accelerations to be derivable from a conservative field.

The key to the alternate derivation is the understanding that the specific internal energy of the plasma is a sum of the perfect–gas thermodynamic specific internal energy, $\mathcal{E}$, and the electromagnetic energy density, per unit mass of fluid, $\mathcal{Q}$. The thermodynamic quantities are

$$\mathcal{E} = (c_p - c_v)T, \tag{78}$$

where c_p, c_v, and T are the specific heat at constant pressure, the specific heat at constant volume, and the temperature, respectively. From eq.(24) the part of $\mathcal{Q}$ that is affected by the fluid flow is represented by the term with $\mathbf{J}$ in it, since the current can be transported by the fluid flow. Using this logic an effective energy density can be written

$$\mathcal{Q}_{eff} = \frac{1}{2\rho}\overline{\mathbf{J}\cdot\mathbf{A}} = -\frac{\epsilon_0 E_0^2}{4\rho}\left(1 - \frac{\alpha^2}{k^2}\right)|\mathbf{M}(\mathbf{r})|^2, \tag{79}$$

with $\mathbf{A}$ and $\mathbf{J}$ defined in Sec I.

Bernoulli's theorem[7] now says that, along every streamline the following physical quantity is conserved:

$$H_B = \frac{1}{2}|\mathbf{v}|^2 + Q_{eff} + \mathcal{E} + \frac{p}{\rho}. \tag{80}$$

Since Q is negative ($k > \alpha$) the logical choice

$$\mathbf{v} = A\mathbf{M}(\mathbf{r}), \tag{69}$$

where A is defined in eq.(77), will exactly cancel out the electromagnetic stresses in eq.(63), a result already derived, to within order ξ^2. This discrepancy to order ξ^2 might have been expected, since the higher order terms arising from the curvature of the fields, in eq.(57), were ignored.

In the above proof, using Bernoulli's theorem, it is assumed that the electronic current is convected by the fluid motion. Since the fluid motion is assumed to be a purely solenoidal mode, no compression takes place, and the electronic density remains uniform. Under these conditions it could, alternatively, be assumed that the electrons (and thereby the currents) are not convected by the fluid, with only the ions and neutral atoms participating in the fluid flow. In this case Q_{eff} will not appear in eq.(80), but the pressure due to the electronic motion would still be transmitted to the ions and would have to be subtracted from the pressure p in eq.(80).

It is then easy to see that exactly the same velocity profile would have to be assumed to balance the electronic contribution, whether the electrons are convected with the fluid or not. From simple energy considerations, it is more likely that electrons are convected with the fluid because there would then be no external static magnetic field due to the motion of the ions. (The kinetic energy of the electrons is ignored here. A full analysis would have to compare the added kinetic energy due to electronic motion to the energy in the magnetic field that would arise if the electrons were not entrained. It is assumed here that the electronic kinetic energy would be smaller.)

With the first two terms in eq.(80) cancelling, the quantity

$$\mathcal{E} + \frac{p}{\rho} = c_p T, \tag{81}$$

the specific enthalpy, is constant along any flowline. It immediately follows that the temperature must be constant along a flowline. If the motion of the compressible gas is to remain steady, then the pressure must be constant in the plasmoid, and assumed equal to the pressure of the "ambient atmosphere". This, then, forces the mass density, ρ, to also be a constant within the plasmoid. Since pressure, density and temperature are constant throughout the plasmoid, and may take on the same values as in the ambient isentropic atmosphere, the flow is isentropic and, therefore, can be initiated by an adiabatic process. The criteria of the flow, listed as assumptions (1–3), above, are met.

As field strength increases, the fluid speed, from eq.(77), must increase in the same proportion. The question naturally arises as to whether turbulence will occur beyond a threshhold velocity. This cannot be answered in full, but if the electrons are entrained in the flow, there is a mechanism that will resist small perturbations that may grow into turbulent flow.

Remember that the term $Q = (1/2\rho)\overline{\mathbf{J}\cdot\mathbf{A}}$, which is proportional to the time-averaged energy of interaction of the magnetic field and the current in the time-harmonic field, is negative for $k > \alpha$. A small perturbation in velocity that does not change $|\mathbf{v}|$, but implies a local change of flow direction without changing $\mathbf{A}$, will decrease the magnitude of Q. The reason for this is that the entrained current will still have the same magnitude, but its component parallel to the original direction, parallel to $\mathbf{A}$ must be smaller. A smaller magnitude of Q means increased energy, which leads to a restoring force proportional to $|E_0|^2$. The fluid therefore would appear to become more rigid as the field strength increases, at least partially offsetting small statistical fluctuations that could grow turbulently. This effect has already been mentioned in association with eqs.(26), (27) and fig.(5).

Note that a force-free vortex cannot occur by itself in a compressible fluid. The pressure gradients would immediately give rise to velocity components that were not solenoidal. Such a situation would lead to rapid destruction of the vortical field. Similarly, a time-harmonic force-free field cannot persist by itself in a compressible fluid because of the stresses due to finite-mass electrons in the presence of electromagnetic fields with non-zero curvature. The remarkable coincidence of force-free vortices being stable in the presence of force-free electromagnetic fields makes the entire theory viable.

There is a very special case that is worth mentioning, even though the possibility of its physical existence may be apocryphal. At the critical frequency, where $\omega = c\alpha$ the current and charge densities become zero. It follows that a force-free electromagnetic excitation could exist in otherwise "empty" space. The energy would be contained within a spherical region, terminated by a set of boundary currents, and could not radiate away. Classically, there is no limit to the energy density that might be confined in such an electromagnetic mode if the outside influences are large enough to contain the stresses on any surface currents that would be needed to match boundary conditions. If such an object could exist, the limit on the field strengths would be near the point where the vacuum polarization due to virtual electron-positron pair production was affected.

Classically, such a "vactoid" (vacuum plasmoid) would be almost undetectable. Assuming the "correct" terminating currents, the electromagnetic fields immediately outside its boundary are nil, and no charges inside it are present to radiate. A more modern view says that the trapped energy represents an $E = mc^2$ mass, and would have a gravitational field. Also, scattering from virtual electron–positron pairs might eventually lead to decay. Unless there was a direct contact with ionizable or charged particles, the presence of an individual vactoid would go unnoticed. The interaction of two vactiods or, for that matter, two force–free plasmoids, is not yet known, but can be investigated.

The major argument against the existence of vactiods is the apparent need for a set of surface currents to match the inside–to–outside boundary conditions, with no mechanism to generate or support them. The Lorentz forces on such currents would amount to a pressure equal to the energy density of the electromagnetic field at the surface. This would tend to force the vactoid to disperse unless there were processes outside the vactoid that balanced the electromagnetic pressure acting at the surface from the inside.

Even apart from this vacuum special case, a reliable process for manufacturing force–free plasmoids is still to be found. In the next section, some of the issues involved will be discussed. Appendix B discusses the case of an energetic (one megajoule), 20 cm. diameter plasmoid, formed in air at sea–level atmospheric pressure.

IX. DISCUSSION

The analysis so far has assumed that force–free plasmoids exist and proceeded to derive some of their properties. This was taken to be a worthwhile intellectual excercise, since the theory appears applicable to otherwise unexplicable observations. As it stands, the weak points in the theory concern prescriptions for the manufacture of force–free plasmoids and the nature of the fields external to, yet compatible with, force–free plasmoids. These last two issues lead to the suspicion that describing autonomous self–confining energetic plasmas, such as ball lightning, with force–free fields is probably not incorrect, but may be incomplete. This section aims at discussing the previous sections, highlighting some of the shortcomings of the theory.

Arguments found at the end of Sec. III accompanying eqs.(28)–(35) were used to show that the fields at the surface of an ideal plasmoid do not radiate. No inferences were made about the transition in field amplitudes at the surface of a plasmoid.

The moot point as to whether there can be a true discontinuity in each, or either, of the electric and magnetic fields, between inside and outside the plasmoid, will not be considered in great detail. If a true discontinuity can arise, the question of how to determine surface currents cannot be answered easily. The assumption will be made that if there are discontinuities the transition to the "outside", fields occurs continuously.

That is, there must be a small region in the neighborhood of the boundary of a plasmoid where the fields, along with their first derivatives, change continuously. If so, then plasmoids must have currents in this region to match the boundary conditions imposed by classical electrodynamics. (If not, the possibility that plasmoids may exist having no surface currents must be investigated. Lack of surface currents leads to the apparently illogical consequence of a standing wave having no sources. As an approximation, however, this may not be an impossible situation. There may be waves whose space–time symmetry severely limits their ability to propagate in the "usual" manner. These waves do have sources, but the compact support of these sources may be a very complicated function of the retarded fields. In the literature such waves have been called Diffractionless/Phase Conjugated Waves, Electromagnetic Missiles, Bessel Beams, etc.[22] A force–free plasmoid unterminated by surface boundary currents may fall into this category. The rate at which a plasmoid disintegrates if no boundary currents exist can be investigated.) As has been seen, these currents are needed to satisfy Maxwell's equations, but since they do not radiate, they are not needed to counteract a steady–state flow of energy from the surface of the inside region.

To make the last point clearer, consider the case of three–dimensional space divided into two regions, "inside" and "outside" by a spherical boundary. On the inside, for $r < R_0$, the electric and magnetic fields are $\mathbf{E}_i$ and $\mathbf{H}_i$. On the outside, for $r > R_0$, the fields are $\mathbf{E}_o$ and $\mathbf{H}_o$. If $\hat{\mathbf{n}}$ is the outwardly pointing normal, it is shown in standard texts[23] that the inside and the outside will evolve independently if the surface, at $r = R_0$ sustains the equivalent electric (J) and magnetic (M) currents

$$\mathbf{J} = \hat{\mathbf{n}} \times (\mathbf{H}_o - \mathbf{H}_i), \tag{82}$$

and

$$\mathbf{M} = -\hat{\mathbf{n}} \times (\mathbf{E}_o - \mathbf{E}_i). \tag{83}$$

For solving problems on computers, the carriers of these currents are irrelevant. Physically, J can be carried by normally charged particles like electrons. The magnetic current M can only be approximated by normally charged particles, since magnetic charges evidently do not exist.

The purpose for this discussion is that if a real, steady–state, force–free plasmoid were to exist, with non–zero fields inside and free fields outside, some mechanism would have to be provided to sustain multiple layers of sheet currents on the surface of the plasmoid if there are discontinuities in the fields. No such mechanism immediately comes to mind.

On the other hand, assume a plasmoid of this type is suddenly created with no surface currents. The jump conditions at the surface can initiate a mode of disintegration that involves transient (one suspects non–harmonic) phenomena that will generate waves whose initial energy densities can be equal to the electromagnetic energy densities at the surface of the plasmoid. These waves would linearly add to the fields inside and outside the original plasmoid. Since they are free–space waves, these transients would not, by themselves, disrupt the internal plasmoid fields.

Very quickly, then, the space around the original, now decaying, plasmoid would fill with ultimately outgoing energy. The exact form of the outgoing radiation then, in principle, can be calculated. It may then, in principle, be possible, knowing the outgoing field quantities on a fixed surface, to establish an externally generated electromagnetic field that can set up a standing wave that exactly compensates for the losses at the surface of the decaying plasmoid. A model like this is very close to that described by Kapitiza[4].

Surprisingly, the spherically symmetric form of all the assumed fields allows straightforward analytic solutions that permit plasmoid properties in a spherical region $0 \leq r < R_0$, standing waves in the "outer" region $R_0 \leq r < \infty$, and do not require any currents on the surface at $r = R_0$. This situation is graphically shown in fig.(12).

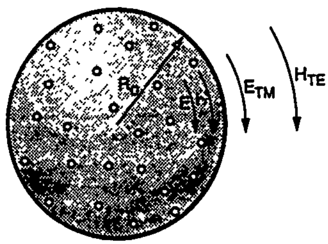

Figure 12. Matching plasmoid boundary conditions. At the surface of a plasmoid the E and H fields are parallel and tangential to the surface. This can be exactly matched by external vacuum fields.

In fig.(12) the region $0 \leq r < R_0$ contains a force–free harmonic plasmoid, at frequency ω, with fields and currents obeying eqs.(9)–(14). The spatial part of the fields in this region are assumed to be proportional to M in eq.(8). Since $r = R_0$ is a zero of the Bessel function $j_l(\alpha r)$, the fields on the surface of the plasmoid are directed along the polar ($\hat{\theta}$) direction.

In addition, $r = R_0$ is a zero of $j_l(kr)$. As a consequence, the magnetic field of a TM cavity mode at frequency ω can have its magnetic field equal to zero at $r = R_0$, and its electric field will only have a $\hat{\theta}$ component. The dual situation of a TE cavity mode can have its electric field equal to zero, and its magnetic field will only have a $\hat{\theta}$ component. It is a simple matter to adjust the amplitudes of a TM and a TE mode so that all the fields are continuous at $r = R_0$. There is, then, no need for currents at $r = R_0$, but a need to maintain an external electromagnetic field with the correct boundary conditions. In principle, this can be done without violating any physical laws.

Alternately, a plasmoid could be created with a single layer of electric current on its surface, which would allow a surface discontinuity in the magnetic field. To obviate extra layers of current the external standing–wave field can be a pure TM cavity mode with zero magnetic field at the surface of the plasmoid. The boundary conditions for such a situation are, in principle, achievable inside a conducting spherical cavity, as shown in fig.(13). The radius of the plasmoid, R_0 satifies eq.(35), while the radius of the cavity, a, is chosen to satisfy

$$\frac{\partial}{\partial r}\left[r j_l(kr)\right]_{r=a} = 0,$$

as can be seen by examining eq.(31).

The latter problem is to create and maintain a single sheet of surface current to neutralize jumps in the surface magnetic field proportional to $(\alpha/k) \ll 1$, see eqs. (11)–(13). This seems more tractable compared to generating and maintaining a double layer of extremely large currents to match a discontinuity in the electric field. The magnitudes of these large currents would have to be inversely proportional to the "skin depth" or transition region between the inside and the outside of the plasmoid, and would have to be out of phase with each other, to simulate a discontinuity in only the electric field. As a result, the two sheets of current would strongly repel each other, tending toward instability.

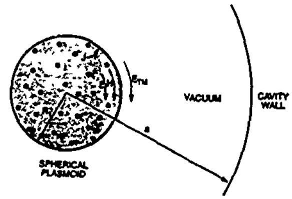

Figure 13. Matching plasmoid boundary conditions in a resonant cavity. By matching only the electric field at the surface of a plasmoid, a current sheet, J, is needed to account for the nominal jump in the magnetic field.

The general question of terminating the surface of a plasmoid probably cannot be answered easily without more work. Anecdotal evidence seems to indicate that the autonomous plasmoids responsible for ball lightning do not have external supportive fields that need to match stringent boundary conditions[24], but measurements are virtually non-existent. On the other hand, workers in Japan have produced plasmoids with exceptionally long lifetimes, in cavities, sustained by microwave generators. Whether either of these examples can be described by force-free time-harmonic waves remains to be seen.

In Appendix C speculative terms are added to Maxwell's equations that would allow for a jump in either the magnetic or the electric field at a surface without the need for a current on that surface.

Apart from surface boundary considerations, a picture emerges concerning the description of a spherical plasmoid with a "trapped" electromagnetic field inside it. Consistent with a current given by eq.(14) and an electric field given by eq.(9) the plasmoid will have the complex (capacitive) conductivity

$$\sigma = i\omega\epsilon_0\left(1 - \frac{\alpha^2}{k^2}\right), \quad i^2 = -1. \tag{84}$$

Ionization levels throughout the plasmoid must be such that the free-electron density is given by eq.(49).

From a practical point of view, the plasmoid can be thought to be a linear system responding to Ohm's law in the form $\mathbf{J} = \sigma(\omega)\mathbf{E}$, where $\mathbf{J}$ and $\mathbf{E}$ are complex. (In eqs.(9) and (10) the sine and cosine functions could have been replaced by the complex exponential $e^{-i\omega t}$, making E and H complex functions. Taking the real parts of these functions would then give the same physically meaningful results derived using real trigonometric time functions[6].)

In the high-frequency limit, $k \to \infty$, a plasmoid has the asymptotic specific capacitance equal to the vacuum permittivity $\epsilon_0 = 8.89 pf/m$. At high frequencies, this is equivalent to a low impedance, meaning that high field strengths might be required to initiate small energetic plasmoids.

Alternatively, the plasmoid could be thought of as a lossless dielectric material. From eqs.(12) and (14), this material would have an effective permittivity,

$$\epsilon_{eff} = \frac{\alpha^2}{k^2}\epsilon.$$

Note that it is possible that $\epsilon_{eff} < \epsilon_0$, implying a superluminal phase velocity within the plasmoid.

The plasmoid region will have internal vortical motion that will cancel the tendency to expand due to electromagnetic pressure.

A remarkable "accident" may therefore exist in Nature wherein a non-propagating electromagnetic field and a special vortical fluid field with the same functional form, neither of which could exist separately, can combine to producing a long-lived high-energy-density plasmoid. The outer supportive field, since it can exist in a vacuum, would have no need for vortical material motion for stability.

No radiation can escape from the surface of the plasmoid, to first order, at the frequency ω. Radiation may occur at harmonics of ω, due to the oscillatory motion described by eq.(57). This radiation may have a detectable, distinct, but perhaps not unique, signature.

To produce the plasmoids described in this note, the following steps must be included:

(1) Closed spherical vortices must be produced;

(2) Desired levels of ionization must be maintained;

(3) Electromagnetic energy, at frequency ω, must be produced within the plasmoid.

An explanation is needed for (3), above. The function being called M(r) in this note corresponds to a sum of the functions $\mathbf{m}_{o01}^{(1)}$ and $\mathbf{n}_{e01}^{(1)}$ on pp. 416–417 of Stratton's book[6]. On page 419 of the same source, a plane wave polarized in the x direction, propagating in the z direction, is expanded in the form

$$\mathbf{a}_x e^{ikz} = \sum_{n=0}^{\infty} i^n \frac{2n+1}{n(n+1)}[\mathbf{m}_{o1n}^{(1)} - i\mathbf{n}_{e1n}^{(1)}]. \tag{85}$$

None of the terms on the right side of eq.(85) can contribute to the fields in the plasmoids described here. That is, the terms in eq.(85) have the wrong spatial symmetries to contribute to the plasmoid fields. The same comments can be shown to be true if the field due to a point source was considered. As a result, external localized sources, which may be expanded in terms of plane waves or point sources, cannot produce energy that will be trapped in a plasmoid.

If, however, the plasmoid region was traversed by a column of current whose axis was parallel to the axis of the plasmoid, the field of the plasmoid can be pumped. Another way of looking at this is that a circular cylinder of current, with its symmetry axis coincident with the plasmoid axis, will emit radiation compatible with being trapped by the plasmoid.

From what has been derived in this note, a possible scenario can be envisioned for the creation of a spherical plasmoid: A very strong discharge occurs in an ionizable gas in the presence of a high–frequency electromagnetic field. The pinch effect due to the discharge initiates a compression, which turns into a swirling vortex. At the right level of ionization, the external time–harmonic field is expelled from the vortical region and radiation from the column of the discharge inside the vortex at the same frequency as the external fields cannot escape. At the end of the discharge, a highly energetic plasmoid emerges that has nearly as much kinetic energy as internal electromagnetic energy. Created in a quiescent ambient gas with a low kinematic viscosity, the plasmoid will slowly come up to a speed that is compatible with its tangential gas velocity. At early times, the plasmoid could appear to move very slowly and may respond to ambient gas flows.

Catastrophic demise can occur if the radiation does not leak away at the same rate the fluid motion dissipates heat via viscous forces. The fluid and electromagnetic forces will not balance and the plasmoid may come apart. On the other hand, variations in the external fields could cause boundary mismatches that might also lead to catastrophic decay.

Alteratively, the viscous dissipation may closely match the electromagnetic losses, and if thermal stresses are not too large, the plasmoid may expire gradually.

The assumption will be that the smaller the value of ζ, in eq.(51), the longer the lifetime of the excitation, all other things being equal. At this time there are no estimates for lifetimes, but some reasonable values for the parameters might be $R_0 = 10$cm, $\lambda = 1.0$mm, and $\zeta = .01$. Under these conditions a 10cm radius spherical excitation may have a total energy density on the order of 10^8 Joules/m^3. This value is larger than average reported energies of ball lightning[24], and these excitations may represent an electromagnetically excited fluid state that could serve as a model for ball-lightning-like phenomena.

If ζ is kept constant, the total energy in any spherical excitation will be proportional to the factor

$$F = \left(\frac{R_0}{\lambda}\right)^4 \cdot R_0. \tag{86}$$

This implies that for a fixed ratio of the radius to the wavelength, for the same value of ζ, the energy content of a spherical excitation is linearly proportional to its radius. Since the volume is proportional to the cube of the radius, the smaller the sphere, the higher the energy density, under the above fixed ratio of radius to wavelength. For the numerical example already given, if the radius were shrunk to one millimeter, the total energy content would be about 15,000 Joules. The wavelength of the harmonic wave would be 10μm; and the energy equivalent to a fifty-watt lightbulb burning for five minutes.

APPENDIX A

THE $l = 1$ FAR FIELD

In this appendix, the far-field radiation from the surface of an ideal force-free plasmoid will be analyzed for the $l = 1$ case. It will be shown that, for appropriate values of R_0, the radius, the fields at the surface of the plasmoid will not radiate, even if the plasmoid is not terminated by surface currents. An important conclusion is that under certain conditions surface currents, which are needed to satisfy Maxwell's equations if field discontinuities occur at the boundary, are not needed to prevent steady-state radiation from an $l = 1$ force-free plasmoid.

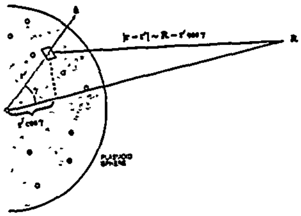

Figure A1. The plasmoid far field. Radiation from a plasmoid involves integrating the fields over the surface with a Green's function linking the surface with the far-field point **R**.

Let the wavenumber be $k = \omega/c$, with c being the speed of light, and ω the angular frequency of the fields. Referring to fig.(A1), the Stratton-Chu formula, on page 467 of Stratton's book[6], for the electric field due to the radiation from the surface of a closed region can be written

$$\mathbf{E}(\mathbf{r}) = i\omega\mu_0 \oint_S [\hat{\mathbf{n}}\times\mathbf{H}(\mathbf{r}')]g(|\mathbf{r}-\mathbf{r}'|)da' + \nabla\times\oint_S [\hat{\mathbf{n}}\times\mathbf{E}(\mathbf{r}')]g(|\mathbf{r}-\mathbf{r}'|)da' + \oint_S [\hat{\mathbf{n}}\cdot\mathbf{E}(\mathbf{r}')]g(|\mathbf{r}-\mathbf{r}'|)da', \tag{A1}$$

where $\hat{\mathbf{n}}$ is the outwardly pointing normal to the region, $\hat{\mathbf{r}}$ is the vector distance from a fixed origin to the oberservation point, $\hat{\mathbf{r}}'$ is the vector distance from a fixed origin to the integration point on the surface, the harmonic time dependence of the field quantities is $e^{-i\omega t}$, and the Green's function is

$$g(r) = \frac{e^{ikr}}{4\pi r}. \tag{A2}$$

At the surface of a plasmoid, the fields have no components normal to the boundary, so that the last integral in eq.(A1) is zero. Since $\mathbf{E}$ and $\mathbf{H}$ are identical, within a constant factor, only the first integral in eq.(A1) has to be evaluated. It will be shown that the value of this integral is also zero if the radius is chosen correctly. Showing that this integral is zero in the far field is equivalent to showing that the term involving the second integral is also zero. The curl of a null function is zero.

The strategy will be to first show, rigorously, that the value of the first integral, call it $\mathbf{V}_1$, can be zeroed in the far field, *i.e.*, as $\mathbf{r} \to \infty$.

In fig.(A1), as $\mathbf{r} \to \infty$, the quantity

$$|\mathbf{r} - \mathbf{r}'| \sim R - \mathbf{r}' \cdot \hat{\mathbf{r}}, \tag{A3}$$

where $R = |\mathbf{r}|$, $\hat{\mathbf{r}}$ is the unit vector in the direction of $\mathbf{r}$, and an angle, γ, is defined by

$$\mathbf{r}' \cdot \hat{\mathbf{r}} = |\mathbf{r}'| \cos\gamma. \tag{A4}$$

Now define an equivalent surface current, $\mathbf{J}$, where

$$\mathbf{J} = \hat{\mathbf{n}} \times \mathbf{H}. \tag{A5}$$

The integral to be evaluated becomes

$$\mathbf{V}_1 = \frac{e^{ikR}}{4\pi R} \oint_S \mathbf{J}(\mathbf{r}') e^{-ik|\mathbf{r}'|\cos\gamma} da'. \tag{A6}$$

In eq.(A6) the integral extends over the entire surface current distribution.

In the $l = 1$ spherical plasmoid case, this equivalent surface current at radius R_0 can be written as a vector function proportional to

$$\mathbf{J}_{surf} = j_0(\alpha R_0) \sin\theta \hat{\phi}, \tag{A7}$$

where $j_0(\alpha R_0) = -.2172\ldots$ corresponds to the value at the first zero of $j_1(\alpha r)$. The first zero was chosen arbitrarily, any other value of r for which $j_1(\alpha r) = 0$ would give the same result.

The value of $\mathbf{V}_1$ becomes proportional to the integral

$$\mathbf{I} = \oint_S \hat{\phi} \sin\theta' e^{-ikR_0 \cos\gamma} d\Omega', \tag{A8}$$

where the distribution is centered at $\mathbf{r}' = 0$, standard spherical coordinate notation is used, and the integration is over the surface of the sphere $|\mathbf{r}'| = R_0$.

The integral in eq.(A8) can be evaluated in closed form by expanding all quantities in the integrand in terms of spherical harmonics[25] $Y_{l,m}(\Omega')$, in the primed coordinates, and $Y_{l,m}(\Omega)$, in the unprimed (far–field) coordinates. First convert the spherical vector $\hat{\phi}'$ into a sum of unit Cartesian vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ in the x and y directions:

$$\hat{\phi}' = -\hat{\mathbf{x}} \sin\phi' + \hat{\mathbf{y}} \cos\phi'. \tag{A9}$$

Next note that

$$\sin\theta' \sin\phi' = i\sqrt{\frac{8\pi}{12}}[Y_{1,1}(\Omega') + Y_{1,-1}(\Omega')], \tag{A10}$$

and

$$\sin\theta' \cos\phi' = \sqrt{\frac{8\pi}{12}}[-Y_{1,1}(\Omega') + Y_{1,-1}(\Omega')], \tag{A11}$$

The exponential in the integral has the expansion[25]

$$e^{-ikR_0 \cos\gamma} = 4\pi \sum_{l=0}^{\infty} i^{-l} j_l(kR_0) \sum_{m=-l}^{l} Y^*_{l,m}(\Omega') Y_{l,m}(\Omega), \tag{A12}$$

where the asterisk (*) denotes complex conjugation. Using the orthogonality property of the spherical harmonics,

$$\oint_S Y^*_{l,m}(\Omega') Y_{l',m'}(\Omega') d\Omega' = \delta_{l,l'} \delta_{m,m'}, \tag{A13}$$

a little algebra shows that, finally,

$$\mathbf{I} = -4\pi j_1(kR_0)\hat{\phi}, \qquad (A14)$$

where $\hat{\phi}$ is the unit spherical vector in the far-field.

If the product kR_0 is one of the zeroes of $j_1(kR_0)$ then the amplitude of the outgoing electric field due to the $l = 1$ surface radiation goes to zero more quickly than the inverse of the distance in the far-field. The fields at the surface of a plasmoid with this radius will therefore not radiate. An exactly analogous argument shows that the $l = 1$ magnetic surface field will not radiate.

There is an interesting interpretation to this result. It was shown in Sec. IX that it is possible to have physically realizable fields outside a plasmoid such that no discontinuities occur at the boundary. In general, if such a situation would arise, the external field would be exchanging energy with the plasmoid, even though the net exchange would be zero. At the radii that correspond to zeroes of the spherical Bessel function, call them resonances, this exchange of energy is halted. It would appear that under such conditions plasmoids would be more mechanically stable. The reasoning for this is that momentum transfer and energy transfer are related.

Away from resonance, the energy and momentum balances are maintained by exchanges between the inside and outside of a plasmoid. Disruption of either the flow from the inside to the outside, or the outside to the inside, will produce a net momentum transfer rate, which is equivalent to a net force. At resonance, forces arising from fluctuations of the fields would thereby be minimized.

APPENDIX B

A NUMERICAL EXAMPLE

On the basis of anecdotal evidence, very energetic plasmoids, known as "Ball Lightning" may have total energy contents of one megajoule with a radius of 10cm. With no further justification, it will be instructive to calculate a set of parameters that could produce such a force–free plasmoid in the Earth's atmosphere at sea level and at $0°C$.

For a sphere with $R_0 = 10$cm, from eq.(8), assuming an $l = 1$ plasmoid, with a radius at the first zero of $j_1(\alpha r)$,

$$\alpha = 44.93. \tag{B1}$$

Assuming k will be large enough to ignore α^2/k^2, then the total energy will be shared nearly equally by the electromagnetic field and the fluid kinetic energy, each having about half a megajoule. Expressed as an integral

$$\frac{\epsilon_0 E_0^2}{4} \int_0^{R_0} |\mathbf{M}(\mathbf{r})|^2 d\mathbf{r} = .5 \times 10^6, \tag{B2}$$

where M(r) is defined by eq.(8), with $l = 1$. Direct integration shows that

$$\int_0^{R_0} |\mathbf{M}(\mathbf{r})|^2 d\mathbf{r} = \frac{35.87}{\alpha^3}, \tag{B3}$$

so that

$$E_0 = 3.566 \times 10^9 \text{volts/meter}. \tag{B4}$$

Taking $\xi = .01$ in eq.(58),

$$\omega = \sqrt{\frac{1.759 \times 10^{11} \times 3.566 \times 10^9}{.01 \times .1}} = 7.920 \times 10^{11}, \tag{B5}$$

is equivalent to a $126 GHz$ field with $k = 2642\text{meters}^{-1}$ To satisfy the second part of eq.(35), this value of k must be changed slightly to

$$k = 2686(\text{meters})^{-1}, \tag{B6}$$

which is equivalent to a 128 GHz field.

From eq.(38) the charge density is

$$\rho_e = -32.63\text{coul/m}^3, \tag{B7}$$

amounting to 8.532×10^{17} electrons. Considering that there are about 1.126×10^{23} diatomic molecules of gas in the plasmoid, on the order of only one to two molecules in 10^5 needs to contribute an electron.

At the surface of the sphere, from eqs.(5), (20) and (27) the speed of the gas u_s is a function of the polar angle θ:

$$u_s = 1.431\times10^3 \sin\theta(\text{meters/sec}). \tag{B8}$$

While u_s represents a very large speed, it is not true that the plasmoid will be traveling at 955 meters/sec, the equivalent solid–sphere speed to match the surface speed. The reason is that the plasmoid was assumed to be created in a quiescent atmosphere. Analysis of the Navier–Stokes equation(7) shows that the boundary layer emerging from the surface of the sphere as a result of the surface motion diffuses like a thermal wave. To grow out to a distance of one extra R_0 takes on the order of

$$\Delta t = R_0^2/\nu = .01/1.32\times10^{-5} = 757\text{sec}, \tag{B9}$$

where ν is the kinematic viscosity of air at 1 Atm. and $0°C$.

In the first few seconds, the plasmoid will therefore hardly be moving relative to the quiescent atmosphere. On the order of tens of seconds some motion will be evident, but it is not clear that the plasmoid has a lifetime long enough to get up to any appreciable speed. Its entire liftime, then, may be spent "floating", or being carried by air currents, or responding to electrodynamic forces.

It is instructive to scale the radius down to $50\mu m$. The total energy, assuming the α/k ratio stays the same, might naively be projected to 500 joules. Since the charge density will go up inversely as the square of the radius, the density of the plasmoid must be increased. Since only about one molecule in 10^5 will be ionized in the larger system, a density corresponding to about forty times sea-level atmospheric air must be assumed, which will be totally ionized. The wavelength of the trapped radiation will be $\lambda = 1.17\mu m$, in the infrared range. The surface speed, eq.(B8), will be $u_s = 7.16 \times 10^4 \sin\theta$, with a maximum internal speed of 2.2×10^5meters/sec, still not relativistic. The number of free electrons will be 4.3×10^{13}. It would take on the order of 8ms for such a plasmoid to come up to speed at $0°C$. At different temperatures, where the kinematic viscosity may be higher, this time could be much shorter. The assumption has been that the ambient atmosphere is also at a density forty times that of sea-level air. It is not clear how the stability will be affected if the plasmoid travels into a lower-density and/or pressure region. The electromagnetic pressure, balanced by the vortical field, is almost 10^{10} atmospheres. If even one joule is trapped the pressure is on the order of 10^7 atmospheres. It would seem that a perturbation of a few atmospheres of pressure may not be immediately destabilizing. More work must be done in this area.

APPENDIX C

AD HOC FIELD TERMS

In this appendix, terms will be added to Maxwell's equations that will allow true discontinuities to arise in the tangential components of electromagnetic fields at surfaces, generating accompanying surface conditions that cannot be derived from the "usual" set of Maxwell's equations. It should be understood that there may be no physical justification for these terms. They serve as an illustration that it is possible to assume new operators in Maxwell's equations that will not affect any solutions that are continuous, yet can give a contribution at a field discontinuity.

The terms to be eventually introduced are classically unobservable, because the electromagnetic field is classically assumed to be analytic everywhere. An objection may be raised that the introduced terms are not Lorentz invariant. In fact, they are. Lorentz invariance will be investigated separately. The important point is that classically unobservable terms can be added to Maxwell's equations, causing observable deviations from the usual formulation at true field discontinuities.

Consider the operator, $\mathcal{O}$, acting on the vector function $\mathbf{M}(\mathbf{r})$, at a point $\mathbf{r}'$,

$$\mathcal{O}\mathbf{M}(\mathbf{r}) \equiv \int d\Omega \hat{\mathbf{r}} \times \hat{\mathbf{r}} \cdot \nabla \mathbf{M}(\mathbf{r}) \Big|_{\mathbf{r}=\mathbf{r}'}, \qquad (C1)$$

where $\hat{\mathbf{r}}$ is the unit radial vector centered at $\mathbf{r}'$, and the integration is over the full 4π stearadians. From the properties of the gradient operator, this can be written,

$$\mathcal{O}\mathbf{M}(\mathbf{r}) \equiv \int d\Omega \hat{\mathbf{r}} \times \frac{\partial}{\partial r} \mathbf{M}(\mathbf{r}) \Big|_{\mathbf{r}=\mathbf{r}'}, \qquad (C2)$$

Integration in eq.(C2) is over the angular parts of the integrand. The operator $\partial/\partial r$ does not affect the angular parts of $\mathbf{M}(\mathbf{r})$. In general, any wavefunction obeying Maxwell's equations can be expanded as a sum of multipole fields in the form[25]

$$\mathbf{M}(\mathbf{r}) = \sum_{J,l,M} \left[a^{l}_{J,M} z^{J}_{l}(r) \mathbf{T}_{J,l,M} + a^{l+1}_{J,M} z^{J}_{l+1}(r) \mathbf{T}_{J,l+1,M} + a^{l-1}_{J,M} z^{J}_{l-1}(r) \mathbf{T}_{J,l-1,M} \right], \qquad (C3)$$

where $\mathbf{T}_{J,l,M}(\Omega)$ is a vector spherical harmonic [25], $a^{l}_{J,M}$'s are complex constants, and

$z_l^J(r)$'s are functions of the radial distance from the center of the expansion. The description of a wavefunction using eq.(C3) is also known as a Helmholtz decomposition.

Vector spherical harmonics are orthogonal on the surface of the unit sphere, *i.e.*,

$$\int d\Omega \mathbf{T}^*_{J',l',M'}(\Omega)\cdot\mathbf{T}_{J,l,M}(\Omega) = \delta_{J,J'}\delta_{l,l'}\delta_{M,M'}, \tag{C4}$$

where the asterisk denotes complex conjugation.

The unit radial vector can be expressed as

$$\hat{\mathbf{r}} = \frac{1}{\sqrt{4\pi}}\mathbf{T}_{0,1,0}(\Omega). \tag{C5a}$$

Similarly, the unit azimuthal vector can be written

$$\hat{\phi} = \frac{1}{\sqrt{4\pi}}\mathbf{T}_{1,1,0}(\Omega). \tag{C5b}$$

From the rules for coupling representations of the rotation group[25], the only terms in the multipole expansion of $\mathbf{M}(\mathbf{r})$ that can contribute to the integral in eq.(C2) are those associated with $\mathbf{T}_{1,1,m}$, $-1 \leq m \leq 1$. When expanded about different origins, the vector function $\mathbf{M}(\mathbf{r})$ will appear to have differing amplitudes associated with the the vector spherical harmonics $\mathbf{T}_{1,1,m}$. The amount of $\mathbf{T}_{1,1,m}$ at a point of expansion of $\mathbf{M}(\mathbf{r})$ is proportional to $\nabla\times\mathbf{M}(\mathbf{r})$ at that point. The integral in eq.(C2) will generally define a different vector at each point $\mathbf{r}'$, depending on the local expansion of $\mathbf{M}(\mathbf{r})$ about the point $\mathbf{r}'$. If this local expansion does not have harmonic terms of the form $\mathbf{T}_{1,1,m}$, there will be no contribution at that point.

It is now clear what form an operator can take to give no contribution when applied to analytic vector fields arising from solutions of the classical Maxwell equations:

$$\overline{\mathcal{O}}\mathbf{M}(\mathbf{r}) \equiv \int d\Omega\hat{\mathbf{r}}\times\frac{\partial}{\partial r}\left[\mathbf{M}(\mathbf{r}) - \sum_{m=-1}^{1}\langle\mathbf{T}_{1,1,m},\mathbf{M}(\mathbf{r})\rangle\mathbf{T}_{1,1,m}\right]_{\mathbf{r}=\mathbf{r}'}, \tag{C6}$$

where

$$\langle \mathbf{T}_{1,1,m}, \mathbf{M}(\mathbf{r})\rangle = \int d\Omega \mathbf{T}^{*}_{1,1,m}(\Omega)\cdot \mathbf{M}(\mathbf{r})\Big|_{\mathbf{r}=\mathbf{r}'}. \qquad (C7)$$

In eq.(C6), $\overline{\mathcal{O}}$ is a linear operator that gives a null result when applied to an analytic multipole field. At a discontinuity in the field $\mathbf{M}(\mathbf{r})$ the radial derivative term, in eq.(C2), will produce a singularity in the neighborhood of the discontinuity. It should also be noted that removing harmonics of the form $\mathbf{T}_{1,1,m}$ does not affect the vector field $\mathbf{M}(\mathbf{r})$ at $|\mathbf{r}| = 0$ because

$$\lim_{r\to 0} z_l^J(r) = 0;\ l \neq 0. \qquad (C8)$$

Therefore

$$\lim_{|\mathbf{r}|\to 0}\left[\mathbf{M}(\mathbf{r}) - \sum_{m=-1}^{1} \langle \mathbf{T}_{1,1,m}, \mathbf{M}(\mathbf{r})\rangle \mathbf{T}_{1,1,m}\right] = \mathbf{M}(\mathbf{r})\Big|_{\mathbf{r}=0}, \qquad (C9)$$

which means that if the function $\mathbf{M}(\mathbf{r})$ has a discontinuity at $\mathbf{r} = \mathbf{r}'$ the integral in eq.(C2) will define a generating function for jumps in the field $\mathbf{M}(\mathbf{r})$, in the sense of the theory of distributions[26]. Less rigorously, the integral in eq.(C2) behaves like a delta function at discontinuities in $\mathbf{M}(\mathbf{r})$, is finite at discontinuities of the gradient of $\mathbf{M}(\mathbf{r})$, and is otherwise zero.

Maxwell's equations can be rewritten for any given time-harmonic frequency ω, in vacuo, incorporating the operator $\overline{\mathcal{O}}$,

$$\nabla\times\mathbf{H}(\mathbf{r}) + G_H\overline{\mathcal{O}}\mathbf{H}(\mathbf{r}) = -i\omega\epsilon_0\mathbf{E}(\mathbf{r}) + \mathbf{J}(\mathbf{r}), \qquad (C10)$$

and

$$\nabla\times\mathbf{E}(\mathbf{r}) + G_E\overline{\mathcal{O}}\mathbf{E}(\mathbf{r}) = i\omega\mu_0\mathbf{H}(\mathbf{r}), \qquad (C11)$$

where the time-harmonic factor is $\exp(-i\omega t)$, and the numerical factors G_H and G_E will be determined below.

The constants G_H and G_E can be determined by assuming that no current will flow on the surface defining a discontinuity in either $\mathbf{E}$ or $\mathbf{H}$. The usual method for deriving the current at a boundary can be modified to find G_H and G_E. Enclose a small portion of the boundary between two regions where either **H** or **E** has a jump discontinuity with a closed curve in a plane that cuts through the boundary. In the usual proof, both sides of eq.(C10), for example, are integrated over the area of the closed curve, Stoke's theorem is applied, and the current flowing through the area is found to be equal to the line integral of the component of **H** tangent to the closed curve[6]. At a discontinuity, the current flowing through the closed curve is proportional to the difference in tangential components of **H** on both sides of the boundary of the discontinuity, *i.e.*,

$$\mathbf{J}_{surf} = \hat{\mathbf{n}} \times (\mathbf{H}_1 - \mathbf{H}_2), \tag{C12}$$

where $\hat{\mathbf{n}}$ is the unit normal to the boundary, pointing from the region with the field $\mathbf{H}_2$ into the region with the field $\mathbf{H}_1$.

At a field discontinuity, the integral of $\overline{\mathcal{O}}\mathbf{H}(\mathbf{r})$ over the surface of the closed curve, discussed above, is proportional to the right–hand side of eq.(C12). It can be shown that if

$$G_H = G_E = -\frac{1}{2\pi}, \tag{C13}$$

no current will flow tangential to *true* field discontinuities.

It can be argued that Maxwell's equations, including $\overline{\mathcal{O}}$ terms, with eq.(C13), are Lorentz invariant. The logic is as follows: All observers moving relative to an analytic electromagnetic field will conclude that the field they see is also analytic, and the $\overline{\mathcal{O}}$ terms will not contribute. On the other hand, any true discontinuity in either the electric or magnetic field will be described differently by different observers, but all observers will agree that no current would be flowing tangentially to the boundary between the regions where the discontinuity occurs.

The reader should again be cautioned that the added $\overline{\mathcal{O}}$ terms in Maxwell's equations may have no physical justification. Experimentally, effects such as ball lightning and anomalous charge–cluster formation occur in the vicinity of very strong local discharges that may cause field discontinuities. The purpose of this appendix is to show that such effects can arise from terms that are classically unobservable, causing no changes whatsoever in any other solutions where true discontinuities do not arise.

Applied to the force–free case, for cases that functionally look like eq.(8) of the main text, at a radius corresponding to a zero of the function $j_l(\alpha r)$ the fields are tangential to the spherical surface. A discontinuity at this surface equivalent to a drop to zero fields outside the plasmoid with large fields inside, will not need a set of boundary currents to sustain the wavefunctions. Indeed, the remarkable special case, called a vactiod in the text, wherein a spherical nonpropagating field is "trapped" in the vacuum, becomes realizable.

If the above theory applied at zero frequency to "perfect" superconductors, where the transition from non-zero field to zero field, outside to inside, occurred as a true discontinuity, there would be no current residing on the surface. Since the quantity $\hat{n} \times H$ appears in the equations for the external field, not the current, per se[6], there would be no obvious contradiction introduced into classical electromagnetism. The logical extension of such assumptions would be the existence of zero-frequency vactoids, having a static electric or magnetic field trapped in, say, a toroidal closed loop with no enclosing currents. Any further speculation concerning such pathological states of the electromagnetic field is beyond the scope of this report.

At the present state of development of technology, it is impossible to either accept such extra terms or reject them out of hand.

APPENDIX D

TOTAL PLASMOID ENERGY

Equation (23) expressed the energy in the electromagnetic field of a plasmoid as having a time dependence. Since energy must be classically be conserved instantaneously, the time variations must be accounted for by the kinetic energy of the electrons oscillating along electric field lines. When this energy is added to $\mathcal{Q}$ in eq.(23), the time dependence of the total energy of the plasmoid at any point should be independent of time.

This is easily proven. The kinetic energy of an individual electron at the point **r** is

$$K_e = \frac{1}{2} m_e |\mathbf{v}|^2, \tag{D1}$$

where m_e is the mass of the electron and **v** is its (non-relativistic) velocity. The kinetic energy of the electron gas can be written in terms of the current density at **r**,

$$\mathcal{T} = \frac{1}{2}\left(\frac{m_e}{e}\right) \int \mathbf{J}(\mathbf{r},t)\cdot\mathbf{v}dv, \tag{D2}$$

where $e = -1.602\times 10^{-}19C$.

Using eq.(37), and the time dependence of the current,

$$\mathcal{T} = \frac{1}{\omega}\cos^2(\omega t + \varphi) \int \mathbf{J}(\mathbf{r})\cdot\mathbf{E}(\mathbf{r})dv. \tag{D3}$$

Since the spatial part of the vector potential can be related to spatial part of the electric field,

$$\mathbf{A}(\mathbf{r}) = \frac{1}{\omega}\mathbf{E}(\mathbf{r}), \tag{D4}$$

then

$$\mathcal{T} = -\frac{1}{2}\cos^2(\omega t + \varphi) \int \mathbf{A}(\mathbf{r})\cdot\mathbf{J}(\mathbf{r})dv. \tag{D5}$$

The quantity $\mathcal{Q} + \mathcal{T} > 0$, which is the total energy of the plasmoid, is now independent of time.

Although the total energy is positive, two parts of the sum, *i.e.*, the second integral term in eq.(23) and the electron kinetic energy, are equal and opposite in sign. Even the local energy densities associated with these terms are equal and opposite in sign, the kinetic energy density always being positive. Only frequencies above the critical frequency will be considered, since electrons then carry the current.

The free electrons making up the current can be considered as moving about the minimum of a potential well whose average depth at any point, $\mathbf{r}$, is $(1/4)\mathbf{A}(Vr)\cdot\mathbf{J}(Vr)$. From eq.(25), the depth of this equivalent potential well increases quadratically as the frequency increases. The argument made in the main text is that if the electronic kinetic energy is not changed, a small local geometric perturbation will raise the energy of the plasmoid. That argument was based on the out-of-phase relationship of $\mathbf{J}(\mathbf{r})$ and $\mathbf{A}(\mathbf{r})$.

Another simple argument can be made that geometrically perturbing the plasmoid raises its energy. Following Gekker[21] (p.80) each oscillating electron can be thought of as an oscillating electric dipole, whose potential energy is at a minimum when its dipole moment is aligned with the fields. If the perturbation does not leave the plasmoid in another force-free state, a torque is needed to turn any given dipole to a different orientation. This torque will have components arising from the electric as well as from the magnetic field. A restoring torque will tend to line up the dipole moment with the electric field and force the electronic motion to follow magnetic field lines. Since both the electric and magnetic fields are parallel, all the contributions to the restoring torque in a small perturbation will add constructively and be opposed to the perturbing motion.

APPENDIX E

NON-RADIATING HARMONIC FIELDS THAT CAN CONFINE PLASMA CURRENTS AND CHARGES

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ABSTRACT

Elementary theory of force-free fields is applied to electromagnetically excited charge densities. It is shown that mechanically stable non-radiating modes may be excited in plasmas by harmonic fields, even in the absence of D.C. magnetic fields. The free-charge density needed to sustain a stable spherical non-radiating mode can be shown to be uniform, and dependent only on the electromagnetic wavenumber and spatial extent of the mode, in a neutral plasma. The time-averaged electric and magnetic fields in such modes are all zero. A virial theorem check verifies the consistency of an intuitive approach and a more rigorous lagrangian formulation. Electromagnetic stress effects, which can be accurately balanced by vortical fluid motion, are ignored.

I. INTRODUCTION

If the vector field $\mathbf{H}(r)$ is a force-free field then

$$\nabla \times \mathbf{H} = \alpha \mathbf{H}, \tag{1}$$

where α is a complex constant. It may seem strange to imagine a vector field that is proportional to its own curl, without the field being identically null. Nonetheless, such fields can be constructed, at least mathematically, having non-zero magnitudes. The properties of such fields may be very valuable in diverse areas of engineering. For example, if the windings of a D.C. electromagnet followed the streamlines of a force-free field the force on the wires, due to the magnetic field they produced, would be zero. (Hence the name for the fields.) In fluid dynamics the contribution of the $\mathbf{v} \times \nabla \times \mathbf{v}$ term in the Navier-Stokes equation is zero in a vortical fluid mode whose flow follows streamlines of a force-free field. (Such modes are known as Beltrami vortices in fluid dynamics.[1])

In electromagnetic engineering, electromagnetic excitations that obey eq.(1) can be shown to exist in vacuo as well as in the presence of charge and currents. These modes are stationary, *i.e.*, provided with the proper boundary conditions, they will not propagate.

In this note, some of the simple properties of force–free non–radiating fields will be derived. Any references to a "non–radiating mode" in this note will always imply reference to a "force–free non–radiating mode".

Applications will be made to the theory of stable electromagnetically self–confining excitations that may have long lifetimes and very high free–charge and energy densities. Many of the phenomena to be described may never have been observed. Some of the results of the simple theory may explain experiments performed by Bostick and his associates, as well as results reported by Puthoff and Shoulders(2).

The idea of applying the force–free mode to confining plasma is not new(3,4). Techniques employing names such as the "Spheromak" and "Compact Toroid", which nominally depend on force–free plasma motion, have been implemented with varying degrees of success(5,6). In these schemes the object is to induce the gross fluid velocity of the plasma to conform to a Beltrami vortical mode. This lowers the magnetic interaction energy and, presumably, raises confinement times.

A common feature of the past applications of force–free modes to plasmas is that the fluid, or ion, velocity is non–zero and directed along the ergotic streamlines of a Beltrami vortex. The stationary (zero–frequency) magnetic field also lies along these streamlines. Since the plasma is assumed to be neutrally charged, Maxwell's equations predict that the zero–frequency magnetic field is not necessarily accompanied by an electric field.

In what follows, the force–free idea is taken one step farther, with the introduction of a high–frequency electromagnetic field. A self–consistent solution will be shown to exist that allows the electric and magnetic fields, as well as the current to have the force–free form. Beyond a critical frequency, the magnetic interaction energy of the oscillating free–electron density is not only minimized, it changes sign and becomes negative.

An initial physical model is postulated that has the ions stationary, *i.e.*, there is no net fluid motion, only the free electrons oscillate very nearly along force–free streamlines. Applying Lagrange's equations to this model forces the free–electron density to have a specific value with respect to the size parameter α and the frequency, but independent of the amplitudes of the fields. It will turn out that if α is a constant, uniform free–charge and ion densities are consistent with force–free high–frequency electromagnetic/fluid modes.

This model is incomplete, owing to the fact that the free–electron gas, with non–zero mass electrons, cannot exactly oscillate along curved electric field lines, and will thereby experience electromagnetic stresses. These stresses are, at high frequencies, equivalent to a positive pressure field proportional to the square of the electric field(20). A direct application of Bernoulli's theorem[21] says that introducing a fluid velocity whose square is proportional to the square of the electric field, *i.e.*, a Beltrami vortex, will counterbalance the major electromagnetic stresses. The introduction of fluid vortical motion for enhanced stability will not be addressed in this note, but will not affect the results derived here.

The virial theorem predicts that no excitation can be confined indefinitely by only classical electromagnetic fields, even in the relativistic limit[7,8,9,10]. The model proposed here does not violate this prediction. Application of the virial theorem, ignoring radiation from surface currents and non-linear effects, yields conditions for long-term stability that are identical to those derived for self-consistent force-free motion of non-zero mass electrons in an electromagnetic field.

The major approximation made is that the massive electrons exactly follow the curved force-free electric field lines. In a sequel, this approximation and an analysis of the virial will be pursued[11]. It will be shown that there may be tradeoffs between energy density, frequency and confinement time, for a fixed-sized excitation.

II. KINEMATICS OF THE FIELDS

Taking the curl of eq.(1),

$$\nabla \times \nabla \times \mathbf{H} = \nabla \nabla \cdot \mathbf{H} - \nabla^2 \mathbf{H}, \tag{2}$$

and realizing that, since **H** is derived from a curl,

$$\nabla \cdot \mathbf{H} = 0, \tag{3}$$

then it follows that **H** obeys the vector Helmholz equation,

$$\nabla^2 \mathbf{H} + \alpha^2 \mathbf{H} = 0. \tag{4}$$

Equation (4) is the key to searching for solutions to eq.(1). Not all solutions of eq.(4) will be solutions of eq.(1), but all solutions of eq.(1) will be solutions of eq.(4). The process of generating solutions to eq.(1) can be approached in several ways. One way is to choose any vector field that obeys eq.(4) and successively take its curl, noting all new terms that arise. If the number of independent terms stops increasing, form a linear combination of the final set of terms such that the linear combination obeys eq.(4).

Starting from

$$\mathbf{F}_l = \sin^l \theta j_l(\alpha r)\hat{\phi}, \tag{5}$$

the resultant force–free vector field is

$$\mathbf{F}_{l,B}(r,\theta,\phi) = (l+1)\cos\theta \sin^{l-1}\theta \frac{j_l(\alpha r)}{\alpha r}\hat{\mathbf{r}} - \left[j_{l-1}(\alpha r) - l\frac{j_l(\alpha r)}{\alpha r}\right]\sin^l \theta\hat{\theta} \pm j_l(\alpha r)\sin^l\theta\hat{\phi}, \tag{6}$$

where l is any integer, $l > 0$, j_l is the familiar spherical Bessel function, and the vectors $\hat{\mathbf{r}}$, $\hat{\theta}$, and $\hat{\phi}$ are the familiar right–handed 3–D polar coordinate unit vectors. The vortical field described by eq.(6) has a useful application, as will be shown below.

It is somewhat tedious, but straightforward, to show that

$$\nabla\times\mathbf{F}_{l,B}(r,\theta,\phi) = \pm\alpha\mathbf{F}_{l,B}(r,\theta,\phi). \tag{7}$$

A somewhat more elegant expression for force–free fields can be derived from the properties of the rotation group. Using the notation in Rose's book[12] for the vector spherical harmonics $\mathbf{T}_{J,L,M}$, force–free fields can be immediately written that correspond to the $(2l+1)$–dimensional representations of the three–dimensional rotation group:

$$\mathbf{B}_{l,m} = \pm j_l(\alpha r)\mathbf{T}_{l,l,m} + \frac{i}{\sqrt{2l+1}}\left[\sqrt{l+1}\,j_{l-1}(\alpha r)\mathbf{T}_{l,l-1,m} - \sqrt{l}\,j_{l+1}(\alpha r)\mathbf{T}_{l,l+1,m}\right], \tag{8}$$

where $\mathbf{B}_{l,m}$ is a force–free field for $l > 0$ and $-l \leq m \leq l$.

It is not clear at this time that the $\mathbf{B}_{l,m}$ form a complete set for the expansion of general force–free fields. Note that the spherical Bessel functions can be replaced by spherical Hankel functions, and that α does not have to be real. Note also that force–free fields can be expanded in cylindrical $(\hat{\rho}, \hat{\phi}, \hat{\mathbf{z}})$ coordinates; and may be synthesized using functions obeying the scalar Helmholz equation, yet possessing singularities appropriate to conical geometries. Neither of these latter solution sets will be pursued in this note.

For constant values of α, Zaghloul and Barajas employ Hansen's straightforward method to synthesize force–free vector fields from arbitrary solutions of the Helmholtz equation. The reader is referred to their paper and Stratton's book for details[13,14,15].

The real fields $\mathbf{F}_{n,B}$, from eq.(6), can be shown to be a subset of the set of fields $\mathbf{B}_{l,m}$ from eq.(8). The explicit forms displayed in eq.(6) for the field components, however, are easier to deal with analytically.

Consider a vortical velocity field with the form $\mathbf{F}_{l,B}$. Within a common constant factor imparting the correct dimensionality, the time rates of change of the coordinates are

$$\frac{dr}{dt} = (l+1)\cos\theta \sin^{l-1}\theta \frac{j_l(\alpha r)}{\alpha r}, \tag{9}$$

$$\frac{d\theta}{dt} = -\frac{1}{r}\left[j_{l-1}(\alpha r) - l\frac{j_l(\alpha r)}{\alpha r}\right]\sin^l\theta, \tag{10}$$

and

$$\frac{d\phi}{dt} = \frac{1}{r} j_l(\alpha r)\sin^{l-1}\theta. \tag{11}$$

Starting with eqs. (9)–(11) it is possible to integrate the trajectory of a point having the above components of velocity. The time, t, can be eliminated from any pair of the expressions. If eq.(10) is divided by eq.(9), the resultant equation

$$\frac{d\theta}{dr} = \frac{1}{l+1}\tan\theta\left[\frac{l}{r} - \alpha\frac{j_{l-1}(\alpha r)}{j_l(\alpha r)}\right], \tag{12}$$

can be integrated analytically to obtain

$$\sin\theta = A\left[\alpha r j_l(\alpha r)\right]^{-\frac{1}{l+1}}, \tag{13}$$

where A is a constant.

Equation (13) coupled with the non-analytic equation for ϕ,

$$\phi = \pm\frac{\alpha A}{l+1}\int_{r_0}^{r}\frac{dr'}{\sqrt{\left[\alpha r' j_l(\alpha r')\right]^{\frac{2}{l+1}} - A^2}}, \tag{14}$$

determine streamlines.

A very important property of eqs. (9)–(11) is that for at every value of r_m for which $j_l(\alpha r_m) = 0$, the radial ($\hat{\mathbf{r}}$) and azimuthal ($\hat{\phi}$) components of the vector fields $\mathbf{F}_{l,B}$ vanish. At any such $r = r_m$ the only component of the field is the longitudinal ($\hat{\theta}$) component. Vortical fields like $\mathbf{F}_{l,B}$ have, then, a layered structure. It follows that if the simplified longitudinal boundary condition at any of the r_m can be met everywhere on the surface of a sphere then the field internal to the sphere will obey eqs. (9)–(11) at all times if they obeyed these equations and eq.(1) for $r \leq r_m$, or if force–free fields represented stable configurations for the physical system under consideration.

For the special case $l = 1$, the longitudinal boundary condition is met by the velocity components of an incompressible inviscid fluid at the surface of a sphere immersed in an infinite uniform flow, or the magnetic field at the surface of a spherical superconductor in an otherwise constant field, or, in short, by any harmonic field whose radial component is zero at the surface of the sphere $r = r_m$. The $l = 1$ case would be indistinguishable from a solid sphere with no internal structure, from the outside.

In the next sections, arguments will be made to show that a conducting plasma may be excited to sustain a non–radiating electromagnetic mode whose electric and magnetic fields obey the force–free conditions. Arguments will also be made to show that these modes can be stable and self–confining, when terminated with the appropriate boundary conditions. The current in the plasma will be forced to flow along the streamlines of a force–free vortex.

Electromagnetic volume stresses will be ignored in this note, but are not negligible. These stresses can be counteracted by fluid motion. A full discussion of the interaction of fluid stresses and electromagnetic stresses will not be needed to understand the principles presented here.

In the final section and in the Appendix, some speculations will be made as to how this theory can be applied to predict the existence and properties of stable "particles" that are described by non–radiating modes trapped in spheres of neutral plasma.

III. INTERACTION WITH ELECTROMAGNETIC FIELDS

Let $\mathbf{H}$ in eq.(1) be a magnetic field. If $\mathbf{E}$ is the accompanying electric field and $\mathbf{J}$ is the current, in otherwise free space Maxwell's equations are

$$\nabla \times \mathbf{H} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}, \tag{15}$$

and

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}, \tag{16}$$

where ϵ_0 and μ_0 are the permittivity and permeability of the vacuum.

Assuming a time-harmonic dependence $e^{-i\omega t}$, eqs. (1), (15) and (16) imply that

$$\mathbf{E} = \frac{i\omega\mu_0}{\alpha}\mathbf{H} - \nabla\psi, \tag{17}$$

where ψ is a scalar potential. The vector potential, $\mathbf{A}$, is then

$$\mathbf{A} = \frac{\mu_0}{\alpha}\mathbf{H}. \tag{18}$$

From the Lorentz condition

$$\nabla \cdot \mathbf{A} + \epsilon_0\mu_0 \frac{\partial \psi}{\partial t} = 0, \tag{19}$$

it follows that

$$\frac{\partial \psi}{\partial t} = -i\omega\psi = 0, \tag{20}$$

because $\nabla \cdot \mathbf{H} = 0$. The above choice for the vector potential was not unique. It was made to force the current to also have the form of a force–free field. To see this, solve for $\mathbf{J}$:

$$\mathbf{J} = \alpha\left(1 - \frac{k^2}{\alpha^2}\right)\mathbf{H}, \tag{21}$$

where k is the free–space wave number $\omega\sqrt{\epsilon_0\mu_0}$. From eq.(21) it follows that $\nabla \cdot \mathbf{J} = 0$, and that $\mathbf{J}$ is also a force–free mode if $\mathbf{H}$ is a force–free mode. This implies, also, that at any frequency greater than zero, a force–free–field current does not produce a scalar potential. Therefore, an excess of charge of a single sign entrained in a force–free mode in a time–harmonic electromagnetic field will have a non–time–harmonic electrostatic field associated with it. In other words, the charge density in a stationary non–radiating mode may not be uniform in space, but it is not a function of time. It will be shown below that the charge density must also be spatially uniform.

Note that if $k = |\alpha|$, with real α, the current becomes zero and the non-radiating mode corresponds to a vacuum excitation of the electromagnetic field having finite energy density and zero mean momentum density. (Momentum density is proportional to $\mathbf{E} \times \mathbf{H}$, whose mean is zero here.) Note also that the relative phase of the current and the magnetic field it generates reverses at $k = |\alpha|$. As the frequency, ω, increases, the value of k and subsequently the absolute value of $\mathbf{J}$ compared to $\mathbf{H}$ increases, beyond the frequency when $k = |\alpha|$.

Equation (21) has some interesting consequences when the interaction energy of the current and the magnetic field is considered. Following the logic in Stratton's book[13], but accounting for time-harmonic variation variations of the field, an expression can be written for the *r.m.s.* energy, $\overline{Q}$, of the current, $\mathbf{J}$, in the presence of the vector potential, $\mathbf{A}$ (the asterisk $*$ means complex conjugation):

$$\overline{Q} = \frac{1}{2}\int \mathbf{J}\cdot\mathbf{A}^* dv. \tag{22}$$

From eqs. (18) and (21), this becomes

$$\overline{Q} = \frac{\mu_0}{2}\left(1 - \frac{k^2}{\alpha^2}\right)\int |\mathbf{H}|^2 dv. \tag{23}$$

For real values of α the magnetic interaction part of the energy density becomes negative at high frequencies ($\omega > c\alpha$, c is the speed of light). The implication is that force-free-mode currents created in short-wavelength electromagnetic fields, residing in local minimum of potential energy, are mechanically stabilized by the electromagnetic fields. From eq.(22), the higher the frequency of the field the more "bound" the current. In order for the above analysis to apply, assuming its interpretation is correct, it has to be possible for the field to couple to the current and the boundary conditions have to be met.

In a way, the streamlines of a force-free vortex seem to act like waveguides, allowing modes to penetrate if they are small enough to "fit". At the critical frequency, $\omega = c\alpha$, an electromagnetic free-space mode can exactly match the size of the force-free mode. At higher frequencies, free-space modes can fit inside the force-free mode and stabilize it.

IV. ENERGY, FORCE AND MOMENTUM CONSIDERATIONS

The ultimate practical question is, "Can this theory be applied to produce stably confined plasma and, if so, how?" First consider the forces on individual "blobs" of charge that make up a current. The Lorentz force on a charge density, ρ, moving at a velocity $\mathbf{v}$ in an electromagnetic field is

$$\mathbf{F} = \rho(\mathbf{E} + \mu_0 \mathbf{v} \times \mathbf{H}). \tag{24}$$

Since $\mathbf{J} = \rho\mathbf{v}$ this reads

$$\mathbf{F} = \rho\mathbf{E} + \mu_0 \mathbf{J} \times \mathbf{H} = \rho\mathbf{E} \tag{25}$$

for force–free modes. In other words, the only non–zero–time–averaged force acting on the charge in an isolated force–free mode arises from the zero–frequency electrostatic field due to the other charges. In a neutral plasma the electrostatic forces are evidently zero. If excess charge is present in the current, an electrostatic field will perturb the trajectories. It will be very instructive to determine the sum of the magnetic interaction energy and the kinetic energy to determine the stability of the overall system. The reason for ignoring the energy in the oscillating electric field is that it does not directly contribute to the stability of the current. *I.e.*, an infinitesimal change in current will affect the magnetic interaction energy and the kinetic energy only.

The electric field will, however, determine the kinetic energy of the equilibrium system. At high frequencies the motion of the charges, ignoring electrostatic and outside mechanical forces, is small–displacement oscillation about an equilibrium, at the frequency of the field. (It is important to realize that there is no net motion of the charges. The background neutralizing ions are also assumed to be motionless.) The motion of the charges, assumed to be carried by electrons, is driven by the electric field, which is parallel to the current. Calculations will all be non–relativistic.

In a simple–minded initial intuitive approach, which admittedly should not be giving the correct results, Newton's law $\mathbf{F} = m\mathbf{a}$ will be applied. The volume force on a charge distribution with density ρ is $\rho\mathbf{E}$. The acceleration is the time derivative of current divided by the charge density, ρ:

$$\rho\mathbf{E} = \rho_m \frac{\dot{\mathbf{J}}}{\rho}, \tag{26}$$

where ρ_m is the mass density of the charge. From eqs. (17), (20) and (21), putting all quantities in terms of E and rearranging,

$$\rho\mathbf{E} = \left(\frac{\rho_m}{\rho}\right)(-i\omega)\frac{(\alpha^2 - k^2)}{i\omega\mu_0}\mathbf{E}. \tag{27}$$

Cancelling the $i\omega$ and assuming that all the current will be carried by electrons, the expression for the electron–cloud charge density is

$$|\rho| = \left(\frac{m}{e}\right)\frac{(k^2-\alpha^2)}{\mu_0}, \quad k > \alpha, \tag{28}$$

where $m/e = 5.69\times10^{-12}\,Kg/coul$ is the mass–to–charge ratio of the electron. An important note is that eq.(28) is only valid for $k > \alpha$. The reason for this is that the sign of the charge–density distribution changes at $k = \alpha$ and for $k < \alpha$ the sign is inconsistent with negative charge carriers. This is an important factor in stability considerations[11].

Substituting for quantities to obtain the *r.m.s.* kinetic energy density,

$$\frac{1}{2}\rho_m\bar{v}^2 = \frac{1}{2}\left|\left(1-\frac{k^2}{\alpha^2}\right)\right|\mu_0|\mathbf{H}|^2. \tag{29}$$

Newton's law, in its simple–minded form applied above, is not strictly valid in this case. A constant charge and mass density was implicitly assumed from the outset if Newton's law is applied. The fact that a spatially uniform charge density results shows that the assumption leads to self–consistent results.

In the Appendix, the lagrangian formulation of the system will be investigated. It will be shown that the result in eq.(28) can be derived without assuming uniform charge density. It is valid under all conditions where overall charge neutrality is maintained and gas–dynamic effects are ignored.

Equations (28) and (29) are extremely important from a practical point of view. From eq.(28) it is evident that the amount of charge participating in the motion is determined only by α and the frequency. Furthermore, as proved in the Appendix, uniform volume charge density is a mandated, variationally correct solution of the lagrangian equations. An interesting point is that the free–charge density is independent of the amplitude of any electromagnetic field quantity. By assuming that an Ohm's law can be written in the form $\mathbf{J} = \sigma(\omega)\mathbf{E}$ for a non–radiating mode, then the conductivity $\sigma(\omega)$ is purely imaginary and can be written

$$\sigma(\omega) = \frac{i(k^2-\alpha^2)}{\omega\mu_0}, \tag{30}$$

so that at high frequencies a non–radiating mode becomes purely capacitive, with an asymptotic specific capacitance equal to $\epsilon_0 = 8.85pf/m$. The endeavor to drive modes at higher and higher frequencies might encounter a difficulty equivalent to trying to match a very small impedance. Very high field strengths might be required in small (order of ten wavelengths on–a–side) volumes. Laser radiation may be a practical solution as a high–power driver.

From eqs.(23) and (29) the sum of the kinetic and potential energies relevant to stability of the overall motion is *zero*, if surface stabilizing currents and vortical fluid motion are neglected. (Note, however, that the total energy of the system is positive.) As the frequency increases beyond the critical frequency, the amount of charge contributing to the current increases, the kinetic energy increases and the potential energy decreases for constant absolute-value field strengths. Beyond the critical wavenumber, the kinetic energy increases asymptotically like the square of the frequency.

For energy and momentum balance, a critical amount of free charge must be available. If not enough, or too much charge is present the conditions of the motion may not be met at the given frequency. In addition, the system cannot radiate from its interior because the **E** and **H** vectors are parallel everywhere internally.

Evidently, there is no restriction on the amplitude of the electromagnetic fields within the mode. Since the theory has been derived here for non-relativistic plasmas, there may be some eventual dependence on field amplitudes in ultra energetic modes. For very small field amplitudes, fluid and thermal forces might disrupt a non-radiating mode. When real systems are considered, massive electrons, which carry the current, cannot constrain their motion to the curved electric field lines. This introduces a contraint on the field amplitudes, such that the excursions of the electron positions from the field lines must be "small".

Field curvature leads to electromagnetic stresses that induce electronic "drift" across field lines. The interaction of the resultant pressure tensor on the free-electron gas with the overall fluid is very important. None of these complications, however, will pursued here.

The consequences of this approximation will be discussed in a sequel[11].

In what follows, it is assumed that a finite spherical region of a short-wavelength non-radiating mode is sustained by matching only the electromagnetic boundary conditions at a fixed radius. As was mentioned above, modes can be chosen such that the boundary conditions reduce to only tangential conditions on the velocities and the fields. This can translate into a requirement for surface currents to produce jumps in the electric and magnetic fields going from the force-free fields inside the spherical region to the free-space fields outside, and/or vortical fluid motion that can counteract the electromagnetic surface and internal stresses.

No mechanism will be described to fully invoke the needed conditions. They may be very difficult to reproduce physically. If such boundary conditions were not provided, however, the non-radiating mode would decay. The speed of this decay has not been calculated, but it seems intuitively reasonable that systems with large densities of kinetic energy would take longer to decay than those with small densities, if fluid-dynamic effects are ignored. Surface currents and vortical motion serve the purpose of counteracting the normal electromagnetic stresses, which would otherwise tend to force the excitation to expand.

Zero total energy of the free-electron cloud, above the critical frequency, means that the electronic motion is in neutral stability, but might show some rigidity in response to compressive forces. Unless "other" mechanical forces, such as internal vortical motion, are involved, a non-neutral, classical force-free system will self-destruct due to internal electrostatic forces. An ionized fluid or neutral plasma will barely be stable, if mechanisms are not present that can lower the overall energy. Surface currents contribute to stability by a "pinch" mechanism, which will effectively be a surface tension.

It must be emphasized that lowered energy due to either the pinch effect of currents or vortical motion are absolutely required for stability. What makes the present model attractive is the minimization of the body forces, induced by the interaction of the current with the force-free electromagnetic field. By themselves, these body forces could tend to severely limit the lifetime of an excitation.

The question of meeting the boundary conditions for the electromagnetic field also has to be addressed carefully. A non-radiating mode with a given harmonic internal current will generate external harmonic electric and magnetic fields that do not have the force-free form. It is a simple matter to say that the "correct" boundary conditions on the field, *i.e.*, those that do not permit radiation to escape from a truncated volume, can be reproduced with surface densities of electric and magnetic current[16]. There are difficulties with such a solution. The first difficulty is that a surface density of magnetic current may not be achievable. Such a current may be approximated, however, by "eddies" at the surface. The pure electric-current boundary condition, on the other hand, can be met by having a physically achievable surface-current distribution. In either case large accumulations of surface charge would be required to produce boundary currents. The stabilizing effects of such currents can also be achieved by setting the supporting fluid into a vortical flow, counteracting the electromagnetic stresses.

This point will be discussed further in a sequel, where it will be shown that the proper choice of an excitation radius for a given frequency can drastically affect the radiation boundary conditions and stability.

Another way of looking at the overall picture is to realize that in a harmonic field, the electric field *can* have a minimum. *I.e.*, the first derivatives of the potentials can all be zero, and their second derivatives can all be positive. Unlike the physical situation giving rise to Earnshaw's theorem[13], the components of the potentials and fields obey the Helmholtz equation instead of the Laplace equation. This allows the possibility of stabilization of charges and currents under the influence of only self-consistent harmonic electromagnetic fields and, perhaps, added vortical fluid motion.

In short, boundary conditions for truncated force-free modes that are met by surface currents and/or internal fluid motion may lead to plasma excitations that are mechanically stable and relatively long lived.

Puthoff has suggested[17] that by introducing the zero-point-field (ZPF), stability may be greatly enhanced. Conducting systems will experience a net outside pressure, analogous to the Casimir force, due to the exclusion of ZPF modes. The more conducting, at higher frequencies, the more ZPF modes will be excluded from the interior, and the higher the pressure differential. Additionally, the energy to stabilize non-radiating modes might be available from the higher frequencies in the ZPF, once the mode has been formed. If the above theory is to be believed, The ZPF may play an important role in stabilizing non-radiating modes.

Detailed calculations would have to be done to determine critical densities of charge versus size and frequency for all the quantities to come together exactly right. Further research would investigate how to inject charge into already-formed modes, and have them stabilized by higher frequencies that come from directed sources, or are present in the ZPF.

A final point relates to the initial construction of non-radiating modes. If these modes are stable, why are they not a common phenomenon? Actually, they might be quite common in Nature, but with short lifetimes and low energies. They also might be very small and hard to detect. Considering that the most logical place to look for charged, high-frequency non-radiating modes would be in powerful electrical discharges, such as found in lightning strokes, the surrounding phenomena might mask their occurance.

In the laboratory, strong microwave or laser (perhaps accompanied with D.C.) discharges, with possible extra-electron injection, might be conducive to non-radiative-mode production. Discharges in a highly-ionizable gas such as cesium or Penning mixtures, under the above conditions may enhance the effect. In any case, detailed calculations and experiments might have to be done to determine optimal conditions such as discharge field strengths, gas density, frequency, electrode size, etc.

The investigation of non-radiative-mode effects may be very fruitful in the future. For example: Rigid non-radiative composite "particles" containing large numbers of fusable ions may be compressible in strong quadrupole fields without undergoing chaotic turbulence-like instabilities, and have lifetimes consistent with producing copious thermonuclear reactions. Large-diameter modes, with significant lifetimes, might be capable of storing large amounts of energy, etc. The field seems to be wide open, hinging on simple experiments that may prove the basic principles.

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APPENDIX

The assumption that a self-consistent charge distribution (neutralized by charges of the opposite polarity) can carry harmonic current in a non-radiating mode leads to equations for the charge density at any point. The lagrangian equations for the system lead to a non-linear first-order vector partial differential equation for the charge density. It will be shown that an exact solution exists wherein the charge density is certainly uniform. Equation (28) will be found to be valid at all field strengths.

It will also be shown that the occurrance of such modes is consistent with a uniform mass density and rigid motion in a plasma or gas whose polarizability is a function of only the gas density.

The *r.m.s.* lagrangian density[18] for a current of massive particles in the presence of a non-radiating harmonic electromagnetic field in a neutral plasma is proportional to

$$\mathcal{L} = \frac{1}{2}\rho_m|\dot{\mathbf{r}}|^2 + \mathbf{J}\cdot\mathbf{A}^* + \frac{1}{2}(\epsilon_0|\mathbf{E}|^2 - \mu_0|\mathbf{H}|^2) - \rho_e\psi, \qquad (A1)$$

where ρ_m is the mass density of the current, $\mathbf{r}$ is the current field's spatial coordinates (the dot means velocities), ρ_e is the charge density, and the scalar electromagnetic potential, ψ, from eq.(20), is zero.

Using the relation $\mathbf{J} = \rho_e\dot{\mathbf{r}}$, with equations (1), (16), (18) and (20), the lagrangian equations for the motion can be written in the vector form, after a little algebra,

$$-i\omega\left[\left(\frac{m}{e}\right)\alpha\left(1-\frac{k^2}{\alpha^2}\right)+\frac{\rho_e\mu_0}{\alpha}\right]\mathbf{H}+\frac{1}{\rho_e^2}\left(1-\frac{k^2}{\alpha^2}\right)\left[\frac{3}{2}\left(\frac{m}{e}\right)(k^2-\alpha^2)-\mu_0\rho_e\right]|\mathbf{H}|^2\nabla\rho_e = 0, \qquad (A2)$$

where $m/e = 5.69\times 10^{-12}\,Kg/coul$ is the electron mass-to-charge ratio.

Equation (A2) immediately yields one solution by inspection:

$$\nabla\rho_e = 0, \qquad (A3a)$$

and

$$\left(\frac{m}{e}\right)\alpha\left(1-\frac{k^2}{\alpha^2}\right)+\frac{\rho_e\mu_0}{\alpha} = 0, \qquad (A3b)$$

which can be solved for the same ρ_e given by eq.(28):

$$|\rho_e| = \left(\frac{m}{e}\right)\frac{(k^2-\alpha^2)}{\mu_0}, \quad k > \alpha, \tag{A4}$$

It is not clear that eqs.(A3) represent a unique exact solution for $k^2 \neq \alpha^2$. The lagrangian equations may be integrable for non–uniform density distributions. Such a case might arise for a shape perturbed away from spherical. Under that assumption, it would also be expected that an entire mode might oscillate about an equilibrium with a frequency that might be much lower than the frequency of the electromagnetic excitation, ω. This possibility was not taken into account in setting up eq.(A2), where it was assumed that the partial time derivatives of the mass and/or charge densities are zero. The total time derivatives of these quantities that appear in the lagrangian equations arise only from the term $\dot{\mathbf{r}} \cdot \nabla \rho_e$.

It can now be shown that the above analysis is consistent with a null virial. The analysis below will be expanded somewhat in a subsequent note[11].

The particles in a high frequency non–radiating plasma mode will all be assumed to perform small–amplitude oscillations about equilibria, so that at any time the position, $\mathbf{s}(\mathbf{r},t)$, of any particle can be described by

$$\mathbf{s}(\mathbf{r},t) = \mathbf{r} + \frac{1}{\rho_e}\int_0^t \mathbf{J}(\mathbf{r})e^{-i\omega t'}dt', \tag{A5}$$

where $\mathbf{J}(\mathbf{r})e^{-i\omega t}$ is the time–dependent current at the space–fixed point $\mathbf{r}$. The force, $\mathbf{F}(\mathbf{r},t)$ acting on the particle comes from the electric field,

$$\mathbf{F}(\mathbf{r},t) = \rho_e \mathbf{E}(\mathbf{r})e^{-i\omega t}. \tag{A6}$$

The virial theorem[18] requires that

$$\overline{\int_V \mathbf{s}(\mathbf{r},t)\cdot\mathbf{F}^*(\mathbf{r},t)dV} = -\rho_m \overline{\int_V |\mathbf{v}|^2 dV}. \tag{A7}$$

By straightforward algebra it is evident that

$$\overline{\int_V \mathbf{s}(\mathbf{r},t)\cdot\mathbf{F}^*(\mathbf{r},t)dV} = \left(\frac{i}{\omega}\right)\int_V \mathbf{J}(\mathbf{r})\cdot\mathbf{E}^*(\mathbf{r})dV = \frac{\mu_0}{k^2-\alpha^2}\int_V |\mathbf{J}(\mathbf{r})|^2 dV, \tag{A8}$$

and

$$\rho_m \int_V \overline{|\mathbf{v}|}^2 dV = \frac{\rho_m}{{\rho_e}^2} \int_V |\mathbf{J}(\mathbf{r})|^2 dV, \tag{A9}$$

where the integrations are over the volume of the mode. For purely spherical modes the "natural" volume for integration might be from the origin to the first zero of the appropriate Bessel function associated with the mode.

Substitution of eqs.(A8) and (A9) into (A7) gives:

$$\frac{\mu_0}{k^2 - \alpha^2} \int_V |\mathbf{J}(\mathbf{r})|^2 dV = \frac{\rho_m}{{\rho_e}^2} \int_V |\mathbf{J}(\mathbf{r})|^2 dV. \tag{A10}$$

Cancelling the integral from both sides of eq.(A10), and solving for ρ_e, obtains

$$\rho_e = -\left(\frac{m}{e}\right) \frac{(k^2 - \alpha^2)}{\mu_0}, \tag{A11}$$

which is identical to the result assumed from a lagrangian formulation, as well as the simple analysis based on simple mechanics preceding eq.(28).

Note that eq.(A11) needs the relationship between the mass density of the electrons and the charge density, *i.e.*, $\rho_m/\rho_e = -m/e = -5.69\times10^{-12} Kg/coul$, where the minus sign arises because the charge density, ρ_e, is negative. The somewhat interesting insight from eq.(A11) is that unless the frequency is high enough, $k^2 > \alpha^2$, the required charge density for the electrons must be positive, clearly an inconsistency. Not only must the frequency be high enough to predict a consistent charge density for the current carriers (the much heavier ions are never considered capable of carrying the high–frequency current), a full virial analysis[11] shows that one of the conditions for stability is that $k^2 > \alpha^2$.

The question can also arise as to what the allowable gas density can be in a non–radiating mode.

Without taking compressible fluid dynamics and its interaction with the electromagnetic stresses into account, the density cannot be fully analyzed. Further work, to be reported on in more detail at a later time, seems to indicate that when vortical motion is introduced, a stable excitation can arise that has uniform density and entropy, with a gas pressure exactly equal to the gas pressure of the surrounding atmosphere at the time of its creation, and is stable against small perturbations of internal velocity[11].

In summary, the assumption that non–radiating modes can exist in a neutral plasma leads to the conclusion that the free–charge density entrained by the current can be a constant throughout the mode and dependent on the free–space electromagnetic wave number for a given modal α. It is also consistent with assuming that the only forces acting in the volume arise from fluid motions and the time–varying electromagnetic field, in a neutral plasma.

APPENDIX F

VIRIAL ANALYSIS OF HARMONIC FORCE-FREE EXCITATIONS

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ABSTRACT

Conditions for a null virial for electrons forming the current in a force-free harmonic mode are investigated. Above a critical frequency, this current lies in a potential well whose relative depth increases as the frequency increases. Conditions for greatest stability, ignoring electromagnetic stresses, are investigated. Fluid-dynamic effects are also ignored. Under appropriate approximations, the conditions for a null electron virial turn out to be identical to conditions for the motion of electrons oscillating along the electric field lines in the force-free electromagnetic field, above the critical frequency. The free-electron density must be uniform, and depends only on the size of an excitation and the frequency of the time-harmonic electromagnetic field. Assuming reasonable limits on physical parameters leads to upper bounds on the energy content of excitations in the megajoule range for a sphere with a 10cm radius. A negative potential well for electrons, above the critical frequency, allows the possibility of quantum effects, which are not investigated. Further areas of research are briefly discussed.

I. INTRODUCTION

If the vector field $\mathbf{H}(r)$ is a Beltrami, or force-free field[1,2] then

$$\nabla\times\mathbf{H} = \alpha\mathbf{H}, \tag{1}$$

where α can be a complex constant. Let $\mathbf{H}$ in eq.(1) be a magnetic field. If $\mathbf{E}$ is the accompanying electric field and $\mathbf{J}$ is the current, in otherwise free space Maxwell's equations are

$$\nabla\times\mathbf{H} = \epsilon_0\frac{\partial\mathbf{E}}{\partial t} + \mathbf{J}, \tag{2}$$

and

$$\nabla\times\mathbf{E} = -\mu_0\frac{\partial\mathbf{H}}{\partial t}, \tag{3}$$

where ϵ_0 and μ_0 are the permittivity and permeability of the vacuum.

Assuming a time–harmonic dependence $e^{-i\omega t}$ for all quantities, eqs. (1), (2) and (3) imply that

$$\mathbf{E} = \frac{i\omega\mu_0}{\alpha}\mathbf{H} - \nabla\psi, \tag{4}$$

where ψ is a scalar potential. The vector potential, **A**, is then

$$\mathbf{A} = \frac{\mu_0}{\alpha}\mathbf{H}. \tag{5}$$

From the Lorentz condition

$$\nabla\cdot\mathbf{A} + \epsilon_0\mu_0\frac{\partial\psi}{\partial t} = 0, \tag{6}$$

it follows that

$$\frac{\partial\psi}{\partial t} = -i\omega\psi = 0, \tag{7}$$

because $\nabla\cdot\mathbf{H} = 0$. The above choice for the vector potential was not unique. It was made to force the current to also have the form of a force–free field. To see this, solve for **J**:

$$\mathbf{J} = \alpha(1 - \frac{k^2}{\alpha^2})\mathbf{H}, \tag{8}$$

where k is the free–space wave number $\omega\sqrt{\epsilon_0\mu_0}$. From eq.(1) it follows that $\nabla\cdot\mathbf{J} = 0$, and that **J** is also a force–free mode if **H** is a force–free mode. From eqs.(4) and (8), **E** can also be written in terms of **J**,

$$\mathbf{E} = \frac{i\omega\mu_0}{\alpha^2 - k^2}\mathbf{J}. \tag{9}$$

It is important to note that eqs.(1–9) will not all necessarily hold if α has any spatial variation. It also can be shown that, starting with a force-free harmonic current that fills all of space, the electric and magnetic fields will be harmonic and also have the force-free form, if α is constant.

A feature of the above equations is that, unlike in a vacuum plane-wave, as the frequency increases, the field energy becomes more concentrated in the electric field. For $k \gg \alpha$ the direct contribution of the magnetic field energy to the total energy becomes negligible compared to the contribution from the electric field. Indeed, for $k > \alpha$, with critical frequency $\omega_{crit} = \alpha c$, the interaction energy of the current and the magnetic field is negative. In other words, the current finds itself in a potential well, whose relative depth increases with frequency, all other things being equal, after the point where $k = \alpha$. This energy well is intriguing, considering the quantum mechanical possibilities and stationary states that may arise. It was already shown[2] that, classically, the sum of the magnetic interaction energy of the free electrons and their kinetic energy is zero. Although quantum and relativisitc effects may alter the picture somewhat, they will not be considered here.

The above equations represent a valid solution of Maxwell's equations that can fill all of space with massless current. A more realistic physical situation realizing finite-size excitations and non-zero-mass current carriers (electrons) will now be considered. Toward that end it will be shown, using the virial, that the stability of finite-size harmonic excitations in a neutral fluid be optimized by adjusting the free-charge density. The optimum free-charge density will be shown to depend on the electromagnetic wavenumber, k, the size parameter of the excitation, α, and the condition $k > \alpha$.

A number of assumptions will be involved in the course of performing the analysis. The first assumption is that the background ions do not participate in the motion of the excitations, nor are they affected by any of the high-frequency fields. It is next assumed that all the temperatures are low, and the gas-dynamic pressures inside and outside the excitations are equal, so that thermodynamic and fluid-dynamic effects will be ignored. In other words, to first approximation the excitations to be studied will "float" in an ambient fluid with no net motion. The gas density and free-electron density will also be assumed uniform in space within the excitation. A major result of this note will be the determination of the optimal uniform free-charge density.

The next assumption involves an approximation. It will be assumed that the electrons, carrying all the current, oscillate harmonically along curved electric field lines. This approximation would be exact if the electrons had no mass or if the amplitudes of oscillation were truly infinitesimal and the temperature was low enough. As the frequency increases and the field amplitudes decrease, this approximation becomes more accurate. Excursion of electronic motion from the curved electric field lines forces the current to have non–force–free components and can thereby lead to losses via synchrotron radiation, so it is important that these excursions be minimized. In addition, internal pressure gradients can arise if the field lines are curved. These pressure gradients, proportional to the square of the field intensities, may become so large as to dominate any stability analysis. Nevertheless, such effects will be ignored in this note.

At the outer surface of the excitation, it will be assumed that the jumps in the electric and magnetic harmonic fields, from large values inside to zero outside, sustain equivalent surface currents [2,3]. Since these harmonic surface currents may be very large, to accomodate their size in a thin layer without large excursions of the electrons might require very large concentrations of (neutralized) free charge near the surface, and therefore large concentrations of mass at the surface. These will be assumed to exist. Surface currents are invoked to prevent radiation losses. Subsequent work will deal with surface radiation in a more realistic manner. To do so here would not be appropriate.

Conditions under which all the above assumptions can be met in a reproducible experimental situation will not be handled in this note, but none of the assumptions appear to violate any physical laws or introduce contradictions. Subsequent work is being directed toward eliminating the need for surface currents, and using vortical fluid modes to counter electromagnetic stresses.

II. VIRIAL CONSIDERATIONS

Consider the mean–square moment of inertia, I, of the electrons in a finite–volume excitation, and its time derivatives:

$$I = \int_V \rho_m \mathbf{r}^* \cdot \mathbf{r}\, dV, \tag{10}$$

$$\frac{dI}{dt} = \dot{I} = \int_V \rho_m \left[\mathbf{r}^* \cdot \dot{\mathbf{r}} + \dot{\mathbf{r}}^* \cdot \mathbf{r}\right] dV, \tag{11a}$$

and

$$\frac{d^2 I}{dt^2} = \ddot{I} = \int_V \rho_m \left[\mathbf{r}^* \cdot \ddot{\mathbf{r}} + 2|\dot{\mathbf{r}}|^2 + \ddot{\mathbf{r}}^* \cdot \mathbf{r}\right] dV, \tag{11b}$$

where ρ_m is the (spatially uniform) mass density of the electrons, $\mathbf{r}$ is the complex position vector associated with an "infinitesimal" blob of electrons, and the integration is over the volume of the excitation. Strictly speaking, the integrations in eqs.(10) and (11) should be sums over individual electrons, but their densities are assumed high enough for the integral to be an appropriate approximation. The quantity $\dot{I}$ is will be called the "virial of the electronic motion"(4,5,6,7). The exclusion of the electromagnetic field from the above integrals will be discussed below.

The quantity $\mathbf{r} \equiv \mathbf{r}(x, y, z, t)$ is the instantaneous position of an infinitesimal volume whose mean position is at the fixed point $\mathbf{r}_0 = \mathbf{r}(x, y, z, t = t_0)$, and oscillates with the same frequency as the applied harmonic field. By definition, the velocity, $\mathbf{v}$, is proportional to $\mathbf{J}$:

$$\mathbf{v} = \mathbf{J}/\rho_e, \tag{12}$$

where ρ_e is the (spatially uniform) charge density of free electrons. The charge density, ρ_e, is simply related to the mass density, ρ_m by $\rho_e = (e/m)\rho_m$, where $(e/m) = 1.759 \times 10^{11}$ $coul/kg$. Since all the quantities are assumed to be harmonic in time, so that $\mathbf{J}(x, y, z, t) \equiv \mathbf{J}(x, y, z)e^{-i\omega t}$, then $\mathbf{r}$ can be written

$$\mathbf{r} = \mathbf{r}_0 + \frac{\mathbf{J}}{\rho_e}\int e^{-i\omega t}dt = \mathbf{r}_0 + \left(\frac{i}{\omega\rho_e}\right)\mathbf{J}e^{-i\omega(t-t_0)}. \tag{13}$$

A sufficient condition for self-confinement is that the time-integrated average of $\dot{I}$ is zero. This condition implies that the moment of inertia of the electron cloud can assume its original value any number of times, and cannot become uniformly unbounded as time increases. Instead of working directly with $\dot{I}$, however, investigation will be made into conditions under which the time-integrated average of $\ddot{I}$ is zero. This condition is more subtle. A necessary condition for the time-integrated average of $\dot{I}$ to be zero is that the time-integrated average of $\ddot{I}$ be zero. What will be shown is that if the uniform charge density has exactly the right size, and the frequency is above a critical value, the time-integrated average of $\ddot{I}$ is exactly zero. A larger value for the charge density would drive the time-integrated average of $\ddot{I}$ negative, leading to collapse and, perhaps, instabilities. This is because the exact value of uniform charge density derived for a null is the same required from classical mechanics to simultaneously satisfy Maxwell's equations and the laws of inertia, with finite-mass electrons in harmonic motion along field lines. For this reason, the conditions to be derived will be considered sufficient for self containment. Gerjouy and Stabler(6) also discuss this point.

Substituting the time-dependent $\mathbf{r}$ from eq.(13) into eq.(11b) and taking the time average automatically gives zero, because of the assumption that the charges oscillate with fixed centers implies confinement. It is far more interesting to substitute for the terms in $\ddot{I}$, assuming harmonic motion for the charges, to calculate conditions for stability. Denote the time-integrated average of a quantity by a bar, *i.e.*,

$$\overline{Q} = \lim_{T\to\infty}\left[\int_0^T Q dt\right]. \tag{14}$$

To obtain new information from eqs.(11), note that the terms in the integrand containing $\ddot{\mathbf{r}}$, the acceleration, are related to the volume force on the electrons[2],

$$\rho_m \ddot{\mathbf{r}} = \rho_e \mathbf{E}(\mathbf{r}) e^{-i\omega t}. \tag{15a}$$

The term with $|\dot{\mathbf{r}}|^2$ can be written

$$\rho_m |\dot{\mathbf{r}}|^2 = \frac{\rho_m}{\rho_e^2}|\mathbf{J}|^2. \tag{15b}$$

From eqs.(9), (11), (13), (14), and (15), with a little algebra,

$$\overline{\frac{d^2 I}{dt^2}} = \left[\frac{\rho_m}{\rho_e^2} + \frac{\mu_0}{\alpha^2 - k^2}\right]\int_V |\mathbf{J}|^2 dV. \tag{16}$$

In order for the right hand side of eq.(16) to be zero the following conditions must hold:

$$k > \alpha, \tag{17a}$$

and

$$|\rho_e| = \frac{(m/e)}{\mu_0}\left(k^2 - \alpha^2\right), \tag{17b}$$

where $(m/e) = 5.69\times10^{-12} kg/coul$ is the mass-to-charge ratio for the electron. Equation (17b) was derived previously[1] using a lagrangian formulation. From variational considerations introduced in classical mechanics[2,8], the result in eq.(17b) arises from only assuming that the quantities **E**, **H**, and **J**, have the force-free form, and that the free electrons carrying the current have non-zero mass. Uniform charge density is not an a-priori assumption in the lagrangian calculation.

A diversion is needed at this point to discuss the effects of the harmonic electromagnetic field assumed in the present model.

The usual analysis involving the virial theorem[4,5,6,7] includes the electromagnetic energy density in the integrand in eq.(10). The assumptions made here are that the electronic velocities are non-relativistic and the harmonic fields $\mathbf{E}$ and $\mathbf{H}$ are non-zero inside the excitation and zero outside, which is equivalent to effective electric and magnetic currents respectively equal to

$$\mathbf{J}_{eff} = -\hat{\mathbf{n}} \times \mathbf{H}, \tag{18a}$$

and

$$\mathbf{M}_{eff} = \hat{\mathbf{n}} \times \mathbf{E}, \tag{18b}$$

where $\hat{\mathbf{n}}$ is the outward-pointing normal to a sharply defined excitation boundary.

The first thing to realize is that if such currents existed, they would exactly counteract the electromagnetic stresses at the excitation boundary. The reasoning is that internally, the electromagnetic field, terminated by the currents in eqs.(18a–b), seems to stretch to infinity with a zero Poynting vector ($\mathbf{E} \times \mathbf{H} = 0$). No energy is lost to the "outside" if the currents are maintained. For the lowest $l = 1$ spherical mode[2], at $\alpha R_0 = 4.493\ldots$, both $\mathbf{E}$ and $\mathbf{H}$ are tangential to the boundary and therefore the stresses due to the internal fields are normal to the surface. The occurance of the currents in eqs.(18a–b) can be thought of as representing pinches, which barely contain the normal stresses due to the electromagnetic field trapped inside. Viewed in this manner, the surface currents represent a lowering of the energy of the overall system.

It is understood that surface currents, while countering the electromagnetic surface stresses, will not null the pressure gradients that arise from variations in the fields, and will tend to produce internal motion. No mechanism can produce ideal surface currents. The present work is primarily aimed at finding the most stable configuration of the charge distribution when electromagnetic stresses and radiation are ignored. Ongoing work appears to indicate that judicious choices of k and α can null the outgoing radiation field and vortical fluid modes can null the electromagnetic stresses. Invoking idealized surface electric and magnetic currents will merely be an a dodge that is temporarily needed to obtain an important result concerning the charge density. These surface currents will not be discussed any further.

Returning to the results in eqs.(17), it might first appear that if the free-charge density were increased beyond the value prescribed in (17b), stability would increase because $\bar{\bar{I}}$, in eq.(16) would decrease. If this were to happen, the electromagnetic fields and mechanical trajectories would no longer follow force-free field lines and would no longer be consistent with the model. From the simplest considerations, is was shown that small perturbations in fluid density did not grow[3]. It will be assumed that if the conditions already discussed are met, with the currents lying in a negative-potential well, the excitations under study are stable and only slowly evanescent.

It is important now to estimate how close a real physical system can be made to approach the conditions for force-free-field approximations on the vector fields **E**, **H**, and **J**. If electrons were massless, the problem would be far simpler, since they would experience no centripedal accelerations and would conform to electric field lines like currents in wires. In reality, the electrons cannot confine their motion to be strictly along the curved field lines of a force-free field.

The assumption that the electrons undergo simple harmonic motion along electric field lines becomes more accurate as the frequency increases and the electric field decreases. This can be seen from eq.(13), where the amplitude of the harmonic motion is $\mathbf{J}/(\rho_e\omega)$. Substituting for **J** in terms of **E**, a condition for small-amplitude oscillations can be written

$$\frac{|\mathbf{E}|_{max}}{(\frac{m}{e})\omega^2} \ll \frac{4.493}{\alpha}, \tag{19}$$

where $|\mathbf{E}|_{max}$ is the maximum absolute amplitude of the harmonic electric field within the excitation. Using R.M.K.S. units and a radius, $R_0 = 4.493/\alpha$ eq.(19) becomes

$$|\mathbf{E}|_{max} = 2.02\times 10^7 \zeta(\frac{R_0}{\lambda^2}), \tag{20}$$

where ζ is an appropriately small factor that satisfies eq.(19), and λ is the free-space wavelength of the harmonic wave.

The assumption will be that the smaller the value of ζ, the longer the lifetime of the excitation, all other things being equal. At this time there are no estimates for liftimes, but some reasonable values for the parameters might be R_0 = 10cm, λ = 1.0mm, and $\zeta = .01$. Under these conditions a 10cm radius spherical excitation will have a total energy density on the order of 10^6 Joules. This value is larger than average reported energies of ball lightning[8], and these excitations may represent an electromagnetically excited fluid state that could serve as a model for ball-lightning-like phenomena.

The total energy in any spherical excitation will be proportional to the factor

$$F = \left(\frac{R_0}{\lambda}\right)^4 R_0. \tag{21}$$

This implies that for a fixed ratio of the radius to the wavelength, for the same value of ζ, the energy content of a spherical excitation is linearly proportional to its radius. Since the volume is linearly proportional to the cube of the radius, the smaller the sphere, the higher the energy density, under the above fixed ratio of radius to wavelength. For the numerical example already given, if the radius were shrunk to one millimeter, the total energy content would be about 15,000 Joules. The wavelength of the harmonic wave would be $10\mu m$; and the energy equivalent to a fifty-watt lightbulb burning for five minutes.

III. CONCLUSIONS

A model for electromagnetic/ionized-fluid excitations predicts the self containment of high-energy modes in small volumes. The virial analysis yields the same results as self-consistent solution of Maxwell's equations with current solved with a lagrangian formulation, allowing the current to be carried by non-zero-mass electrons. The important result is that, excluding electromagnetic stess effects, the most stable configuration must have a uniform free-charge density that is determined only by the size of the excitation and the frequency of the harmonic electromagnetic field.

The artifice of idealized surface currents was used to isolate the internal fields from the outside and prevent radiation. Internal electromagnetic volume stresses were ignored in the analysis. Ongoing work is indicating that if k and α are chosen correctly, then the outgoing radiation field can be nullified. Indications are also that the internal electromagnetic stresses can be cancelled by vortical fluid modes.

The excitations will have finite lifetimes, but may persist for a considerable amount of time if losses are not large. Upper limits were set on the maximum electric field strength within an excitation, to confine the electronic motion to electric field lines. It is expected that there may be a tradeoff between maximum field strength, i.e. maximum energy density, frequency, and lifetime. Larger field strengths and lower frequencies imply larger excursions of the electrons from electric field lines, leading to volume losses as well as surface losses, with synchrotron radiation probably being very important. Vortical modes may greatly affect lifetimes.

Using reasonable input numbers, maximum field strengths and subsequent stored energies are very high, making the production of such excitations look very attractive. It is not clear, however, what a production facility for harmonic force-free excitations might look like, since the internal vortical structure of these excitations is still under investigation. "Fireballs" reported in the literature[9] may be candidates for investigating their internal fields to determine whether they fit the harmonic force-free model.

Theoretical areas still need to be researched. Radiation rates due to non-linear fluid effects to be calculated. These rates might set an upper bound to the lifetime of a classical excitation. Although a rough stability analysis was done in a previous paper[2], a more rigorous analysis involving perturbations of the current remains to be done. This would involve the calculation of restoring forces, if any, that would arise from a physical deflection, as well as other mechanisms that would return the excitation to its original state after a perturbation. Energy losses due to perturbations of the conditions from ideal should be calculated. Much of this depends on the work being done in connection with electromagnetic stresses influencing fluid motion.

The idea of applying quantum mechanics to these excitations can be investigated. As the size of the excitations shrinks, quantum effects may become very important, according to Nardi[10]. If a self-consistent potential is formed by the currents in an excitation, there may be a collective electronic wavefunction that exactly confines its flux-field vectors to the streamlines of a force-free electric field. Quantized in this manner, lifetimes of states may be much larger than would be expected from purely classical considerations. Maximum energy levels may also be much larger. The form of the calculations would probably resemble Hartree-Fock atomic procedures, and would ultimately have to include fluid-dynamic effects.

Further investigations seem to hold great promise.

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APPENDIX G

SURFACE RADIATION FROM IDEAL FORCE–FREE HARMONIC ELECTROMAGNETIC EXCITATIONS

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ABSTRACT

Spherically symmetric force–free harmonic solutions to Maxwell's equations may be truncated at a radius where the fields are tangential to a sphere. Equivalent currents, which satisfy the jump boundary conditions between the inner fields and the vacuum, maintain the excitation as though the inner solution extended to infinity and prevent radiation flow. Since the Poynting vector is identically zero inside the spherical boundary, no energy can escape from the inner volume even if the surface currents were not present. The surface boundary fields can radiate away the energy, however, which is exactly cancelled by the radiation from the surface currents. It will be shown that for any given frequency an infinite number of choices for the force–free parameter are possible such that the far–field radiation from the surface currents is zero. Each of these choices implies a different spherical truncation radius. For large wavenumbers, successive possible choices of truncation radii differ by approximately a half wavelength. For any of these choices, no surface currents are needed to prevent steady–state radiation.

I. INTRODUCTION

It has been shown that Maxwell's equations for harmonic electromagnetic waves, with angular frequency ω, admit solutions having the electric and magnetic fields, **E** and **H**, as well as the current, **J**, in the force–free form[1]. A field, **F**, has the force–free form if

$$\nabla \times \mathbf{F} = \alpha \mathbf{F}, \tag{1}$$

where α, the force–free constant, is not a function of either space or time.

Consider the following solution to eq.(1) in spherical coordinates (r, θ, ϕ)

$$\mathbf{F}_l(r,\theta,\phi) = (l+1)\cos\theta\sin^{l-1}\theta\frac{j_l(\alpha r)}{\alpha r}\hat{\mathbf{r}} - \left[j_{l-1}(\alpha r) - l\frac{j_l(\alpha r)}{\alpha r}\right]\sin^l\theta\hat{\theta} \pm j_l(\alpha r)\sin^l\theta\hat{\phi}, \quad (2)$$

where l is any integer, $l > 0$, j_l is the familiar spherical Bessel function, and the vectors $\hat{\mathbf{r}}$, $\hat{\theta}$, and $\hat{\phi}$ are the familiar right-handed 3–D polar coordinate unit vectors. This is not the only solution to eq.(1), but it has interesting properties(1). To be specific, let $l = 1$ and α be real. At $\alpha r = 4.493\ldots$, $j_1(\alpha r) = 0$ and the field becomes tangential to a spherical surface, with radius

$$R_0 = 4.493\ldots/\alpha. \quad (3)$$

Both E and H can be written as vector fields proportional to F in eq.(2), multiplied by the time factor $\exp(-i\omega t)$. This time factor will henceforth be suppressed. If the solution to Maxwell's equations is truncated at $r = R_0$, with zero fields for $r \geq R_0$, the resultant field will continue to be a solution throughout all of space if currents are made to flow on the surface of the sphere at $r = R_0$. The equivalent of an electric current, $\mathbf{J}_s$, and a magnetic current, $\mathbf{M}_s$ must flow to satisfy the jump conditions in the fields at $r = R_0$. These currents are defined to be

$$\mathbf{J}_s = -\hat{\mathbf{n}}\times\mathbf{H}_s, \quad (4)$$

and

$$\mathbf{M}_s = \hat{\mathbf{n}}\times\mathbf{E}_s, \quad (5)$$

where $\mathbf{E}_s$ and H_s are the surface electric and magnetic fields and $\hat{\mathbf{n}}$ is the outward normal at $r = R_0$(2). The physical significance of these currents is discussed elsewhere(3).

From eqs.(2–5), the surface currents, $\mathbf{J}_s$ and $\mathbf{M}_s$, will be proportional to

$$\mathbf{J}_{surf} = j_0(\alpha R_0)\sin\theta\hat{\phi}, \quad (6)$$

where $j_0(\alpha R_0) = -0.2172\ldots$

Under ideal situations no radiation can escape from the spherical surface if the currents $\mathbf{J}_s$ and $\mathbf{M}_s$ are present. The radiation from these currents exactly cancels the radiation from the surface of the excitation. The reason is that the Poynting vector,

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}, \tag{7}$$

is identically equal to zero everywhere for $r < R_0$.

In the next section, the far–field radiation from the surface currents will be analyzed for the $l = 1$ case. It will be shown that, for appropriate values of R_0, the surface currents will not radiate. The conclusion will be that under certain conditions, surface currents are not needed to prevent steady–state radiation from an excitation.

II. NULL SURFACE RADIATION CONDITIONS

Defining the wavenumber $k = \omega/c$, with c being the speed of light, the vector potential in the far field for a volume electric current, $\mathbf{J}(\mathbf{r})$, approaches the expression[2]

$$\mathbf{A}(\hat{\mathbf{r}}) = \frac{e^{ikR}}{4\pi R} \int_v \mathbf{J}(\mathbf{r}') e^{-ik|\mathbf{r}'|\cos\gamma} dV', \tag{8}$$

where $\hat{\mathbf{r}}$ is a unit vector in the direction of the point in the far field, R is the distance of the far–field point from an arbitrarily chosen center of the current distribution, $\mathbf{r}'$ is the vector from the center of the distribution to the integration point, and

$$|\mathbf{r}'| \cos\gamma = \hat{\mathbf{r}} \cdot \mathbf{r}'. \tag{9}$$

In eq.(8) the volume integral extends over the entire current distribution volume, V.

When the current distribution collapses to a surface distribution like that in eq.(6), the integral in eq.(8) becomes proportional to the integral

$$\mathbf{I} = \oint_S \hat{\phi} \sin\theta' e^{-ikR_0 \cos\gamma} d\Omega', \tag{10}$$

where the distribution is centered at $\mathbf{r}' = 0$, and the integration is over the surface of the sphere $|\mathbf{r}'| = R_0$.

The integral in eq.(8) can be evaluated in closed form by expanding all quantities in the integrand in terms of spherical harmonics[4] $Y_{l,m}(\Omega')$, in the primed coordinates, and $Y_{l,m}(\Omega)$, in the unprimed (far–field) coordinates. First convert the spherical vector $\hat{\phi}'$ into a sum of unit Cartesian vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ in the x and y directions:

$$\hat{\phi}' = -\hat{\mathbf{x}} \sin\phi' + \hat{\mathbf{y}} \cos\phi'. \tag{11}$$

Next note that

$$\sin\theta' \sin\phi' = i\sqrt{\frac{8\pi}{12}}[Y_{1,1}(\Omega') + Y_{1,-1}(\Omega')], \tag{12}$$

and

$$\sin\theta' \cos\phi' = \sqrt{\frac{8\pi}{12}}[-Y_{1,1}(\Omega') + Y_{1,-1}(\Omega')], \tag{13}$$

The exponential in the integral has the expansion[4]

$$e^{-ikR_0 \cos\gamma} = 4\pi \sum_{l=0}^{\infty} i^{-l} j_l(kR_0) \sum_{m=-l}^{l} Y^*_{l,m}(\Omega') Y_{l,m}(\Omega), \tag{14}$$

where the asterisk (*) denotes complex conjugation. Using the orthogonality property of the spherical harmonics,

$$\oint_S Y^*_{l,m}(\Omega') Y_{l',m'}(\Omega') d\Omega' = \delta_{l,l'} \delta_{m,m'}, \tag{15}$$

a little algebra shows that, finally,

$$\mathbf{I} = -4\pi j_1(kR_0)\hat{\phi}, \tag{16}$$

where $\hat{\phi}$ is the unit spherical vector in the far-field.

If the product kR_0 is one of the zeroes of $j_1(kR_0)$ then the vector potential for outgoing waves due to the $l = 1$ surface electric current goes to zero more quickly than the inverse of the distance in the far-field. This current will therefore not radiate[2]. An exactly analogous argument shows that the $l = 1$ magnetic surface current will not radiate either, if the same conditions are met.

In the next section, proof will be given for the general result that the outgoing radiation field from $l > 1$ cases can also be nullified.

III. GENERAL CONDITIONS FOR NULL RADIATION

In the $l = 1$ case the integrations needed to evaluate the far field simplified. For any other allowed value of l, *i.e.*, $l > 1$, the integrals involve sums of terms of the type $j_L(kR_0)Y_{L,\pm 1}(\Omega)$, with L odd. Only for $l = 1$ is there a single term in the sum. For higher values of l, then, it is not clear whether the far-field radiation may be only be minimized, or nulled in any one direction, but cannot rulled out completely in all directions. The azimuthally symmetric $l = 1$ mode in eq.(2), and modes derived from it by rotations and subsequent linear superposition, are not the only ones for which the surface radiation of a truncated force-free harmonic excitation can be exactly nullified by a judicious choice of the force-free constant α and the free-space wavenumber k without invoking surface currents. This will now be shown by comparing the fields in the above excitations with the internal electromagnetic oscillations of a resonant spherical cavity[5]

The strategy will now be to choose excitations that obey eq.(1), choosing R_0 and k such that

$$j_l(\alpha R_0) = 0, \tag{17}$$

and

$$j_l(kR_0) = 0. \tag{18}$$

Define, now, the complementary vector fields

$$\mathbf{M}_l = j_l(kr)\sin^l\theta\hat{\phi}. \tag{19}$$

and

$$\mathbf{N}_l = (l+1)\cos\theta\sin^{l-1}\theta\frac{j_l(kr)}{kr}\hat{\mathbf{r}} - [j_{l-1}(kr) - l\frac{j_l(kr)}{kr}]\sin^l\theta\hat{\theta}. \tag{20}$$

A solution of Maxwell's equations in a perfectly conducting hollow sphere of radius $r = R_0$ is

$$\mathbf{E} = \mathbf{M}_l, \quad \mathbf{H} = -i\frac{k}{\omega\mu_0}\mathbf{N}_l. \tag{21}$$

If eq.(18) holds, then eqs.(21) represent a resonant excitation of a spherical cavity. The fields represented by eq.(21) are confined to to the region $r \leq R_0$. The electric field at $r = R_0$ is zero. The magnetic field induces a current $\mathbf{I} = -\hat{\mathbf{n}}\times\mathbf{H}$ at $r = R_0$, where $\hat{\mathbf{n}}$ is the outward-pointing normal to the spherical surface. This current, considered by itself in unbounded free space, is the only source of the electromagnetic field both inside and outside the spherical cavity. As such, this current cannot radiate into the vacuum region $r > R_0$, so that the electromagnetic field external to **I** will remain zero if it is initially zero.

Now note that at $r = R_0$

$$\mathbf{F}_l(R_0, \theta, \phi) = \mathbf{N}_l(R_0, \theta, \phi). \tag{22}$$

Consider first the magnetic field in a force-free excitation where eqs.(17) and (18) hold. Radiation from the surface at $r = R_0$, due only to the magnetic field, will entirely depend on the contribution from the term $\mathbf{I}'(\theta, \phi) = \hat{\mathbf{n}}\times\mathbf{H}(R_0, \theta, \phi)$, where $\hat{\mathbf{n}}$ is the outward normal to the sphere at (R_0, θ, ϕ). Since $\hat{\mathbf{n}}\cdot\mathbf{H}(R_0, \theta, \phi) = 0$, it does not contribute[5]. From the above argument involving **I**, if the external radiation field due to the surface contribution from the magnetic component of a force-free excitation is initially zero, it will remain zero. Exactly analogous arguments apply to the electric field contributions to the external radiation.

IV. DISCUSSION

The conclusion to be drawn is that ideal spherical electromagnetic force-free excitations can exist with no external radiation field for all values of l in eq.(2). Equation (2) has no azimuthal variation. It is not difficult to generalize $\mathbf{M}_l$ and N_l in eq.s(19) and (20), and thereby $\mathbf{F}_l$ using the formulas on page 416 of Stratton's book[5]. None of the arguments change when azimuthal variation is introduced, so that all the results derived for modes with no azimuthal dependence carry over to the modes having any consistent dependence on the angle ϕ.

For reasons of simplicity, all comments will henceforth be directed toward the $l = 1$ case having no azimuthal dependence. This is the mode with the lowest symmetry and least complicated geometry. The surface fields topologically are eqivalent to inviscid-fluid flow-lines over a rigid sphere.

Since the mode was truncated at the first zero of the $j_1(\alpha r)$ function, it is easy to see that a necessary condition for nullifying surface radiation is

$$k \geq \alpha. \tag{21}$$

Below a certain frequency, for a given α, the surface radiative losses cannot be nulled (unless the $k = 0$, or zero-frequency case, is allowed). It has already been shown that for $k > \alpha$ the interaction energy of the volume harmonic current and the oscillating magnetic field has a negative value everywhere within the excitation[3]. In other words, the current lies in a negative potential well whose relative depth increases with frequency for $k > \alpha$. This potential is positive for $k < \alpha$. Although it has not yet been tested, this presumably means that the $k > \alpha$ case is more stable than the $k < \alpha$ case. It is interesting that surface radiation can only be nulled in what is presumably the stable region.

Outside the excitation there are null fields.

When non-zero-mass electrons are introduced as harmonic current carriers, as the actual physical situation demands, certain energy-loss mechanisms can be identified[3]. It is evident that if force-free harmonic excitations can be manufactured in a form that resembles spherical plasmoids, it is most likely that α and k will have to be tailored in the manner described in this note, to prevent radiation losses.

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