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6. AUTHOR(S)
Michael S. Pilant and
William Rundell

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Texas A&M Research Foundation
Box 3578
College Station, Texas 77843

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R.W. Prestridge
ONR Resident Representative
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13. ABSTRACT *(Maximum 200 words)*

Our research concerns undetermined coefficient problems in partial differential equations, in particular those problems where the unknown coefficients depend only on the dependent variables. The problems modeled by these equations are related to the determination of unknown physical laws or relationships. The nonlinear terms which we seek to recover in our model problems correspond to material properties that have physical significance. These include temperature dependent specific heats, conductivities, and reaction terms.

The main analytical tool is the Fixed Point Projection method, which the investigators have developed for use in elliptic and parabolic inverse problems. This method involves projecting the underlying differential operator onto that subset of the domain where the overposed data is given, and reformulating the inverse problem as an equivalent fixed point problem, which is then solved by iteration. Analytical as well as numerical results have been obtained by the investigators, and the FPP method is currently being extended to hyperbolic inverse problems.

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William Rundell and Michael Pilant

Principal Investigators

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Introduction.

Our research efforts are concerned with undetermined coefficient problems in partial differential equations, in particular those problems where the unknown coefficients depend *only* on the dependent variables. The problems modeled by these equations are related to the determination of unknown physical laws or relationships. The nonlinear terms which we seek to recover in our model problems correspond to material properties that have physical significance; temperature dependent specific heats, conductivities, reaction terms, to name a few. Examples of such problems are – the determination of an unknown reaction term $f(\cdot)$ in $u_t - u_{xx} = f(u)$, or the conductivity $k(\cdot)$ in the equation $\nabla . k(u)\nabla u = 0$. We seek to determine these functions by giving *only* overposed boundary data.

This type of problem is distinct from those that involve media with unknown inhomogeneities; that is, the differential equations contain an unknown coefficient that depends on the independent spatial variable. In a given physical problem both situations may occur, that is, the unknowns have both spatial as well temperature dependence. This is a considerably more difficult problem and has received little attention in its full generality; instead both the limiting cases of dependence on a single one of these variables have received the vast majority of the research efforts. For those cases where the media is isotropic, the assumption that the unknowns depend only on the independent variable may be a very reasonable one. From a mathematical standpoint the two types of problem pose different mathematical difficulties. Both types of inverse problem leads to nonlinear equations, but in the case when the unknown coefficient depends on the spatial variables the direct problem is frequently linear. In contrast, when the coefficient is a function of the dependent variable, the direct problem is nonlinear.

Examples:

(I) The recovery of the one (or more than one) of the functions c, k, f or h from the parabolic equation

$$c(u)u_t - \nabla . k(u)\nabla u = f(u) + \gamma(x,t) \qquad \text{in } \Omega \times [0,T] \tag{1}$$

with initial data $u(x,0) = u_0$, and boundary data

$$\frac{\partial u}{\partial \nu} = h(u) + \alpha(x,t) \qquad x \in \partial\Omega \tag{2}$$

We prescribe as overposed data the value of u at a specified point x_0 on $\partial\Omega$. Of course, if more then one function is unknown additional overposed data must be prescribed. This can be either the value of u at other boundary points, or the value of u at the same point for different experiments, that is, for values of the known functions γ, α or u_0. The functions c, k, f and h represent the specific heat, conductivity, reaction term, and radiation boundary condition respectively. An alternative type of overposed condition would be to prescribe the total energy of the system as a function of time. This leads to recovery problems with non-local boundary conditions.

(II) The recovery of one of the functions k or f in

$$-\nabla . k(u)\nabla u = f(u) + \gamma(x,y) \qquad \text{in } \Omega \tag{3}$$

given Neumann data on $\partial\Omega$, plus the value of $u(x) = \theta(x)$ on some subset of the boundary as overposed data. Once again, if both k and f are unknown then we should prescribe additional overposed data.

(III) The recovery of the functions $f_1(\phi)$ and $f_2(\phi)$ in the parabolic system

$$u_t - D_1 \Delta u = \alpha u + f_1(\phi)$$
$$v_t - D_2 \Delta v = \beta v + f_2(\phi)$$

where ϕ is a known function of u and v and represents the interaction between these functions. A typical example might be $\phi = uv$.

(IV) If it is known that the coefficient may depend on the gradient of u as well u itself,

$$u_t - u_{xx} = f(u, u_x) \qquad \text{in } \Omega \times [0, T] \tag{4}$$

then we might try to use a finite expansion of the function in the variable ∇u, and look for an approximation to f in one of the forms $f(u, u_x) = f_1(u) + f_2(u)u_x$ or $f(u, u_x) = \{f_1(u) + f_2(u)u_x\}/\{1 + f_3(u)u_x\}$. The first is a linear approximation in the variable u_x, the second a rational function approximation. In either case we must recover a vector of unknowns $(f_1(u),\ f_2(u), \dots)$

Our interest in these problems is broad. We seek to identify the correct types of data under which the problem has a solution. A uniqueness result indicates sufficiency of data, an existence result indicates that the problem is not overdetermined. The extent to which the solution depends continuously on the data is clearly important information for the implementation of any practical scheme. In addition to these questions we are interested in constructibility of the solution and in the development of algorithms that lead to efficient numerical recovery.

Unknown coefficient problems are notorious for several reasons; they tend to be difficult mathematical questions, they are often severely ill-posed, and when a solution is found it tends to be ad hoc. While not much can be done about the first two of these, numerous attempts have been made to resolve the third. The lack of success in this endeavor, even for seemingly restricted cases (parabolic equations with a single unknown, $u-$dependent coefficient) is well known. A major theme at the recent Arcata conference* was the discussion of various attempts at providing "general" solution techniques for such problems.

Another feature of these inverse problems, due in part to their intrinsic nonlinearity, is that existence of a solution does not imply uniqueness or vice-versa. Indeed there are many problems for which uniqueness has been shown but existence questions remain completely open. Even in those cases where there has been successes, methods yielding existence are often entirely different form those that gave uniqueness, and the time span between the proofs of uniqueness and that of existence can be considerable. It was 20 years in the case of the recovery of a potential in an ordinary differential equation from spectral data.

The research proposed and our current endeavors, focus around the examining of several approaches that have been used with success in other areas and attempt to apply them to the types of problem mentioned in the examples above. Each method suggests a possible algorithm for the inverse problem, our main interset is to determine those that are feasible, attempt to prove convergence of the method, and if possible compare competing methods for computational efficiency.

* *Inverse problems in Partial Differential Equations* AMS/SIAM/IMS Summer Conference Series, Arcata California, July 26–August 4, 1989

Work by the investigators prior to the grant period.

In a series of papers the investigators have developed computational algorithms based on iteration techniques which are applicable to a wide variety of problems of both elliptic and parabolic type. The main strategy of this procedure can be outlined as follows. Using an initial approximation to the unknown function, one solves the direct problem to obtain the solution of the differential equation, $u(x,t)$. Evaluating the differential operator on the overposed boundary, then using the solution of the direct problem and the overposed data, one obtains a nonlinear equation for the unknown coefficient or term that is to be reconstructed. Solving this nonlinear equation, a new approximation to the unknown term is obtained. This procedure is repeated until convergence results. Since this method involves projecting the differential operator onto that subset of the domain where the overposed data is given, it has been referred to as the *Fixed Point Projection* or *FPP* method. The terminology "fixed point" refers to the reformulation of the inverse problem as an equivalent fixed point problem.

As an illustration of the technique, consider the recovery of the solution pair $\langle u, f\rangle$ in $u_t - u_{xx} = f(u)$ with $u_x(0,t) = g_0(t)$, $u_x(1,t) = g_1(t)$ and $u(x,0) = u_0$. We denote the solution of this direct problem by $u(x,t;f)$ in order to show the dependence of u on the function f. Let the overposed data be $u(0,t) = \theta(t)$. If the differential equation is evaluated on the section of boundary where the overposed data is prescribed ($x = 0$), then we obtain $f(\theta(t)) = \theta'(t) - u_{xx}(0,t;f) \equiv \mathbf{T}[f]$. We can show that the inverse problem is equivalent to proving the existence of a unique fixed point for $\mathbf{T}$, and give conditions on the data for which such a fixed point exists. The solution can then be obtained by the iteration scheme $f_{n+1} = \mathbf{T}[f_n]$.

With this method the investigators have been able to prove uniqueness and convergence results for the following inverse problems.

(a) The recovery of the solution pair $\langle u, f\rangle$ in $u_t - \Delta u = f(u) + \gamma$ given the initial value $u(x,0)$ and Neumann data on $\partial\Omega$, plus the value of $u(x_0,t) = \theta(t)$, for some point $x_0 \in \partial\Omega$, as overposed data, [4].

(b) The recovery of the solution pair $\langle u, f\rangle$ in $\Delta u = f(u)$ given Neumann data on $\partial\Omega$, plus the value of $u(x) = \theta(x)$ on some subset of the boundary, [3].

(c) The recovery of the solution pair $\langle u, k\rangle$ in $u_t - \nabla . k(u)\nabla u = 0$ given the initial value $u(x,0)$ and the flux, $k(u)\partial u/\partial\nu$ on $\partial\Omega$, plus the value of $u(x_0,t) = \theta(t)$, $x_0 \in \partial\Omega$ as overposed data, [7].

(d) The recovery of the solution pair $\langle u, c\rangle$ in $c(u)u_t - \Delta u = \gamma$ given the initial value $u(x,0)$, Neumann data on $\partial\Omega$, with $u(x_0,t) = \theta(t)$, $x_0 \in \partial\Omega$ forming the overposed data, [7].

(e) The recovery of the solution pair $\langle u, h\rangle$ in the one dimensional heat equation subject to the nonlinear boundary conditions $u_x = h(u)$ on $\partial\Omega$. The value of $u(0,t) = \theta(t)$, is given as the overposed data, [5].

Results of some of the numerical computations can be found in [1].

In addition to this work we have used the method to obtain numerical algorithms for the solution of the multiple unknown coefficient problems in items (III) and (IV) above as well as the recovery of both $c(u)$ and $f(u)$ in $c(u)u_t - u_{xx} = f(u)$, [8]. These problems are considerably more complex, and to date we have not been able to answer uniqueness or convergence questions. Some of the complications include the differing spaces in which the various coefficients lie, and in obtaining physically meaningful conditions on the data that allow one to solve the update equations. We have been able to provide necessary conditions on the data, and to demonstrate the feasibility of the approach for these problems. The last part of this work was completed under the present ONR grant period.

For certain problems, the recovery of $k(u)$ in $\nabla . k(u)\nabla u = \gamma$ and the recovery of the function $h(u)$ in the nonlinear boundary condition $\frac{\partial u}{\partial \nu} = h(u)$ for the heat equation in n spatial dimensions, we have been able to use direct approaches and obtain uniqueness results under quite general assumptions on the data, [6, 13]. Neither of these problems appear to be suitable for application of the *FPP* method. The second of these problems was done during the ONR grant period.

The *FPP* method used in the above problems has the advantage that it is applicable to a wide class of undetermined coefficient problems of parabolic and elliptic type, and is easy to implement provided that the boundary conditions are in a particular form (specifically, the overposed data should consist of Dirichlet data on some subset of the boundary). In addition, as long as the data is sufficiently "noise-free," the method is quite robust. Convergence is rapid, and takes place in the optimal space for the problem, that is the space with the maximum regularity permitted by the data.

On the other hand, the approach has its drawbacks: It is only applicable when the differential equation (or that part of the differential operator containing the unknown coefficient) is evaluated on the overposed boundary, one must be able to solve the resulting equation for the unknown coefficient on that section of the boundary where the overposed data is specified. The overposed data has to be monotone on that section of the boundary where it is specified, and since implementation of the algorithm *always* requires the differentiation of the overposed data, any "noisy" data should be smoothed before use in the numerical inverse solver. The algorithm is sensitive to the amount and type of smoothing employed. Finally, it is often difficult to prove the convergence of the algorithm.

Overcoming some (or all) of these restrictions is the primary motivation for our further research on these problems. There are alternative methods, some of which have been used to solve other types of inverse problems, but to our knowledge none of these have been applied to the recovery of terms containing the dependent variable in quasilinear equations.

Research under the grant period.

Our current research program consists of a continuation of our previous work in parabolic and elliptic equations, and also an extension of these ideas and techniques to undetermined coefficient problems for first order hyperbolic equations.

For the elliptic/parabolic problems we are currently investigating several approaches – Least Squares, Conjugate Gradient, Newton and Quasi-Newton, Homotopy, and Collocation methods. These methods can be broadly classified as either *residual* or *non-residual* based recovery schemes.

We will once again use our one spatial dimension reaction-diffusion example to illustrate the main ideas. The inverse problem for the recovery of f is equivalent to solving the equation $u(0,t;f) = \theta(t)$. More generally we can write the overposed data as $B[u] = \theta(t)$ where B is some functional of u. The case $B[u] = u(0,t)$ corresponds to the Dirichlet condition above, and $B[u] = \int_0^1 u(x,t)dx$ would correspond to a specification of energy. Note that B need not be a local operator.

The residual methods are iterative schemes that have the general form:

$$f^{(n+1)} = f^{(n)} - [D_n]\left(B[u(x,t;f^{(n)}) - \theta(t)]\right) \tag{5}$$

where D_n is linear operator. The *FPP* method is a special case of the above, [4], as are the Newton, Quasi-Newton and Homotopy methods.

If one knows a priori information about f (for example it is positive or monotone or very smooth), then one may be able to incorporate this information into the choice of D_n and regularize an otherwise poorly behaved method. Smoothing splines, moving averages, spline interpolation, non-uniform weight methods are some operations which are typically incorporated into the operator D_n. Slight changes in the form of D_n can result in substantial changes in the performance of the algorithm.

We have looked at general residual schemes and have shown that one of them is equivalent to the *FPP* method for the recovery of f in in the reaction-diffusion case when the overspecified data is of Dirichlet type. It is interesting to note that although the *FPP* method is not applicable to other boundary conditions such as the prescription of total energy $E(t) = \int_\Omega u(x,t)dx$, the residual update scheme can in fact be used in this case. We have obtained some numerical results for this modified scheme that are extremely encouraging.

An example of a non-residual update scheme are collocation methods. In one particular implementation, we use the evolutionary nature of the equations to sequentially determine the coefficients α_i in the basis expansion $f = \sum_{i=0}^{i=N} \alpha_i \phi_i$ by solving the single nonlinear equation

$$\theta(t_{j+1}) = B[u(x,t_j;\sum_{i=0}^{i=j+1} \alpha_i \phi_i)]$$

for a monotone sequence of points $\{t_j\}$. We assume that the coefficients $\{\alpha_i\}$, $1 \le i \le j$, have been determined for $\{t_j\}$, $1 \le i \le j$. This leaves a single unknown α_{j+1} to be determined by the remaining equation.

Another example of a non-residual update scheme is the least squares method. This is a popular approach especially for recovering a discrete set of parameters. Output least squares has been used extensively in control theory.

A survey of all of these methods as they apply to the above type of undetermined coefficient problem was presented by the investigators in the proceedings of the Arcata conference.

In this last half year we have concentrated on the collocation method, and applied it to two particular problems: one for the recovery of an unknown boundary condition in the heat equation, and the other for an interior coefficient, (forcing term).

For the first problem considerable insight has been obtained. We have been able to show that there is a unique piecewise linear approximant $\bar{f}_N$ to the function f such that the solution of the direct problem $u(0,t;\bar{f}_N)$ agrees with the overposed data at each of the N collocation points. In addition, we have given conditions under which the function $\bar{f}_N$ will converge to the function f in the supremum norm, and proved that an effective rate of convergence of $N^{-1/2}$ results. This work has been recently submitted [9], and details on the numerical implementation of the algorithm as well as its application to boundary control will appear in the 1989 Tampa CDC conference proceedings.

For the recovery of the forcing term $f(u)$ in $u_t - \Delta u = f(u)$ we have been able to show uniqueness of the approximating piecewise linear function $\bar{f}_N$. We can show that $\bar{f}_N$ *cannot* be expected to converge to f in the supremum norm, requiring that $u(0,t;\bar{f}_N)$ agrees with the overposed data at the N collocation points. We believe that one can obtain convergence, if instead of collocating $u(0,t;\bar{f}_N) - \theta(t)$, one collocates the derivatives of these quantities, $u_t(0,t;\bar{f}_N) - \theta'(t)$. This situation would be entirely in keeping with the requirements of our previous results using the *FPP* method, but we have not as yet been able to prove a positive convergence result for this case. Numerical experiments bear out the conjecture.

In light of the success of the collocation method, we intend to investigate the interior problem further, and also to use the method for resolving other inverse problems. Incremental collocation schemes have several attractions; in theory they are widely applicable and offer a way to gain insight into problems with multiple unknowns. The method is fairly straightforward to implement, and when applicable, is extremely computationally efficient. Indeed, in some cases the cost of the inverse recovery is roughly the same order as that for a single direct solve of the nonlinear problem, [9]. This is quite rare in undetermined coefficient problems.

Our intention is to continue on with this program, looking for various ways to formulate these inverse coefficient problems. We have begun to look at several methods that offer promise, these include the Newton, Quasi-Newton and homotopy methods. We will continue our work on the *FPP* and collocation schemes.

We have been looking at undetermined coefficient problems for first order hyperbolic equations

$$u_t + Au_x + Bu = C \tag{6}$$

subject to initial and boundary conditions. These equations are ubiquitous in applied mathematics and occur in such diverse areas as hydrodynamics, traffic flow, neutron transport theory and population dynamics. While the direct problem corresponding to these models are reasonably well understood, the same cannot be said for inverse problems. The coefficients A, B, and C as well as terms that occur from the boundary may very well be unknown, and additional data must be made available for their recovery.

In order to get a better feel for the more complex problems, we have looked at several special cases, in particular when A is a constant. In this case the characteristics are known and the differential equation is simpler to analyze. We are unwilling to look at problems that are generated entirely by mathematical considerations, and thus have worked on specific questions that arise in a particular area, in this case, population dynamics. The main reason for this is that the mathematical models almost always lead to equations with linear characteristics and thus avoid some obvious technical difficulties that would be associated with the direct problem. Although the equations we consider model the growth of an age-structured population, where the birth and death rates depend on age (as well as other variables), they are certainly not limited to this application.

If $\rho(a,t)$ denotes the number of individuals of age a alive at a particular time t, and λda is the probability of a death occurring to a given individual in the period a to $a+da$. Since this quantity should depend on the age of the individual λ depends on a, and in the case of competition, should also depend on $\rho(a,t)$. This dependence may be in fact be on the ***total population*** at time t, $P(t) = \int \rho(a,t)da$, and there certainly are circumstances where $\lambda = \lambda(a,P)$ is the appropriate model. This leads to the following equation,

$$\rho_t(a,t) + \rho_a(a,t) + \lambda\rho(a,t) = 0 \tag{7}$$

In addition to this we may prescribe the initial condition

$$\rho(a,0) = \phi(a) \tag{8}$$

and a boundary condition at $a = 0$ which corresponds to the renewal process. This may take either of the forms

$$\rho(0,t) = B(t) \tag{9}$$

or

$$\rho(0,t) = \int \beta\rho(a,t)\,da \tag{10}$$

where βda is the probability of a individual of age a giving birth (per unit time) in the interval a to $a+da$. β will depend on a and probably also on $\rho(a,t)$. Equations (7), (8) and (10) constitute the standard model for age-structured population dynamics. See the paper [10] and its references within for more details. We will refer to this as the ***direct problem.*** It corresponds to solving a (possibly nonlinear) first order hyperbolic equation with non-local boundary conditions. However, our " species" could be also be mechanical parts; $\rho(a,t)$ would denote the number of parts of age a still in service at time t, and λ would be an age dependent rate of failures.

In order to solve this direct problem we have to know the values of λ, ϕ and either B or β, and this is certainly not reasonable in many cases. If this is so, how can these quantities be determined? What constitutes reasonable data that might be measured in order to identify one or more of these quantities? As expected, the answer to this question depends on which quantity is considered to be unknown, and on what variables it depends. There are certainly situations where one could evaluate $\rho(a,T)$ at some fixed time T, this corresponds to the taking of a census or of an inventory that accounts for the age of each object. In other situations this may not be possible, but one is able to measure the value of $P(t)$, the total population or number of parts in service.

There is now a considerable number of physically reasonable inverse problems that might be posed. Some of these include:

(1) Given the function $\lambda = \lambda(a,P)$ the initial data $\rho(a,0) = \phi(a)$, and the values of $P(t)$ for some range $0 \le t \le L$ where L is the maximum lifespan, determine the function $\beta(a)$. This corresponds to the determination of a nonlocal boundary coefficient in a quasilinear hyperbolic equation.

(2) Given the functions λ, and β determine the the initial data $\rho(a,0) = \phi(a)$. The overposed data could be either the total population over some time interval $[0,T]$ or the values of the age-structure at a later time T, $\rho(a,T)$.

(3) Given the functions ϕ, and β determine the the death function λ. There are several cases. The first is when λ is a function of a only and the overposed data consists of the values of the age-structure at

a later time T. A second is when λ depends on both a and P but is a simple sum of the natural and environmental death rates, that is, λ has the form $\lambda_N(a) + \lambda_E(P)$, where λ_N is the death rate due to natural causes and λ_E is the death rate due to the environment. We assume that $\lambda_N(a)$ is known, and the overposed data consists of the total population over some time interval $[0,T]$ Another possible form for λ might be $\lambda(a,P) = \lambda_N(a)[1 + \lambda_E(P)]$. The first form assumes that the environmental pressure is the same for all ages, the second assumes that it is coupled to age so that environmental pressure is greater on the age group with a high natural death rate.

Problem one was solved in [10], where conditions were imposed for unique recovery. These *required* a certain incompatibility between the initial population and the initial birth rate. Without this condition, both non-uniqueness and non-existence is possible, and this was shown by counterexamples. In addition it was proved that when a solution exists it is unique and continuous dependence result could be obtained. These stated that the function β can be controlled in the supremum norm provided the overposed data is controlled in the C^2 norm, proving that the inverse problem is ill-conditioned. A counterexample showed this result to be essentially the best possible. Some numerical examples were presented.

Problem two was considered in [11]. It was shown that uniqueness of the problem followed under certain conditions. The degree of ill-conditioning of the inverse problem depended critically on the behavior of the function λ in a neighborhood of the maximum life span, L for the case of overposed data $P(t)$, and on the support of the birth function β for the case of overposed final data. Numerical reconstructions were presented.

Problem three is discused in [12]. Uniqueness, existence and continuous dependence results are obtained and the recovery of λ_E is obtained by an iteration procedure. Under certain cases this scheme may converge monotonically.

These last two papers was work performed under the ONR contract.

Clearly, the above represent only a fraction of the reasonable questions. The recovery of both the birth and death functions would be desirable, as would the recovery of one of these coefficients as a function of two variables, a and P (or possibly a and ρ). The case when the coefficient A in (6) is non-constant but depends on x, t or u is the important one for many applications. Much work remains to be done, actually we have only scratched the surface. We propose to continue this research into first order hyperbolic equations, and we have two graduate students who are starting to work on these problems.

Papers published

During the past year we have submitted five papers for publication, [7, 9, 11, 12, 13]. One of these, [13] has been accepted by the *Journal of Differential Equations*, and a second, [7] by *Numerical Methods in P.D.E.* Another paper, [8], was revised during the grant period. Several papers incorporating our most recent results are in preparation. In addition, there are two conference proceedings resulting from this work. One of these is to the I.E.E.E conference in Control [1], and we have provided a copy of the paper that will be published in the proceedings. The other was

[1] December 13–15, Tampa Florida.

presented at the Arcata conference mentioned earlier.

Invited Talks

1. *Inverse problems in Partial Differential Equations*, AMS/SIAM/IMS Summer Conference Series, Arcata California, July 26, August 4 1989. Invited hour talk, presented by Michael Pilant. The talk will appear in the conference proceedings to be published by SIAM.
2. "A collocation scheme for the identification of coefficients in nonlinear parabolic equations" I.E.E.E conference in Control, Tampa, Florida December 13–15, 1989. Invited talk, to be given by Michael Pilant. A conference proceedings will result.
3. "Using Domain Decomposition Methods for Computing Singularities of Equations of Mixed Type," SIAM Conference on Domain Decomposition Methods, Houston, March 20-22, 1989. Presented by Michael Pilant, also chaired session.
4. "Undetermined Coefficient Problems for First Order Hyperbolic Equations," European Conference on Inverse Problems, Montpelier, France, November 26–December 2, 1989. Presented by William Rundell.
5. AMS Regional Meeting "Special Session on Inverse problems," Manhattan, Kansas March 16–17, 1990. Invited talk (William Rundell).
6. SIAM Symposium on "Invariant Imbedding and Inverse problems," Albuquerque, New Mexico April 19–22, 1990. Invited talk (William Rundell).

Other activities during the contract period

William Rundell was the Chairman for the AMS/SIAM/IMS Summer Conference on "Inverse problems in Partial Differential Equations" held at Arcata California, July 26, August 4 1989. Approximately 70 people attended the workshop style conference, there were 15 main speakers covering as wide a range of topics as possible within the framework of the title. Funding for the conference was by NSF and ARO.

Equipment and personnel available for continued research.

This fall we were fortunate in being able to hire two new assistant professors who will actively participate in the general area of inverse problems, and will add considerable expertise to our efforts.

Richard Fabiano, a student of John Burns, spent 3 years at Brown working with Tom Banks. His speciality is in control theory, in particular for viscoelastic equations, but he is interested in pursuing the not unrelated area of undetermined coefficient problems. His knowledge of the output least squares methods for distributed parameter systems will be very useful in this regard.

Bruce Lowe was a student of Robert Kohn and graduated from Courant. He works in numerical schemes for undetermined coefficient problems using variational methods.

There are three Ph.D. students beginning work in this area under our direction.

We have excellent computational resources at our disposal. These include an IRIS 3130, MIPS 120, MIPS M2000, two SUN and two VAX workstations. The IRIS graphics workstation and the M2000 were obtained through NSF equipment grants.

The current ONR grant is the primary means of funding for this research effort. We believe substantial progress has been, and will continue to be achieved in the area of inverse coefficient recovery problems for partial differential equations.

Referenced publications by the investigators.

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Recovery of an Unknown Specific Heat by Means of Overposed Data

Recovery of an unknown specific heat by means of overposed data

Michael Pilant *

William Rundell *

Department of Mathematics
Texas A&M University
College Station Tx. 77843.

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Abstract:

In this paper we consider a class of inverse problems in which an unknown function, $c(\cdot)$, is to be determined from a parabolic initial-value problem, with overposed Dirichlet data along a portion of the boundary. A mapping between the overposed data and the unknown coefficient is obtained in the form of a singular integral equation. This is solved by iteration, and the resulting fixed point is shown to be the solution of the inverse problem. Sufficient conditions for convergence of this method, as well as an extension to the case of an unknown thermal conductivity, are given.

I. Introduction

In this paper we consider a class of inverse problems in which an unknown function, $c(\cdot)$, is to be determined from

$$c(u)u_t - u_{xx} = \gamma(x,t) \qquad \text{in } [0,1]\times[0,T] \tag{1.1}$$

with initial data given by

$$u(x,0) = u_0(x) \qquad \text{in } [0,1] \tag{1.2}$$

and boundary data given by

$$u_x(0,t) = g_0(t) \qquad u_x(1,t) = g_1(t) \tag{1.3}$$

The functions γ, u_0, and g_i are given data.

If c were known (and sufficiently regular), then (1.1) - (1.3) would be a well posed initial boundary-value problem for the determination of $u(x,t)$. In order to recover the pair $\langle u, c\rangle$ we need to give additional information and we do this in the form of Dirichlet data along part of the boundary; specifically, one is given the values of $u(0,t)$ for $0 < t < T$.

The recovery of unknown coefficients whose argument is the dependent variable of the equation has many important applications and considerable recent effort has been invested in these problems. In particular, the question of when a given amount of overposed data is sufficient to uniquely determine the coefficient has received a fair amount of recent attention, [2,3,4,6–12]. There are fewer constructive existence results, not because of their lack of importance, but rather because of their inherant difficulty. In previous work [8], the authors have shown how to recover the function f in $u_t - u_{xx} = f(u)$ by giving overposed Dirichlet data in the same fashion, and have shown how to recover h in a nonlinear radiation boundary condition $\partial u/\partial\nu = h(u)$ for the heat equation by prescribing the value of u at a point on the boundary [9]. Numerical implementations of these algorithms appear in [10].

In this paper the method of attack on the problem will be similar to that used in [8]. The main complication is the fact that the unknown coefficient multiplies one of the principal terms in the operator. We must ensure, for example, that $c > 0$ at all times. We shall prove that this is indeed the case, under suitable conditions on the data. Furthermore, we shall demonstrate that $c(u)$ can be recovered by an iteration scheme, and give sufficient conditions for the unique solution of the inverse problem.

Equation (1.1) describes the evolution of the temperature in a homogeneous rod of constant thermal conductivity, but in which the specific heat depends on the temperature. The forcing term, γ, corresponds to known temperature sources (or sinks). Thus the *direct problem*, namely (1.1) - (1.3) for a given coefficient c is inherently a quasilinear evolution equation for $u(x,t)$. For the one-dimensional case discussed above, the temperature flux $\partial u/\partial x$ is known at both ends of the rod. The temperature at one end is measured and provides the additional (overposed) data for the problem. This type of problem is typical of problems where the dependence of material properties is not known in advance, and must be deduced by experiment. We set up a experiment in which the thermal flux is controlled at the boundaries and then watch the resulting temperature behavior. Recovering the material property is equivalent to showing that there is at most one specific heat which can give rise to the overposed data, and that one can in fact construct the unknown function from knowledge of the overposed data.

In fact the methods of this paper go through formally for the case of n space variables since the resulting estimates have multidimensional parallels, but the analysis is more cumbersome. If Ω is a fixed bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$ and x_0 is some fixed point in $\partial\Omega$ then the corresponding problem is to recover the pair c and u from

$$c(u)u_t - \Delta u = \gamma(x,t) \qquad \text{in } \Omega \times [0,T]$$

$$u(x,0) = u_0(x) \qquad \text{in } \Omega$$

$$\frac{\partial u}{\partial \nu} = g \qquad \text{on } \partial\Omega.$$

with a measurement of overposed data consisting of the temperature at the point x_0

$$u(x_0,t) = \theta(t).$$

In section 5. we consider a similar problem to the above, namely the recovery of the thermal conductivity $k(u)$ in

$$u_t - \nabla \cdot (k(u)\nabla u) = \gamma(x,t).$$

II. Mathematical Preliminaries

We shall consider the problem

$$c(u)u_t - u_{xx} = \gamma(x,t) \quad 0 < x < 1,\ t > 0 \tag{2.1}$$

$$u_x(0,t) = 0, \quad u_x(1,t) = g_1(t) \tag{2.2}$$

$$u(x,0) = u_0(x) \tag{2.3}$$

with overposed Dirichlet data

$$u(0,t) = \theta(t). \tag{2.4}$$

We define the *direct* problem to be (2.1)-(2.3) for a known c, and denote by $u(x,t;c)$ the solution of this initial-value problem.

We will make the assumption that the overposed data $\theta(t)$ is a strictly montone function of t for all $t > 0$. In addition we have the following constraint, arising from the consistency of the data

$$c(\theta(0))\theta'(0) - u_0''(0) = \gamma(0,0) \tag{2.5}$$

which determines the value of $c(\theta(0))$ uniquely. By rescaling the time variable, we may assume that $c(\theta(0)) = 1$. Denote by $\psi(x,t)$ the function $u(x,t;1)$, so that ψ is the solution of

$$\psi_t - \psi_{xx} = \gamma(x,t) \tag{2.6}$$

$$\psi_x(0,t) = 0, \quad \psi_x(1,t) = g_1(t) \tag{2.7}$$

$$\psi(x,0) = u_0(x). \tag{2.8}$$

If $c(u) \approx 1$, then $\theta(t) \approx \psi(0,t)$. The deviation of $u(x,t;c)$ from ψ measures, in some sense, the size of the nonlinear term $c(u)$.

Some comments might be made at this point. If u_{xx} is negligible, then we have from (2.1) $c \approx \gamma/u_t$ which implies that as c increases then u_t decreases. We can recover, to leading order, the behavior of c by computing $\gamma(0,t)/\theta'(t)$. In fact if the boundary data is such that $u_{xx}(0,t) = 0$ then the function c can be recovered directly from $c(\theta(t)) = \gamma(0,t)/\theta'(t)$. Although the idea of a perturbation approach about the "flat data" case is appealing, numerical results indicate that unknown coefficients with large variations (even discontinuities) can be recovered efficiently.

We will need some notation. With

$$D_T := \{\, (x,t) \mid \ 0 < x < 1,\ 0 < t < T \,\} \tag{2.9}$$

we define several norms and seminorms

$$\| u \|_\infty = \sup_{D_T} |u(x,t)| \tag{2.10}$$

$$|u(\cdot,t)|_1 = \sup_{\xi \neq \eta} \frac{|u(\xi,t) - u(\eta,t)|}{|\xi - \eta|} \tag{2.11}$$

$$\| f \|_\infty = \sup |f(\xi)|, \quad \xi \in \mathrm{dom}(f) \tag{2.12}$$

$$|f|_1 = \sup_{\xi \neq \eta} \frac{|f(\xi) - f(\eta)|}{|\xi - \eta|} \quad \xi, \eta \in \mathrm{dom}(f) \tag{2.13}$$

$$\| f \| = \| f \|_\infty + |f|_1 \tag{2.14}$$

The set of all functions f defined on the set X with $\|f\| < \infty$ we denote by $Lip_1(X)$. We will also use the notation $f \in Lip_1$ if the set X is understood.

The role of the time variable is one of the complicating factors. We in fact avoid the usual semi-norm for solutions of parabolic equations

$$|u|_\alpha = \sup_{\substack{\xi \neq \eta \\ t \neq \tau}} \frac{|u(\xi,t) - u(\eta,\tau)|}{[|\xi - \eta|^2 + |t - \tau|]^{\alpha/2}},$$

since the appearance of t in such an asymmetric way is one of the major obstacles faced in formulating the mapping between θ and f.

We will also have occasion to use the notation

$$u(\cdot,t)\Big|_0^y \equiv u(y,t) - u(0,t).$$

For a fixed $E < 1$, define the subset B_E to be the ball of radius E, in Lip_1, centered on the function identically 1. That is,

$$B_E = \{\, c \mid c \in Lip_1[0,\infty),\ \|1 - c\| < E \,\}. \tag{2.15}$$

We remark that any function, c, defined on $Lip_1[0,T]$ can be extended to a function on $Lip_1[0,\infty)$ by continuation as a constant, without increase in norm. This continuation will be used later.

If $c \in B_\delta$ for some $\delta < 1$, we have $|1 - c| < \delta$ and hence $0 < 1 - \delta < |c| < 1 + \delta < 2$ and c is uniformly bounded away from $c = 0$, which is necessary for the iteration scheme to remain well defined.

We shall denote by C a generic constant which depends on the domain, but is independent of the solution $u(x,t;c)$, and we denote by $C(E)$ a generic constant which depends on u only through E.

Finally, we say the $c \in B_E$ has property $\mathcal{S}$ if

$$\|c(u) - c(v)\| \le C_E \|u - v\|$$

for some constant $C_E < \infty$. This property holds, for example, if $c \in C^2$.

We will also require the following assumptions on the data and on c

A1. (i) $u_0 \in C^3[0,1]$, (ii) $g_1(t) \in C^2[0,\infty)$, (iii) $\gamma(x,t) \in C^1[0,1] \times C^1[0,\infty)$.

A2. $\theta(t)$ is a monotone function, whose derivative lies in $C^1[0,\infty)$ and $\inf_{t>0} |\theta'(t)| \ge \delta > 0$.

A3. $c \in B_E$ for some $E < 1$.

A4. The function θ is such that there exists a $c \in B_E$ with $u(0,t;c) = \theta(t)$.

Some remarks on these assumptions are in order.

Remark 1: Assumption **A1** is sufficient to ensure that u is sufficiently regular and generates smooth boundary data. **A2** is necessary to recover c from knowledge of $c(\theta(t))$, and to ensure compatibility with some $c \in Lip_1$ function generating the data. **A3** is sufficient to ensure that the problem remains strictly parabolic. Finally, **A4** is necessary in order to ensure that a solution to the problem exists.

Remark 2: In the case where u is spatially independent, the direct problem becomes

$$u_t = [c(u)]^{-1}\gamma(t)$$

and this would not necessarily possess a unique solution unless $c(\cdot)$ were, for example, Lipschitz. Uniquess of the direct problem is necessary in order that our algorithm be well defined. To show uniqueness of the inverse problem we actually need to impose further constraints on the solution c, namely that it have property $\mathcal{S}$.

III. The Boundary Coefficient Mapping

Evaluating (2.1) on the overposed boundary, and using the overposed data (2.4), we observe that any strong solution of (2.1)–(2.3) and (2.4) must satisfy

$$c(\theta(t))\theta'(t) - u_{xx}(0,t;c) = \gamma(0,t). \tag{3.1}$$

Define the mapping $\mathbf{T}[c]$ by

$$\mathbf{T}[c] = \frac{u_{xx}(0,t;c) + \gamma(0,t)}{\theta'(t)}, \tag{3.2}$$

where $u(x,t;c)$ is the solution of (2.1)–(2.3) for a prescribed $c(\cdot)$. Any solution of the inverse problem therefore satisfies

$$\mathbf{T}[c](t) = c(\theta(t)). \tag{3.3}$$

We call such a function a θ–fixed point of T. The natural iteration scheme introduced is clearly

$$c^{(n+1)}(\theta(t)) = \mathbf{T}[c^{(n)}](t). \tag{3.4}$$

If $u^{(n)} \equiv u(x,t;c^{(n)}) > \theta(T)$ then we define $c^{(n)}(u^{(n)}) = c^{(n)}(\theta(T))$ and similarily if the argument is less than $\theta(0)$. This extends $c^{(n)}$ in a Lipschitz continuous manner without increasing its norm. This guarantees that the scheme is well defined even if it should happen that one of the iterates $u^{(n)}$ should lie outside of the range of values of the overposed data.

Note that $\theta(t)$ must be monotone for one to recover $c^{(n+1)}$ from (3.4). As we remarked previously, in order for this scheme to be well defined, we must have uniqueness for the forwards or direct problem. We will show that $\mathbf{T}[c]$ maps a certain ball in B_E into itself. If the data is sufficiently smooth (satisfying Assumptions **A1, A2, A3**), and the initial starting value $c^{(0)}$ lies in B_E, then it is guaranteed that each iterate $c^{(n)}$ will yield a unique $u(x,t;c(^n))$ to the direct problem.

The outline of the rest of the paper is as follows. Lemma 1 shows that the inverse problem can be reduced to finding a fixed point of the mapping $\mathbf{T}[\,]$. Theorem 1 shows that $\mathbf{T}[\,]$ maps B_E to itself which implies that the iteration scheme is well defined. In Theorem 2, using an additional regularity assumption, we show that there is a unique fixed point.

The iteration scheme (3.4) can be recast in the form

$$\begin{aligned} c^{(n+1)}(\theta(t)) &= \frac{u_{xx}(0,t;c^{(n)}) + \gamma(0,t)}{\theta'(t)} \\ &= \left[\frac{u_t^{(n)}(0,t;c)}{\theta'(t)}\right] c^{(n)}(u^n(0,t)) \\ &= c^{(n)}(\theta(t)) + \left[\frac{u_t^{(n)}(0,t;c) - \theta'(t)}{\theta'(t)}\right] c^{(n)}(u^{(n)}) \\ &\quad + \left[c^{(n)}(u^{(n)}(0,t)) - c^{(n)}(\theta(t))\right] \\ &= c^{(n)}(\theta(t)) + \mathcal{F}_n(u^{(n)}(0,t) - \theta(t); c^{(n)}) \end{aligned} \tag{3.5}$$

where it is seen to be equivalent to a nonlinear, nonstationary residual update scheme. In order to show the equivalence of θ-fixed points and solutions to the inverse problem we prove the following Lemma.

Lemma 1. *If $\langle u, c\rangle$ is a solution of (2.1)–(2.3), then u satisfies the overposed boundary conditions (2.4) if and only if c is a θ-fixed point of $\mathbf{T}[\,]$, defined in (3.2).*

Proof: If $\langle u, c\rangle$ is a solution of the equations (2.1)–(2.4) c is a θ-fixed point. This is clear, since the identity $c(\theta(t))\theta'(t) - u_{xx}(0,t) = \gamma(0,t)$, which follows from (3.1), implies that

$$c(\theta(t)) = \frac{u_{xx}(0,t) + \gamma(0,t)}{\theta'(t)} = \mathbf{T}[c](t).$$

On the other hand, suppose c is a θ-fixed point, then

$$c(\theta(t))\theta'(t) - u_{xx}(0,t) = \gamma(0,t)$$

but from (2.1) the solution u satisfies

$$c(u(x,t))u_t(x,t) - u_{xx}(x,t) = \gamma(x,t).$$

Evaluting this last expression at $x = 0$, and subtracting from the previous expression, we obtain

$$c(\theta(t))\theta'(t) - c(u(0,t))u_t(0,t) = 0.$$

Therefore,

$$[c(\theta(t)) - c(u(0,t))]\,\theta'(t) + c(u(0,t))\left[\theta'(t) - u_t(0,t)\right] = 0.$$

Solving for $\theta'(t) - u_t(0,t)$ we have

$$\theta'(t) - u_t(0,t) = [c(u(0,t)) - c(\theta(t))]\frac{\theta'(t)}{c(u(0,t))}.$$

By Assumption **A3**, $|c(u(0,t))| > 0$ and by **A2** $|\theta'(t)| < \infty$. Consequently, if c is uniformly Lipschitz;

$$|\theta'(t) - u_t(0,t)| \leq \|c\|\frac{|\theta'|_\infty}{|c(u(0,t))|}|\theta(t) - u(0,t)|.$$

Setting $\alpha(t) = \theta(t) - u(0,t)$

$$|\alpha'(t)| \leq M|\alpha(t)|. \tag{3.6}$$

Since $\alpha(0) = 0$ (the initial boundary data and overposed data are consistent), Gronwall's inequality yields $|\alpha(t)| = 0$, and the Lemma is proved.

The equivalence of θ-fixed points and solutions of the inverse problem appears to be a general result for this method of solving single inverse coefficient problems.

IV. Iteration Procedure

The Green's function for the heat equation on the unit interval, with homogeneous Neumann boundary conditions, has the form

$$K(x,y,t) = G(x,y,t) + H(x,y,t)$$

where

$$G(x,y,t) = \frac{1}{2\sqrt{\pi t}}\left\{e^{-(x-y)^2/4t} + e^{-(x+y)^2/4t}\right\} \tag{4.1}$$

and H is bounded, C^∞-smooth kernel for $t \geq 0$, [1].

Rewriting (1.2) in the form

$$u_t - u_{xx} = \gamma(x,t) + \big(1 - c(u)\big)u_t \tag{4.2}$$

and subtraction from (2.6)-(2.8) implies that the function $v \equiv u - \psi$ must satisfy

$$\begin{aligned} v_t - v_{xx} &= [1 - c(u)]\,u_t \\ v_x(0,t) = 0, &\quad v_x(1,t) = 0 \\ v(x,0) &= 0 \end{aligned}$$

Consequently,

$$u - \psi = v = \int_0^t\int_0^1 K(x,y,t-\tau)[1 - c(u(y,\tau))]u_t(y,\tau)dyd\tau \tag{4.3}$$

Differentiating, with respect to x twice, and setting $x = 0$, we obtain

$$u_{xx}(0,t;c) = \psi_{xx}(0,t) + \int_0^t\int_0^1 K_{xx}(0,y,t-\tau)\,[1 - c(u)]\,u_t\,dyd\tau \tag{4.4}$$

and consequently

$$\begin{aligned} \mathbf{T}[c](t) &= \theta^{-1}(t)\,[\gamma(0,t) + u_{xx}(0,t)] \\ &= \theta^{-1}(t)\left[\gamma(0,t) + \psi_{xx}(0,t) + \int_0^t\int_0^1 K_{xx}(0,y,t-\tau)[1 - c(u)]u_t dyd\tau\right] \\ &= \theta^{-1}(t)\left[\psi_t(0,t) + \int_0^t\int_0^1 K_{xx}(0,y,t-\tau)[1 - c(u)]u_t\,dyd\tau\right]. \end{aligned} \tag{4.5}$$

The next Lemma is crucial to the estimates that will follow.

Lemma 2. *The kernels K_x and K_{xx} have the property that for any fixed x_0,*

$$\int_0^t \left[\int_0^1 K_x(x,y,t-\tau)dy\right] h(x_0,\tau)d\tau = 0 \quad t>0 \tag{4.6}$$

$$\int_0^t \left[\int_0^1 K_{xx}(x,y,t-\tau)dy\right] h(x_0,\tau)d\tau = 0 \quad t>0 \tag{4.7}$$

for $x \in [0,1]$ and $h \in Lip_1$.

Proof: Fix x_0, and set $f(t) = h(x_0,t)$. The function

$$w(x,t) = \int_0^t \int_0^1 K(x,y,t-\tau)f(\tau)dyd\tau$$

satisfies the following boundary value problem:

$$w_t - w_{xx} = f(t)$$

$$w_x(0,t) = 0 \qquad w_x(1,t) = 0$$

$$w(x,0) = 0.$$

w is therefore a function of t alone, hence $w_x \equiv 0$ and $w_{xx} \equiv 0$.

This lemma implies the usefu identities

$$\int_0^t \int_0^1 K_x(x,y,t-\tau)\left[h(y,\tau) - h(x_0,\tau)\right]dyd\tau = \int_0^t \int_0^1 K_x(x,y,t-\tau)h(y,\tau)dyd\tau$$

and

$$\int_0^t \int_0^1 K_{xx}(x,y,t-\tau)\left[h(y,\tau) - h(x_0,\tau)\right]dyd\tau = \int_0^t \int_0^1 K_{xx}(x_0,y,t-\tau)h(y,\tau)dyd\tau$$

which will be used to simplify the nonlinear estimates which follow.

The first step will be to show that $\mathbf{T}$ maps a ball in Lip_1 to itself on the space B_E, that is, we will show that if $E < 1$ is sufficiently small

$$\|1 - \mathbf{T}[c]\| \le \|1 - c\| \tag{4.8}$$

for $c \in B_E$. Note that from (4.5),

$$\begin{aligned} \mathbf{T}[c] - 1 = {} & [\psi_t(0,t) - \theta'(t)] \,/\, [\theta'(t)] \\ & + \int_0^t \int_0^1 K_{xx}(0,y,t-\tau)[1-c(u)]u_t \; dyd\tau \,/\, [\theta'(t)] \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} \|\mathbf{T}[c] - 1\| \le {} & \|1/\theta'(t)\| \|\| [\psi_t(0,t) - \theta'(t)] \| \\ & + \|1/\theta'(t)\| \left\| \int_0^t \int_0^1 K_{xx}(0,y,t-\tau)(1-c(u))u_t \; dyd\tau \right\| \end{aligned}$$

We must carefully estimate the Lip_1 norm of the operator $\mathcal{K}$, defined by

$$\tilde{h}(x,t) \equiv \mathcal{K}[h] = \int_0^t \int_0^1 K_{xx}(x,y,t-\tau)h(y,\tau)dyd\tau \tag{4.10}$$

on the boundary $x = 0$.

Suppose that $h(x,t) = x$, we have (with some easy computations)

$$\begin{aligned}\int_0^t\int_0^1 K_{xx}(x,y,t-\tau)h(y,\tau)dyd\tau &= \int_0^t\int_0^1 K_{xx}(x,y,t-\tau)ydyd\tau \\ &= \int_0^t\int_0^1 K_{xx}(x,y,t-\tau)\frac{y}{|t-\tau|^{1/2}}|t-\tau|^{1/2}dyd\tau \\ &\le C|t|^{1/2}\end{aligned}$$

We see that Lipschitz functions in x are mapped to functions in $C^{1/2}$ with respect to t. This implies that a function which is Lip_1 with respect to x and t is mapped to one which is Lip_1 in x and only $C^{1/2}$ in t, apriori.

It is precisely this loss of regularity, in t, which makes the analysis so much more delicate. The iteration scheme is not a priori well defined unless there are additional conditions present which restore the range of K to be a subset of Lip_1 in the time variable. The interaction between the nonlinearity and the overposed boundary has the effect of restoring the smoothness. A similar phenomenon may be found in [8].

Using (4.7) we have the identity

$$\begin{aligned}&\int_0^t\int_0^1 K_{xx}(0,y,t-\tau)[1-c(u)]u_t(y,\tau)dyd\tau \\ &\quad= \int_0^t\int_0^1 K_{xx}(0,y,t-\tau)\left\{[1-c(u(y,\tau))]u_t(y,\tau) - [1-c(u(0,\tau))]u_t(0,\tau)\right\}dyd\tau \\ &\quad= \int_0^t\int K_{xx}(0,y,t-\tau)[1-c(u(y,\tau))]\left[u_t(y,\tau)-u_t(0,\tau)\right]dyd\tau \\ &\quad\quad+ \int_0^t\int_0^1 K_{xx}(0,y,t-\tau)\left[c(u(0,\tau))-c(u(y,\tau))\right]u_t(0,\tau)dyd\tau. \end{aligned} \tag{4.11}$$

To estimate the three differences appearing in the integrands of (4.11) we shall require Lemmas 3–5:

Lemma 3. $|c(u(x,t)) - c(u(0,t))| \le C|c|_1\|u_{xx}\|_\infty|x|^2$.

Proof: The proof of this Lemma is a direct application of Taylor's Theorem, and the fact that $u_x(0,t) = 0$.

$$\begin{aligned}|c(u(x,t)) - c(u(0,t))| &\le |c|_1|u(x,t) - u(0,t)| \\ &\le |c|_1|u_x(0,t)\,x + u_{xx}(\xi,t)x^2/2| \text{ for some } 0<\xi<1 \\ &\le C|c|_1\|u_{xx}\|_\infty|x|^2.\end{aligned}$$

Lemma 4. $|1 - c(u(x,t))| \le C\|1-c(u)\|\left\{\|u_{xx}\|_\infty|x|^2 + \|u_t\|_\infty|t|\right\}$.

Proof: Using a Taylor series expansion of $c(u(x,t))$, we conclude

$$\begin{aligned}|1-c(u(x,t))| &= |1-c(u(0,t)) + c(u(0,t)) - c(u(x,t))| \\ &\le |1-c(u(0,t))| + |c(u(0,t)) - c(u(x,t))| \\ &= |c(u(0,0)) - c(u(0,t))| + |c(u(0,t)) - c(u(x,t))| \\ &\le |c|_1|u_t(0,\cdot)t| + |c|_1|\tfrac{1}{2}x^2u_{xx}(\cdot,t)| \\ &\le C|c|_1\left\{|u_t|_\infty t^\alpha + |u_{xx}|_\infty x^2\right\}\end{aligned}$$

which proves the lemma since

$$|c|_1 \le |c|_1 + \|1-c\|_\infty = \|1-c\|_\infty + |1-c|_1 = \|1-c\|$$

Lemma 5 is a fundamental a priori estimate for the direct problem.

Lemma 5. *Given $c \in B_E$, we have for E sufficiently small:*

$$|u_t(x,t)| \leq C(E)$$

$$|u_{xx}(x,t)| \leq C(E)$$

and

$$|u_t(x,t) - u_t(0,t)| \leq C(E)|x|.$$

Proof: From [5, Section 4.5], we have the fundamental estimates that the solution of

$$u_t - u_{xx} = f(x,t)$$

satisfies

$$||u_t, u_{xx}|| \leq C_1 + C_2||f||$$

where C_1 depends only on the initial–boundary data. With $f = (1 - c(u))u_t$, we are led to the estimate

$$||u_t|| \leq C_1 + C_2||(1-c(u))u_t|| \leq C_1 + C_2||1-c(u)||||u_t|| \leq C_1 + C_2E||u_t||$$

which leads to

$$||u_t|| \leq \frac{C_1}{1 - C_2E}.$$

The Lemma follows immediately from the definition of the norm.

Using these lemmas and referring to remark 2, we conclude that

Theorem 1. *Under the assumptions on the data given by* **A1—A4**, *and if*

$$\varepsilon \equiv \frac{||\psi_t - \theta'||}{|\theta'(0)|^2} \cdot [1 + |\theta'(0)|]$$

and

$$\delta \equiv C(E)\frac{|u_0''|_\infty}{|\theta'(0)|^2}$$

satisfy $\varepsilon < (1-\delta)E$, then for $t < T$ and E sufficiently small, **T**[ ] *maps B_E into itself. $C(E)$ is a constant which depends continuously on E.*

Proof: Estimate each of the integrands in (4.11) using Lemmas 4, 5, and 3, respectively to obtain:

$$\begin{aligned} &\left|\int_0^t \int_0^1 K_{xx}(0,y,t-\tau)[1-c(u)]u_t(y,\tau)dyd\tau\right| \\ &\leq \int_0^t \int_0^1 |K_{xx}(0,y,t-\tau)| \Big\{ |c|_1 \cdot C \left[||u_{xx}||_\infty y^2 + ||u_t||_\infty \tau\right] C(E)|y| \\ &\qquad + C(E)|c|_1||u_{xx}||_\infty |y|^2 \Big\} dyd\tau. \end{aligned} \tag{4.12}$$

In order to simplify this expression we use the estimates:

$$\int_0^t \int_0^1 \left|K_{xx}(0,y,t-\tau)\right| |y|^\alpha dyd\tau \leq c(\alpha)t^{\alpha/2}$$

$$\int_0^t \int_0^1 \left|K_{xx}(0,y,t-\tau)\right| \tau^\beta |y|^\alpha dyd\tau \leq c(\alpha)t^{\beta+\alpha/2}$$

Taking $\alpha = 1$ and $\beta = 1$ in the above estimates, (4.12) reduces to

$$\begin{aligned}
&\Big| \int_0^t \int_0^1 K_{xx}(0, y, t-\tau)[1 - c(u)]u_t(y,\tau)dyd\tau \Big| \\
&\quad \le |c|_1 \cdot C(E)\,[\|u_{xx}\|_\infty]\left[t^{3/2} + t\right] + |c|_1 C(E)\|u_t\|_\infty t^{3/2} \\
&\quad \le |c|_1 C(E)\|u_{xx}\|_\infty t + |c|_1 C(E)(\|u_{xx}\|_\infty + \|u_t\|_\infty)t^{3/2} \\
&\quad \le \|1 - c\| \left[C(E)\|u_{xx}\|_\infty t + O(t^{3/2})\right].
\end{aligned}$$

Hence, from (4.9), and the above inequality, we have

$$\begin{aligned}
\|1 - \mathbf{T}[c]\|_\infty \le &\frac{\|\psi_t(0,t) - \theta'(t)\|_\infty}{\inf_{t>0}|\theta'|} \\
&+ \frac{\|1 - c\|}{\inf_{t>0}|\theta'|}\left[C(E)\|u_{xx}\|_\infty t + O(t^{3/2})\right].
\end{aligned} \tag{4.13}$$

In order to compute the Lipschitz norms of $c(\theta(t))$, we use the fact that

$$\begin{aligned}
|c|_1 &= \sup_{t_1 \ne t_2} \frac{|c(\theta(t_1)) - c(\theta(t_2))|}{|\theta(t_1) - \theta(t_2)|}\frac{|t_1 - t_2|}{|t_1 - t_2|} \\
&\le \frac{1}{\inf_{t>0}|\theta'|} \sup_{t_1 \ne t_2} \frac{|c(\theta(t_1)) - c(\theta(t_2))|}{|t_1 - t_2|}
\end{aligned}$$

Using a similar breakup of the integrals as in (4.11), one can show that

$$\begin{aligned}
|1 - \mathbf{T}[c]|_1 \le &\frac{\|[\psi_t - \theta']\|}{(\inf_{t>0}|\theta'|)^2} \\
&+ \frac{\|1 - c\|}{\inf_{t>0}|\theta'|^2}\left[C(E)\|u_{xx}\|_\infty + O(t^{1/2})\right].
\end{aligned} \tag{4.14}$$

Combining (4.13), (4.14) we obtain

$$\begin{aligned}
\|1 - \mathbf{T}[c]\| \le &\frac{\|\psi_t - \theta'\|}{\inf_{t>0}|\theta'|)^2} \cdot \left[1 + \inf_{t>0}|\theta'|\right] \\
&+ \left\{\left[C(E)t\inf_{t>0}|\theta'| + C(E)\right]\frac{\|u_{xx}\|_\infty}{\inf_{t>0}|\theta'|^2}\right\}\|1 - c\| \\
\equiv &\ \hat{\varepsilon} + \hat{\delta}\|1 - c\|
\end{aligned} \tag{4.15}$$

For $t < T$ sufficiently small, we have $\hat{\varepsilon} \approx \varepsilon$ and $\hat{\delta} \approx \delta$, and by the assumptions on ε and δ, $\|1 - \mathbf{T}[c]\| \le E$ and the Theorem is proved.

This is sufficient for showing that $T : B_E \to B_E$. We therefore have a bounded sequence of iterates in $B_E \subset Lip_1$ with a subsequence converging in C^β, strongly, for $\beta < 1$.

We must show that the limit is unique in order to obtain a solution to the inverse problem. We do this by showing that B_E contains an *attractor* for $\mathbf{T}$, that is, there exists an element $\bar{c} \in B_E$ such that $\bar{c}$ is a θ-fixed point and

$$\|\mathbf{T}[c] - \mathbf{T}[\bar{c}]\| \le \delta\|c - \bar{c}\| \qquad \delta < 1$$

for all $c \in B_E$. We note that

$$\begin{aligned}
\mathbf{T}[c] - \mathbf{T}[\bar{c}] &= [\theta'(t)]^{-1} \int_0^t \int_0^1 K_{xx}(0,y,t-\tau)\,[\bar{c}(\bar{u}) - c(u)]\, u_t dy d\tau \\
&\quad + [\theta'(t)]^{-1} \int_0^t \int_0^1 K_{xx}(0,y,t-\tau)[1-\bar{c}(\bar{u})](u_t - \bar{u}_t) dy d\tau \\
&= [\theta'(t)]^{-1} \left\{ \int_0^t \int_0^1 K_{xx} \left([\bar{c}(\bar{u}) - \bar{c}(u)]\, u_t(\cdot,\tau) \right) \Big|_0^y dy d\tau \right. \\
&\quad + \int_0^t \int_0^1 K_{xx} \left([\bar{c}(u) - c(u)]\, u_t(\cdot,\tau) \right) \Big|_0^y dy d\tau \\
&\quad \left. + \int_0^t \int_0^1 K_{xx} \left([1-\bar{c}(\bar{u})]\,[u_t(\cdot,\tau) - \bar{u}_t(\cdot,\tau)] \right) \Big|_0^y dy d\tau \right\}.
\end{aligned}$$

The only difficult term to estimate is the second integral in the last equality. If $\bar{c}$ is smooth enough so that $\|\bar{c}(\bar{u}) - \bar{c}(u)\| \leq C_E \|\bar{u} - u\|$, that is if Property $\mathcal{S}$ of section 2 holds, then we may conclude that

$$\|\mathbf{T}[c] - \mathbf{T}[\bar{c}]\| \leq \left[C(E) \frac{\|u_{xx}\|_\infty}{[\inf_{t>0} |\theta'|]^2} + O(t) \right] \|c - \bar{c}\| \tag{4.16}$$

where $C(E)$ is a bounded constant, which depends continuously on E.

Consequently, if $t < T$ is sufficiently small, and $|u_0''|_\infty / \theta'(0)$ is sufficiently small, then $\mathbf{T}[\,]$ is a contraction on B_E, with respect to $\bar{c}$.

If we assume

A4*. The function $\theta(t)$ is such that there exists a $c \in B_E$ with Property $\mathcal{S}$ such that $u(0,t;c) = \theta(t)$.

We then have,

Theorem 2. *If* **A1–A3** *and* **A4*** *hold then there is a unique solution to the inverse problem. Furthermore, this solution c can be obtained as the limit of the iteration scheme (3.4).*

We remark at this point that although this proof of a unique fixed point does not constitute an existence proof (since we have assumed apriori the existence of a c giving rise to the overposed data), it is nevertheless constructive and yields geometric convergence to the solution. An existence proof would require a characterization of the class of overposed data which is the image (under the heat operator) of B_E. This is known to be a difficult problem. A partial explanation is that C^3 initial-data, with homogenous boundary data and forcing functions, leads to data $u(0,t) = \theta(t)$ which is analytic in t, and can be represented as a Dirichlet series for $t > 0$.

V. Recovering an unknown thermal conductivity.

A related problem to the one considered in this paper is the recovery of the conductivity $k(u)$ from the nonlinear parabolic equation

$$u_t - \nabla \cdot (k(u) \nabla u) = \gamma(x,t). \tag{5.1}$$

The direct problem consists of (5.1) along with the same boundary conditions as for the determination of $c(u)$, that is initial data together with the specification of the flux on the lateral boundaries.

$$k(u) \frac{\partial u}{\partial \nu} = g(t) \tag{5.2}$$

$$u(x,0) = u_0(x) \tag{5.3}$$

The overposed data needed to recover $k(u)$ will again be given by

$$u(x_0,t) = \theta(t) \qquad x_0 \in \partial\Omega \tag{5.4}$$

In one space variable several results are known for this problem. Uniqueness of a solution pair $\langle u, k \rangle$ has been shown by Cannon and Yin, [3], and Cannon and DuChateau, [2], have provided conditions that guarantee a minimum to the quantity $\|u(0,t;k) - \theta(t)\|$ in an appropriate setting.

The problem (5.1)–(5.4) has many similarities to (1.1)–(1.4). However the strategy of evaluating (5.1) on the overposed section of the boundary, $x = x_0$ leads to a difficult scheme to analyze since $k(u)$ is implicitly defined in the equation. A common strategy (see for example, [2]), in dealing with this situation problem is to introduce the change of variables $v = \int_0^u k(r)dr$. In the special case $\gamma = 0$ and $u_0 = 0$ this leads to the boundary value problem

$$\begin{aligned} c(v)v_t - \Delta v &= 0 \\ \frac{\partial v}{\partial \nu} &= g(t) \\ v(x,0) &= 0 \end{aligned} \tag{5.5}$$

with the overposed data

$$v(x_0,t) = \int_0^{\theta(t)} k(r)dr \equiv \psi_k(\theta)$$

where $c(v) \equiv [k(\psi_k^{-1}(v)]^{-1}$. Implementing the iteration scheme (3.4) by evaluating (5.5a) on the boundary point x_0 gives

$$c_{n+1}(\psi_{k_n}(\theta)) = \mathbf{T}^{(1)}[c_n] = \frac{\Delta v(x_0,t;c_n)}{v_t(x_0,t;c_n)} \tag{5.6}$$

and the problem is reduced to a problem very similar to (1.1)-(1.4).

In the case of one space dimension, another strategy is possible. Let $u(x,t)$ satisfy

$$u_t - (k(u)u_x)_x = \gamma(x,t) \tag{5.7}$$

with

$$-k(u(0,t))u_x(0,t) = g_0(t) \tag{5.8}$$

$$k(u(1,t))u_x(1,t) = g_1(t) \tag{5.9}$$

$$u(x,0) = u_0(x) \tag{5.10}$$

and let

$$u(0,t) = \theta(t) \tag{5.11}$$

be given as overposed data. If we define $u(x,t;k)$ to be the solution of (5.7, 5.8, 5.10, 5.11), then for $k \in C^2$ there is a unique strong solution to the direct problem. In this formulation, we are considering the condition (5.9) as the "overposed data" and incorporating the measured Dirichlet data (5.11) into the direct problem. On the boundary $x = 0$, we have from (5.9) that

$$k(\theta(t))u_x(0,t;k) = g_0(t) \tag{5.12}$$

and this leads to the iteration scheme generated by

$$k(\theta(t)) = \mathbf{T}^{(2)}[k] \equiv -\frac{g_0(t)}{u_x(0,t;k)}. \tag{5.13}$$

Of course, in order for (5.13) to make sense, we must ensure that $u_x(0,t;k)$ is uniformly bounded away from zero, but this can be achieved by giving appropriate conditions on the data. Numerically, (5.13) has superior convergence properties with respect to (5.6), partly due to the compactness of $\mathbf{T}^{(2)}$. Many numerical experiments indicate that effective convergence of the scheme $k_{n+1} = T_\theta^{(2)}[k_n]$ is obtained in a few iterations. This last scheme has many similarities to that used in [9] to recover the form of the nonlinear boundary condition $u_x = h(u)$ from overposed Dirichlet data, in that the boundary operator rather than the second order partial differential equation itself, is used to provide the update scheme. The fact that lower order differential operators are involved in the update scheme appears to be responsible for its superior convergence properties.

In a higher number of spatial dimensions, the interchanging of the overposed and primary data is not possible as implemented above, but the following overposed boundary value problem case can be used to obtain recovery of $k(u)$

$$\begin{aligned} u_t - \nabla\cdot(k(u)\nabla u) &= \gamma \\ u(x,t) = h(x,t) &\qquad x\in\partial\Omega \\ u(x,0) &= u_0(x) \end{aligned} \tag{5.14}$$

as the direct problem, and oveposed data given as the value of the heat flux at a given boundary point $x_0 \in \partial\Omega$

$$k(u)\frac{\partial u}{\partial \nu} = g(t)\Big|_{x=x_0} \tag{5.15}$$

Equation (5.15) leads to the update scheme

$$k(\theta(t)) = \frac{g(t)}{\frac{\partial u}{\partial \nu}(x_0,t;k)} \tag{5.16}$$

where $\theta(t) = h(x_0,t)$ is known. The nonvanishing of the normal derivative $\partial u/\partial\nu$ at the point $x = x_0$ can be guaranteed by imposing the condition that $h(x,t) \le h(x_0,t)$ (at least in the case of $\gamma = 0$ and $u_0 = 0$ and with suitable modifications otherwise). This condition is necessary to guarantee that the entire range of values of $u(x,t)$ is contained on the line $x = x_0$ where the overposed boundary condition is imposed.

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Identification of Nonlinear Flux Terms in a Parabolic Initial-Boundary Value Problem

Identification of nonlinear flux terms in a parabolic initial - boundary value problem

Michael Pilant*

William Rundell*

Department of Mathematics

Texas A&M University

College Station Tx. 77843.

Abstract. *The identifiability of a function describing nonlinear radiation in the one dimensional heat equation is considered. It is shown that a monotone sequence of temperature measurements on one boundary is sufficient to determine a unique piecewise linear flux function which yields the desired temperature response. As a consequence, we show that the temperature at one end can be controlled. Assuming the data arises from a sufficiently smooth flux function, we obtain* a posteriori *convergence results. The results of a numerical simulation are given.*

Keywords. Inverse problem, heat equation, radiation boundary conditions, temperature control.

AMS(MOS) subject classifications. 35R30, 35K05, 93B30, 93B40

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1. Introduction.

Consider the following initial - boundary value problem

$$u_t - u_{xx} = g(x,t) \qquad 0 < x < 1, \quad 0 < t < T \tag{1.1}$$

$$u(x,0) = u_0(x) \qquad 0 \le x \le 1 \tag{1.2}$$

$$u_x(0,t) = f\big(u(0,t)\big) \tag{1.3a}$$

$$-u_x(1,t) = f\big(u(1,t)\big) \tag{1.3b}$$

Given sufficient smoothness on and knowledge of the function $f(u)$, the above *direct problem* has a unique solution for $u(x,t)$. However, we envision the situation where the function f is unknown, save that it depends only on u, and wish to determine both $u(x,t)$ and $f(u)$ by making additional boundary measurements. The solution of the direct problem for a given function f will be denoted by $u(x,t;f)$. For further discussion of the direct problem and the modeling of such boundary conditions, see the monograph [1].

One of our assumptions will be that $f(u)$ has fixed sign over its range, and without any loss of generality, we assume that it is nonnegative. Given this, additional assumptions are made on the data so that heat is being lost through the boundaries at $x = 0$ and $x = 1$, and the temperature $u(x,t)$ is decreasing in time for each position x.

Finally, *overposed data*

$$u(0,t) = h(t) \qquad 0 \le t \le T. \tag{1.4}$$

is prescribed, and for compatibility this must be monotone in t. It will, in fact, be monotonically decreasing.

Equations (1.1) - (1.3) describe, for example, the diffusion of heat in a uniaxial bar with nonlinear radiation boundary conditions at the ends, our problem is therefore to determine the unknown temperature - dependent radiation function $f(u)$ from a knowledge of the initial temperature distribution, and a measurement of the temperature $u(x,t)$ at one boundary.

If the heat loss occurs through boundary conduction, we may assume that the flux is a function of the temperature difference. One commonly used model specifies a linear relationship, (Newton's law of cooling)

$$\frac{\partial u}{\partial \nu} = -k(u - u_\infty).$$

(Here, $\partial u/\partial \nu$ denotes the derivative normal to the boundary.) On the other hand, black body radiation into a vacuum (with zero ambient temperature) is governed by Stefan's law

$$\frac{\partial u}{\partial \nu} = \sigma u^4$$

where σ is Boltzmann's constant, If the body is partially absorbing, then we have

$$\frac{\partial u}{\partial \nu} = \epsilon \sigma u^4$$

where ϵ is the emissivity of the surface. If the emissivity is unknown, but temperature dependent, we have boundary conditions of the form (1.3). If heat is lost through the boundary through both radiation and conduction, then we must assume a more general form of the law – as in (1.3).

This leads to an inverse problem for the unknown dependence of the flux on temperature. We can also view this as a problem in nonlinear boundary control. If the boundary satisfies a *known* radiation law

$$\frac{\partial u}{\partial \nu} = \bar{h}(u), \tag{1.5}$$

but the incoming flux Q is temperature dependent and unknown, we have

$$\frac{\partial u}{\partial \nu} = \bar{h}(u) + Q(u) \equiv f(u). \tag{1.6}$$

We now wish to control the temperature of the rod to the extent permitted by controlling the temperature at one end. We seek to do this by controlling the total flux there. The *target* set is the desired temperature response $\{h(t_j)\}$ and the unknown function $Q(\cdot)$ is the control. We show that for any target set of monotone data, a piecewise linear function $\bar{f}$ exists with the property that the solution $u(x,t,\bar{f})$ to (1.1)–(1.3) with $f = \bar{f}$ satisfies $u(0,t_j,\bar{f}) = h(t_j)$.

At best we will only be able to recover the function $f(u)$ for those values taken on by $u(0,t)$ for $0 \le t \le T$. It is thus necessary that the range of values of the function $u(x,t)$ for $x \in [0,1]$ be contained in the range of values of $u(0,t)$ for $0 \le t \le T$.

There are several special cases of the inverse recovery problem. First, if the boundary conditions on $x = 1$ are known (and are independent of f), for example $\alpha u_x(1,t) + \beta u(1,t) = g_1(t)$, then the direct problem may be solved by using the overposed data (1.4) and the known boundary condition on $x = 1$. The flux boundary conditions on $x = 0$

can then be used to recover f. Similarly, if two overposed boundary measurements are used, one can recover two possibly different flux functions f_1 and f_2. This also shows that the problem of recovering a single flux function by means of two measurements is overdetermined.

In addition, we require the overposed data to be monotone in t. This monotonicity property is essential to recover the function f from a knowledge of $f(h)$. No attempt to completely characterize the class of allowable overposed data $h(t)$ will be made. Instead we assume that for some function $f(u)$, the data $u(0,t;f)$ is given, and show that the unknown radiation function can be reconstructed from the overposed data. More precisely, we prove under certain conditions that if the value of h is given at a sequence of points t_j for $j = 1, 2, \ldots, N$, then there exists a piecewise linear function $\bar{f}_N$ with slope changes at the points $h(t_j)$, such that $u(0,t_j;\bar{f}_N) = h(t_j)$, and $\lim_{N\to\infty} \|f - \bar{f}_N\|_\infty = 0$. We define the *residual* of the mapping from $h \to f$ as $u(0,t;f) - h(t)$. Requiring the residual to vanish at a discrete set of points t_j, generates a *collocation* scheme.

In a recent paper [6], the authors proved the existence of a unique solution to (1.1) – (1.4). The solution of the inverse problem was obtained by an iteration scheme using the boundary condition itself as the update algorithm. Although the update scheme required little computational cost to implement, each iteration involved solving a (nonlinear) direct problem of the form (1.1) – (1.3), and required the data to be fairly noise free. The method proposed here is entirely different, in that the algorithm needs only solve a *linear* partial differential equation at each stage. Since the problem is clearly nonlinear in f, the calculation of the slopes requires an iterative procedure such as the secant method. We have performed several numerical experiments which indicate that the scheme is robust under noisy data.

Our approach in this paper will be to first present some preliminary technical details needed to set up the problem. We then state the main results in the form of several lemmas and theorems. A thorough discussion of the algorithm and a numerical example is then presented, followed by the proofs of all the lemmas and theorems in a separate appendix.

2. Preliminaries.

In this section our assumptions on the class of admissible data and radiation functions f are given, and we introduce some notation that will be useful in the sections to follow.

In order to obtain an integral representation of the solution to (1.1) - (1.3), it is convenient to consider the function, $w(x,t)$ that satisfies the initial boundary value problem

$$\begin{aligned} &w_t - w_{xx} = g(x,t) && 0 < x < 1, \quad 0 < t < T \\ &w_x(0,t) = 0 \qquad -w_x(1,t) = 0 && 0 \leq t \leq T \\ &w(x,0) = u_0(x) && 0 \leq x \leq 1. \end{aligned} \tag{2.1}$$

We note that $w(x,t) = u(x,t;0)$ and the solution of (1.1) - (1.3) can be written as

$$\begin{aligned} u(x,t) = w(x,t) &- 2\int_0^t \theta(x,t-\tau) f\big(u(0,\tau)\big)d\tau \\ &- 2\int_0^t \theta(x-1,t-\tau) f\big(u(1,\tau)\big)d\tau \end{aligned} \tag{2.2}$$

where $\theta(x,t)$ is the theta function defined for $t > 0$ by

$$\theta(x,t) = \sum_{m=-\infty}^{m=\infty} K(x-2m,t) \tag{2.3}$$

and $K(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$ is the fundamental solution of the free space heat equation. See [1].

The first lemma gives a regularity result for the forward or direct problem, that is when f is a known function.

Lemma 1. *Let $u(x,t)$ satisfy (1.1) - (1.3) with $f' \in L^\infty$ and u_0, $g(x,t)$ sufficiently smooth. Then for $0 \leq x \leq 1$, $0 \leq t \leq T$, the function $u = u(x,t;f)$ given by (2.2) is twice continuously differentiable in x and $u_t(x,t)$ is C^α in t for $0 \leq \alpha < 1/2$.*

Since we evaluate solutions of the direct problem on the boundaries $x = 0$ and $x = 1$, the following properties of the theta function will be needed,

(1) $\theta(-x,t) = \theta(x,t) \quad -\infty < x < \infty, \quad t > 0$

(2) $\theta(-1,t) = \frac{1}{\sqrt{4\pi t}} \sum_{k=1}^{\infty} e^{-(k-1/2)^2/t} \equiv H_1(t)$. $H_1(t)$ is a C^∞ function on $[0,\infty)$ whose n^{th} partial derivative in t, evaluated at $t = 0$, vanishes.

(3) $\theta(0,t) = \frac{1}{\sqrt{4\pi t}} + \frac{1}{\sqrt{\pi t}} \sum_{k=1}^{\infty} e^{-k^2/t} = \frac{1}{\sqrt{4\pi t}} + H_0(t)$, where $H_0(t)$ is C^∞ on $[0, \infty)$ with $H_0^{(n)}(0) = 0$ for all n.

We introduce the notation

$$\mathcal{S}\phi(t) = 2\int_0^t \theta(-1, t-\tau)\phi(\tau)\, d\tau \tag{2.4}$$

$$\mathcal{A}\phi(t) = -2\int_0^t \theta(0, t-\tau)\phi(\tau)\, d\tau \tag{2.5}$$

The kernel of $\mathcal{S}$ is a C^∞ function and $\mathcal{S}$ is an infinitely smoothing operator. The kernel of the operator $\mathcal{A}$ has a singularity of order $1/2$ and takes functions in C^α into functions in $C^{\alpha+1/2}$. If $\psi(t)$ has a bounded L^∞ derivative then the equation $\mathcal{A}\phi(t) = \psi(t)$ can be written in the equivalent form, (c.f. [1]).

$$\phi(t) = \frac{2}{\sqrt{\pi}}\left\{\frac{\psi(0)}{\sqrt{t}} + \int_0^t \frac{\psi'(\tau)d\tau}{\sqrt{t-\tau}} + \int_0^t H(t,\tau)\phi(\tau)\, d\tau\right\} \tag{2.6}$$

where H is a C^∞ function. This equation is of second kind Volterra type and can be uniquely solved for the function ϕ. Thus

$$\phi = \mathcal{A}^{-1}\psi = (I + \mathcal{R})\tilde{\psi} \tag{2.7}$$

where $\mathcal{R}$ is the resolvent operator corresponding to the Volterra integral equation (2.6) with smooth kernel H and

$$\tilde{\psi}(t) = \left\{\frac{\psi(0)}{\sqrt{t}} + \int_0^t \frac{\psi'(\tau)d\tau}{\sqrt{t-\tau}}\right\} \tag{2.8}$$

We will need some properties of the operators $\mathcal{A}$ and $\mathcal{S}$.

Lemma 2. *For any function $\phi(t)$ with $\phi(0) = 0$ and $\phi'(t)$ in L^∞,*

$$\sup_{0\le t\le T} \|\mathcal{A}^{-1}\phi\|_{\infty,[0,t]} \le C\sqrt{T}\,\|\phi'\|_{\infty,[0,T]} \tag{2.9}$$

$$\sup_{0\le t\le T} \|\mathcal{A}^{-1}\mathcal{S}\phi\|_{\infty,[0,t]} \le C\,T\,\|\phi\|_{\infty,[0,T]} \tag{2.10}$$

The values of the overposed data $h(t)$ are assumed to be given at a discrete set of points $\{t_j\}_0^N$, where $t_0 = 0$ and $t_N = T$. Let

$$\Delta t = \sup_j (t_{j+1} - t_j), \qquad 0 \le j \le N-1.$$

The notation

$$\|f\|_{\infty,[v_1,v_2]} = \sup_{v_1 \le v \le v_2} |f(v)|,$$

$$\|f\|_{\infty,S} = \sup_{v \in S} |f(v)|,$$

and

$$\|\phi\|_{\infty,[t_1,t_2]} = \sup_{t_1 \le t \le t_2} |\phi(t)|$$

will be used. The interval will be ommitted when no confusion can arise. We use C to denote a generic constant that may depend on the θ function, the data, and various Lipschitz constants, but will be independent of Δt or the function approximation to f that will be obtained by collocation.

The following assumptions on f and the data are made:

A1. The function $f(\cdot)$ is nonnegative, $f' \in L^\infty[0,\infty)$ with $\|f'\|_\infty \le M$ for some fixed constant M.

A2. The initial data is such that u_0'' is in $C^{1/2}$, and the source term $g(x,t)$ is $C^{1/2}$ in both x and t. Thus $w(x,t)$ is a classical solution to (2.1).

A3. The solution $u(x,t)$ to the direct problem (1.1) - (1.3) satisfies

$$\text{range}_{0 \le s \le t}\{u(x,s)\} \subseteq \text{range}_{0 \le s \le t}\{u(0,s)\}$$

for all $x \in [0,1]$ and $t \in [0,T]$.

In order to guarantee the a priori existence of a solution to the inverse problem (1.1)-(1.4), we assume the overposed data $h(t)$ actually arises from some admissible function f.

A4. There exists a function $f(\cdot)$ satisfying **A1** such that the solution $u(x,t;f)$ to (1.1) - (1.3) satisfies $u(0,t;f) = h(t)$, where h is a decreasing function of t

3. The Collocation Method.

The collocation approach to the solution of (1.1) - (1.4) can be described as follows. We consider a function $\bar{f}$, piecewise linear on each of the intervals $[t_j, t_{j+1}]$, with the corresponding function $\bar{u}(x,t) \equiv u(x,t;\bar{f})$ defined on $0 < x < 1, \quad 0 < t < T$ and satisfying

$$\bar{u}_t - \bar{u}_{xx} = g(x,t)$$
$$\bar{u}_x(0,t) = \bar{f}(\bar{u}(0,t)), \quad -\bar{u}_x(1,t) = \bar{f}(\bar{u}(1,t)) \tag{3.1}$$
$$\bar{u}(x,0) = u_0(x).$$

Assume that $f(v)$ is either known or has been approximated by a function $\bar{f}(v)$ for $v > h(t_j)$. On the interval $[h(t_{j+1}), h(t_j)]$ extend $\bar{f}$ by the linear function $\bar{f}(v) = c_j(v - h(t_j)) + \bar{f}(h(t_j))$, for some constant c_j which should be chosen in order to satisfy (1.4) at the next collocation point $t = t_{j+1}$, that is to satisfy $u(0, t_{j+1}; \bar{f}) = h(t_{j+1})$.

One particular difficulty is the lack of knowledge of a starting value for $\bar{f}(h_0)$. We denote the starting error $f(h_0) - \bar{f}(h_0)$ by E_0. We define ψ by

$$\psi_t - \psi_{xx} = g(x,t) \qquad 0 < x < 1, \quad 0 < t < T \tag{3.2}$$
$$\psi(x,0) = 0 \qquad 0 \leq x \leq 1 \tag{3.3}$$
$$\psi_x(0,t) = \varepsilon(t) \tag{3.4}$$
$$\psi_x(1,t) = 0 \tag{3.5}$$

where $\varepsilon(t)$ is chosen to satisfy $\varepsilon(0) = E_0$ and so that ψ vanishes at the collocation points. Essentially, the function ψ measures the propagation of an initial error in $\bar{f}$.

The main results of this paper are:

Theorem 1. *(Stability Theorem) If assumptions* **A1 - A4** *hold, and the initial error* $|f(h_0) - \bar{f}(h_0)| = E_0$ *then for sufficiently small* T, *we have the following estimate*

$$\sup_{0 \leq t \leq T} \|f - \bar{f}\|_{\infty, \bar{u}(0,t)} \leq C\big[\|u - \bar{u}\|_{\infty, x=0} + E_0 + \|A^{-1}[u - \bar{u} - \psi]\|_{\infty, x=0}\big]$$

Theorem 2. *(Convergence Theorem) If assumptions* **A1 - A4** *hold, then there exists a unique piecewise linear function* $\bar{f}$ *such that the solution* $\bar{u}(x,t;\bar{f})$ *to (1.1) - (1.3) with* f *replaced by* $\bar{f}$ *satisfies (1.4) at each of the observation points* $t = t_j$, $j = 0, 1, \ldots, N$. *Furthermore, if the set of slope constants* $\{c_j\}_{j=1}^{j=N}$ *remain bounded then*

$$\sup_{0 \leq t \leq T} \|f - \bar{f}\|_{\infty, \bar{u}(0,t)} \leq C[E_0 + \sqrt{T}\Delta t^{\alpha}] \qquad \text{for } 0 \leq \alpha < 1/2$$

In the next section, we show that the initial error can be made $\mathcal{O}(\Delta t)$ by choosing a constant approximation to f on the initial interval. Consequently, we have

Corollary. *(Startup Error) If the set of slope constants* $\{c_j\}_{j=1}^{j=N}$ *remains bounded, and the startup error is* $\mathcal{O}(\Delta t)$ *then the collocation scheme converges globally of order* p *where* $p < \frac{1}{2}$.

The proofs of these theorems are presented in the Appendix.

The next lemma is crucial to the well posedness of the collocation method. We show that the boundary values of the direct problem depends monotonically on the radiation function f

Lemma 3. *(Monotonicity) Let* (v_1, f_1) *and* (v_2, f_2) *satisfy (1.1) - (1.3) where assumptions* **A1 - A4** *hold for each solution pair. If* $f_1 \geq f_2$, *then* $v_1(x,t) \leq v_2(x,t)$ *for all* (x,t). *In addition, if for some* t', $f_1\big(v(1,t')\big) > f_2\big(v(1,t')\big)$, *then* $v_1(x,t') < v_2(x,t')$.

Let us define the functions $\phi_0(t)$ and $\phi_1(t)$ by

$$\phi_0(t) = u(0,t) - \bar{u}(0,t) \tag{3.6}$$

$$\phi_1(t) = u(1,t) - \bar{u}(1,t) \tag{3.7}$$

$$\Delta f(t) = f\big(h(t)\big) - \bar{f}\big(h(t)\big) \tag{3.8}$$

where $u(x,t)$ and $\bar{u}(x,t)$ are the solutions of the direct problems (1.1) – (1.3) and (3.1) respectively.

We have,

Lemma 4. *Given the assumptions* **A1** – **A4**,

$$\|\phi_1\|_{\infty,[0,t]} \leq C\left(t\|\phi_0(t)\|_{\infty,[0,t]} + \sqrt{t}\|\Delta f(t)\|_{\infty,[0,t]}\right). \tag{3.9}$$

This says that the error in the approximation of the solution of (1.1) – (1.3) by that of (3.1) is no greater on the "trailing boundary" $x = 1$, than it is on the "leading boundary" $x = 0$ plus some contribution from the radiation at the boundary. On the other hand the difference in the functions f and $\bar{f}$ must satisfy,

Lemma 5.

$$\|\Delta f\|_{\infty,[0,t]} \leq C\left(\|\phi_0\|_{\infty,[0,t]} + \varepsilon + \|\mathcal{A}^{-1}[\phi_0 - \psi]\|_{\infty,[0,t]} + t\,\|\phi_1\|_{\infty,[0,t]}\right). \tag{3.10}$$

Given the collocation condition that $\phi_0(t_j) = 0$ for $j = 0, 1 \ldots N$ we have the estimates,

Lemma 6. *If f' and $\bar{f}' \in L_\infty$, then for any $0 < \alpha < 1/2$ the function $\phi_0(t)$ satisfies*

$$\sup_{0 \le t \le T} |\phi_0(t) - \psi| \le C\,(\Delta t)^{1+\alpha} \tag{3.11}$$

$$\sup_{0 \le t \le T} |\phi_0'(t) - \psi_t| \le C\,(\Delta t)^{\alpha}. \tag{3.12}$$

We remark that if u and $\bar{u}$ are smooth enough, then we obtain estimates of the form

$$\sup_{0 \le t \le T} \phi_0(t) \le C\,(\Delta t)^2 \tag{3.9a}$$

$$\sup_{0 \le t \le T} \phi_0'(t) \le C\,(\Delta t) \tag{3.10a}.$$

Lemma 3 is the key to the method in that it guarantees that the piecewise-linear collocation scheme is well defined. For suppose (u, f) satisfies (1.1) - (1.3), and that $u(x,t)$ for $t \le t'$ and $f(v)$ for $v \ge u(1,t')$ are assumed known. Then there is at most one linear function $\bar{f} \equiv c(v - u(1,t')) + b$ that agrees with f at the point $v = u(1,t')$ and such that $u(1,t'')$ takes on a given value for some $t'' > t'$. For if $f_1 = c_1(v - u(1,t')) + b > f_2 = c_2(v - u(1,t')) + b$ with say $c_1 > c_2$, then from lemma 3, $u(x,t;f_1) > u(x,t;f_2)$ for all $t > t'$. In addition, if over the interval $[t', t'']$, either $f > \bar{f}$ or $f < \bar{f}$, then it could not be that $u(1,t'';f) = u(1,t'';\bar{f})$.

Besides the simplicity of this method, the collocation procedure is quite versatile and offers advantages over global recovery schemes. The fixed point recovery method for (1.1) - (1.4) described in [6] is difficult to formulate for overposed data that is measured at an *interior* point, or for the case of more than one spatial variable. Given any situation where the overposed data depends on the function f in a monotone manner, the collocation method can be carried out in principle.

Theorem 3. *(Uniqueness Theorem) If assumptions* **A1 – A4** *hold then the difference between any two solutions f_1 and f_1 is a function $\tilde{f}(v)$ with an infinite number of zeroes in any interval $[v_1, v_2] \subset [h(T), h(0)]$,*

Corollary. *If assumptions* **A1 – A4** *hold, where f is analytic, then there is at most one solution pair (u, f) to (1.1) - (1.4).*

We remark that this condition on the difference of any two solutions f_1 and f_2 being a function with an infinite number of zeroes is identical with the situation obtained by several authors for the reaction-diffusion equation $u_t - \Delta u = f(u)$ where the function $f(u)$ has to be determined from overspecified boundary data, [2, 3]. Clearly, an infinitely oscillatory function cannot be distinguished by a discrete set of boundary measurements.

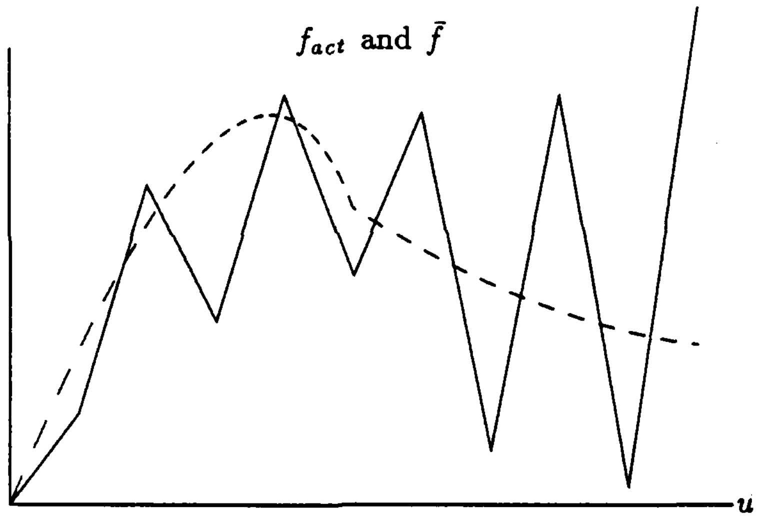
4. Analysis of the Algorithm and a Numerical Example.

In this section we describe a numerical implementation of the collocation scheme.

(a) Generate "'experimental data." Select a function f_{act} and obtain the solution $u(\cdot; f_{act})$ to (1.1) - (1.3). The value of this function on the line $x = 0$ forms the overposed data, and is passed to the inversion algorithm as $h(t)$ at the discrete points t_j.

(b) Assume that $f(v)$ is either known or has been approximated by a function $\bar{f}(v)$ for $v < h(t_j)$. On the interval $[h(t_j), h(t_{j+1})]$ extend $\bar{f}$ using the linearized approximation $\bar{f}(v) = \lambda_j\big(v - h(t_j)\big) + \bar{f}\big(h(t_j)\big)$, for some constant λ_j which should be chosen in order to satisfy the collocation condition $u(0, t_{j+1}; \bar{f}) = h(t_{j+1})$. We use a secant method to calculate the slope λ_j from the resulting nonlinear equation

(c) Step (a) is continued for each interval $[t_j, t_{j+1}]$.



There are some remarks to be made on the above procedure.

(1) How does one obtain the starting value $f(h(0))$? If the initial data and boundary data are compatible at $t = 0$ then we have $f(h(0)) = u_0'(0)$ which determines $\bar{f}$ initially. This will be the case if the process has been evolving for some positive time interval. If the data are incompatible, we then use collocation on $[h_0, h_1]$ to determine a *constant* approximation to f on this interval. This results in an error which is $\mathcal{O}(\Delta h)$.

(2) The number of iterations required to obtain the slope λ on a given collocation interval will obviously depend on the tolerance required, and to the deviation of the function f_{act} from linearity on this interval, For most of the step sizes and values of an actual function f, f_{act} we chose, about 3 iterations sufficed to obtain λ to within the same accuracy

as the forwards solution $u(x,t;f)$. The cost of the method is therefore 2 or 3 times the cost of solving a *linear* direct problem, since the approximation is by a linear function on each of the intervals $[t_j, t_{j+1}]$.

(3) An alternative method for this problem would be to let $\bar{f}$ depend on M parameters $\{(c_1, \ldots, c_j)\}$, and impose N constraints to determine the c_j's by a least squares procedure. In our problem this is not necessary because $M = N$, and for a parabolic equation the values of $u(x_0, t_1)$ do not affect the values of $u(x_0, t_2)$ if $t_1 > t_2$. Thus the value of c_j does not depend on the value of c_{j+1}. This allows the coefficients to be solved sequentially. This gives the collocation method a computational edge with respect to other global recovery schemes.

(4) The collocation method for this problem leads to an efficient scheme for obtaining an approximating function $\bar{f}$. The (almost) square root convergence of this approximation to f, as a function of the stepsize Δt is a worst case analysis. The observed rate of convergence for a wide variety of test examples was nearly linear. To find a value for the slope of $\bar{f}$ over the interval $[t_j, t_{j+1}]$ that matched $\bar{u}(0,t)$ to $h(t)$ to within sufficient accuracy to be compatible with the rest of the numerical scheme required only a few iterations of a procedure like the secant method. At each stage of this process we must of course solve the direct problem over the interval $[t_j, t_{j+1}]$ with our guessed value of $\bar{f}$. Since this is a linear function, we are only required to solve the heat equation with linear boundary conditions.

(5) In [6], an iteration scheme was used to construct an approximating sequence to $f(u)$. Although convergence was quite rapid, (effective approximation was obtained within a few iterations), each stage of this scheme required the solution of the heat equation with nonlinear boundary conditions. This iteration approach did however allow us to obtain an existence, uniqueness and stability result for the inverse problem (1.1) - (1.4). The collocation method is not suited to this task. However, since it is based on monotonicity of the data $u(0,t;f)$ on the function f, we can use this to obtain a uniqueness theorem, although this is weaker than that obtained by the methods of [6].

We choose $f_{act} = 3u^2 - 2u^3$. The function $u(x,t;f_{act})$ was calculated numerically and the values of $h(t) \equiv u(1, t_j; f_{act})$ at the points $t_j = j/N$ for $j = 0,\ 1,\ \ldots,\ N$, used as data for the collocation scheme. We used a step size of $k = 0.005$ in time and $h = 0.04$ in the spatial direction. The table below shows the difference of f and $\bar{f}$ in the supremum and L^2 norms for various values of N.

Convergence rate of $\bar{f}$ to f_{act}.		
N	$\|f_{act} - \bar{f}\|_\infty$	$\|f_{act} - \bar{f}\|_{L^2}$
2	0.0889	0.0410
5	0.0389	0.0130
10	0.0173	0.0036
25	0.0080	0.0011
50	0.0051	0.0010

Notice that the convergence of $\bar{f}$ to f_{act} appears to be nearly linear in $\Delta t \equiv 1/N$, rather than the square root dependence obtained in the proof of Theorem 1. This is due to the fact that the f_{act} generating the data is analytic, which implies that the exact solution is infinitely smooth. The apparent linear convergence is only asymptotic, and the last entries of the above table are close to the limitations imposed by truncation error.

5. Appendix: Proofs of the lemmas.

We collect here the proofs of the lemmas used in the proof of theorem 1.

Proof of Lemma 1: Let $f(0)=0$ with $f' \in L_\infty$ and

$$u(t) = \int_0^t \frac{f(\tau)}{\sqrt{t-\tau}} d\tau = \int_0^t \frac{f(t-\tau)}{\sqrt{\tau}} d\tau$$

Differentiating once, in t, we get

$$U(t) \equiv u'(t) = \int_0^t \frac{f'(t-\tau)}{\sqrt{\tau}} d\tau$$

Since

$$U(t+h) - U(t) = \int_0^t f'(\tau)[\frac{1}{\sqrt{t+h-\tau}} - \frac{1}{\sqrt{t-\tau}}] d\tau + \int_t^{t+h} f'(\tau) \frac{1}{\sqrt{t+h-\tau}} d\tau$$

Therefore

$$|U(t+h) - U(t)| \le \|f'\|_\infty [\int_0^t |\frac{1}{\sqrt{t+h-\tau}} - \frac{1}{\sqrt{t-\tau}}| d\tau + \int_t^{t+h} \frac{1}{\sqrt{t+h-\tau}} d\tau]$$

$$\le C\|f'\|_\infty h^{1/2}(1 + \log(h))$$

We lose half a derivative due to the singularity of order 1/2 in the kernel. This carries over to the solution of (1.1)-(1.3) via the representation (2.2) □.

Proof of Lemma 2: From equations (2.7) and (2.8) we have using $\psi(0) = 0$,

$$\mathcal{A}^{-1}\psi = \int_0^t \frac{\psi'(\tau)\,d\tau}{\sqrt{t-\tau}} + \mathcal{R}\psi'$$

where $\mathcal{R}$ is an integral operator with smooth kernel. The first equation, (2.9) now follows. For the second part, we have

$$\mathcal{A}^{-1}S\psi(t) = \int_0^t \frac{1}{\sqrt{t-\tau}} \int_0^\tau H_1'(\tau - s)\psi(s)\,ds\,d\tau + \mathcal{R}\Big\{\int_0^t H_1'(t-\tau)\psi(\tau)\,d\tau\Big\}$$

since $H_1(0) = 0$. The function $H_1'(t)$ is continuous and hence the kernel of the first integral operator is integrable. Equation (3.3) is now immediate. □

Proof of Lemma 3: The function $v = v_1 - v_2$ must satisfy

$$v_t - v_{xx} = 0$$
$$v(x,0) = 0$$
$$v_x(0,t) - f_1'(\xi_0(t))v(0,t) = f_1(v_2(0,t)) - f_2(v_2(0,t))$$
$$-v_x(1,t) - f_1'(\xi_1(t))v(1,t) = f_1(v_2(1,t)) - f_2(v_2(1,t))$$

for some functions $\xi_0(t)$ and $\xi_1(t)$. Let $\alpha(x,t)$ be any function in $C^2([0,1]) \times C^1([0,T])$ such that

$$\alpha_t - \alpha_{xx} = {\alpha_x}^2$$

and

$$\alpha_x(0,t) = -f_1'(\xi_0(t)), \qquad \alpha_x(1,t) = f_1'(\xi_1(t))$$

and put $w(x,t) = v(x,t)e^{\alpha(x,t)}$. Then $w(x,t)$ satisfies

$$w_t - w_{xx} + 2\alpha_x w_x = 0$$

$$w(x,0) = 0$$
$$w_x(0,t) = e^{-\alpha(0,t)}\Big\{f_1(v_2(0,t)) - f_2(v_2(0,t))\Big\} \geq 0$$
$$-w_x(1,t) = e^{-\alpha(1,t)}\Big\{f_1(v_2(1,t)) - f_2(v_2(1,t))\Big\} \geq 0$$

since $f_1 \geq f_2$. The maximum principle for the principal part of the operator now shows that $w \leq 0$, and thus $v_1 \leq v_2$. The sharper conclusion of the second part follows from the fact that if the boundary conditions in the above boundary value problem are inhomogeneous, w cannot identically vanish. □

Proof of Lemma 4: If we substitute the pairs $\langle u\,,\, f\rangle$ and $\langle \bar{u}\,,\, \bar{f}\rangle$ into equation (2.2) and subtract we obtain

$$\begin{aligned} u(x,t) - \bar{u}(x,t) &= 2\int_0^t \theta(x,t-\tau)\left\{f\big(u(0,\tau)\big) - \bar{f}\big(\bar{u}(0,\tau)\big)\right\}d\tau \\ &\quad - 2\int_0^t \theta(x-1,t-\tau)\left\{f\big(u(1,\tau)\big) - \bar{f}\big(\bar{u}(1,\tau)\big)\right\}d\tau. \end{aligned} \tag{5.1}$$

Evaluating this equation on the line $x = 1$, and expanding terms, gives

$$\begin{aligned} \phi_1(t) = 2&\int_0^t \theta(1,t-\tau)\left\{f\big(u(0,\tau)\big) - f\big(\bar{u}(0,\tau)\big)\right\}d\tau \\ +2&\int_0^t \theta(1,t-\tau)\left\{f\big(\bar{u}(0,\tau)\big) - \bar{f}\big(\bar{u}(0,\tau)\big)\right\}d\tau \\ -2&\int_0^t \theta(0,t-\tau)\left\{f\big(u(1,\tau)\big) - f\big(\bar{u}(1,\tau)\big)\right\}d\tau \\ -2&\int_0^t \theta(0,t-\tau)\left\{f\big(\bar{u}(1,\tau)\big) - \bar{f}\big(\bar{u}(1,\tau)\big)\right\}d\tau \end{aligned}$$

and thus

$$\begin{aligned} |\phi_1(t)| \leq 2M&\int_0^t |\theta(1,t-\tau)|\,|\phi_0(\tau)|\,d\tau + \int_0^t |\theta(1,t-\tau)|\big|f\big(\bar{u}(0,\tau)\big) - \bar{f}\big(\bar{u}(0,\tau)\big)\big|\,d\tau \\ +2M&\int_0^t |\theta(0,t-\tau)|\,|\phi_1(\tau)|\,d\tau + \int_0^t |\theta(0,t-\tau)|\big|f\big(\bar{u}(1,\tau)\big) - \bar{f}\big(\bar{u}(1,\tau)\big)\big|\,d\tau. \end{aligned}$$

By assumption the range of values of $\bar{u}$ on the line $x = 1$ is contained in the range over $x = 0$ for any interval $0 \leq t \leq T$ and by the collocation assumption $\bar{u}(0,t) = u(0,t) = h(t)$ at each of the points t_j, we have

$$\begin{aligned} |\phi_1(t)| \leq 2M&\int_0^t |\theta(1,t-\tau)|\,|\phi_0(\tau)|\,d\tau + \int_0^t |\theta(1,t-\tau)|\Delta f(\tau)\,d\tau \\ +&\int_0^t |\theta(0,t-\tau)|\Delta f(\tau)\,d\tau + 2M\int_0^t |\theta(0,t-\tau)|\,|\phi_1(\tau)|\,d\tau. \end{aligned}$$

Applying Gronwall's inequality to the above inequality, noting that the function $\theta(0,t)$ has an integrable singularity at $t = 0$, gives the statement of the lemma. □

Proof of Lemma 5: We have the identity

$$u(x,t)-\bar{u}(x,t)=\psi(x,t)-2\int_0^t \theta(x,t-\tau)\left\{f\big(u(0,\tau)\big)-\bar{f}\big(\bar{u}(0,\tau)\big)-\varepsilon(t)\right\}d\tau$$
$$-2\int_0^t \theta(x-1,t-\tau)\left\{f\big(u(1,\tau)\big)-\bar{f}\big(\bar{u}(1,\tau)\big)\right\}d\tau. \qquad (5.2)$$

Evaluating on the line $x=0$ gives

$$\phi_0(t)=\mathcal{A}\left\{f\big(u(0,t)\big)-\bar{f}\big(\bar{u}(0,t)\big)-\varepsilon(t)\right\}+\psi(0,t)$$
$$+\mathcal{S}\left\{f\big(u(1,t)\big)-\bar{f}\big(\bar{u}(1,t)\big)\right\}$$

expanding the arguments, and rearranging terms, we obtain the equivalent expression

$$\mathcal{A}\{f\big(\bar{u}(0,t)\big)-\bar{f}\big(\bar{u}(0,t)\big)-\varepsilon(t)\}=\mathcal{A}\{f\big(\bar{u}(0,t)\big)-f\big(u(0,t)\big)\}+\phi_0-\psi(0,t)$$
$$+\mathcal{S}\{f\big(u(1,t)\big)-f\big(\bar{u}(1,t)\big)+f\big(\bar{u}(1,t)\big)-\bar{f}\big(\bar{u}(1,t)\big)\}$$

or

$$f\big(\bar{u}(0,t)\big)-\bar{f}\big(\bar{u}(0,t)\big)=\varepsilon(t)+f\big(\bar{u}(0,t)\big)-f\big(u(0,t)\big)+\mathcal{A}^{-1}[\phi_0-\psi]$$
$$+\mathcal{A}^{-1}\mathcal{S}\{f\big(u(1,t)\big)-f\big(\bar{u}(1,t)\big)\}$$
$$+\mathcal{A}^{-1}\mathcal{S}\{f\big(\bar{u}(1,t)\big)-\bar{f}\big(\bar{u}(1,t)\big)\}.$$

Since $\mathcal{A}^{-1}\mathcal{S}$ is an integral operator with an integrable kernel, we can apply Gronwall's inequality to the function

$$f\big(\bar{u}(0,t)\big)-\bar{f}\big(\bar{u}(0,t)\big)$$

and use the range assumption that

$$[\bar{u}(1,0),\bar{u}(1,t)]\subset[\bar{u}(0,0),\bar{u}(0,t)]$$

to obtain

$$\left|f\big(\bar{u}(0,t)\big)-\bar{f}\big(\bar{u}(0,t)\big)\right|\le\left|f\big(u(0,t)\big)-f\big(\bar{u}(0,t)\big)\right|+|\mathcal{A}^{-1}[\phi_0-\psi]|+|\varepsilon(t)|$$
$$+\left|\mathcal{A}^{-1}\mathcal{S}\{f\big(u(1,t)\big)-f\big(\bar{u}(1,t)\big)\}\right|.$$

Using the uniform Lipschitz assumption on f, and (2.10), we obtain

$$\left|f\big(\bar{u}(0,t)\big)-\bar{f}\big(\bar{u}(0,t)\big)\right|\le M\left|\phi_0(t)\right|+|\mathcal{A}^{-1}[\phi_0-\psi]|+CMt\|\phi_1\|_{\infty,[0,t]}+|\varepsilon(t)|.$$

Taking the supremum of the right hand side (over t), we obtain the conclusion of the lemma. $\square$

Proof of Lemma 6: Since $\bar{f}$ is piecewise linear, with slopes c_j we have $\|\bar{f}'\|_\infty = \max_{j=1,\dots,N}\{c_j\}$. Therefore, since $(f - \bar{f} - \varepsilon)' \in L_\infty$ and $(f - \bar{f} - \varepsilon) = 0$ at $t = 0$, $(u_t - \bar{u}_t - \psi_t) \in C^{1/2}[0,T]$. By the collocation condition, $u - \bar{u} - \psi$ vanishes at the collocation points, and by the intermediate value theorem the derivative vanishes at some point in the interior of each interval. This immediately yields the result

$$\sup_{0\le t\le T} |u - \bar{u} - \psi| \le C\left(\Delta t\right)^{1+\alpha} \tag{5.3}$$

$$\sup_{0\le t\le T} |u_t - \bar{u}_t - \psi_t| \le C\left(\Delta t\right)^{\alpha} \tag{5.4}$$

for any $0 < \alpha < 1/2$. Note that it is only the smoothness of $\bar{f}$ and the initial error (propagated through ψ) that limit the convergence rate. $\square$

Proof of Theorem 1: To prove Theorem 1, we substitute (3.9) into (3.10), collect the terms involving $\|\Delta f\|_{\infty,[0,T]}$, and sup over $t \in [0,T]$ with T sufficiently small. The estimate essentially follows from the representation theorem, and is valid for any $f \in C^\alpha$. The procedure can be continued for large T by bootstrapping. $\square$

Proof of Theorem 2: The fact that there is a unique piecewise-linear function $\bar{f}$ follows from monotonicity. We use Lemmas 2 and 6 to obtain the stated bound. $\square$

Proof of Theorem 3: Let (u_1, f_1) and (u_2, f_2) be two solutions to (1.1) – (1.4). We make the assumptions that the data $g(x,t)$ and $u_0(x)$ satisfy **A3** and **A4**, and assume that both f_1 and f_2 lie in C^α for $\alpha > 1/2$. This last condition is to guarantee the existence of a strong solution to the direct problem.

If $f_1(h(0)) > f_2(h(0))$ then for some $t' > 0$, $f_1(v) > f_2(v)$ for $h(t') \le v \le h(0)$. Lemma 3 then gives $u_1(x,t;f_1) < u_2(x,t;f_2)$ for $0 < t \le t'$ and in particular $u_1(1,t) < u_2(1,t)$ which is in violation of (1.4), since both these functions must agree with the overposed data $h(t)$ at this point.

If $f_1(h(0)) = f_2(h(0))$, and $f_1 - f_2$ does not vanish identically in a neighborhood of $h(0)$, then there is a $t' > 0$, such that for $h(t') \le v \le h(0)$ either $f_1(v) > f_2(v)$, or $f_1(v) < f_2(v)$, or $f_1(v) = f_2(v) + \tilde{f}(v)$ where $\tilde{f}$ is a C^α function with an infinite number of zeroes in any neighborhood of $h(0)$. In the first two cases lemma 3 once again gives a contradiction with

(1.4). The last case is precisely the statement of theorem 2, and the corollary is a direct consequence. □

Note that if the data and f are sufficiently smooth then the solution u will be C^2 in time, globally. For $\bar{f}$ piecewise linear, $\bar{u}$ will be C^2 in time on the boundary *between* collocation points $\{t_i\}$, but not globally. (The discontinuities of the derivatives of $\bar{u}$ on the boundary can occur only at the collocation points.)

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Determining the Initial Age Distribution for an Age Structured Population

Determining the initial age distribution for an age structured population

*Michael Pilant**
*William Rundell **

Department of Mathematics
Texas A&M University
College Station Tx. 77843.

Abstract: This paper deals with an inverse problem in age-structured population dynamics; the recovery of an unknown initial age distribution. We attempt to recover this function from overposed data which consists of either the total population over a time interval equal to the maximum life span of the species or the age structure of the population at a fixed later time. Existence, uniqueness and continuous dependence of the initial distribution function on the data are addressed. Some numerical simulations are presented to illustrate the feasibility of recovery using the methods of the paper.

Key Words: Inverse problem, initial function, age structured population, ill-posed.

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1. Introduction.

In population dynamic models that consider the age structure of the species, we are interested in finding $\rho(a,t)$, the number of individuals of age a alive at time t. The total population $P(t)$ of the species is then given by $P(t) = \int_0^L \rho(a,t)\,da$, where L is the *maximum life span* of the species. The first mathematical model along these lines was proposed by Lotka. In this treatment one assumes as known, a *birth function* β and a *death function* λ. $\beta(a)da$ and $\lambda(a)da$ are the probability of an individual of age a giving birth or dying in the interval a to $a + da$. The functions β and λ depend on the age a of the individual. Further, to incorporate cooperation or competition between individuals of the species, they should be considered to depend on the total population $P(t)$ or possibly on $\rho(a,t)$. In this paper we restrict our attentions to the linear model.

One standard problem is the following; given an initial population with age distribution $\phi(a) \equiv \rho(a,0)$, and the functions β and λ, determine the time evolution of the population $\rho(a,t)$. We shall refer to this problem as the *direct problem*. In the case where λ and β are functions only of age, the models of Lotka, [Lotka, 1956], and McKendrick [McKendrick, 1926] leads to respectively a linear integral or a linear differential equation to be solved for the function $\rho(a,t)$. A nonlinear model where the birth and death processes depend on the total population was introduced by Gurtin and MacCamy [Gurtin and MacCamy, 1974], and Hoppensteadt [Hoppensteadt, 1974].

In practice, it may not be possible to monitor a species sufficiently closely in order to obtain a good approximation to some of the parameters required for the solution of the direct problem. In a recent paper, [Rundell, 1989], it was shown that one could recover an unknown birth function from overposed data consisting of the total population, a quantity that may be much more easily determined by an experimenter, over an appropriate time interval. Problems such as these are usually referred to as *inverse problems*. Another situation that can occur is when one is unable to accurately compute the value of the initial age structure, that is, when it is difficult or impossible to take a census. In this paper we investigate the possibility of determining the initial age structure $\phi(a)$ from a knowledge of the birth and death functions and the total population $P(t)$ measured over an interval of time equal to a lifespan of the species. We show that conditions can be imposed on the data that will guarantee a unique solution of the inverse problem, and that the stability of the recovery scheme depends on the behaviour of the death function near the maximum life span. We also investigate the problem of recovering an "initial" age structure $\phi(a)$, assuming as known λ and β, from a knowledge of the age structure $\rho(a,T)$ at a later time T. Thus we attempt to recover unknown census data at a previous time by using information on the birth and death process and census data taken at the present.

We have performed numerical experiments to investigate the feasibility of the methods we introduce, and some of these are presented in this paper.

2. Notation and the solution of the Direct Problem.

We let L denote the maximum life span of the species, and use $\rho(a,t)$ to represent the number of individuals of age a alive at time t. We use $\lambda = \lambda(a)$ to represent the *death function*, and $\beta = \beta(a)$ for the *birth function*. The differentials λda and βda represent the probabilities of an individual dying or giving birth in the interval $(a, a+da)$ respectively. The function

$$\pi(a) = \exp\left\{-\int_0^a \lambda(r)\,dr\right\} \tag{1}$$

is called the *life table* of the species, and is the probability of an individual living to age a. $\pi(a)$ is a smooth monotone non-increasing function of age, satisfying $\pi(0) = 1$ and $\pi(L) = 0$. $B(t) = \rho(0,t)$ is the *birth rate* at time t and can be written in terms of the birth function as

$$B(t) = \int_0^L \rho(a,t)\beta(a)\,da\,. \tag{2}$$

The *initial age distribution* $\phi(a)$ is given by

$$\rho(a,0) = \phi(a),. \tag{3}$$

It will be convenient to introduce the weighted variables $\tilde{\phi}(a) = \phi(a)/\pi(a)$, and the *net maternity function* $b(a) = \beta(a)\pi(a)$. It can easily be shown that if the life table, birth rate and initial age distribution are known then $\rho(a,t)$ can be computed and is given by, ([Keyfitz, 1968] or [Webb, 1985])

$$\rho(a,t) = \begin{cases} \pi(a)\tilde{\phi}(a-t) & \text{if } t \le a; \\ \pi(a)B(t-a) & \text{if } t > a. \end{cases} \tag{4}$$

In fact (4) can be used to show that $B(t)$ must satisfy the *Sharp-Lotka* equation

$$B(t) = \int_0^t b(a)B(t-a)\,da + \int_t^L b(a)\tilde{\phi}(a-t)\,da \tag{5}$$

valid for $t \le L$. We also note that if b vanishes on the interval $a_2 < t \le L$ then we have on this interval

$$B(t) = \int_0^t b(a)B(t-a)\,da\,. \tag{6}$$

If $t > L$, $B(t)$ satisfies

$$B(t) = \int_0^L b(a)B(t-a)\,da\,. \tag{7}$$

The first integral in (5) represents the births from those individuals born in the interval $(0,t]$, while the second represents the births at time t from the initial population. Given β, π and ϕ, this linear Volterra integral equation of the second kind can be uniquely solved for $B(t)$. The *total population* alive at time t, $P(t) = \int_0^L \rho(a,t)da$, is given by the relation

$$P(t) = \int_0^t \pi(a)B(t-a)\,da + \int_t^L \pi(a)\tilde{\phi}(a-t)\,da \tag{8}$$

It can be shown that if β, λ and ϕ are nonnegative, continuous functions on $[0, L)$, then $\rho(a,t)$ is continuous on $[0,t)$ and (t, L), with at most a jump discontinuity at $a = t$. From equation (4) the function $\rho(a,t)$ will be continuous at $a = t$ if and only if $\phi(0) = B(0) \equiv \int_0^L \beta(a)\phi(a)da$. We do not impose this requirement on our model.

In McKendrick's equivalent formulation of the direct problem [McKendrick, 1926], $\rho(a,t)$ is given by the solution of the first order hyperbolic equation

$$\frac{\partial \rho}{\partial a} + \frac{\partial \rho}{\partial t} + \lambda(a)\rho = 0\,, \tag{9}$$

with the initial-boundary conditions formed by equation (3) and

$$\rho(0,t) = \int_0^L \beta(a)\rho(a,t)\,da\,. \tag{10}$$

This approach will be useful in the reconstruction of the initial distribution from the values of $\rho(a,T)$ for some later time T.

Some remarks on notation.

The r^{th} derivative of a function $f(t)$ we denote by $f^{(r)}(t)$.

We let $\mathcal{L}[f]$ denote the Laplace transform of a function and we denote by $f * g$ the convolution $\int_0^t f(t-s)g(s)ds$.

For a function $f(s)$ defined on the interval $0 \leq s \leq L$ we define $\bar{f}(s) = f(L-s)$.

The following result is elementary, but will be used sufficiently often in this paper to merit separate attention.

Lemma 1. *For any continuous functions k and g defined on a subset of $[0,\infty)$ with k not identically zero there is at most one solution of the convolution equation $k * f = g$. If k and g have m continuous derivatives with $k^{(r)}(0) = 0$ for $r < m$ while $k^{(m)}(0) \neq 0$, then there exists a continuous solution f of the convolution equation and this solution depends continuously on the functions $k^{(m)}$ and $g^{(m)}$ in the supremum norm.*

Proof: For the first part, note that by extending as a constant to all $[0,\infty)$ we can form the Laplace transform of the functions and obtain the relation $\mathcal{L}[k].\mathcal{L}[f] = \mathcal{L}[g]$. If the convolution equation had two solutions f_1 and f_2 then their difference f would satisfy $\mathcal{L}[k].\mathcal{L}[f] = 0$. Since the product of the two analytic functions $\mathcal{L}[k]$ and $\mathcal{L}[f]$ cannot vanish identically unless one of the is identically zero, we obtain that $\mathcal{L}[f] = 0$ and hence $f = 0$.

For the second part, if we differentiate the convolution m times, using the fact that $k^{(r)}(0) = 0$ if $r < m$, we have the second kind Volterra equation $k^{(m)}(0)f(t) + \int_0^t k^{(m)}(t-s)f(s)ds = g^{(m)}(t)$.

A continuous solution to this type of equation exists provided that the kernel and right-hand side are continuous, [Linz, 1985].

This result cannot be improved upon, that is, we cannot expect continuous dependence of the solution f to the convolution equation $k*f = g$ unless the above conditions hold. A counterexample will be provided later in the context of the inverse problem.

Recovery of initial distribution from total population

We shall make the following assumption on the birth and death processes,

A1. For some positive integer $m \geq 1$, $\pi(a)$ and $\beta(a)$ are $2m+1$ times continuously differentiable on $[0, L]$. $\beta(a)$ is nonnegative and has support contained in the interval $[a_1, a_2]$ where $0 < a_1 < a_2 < L$. $\pi(a)$ is a monotonically nonincreasing function with $\pi(0) = 1$ and whose r^{th} derivative $\pi^{(r)}(L)$ is zero for $0 \leq r < m$, but $\pi^{(m)}(L) \neq 0$.

These assumptions should be valid for a wide range of species. The behaviour of $\pi(a)$ at the maximum life span L will be crucial to our analysis. By definition this function must vanish at $a = L$, but the issue is whether its derivatives do likewise. For humans in a developed society it may very well be that $\pi^{(r)}(L) = 0$ for all values of r, but this is unlikely to be true for any species for which the aged are under increased environmental pressure. As we will see, the larger the value of m, the more ill-conditioned the inverse problems will be. In our numerical runs we have used $m = 1$.

We assume that the function $P(t)$ is known for $0 \leq t \leq L$ and satisfies

A2. $P(t)$ is $m+2$ times continuously differentiable on the interval $[0, L]$

We can rewrite equations (5) and (8) in the form

$$B(t) = (b * B)(t) + \Psi_1(t) \tag{11}$$

$$P(t) = (\pi * B)(t) + \Psi_2(t) \tag{12}$$

where

$$\Psi_1(t) = \int_t^L b(a)\tilde{\phi}(a - t)\, da \tag{13}$$

$$\Psi_2(t) = \int_t^L \pi(a)\tilde{\phi}(a - t)\, da \tag{14}$$

Lemma 2. *The functions Ψ_1 and Ψ_2 are related by $\Psi_1(t) = \int_t^L \gamma(a - t)\Psi_2(a)\, da$ where $\gamma(t)$ is the solution of*

$$b(t) = \int_0^{L-t} \pi(a + t)\gamma(a)\, da\,, \tag{15}$$

or equivalently

$$\bar{b}(t) = \int_0^t \bar{\pi}(t - a)\gamma(a)\, da\,. \tag{16}$$

The function γ is m times continuously differentiable on $[0, L]$ and γ vanishes identically on $[0, L - a_2]$.

Proof: We have $\bar{\Psi}_1(t) = \int_0^t \bar{b}(t - a)\bar{\phi}(a)da$ and $\bar{\Psi}_2(t) = \int_0^t \bar{\pi}(t - a)\bar{\phi}(a)da$, that is, $\bar{\Psi}_1 = \bar{b} * \bar{\phi}$ and $\bar{\Psi}_2 = \bar{\pi} * \bar{\phi}$.

Now

$$\begin{aligned}\int_0^t \gamma(t-s)\bar{\Psi}_2(s)\,ds &= \int_0^t \gamma(t-s)\int_0^s \bar{\pi}(s-a)\tilde{\phi}(a)\,da\,ds\\ &= \int_0^t \int_a^t \gamma(t-s)\bar{\pi}(s-a)\,ds\,\tilde{\phi}(a)\,da\\ &= \int_0^t \int_0^{t-a} \gamma(t-a-s)\bar{\pi}(s)\,ds\,\tilde{\phi}(a)\,da\\ &= \int_0^t \bar{b}(t-a)\tilde{\phi}(a)\,da\\ &= \bar{\Psi}_1(t)\end{aligned}$$

where we have used (16). This gives

$$\begin{aligned}\bar{\Psi}_1(t) &= \int_0^t \gamma(t-s)\bar{\Psi}_2(s)\,ds\\ &= \int_{L-t}^L \gamma(t+a-L)\Psi_2(a)\,da\end{aligned}$$

or

$$\Psi_1(t) = \int_t^L \gamma(a-t)\Psi_2(a)\,da$$

Note that since $\pi(L) = 0$, a differentiation of (15) will lead to another Volterra integral equation of the first kind for γ. Since $\pi^{(m)}(L) \neq 0$, $m+1$ differentiations will convert this to a second kind Volterra equation

$$b^{(m+1)}(t) = (-1)^m\left\{\pi^{(m)}(L)\gamma(L-t) + \int_0^{L-t} \pi^{(m+1)}(a+t)\gamma(a)\,da\right\} \tag{17}$$

from which γ can be uniquely recovered. The fact that $\gamma(t)$ is zero for $0 \leq t \leq L - a_2$ follows from assumption **A1** and (16).

This completes the proof of the lemma.

From (11) and (12) we obtain

$$\begin{aligned}\pi * B &= \pi * b * B + \pi * \Psi_1\\ b * P &= b * \pi * B + b * \Psi_2\end{aligned}$$

and hence

$$\Psi_2 - b * \Psi_2 + \pi * \Psi_1 = P - b * P.$$

Using lemma 2 and some amount of rearrangement we can write this as

$$\Psi_2(t) + \int_0^L K(t,a)\Psi_2(a)\,da = d(t) \tag{18}$$

where

$$d(t) = P(t) - \int_0^t b(t-a)P(a)\,da$$

and

$$K(t,a) = \begin{cases} \int_0^a \pi(t-a+s)\gamma(s)ds - b(t-a) & a < t, \\ \int_{a-t}^a \pi(t-a+s)\gamma(s)ds & a > t. \end{cases}$$

The continuity assumptions on γ and π show that $K(t,a)$ is $m+1$ times continuously differentiable in a and t for $t \neq a$.

For $a < t$ and $1 \le r \le m+1$ we have

$$\frac{\partial^r K}{\partial t^r} = \int_0^a \pi^{(r)}(t-a+s)\gamma(s)\,ds - b^{(r)}(t-a)$$

Consequently,

$$\lim_{a \to t^-} \frac{\partial^r K}{\partial t^r} = \int_0^t \pi^{(r)}(s)\gamma(s)\,ds. \tag{19}$$

For $a > t$ and $1 \le r \le m+1$ we obtain by using the properties of γ and its derivatives at $t = 0$ that

$$\frac{\partial^r K}{\partial t^r} = \sum_{k=0}^{r-1} \pi^k(0)\gamma^{(r-1-k)}(a-t) + \int_{a-t}^a \pi^{(r)}(t-a+s)\gamma(s)\,ds$$

and hence

$$\lim_{a \to t^+} \frac{\partial^r K}{\partial t^r} = \int_0^t \pi^{(r)}(s)\gamma(s)\,ds. \tag{20}$$

Equations (19) and (20) now show that $K(t,a)$ and its partial derivatives up to order $m+1$ are continuous for $0 \le t \le L$.

The operator $\mathcal{K}f = \int_0^L K(t,a)f(a)da$ is compact and thus has only a finite number of eigenvalues in the exterior of any neighborhood excluding the origin. Furthermore $\mathcal{K}$ is neither symmetric nor definite. If the value -1 is not in the spectrum of the operator then there will be a unique solution to (18). This condition depends only on the birth and death functions. It is difficult to give conditions on these functions that will guarantee the solvability of (18). Numerically one can compute the spectrum of $\mathcal{K}$, in particular the eigenvalue with largest negative real part. For all reasonable functions π and β that we investigated, this eigenvalue had significant complex component. We will therefore assume that

A3. The functions π and β are such that the spectrum of $\mathcal{K}$ does not contain the point -1.

We now show how to recover $\tilde{\phi}$. The regularity assumption **A2** shows that $d(t)$ is $m+1$ times differentiable. Thus a solution of (18) will possess $m+1$ continuous derivatives in view of what we have shown about $K(t,a)$

By differentiating (14) $m+1$ times and using the fact that $\pi^{(r)}(L) = 0$ for $r < m$, we obtain

$$\Psi_2^{(m+1)}(t) = (-1)^m \Big\{ \pi^{(m)}(L)\tilde{\phi}(L-t) + \int_0^{L-t} \pi^{(m+1)}(a+t)\tilde{\phi}(a)\,da \Big\}$$

or

$$\Psi_2^{(m+1)}(L-t) = -\pi^{(m)}(L)\tilde{\phi}(t) + \int_0^t \pi^{(m+1)}(L-t+a)\tilde{\phi}(a)\,da \tag{21}$$

From assumptions **A1** it follows that the second kind Volterra equation (18) has a unique continuous solution $\tilde{\phi}$ on $[0, L]$ that depends continuously on the functions $\Psi_2^{(m+1)}(t)$ and $\pi^{(m+1)}(a)$ in the supremum norm. From (18) and Lemma 1, this means that $\tilde{\phi}$ depends continuously on the functions $\pi^{(m+1)}(a)$, $\beta^{(m+1)}(t)$ and $P^{(m+1)}(t)$, in the supremum norm.

We thus have shown

Theorem 1. *If the birth function and life table satisfy assumption* **A1**, **A2** *and* **A3**, *then there exists a unique initial age distribution* $\phi(a)$ *which gives rise to an observed total population on the interval* $[0, L]$.

Remarks:

1. The assumptions **A1** – **A3** are not sufficient to guarantee that the initial function $\phi(a)$ will be nonnegative, which is the only admissible class of such functions. From a practical standpoint this is not a serious objection since any physically observed $P(t)$ must come from a nonnegative initial distribution.
2. From the above it is clear that one must provide the values of the overposed data on an interval of length L. If this interval is in $[T, L+T]$ then this analysis shows that one can recover $\rho(a, T)$. When $T = 0$ this recovers the initial population structure $\rho(a, 0) \equiv \phi(a)$. In the next section we will discuss the problem of determining $\phi(a)$ from overposed data $\rho(a, T)$.

Recovery of initial distribution from census data

Here we assume that the functions $\pi(a)$, $\beta(a)$ are known as well as the function $\psi(a)$ which is equal to $\rho(a, T)$ for some fixed $T > 0$.

For this inverse problem we will require slightly different conditions on the birth and death processes.

- **B1.** For some positive integer $m \geq 1$, $\pi(a)$ and $\beta(a)$ are $m+1$ times continuously differentiable on $[0, L]$. $\pi(a)$ is a monotonically nonincreasing function with $\pi(0) = 1$ and $\pi(L) = 0$. $\beta(a)$ is nonnegative, has support contained in the interval $[a_1, a_2]$ where $0 < a_1 < a_2 < L$, and $\beta^{(m)}(a_2) \neq 0$.

For the overposed data $\psi(a)$, we require

- **B2.** $\psi(a)$ is an $m+1$ times continuously differentiable function on $[0, T)$ and on $(T, L]$. It has at most a jump discontinuity at $a = T$ in the case that $T < L$. If $T \leq a_2$ then the compatibility condition $\tilde{\psi}^{(r)}(0) = \int_0^T \beta(a)\tilde{\psi}^{(r)}(a)da + \int_T^L \beta(a)\tilde{\psi}^{(r)}(a)da$ holds.

Note that we are imposing a boundary condition of the support of the birth function, rather than on the death function as we did in the previous section.

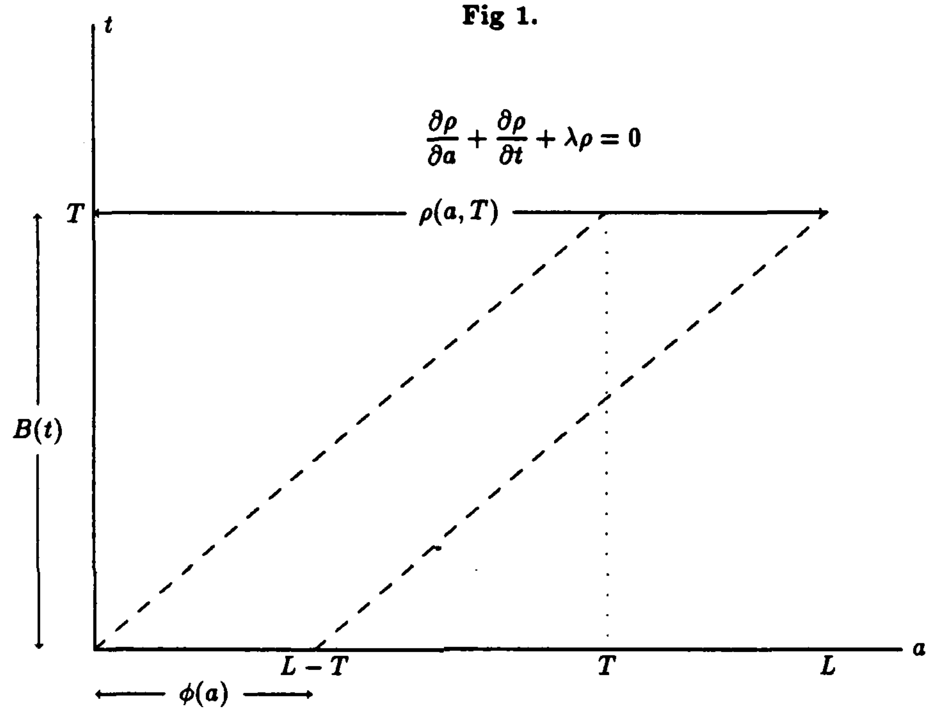
We first show that we can construct the birth function $B(t)$ over the range $0 \leq t \leq T$.

In the case $T < L$ we have from (4) that

$$\psi(a) = \rho(a, T) = \begin{cases} \pi(a)\tilde{\phi}(a - T) & \text{if } a \geq T, \\ \pi(a)B(T - a) & \text{if } a < T, \end{cases} \tag{22}$$

and thus knowledge of $\psi(a)$ implies that we may simply read off the values of $\tilde{\phi}(a)$ for the range $0 \leq a < L - T$. To recover the initial distribution for ages $a > L - T$ we must use the information provided by ψ to obtain the birth function $B(t)$ over the range $0 < t \leq T$. We assume that B is continuous at $t = 0$, so that $B(0)$ can be recovered by its limit as $t \to 0$. The occurence of the term $\pi(a)$ in equation (22) suggests the utility of working with $\tilde{\psi}$ defined by $\tilde{\psi}(a) = \psi(a)/\pi(a)$.
We illustrate the above ideas in Figure 1 below.

Fig 1.



The Sharp-Lotka equation gives for $T \leq t \leq L$

$$B(t) = \int_0^T b(t-a)B(a)\,da + \int_T^t b(t-a)B(a)\,da + \int_t^L b(a)\tilde{\phi}(a-t)\,da$$

and since $\int_t^L b(a)\tilde{\phi}(a-t)da = \int_0^{L-t} b(a+t)\tilde{\psi}(a+T)da$ is known for $T \leq t \leq L$ we have that $B(t)$ satisfies the equation

$$B(t) - \int_T^t b(t-a)B(a)\,da = \alpha_1(t) \qquad \text{for } t \geq T \tag{23}$$

where $\alpha_1(t) = \int_t^L b(a)\tilde{\psi}(T+a-t)da + \int_{t-T}^t b(a)\tilde{\psi}(T-t+a)da$ is defined for $T \leq t \leq L$ and can be computed from π, β and $\psi(a)$.

By lemma 1, equation (23) can be solved uniquely for the function $B(t)$ for $t \geq T$ and the continuity of $\alpha_1(t)$ shows that this solution will be continuous on $(T, L]$.

From the fact that $b(0) = 0$, we can show that

$$\lim_{t \to T^+} B^{(r)}(t) = \lim_{t \to T^+} \alpha_1^{(r)}(t) = (-1)^r \{ \int_0^T \beta(a)\tilde{\psi}^{(r)}(a)da + \int_T^L \beta(a)\tilde{\psi}^{(r)}(a)da \}$$

From (22) it follows that

$$\lim_{t \to T^-} B^{(r)}(t) = (-1)^r \tilde{\psi}^{(r)}(0)$$

From the Sharp-Lotka equation it follows that if ϕ is continuous (actually even if ϕ is piecewise continuous) then the birth rate $B(t)$ must have the same differentiability as the net maternity function $b(t)$. Thus the condition

$$\tilde{\psi}^{(r)}(0) = \int_0^T \beta(a)\tilde{\psi}^{(r)}(a)da + \int_T^L \beta(a)\tilde{\psi}^{(r)}(a)da$$

on the overposed data is necessary if it actually arises from a physically reasonable initial distribution and a smooth birth function.

Suppose now that $nL \leq T \leq (n+1)L$ for $n \geq 1$. Then $\psi(a) = \rho(a,T)$ being given, implies that $B(t)$ is known for $T - L \leq t \leq T$. We show this implies that $B(t)$ can be recovered over the range $\max\{0, T-2L\} \leq t \leq T - L$. For $t \geq L$,

$$\begin{aligned} B(t) &= \int_0^L b(a)B(t-a)\,da \\ &= \int_{t-L}^t b(t-a)B(a)\,da \\ &= \int_{t-L}^{T-L} b(t-a)B(a)\,da + \int_{T-L}^t b(t-a)B(a)\,da\,. \end{aligned}$$

If $t \geq \max\{L, T-L\}$ then both $B(t)$ and $\int_{T-L}^t b(t-a)B(a)da$ are known. This implies that the function $\alpha_2(t)$ which is equal to their difference is known for the range $\max\{L, T-L\} \leq t \leq T$. This leads to the convolution equation

$$\alpha_2(t) = \int_{t-L}^{T-L} b(t-a)B(a)\,da \tag{24}$$

which can be solved uniquely for $B(t)$ for t in the range $\max\{0, T-2L\} \leq t \leq T-L$. By continuing this argument we can reconstruct the birth function $B(t)$ over the range $0 \leq t \leq T$.

Note however that each time we compute $B(t)$ over an interval $T-(n+1)L \leq t \leq T-nL$ from previously computed values on the interval $T - nL \leq t \leq T-(n+1)L$ we must solve a first kind convolution integral equation whose kernel vanishes to order m at $a = 0$. Clearly the more times we have to solve this ill-posed problem the more inaccurate our knowledge of the function $B(t)$ will be. Thus we should expect that if $T < L$ then recovery of $B(t)$, which entails solving the second kind equation (23), should be straightforward and the solution obtained should depend continuously on the data π, β and ψ. If $T > nL$ then we must solve n first kind equations of the form (24) in order to recover $B(t)$ for $0 \leq t \leq L$. Thus the longer we wait to take the census that provides the values of $\psi(a)$, the poorer we should expect any reconstruction of the initial distribution to be, and that a significant difference should occur if T is chosen to be greater than the lifespan L, and a further degradation should be encountered as T is allowed to exceed each subsequent generation.

Having recovered the function $B(t)$ for all $t \geq 0$, the Sharp-Lotka equation gives the following representation for $\tilde{\phi}$ on $0 \leq t \leq L$

$$\int_0^{L-t} b(t+a)\tilde{\phi}(a)\, da = B(t) - \int_0^t b(t-a)B(a)\, da \equiv \alpha_3(t).$$

This can be rewritten as the convolution equation $\int_0^t b(L-t+s)\tilde{\phi}(s)\, ds = \alpha_3(L-t)$ and since β vanishes identically on the interval $[a_2, L]$ this is equivalent to

$$\int_0^{t-(L-a_2)} b(L-t+s)\tilde{\phi}(s)\, ds = \alpha_3(L-t) \tag{25}$$

from which it follows that there is at most one function $\tilde{\phi}$ defined on the interval $[0, a_2)$ that solves the inverse problem. A necessary condition for a bounded solution of (25) is that the function $\alpha_3(t)$ vanish with its $m+1$ derivatives at the point $t = a_2^+$. This is a further condition on $B(t)$ which corresponds to a condition on the overposed data.

As in the previous problem we can convert (25) to a second kind equation by $m+1$ differentations.

$$-b^{(m)}(a_2)\tilde{\phi}(t) + \int_0^t b^{(m+1)}(a_2-t+s)\tilde{\phi}(s)\, ds = \alpha_3^{(m+1)}(a_2-t) \tag{26}$$

and obtain the uniqueness of a continuous solution $\phi(a)$. Here the fact that $\beta^{(m)}(a_2) \neq 0$ is crucial. Once again we have not given any conditions that guarantee a nonnegative solution.

In the case that $T < L$ we may utilize the information that $\tilde{\phi}(a) = \tilde{\psi}(a+T)$ for $a \leq L-T$ and rewrite (25) as

$$\int_{L-T}^t b(a_2-t+s)\tilde{\phi}(s)\, ds = \alpha_3(a_2-t) - \int_0^{L-T} b(a_2-t+s)\tilde{\psi}(a+T)\, da \qquad \text{for } L-T \leq t \leq a_2 \tag{27}$$

The above analysis demonstrates the recovery of the initial age distribution for the range $0 \leq a \leq a_2$. What about the interval (a_2, L)? If $T > L - a_2$ then we can never recover these values. Those individuals in the initial population whose age is greater then a_2 do not participate in the birth process and have all died within a time period of length $L - a_2$.

Given the remarks following lemma 1, we would not expect that the initial data $\phi(a)$ would depend continuously in the supremum norm on the function $\psi(a)$ or any of its derivatives. That this is so can be seen by taking $\tilde{\phi}_n(a) = \sqrt{n}\chi[0, 1/n]$, and considering the values it imposes on the final data $\psi_n(a) = \rho(a, L)$ at a time equal to the lifespan. We assume that β and π are C^∞ functions. $\Psi_n(t) = \int_0^{L-t} b(a+t)\tilde{\phi}_n(a)da$, and thus for any integer r

$$\begin{aligned}\Psi_n^{(r)}(t) &= \int_0^{L-t} b^{(r)}(a+t)\tilde{\phi}_n(a)\, da \\ &= \sqrt{n}\int_0^{1/n} b^{(r)}(a+t)\, da\end{aligned}$$

and thus

$$\sup_{0\le t\le L} |\Psi_n^{(r)}(t)| \le \frac{1}{\sqrt{n}} \sup_{0\le a\le L} b^{(r)}(a).$$

For any r, the function $\Psi_n^{(r)}(t)$ tends to zero in the supremum norm as $n \to \infty$, but $\bar{\phi}_n$ does not tend to a continuous function. The Sharp-Lotka equation gives $B_n(t) = \int_0^t b(a)B_n(t-a)da + \Psi_n(t)$ and by lemma 1 we see that $B_n^{(r)}(t)$ will also tend to zero as $n \to \infty$ for any r. Since $\psi_n(a) = \rho(a, L) = \pi(a)B_n(L-a)$ this shows that the final data, $\psi_n(a)$, and all its derivatives can tend uniformly to zero as $n \to \infty$, yet the initial distribution fails to converge to a continuous function.

Some numerical experiments

In order to demonstrate the feasibility of recovering the initial age distribution from ϕ, λ and either $P(t)$ or $\rho(a,T)$, we give the results of some numerical experiments. By this means we hope to show the extent to which numerical recovery of the initial age distribution can be recovered and some of the complicating factors in the process.

It was assumed that the birth function β and the life table function π were known. The maximum life span L was chosen to be 10. We used a life table $\pi(a)$ as shown in figure 2. This was constructed by taking a cubic function passing through the points $(0,1)$, $(1,0.8)$, $(7,0.3)$ and $(10,0)$. If desired, a death function $\lambda(a)$ can then be obtained by setting $\lambda(a) = \pi'(a)/\pi(a)$. We used two slightly different birth functions β_1 and β_2 as we show in figure 3. The first of these satisfies the conditions of **A1**, and is a a twice continuously differentiable function on $[0, L]$ with support on the interval $[2,8]$ The function β_2 is similiar except it has a discontinuity in its first derivative at $a = \beta_2$ as required by **B1**.

In order to procure data for the inversion process we took an initial function $\phi_{act}(a)$ as shown in figure 2 and solved the direct problem numerically to obtain either the value of the total population $P(t)$ over an interval $[0, L]$, or $\rho(a,T)$ for some fixed time $T > 0$. This overposed data, as well as $\beta(a)$ and $\pi(a)$ were passed to the inversion routines to recover $\phi(a)$.

Fig 2. Life Table and Actual Initial Distribution

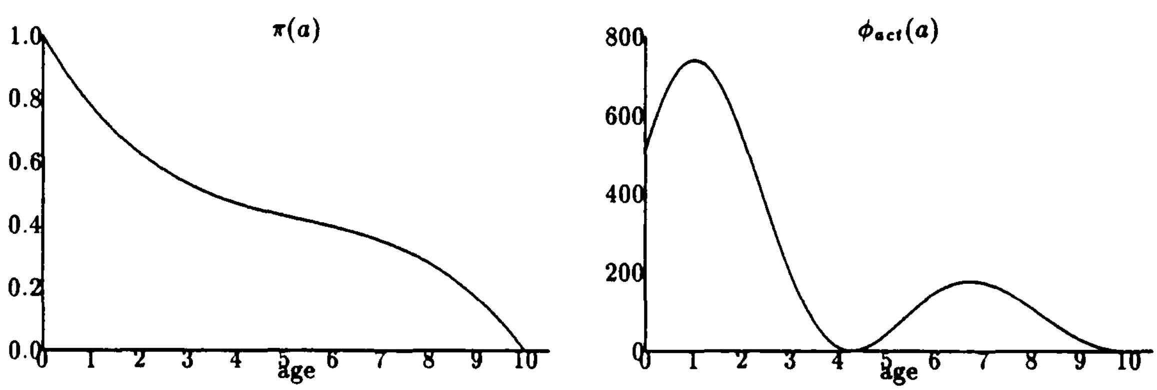
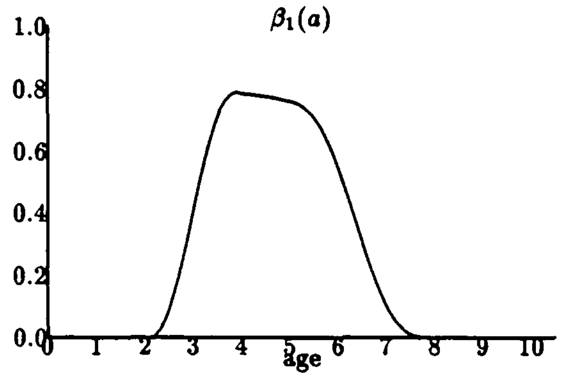


Fig 3 Birth Functions

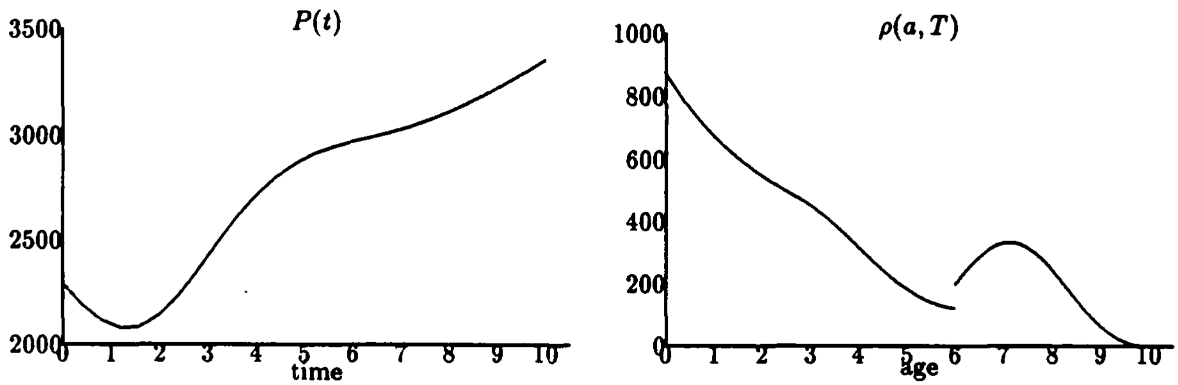


These functions are not meant to represent actual values for any particular species, but they contain some of the features common to a large class of population structures.

Some details on these numerical procedures will now be given.

For the first numerical test, we used the birth function $\beta_1(a)$, solved the direct problem using $\phi_{act}(a)$ to obtain a total population function $P(t)$ at a discrete series of gridpoints t_i on the time interval $0 \leq t \leq L$. A plot of the function $P(t)$ resulting from these values of π, β_1 and ϕ_{act} is shown in figure 4.

Fig 4 Plots of the Overposed Data



The values of $P(t_i)$ were then passed to our inverse solving code to attempt to recover the values of ϕ. To achieve this we recovered in succession the functions γ, Ψ_2 and finally ϕ. For the first and third of these we must solve a first kind Volterra equation (16), or convert it to second kind by differentation, and for the computation of Ψ_2 we need to solve a second kind Fredholm equation (18).

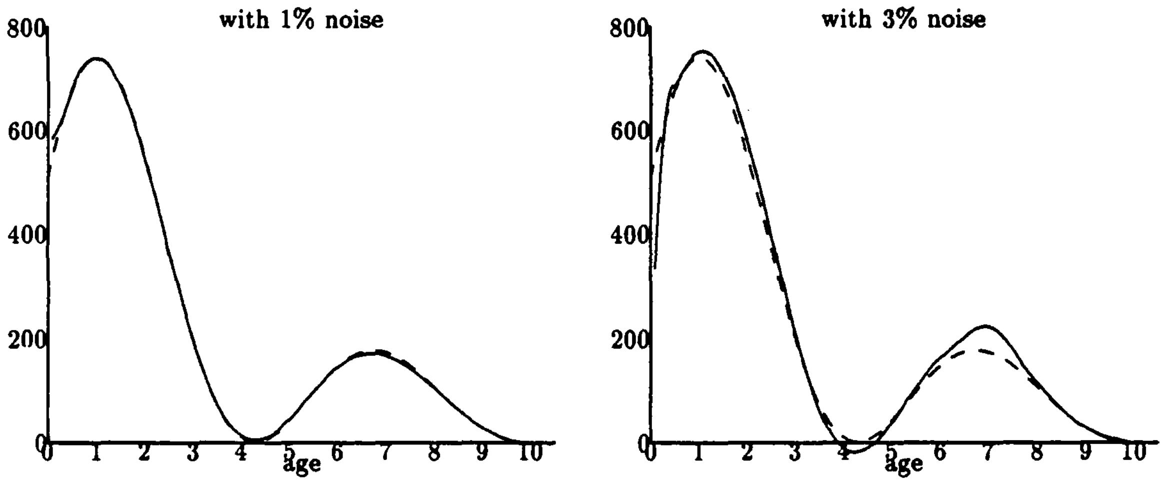
Where do the difficulties lie? Throughout we assume that both π and β are able to be computed exactly. Thus in theory we can solve for γ to within any precision demanded. Numerically, the question is whether to solve the Volterra equation as a first kind equation, (16), or second kind form, (17). The conventional wisdom is to take the former approach, especially if the evaluation of the kernel and free term may be subject to error. While we have assumed this is not the case here, there is another factor to consider. Virtually all algorithms for solving first kind Volterra equations of the form $\int_0^x k(x,t)f(t)dt = g(x)$ assume that the kernel $k(x,t)$ does not vanish on the diagonal $x = t$. Violation of this condition leads to instabilities. Rather than obtaining increased acccuracy with decreasing stepsize, this case gives increase in accuracy with diminishing stepsize only for sufficiently large stepsize. As the stepsize is further reduced the solution loses accuracy rapidly. This phenomenon is typical of ill-posed problems. In our problem if we use equation (16), $k(x,t) = \pi(L - x + t)$, and the non-vanishing condition is violated. In fact $k(x,x) = 0$ for all x. We therefore took the compromise approach of repeatedly differentiating the equation until the resulting kernel had a non-vanishing diagonal, but stopping short of the full conversion to second kind which would require differentiating the equation one more time. For our situation, since we are assuming that $m = 1$ in **A1**, we therefore differentiate equation (16) once and determine γ as the solution of

$$b'(L-t) = \int_0^t \pi'(L-t+a)\gamma(a)\,da. \tag{28}$$

Here the kernel satisfies $k(t,t) = \pi'(L) \neq 0$. The value of $\pi'(L)$ is approximately -0.2. We solved the above equation using a discretization method based on the midpoint rule, [Linz, 1985]. For the second stage we must recover Ψ_2 as the solution to (18). We solved this equation by the Nystrom method [Baker, 1979], which converts the integral equation to a linear system of equations. The kernel of the integral equation depends only on π and γ, and with our assumptions we should be able to evaluate this accurately. The free term $d(t)$ depends on π, β and also the measured population $P(t)$. Depending on the quality of this data we may or may not be able to accurately compute the free term. Any inaccuracies in this will be magnified by the condition number of the matrix of the system. Of course, if there is an eigenvalue of this matrix near the value -1 then this condition number could be very large. We investigated a wide selection of reasonable values for the birth and death functions and in no case was there a problem with the matrix inversion. For the values of π and β shown above the resulting matrix has condition numbers of approximately 9.5 for the case of β_1 and 2.8 for the case of β_2. Thus the error made in the recovery of ψ should be of a similar order of magnitude to the error in the observed $P(t)$. There are still some difficulties however. It follows from equation (14) that the function $\Psi_2(t)$ should satisfy the necessary conditions $\Psi_2(L) = \Psi_2'(L) = 0$ for recovery of a bounded initial distribution $\phi(a)$. From the integral equation (18), since $K(L,a) = K_t(L,a) = 0$, it follows that the function $d(t)$ should satisfy $d(L) = d'(L) = 0$. Given errors in $P(t)$ this will, in general, not be true. Thus although the magnitude of the error on the computed values of Ψ_2 may be small, and the average value of the relative error small, the relative error in the vicinity of the point $t = L$ may be considerable. This will significantly affect the computation in the third stage. We must solve equation (14), and once again we are faced with the question of whether to differentiate this equation prior to numerical solution, and if so, how many times. The issues are slightly different from those encountered in the first stage. The kernel is identical in both cases, but whereas the free term in (16) was assumed to be known to high accuracy, this may not be the case with equation (14). Since errors made in solving a Volterra equation for small values of t will always contribute to the solution for later t, it is imperative to have good starting values. However, the values of $\phi(a)$ for a small will depend on the values of Ψ_2 near L, which is precisely the place where this function has the largest relative error. This factor gives the greatest contribution to the error in effectively computing the initial distribution ψ.

With sufficiently accurate data on $P(t)$ we were able to recover the values of the function ϕ to within any desired accuracy. We then ran the inverse code with a value of $P(t)$ that had a certain percentage of noise added. To achieve this we added normally distributed random values to the (accurate) overposed data computed by the direct solver. This noisy data was then smoothed by a smoothing spline routine before being passed to the inverse solver. In figure 5 we show the results for 1% and 3% noise in the overposed data $P(t)$. The function ϕ_{act} is shown as a dashed curve.

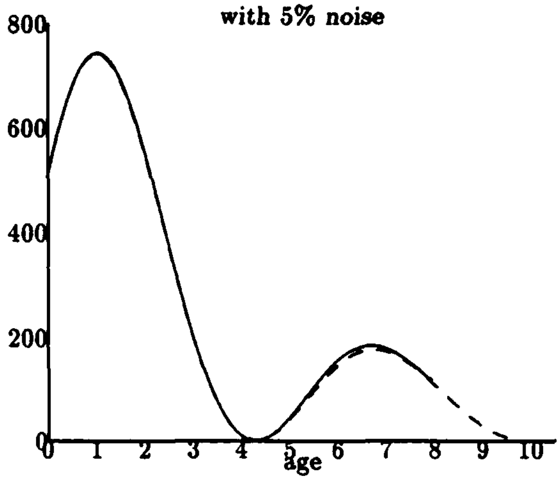
Fig 5. Computed ϕ with noise in $P(t)$

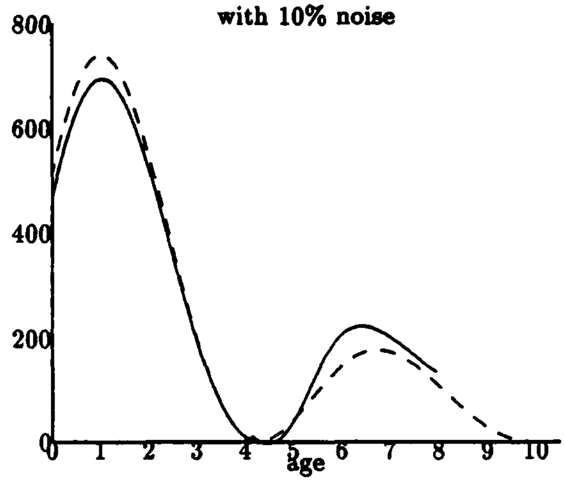


We also did similar computations for the second inverse problem. Results are shown in figure 6 for 5% and 10% error in the overposed data. We used β_2 as the birth function, and used a value of 6 for the final time T. The results are shown in figure 6

Clearly, the second problem is less sensitive to noise in the overposed data than the first. There are several reasons for this. First, we are computing the function $\phi(a)$ over a shorter range, $4 \leq a \leq 8$ in this case. Second, we have to solve one first kind and one second kind Volterra equation when $\rho(a, T)$ is the overposed data. The overposed data only appears in the free term of the first of these equations, (26). The values obtained from this equation are used to compute the free term for the second equation (23). In both equations the kernels can be computed directly in terms of π and β. In contrast, when $P(t)$ is the overposed data, the solution of a first kind Volterra equation is used to compute the kernel of a second kind Fredholm equation, and the result used in another first kind equation. Thus the errors tend to amplify with each sucessive computation, and there are simply more of them (and more ill-posed ones) in the first inverse problem. Third, although (27) is first

Fig 6. Computed ϕ with noise in $\rho(a,T)$





kind with the kernel $k(t,a)$ such that $k(t,t) = 0$, when we differentiate this equation the resulting equation has a kernel that does not vanish on the diagonal, and in fact for the function β_2 we have $\beta_2'(a_2) = -0.8$, which is a relatively large value, and allows an accurate and stable scheme to be used in the recovery of $\phi(a)$.

What would happen if we used β_2 in the recovery of ϕ with $P(t)$ as the overposed data or the function β_1 when $\rho(a,T)$ was overposed data?

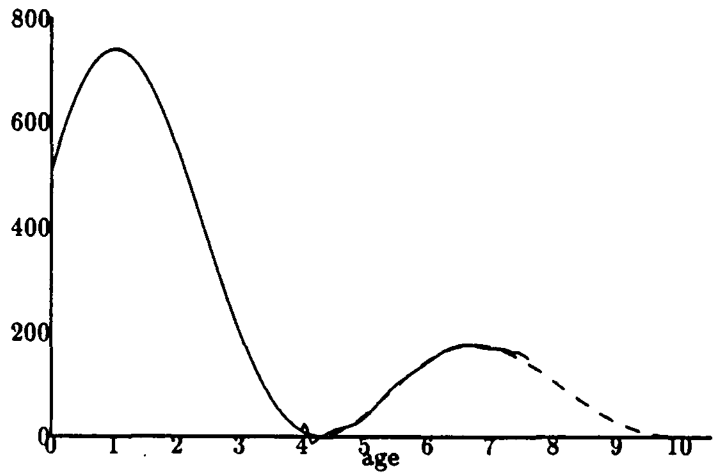
In the analysis of section 2 we assumed that the birth function had $2m+1$ continuous derivatives, which in our examples requires that β be thrice continously differentiable on $[0,L]$. This is true of β_1, but certainly not the case for β_2 which has a discontinuous derivative at $a = a_2$. To test how lack of regularity in the birth and death processes affects the numerical recovery of $\phi(a)$ we ran the above codes with accurate values of $P(t)$ but using β_2 in place of β_1. The values obtained had no relation to ϕ_{act} whatsoever. A similar result was obtained when we used β_1 in the second inverse problem.

The explanation for this is as follows. With the function β_1, (which was built in such a manner to have 3 continuous derivatives) we had a continuous free term in equation (15), which was used to recover the function γ. With β_2 the free term is discontinuous and mathematically one must expect the same to be true of the γ obtained as a solution. This has significant effects on the accuracy of the quadrature schemes used to solve the Fredholm equation (18) the kernel of which depends on integrals of γ. Thus even although we may compute the free term $d(t)$ (depending on π and P) accurately there will be numerical errors in the evaluation of the kernel of this equation. Indeed $K(t,a)$ will not be continuously differentiable in t and therefore neither will be the solution Ψ_2. Once again we will have errors in Ψ_2, and these errors will manifest themselves as oscillations due to behaviour of the derivative. When we attempt to recover $\phi(a)$ from (14) it may in fact be better to solve this first kind equation without prior differentiation since we prefer not to differentiate the free term Ψ_2. However, without differentation, the numerical schemes for solving the first kind equation will be limited to low accuracy by the restriction on the stepsize. In practice we found that when we replaced β_1 by a virtually identical function except that we joined its component parts so that only the first 2 derivatives were continuous, we were able to effectively recover $\phi(a)$ from accurate overposed data. However, for a given set of grid sizes the error in computing ϕ was more than ten times the error in computing the initial data with β_1. From a numerical as well as a mathematical standpoint the regularity assumption **A1** on the birth and death functions are therefore necessary.

As noted above, we were unable to recover $\phi(a)$ from $\rho(a,T)$ when β_1 was the birth function. The condition $\beta'(a_2) \neq 0$ is essential for the stability of the numerical schemes. In order to determine ϕ from π, β_1, and $\rho(a,T)$ we modified our procedure slightly.

There are two different, but closely related, possibilities in dealing with the fact that both $\beta_1(a_2)$ and $\beta_1'(a_2)$ are zero. First, one could modify the function β_1 in a small neighborhood of the point $a = a_2$ so that it was continuous on $[0,L]$, twice continuously differentiable on $[0,a_2)$, and the derivative at $a = a_2$ was non zero. Second, one could retard the argument of β_1 in equation (27), effectively solving this equation with the value of a_2 replaced by $a_2 - \epsilon$ in both the kernel and free term. This gives $k(t,t) = b(a_2 - \epsilon)$ which is now nonzero for $\epsilon > 0$. How is ϵ chosen, or the modification near $a = a_2$ selected for maximum performance? How would you know you had obtained such a maximum? One possible method is to fine tune the choice of parameters using known solutions. The result obtained by retarding the argument in the kernel, choosing $\epsilon = 0.3$, is shown in figure 7.

Fig 7. Computed ϕ using modified β_2



The relatively large errors near the point $L - T = 4$ are due to inaccuracies in the Volterra solver from inaccurate evaluation of the kernel (which depends on $b(a)$) near the point $a = a_2$ where $\beta(a)$ was modified. Of course, we could smooth the function ϕ near this point, using the accurately determined values for $a < L - T$ and the reasonable well computed values for a larger than and away from $L - T$. In addition, we could modify the result to guarantee a postive solution. These would significantly improve the results obtained.

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Determining a Coefficient in a First Order Hyperbolic Equation

Determining a coefficient in a first order hyperbolic equation

Michael Pilant *
William Rundell *

Department of Mathematics
Texas A&M University
College Station Tx. 77843–3368.

Abstract: This paper deals with an inverse problem for a first order hyperbolic equation: namely the determination of the coefficient $\lambda(a,t,\rho)$ in $\rho_t + \rho_a + \lambda\rho = 0$. Conditions on the imposed data which lead to a unique solution are presented and an algorithm for the reconstruction of the coefficient is given.

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1. Introduction.

This paper considers the problem of determining the unknown coefficient λ in the first order hyperbolic equation

$$\rho_t(a,t) + \rho_a(a,t) + \lambda\rho(a,t) = 0. \tag{1.1}$$

Equations such as (1.1) are ubiquitous, but our choice of independent variables indicates a particular application we have in mind; a stands for *age*, t for *time* and the density $\rho(a,t)$ denotes the number of individuals of age a at a particular time t. We assume $\lambda \geq 0$ and in this case λ denotes a *death function*, λda being the probability of a death occurring to a given individual in the time period a to $a + da$. Since this quantity should depend on the age of the individual λ depends on a, and in the case of competition, should also depend on $\rho(a,t)$. This dependence may be in fact be on the *total population* at time t, $P(t) = \int \rho(a,t)da$, and there certainly are circumstances where $\lambda = \lambda(a, P)$ is the appropriate model. While we are using the language of *age structured population dynamics*, our " species" could be also be mechanical parts, $\rho(a,t)$ would denote the number of parts of age a still in service at time t, and λ would be age dependent rate of failures.

In addition to (1.1) we would normally be given initial-boundary data; typically the initial age distribution $\rho(a,0)$ together with information on the boundary $a = 0$. This may be the actual value of $\rho(0,t)$, (the number of new parts put into service at time t) or an equation involving this quantity, such as $\rho(0,t) = \int \beta(a)\rho(a,t)da$. The latter case of a non-local boundary condition is the usual one in population dynamics, $\beta(a)$ is the birth function and βda is the probability of an individual of age a giving birth in the interval a to $a + da$. Thus the usual *direct problem* (or a simplification of such) is: given the value of $\rho(a,0)$, boundary data along $a = 0$, and the death function λ; compute the value of $\rho(a,t)$.

In practice our knowledge of the death process may be incomplete. Even in the case when λ depends only on a we may not be able to obtain accurate data on the age of death or failure. The case $\lambda = \lambda(a)$ is the age structured equivalent of the familiar Malthus law, and to obtain a more accurate picture some dependence of λ on either ρ or P must be assumed. How can such a law be determined, and what are the reasonable experiments that would allow the recovery of the death function? In the language of inverse problems, what *overposed data* can be prescribed that allows the determination of the coefficient λ?

In the second section of this paper we shall show that one can recover the value of $\lambda \equiv \lambda(a)$ from a measurement of the age structure $\rho(a,T)$, at a later time T and the specification of $\rho(0,t)$. For the case of the boundary condition $\rho(0,t) = \int \beta(a)\rho(a,t)da$, we show that the inverse problem can be reformulated as a nonlinear integral equation for the death function, and indicate how the equation may be used to recover this function.

In the third section we consider the problem of determining the dependence of the death process on total population. We will not attempt to recover an unknown function λ of the two independent variables a and P. As a minimum this would require giving overposed data that corresponds to a function of two variables, and this is more than we envision as being reasonable. Instead we shall assume that $\lambda(a, P)$ has a known form and that the dependence on age has already been determined. Two possibilities give reasonable physical models: In the first we assume that the death rate is a simple sum of the natural and environmental rates, that is, λ has the form

$\lambda_N(a) + \lambda_E(P)$, where λ_N is the death rate due to natural causes and λ_E is the death rate due to the environment. In the second case we assume that $\lambda(a, P) = \lambda_N(a)[1 + \lambda_E(P)]$. The first form assumes that the environmental pressure is the same for all ages, the second assumes that it is coupled to age so that environmental pressure is greater on the age group with a high natural death rate. We expect that λ_E is small for relatively small values of P, so that $\lambda \approx \lambda_N(a)$ and that the process is well understood in this case. The problem is to extrapolate this knowledge to attempt to recover the dependence of λ on P when the latter may strongly affect the death process.

Here the appropriate overposed data could be the value of $P(t)$ itself since frequently this is a quantity that is easy to measure. We shall show that one can in fact recover λ_E from knowledge of initial-boundary data and $P(t)$ over some range. For each of these recovery problems we consider the two types of boundary condition, $B(t) \equiv \rho(0, t)$ given and $\rho(0, t) = \int \beta(a)\rho(a, t)da$. As expected, the second of these leads to a more complex recovery procedure.

The direct problem involving (1.1) can be considered to be well understood, even when λ and β depend on ρ, and possibly in a non-local way. On the other hand, comparatively little work has been done even for the most obvious inverse problems. This is not to say that the importance has of identifying the birth and death processes has not been acknowledged in the literature. This is far from the case, but most of the model testing has been confined to the identification of parameters that appear in assumed forms rather than the recovery of the actual functional relationship. In previous work the authors have investigated the problems of recovering the initial distribution $\rho(a, 0)$ from certain overposed data, [6] and the determination of the birth function β from a measurement of total population, [7].

We introduce our notation and some of the ideas used in the solution of the direct problem for (1.1). We shall take as our vocabulary the language of population dynamics.

By $C[a, b]$ we mean the usual space of functions continuous on $[a, b]$ and the norm on this space we denote by $\|\ldots\|_{[a,b]}$. Several subsets will be used. C_M is the subset consisting of those functions whose range lies in the interval $[0, M]$. $C_+[a, b]$ is the subset of non-negative valued functions on $[a, b]$, and $C_0[a, b]$ is the subspace of functions that vanish at the left endpoint a.

We shall drop the interval indication when no confusion can arise. We assume that the species has a maximum life span of L, although this quantity may be infinite. In order to ensure that death must occur by age L, the (strictly positive valued) death function must satisfy $\int_0^L \lambda(a, \cdot)da = \infty$. Consequently λ will be singular if L is finite. In the case of a linear model where the death function depends only on the age a, it is useful to use the *life table* function $\pi(a)$ defined by $\pi(a) = \exp(-\int_0^a \lambda(s)ds)$. This is a strictly decreasing function of a with $\pi(0) = 1$ and $\pi(L) = 0$. It is the probability that an individual will live to age a.

2. Recovery of an age-dependent death rate.

We seek to recover the pair of functions (λ, ρ) from

$$\rho_t + \rho_a + \lambda(a)\rho = 0 \tag{2.1}$$

with the initial condition

$$\rho(a, 0) = \phi(a)\,, \tag{2.2}$$

one of the boundary conditions

$$\rho(0,t) = B(t) \tag{2.3a}$$

$$\rho(0,t) = \int_0^L \beta(a)\rho(a,t)\,da \tag{2.3b}$$

In order to achieve this we must give some additional information, which we refer to as overposed data, since if λ were known, the prescription of other information would lead to an overdetermined problem. We shall use as overposed data

$$\psi(a) = \rho(a, T) \tag{2.4}$$

which corresponds to the value of $\rho(a,t)$ at a later time T. Thus, we are given the conservation equation (2.1), one of the renewal laws (2.3), and the results of two censuses taken a time period T apart.

Instead of the unknown function λ, we use the life table π. Once π has been determined we can recover λ as the logarithmic derivative of this function.

Of the two boundary conditions, (2.3a) leads to a considerably easier recovery problem, and we shall consider this case first.

If we integrate equation (2.1) with $\lambda = \lambda(a)$ along characteristic coordinates, we obtain

$$\rho(a,t) = \begin{cases} \rho(0, t-a)\, e^{-\int_0^a \lambda(s)ds} & \text{if } a < t; \\ \phi(a-t)\, e^{-\int_{a-t}^a \lambda(s)ds} & \text{if } t \le a. \end{cases} \tag{2.5}$$

Note that if (2.3a) is used, that is the Dirichlet data $\rho(0,t) = B(t)$ is prescribed, then (2.5) with $t = T$ can be solved directly to recover $\lambda(a)$. If $T < L$ then we have the explicit solution

$$\lambda(a) = \begin{cases} \frac{d}{da}\left[\log \frac{B(T-a)}{\psi(a)}\right] & \text{if } a < T; \\ \lambda(a-T) + \frac{d}{da}\left[\log \frac{\phi(a-T)}{\psi(a)}\right] & \text{if } a \ge T; \end{cases} \tag{2.6}$$

and if $T \ge L$ then only the first of these equations is required. It is therefore clear that if a physically meaningful solution $\lambda(a)$ is to obtained, then the above equations constrain the admissible data, B, ψ. In fact in order to obtain a continuous solution $\lambda(a)$ to (2.6) it is necessary that ϕ, ψ and $B(t)$ be positive and continuously differentiable on $[0,T)$ with ψ having these properties on $[0,L]$. The nonintegrability condition $\int_0^L \lambda da = \infty$ will in fact follow from $\psi(L) = 0$. The positivity of λ depends on the interaction between the overposed data ψ and the data for the direct problem

ϕ and B. Even in the relatively straightforward case of (2.3a) there are enormous compatibility constraints, and these will be greater when boundary condition (2.3b) is used.

The second equation in (2.6) shows that if $T < L$ and if $\lambda(a)$ has been recovered for $a < T$, then we can obtain the values of $\lambda(a)$ over the remaining interval (T, L) in terms of the initial and final data. This is true regardless of which of the boundary conditions (2.3a) or (2.3b) are used.

The second case, where the renewal law uses the specification of the function β, is more complex. If we integrate (2.5) in a from $a = 0$ to $a = L$, then we obtain the *Sharp-Lotka Equation*

$$B(t) = \int_0^t B(t-a)\beta(a)\pi(a)\,da + \int_t^L \beta(a)\pi(a)\frac{\phi(a-t)}{\pi(a-t)}\,da$$

where $\pi(a) = \exp(-\int_0^a \lambda(s)ds)$. The birth rate B is a function of π rather than prescribed data, and we use the notation $B = B(t;\pi)$. Let us assume that $T \le L$ for the moment. For a given function $\pi(a)$ defined on $[0, L]$ we can define the mapping $\pi \mapsto B(\,\cdot\,;\pi)$ where B is the solution of the second kind Volterra equation

$$B(t) = \int_0^t B(t-a)\beta(a)\pi(a)\,da + \int_t^T \beta(a)\phi(a-t)\frac{\pi(a)}{\pi(a-t)}\,da + \int_T^L \beta(a)\phi(a-t)\frac{\mathcal{S}\pi(a)}{\mathcal{S}\pi(a-t)}\,da. \quad (2.7)$$

Here $\mathcal{S}$ is that operator that maps the values of a function with domain $[0, L]$ to one defined on $[0, T]$ by

$$\mathcal{S}\pi(a) = \begin{cases} \pi(a) & \text{if } 0 \le a < T; \\ \sigma\{\pi(a)\} & \text{if } T \le a < 2T; \\ \sigma^2\{\pi(a)\} & \text{if } 2T \le a < 3T; \\ \vdots & \end{cases} \quad (2.8a)$$

and σ is defined by

$$\sigma\{\pi(a)\} = \frac{\psi(a)}{\phi(a-T)}\pi(a-T) \quad (2.8b)$$

and uses the second line of equation (2.5) to map values of π on $[T, 2T)$ onto values on $[0, T]$. Using this we can now write the first line of equation (2.5) as

$$\pi(a) = \frac{\psi(a)}{B(T-a;\pi)} \equiv \mathbf{T}[\pi] \quad (2.9)$$

We can view (2.9) as a nonlinear equation for the restriction of the function $\pi(a)$ defined on the interval $[0, T]$. If we have obtained a solution of this equation, then we can extend it to successive intervals $[T, 2T]$, $[2T, 3T]$, ..., using (2.8). It follows directly that the function $\pi(a)$ must satisfy the endpoint conditions $\pi(0) = 1$ and $\pi(T) = \psi(T)/\phi(0)$. Can we show that the mapping $\mathbf{T}$ has a (unique) fixed point, and if so, can this fixed point be obtained by the usual iteration scheme

$$\pi_{n+1} = \mathbf{T}[\pi_n]? \quad (2.10)$$

An alternative formulation that eliminates the function B can be obtained by using the first line of (2.5) to get $B(t) = \psi(T-t)/\pi(T-t)$ and then from (2.7) we have, writing $p(t) = 1/\pi(t)$

$$\psi(t)p(t) = \int_0^t \psi(T-t+a)\beta(a)\frac{p(T-t+a)}{p(a)}\,da + \int_t^T \beta(a)\phi(a)\frac{p(a-t)}{p(a)}\,da + \int_T^L \beta(a)\phi(a-t)\frac{\mathcal{S}p(a-t)}{\mathcal{S}p(a)}\,da.$$

We can consider this as a nonlinear, second kind integral equation for the unknown $p(t)$. It is to be noted that one can show that both sides of the abobe equation or monotone functions of λ, but not necessarily monotone in the variable p.

The difficulty in any analysis of the mapping $\mathbf{T}$ is in the relationships between the overposed data ψ and the functions β and ϕ. Even in the case of given $B(t)$ this was a complex relationship. It can be shown that $|B(a,\pi_1) - B(a,\pi_2)| \leq C|\pi_1(a) - \pi_2(a)|$ for some C that depends on the data ϕ, β and ψ. This leads to the inequality $|\mathbf{T}[\pi_1] - \mathbf{T}[\pi_2]| \leq \tilde{C}|\pi_1(a) - \pi_2(a)|$, and if we can impose conditions on the data that guaranteed $\tilde{C} < 1$ then we could invoke the contraction mapping theorem to answer our questions. We were not able to find any reasonable conditions on the data that lead to this condition. We can also show that $\mathbf{T}$ is a compact map on the space of functions that are continuous on $[0,T]$ and satisfy the bounds $\psi(T)/\phi(0) \leq \pi \leq 1$. If it could be shown that $\mathbf{T}$ mapped this set to itself we would obtain an existence result using Schauder's theorem. Once again we are prevented from carrying out the idea by the interrelationship of the data.

However, we have attempted several numerical implementations of the algorithm defined by the iteration scheme (2.10) and convergence to the actual solution was rapid, if the starting value was taken as the straight line joining the known endpoint values at $a = 0$ and $a = T$, the relative error decreased to within a few percent in 3 or 4 iterations.

3. Recovery of a Nonlinear death rate.

We consider the problem of recovering a population dependent death function $\lambda(a,P)$. As mentioned in the introduction, we shall look for a λ in one of two forms; either $\lambda(a,P) = \lambda_N(a)\{1 + f(P(t))\}$ or $\lambda(a,P) = \lambda_N(a) + f(P(t))$, where $\lambda_N(a)$ is a known function. Thus we seek to determine the dependence of λ on the total population under the assumption that we know the age dependence of this function. In many physical models this is realistic since, if the total population is much smaller than the carrying capacity of the environment, it is to be expected that overcrowding would play a negligible role in the death process, that is $\lambda \approx \lambda_N(a)$. The values of such a λ_N may be known, or could have been obtained from the methods of the previous section. From physical principles we expect $f(P)$ to be a nonnegative function, monotonically increasing in P with $f(0) = 0$. In our mathematical model we shall require f to satisfy only the first of these conditions. It will turn out that both of the above forms for λ lead to similar mathematical problems, so that we shall present the details for one and only briefly point out the differences presented by the other.

We thus seek to recover the pair of functions (f,ρ) from

$$\rho_t + \rho_a + \lambda(a,f)\rho = 0 \tag{3.1}$$

with the initial condition

$$\rho(a,0) = \phi(a) \tag{3.2}$$

and either of the boundary conditions

$$\rho(0,t) = B(t) \tag{3.3a}$$

or

$$\rho(0,t) - \int_0^L \beta(a)\rho(a,t)da = 0 \tag{3.3b}$$

and the overposed data

$$P(t) = \int_0^L \rho(a,t)da \tag{3.4}$$

which are prescribed over some interval $0 \le t \le T$.

Since the unknown f only appears in the form $f(P(t))$, we shall make the change of variable $\tilde{f}(t) = f(P(t))$. Thus f is uniquely recoverable from $\tilde{f}$ provided that P is monotonic, a condition we shall require. In order to indicate the dependence of ρ on f we shall write the solution of the direct problem (3.1) – (3.3) by $\rho(a,t;f)$.

For a solution pair $(\rho, \tilde{f})$ we require that $\tilde{f} \in C_M$ for some $M > 0$, that $\rho(a,t)$ have continuous partial derivatives on $(0,L) \times (0,T)$ except on the line $a = t$ and that (3.1), (3.2), (3.4) and one of (3.3) holds.

We shall deal with the case $\lambda(a,P) = \lambda_N(a)\{1 + f(P(t))\}$ for the moment and make the following assumptions on the prescribed data:

A1. $\lambda_N(a)$ is a continuous, positive function on $[0,L)$ with $\int_0^L \lambda_N(a)da = \infty$. $\beta(a)$ is continuous and non-negative on $[0,L]$.

A2. $\phi(a)$ is a continuous, non-negative function with $\phi(a)\exp(\int_0^a \lambda_N(s)ds)$ uniformly bounded on $[0,L]$.

The condition on λ_N ensures that all members of the population must die by age L. The condition on ϕ states that the initial population be consistent with the death process $\lambda = \lambda_N$.

We shall denote by $D(t;f)$ the function $\int_0^L \lambda_N(a)\rho(a,t;f)da$. Thus $D(t;0)$ corresponds to the case when the death process is independent of the population, and in the case of this linear model we can interpret $D(t;0)$ as the *death rate* at time t.

We consider the case of the boundary condition (3.3a) first. In this case we assume that the prescribed birth function and the overposed data $P(t)$ satisfy

A3. $B(t)$ is a continuous, positive function on $[0,T]$.

A4a $P(t)$ is continuously differentiable, positive and strictly increasing function on $[0,T]$ such that for some $M > 0$, $B(t) - (M+1)D(t;0) \le P'(t) \le B(t) - D(t;M)$ for $0 \le t \le T$.

We shall also assume that the birth rate at $t = 0$ shall be compatible with the initial data, $B(t) = \int_0^L \beta(a)\phi(a)da = \phi(0)$. This condition will hold in the case of an already evolving process, and is made to simplify the analysis. At the end of this section we indicate how this (technical) assumption may be removed.

We show that the solution to the inverse problem can be reduced to showing that a certain mapping of the function f has a fixed point. We do this by showing that the map is contractive. This also leads to a natural iteration scheme that may be used to compute the function f. In setting up this structure we will also show that the map is a compact, isotone map on C_M where M is as in **A4a**. This not only leads to an alternative existence proof but shows that the sequence of iterates converges monotonically to the solution of the inverse problem.

Differentiating (3.4), we have

$$\begin{aligned} P'(t) &= \int_0^L \rho_t(a,t;\tilde{f})da \\ &= \int_0^L -\rho_a(a,t;\tilde{f}) - \lambda\rho(a,t;\tilde{f})\,da \\ &= \rho(0,t;\tilde{f}) - \int_0^L \lambda_N(a)\{1+\tilde{f}(t)\}\rho(a,t;\tilde{f})\,da \\ &= \rho(0,t;\tilde{f}) - \{1+\tilde{f}(t)\}\int_0^L \lambda_N(a)\rho(a,t;\tilde{f})da \end{aligned}$$

where $\rho(L,t;\tilde{f}) = 0$ by assumption. Hence

$$\tilde{f}(t) = \frac{\rho(0,t;\tilde{f}) - P'(t)}{\int_0^L \lambda_N(a)\rho(a,t;\tilde{f})da} - 1 \equiv \mathbf{T}[\tilde{f}]. \tag{3.5}$$

In the case of boundary condition (3.3a) this gives

$$\mathbf{T}_a[\tilde{f}] = \frac{B(t) - P'(t)}{\int_0^L \lambda_N(a)\rho(a,t;\tilde{f})da} - 1 \tag{3.6}$$

and for condition (3.3b)

$$\mathbf{T}_b[\tilde{f}] = \frac{\int_0^L \beta(a)\rho(a,t;\tilde{f})da - P'(t)}{\int_0^L \lambda_N(a)\rho(a,t;\tilde{f})da} - 1 \tag{3.7}$$

In either case, this suggests that we look for a fixed point of the mapping $\mathbf{T} = \mathbf{T}_a$ or $\mathbf{T}_b$, and seek a solution of the inverse problem by the iteration scheme

$$\tilde{f}^{(n+1)} = \mathbf{T}[\tilde{f}^{(n)}] \tag{3.8}$$

for some suitably chosen initial approximation f_0. It is straightforward to show that $\tilde{f}$ will be solution of (3.1) – (3.4) if and only if $\tilde{f}$ is a fixed point of the map $\mathbf{T}$ on a suitably defined subspace of $C[0,T]$.

In the case of condition (3.3a) we can in fact prove the following theorem

Theorem 1. *If assumptions* **A1** - **A4a** *hold, then there exists a unique solution* $f \in C_M$ *to (3.1), (3.2), (3.3a) and (3.4). This solution can be obtained as the limit of a monotonicaly increasing sequence* $\{\tilde{f}_n\}_{n=0}^{\infty}$ *where* $\tilde{f}_n$ *is defined by (3.8) with* $\mathbf{T} = \mathbf{T}_a$ *and* $\tilde{f}_0 = 0$.

Proof. The solution to the direct problem (3.1), (3.2) and (3.3a) can be written in the form

$$\rho(a,t;\tilde{f}) = \begin{cases} B(t-a)e^{-\int_0^a \lambda_N(s)[1+\tilde{f}(t-a+s)]ds} & \text{if } a < t; \\ \phi(a-t)e^{-\int_0^t \lambda_N(a-t+s)[1+\tilde{f}(s)]ds} & \text{if } t \le a. \end{cases} \tag{3.9}$$

Thus for all $(a,t) \in [0,L] \times [0,T]$ and for $\tilde{f} \geq 0$

$$\rho(a,t;f) \le \max\{\|B\|_{[0,T]}, \|\phi\|_{[0,L]}\}.$$

Also if $\tilde{f}_1(t) \geq \tilde{f}_2(t)$ for all $t \in [0,T]$ then it easily follows from (3.9) that $\rho(a,t;f_1) \leq \rho(a,t;f_2)$ for all $(a,t) \in [0,L] \times [0,T]$. The positivity of λ_N now shows that $D(t;\tilde{f}_1) \leq D(t;\tilde{f}_2)$ for $0 \leq t \leq T$, and hence from (3.6) $\mathbf{T}_a[\tilde{f}_1] \geq \mathbf{T}_a[\tilde{f}_2]$. Thus $\mathbf{T}_a$ is an isotone map on $C_+[0,T]$ and $\mathbf{T}_a[\tilde{f}]$ is continuous on $[0,T]$ for any $\tilde{f}$ in C_M. Also

$$\frac{B(t)-P'(t)}{D(t;0)} - 1 \leq \mathbf{T}_a[\tilde{f}] \leq \frac{B(t)-P'(t)}{D(t;M)} - 1$$

Assumption **A4a** now shows that $0 \leq \mathbf{T}_a[\tilde{f}] \leq M$ and thus $\mathbf{T}_a : C_M \to C_M$.
We shall use the notation, $\Lambda(a) = \int_0^a \lambda_N(s)ds$, $\pi(a) = \exp(-\int_0^a \lambda_N(s)ds)$ and $\tilde{\phi}(a) = \phi(a)/\pi(a)$. By assumption **A2** this last quantity is uniformly bounded on $[0,L]$.
Let f_1, f_2 be in C_M and restrict t for the moment so that $t < L$. We have that

$$D(t;\tilde{f}_1) - D(t;\tilde{f}_2) = \int_0^t \lambda_N(a)B(t-a)\pi(a)[e^{-\int_0^a \lambda_N(s)\tilde{f}_1(t-a+s)ds} - e^{-\int_0^a \lambda_N(s)\tilde{f}_2(t-a+s)ds}]$$
$$+ \int_t^L \lambda_N(a)\tilde{\phi}(a-t)\pi(a)[e^{-\int_{a-t}^a \lambda_N(s)\tilde{f}_1(t-a+s)ds} - e^{-\int_{a-t}^a \lambda_N(s)\tilde{f}_2(t-a+s)ds}]da.$$

Using the inequality $e^{-x} - e^{-y} \leq x - y$ gives

$$|D(t;\tilde{f}_1) - D(t;\tilde{f}_2)| \leq \|B\|_{[0,T]} \int_0^t \lambda_N(a)\pi(a)\Lambda(a)\,da \cdot \|\tilde{f}_1 - \tilde{f}_2\|_{[0,T]}$$
$$+ \int_t^L \lambda_N(a)\tilde{\phi}(a-t)\pi(a)[\Lambda(a) - \Lambda(a-t)]\,da \cdot \|\tilde{f}_1 - \tilde{f}_2\|_{[0,T]}$$

Now

$$\int_0^t \lambda_N(a)\pi(a)\Lambda(a)\,da = -\int_0^t \pi'(a)\Lambda(a)\,da$$
$$= 1 - \pi(t) - \Lambda(t)e^{-\Lambda(t)}$$

and

$$\int_t^L \lambda_N(a)\tilde{\phi}(a-t)\pi(a)[\Lambda(a) - \Lambda(a-t)]\,da \leq \|\tilde{\phi}\|_{[0,L]} \int_t^L \lambda_N(a)\pi(a)[\Lambda(a) - \Lambda(a-t)]\,da$$
$$= \|\tilde{\phi}\|_{[0,L]}\left\{\pi(t)\Lambda(t) + \int_t^L [\lambda_N(a) - \lambda_N(a-t)]\pi(a)\,da\right\}$$

where we have used $\pi(0) = 1$, $\pi(L) = 0$ and $\Lambda(0) = 0$ and the fact that Λ is an increasing function. Let

$$C_1(t,\lambda_N) = 1 - \pi(t) - \Lambda(t)e^{-\Lambda(t)}$$
$$C_2(t,\lambda_N) = e^{-\Lambda(t)}\Lambda(t) + \int_t^L [\lambda_N(a) - \lambda_N(a-t)]e^{-\Lambda(a)}\,da$$

It follows that $0 \leq C_1 \leq 1$ for $t \in [0,L]$ and $\lim_{t\to 0} C_1 = 0$, while C_2 satisfies

$$C_2(t,\lambda_N) \leq 1 + \int_t^L \lambda_N(a)e^{-\Lambda(a)}\,da + \int_t^L \lambda_N(a-t)e^{-\Lambda(a-t)}\,da$$
$$\leq 1 + \pi(t) - \pi(L-t)$$

so that C_2 is uniformly bounded on $[0, L]$. The above estimates show that $[\lambda_N(a) - \lambda_N(a - t)]e^{-\Lambda(a)}da$ is integrable on $[0, L]$ and thus $\lim_{t\to 0} C_2 = 0$.

We therefore have

$$|D(t;\tilde{f}_1) - D(t;\tilde{f}_2)| \leq C(t, B, \phi, \lambda_N)\|\tilde{f}_1 - \tilde{f}_2\|_{[0,T]} \tag{3.10}$$

where

$$C(t, B, \phi, \lambda_N) = C_1(t, \lambda_N)\|B\|_{[0,T]} + C_2(t, \lambda_N)\|\tilde{\phi}\|_{[0,L]}$$

Note that C is independent of $\tilde{f}_1$ and $\tilde{f}_2$, is bounded on $[0, L]$ and $\lim_{t\to 0} C = 0$. Thus

$$\begin{aligned}|\mathbf{T}_a[\tilde{f}_1] - \mathbf{T}_a[\tilde{f}_2]| &\leq \frac{|B(t) - P'(t)|}{D(t;\tilde{f}_1)D(t;\tilde{f}_2)}.|D(t;\tilde{f}_1) - D(t;\tilde{f}_2)| \\ &\leq \frac{|B(t) - P'(t)|}{D^2(t;M)}.|D(t;\tilde{f}_1) - D(t;\tilde{f}_2)| \\ &\leq C\|\tilde{f}_1 - \tilde{f}_2\|_{[0,T]}\end{aligned}$$

where $C = C(t, \lambda_N, B, \phi, M, P)$ is independent of $\tilde{f}_1$ and $\tilde{f}_2$ and $\lim_{t\to 0} C = 0$. Thus for some fixed τ, depending only on M, B, ϕ, λ_N and P

$$\sup_{0\leq t\leq \tau} |\mathbf{T}_a[\tilde{f}_1] - \mathbf{T}_a[\tilde{f}_2]| \leq \alpha\|\tilde{f}_1 - \tilde{f}_2\|_{[0,T]} \tag{3.11}$$

where $\alpha < 1$. Thus on the space $C[0, \tau]$ $\mathbf{T}_a[\tilde{f}]$ will be a contraction mapping, and will have a unique fixed point in $C[0, \tau]$. By the bootstrap procedure we may extend this solution to all of $[0, T]$ by progressing in increments of τ.

This shows that the iteration scheme defined by (3.8) will converge to the fixed point of $\mathbf{T}_a$. To show monotonicity of this sequence we first observe that $\mathbf{T}_a$ is an isotone mapping on C_M and that **A4a** implies $\mathbf{T}_a[0] \geq 0$ and $\mathbf{T}_a[M] \leq M$. Now if $\tilde{f}_k$ is any bounded sequence in C_M it is easily seen that $D(t;\tilde{f}_k)$ is an equicontinuous family since $D(t;\tilde{f}_k) \leq D(t;0)$. The Ascoli-Arzela theorem shows that $\mathbf{T}_a$ is a compact map on C_M. These facts provide a separate existence proof and the further property that the sequence $\{\tilde{f}_n\}$ defined by $\tilde{f}_{n+1} = \mathbf{T}_a[\tilde{f}_n]$, $\tilde{f}_0 = 0$ converges monotonically to a fixed point of $\mathbf{T}_a$, [1]. If instead, $\tilde{f}_0 = M$, then $\tilde{f}_n$ is a decreasing sequence converging to a fixed point of $\mathbf{T}_a$. Of course, from (3.11) these fixed points must coincide. This completes the proof of the theorem.

For the case of boundary condition (3.3b) our program is the same, except that $\mathbf{T}_b$ as defined by (3.7) now has the additional factor of the unknown "birth function" $\int_0^L \beta(a)\rho(a, t; \tilde{f})da$, which also depends on $\tilde{f}$. We shall still be able to show that $\mathbf{T}_b$ is a contraction on a subset of $C[0, T]$, but we cannot expect that $\mathbf{T}_b$ will be an isotone map, since this will depend on the relationship between $\lambda_N(a)$ and $\beta(a)$. Thus, although we will be able to show that $\mathbf{T}_b$ has a unique fixed point which can be approximated by the sequence generated by (3.8), the sequence need not converge monotonically and perhaps not for all f_0 in our admissible class $C_+[0, T]$.

We shall have to modify our conditions on $P(t)$; there is a further complication added to this case by the fact that $\mathbf{T}_b$ is not monotone. Now it is no longer sufficient to verify that $\mathbf{T}[0] \geq 0$ and

$\mathbf{T}_b[M] \leq M$ in order to claim that $\mathbf{T}$ maps C_M into itself. Actually we will not attempt to obtain an upper bound for the solution f; we will be content to show that $\mathbf{T}_b[f] \geq 0$ if $f \geq 0$, for then we can show that $\mathbf{T}_b$ maps the subset $C_+[0,T]$ into itself and if $\mathbf{T}_b$ is a contraction, then the existence of a unique fixed point is guaranteed. For this problem we make the assumption

A4b $P(t)$ is continuously differentiable, positive and strictly increasing function on $[0,T]$ such that $P'(t) \leq \int_0^L (\beta(a) - \lambda_N(a))\rho(a,t;f)da$ for all functions $f \in C_+[0,T]$.

Assumption **A4b** says that the rate of change of the total population cannot exceed the difference of the birth and death rates, for any admissible functions f. Since this condition involves the unknown f, it is not particularly satisfactory and will be difficult to verify in practice. However, it is not easy to give an easily-checked condition on $P(t)$ that will guarantee that $\mathbf{T}_b$ will map $C_+[0,T]$ into itself. Although the function $\rho(a,t;f)$ will be a monotone function of f, we cannot use this information to obtain correct bounds on $P'(t)$ since the function $\beta(a) - \lambda_N(a)$ must change sign on $[0,L]$. To see this last fact, note that in order for $P(t)$ to be increasing the net birth rate must be positive. This cannot be the case if $\beta < \lambda_N$ for all $a \in [0,L)$. On the other hand, β is bounded and λ_N is unbounded at $a = L$. We are thus left with the "average condition" indicated in **A4b**. However, assuming the model is exact, the practitioner can be satisfied that any measured data $P(t)$ *must* have come from a physically reasonable, and hence positive f. Thus there will be a fixed point in $C_+[0,T]$ to which the iteration scheme will converge. The monotonicity assumption on P is to enable the function f to be recovered from $\tilde{f}$.
Given this we can prove:

Theorem 2. *If assumptions* **A1**, **A2** *and* **A4b** *hold, then there exists a unique solution* $f \in C_+[0,T]$ *to (3.1), (3.2), (3.3a) and (3.4). This solution can be obtained as the limit of the sequence* $\{\tilde{f}_n\}_{n=0}^{\infty}$ *where* $\tilde{f}_n$ *is defined by (3.8) with* $\mathbf{T} = \mathbf{T}_b$ *and* $\tilde{f}_0 = 0$.

Proof. First observe that since $\beta(a)$, $\phi(a)$ and $\lambda_N(a)$ are positive it is easily shown that $\rho(a,t;f)$ will be non-negative for $f \geq 0$. For f_1 and f_2 in $C_+[0,T]$ we let $\lambda_i(a,P) = \lambda_N(a)\{1 + f_i(P(t))\}$, $i = 1,2$. If $\rho(a,t;f_1)$ and $\rho(a,t;f_2)$ are solutions to (3.1), (3.2) and (3.3b) then $\rho(a,t) = \rho(a,t;f_1) - \rho(a,t;f_2)$ is a solution of

$$\rho_t + \rho_a + \lambda_1(a,P)\rho = F(a,t) \tag{3.12}$$

$$\rho(a,0) = 0 \tag{3.13}$$

and

$$b(t;f_1,f_2) := \rho(0,t) = \int_0^L \beta(a)\rho(a,t)da\,. \tag{3.14}$$

where

$$F(a,t) = \lambda_N(a)\rho(a,t;f_1)[\tilde{f}_2(t) - \tilde{f}_1(t)].$$

Using integration along characteristics we can easily show that $\rho(a,t)$ satisfies

$$\rho(a,t) = \begin{cases} \int_0^t e^{-\int_s^t \lambda_1(a-t+\eta,P(\eta))d\eta} F(a-t+s,s)ds & \text{if } t \leq a. \\ b(t-a)e^{-\int_0^a \lambda_1(s,P(t-a+s))ds} \int_0^a e^{-\int_s^a \lambda_1(\eta,P(t-a+\eta))d\eta} F(s,t-a+s)ds & \text{if } a < t; \end{cases} \tag{3.15}$$

where $b(\cdot) = b(\cdot; f_1, f_2)$ is given by (3.14). Substitution of (3.15) into (3.14) leads to the following second kind Volterra equation for $b(t)$

$$b(t) = \int_0^t b(t-a) e^{-\int_0^a \lambda_1(s, P(t-a+s))ds} \beta(a) da + \Psi(t) \tag{3.16}$$

where

$$\Psi(t) = \int_t^L \Big\{ \int_0^t e^{-\int_s^t \lambda_1(a-t+s, P(s))ds} \lambda_N(s) \rho(a-t+s, s; f_1)[\tilde{f}_2(s) - \tilde{f}_1(s)] ds \Big\} da$$
$$+ \int_0^t \Big\{ \int_0^a e^{-\int_s^a \lambda_1(s, P(t-a+s))ds} \lambda_N(s) \rho(s, t-a+s; f_1)[\tilde{f}_2(t-a+s) - \tilde{f}_1(t-a+s)] ds \Big\} da$$

We can draw two conclusions from this. First if $\tilde{f}_1(t) \leq \tilde{f}_2(t)$ on $[0, T]$, then $\Psi(t) \geq 0$. This in turn shows that $b(t) \geq 0$ and thus from (3.15) that $\rho(a, t; f_1) \geq \rho(a, t; f_2)$, so that $\rho(a, t; f)$ is again a decreasing function of f. Thus $\rho(a, t; f) \leq \rho(a, t; 0)$ for $f \in C_+[0, T]$. Second,

$$\sup_{0 \leq s \leq t} |\Psi(s)| \leq C(t, \lambda_N, \phi, P) \sup_{0 \leq s \leq t} |\tilde{f}_2(s) - \tilde{f}_1(s)|$$

where $C(\cdot) \to 0$ as $t \to 0$. Since Ψ is the free term of the Volterra equation (3.16), the solution $b(t)$ of this equation must satisfy the estimate

$$\sup_{0 \leq s \leq t} |b(s)| \leq C(t, \lambda_N, \phi, P, \beta) \sup_{0 \leq s \leq t} |\tilde{f}_2(s) - \tilde{f}_1(s)|. \tag{3.17}$$

We can use this estimate to show in a manner similar to that leading to (3.10) that

$$|D(t; \tilde{f}_1) - D(t; \tilde{f}_2)| \leq C(t, \phi, \lambda_N, \beta) \|\tilde{f}_1 - \tilde{f}_2\|_{[0,T]} \tag{3.18}$$

where $C(t, \phi, \lambda_N, \beta) \to 0$ as $t \to 0$.

Now for $\tilde{f}_1$ and $\tilde{f}_2$ in $C_+[0, T]$ we have

$$\left|\mathbf{T}_b[\tilde{f}_1] - \mathbf{T}_b[\tilde{f}_2]\right| \leq \left| \frac{D(t; \tilde{f}_2)[\rho(a, t; \tilde{f}_1) - P'(t)] - D(t; \tilde{f}_1)[\rho(a, t; \tilde{f}_2 - P'(t)]}{D(t; \tilde{f}_1) D(t; \tilde{f}_2)} \right|$$
$$\leq \frac{|\rho(a, t; \tilde{f}_1) + P'(t)||D(t; \tilde{f}_1) - D(t; \tilde{f}_2)| + D(t; \tilde{f}_1)|b(t; f_1, f_2)|}{D^2(t; M)}$$
$$\leq \frac{|\rho(a, t; 0) + P'(t)||D(t; \tilde{f}_1) - D(t; \tilde{f}_2)| + D(t; 0)|b(t; f_1, f_2)|}{D^2(t; M)}.$$

Combining the above with the estimates (3.17) and (3.18), we obtain

$$\sup_{0 \leq s \leq t} \left|\mathbf{T}_b[\tilde{f}_1] - \mathbf{T}_b[\tilde{f}_2]\right| \leq C \|\tilde{f}_1 - \tilde{f}_2\|_{[0,T]}$$

where C depends on the known functions λ_N, β and ϕ and $C \to 0$ as $t \to 0$. Thus for some fixed τ, depending only on the given data, there is an $\alpha < 1$ such that

$$\sup_{0 \leq t \leq \tau} \left|\mathbf{T}_a[\tilde{f}_1] - \mathbf{T}_a[\tilde{f}_2]\right| \leq \alpha \|\tilde{f}_1 - \tilde{f}_2\|_{[0,T]}$$

As in the previous case, we may bootstrap this local fixed point to one obtained on the interval $[0, T]$. This completes the proof of the theorem.

The above analysis shows that we have in fact proved the following

Theorem 3. *If assumptions* **A1** *and* **A2** *hold, with $P'(t)$ a positive, continuous function on $[0,T]$ and if there exists a solution $f \in C_+[0,T]$ to (3.1), (3.2), (3.3b) and (3.4), then this solution can be obtained as the limit of the sequence $\{\tilde{f}_n\}_{n=0}^{\infty}$ where $\tilde{f}_n$ is defined by (3.8) with $\mathbf{T} = \mathbf{T}_b$ for $\tilde{f}_0$ sufficiently close to $\tilde{f}$. Moreover there is at most one solution in $C_+[0,T]$.*

Let us now look at the analysis when the death function λ has the form $\lambda(a,P) = \lambda_N(a) + f(P(t))$.

In this case the mapping $\mathbf{T}$ in (3.5) takes the form

$$\mathbf{T}[\tilde{f}] = \frac{\rho(0,t;\tilde{f}) - P'(t) - D(t;\tilde{f})}{P(t)} \tag{3.19}$$

where as before, $D(t;f)$ is given by $\int_0^L \lambda_N(a)\rho(a,t;\tilde{f})da$, and $\rho(a,t;\tilde{f})$ represents the solution to the direct problem (3.1) - (3.3), which can be written in the form

$$\rho(a,t;\tilde{f}) = \begin{cases} \rho(t-a,0)e^{-\int_0^a \lambda_N(s)}e^{-\int_{t-a}^t \tilde{f}(s)ds} & \text{if } a < t; \\ \phi(a-t)e^{-\int_{a-t}^a \lambda_N(s)ds}e^{-\int_0^t \tilde{f}(s)ds} & \text{if } t \leq a. \end{cases} \tag{3.20}$$

With an analysis much like the previous case we can show that $\mathbf{T}[\tilde{f}]$ is a contraction on the space $C_+[0,T]$, and that for $\tilde{f}_1$ and $\tilde{f}_2$ in $C_+[0,T]$ with $\tilde{f}_1 \geq \tilde{f}_1$, $\rho(a,t;\tilde{f}_1) \leq \rho(a,t;\tilde{f}_1)$ so that $D(t;\tilde{f}_1) \leq D(t;\tilde{f}_1)$. For the case of boundary condition (3.3a) where $\rho(0,t)$ is prescribed data $B(t)$, this shows that $\mathbf{T}$ is an isotone mapping on $C_+[0,T]$.

We assume that $P(t)$ is a continuously differentiable, positive and strictly increasing function on $[0,T]$. If, in addition, it satisfies

$$P(0)e^{-Mt} + \int_0^t e^{-M(t-\tau)}[B(\tau) - D(\tau;M)]d\tau \leq P(t) \leq P(0) + \int_0^t [B(\tau) - D(\tau;0)]d\tau \tag{3.21}$$

for some $M > 0$, then it is easily shown that under condition (3.3a) $\mathbf{T}][0] \geq 0$ and $\mathbf{T}][M] \leq M$, with the consequence that $\mathbf{T}$ maps $C_M[0,T]$ into itself. These results can be summarized in

Theorem 4. *Let $\lambda(a,P) = \lambda_N + f(P)$. If assumptions* **A1 - A3** *and (3.21) hold, then there exists a unique solution $f \in C_M$ to (3.1), (3.2), (3.3a) and (3.4). This solution can be obtained as the limit of a monotonicaly increasing sequence $\{\tilde{f}_n\}_{n=0}^{\infty}$ where $\tilde{f}_n$ is defined by (3.19) with $\rho(0,t) = B(t)$ and $\tilde{f}_0 = 0$.*

Theorem 5. *Let $\lambda(a,P) = \lambda_N + f(P)$. If assumptions* **A1** *and* **A2** *hold, with $P'(t)$ a positive, continuous function on $[0,T]$ and if there exists a solution $f \in C_+[0,T]$ to (3.1), (3.2), (3.3a) and (3.4), then this solution is unique and can be obtained as the limit of the sequence $\{\tilde{f}_n\}_{n=0}^{\infty}$ where $\tilde{f}_n$ is defined by (3.19) with $\rho(0,t;\tilde{f}) = \int_0^L \beta(a)\rho(a,t;\tilde{f})da$ for $\tilde{f}_0$ sufficiently close to $\tilde{f}$.*

In the case of Theorem 4 where $\rho(a,t;f) = B(t)$ is prescribed, the recovery of $\tilde{f}$ is particularly easy. It is straightforward to show that the equation $\mathbf{T}[\tilde{f}] = \tilde{f}$ is equivalent to a linear Volterra integral equation of the second kind for the function $\exp(\int_0^t \tilde{f}(s)ds)$.

If we do not make the compatibility assumption that $\int_0^L \beta(a)\phi(a)da = \phi(0)$ then we have to modify our argument to include the possibility that $\rho(a,t)$ might be discontinuous along the line $a = t$. We should now write (3.4) in the form

$$P(t) = \int_0^t \rho(a,t)\,da + \int_t^L \rho(a,t)\,da$$

and when we differentiate this equation in t we obtain an additional term due to the jump discontinuity. The new version of (3.5) becomes

$$\tilde{f}(t) = \frac{\rho(0,t;\tilde{f}) - P'(t) + [B(0) - \phi(0)]e^{-(1+\tilde{f}(t))\int_0^t \lambda_N(s)ds}}{\int_0^L \lambda_N(a)\rho(a,t;\tilde{f})da} - 1 \equiv \mathbf{T}[\tilde{f}] \tag{3.22}$$

where $B(0)$ is data in the case of condition (3.3a) and equal to $\int_0^L \beta(a)\phi(a)da$ in the case of (3.3b). We can, as before, show that $\mathbf{T}$ is a contraction provided $|B(0) - \phi(0)|$ is sufficiently small. With condition (3.3a) $\mathbf{T}$ will still be an increasing map if $B(0) \le \phi(0)$ and Theorem 1 will go through as before.

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A Parabolic Inverse Problem with an Unknown Boundary Condition

A Parabolic Inverse Problem with an Unknown Boundary Condition

William Rundell*
Department of Mathematics
Texas A&M University
College Station, TX 77843

and

Hong-Ming Yin **
Department of Mathematics and Statistics
McMaster University
Hamilton, Ontario, Canada, L8S 4K1

Abstract: In this paper we consider a parabolic inverse problem in which an unknown function is involved in the boundary condition, and we attempt to recover this function by measuring the value of the temperature at a fixed point onthe boundary. The motivation for studying this problem arises from some physical models such as a heat conduction system where the heat exchange between the system and its surrounding is unknown. We apply the singularity estimates for the fundamental solution of a parabolic equation along with the generalized Bellman-Gronwall inequality to obtain the continuous dependence of the solution upon the known data. Uniqueness of the solution is established as a direct corollary.

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1. Introduction

Consider a physical heat conduction process in a dynamical system Where we assume that the heat flux across the boundary varies with the temperature but do not know the actual law of heat exchange between the system and its surrounding. This situation could occur when such exchange is a combination of both heat convection and radiation. It is therefore of physical interest to be able to recover such a law of exchange, and this paper addresses the question of whether the law can be uniquely determined by a measurement of the temperature at a certain fixed point on the boundary. We show that this is the case, and in addition prove that, in a sense to be defined, depends continuously on the values of the temperature measurement.

The precise mathematical statement of this physical description is as follows:

Let $T > 0$ and $Q_T = \Omega \times (0, T]$, where Ω is a bounded region in R^n with a smooth boundary $S = \partial\Omega$. Find a pair of functions $u(x,t)$ and $p(s)$ defined on $\bar{Q}_T$ and $[A, B]$, respectively, which satisfy the following equations:

$$u_t - \Delta u = f(x,t), \qquad \text{in } Q_T, \tag{1.1}$$

$$\frac{\partial u}{\partial N} + p(u) = g(x,t), \qquad \text{on } S_T = \partial\Omega \times [0,T], \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad \text{on } \bar{\Omega} \tag{1.3}$$

and the additional condition

$$u(x_0, t) = h(t), \qquad t \in [0, T].$$

where x_0 is a fixed point of $\partial\Omega$, N is the inner normal to $\partial\Omega$ and $A = \min_{Q_T} u(x,t)$ and $B = \max_{Q_T} u(x,t)$.

Recently, considerable attention has been paid to the recovery of one or more coefficients for a parabolic initial-boundary problem from over-specified conditions, see [1, 3, 7, 10] for example. Several authors have considered inverse parabolic problems in which an unknown coefficient appears on the boundary. One model makes the assumption that the heat flux across the boundary is a linear function of the temperature, that is the relation

$$\frac{\partial u(0,t)}{\partial x} + h(t)u(0,t) = g(t), \qquad t \in [0, T],$$

holds, but where $h(t)$ is unkown. T. Suzuki, [13, 14] proved that one can uniquely determine the function $h(t)$ from spectral data using the method of Gel'fand-Levitan. For additional work in this direction see [9], [11]. For the semilinear problem (1.1) – (1.3), Scheglov, [12], used integral estimates to prove uniqueness of the solution in the class of piecewise analytic functions. More recently, Pilant and Rundell, [10] studied the problem (1.1)-(1.4) in one space dimension and established a local existence and uniqueness result using the contract mapping principle.

Many of the uniqueness results obtained for inverse problems for semilinear parabolic equations rely on the monotonicity of the solution in the time variable. Not only does this restrict the class of problems considered, but uniqueness results are obtained only within a restricted class (essentially analytic functions) of coefficients. Such was the case for the recovery of unknown, temperature dependent conductivities or forcing functions from over-specified boundary data in [4], [7], [8]. The aim of this paper is to show the uniqueness and the continuous dependence of the classical solution for (1.1)-(1.4) without making aproiri assumptions on the monotonicity of u. Our technique has the additional advantage of showing uniqueness in what is essentially the largest class of coefficients for which a strong solution of the direct problem exists. The proof is based on singularity estimates for the fundamental solution of the heat equation and an application of the generalized Bellman-Gronwall inequality.

2. Notation and Statement of Main Results

We follow the notation of [5] for the spaces and their norms. By a *solution* to the problem (1.1)-(1.4) we mean,

Definition: A pair of functions $u(x,t)$ and $p(s)$ defined on $\bar{Q}_T$ and $[A,B]$, respectively, is called a classical solution of the problem (1.1)-(1.4), if

(1) $u(x,t) \in C^{3,\frac{3}{2}}(\bar{Q}_T)$ and $p(s) \in C^1[A,B]$;

(2) the inequality $h(0) \leq u(x,t) \leq h(t)$ for $x \in S$ and $t \in [0,T]$ holds;

(3) the equations (1.1)-(1.4) are satisfied in the classical sense, where $A = \min_{Q_T} u(x,t)$, and $B = \min_{Q_T} u(x,t)$.

Remark: As was pointed out in [10], one can not in general expect that a solution of problem (1.1)-(1.4) has the property (2), and it is difficult to give conditions on the data to guarantee this inequality since $u(x,t)$ depends on the unknown function $p(s)$. In addition, if $p > 0$, the physical interpretation is that heat is being pumped into the region through the boundary and if $p(u)$ is not uniformly Lipschitz then the temperature may "blow up" in finite time. In this case we could not expect to solve even the direct problem for all values of T.

We assume the following basic regularity assumptions hold throughout this paper:

R: The functions $g(x,t) \in C^{2,2}(S_T)$ and $h(t) \in C^2[0,T]$. Moreover, the function $h(t)$ is montonic increasing on $[0,T]$.

Our main result is:

Theorem. *Assume the condition* **R** *is satisfied. Let* $\langle u(x,t), p(s)\rangle$ *and* $\langle \bar{u}(x,t), \bar{p}(s)\rangle$ *be two solutions of the problem (1.1)-(1.3) satisfying the overposed conditions*

$$u(x_0,t) = h(t),\ 0 \le t \le T \tag{2.1}$$

and

$$\bar{u}(x_0,t) = \bar{h}(t)\ 0 \le t \le T, \tag{2.2}$$

respectively. Let $H(s)$ *and* $\bar{H}(s)$ *denote the inverse functions of* $h(t)$ *and* $\bar{h}(t)$. *Then*

$$\sup_{A \le s \le B} |p(s) - \bar{p}(s)| \le C \sup_{A \le s \le B} |H(s) - \bar{H}(s)|,$$

where C *is a constant dependent upon* $u(x,t), \bar{u}(x,t)$ *and the known data.*

This shows the continuous dependence of the function $p(s)$ on the overposed data $h(t)$ int supremum norm. From this result we immediately obtain uniqueness of the solution of the inverse problem

Corollary. *Under the assumption* **R**, *the problem (1.1)-(1.4) can possess at most one solution.*

3. Proof of the Theorem

We will need two results which we state as lemmas. The proof of the first is based on an iteration argument and can be found in [6]. The verification of the second lemma, which appears in [5], consists of a straightforward calculation.

Lemma 3.1. *Let* $f(t)$ *be a nonnegative non-decreasing function on* $[0,T]$ *and let* $k \in (0,1)$. *If*

$$y(t) \le f(t) + \int_0^t \frac{y(\tau)}{(t-\tau)^k}\, d\tau,$$

then for some constant $C = C(k,T)$,

$$y(t) \le Cf(t),$$

Lemma 3.2. *Let* $0 \le a, b \le n-1$, *then*

$$\int_S \frac{dy}{|x-y|^a |y-\xi|^b} \le \begin{cases} C|x-\xi|^{n-1-a-b}, & \text{if } a+b > n-1; \\ C & \text{if } a+b < n-1. \end{cases}$$

Let $\langle u(x,t), p(s)\rangle$ and $\langle \bar{u}(x,t), \bar{p}(s)\rangle$ be two solutions of the problem (1.1)-(1.3) with the overposed data (2.1) and (2.2), respectively, and let $W(x,t) = u(x,t) - \bar{u}(x,t)$ for $(x,t) \in \bar{Q}_T$. Then $W(x,t)$ must satisfy the equations:

$$W_t - \triangle W = 0, \qquad \text{in } Q_T, \tag{3.1}$$

$$\frac{\partial W}{\partial N} + p(u) - \bar{p}(\bar{u}) = 0, \qquad \text{on } S_T \tag{3.2}$$

$$W(x,0) = 0, \qquad \in \bar{\Omega}. \tag{3.3}$$

Proof of Theorem: From the conditions (1.2) and (2.1), we have

$$\frac{\partial u(x_0,t)}{\partial N_0} + p\big(h(t)\big) = g(x_0,t), \quad 0 \le t \le T,$$

and it follows by assumption **R** that

$$p(s) = g(x_0, H(s)) - \frac{\partial u\big(x_0, H(s)\big)}{\partial N_0}, \quad A \le s \le B, \tag{3.4}$$

where $H(s) = h^{-1}(s)$ is the inverse function of $h(t)$. Similarly, we have

$$\bar{p}(s) = g\big(x_0, \bar{H}(s)\big) - \frac{\partial \bar{u}\big(x_0, \bar{H}(s)\big)}{\partial N_0}, \quad A \le s \le B, \tag{3.5}$$

where $\bar{H}(s) = \bar{h}^{-1}(s)$ is the inverse of $h(t)$. Applying the mean value theorem, we find

$$\begin{aligned} p\big(u(x,t)\big) - \bar{p}\big(\bar{u}(x,t)\big) &= [g(x_0, H(u(x,t)) - g(x_0, \bar{H}(\bar{u}(x,t))] \\ &\quad - \frac{\partial u\big(x_0, H(u(x,t))\big)}{\partial N_0} - \frac{\partial \bar{u}\big(x_0, \bar{H}(\bar{u}(x,t))\big)}{\partial N_0} \\ &= \beta(x,t)W(x,t) - \frac{\partial W\big(x_0, H(\bar{u}(x,t))\big)}{\partial N_0} \\ &\quad + \gamma(x,t)[H(\bar{u}) - \bar{H}(\bar{u})] \end{aligned} \tag{3.6}$$

where

$$\beta(x,t) = \big[g_t\big(x_0, \theta_1(x,t)\big) - \frac{\partial^2 u\big(x_0, \theta_2(x,t)\big)}{\partial t \partial N_0}\big] H'\big(\theta_3(x,t)\big), \qquad (x,t) \in S_T,$$

$$\gamma(x,t) = g_t\big(x_0, \theta_4(x,t)\big) - \frac{\partial^2 u\big(x_0, \theta_5(x,t)\big)}{\partial t \partial N_0}, \quad (x,t) \in S_T,$$

while $\theta_1(x,t)$ and $\theta_2(x,t)$ are between $H(u(x,t))$ and $H(\bar{u}(x,t))$, $\theta_3(x,t)$ is between $u(x,t)$ and $\bar{u}(x,t)$ and $\theta_4(x,t)$ and $\theta_5(x,t)$ and between $H(\bar{u}(x,t))$ and $\bar{H}(\bar{u}(x,t))$.

It is clear by its definition that $\beta(x,t)$ is continuous on S_T. We now substitute $p(u)-\bar{p}(\bar{u})$ in equation (3.2) by its expression in (3.6) and employ the representation of the solution for a second kind boundary value problem, (see [5]). This gives

$$W(x,t)=\int_0^t\int_S \Gamma(x,t;\xi,\tau)\varphi(\xi,\tau)\,dS_\xi d\tau, \quad (x,t)\in\bar{Q}_T, \tag{3.7}$$

where the function $\Gamma(x,t;\xi,\tau)$ is the fundamental solution of the heat equation (3.1), and the function $\varphi(x,t)$ is the solution of the integral equation

$$\varphi(x,t)=2F(x,t)+\int_0^t\int_S\Big[\frac{\partial\Gamma(x,t;\xi,\tau)}{\partial N(x,t)}+\beta(x,t)\Gamma(x,t;\xi,\tau)\Big]\varphi(\xi,\tau)\,dS_\xi d\tau \tag{3.8}$$

for $(x,t)\in S_T$ and where

$$F(x,t)=\frac{\partial W(x_0,H(\bar{u}(x,t))}{\partial N_0}+\gamma(x,t)\big[H(\bar{u}(x,t))-\bar{H}(\bar{u}(x,t))\big], \qquad (x,t)\in S_T.$$

It is well-known, [5], that

$$|\Gamma(x,t;\xi,\tau)|\le\frac{C}{|t-\tau|^{\mu}\,|x-\xi|^{n-2\mu}}, \qquad x,\ \xi\in\Omega,\ 0\le\tau<t\le T, \tag{3.9}$$

where μ is an arbitrary number in the interval $(0,1)$. Moreover, since S, as well as the coefficients of the equation (2.1), are smooth, we apply the estimate (2.12) of Chapter 5 in [5] with $\beta=1$ to find that

$$\Big|\frac{\partial\Gamma(x,t;\xi,\tau)}{\partial N(x,t)}\Big|\le\frac{C}{|t-\tau|^{\mu}\,|x-\xi|^{n-2\mu}}, \qquad t>\tau, \tag{3.10}$$

where μ is an abitrary value in the interval $(\frac{1}{2},1)$.

If we restrict μ in the interval $(\frac{1}{2},1)$, and use the estimates (3.9) and (3.10) in (3.8), we obtain

$$\begin{aligned}|\varphi(x,t)|&\le 2|F(x,t)|+2C\int_0^t\int_S\frac{1}{|t-\tau|^{\mu}}\,\frac{|\varphi(x,t)|}{|x-\xi|^{n-2\mu}}\,dS_\xi d\tau\\&\le 2|F(x,t)|+2C\int_0^t\frac{\|\varphi(\,\cdot\,,\tau)\|_{L^\infty(\Omega)}}{|t-\tau|^{\mu}}\,d\tau, \qquad (x,t)\in S_T. \end{aligned}\tag{3.11}$$

where we have used Lemma 3.2 to obtain the final inequality.

Let $F^*(t) = \sup_{0 \le \xi \le t} \|F(\cdot,\xi)\|_{L^\infty(S)}$. Then using (3.9), for $t \in [0,T]$

$$\|\varphi(\cdot,t)\|_{L^\infty(S)} \le 2\|F(\cdot,t)\|_{L^\infty(S)} + C\int_0^t \frac{\|\varphi(\cdot,\tau)\|_{L^\infty(S)}}{|t-\tau|^\mu}\,d\tau,$$
$$\le 2F^*(t) + C\int_0^t \frac{\|\varphi(\cdot,\tau)\|_{L^\infty(S)}}{|t-\tau|^\mu}\,d\tau\,,$$

and an application of Lemma 3.1 gives

$$\|\varphi(\cdot,t)\|_{L^\infty(S)} \le CF^*(t), \qquad t \in [0,T]. \tag{3.12}$$

If we now differentiate (3.7) in the direction N_0 and evaluate at x_0, we obtain

$$\frac{\partial W(x_0,t)}{\partial N_0} = \int_0^t \int_S \frac{\partial \Gamma(x_0,t;\xi,\tau)}{\partial N_0}\varphi(\xi,\tau)\,dS_\xi dt, \qquad t \in [0,T]. \tag{3.13}$$

From the assumptions on a solution, one has

$$h(0) \le \bar{u}(x,t) \le h(t), \qquad x \in S, \quad t \in [0.T].$$

and thus

$$0 \le H\big(\bar{u}(x,t)\big) \le t, \qquad t \in [0,T], \quad x \in \partial\Omega.$$

Consequently,

$$\sup_{0\le r\le t}\Big|\frac{\partial W\big(x_0,H(\bar{u}(x,r)\big)}{\partial N_0}\Big| \le \sup_{0\le r\le t}\Big|\frac{\partial W(x_0,r)}{\partial N_0}\Big|.$$

Let us define

$$S(t) = \sup_{0\le r\le t}\Big|\frac{\partial W(x_0,r)}{\partial N_0}\Big|, \qquad t \in [0,T].$$

Now for $t \in [0,T]$,

$$\begin{aligned}
F^*(t) &\equiv \sup_{0\le r\le t} \|F(\cdot,r)\|_{L^\infty(S)} \\
&\le \sup_{0\le r\le t} \Big\|\frac{\partial W\big(x_0,H(\cdot,r)\big)}{\partial N_0}\Big\|_{L^\infty(S)} \\
&\qquad + \sup_{0\le r\le t} \|\gamma(\cdot,r)\{H\big(\bar{u}(\cdot,r)\big) - \bar{H}\big(\bar{u}(\cdot,r)\big)\}\|_{L^\infty(S)} \\
&\le \sup_{0\le r\le t} \Big\|\frac{\partial W(x_0,r)}{\partial N_0}\Big\|_{L^\infty(S)} + C\sup_{0\le r\le t}\|H\big(\bar{u}(\cdot,r)\big) - \bar{H}\big(\bar{u}(\cdot,r)\big)\|_{L^\infty(S)} \\
&\equiv S(t) + C\sup_{0\le r\le t}\|H\big(\bar{u}(\cdot,r)\big) - \bar{H}\big(\bar{u}(\cdot,r)\big)]\|_{L^\infty(S)}
\end{aligned} \tag{3.14}$$

and we can use the estimates (3.10), (3.12) and (3.14) to obtain

$$\begin{aligned}
S(t) &\le \sup_{0\le r\le t}\int_0^r\int_S\left|\frac{\partial\Gamma(x_0,r;\xi,\tau)}{\partial N_0}\right||\varphi(\xi,\tau)|\,dS_\xi d\tau \\
&\le \sup_{0\le r\le t}\int_0^r\int_S\frac{C}{|r-\tau|^\mu}\cdot\frac{\|\varphi(\cdot,\tau)\|_{L^\infty(S)}}{|x_0-\xi|^{n-2\mu}}dS_\xi d\tau \\
&\le C\sup_{0\le\le r}\int_0^r\frac{\|\varphi(\cdot,\tau)\|_{L^\infty(S)}}{|r-\tau|^\mu}\,d\tau \\
&\le C\sup_{0\le r\le t}\int_0^r\Big\{S(\tau)+C\sup_{0\le\xi\le\tau}\|H(\bar u(\cdot,\xi))-\bar H(\bar u(\cdot,\xi))\|_{L^\infty(S)}\Big\}|r-\tau|^{-\mu}\,d\tau \\
&\le C\big[S(t)+\sup_{0\le r\le t}\|H(\bar u(\cdot,r))-\bar H(\bar u(\cdot,r))\|_{L^\infty(S)}\big]\sup_{0\le r\le t}\int_0^r\frac{d\tau}{|r-\tau|^\mu} \\
&= \frac{Ct^{1-\mu}}{1-\mu}\cdot\big\{S(t)+\sup_{0\le r\le t}\|H(\bar u(\cdot,r))-\bar H(\bar u(\cdot,r))\|_{L^\infty(S)}\big\} \\
&\le C^* t^{1-\mu}\big\{S(t)+\sup_{0\le r\le t}\|H(\bar u(\cdot,r))-\bar H(\bar u(\cdot,r))\|_{L^\infty(S)}\big\}.
\end{aligned}$$

If we restrict T by the condition $0<T\le T_0=[\frac{1}{2C^*}]^{\frac{1}{1-\mu}}$, we have

$$S(t)\le C\sup_{0\le r\le t}\|H(\bar u(\cdot,r))-\bar H(\bar u(\cdot,r))\|_{L^\infty(S)},\qquad t\in[0,T_0].$$

We can repeat the above procedure and obtain the estimate (3.15) for all $T_0<T$ provided that the direct problem has a solution over this range.

From (3.4) and (3.5) one has

$$\begin{aligned}
|p(s)-\bar p(s)| &= \left|a(s)[H(s)-\bar H(s)]-\frac{\partial W(x_0,\bar H(s))}{\partial N_0}\right| \\
&\le C\,|H(s)-\bar H(s)|+C\,S(\bar H(s)),
\end{aligned}\tag{3.16}$$

where

$$a(s)=g_t(x_0,\sigma_1(s))-\frac{\partial^2u(x_0,\sigma_2(s))}{\partial N_0\partial t},$$

while $\sigma_1(s)$ and $\sigma_2(s)$ are the respective mean values of the functions $g(x_0,t)$ and $\frac{\partial u(x_0,t)}{\partial N_0}$ between $H(s)$ and $\bar H(s)$. Noting that $A\le\bar u(x,t)\le B$, we have from (3.15) that

$$\begin{aligned}
S(\bar H(s)) &\le C\sup_{0\le r\le\bar H(s)}\|H(\bar u(\cdot,r))-\bar H(\bar u(\cdot,r))\|_{L^\infty(S)} \\
&\le C\sup_{A\le s\le B}|H(s)-\bar H(s)|.
\end{aligned}$$

Finally, we apply the above estimate to the inequality (3.16) and obtain

$$\sup_{0\le s\le B} |p(s) - \bar{p}(s)| \le C \sup_{A\le s\le B} |H(s) - \bar{H}(s)|$$

to complete the proof of the theorem.

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Undetermined Coefficient Problems
for Quasilinear Parabolic Equations

[Arcata AMS Conference Proceedings]

CHAPTER 11

Undetermined Coefficient Problems for Quasilinear Parabolic Equations*

Michael S. Pilant†
William Rundell†

1. Introduction

We consider the problem of recovering an unknown coefficient (or coefficients) from a parabolic initial boundary value problem of the form

$$u_t = L[u] \text{ in } \Omega \times [0,T] \tag{1.1}$$

$$u(x,0) = u_0(x) \text{ on } \Omega \tag{1.2}$$

$$G[u] = 0 \text{ on } \partial\Omega \times [0,T] \tag{1.3}$$

subject to additional information

$$B[u] = 0. \tag{1.4}$$

The domain Ω is an open, simply connected region in $\mathbf{R}^n$ – although in this talk, for simplicity, we primarily consider $\Omega = (0,1)$, and $n = 1$. The overposed data will usually be in the form of additional boundary measurements. The unknown coefficients may be present in L or G (or both), and we assume the unknown coefficients are functions of the single dependent variable u. Cases where the unknown coefficients depend only on x or t have been extensively studied and will only be mentioned briefly. Reference [7] contains a summary of many of these results and, in addition, has an extensive bibliography of over 1000 articles pertaining to the one-dimensional heat equation. Several recovery methods are considered, and the relationship between the methods is explored. Various issues arising in

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†Department of Mathematics, Texas A&M University, College Station, Texas 77843.

discretization, numerical solution, and treatment of noisy data are also discussed. Finally, a class of fixed point schemes based on residual updating are reviewed.

The main goal of this talk is a brief introduction to some of the methods used for nonlinear coefficient recovery, and an examination of their interrelationships. A survey of some of the literature in this subject is included in the bibliography.

The outline of this talk is as follows:

1. Introduction
2. Background
3. Model Equations and Inverse Problems
4. Meta-Theorem
5. Overposed Data
6. Existence, Uniqueness, and Monotonicity Results
7. Numerical vs. Analytic Methods
8. Discrete vs. Continuous Methods
9. Multiple Unknown Coefficients, Problems with High Dimension
10. Further Questions and Open Problems
11. Summary

2. Background

In order to motivate the discussion of this particular class of inverse problems, we begin with some examples of models and applications which have appeared in the literature.

Physical Models: Classical examples of the equations we consider are

Heat Conduction:

$$u_t - \nabla.(D\nabla u) = f$$

Chemical Kinetics:

$$u_t - D_1 u_{xx} = f_1(u,v)$$
$$v_t - D_2 v_{xx} = f_2(u,v)$$

Population Dynamics:

$$\frac{dP}{dt} = \alpha P$$

$$\text{or,}\quad \frac{\partial P}{\partial t} - DP_{xx} = f(P)$$

where D is a diffusion coefficient and f is a source term. (In all the above cases, standard initial conditions and boundary conditions are imposed). The models may be in the form of ordinary or partial differential equations, single equations or systems, and possess one or more spatial dimensions.

The physical models are initially *black boxes*, in the sense that the details of the underlying process are imperfectly understood:

initial + boundary conditions	⟶	Physical process	⟶	interior + boundary information.

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The physical models are initially *black boxes*, in the sense that the details of the underlying process are imperfectly understood:

initial + boundary conditions ⟶ [Physical process] ⟶ interior + boundary information.

We may have some a priori information on the form of the mathematical model by imposing conservation laws. For example, one often applies a conservation of energy balance in the form

Change in Total Energy	=	Energy Lost Through Boundary	+	Energy from Internal Sources

For a model of heat flow, if E denotes the specific energy, Q the heat flux, and γ the heat generated in the interior of the region Ω, we have

$$\frac{\partial}{\partial t}\iint_{\Omega} E\,dV = \oint_{\partial\Omega} Q \cdot n\,ds + \iint \gamma dV = \iint \nabla.Q\,dV + \iint \gamma dV.$$

Given sufficient regularity, this leads to

$$\frac{\partial E}{\partial u}\cdot\frac{\partial u}{\partial t} = \nabla.Q + \gamma.$$

We denote by $c \equiv \frac{\partial E}{\partial u}$ the specific heat. If c, γ, and Q depend on the unknown u and, in addition, Q depends monotonely on the gradient ∇u, we have the quasilinear parabolic equation

$$c(x,t,u)u_t - \nabla.Q(x,t,u,\nabla u) = \gamma(x,t,u).$$

Assuming a linear relationship between Q and ∇u leads to $Q = D\nabla u$ and to the relation

$$c(x,t,u)u_t - \nabla.D(x,t,u)\nabla u = \gamma(x,t,u).$$

How good is this model? This depends on the physics included, the nature of the materials involved, and *a priori assumptions* on the nature of physical laws.

The *direct* problem is to prescribe the coefficients $\{c, Q, \gamma\}$, impose appropriate initial and boundary conditions (*primary data*), and use them to solve for the dependent variables. The *inverse problem* is to recover one or more of the unknown coefficients as well as u by prescribing additional data. If, for example, we wish to deduce the relationship of D to its arguments, we must perform experiments. We know the input, and resulting output, and want to find D. This is a classic example of an inverse problem.

In the general case, we may expect D to be a complicated function of x, t, u, and ∇u. Under special circumstances, it may have a simpler dependence. If the material is very uniform, we do not expect material coefficients to depend on the spatial variable x. If, in addition, the material properties do not change in time when the dependent variables are held fixed, then we may expect that the coefficients do not depend strongly on t. In such cases, a functional dependence on only the dependent variables, and perhaps their gradients, is not an unreasonable assumption.

The case where the unknown coefficient D is a function of t only, has been investigated in [3, 4, 7, 17, 23]. In some sense, this was a first approximation to the nonlinear coefficient recovery case, in that the unknown coefficient may depend much more strongly on t than on x. If $u(x,t)$ is slowly varying spatially, then $D(u(x,t)) \sim D(t)$.

If the diffusion D is a function of x, we generally need interior measurements, or many (probably infinite) boundary measurements. [The case of recovering the coefficient $D(x)$ in $\nabla.(D(x)\nabla u) = 0$ is discussed in the papers by Isakov and by Sylvester and Uhlmann in this proceedings]. If D is a function of u alone, one can show in some cases that a single boundary measurement suffices [35]. The methods used in each of these problems are quite different.

If we assume an *a priori* dependence $D = D(t)$, $D = D(x)$, or $D = D(u)$, we can match the model to data (output). What if we do not know the nature of dependence? For a first approximation we might, perhaps, set

$$D(x,t,u) \approx D_1(x) + D_2(u) + D_3(t)$$

to correspond to the first few terms in an expansion for D. If one term dominates the others, we can recover the primary dependence, and possibly recover the other terms after fixing the form of the main dependence. However, are we entirely sure that the model is correct? In any mathematical model of a real physical process, terms invariably are neglected. The accuracy of the simplified model can only be tested by experimentation or fitting to data. In certain parameter regions, the simplified model may be adequate, but as the limits of validity are reached, one must return to the underlying process and re-examine the model.

If large gradients are present, Q may depend on ∇u in a nonlinear fashion and, therefore, D will depend on ∇u. An even more complicated dependence may occur if the process is not in equilibrium. As an example of this, consider the case where there is *rate dependence* in the flux law; that is, where the flux Q does not instantaneously respond to the gradient ∇u. Thus, there is a characteristic relaxation time ϵ. Assuming to a first approximation that

$$\epsilon\partial_t Q + Q = D_0\nabla u \qquad D_0 = \text{constant}, \qquad \epsilon << 1$$

we have the convolution

$$\begin{aligned} Q &= D_0 \int_{-\infty}^{t} \exp\big(-(\frac{1}{\epsilon})(t-\tau)\big)\nabla u(x,\tau)d\tau \\ &= k * D_0\nabla u, \end{aligned}$$

and

$$u_t - k * \nabla(D_0\nabla u) = \gamma,$$

which is an example of a heat equation with memory. Determining which of the many types of nonlinear dependencies is present is precisely the modeling process. Determining the form of the linear or nonlinear equations by actually matching the output of the model to the *observed* data is the ultimate goal of solving the inverse problem.

Applications: In general, coefficients reflect scalings and parameters (e.g. amplitudes, frequencies) of physical importance. One tries to set up experiments in which only one (or possibly a few) parameters vary and then seeks to recover the form of the dependency of the unknown coefficients by repeated observation. Unfortunately (at least from the perspective of those who must solve them) most equations which describe physical laws are in fact nonlinear. The forms of nonlinearity may be difficult to determine. Consider the following classical problem in population dynamics. A constant growth rate λ for a species leads to the growth law

$$\frac{du}{dt} = \lambda u.$$

This is unrealistic in that it leads to purely exponential growth (or decay). A more reasonable model is the *logistic equation*

$$\frac{du}{dt} = \lambda(u - \alpha u^2) = (\lambda - \alpha u)u$$

which leads to bounded growth. In reality, the population may have a more complicated growth law of the form

$$\frac{du}{dt} = f(u).$$

If we could monitor the solution, u, of this ordinary differential equation over a time interval $t_1 < t < t_2$, we can recover f immediately. For example, if $u(t) = h(t)$ on the interval $t_1 < t < t_2$, then by substitution into the differential equation we have $h'(t) = f(h(t))$. This implies that $f(\xi) = h'(h^{-1}(\xi))$. This observation underlies much of the convergence results in the parabolic case when the time dependence dominates spatial dependence near the overposed boundary [32, 34], that is, when $|u_{xx}(x_0,t)| << |u_t(x_0,t)|$.

Another natural application is control by means of boundary measurements. If the flux on a particular boundary is under our control, and depends on the instantaneous value of the dependent variable there, we then have a relationship of the form

$$\frac{\partial u}{\partial \nu} = f(u).$$

Given a desired response at a point x_0 on the boundary, $u(x_0,t) = h(t)$, we wish to find the unknown control f which achieves this target state. We note that, in practice, inverse problems and identifiability problems in control tend to be very closely related [39].

3. Model Equations and Inverse Problems

We will discuss various inverse problems and methods for the parabolic equation

$$u_t = L(x,t;u) \text{ on } \Omega \times [0,T]$$

$$G[u] = 0 \text{ on } \partial\Omega \times [0,T]$$

where the operator L and/or the boundary operator G contains an undetermined coefficient.

Classical examples of such problems are

$\mathbf{c}(\mathbf{u})u_t = u_{xx} + \gamma(x,t) \quad c \geq c_0 > 0$	unknown specific heat
$u_t = \partial_x(\mathbf{k}(\mathbf{u})\partial_x u) + \gamma(x,t)$	unknown conductivity
$u_t = u_{xx} + \mathbf{f}(\mathbf{u}) + \gamma(x,t)$	unknown reaction term
$\frac{\partial u}{\partial x}(0,t) = \mathbf{F}(u(0,t))$	unknown boundary condition

These are canonical single coefficient inverse problems. The coefficients are functions from $\mathbf{R}^1$ to $\mathbf{R}^1$. In order to recover the functional dependency we should prescribe overposed data with the correct dimensionality. The overposed data should map a subset of $\Omega \times [0,T] \to \mathbf{R}^1$. These problems are in contrast to other inverse problems where the unknown coefficient depends on x or t but not u:

$$\begin{aligned} u_t &= \mathbf{a(t)}\Delta u + \gamma(x,t) \\ u_t &= \nabla(\mathbf{a(x)}\nabla u) + \gamma(x,t) \\ u_t &= \Delta u + \mathbf{f(x,t)} \\ u_t &= \Delta u + \mathbf{p(x)}u. \end{aligned}$$

If $x \in \Omega \subset \mathbf{R}^n$, these are problems where the unknown coefficient is a function of more than one variable. Examples of such problems are given in [7].

4. A Meta-Theorem

Finding exactly the right space of overposed data is important in order to guarantee existence of a unique solution to the inverse problem. If too much data is overposed, there may not be a solution to the inverse problem. If too little data is overposed, there may not be a unique solution to the inverse problem. Constraints on parameters lead to compatibility conditions between the primary and overposed data, which can be quite complicated. This has led to a dichotomy between existence results and uniqueness results. Consider the problem of recovering $a(x)$ in

$$u_t - \frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right) = 0$$

with $u(x,t)$ satisfying initial data, Neumann boundary conditions, and $u(0,t) = h(t)$ given as overposed data. Furthermore, suppose the $a(x)$ is smooth enough so that $Lu \equiv \frac{\partial}{\partial x}(a(x)\frac{\partial u}{\partial x}) = \lambda u$ has a complete set of L_2 eigenfunctions, satisfying

$$L\phi_j(x) = \lambda_j \phi_j(x).$$

Writing $u(x,t) \equiv \sum_{j=0}^{j=\infty} u_j(t)\phi_j(x)$, one can solve for the coefficients $u_j(t)$ and easily obtain

$$u(x,t) = \sum_{j=0}^{j=\infty} u_j(0)e^{\lambda_j t}\phi_j(x)$$

The $u_j(0)$ are determined by the initial data. Evaluating this expression on $x = 0$, we obtain a Dirichlet series for $h(t)$. This implies that any overposed data $h(t) = u(0,t)$ for reasonable a must be a (very small) subset of the class of analytic functions. This is a very strong compatibility constraint. Under certain conditions, one obtains uniqueness easily, but existence only if the overposed data is in an extremely small subclass of functions [30].

One could argue that the difficulty is that the overposed data is in the *wrong direction*. We are trying to recover a function of x by giving data in the t direction. The following *meta-theorem* was formulated in the late 1960's for parabolic inverse problems [6].

META-THEOREM: *The overposed data and the unknown function should lie in the same direction.*

If the data is in the "correct" direction, the problem will usually be well-posed. If data is not in the "correct" direction, the problem will be ill-posed. For example, recovering $a(x)$ by overposing data on $x = 0$ leads to an extremely ill behaved problem, but recovering $a(t)$ by the same data is a relatively well-behaved problem [3, 4, 23, 24]. Overposing final data $u(x,T)$ leads to a much better behaved problem for coefficients which depend on x, [38].

5. Overposed Data

There are many possible measurements one can perform on an evolving process. In some cases the experimenter can choose the form of the overposed (i.e., measured) data. This is often in the form of local (pointwise) data. Two common examples are

i) Dirichlet type, $u(x_0,t) = h(t) \quad 0 \le t \le T$;

ii) Neumann type, $\partial u/\partial x(x_0,t) = h(t)$.

In some cases, one has only nonlocal observations – for example, measurements of spectra, measurements on the total energy in the system, periodic data, or time-integrated data. Examples of nonlocal data relevant to parabolic initial value problems are

$$\alpha u(0,t) + \beta u(1,t) = h(t),$$
$$\int_0^1 u(x,t)dx = h(t),$$
$$\int_0^1 |u(x,t)|^p dx = h(t),$$
$$\int_0^t k(t-\tau)u(x_0,\tau)d\tau = h(t).$$

For examples of some of these, see [2,7] and the references contained therein.

Overposed Dirichlet Data: The simplest type of overposed data is Dirichlet data. We will consider overposed Dirichlet type at first, and return to the other types later.

As a special case of the inverse problem (1.1)-(1.4), we examine the initial boundary value problem

$$u_t - u_{xx} = f(u) \tag{5.1}$$

$$u_x(0,t) = g_0(t) \qquad u_x(1,t) = g_1(t) \tag{5.2}$$

$$u(x,0) = u_0(x) \tag{5.3}$$

with overposed data

$$u(0,t) = h(t). \tag{5.4}$$

Other coefficient problems considered can be found in references given at the end of this paper [5, 3 – 13, 31 – 36]. In general, u depends on f in a complicated way, and we must use the direct problem (5.1)-(5.3) to obtain $u(x,t;f)$. Solving the inverse problem reduces to solving the "residual equation"

$$R[f] = u(0,t;f) - h(t) = 0 \tag{5.5}$$

for f, given h. In other words, the direct problem yields $f \longmapsto h$, and we must invert this to find $h \longmapsto f$. The analysis of this mapping, which we call the "data-coefficient mapping" is fundamental to the understanding of both the inverse problem and solution strategies.

Properties of the Dirichlet Data–Coefficient Mapping: For the inverse problem (5.1)-(5.4), the basis of many of our residual algorithms is the Green's function representation

$$u(x,t) = \psi(x,t) + \iint K(x,y,t-\tau)f(u(y,\tau))dyd\tau \tag{5.6}$$

where ψ satisfies (5.1)-(5.3) with $f=0$; that is, $\psi(x,t) = u(x,t;0)$. A necessary condition for solving (5.5) is that $\frac{\partial R}{\partial u}\cdot\frac{\partial u}{\partial f}$ be nonsingular. Restricting (5.6) to the overposed boundary $x = x_0$ leads to

$$u(x_0,t) = \psi(x_0,t) + \iint K(x_0,y,t-\tau)f(u(y,\tau))dyd\tau. \tag{5.7}$$

Equations (5.6) and (5.7) form a coupled set of first kind nonlinear Volterra integral equations. They contain all the information of (5.1)-(5.4), but are difficult to solve in this form. Written in terms of partial differential equations (and evaluating on $x = x_0 = 0$) we have

$$u_t(0,t) - u_{xx}(0,t;f) = f(u(0,t)) \tag{5.8}$$

or since $u(0,t) = h(t)$,

$$\begin{aligned} f(h(t)) &= h'(t) - u_{xx}(0,t;f) \\ &= h'(t) - \psi_{xx}(0,t) - \iint K_{xx}(0,y,t-\tau)f(u(y,\tau))dyd\tau. \end{aligned}$$

This amounts to converting (5.7) to an integral equation of second kind by differentiation. One can show by Gronwall arguments that if f is Lipschitz, then f is a solution of the inverse problem if and only if it is a solution of the above mapping [34].

Historical Methods for solving the "Residual Equation": The first two methods for solving equations of the form (5.5) are special cases of parameter identification. In these methods, we assume that f can be described by a finite set of parameters $\{\alpha_1, \alpha_2, \ldots, \alpha_N\}$; that is, $f = f(\alpha_1, \alpha_2, \ldots, \alpha_N)$. An example of this would be $f(\xi) = \sum_{i=1}^{N} \alpha_i \phi_i(\xi)$ where the $\phi_i(x)$ are *known* basis functions. The first method we consider is

a) WEIGHTED LEAST SQUARES. Here the number of measurements (M) may exceed the dimension of the parameter space (N), and consequently the model may be overdetermined. For f's of the above form, we solve the problem

$$\min_{\alpha_j}\Big\{\sum_{i=1}^{M} w_i R[t_i]^2\Big\} = \min_{\alpha_j}\Big\{\sum_{i=1}^{M} w_i \big|u(x_0,t_i,f(\alpha_j)) - h(t_i)\big|^2\Big\}$$

where the $w_i > 0$ are weights. This presents a difficult optimization problem, since the functional may have many local minima and be very flat in the neighborhood of the minima. "Output least squares" methods sometimes have these types of difficulties.

b) COLLOCATION METHOD. This method fixes the basis functions and reduces the problem to a nonlinear algebraic system of equations.
We solve the equations $R[f(\alpha_j)](t_i) = u(0, t_i, f(\alpha_j)) - h(t_i) = 0 \quad i = 1, \ldots, N$. This is called a "collocation method," since we set the residual to zero at a finite set of collocation points.

Residual Update Methods: We now discuss several mappings (Newton, Quasi-Newton, Homotopy) which are related to one another. These methods depend on the Gateâux derivative of u, with respect to f. They rely on the fact that the Gateâux derivative satisfies a linear partial differential equation. To see how the function u depends on the coefficient f, we form the quantity $u(x, t; f + s\phi)$.

The function $u(x, t; f + s\phi)$ satisfies the following boundary value problem:

$$u_t - u_{xx} = f(u) + s\phi(u) \tag{5.9}$$

$$u_x(0, t) = g_0(t), \qquad u_x(1, t) = g_1(t)$$

$$u(x, 0) = u_0(x).$$

Defining the Gateâux derivative by $\hat{u} \equiv \frac{\partial}{\partial s} u(x, t; f + s\phi)\Big|_{s=0} \equiv J[f].\phi$, *it is easily seen that* $\hat{u}$ satisfies the equation

$$\hat{u}_t - \hat{u}_{xx} = f'(u)\hat{u} + \phi(u) \tag{5.10}$$

$$\hat{u}_x(0, t) = 0, \qquad \hat{u}_x(1, t) = 0 \tag{5.11}$$

$$\hat{u}(x, 0) = 0 \tag{5.12}$$

where $u = u(x, t; f)$. We assume sufficient smoothness so that the Gateâux derivative is equivalent to the Fréchet derivative. Knowledge of the functions f and ϕ determine the partial differential equation, (5.8), and its solution, completely. In fact, we have the representation formula

$$\hat{u}(x, t) = \int_0^t \int_\Omega \hat{K}(x, y, t - \tau)\phi(u(y, \tau))dyd\tau$$

where $\hat{K}$ is the Green's function for the operator $\partial_t - \partial_{xx} - f'(u)$.

Based on the fact that the residual $R[f]$ is the primary source of information about model error, it is reasonable to believe most effective iterative recovery methods are equivalent to some type of residual update scheme. For the problem (5.1)-(5.4), these schemes are generally of the form

$$\begin{aligned} f^{(n+1)} &= f^{(n)} + \mathcal{F}[R(\ldots; f^{(n)})] \\ &= f^{(n)} + \mathcal{F}[u(0, t; f^{(n)}) - h(t)] \end{aligned}$$

where $\mathcal{F}$ vanishes only at the origin. We have a solution (and convergence) if and only if $R \equiv 0$. The iteration sequence is equivalent to finding a fixed point of the mapping

$$f = T[f] = [I + \mathcal{F} \circ R \circ u \circ]f$$

defining the iteration scheme

$$f^{(n)} \longmapsto u^{(n)} \longmapsto R^{(n)} \longmapsto \mathcal{F} \longmapsto f^{(n+1)}.$$

This is the function space analog of the "Matrix splitting methods" used in numerical linear algebra. Convergence depends on the Fréchet differentiation of the map $\mathcal{F}$. If the operator $I + \frac{\partial \mathcal{F}}{\partial u} \cdot \frac{\partial u}{\partial f}$ is contractive, we may expect some type of convergence. $\mathcal{F}$ should be chosen, in some sense, to guarantee the convergence of the iteration scheme (as a contraction). It is precisely the choice of $\mathcal{F}$ which distinguishes the various iteration schemes from one another.

As an example of this general method, we have the following schemes:

a) NEWTON-RAPHSON. In this method we seek to solve the nonlinear equation

$$R[f] = u(0, t; f) - h(t) = 0$$

by the iteration scheme

$$f^{(n+1)} = f^{(n)} - \left(\frac{\partial R}{\partial f}\right)^{-1} (f^{(n)}) \cdot R[f^{(n)}].$$

Letting $J \equiv \frac{\partial R}{\partial f}$ we have

$$f_{n+1} = f_n - J[f_n]^{-1}.R[f_n] = f_n - J[f_n]^{-1}.(u(0, t; f_n) - h(t)). \tag{5.13}$$

Rearranging, this becomes

$$J[f_n].(f_{n+1} - f_n) = h(t) - u(0, t; f_n).$$

We seek to recover f_{n+1} from this equation. Letting $\hat{u} \equiv J[f_n].(f_{n+1} - f_n)$, one can show that $\hat{u}$ satisfies

$$\hat{u}_t - \hat{u}_{xx} = f_n'(u_n)\hat{u} + f_{n+1}(u_n) - f_n(u_n) \tag{5.14}$$

$$\hat{u}_x(0, t) = 0, \qquad \hat{u}_x(1, t) = 0 \tag{5.15}$$

$$\hat{u}(x, 0) = 0 \tag{5.16}$$

and on the overposed boundary $x = 0$,

$$\hat{u}(0, t) = h(t) - u(0, t; f_n). \tag{5.17}$$

We therefore have

$$\hat{u}(0, t; f) = \int_0^t \int_0^1 \hat{K}(0, y, t - \tau)\left[f_{n+1}\left(u_n(y, \tau)\right) - f_n\left(u_n(y, \tau)\right)\right] dy d\tau,$$

where $\hat{K}$ is the Green's function for the operator $\partial_t - \partial_{xx} - f_n'(u_n)\, I$. This is a first-kind Volterra integral equation, with convolution kernel.
We can also recover f_{n+1} by evaluating (5.14) on the overposed boundary $x = 0$, using the boundary condition (5.17) to get

$$h'(t) - u_t(0,t;f_n) - \hat{u}_{xx}(0,t;f^{(n+1)})$$

$$= f_n'(u_n(0,t))(h(t) - u_n(0,t)) + f_{n+1}(u_n(0,t)) - f_n(u_n(0,t)). \tag{5.18}$$

This is itself a nonlinear equation for $f^{(n+1)}$, and must be solved by iteration.

b) QUASI-NEWTON. In this method we replace $J[f_n]^{-1}$ by a fixed operator K

$$f^{(n+1)} = f^{(n)} - K \cdot R[f^{(n)}].$$

A natural Quasi-Newton method may be defined by replacing $f_n'(u_n)$ by $f_n'(h)$ in (5.13). This has the effect of changing the linear operator J, and modifying the update scheme. Another possible scheme would be to keep $J[f_n]$ fixed for several iterations. The success of these schemes is highly problem dependent. For an arbitrary choice of K, one can not necessarily reduce this to a projection scheme for a local partial differential operator.

c) HOMOTOPY. In this method we have

$$f^{(s+\Delta s)} = f^{(s)} + \left(\frac{\partial R}{\partial f}\right)^{-1} (f^{(n)}) \cdot [h^{(s+\Delta s)} - h^{(s)}]$$

where $h^{(s)}$ parameterizes the overposed data and $h^{(1)}$ is the actual observed data.
This method can be thought of as a continuation scheme. An initial guess f_0 is selected and the direct problem solved to recover the corresponding boundary values $h_0(t) \equiv u(0,t;f_0)$. We let $h_1(t) = h(t)$, the actual overposed data, and form the parameterized values $h_s(t) = (1-s)h_0(t) + sh_1(t)$, moving from the initial state to the desired state as s moves from 0 to 1. We wish to find the corresponding reaction terms $f_s(t)$, that accomplish this. The function f_1 corresponds to the overposed data $h_1 = h$ and therefore is the homotopy solution of our problem. At each value of the parameter we linearize the problem about the function f_s and move in the direction dictated by the overposed data. This corresponds to implementing the scheme

$$J[f_s].(f_{s+\Delta s} - f_s) = h_{s+\Delta s} - h_s$$

which can be written symbolically as

$$\frac{\partial u}{\partial f} . \Delta f = \Delta h$$

and is therefore equivalent to Euler's method in parameter space $s \in [0,1]$.
In a similar manner to Newton's method, we can obtain a partial differential equation for the updates $\hat{u}$. The result is

$$\hat{u}_t - \hat{u}_{xx}(0,t;f^{(s)}) = f_s'(u_s)\hat{u} + f_{s+\Delta s}(u_s) - f_s(u_s) \tag{5.19}$$

$$\hat{u}_x(0,t) = 0, \qquad \hat{u}_x(1,t) = 0 \tag{5.20}$$

$$\hat{u}(x,0) = 0 \tag{5.21}$$

and

$$\hat{u}(0,t) = h_{s+\triangle s}(t) - h_s(t). \tag{5.22}$$

Using the representation

$$\hat{u}(0,t;f) = \int_0^t \int_0^1 \hat{K}(0,y,t-\tau)\left[f_{s+\triangle s}(u_s(y,\tau)) - f_s(u_s)(y,\tau))\right] dyd\tau$$

we again obtain a first kind Volterra equation for $f_{s+\triangle s}$, of convolution type, where $\hat{K}$ is the Green's function for the operator $\partial_t - \partial_{xx} - f_s'(u_s)I$. Using the additional boundary data, we can obtain an update scheme for $f_{s+\triangle s}$

$$h'_{s+\triangle s}(t) - h'_s(t) - \hat{u}_{xx}(0,t;f_{s+\triangle s} =$$

$$f_s'(u_s(0,t))(h_{s+\triangle s}(t) - h_s(t)) + f_{s+\triangle s}(u_s(0,t)) - f_s(u_s(0,t)) \tag{5.23}$$

The common theme for the previous three methods is the fact that each of the update schemes computes an update $f^{(n)}$ by means of the solution to a linearized partial differential equation. By projecting this onto the overposed boundary one can recover the update function from (5.18) or (5.23).

Non-Residual Schemes: We now approach the inverse problem (5.1)-(5.4) by a different method. Solving (5.8) for f in terms of h, we obtain

$$f(s) = h'(h^{-1}(s)) - u_{xx}(0, h^{-1}(s)). \tag{5.24}$$

If we eliminate f from (5.1), using (5.24), we obtain the *Trace Type Functional* (or *TTF*) equation

$$u_t(x,t) - u_{xx}(x,t) = h'(h^{-1}(u(x,t))) - u_{xx}(0, h^{-1}(u(x,t))) \tag{5.25}$$

In order for this equation to be well defined, the overposed data must satisfy the constraint $0 \leq h^{-1}\big(u(x,t)\big) \leq t$. The direct problem – that is, (5.25) along with the primary data – is to be solved for $u(x,t)$, with f recovered by (5.24).

The difficulty with this method is the form of the nonlinearity. Some results have been obtained for the time dependent coefficient case by finite differencing the nonlocal u_{xx} term [13]. Recent results for the *TTF* method are contained in [12, 14–16]. [For a more thorough survey of the *TTF* method and its application to various inverse problems, see the paper by Cannon and DuChateau in this Proceedings.]

If we eliminate u from (5.8) by using the representation formula

$$u(x,t) = \psi(x,t) + \iint K(x,y,t-\tau) f(u(y,\tau))dyd\tau$$

we obtain the *Fixed Point Projection* (or *FPP*) method

$$f(h(t)) = h'(t) - \psi_t(0,t) - \iint K_{xx}(0,y,t-\tau) f(u(y,\tau))dyd\tau. \tag{5.26}$$

We remark that there are some deep underlying connections between the *FPP*, *TTF*, and collocation methods. We actually solve the *FPP* equation by the equivalent iterative method in which we update f by

$$f^{(n+1)}(h(t)) = h'(t) - u_{xx}^{(n)}(0,t) = h'(t) - u_{xx}(0,t;f^{(n)}),$$

and update u by solving the nonlinear partial differential equation

$$u_t^{(n+1)} - u_{xx}^{(n+1)} = f^{(n+1)}(u^{(n+1)}).$$

Eliminating $f^{(n+1)}$ from the above two equations, we see this is equivalent to the iteration scheme

$$u_t^{(n+1)} - u_{xx}^{(n+1)} = h'(h^{-1}(u^{(n+1)}(x,t))) - u_{xx}^{(n)}(0,h^{-1}(u^{(n+1)}(x,t))) \tag{5.27}$$

which corresponds to a semi-implicit numerical strategy for the *TTF* equation (5.25). Equation (5.27) is implicit in u and explicit in u_{xx}, and is solved in time by a Crank-Nicholson or Tri-level scheme [31, 32, 34]. A fully explicit solution of (5.25) would correspond to the equation

$$\begin{aligned} u_t^{(n+1)} - u_{xx}^{(n+1)} &= h'(h^{-1}(u^{(n)}(x,t))) - u_{xx}(0,h^{-1}(u^{(n)}(x,t))) \\ &= f^{(n)}(u^{(n)}(x,t)). \end{aligned} \tag{5.28}$$

A fully implicit solution of the *TTF* equation would correspond to the equation

$$u_t^{(n+1)} - u_{xx}^{(n+1)} = h'(h^{-1}(u^{(n+1)}(x,t))) - u_{xx}^{(n+1)}(0,h^{-1}(u^{(n+1)}(x,t))) \tag{5.29}$$

which corresponds in turn to the coupled scheme

$$u_t^{(n+1)} - u_{xx}^{(n+1)} = f^{(n+1)}(u^{(n+1)})$$

$$f^{(n+1)}(h(t)) = h'(t) - u_{xx}^{(n+1)}(0,t),$$

and is iterated back and forth at *each* time step. Suppose we have solved the *TTF* equation by this method for $0 \le t \le t_j$. Because of the monotonicity of the data and the range condition, it can be seen that the $f^{(n+1)}$ are changed only on the interval $[h(t_j), h(t_{j+1})]$. Having determined the f over some range, it is forever fixed. This may also be interpreted as solving the nonlinear Volterra integral equation (5.26) for $t > t_j$:

$$\begin{aligned} f(h(t)) = h'(t) - \psi_t(0,t) &- \int_0^{t_j}\int_0^1 K_{xx}(0,y,t_{j+1}-\tau)f(u(y,\tau))dyd\tau \\ &- \int_{t_j}^{t}\int_0^1 K_{xx}(0,y,t_{j+1}-\tau)f(u(y,\tau))dyd\tau \end{aligned} \tag{5.30}$$

which requires inverting the kernel $K_{xx}(0,y,t)$ for small t. The first three terms are known, since $u(x,t) \in [h(t_0), h(t_j)]$ for $0 \le t \le t_j$ by the range condition.

If we collocate on $h'(t)$ – that is, we choose an $\bar{f}$ satisfying $u_t(0,t_j,\bar{f}) = h'(t_j)$ – we have

$$\bar{f}(\bar{u}(t_j)) = h'(t_j) - \psi_t(0,t_j) - \int_0^{t_j}\int_0^1 K_{xx}(0,y,t_j-\tau)\bar{f}(\bar{u}(y,\tau))dyd\tau.$$

This is equivalent to solving (5.30) by a piecewise linear function $\bar{f}$ if we also collocate on $h(t)$ (setting $\bar{u}(0,t_j) = h(t_j)$) and choose the collocation points to correspond to the actual time steps. Collocating only on $h'(t)$ leads to small differences between the methods, as long as a piecewise linear $f = \bar{f}$ is used to solve (5.30). If there is a monotone relationship between the overposed data and the unknown coefficient, collocation methods can be used to solve the *TTF* equation. On the other hand, implementing the *FPP* method corresponds to a particular numerical strategy for solving the *TTF* equation.

Other Types of Overposed Data: In one dimension, it is often possible to switch the roles of overposed and primary data. (This amounts to showing that the overposed data is admissible as primary data). In higher dimensions, this strategy may not work. Consider the inverse problem

$$u_t - \Delta u = f(u) \quad \text{in } \Omega \times [0,T] \tag{5.31}$$

$$u = h \quad \text{on } \partial\Omega \times [0,t] \tag{5.32}$$

with the overposed Neumann data

$$\frac{\partial \psi(x_0,t)}{\partial \nu} = g_0(t). \tag{5.33}$$

The overposed data lie on a curve of dimension one. Using the representation theorem

$$u(x,t) = \psi(x,t) + \iint K(x,y,t-\tau) f(u(y,\tau)) dy d\tau$$

we immediately obtain the "residual equation"

$$g_0(t) = \frac{\partial u}{\partial \nu}(x_0,t) = \frac{\partial \psi(x_0,t)}{\partial \nu} + \iint \frac{\partial K(x_0,y,t-\tau)}{\partial \nu} f(u(y,\tau) dy d\tau$$

which is a Neumann data–coefficient map. This is a first-kind nonlinear integral equation for $f(u)$. It is not clear how to convert it to second kind by any simple means. By analogy to the Jacobi iteration method, one possibility is to subtract off the diagonal terms to obtain the second-kind equation

$$\int_0^t k(t-\tau) f(h(t)) d\tau =$$

$$= \frac{\partial \psi}{\partial \nu} - g(t) + \iint \frac{\partial K}{\partial \nu}(f(h(t)) - f(u(y,\tau)) dy d\tau.$$

where $k(t) = \int_0^1 K(x_0,y,t)dy$. In general this leads to inversion of integral equations with fractional order kernels, whose inverses are not local differential operators.

Another phenomenon manifests itself with nonlocal overposed data. Consider (5.1)-(5.3) with overposed data of the form

$$E(t) = \int_0^1 u(x,t) dx.$$

Integrating (5.1) over [0, 1], we immediately obtain

$$E'(t) = g_1(t) - g_0(t) - \int_0^1 f(u(y,t)dy. \tag{5.34}$$

This is a first-kind nonlinear Fredholm integral equation, with a smooth kernel ($K = 1$). This is not a pleasant situation; however, one can easily show that $f_1 > f_2 \Rightarrow u_1 > u_2 \Rightarrow E_1 > E_2$, that is, E is a monotone function of f. This will therefore yield a unique piecewise linear function $\bar{f}$, by the collocation method, satisfying $E(t_j) = \int_0^1 \bar{f}(u(x,t_j)dx$.

In both of these cases, it is not immediately clear how to implement either the *TTF* or *FPP* methods.

6. Existence, Uniqueness, Regularity

Monotonicity Methods: If u is a monotone increasing function of the unknown coefficient f, then $u(x_0,t;f_1) > u(x_0,t;f_2)$ if $f_1 > f_2$. As a result, collocation methods are readily available. If the initial value f_0 is known (usually from consistency arguments), then on the first interval $u_0 < u < u_1$ we have $f(u) \approx f_0 + M_0(u - u_0)$, where M is the slope to be determined by matching $u(0,t_1;f) = h(t_1)$. Because of the monotonicity, one can easily show that there is at most one value of M_0 which satisfies this equation. In order for this to work we must have the range condition $u(x,t) \in \{u(0,t)\} = \{h(t)\}$. We may compute the successive values of M_i by this process, as long as the overposed data $h(t)$ is monotone. Monotonicity methods have been applied to inverse coefficient problems in [18].

Uniqueness: Uniqueness of undetermined analytic coefficients has been shown in a wide variety of instances. These rely on the property that any analytic function with an infinite set of zeros on a finite interval has an accumulation point and therefore must vanish identically. In one instance, this follows from the result that the difference between two solutions of the inverse problem must vanish infinitely often on an arbitrarily small interval if they agree at the left endpoint, [19]. Uniqueness results have also been obtained by utilizing contraction mapping arguments [31-36], and monotonicity [18].

Energy estimates on the difference of two solutions have been obtained by [14, 27 – 29]. These lead to uniqueness and stability (identifiability) results in various cases.

Existence: Existence questions are still a major source of difficulty. In fact, existence may not be the appropriate question. Existence requires that the overposed data lie in the range of the nonlinear map $f \longmapsto u(x_0,t;f) = h$. This consequently requires that the model (and hence the mapping) be known exactly, and that the data $h(t)$ be known exactly for a continuum of values $0 \le t \le T$.

The fixed point methods that we use (based on the *FPP* algorithm) do not yield existence at the present time because we assume the existence of a solution to the inverse problem, and show that with sufficient smoothness, it is a regular attractor. We then use the contraction properties of the mapping to show uniqueness [31-36]. Although under certain assumptions $f \in Lip_1$ yields $h' \in Lip_1$, the range of the nonlinear map may not in fact be onto a ball in Lip_1. If it is not onto, then we may expect that further constraints on the overposed data are necessary. Because of the discrete nature of overposed data, numerical errors associated with actual implementations, and noise in the data, determining whether the overposed data actually arises from a function in the admissible class is less important than whether the overposed data can be approximated in the limit (in some norm) by a sequence of approximations (which are themselves admissible). The robustness of the algorithm (that is, the sensitivity of the algorithm to noisy data) can therefore be checked by perturbing the primary and overposed data.

7. Numerical vs. Analytic Methods

Because of the intrinsic nonlinearity of the *direct* problems, one must get very precise estimates on the forward map $f \rightarrow u[x,t;f]$, [20,26]. The whole arsenal of nonlinear partial differential equation methods, boundary estimates, and singular integral equation theory must be used. Because of this complexity, numerical simulations are very useful. Reducing problems to finite dimension is not without its drawbacks, but "real" data is always discrete anyway.

Inverse methods based on modifications of the *FPP* method usually have little overhead in update strategy with respect to direct solver. This results from the fact that each iteration of the *FPP* method is equivalent to one direct solve, and the *FPP* method is observed to converge in a very small number of iterations (typically less than five). The update consists of differentiating the numerical solution near the boundary, and smoothing the result with a smoothing spline. This costs very little, compared to the cost of obtaining the numerical solution (via a direct solve). This implies that speed of convergence and stability are more important than cost of update strategy (at least for evolution equations). To summarize, we observe that:

RULE OF THUMB: *Good, efficient, accurate direct solvers are essential before one tackles inverse problems numerically.*

8. Discrete vs. Continuous Methods

Once one has discretized the inverse problem (with finite difference or finite element methods), smoothness is no longer really the issue. All functions in this setting are piecewise linear, or piecewise polynomials of low order. We must approximate the "exact" data by discrete data, therefore introducing errors into the overposed data. Noise and model error add to the difficulties. It is important to check the "robustness" of the inverse method. Numerically, methods may be very stable, in spite of the lack of an existence or uniqueness theorem. In fact:

RULE OF THUMB: *Good, stable numerical results usually indicate the existence of a mathematical theorem!*

The role of "numerical experimentation" has proven to be a valuable tool for testing hypotheses and conjectures. The "best" space for the posing of the discrete problem is not always clear from the problem itself, but it can be crucial to the performance of the numerical algorithm. For example if one applies the collocation method to (5.1)-(5.4), and uses only the overposed data to collocate, i.e. $u(0,t_i;f) = h(t_i)$, very poor results are obtained. It is much better to minimize $|u(0,t_i;f) - h(t_i)| + \lambda|u_t(0,t_i;f) - h'(t_i)|$ or even to collocate on $h'(t)$, that is set $u_t(0,t_i;f) = h'(t_i)$. This results from the fact that the mapping from $h \to f$ is not bounded but $h' \to f$ is.

Regularizations: Often, one must perform certain "regularizations" of the algorithms in order to maintain stability. For example,

a) A priori bounds (e.g., on the specific heat, $c_1 \geq c \geq c_0$) may have to be imposed by truncating the iterates. If the iterations settle down so that truncation is unnecessary, then truncation appears only as part of the "startup" cost for the algorithm (that is, in obtaining a sufficiently good initial guess).

b) The derivatives of the discrete data will usually have to be smoothed and interpolated (e.g., by a smoothing spline). Again, if the iterates become smooth enough so that this is unnecessary in the limit, it becomes part of the startup cost.

c) Nonuniform weighting of the residuals may be imposed. Initial errors may be larger than errors at a later time. Requirements on smaller time steps or more closely spaced overposed data are typical startup costs.

There are numerous ways to "regularize" inverse problems. If the regularizations disappear in the limit, they are fairly harmless. If they persist, then one is solving a "perturbed problem."

9. Multiple Unknowns

Many of the methods discussed in this survey can be extended to multiple coefficient inverse problems. An example of a multiple coefficient problem is the recovery of the solution triple, $\langle u, c, f \rangle$, in $c(u)u_t - \Delta u = f(u)$ from (additional) overposed data. Another situation occurs in the equation $u_t - \Delta u = f$, when the forcing function f might be known to be a function of both u and ∇u. The projection methods are not able to handle the true multivariable case directly, but some further restrictions on the nature of the multivariable dependence allow the projection methods to be applied. One method, which may be feasible under a wide variety of circumstances, is to assume that the forcing function can be approximated by the two-term Taylor series $f(u,\nabla u) \approx f^{(0)}(u) + \vec{f}(u).\nabla u$. The problem can then be treated as an attempt to recover the set of functions $\langle u, f^{(0)}, \vec{f} \rangle$.

One can visualize problems of the form

$$\begin{aligned} \mathbf{c(u)}u_t - \nabla.(\mathbf{k(u)}\nabla u) &= \mathbf{f(u)} + \gamma \\ \frac{\partial u}{\partial \nu} &= g(u) \\ u(x,0) &= u_0(x) \end{aligned}$$

with $n > 1$ unknowns. For a single dependent variable, we have essentially one observation point where Range $\{u\} \in \{u(x_0,t)\}$. In order to distinguish between coefficients we need multiple runs in general (choose different γ's or $u_0(x)$'s). There is much more opportunity for degeneracy, however. If the data is very flat ie $|\nabla^2 u| \approx 0$ then $c(u)u_t \sim f(u)$ and we get $h'(t) \sim f(h)/c(h)$ and only (f/c) is determined, unless $\gamma \neq 0$. We obtain conditions on the Jacobian of the matrix which are more difficult to enforce or verify.

Recovering $\langle \vec{u}, \vec{f} \rangle$ from the system of equations

$$\vec{u}_t - \Delta\vec{u} = \vec{f}(\vec{u}) + \vec{\gamma}$$

is one of the best type of multiple unknown problems, since the matrix multiplying the unknowns $\vec{f}$ is diagonal. Issues in multi-coefficient problems are still poorly understood, although some work has been done [31].

10. Further Questions and Open Problems

There are many function-analytic methods for solving the nonlinear equations of the form

$$R[u[\ldots; f]] = 0$$

in various kinds of spaces. Some of these, as we have shown, can be recast as iterated solutions of various perturbed equations and therefore lend themselves to schemes like the fixed point projection methods. Others are more direct attacks using the integral representations for the solution, or rely on properties of the data-coefficient mapping such as monotonicity. Still others, based on the *TTF* method, rely on existence theory for abstract Cauchy problems. Classifying the various methods in a more unified framework is yet to be done. Classifying ill-posed problems as to the degree of ill-posedness has been an ongoing research topic for many years [see the talk by Seidman in this Proceedings].

If the unknown coefficient involves the gradients of the dependent variables, much less is known. The update schemes are less stable, and more prone to error, particular in the discrete case. Numerical differentiation of data is always tricky. When the overposed data involves gradients in higher dimensions, or is nonlocal, it is not immediately clear how to apply either the *FPP*, collocation, or *TTF* method.

11. Summary

We have observed that for many inverse coefficient problems in parabolic equations, recovery methods generally fall into two categories – residual update and non-residual update strategies. The residual update strategies we have examined are the *FPP*, Newton, Quasi-Newton, and Homotopy methods. The convergence of these methods relies on the Fréchet derivative of the Data-Coefficient mapping. The non residual schemes considered were least squares, collocation, and the *TTF* method. These last two were shown to be closely related to the *FPP* method when actually implemented numerically. The *TTF* method is formulated in the terminology of an abstract Cauchy problem, while the *FPP* method is formulated in terms of a fixed point mapping.

The question of classifying inverse problems as to the degree of ill-posedness is at least subjectively possible, through the existence of a Meta-Theorem, examination of the regularity of the data-coefficient mapping, and existence and uniqueness results for abstract Cauchy problems of *TTF* form.

Existence of solutions to the inverse problems is a difficult issue, and is closely tied to the issue of compatibility of overposed data with the actual model considered. Of more practical importance is *constructibility*, *stability*, and *uniqueness* for solutions, given a definite form for the model. When the model itself is in question, existence is replaced by constructibility of approximations in Hadamard's triumvirate, which historically has defined well-posedness.

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A Collocation Scheme for the Identification of Coefficients in Nonlinear Parabolic Equations

[IEEE CDC Conference Abstract]

A collocation scheme for the identification of coefficients in nonlinear parabolic equations

*Michael Pilant**
*William Rundell **

Department of Mathematics
Texas A&M University
College Station Tx. 77843.

Abstract. *The identifiability of functions describing nonlinear radiation and source terms in the one dimensional heat equation is considered. It is shown that a monotone sequence of temperature measurements on one boundary is sufficient to determine a unique piecewise linear function which yields the desired temperature response. As a consequence, we show that the temperature can be controlled via this piecewise linear function. Numerical results are presented.*

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Introduction.

Consider the initial-boundary value problems on $[0,1]\times[0,T]$,

$$(\mathbf{B})\quad \begin{aligned} &u_t - u_{xx} = \gamma(x,t)\\ &u(x,0) = u_0(x)\\ &u_x(0,t) = f\big(u(0,t)\big)\\ &u_x(1,t) = -f\big(u(1,t)\big)\end{aligned} \qquad\qquad (\mathbf{I})\quad \begin{aligned} &u_t - u_{xx} = \gamma(x,t) + f(u)\\ &u(x,0) = u_0(x)\\ &u_x(0,t) = g_0\big(u(0,t)\big)\\ &u_x(1,t) = -g_1\big(u(1,t)\big)\end{aligned}$$

The functions u_0, γ, g_1 and g_2 are assumed known. Given sufficient smoothness on and knowledge of the function $f(u)$, the above ***direct problems*** have a unique solution for $u(x,t)$. However, we envision the situation where the function f is unknown, save that it depends only on u, and wish to determine both $u(x,t)$ and $f(u)$ by making additional boundary measurements. We shall denote by $u(x,t;f)$ the solution of the direct problem for a given function f. For further discussion of the direct problem and the modeling of such boundary conditions see the monograph [1].

Equations (**B**) describe, for example, the diffusion of heat in a uniaxial bar with nonlinear boundary conditions at the ends. Equations (**I**) describe, for example, the diffusion of heat in a uniaxial bar with nonlinear source terms. Our problem is therefore to determine the unknown temperature - dependent source term (or flux-temperature function) $f(u)$ from a knowledge of the initial temperature distribution, and a measurement of the temperature $u(x,t)$ at one boundary.

We shall first consider the case of recovering the pair $\langle u,f\rangle$ in (**B**). Why should one be interested in recovering the boundary term f? In many materials, the actual radiation mechanism is not known perfectly, and one must use an approximation. Many such approximations are possible. Perfect black body radiation into space (with zero ambient temperature) is governed by the law $\partial u/\partial\nu = \sigma u^4$ where σ is Boltzmann's constant, and $\partial u/\partial\nu$ denotes the derivative normal to the boundary. In a linear model (Newton's law of cooling), one assumes that the heat leaves the ends of the bar with the flux proportional to a temperature difference, $\partial u/\partial\nu = k(u-u_\infty)$. We propose a method for obtaining the exact form of the dependence of the flux on temperature in the nonlinear case.

If the boundary satisfies a ***known*** radiation law $\partial u/\partial\nu = \bar h(u)$ but the incoming flux Q is temperature dependent and unknown, we have $\partial u/\partial\nu = \bar h(u) + Q(u) \equiv f(u)$. This can be viewed as a problem in nonlinear control theory. The ***target*** set is the desired temperature response $\{h(t_j)\}$ and the unknown function $Q(\cdot)$ is the control. We will show that for any target set of monotone data, a piecewise linear function $\bar{f}$ exists with the property that the solution $u(x,t,\bar{f})$ to (**B**) with $f=\bar{f}$ satisfies $u(0,t_j,\bar{f})=h(t_j)$.

At best we will only be able to recover the function $f(u)$ for those values taken on by $u(0,t)$ for $0\le t\le T$. It is thus necessary that the range of values of the function $u(x,t)$ for $x\in[0,1]$ be contained in the range of values of $u(0,t)$ for $0\le t\le T$. In addition, we require the overposed data to be monotone in t. This monotonicity property is essential to recover the function f from a knowledge of $f(h)$. We shall make no attempt to completely characterize the class of allowable overposed data $h(t)$. Instead we shall assume that for some function $f(u)$, the data $u(0,t;f)$ is given, and show that the unknown radiation function can be reconstructed from the overposed data. We define the *residual* of the mapping from $h\mapsto f$ as $u(0,t;f)-h(t)$. Requiring this to vanish at a discrete set of points t_j, generates a *collocation* scheme.

In a recent paper [2], the authors proved the existence of a unique solution to (**B**) with the overposed data $u(0,t)=h(t)$. The solution of the inverse problem was obtained by an iteration scheme using the boundary condition itself as the update algorithm. This method can be summarized as follows. For a given function f, let $U(x,t;f)$ denote the solution to (**B**) except that the overposed data $U(0,t)=h(t)$ is used in place of the condition $u_x(0,t)-f\big(u(0,t)\big)$. If in addition the function U also satisfies this condition, then we must have the identity $f(h(t)) = U_x(0,t;f)\equiv \mathbf{T}[f]$. Indeed, one can show that under suitable conditions, there exists a unique fixed point of the mapping **T**, and that this solves the inverse problem. A similar approach can be used for the problem (**I**). Evaluating the differential equation $u_t-u_{xx}=g+f$ on the line $x=0$, we obtain $f\big(h(t)\big)=\{h'(t)-\gamma(0,t)\}-u_{xx}(0,t;f)\equiv\mathbf{T}[f]$.

Since we have interchanged the primary and overposed data along one boundary, this method for solving the boundary recovery problem (**B**) is limited to one spatial dimension. The fixed point formulation of (**I**) can be carried out in higher dimensions. Both methods involve numerical differentiation of data, and hence require the boundary measurements to be fairly noise-free. Although the fixed point formulations have limitations, they are constructive, and offer the possibility of proving existence and uiqueness results for the inverse problem. In practice they converge very rapidly (typically to within a few percent in less than 5 iterations),

How expensive are these methods? One normally expects an inverse problem to be more expensive than the corresponding direct problem. There are several reasons for this expectation. First, although the direct problem may be linear, the inverse problem will often be nonlinear. Second, inverse problems tend to be ill-posed, which contributes to their difficulty.

In the case of the fixed point methods, one can see that the update algorithm for f_{n+1} in terms of f_n has minimal cost, provided one has obtained the solution of the direct problem $U(x,t,f_n)$. Thus, the cost of the fixed point algorithms is the number of iterations required for acceptable convergence multiplied by the cost of solving the direct problem. As mentioned above, the actual cost of the inverse problem is on the order of a few times the cost of a single direct solve. While this is certainly acceptable, could further improvements be made? The collocation method outlined in this paper offers just such a possibility. Since the direct problem is nonlinear, if iterative methods are required for accuracy (for example Crank-Nicolson), then the cost of the inverse collocation method is comparable to the cost of a single nonlinear direct solve. We have performed several numerical experiments which indicate that the scheme is fairly robust under noisy data.

The Collocation Method.

We shall assume that the maximum values for u at any time occurs on the observed boundary and that the boundary data is monotone. Without loss of generality, we shall assume that it is increasing.

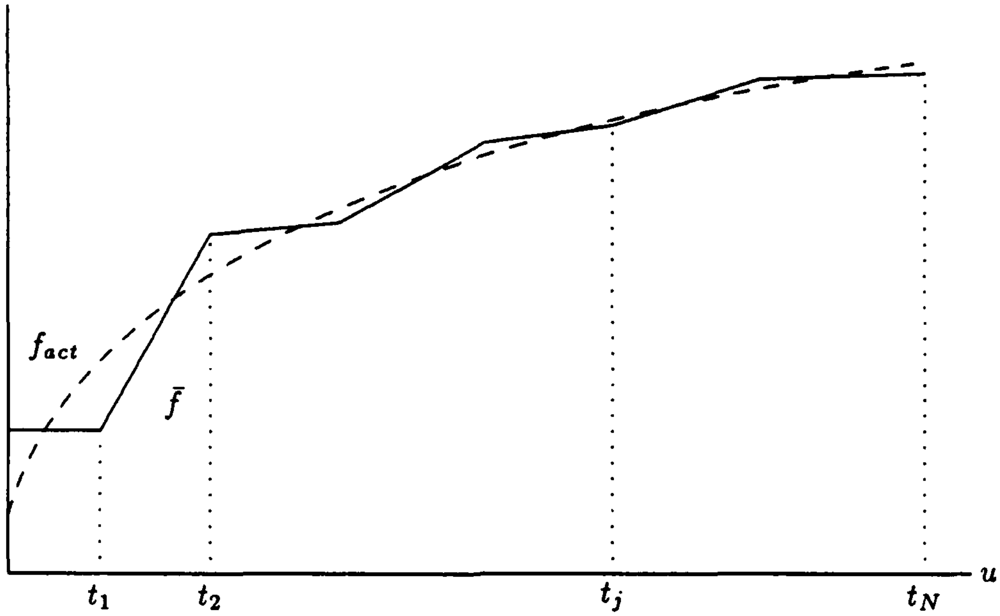
The collocation approach to the solution of problem (**B**) can be described as follows. We consider a function $\bar{f}$, piecewise linear on each of the intervals $[t_j, t_{j+1}]$, with the corresponding function $u(x,t;\bar{f})$.

(a) Assume that $f(v)$ is either known or has been approximated by a function $\bar{f}(v)$ for $v < h(t_j)$. On the interval $[h(t_j), h(t_{j+1})]$ extend $\bar{f}$ using the linearized approximation $\bar{f}(v) = \lambda_j(v - h(t_j)) + \bar{f}(h(t_j))$, for some constant λ_j which should be chosen in order to satisfy the collocation condition $u(0, t_{j+1}; \bar{f}) = h(t_{j+1})$. We use a secant method to calculate the slope λ_j from the resulting nonlinear equation

(b) Step (a) is continued for each interval $[t_j, t_{j+1}]$.

(c) How does one obtain the starting value $f(h(0))$? If the initial data and boundary data are compatible at $t = 0$ then we have $f(h(0)) = u_0'(0)$ which determines $\bar{f}$ initially. This will be the case if the process has been evolving for some positive time interval. If the data are incompatible, we then use collocation on $[h_0, h_1]$ to determine a *constant* approximation to f on this interval. This results in an error which is $\mathcal{O}(\Delta h)$.

A typical graph of f_{act} and $\bar{f}$ is pictured below.



There are some remarks to be made on the above procedure.

(1) The number of iterations required to obtain the slope λ on a given collocation interval will obviously depend on the tolerance required, and to the deviation of the function f_{act} from linearity on this interval, For most of the step sizes and values of an actual function f, f_{act} we chose, about 3 iterations sufficed to obtain λ to within the same accuracy as the forwards solution $u(x,t;f)$. The cost of the method is therefore 2 or 3 times the cost of solving a *linear* direct problem, since the approximation is by a linear function on each of the intervals $[t_j, t_{j+1}]$.

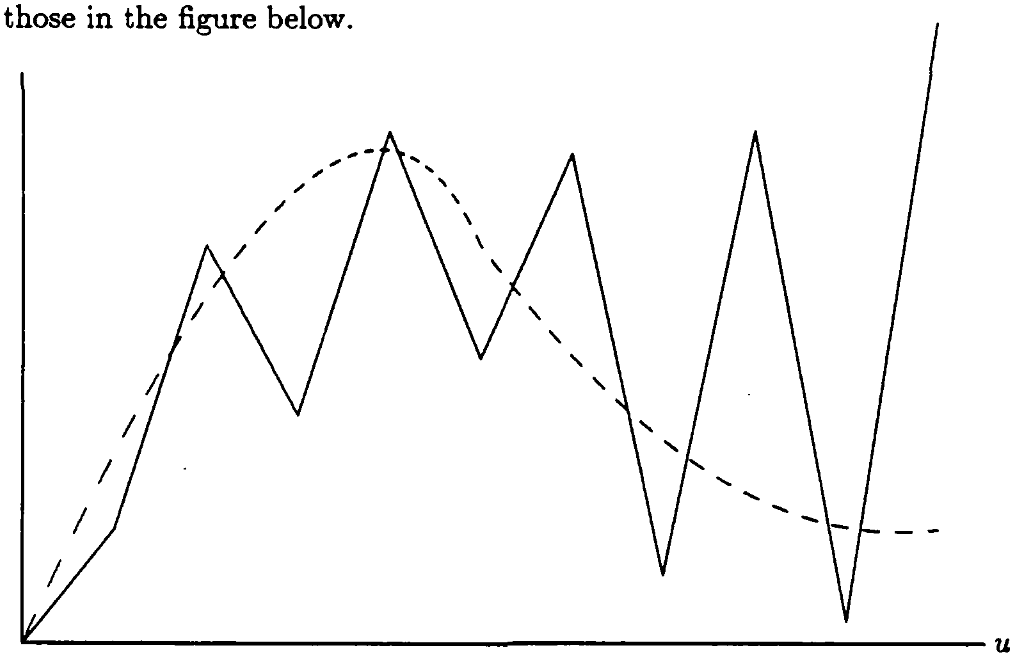
(2) An alternative collocation method for this problem would be to let $\bar{f}$ depend on N parameters $\{(c_1, \ldots, c_N)\}$, and impose N constraints to determine the c_j's by a least squares procedure. In our situation this is not necessary because for a parabolic equation the values of $u(x_0, t_1)$ do not affect the values of $u(x_0, t_2)$ if $t_1 > t_2$. Thus the value of c_j does not depend on the value of c_{j+1}. This allows the coefficients to be solved sequentially.

Let us consider the particular example of problem (**B**), $u_t - u_{xx} = 0$ for $0 < x < 1$, $0 < t < 1$ with $u_x(0,t) = f(u(0,t))$, $-u_x(1,t) = f(u(1,t))$ and $u(x,0) = 10x^3(1-x)^2$. We choose $f_{act}(u) = 3u^2 - 2u^3$. The choice of initial data was simply to ensure compatibility with the boundary data at $x = 0$ and $x = 1$.

The function $u(x,t;f_{act})$ was calculated numerically and the values of $h(t) \equiv u(1,t_j;f_{act})$ at the points $t_j = j/N$ for $j = 0,\ 1,\ \ldots\ ,\ N$, used as data for the collocation scheme. We used a step size of $k = 0.005$ in time and $h = 0.04$ in the spatial direction. The table below shows the difference of f and $\bar{f}$ in the supremum and L^2 norms for various values of N.

Convergence rate of $\bar{f}$ to f_{act}.		
N	$\|f_{act} - \bar{f}\|_\infty$	$\|f_{act} - \bar{f}\|_{L^2}$
2	0.0889	0.0410
5	0.0389	0.0130
10	0.0173	0.0036
25	0.0080	0.0011
50	0.0051	0.0010

For the interior f(u) case, problem (**I**), let us consider the particular example $u_t - u_{xx} = 1 + f(u)\ \ u(x,0) = 0,\ u_x(0,t) = t(t-2)\ \ u_x(1,t) = 0$, with $f_{act}(u) = ue^{1-u}$. Again we assume $x \in [0,1]$, $t \in [0,1]$. The maximum principle guarantees that the maximum value of $u(x,t)$ will, for each fixed t, occur at $x = 0$. If we apply the collocation scheme outlined above to this problem then we might well obtain results like those in the figure below.



Clearly something has gone wrong. The above picture strongly suggests that the inverse problem as defined is ill-posed and stability can only be recovered if we can find the correct dependence of the overposed data on f. The fixed point formulation, which requires one to evaluate the differential operator on the overposed boundary indicates the correct space. In this case using equation (1.8) we see that γ

and h' should have the same regularity as f if a fixed point is to be obtained. This strongly suggests that we should expect that the mapping $h \mapsto f$ is not bounded but that $h' \mapsto f$ might be. This can in fact be shown, [3]. Thus we should expect to include information on the derivative of the overposed data in our collocation scheme.

Instead of setting the value of $u(0, t_j; \bar{f})$ equal to $h(t_j)$ at each of the collocation points we can find the linear function $\bar{f}$ that minimizes the value of $|u(0, t_j; \bar{f}) - h(t_j)| + \alpha |u_t(0, t_j; \bar{f}) - h'(t_j)|$. In practice we have found that the value of α has little effect on the convergence of the scheme provided it is sufficiently large. Instead of this minimization which is more expensive than the direct collocation we can take the limiting of large α which is the same is collocation on $|u_t - h'|$.

We obtained the results in the table below with this procedure.

Convergence rate of $\bar{f}$ to f_{act}.

N	$\|f_{act} - \bar{f}\|_\infty$	$\|f_{act} - \bar{f}\|_{L^2}$
5	0.0531	0.0533
10	0.0253	0.0173
20	0.0145	0.0089
40	0.0116	0.0045
80	0.0065	0.0034

We have been able to prove the following result for the recovery of the boundary unknown, and we are currently investigating analogous results for problem (I).

Theorem. *(Convergence Theorem) Given monotonicity of u and sufficent regularity of the data, there exists a piecewise linear function $\bar{f}$ such that the solution $\bar{u}(x, t; \bar{f})$ to* (**B**) *satisfies the collocation condition $u(0, t_j; \bar{f}_N) = h(t_j)$ for $j = 0, 1, \ldots, N$. Furthermore, $\lim_{N \to \infty} N^p \|f - \bar{f}_N\|_\infty = 0$, for $p < 1/2$.*

Corollary. *If the target set of boundary measurements, $\{h(t_j)\}$ is monotone, there exists a piecewise linear boundary control, $\bar{f}$, such that $u(0, t_j; \bar{f}) = h(t_j)$.*

Besides the simplicity of this method, the collocation procedure offers advantages over global recovery schemes. Given any situation where the overposed data depends on the function f in a monotone manner, the collocation method can be carried out in principle.

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