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OPTIMAL STOCHASTIC MODELING AND CONTROL OF FLEXIBLE STRUCTURES

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Modeling and control design of large flexible space structures under high uncertainty is considered. Linear stochastic system models with multiplicative and additive noises that are state -, control -, and measurement - dependent are treated extensively. Controllability, observability, and robustness issues are also discussed under various optimality considerations. An optimal stochastic controller is derived under perfect information and a sub-optimal compensator is formulated under partial and noisy information. The intent of the report is to treat the modeling of uncertainties and to develop the appropriate stochastic control that is robust.

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FOREWORD

The present monogram is an extensive treatment of modeling and control design of large flexible space structures (LFSS) under high uncertainty. The uncertainties considered are of the multiplicative type, that are state-dependent, control-dependent, and measurement-dependent, in addition to regular additive noise. The theory of such systems involves complex mathematical developments and probabilistic approaches. The motive behind such an approach for the accurate modeling and control design of LFSS entails the fact that future LFSS, that are intended to be deployed in space under virtually no gravitation, have inherently uncertain characteristics. Moreover, testing, validation, and verificatin under realistic conditions is almost impossible, and scaled-down modal testing is still not advanced enough to present a viable means of reducing uncertainties. Thus, a probabilistic approach that incorporates the uncertainties into the analytical representation of LFSS is appropriate and it provides means of robust control design.

The report consists of seven chapters. The first chapter presents an overview of linear stochastic control systems with multiplicative and additive noise and the stochastic modeling and control of LFSS under a high degree of uncertainty. A review of research performed in these areas is followed by a unified theoretical treatment of the subject under perfect information considerations. The second chapter formulates the problem in a mathematical setting as applied to LFSS modeling and control design under uncertainties in the frequencies, damping ratios, and modal vectors. It is conjectured that knowing the statistics of the frequency and damping ratio uncertainties, is sufficient for generating the statistical characteristics of the elements of the modal matrix, under the assumption of Gaussian white noise. Also in this chapter the designation of finite element models and the subsequent model-order reduction, that is necessary for analysis, control design, and implementation purposes, is discussed in fair detail. Various techniques are briefly reviewed and commented upon, with specific application to LFSS. Uncertainties and the performance index are discussed in Chapter 3. Methods of generating the uncertainties and the expected disturbances are presented and ways of incorporating these into the system model are briefly reviewed. In Chapter 4 the optimal feedback control of linear stochastic systems under multiplicative and additive noise is derived and the stability characteristics of such systems are treated under various conditions. The relevance of such systems to LFSS control is commented upon and disadvantages of such a theoretical approach are underlined. The fifth chapter is concerned with appropriate realistic measurement systems. Specifically chosen for LFSS control, linear measurement systems with measurement-dependent noise and additive noise are presented and discussed. Moreover, controllability, detectability, and observability of such systems are defined and related to the regular linear quadratic Gaussian (LQG) case. The realistic state estimation problem is covered in Chapter 6. Linear minimum-mean-square unbiased estimator algorithms and other Kalman-Bucy type filtering algorithms are presented and discussed. A fixed structure linear unbiased estimation procedure is developed and its characteristic features are discussed briefly. The last chapter, Chapter 7, is devoted to the closed-loop stochastic linear compensator. It is shown that, under multiplicative and additive noise contaminating a control system, the certainty-equivalence principle breaks down and the overall estimaton and control should be treated simultaneously.

There is a great deal of interest in this subject presently, and many new articles have appeared treating various aspects of multiplicative noise. Moreover, many researchers have realized the advantages of such probabilistic approaches, not withstanding the difficulties involved. The robustness it inherently provides to a control system due to the adaptabilty of the controller to system and parametric variations cannot be overemphasized. Computational problems have still to be overcome in most applications. However, in the analysis, synthesis, and control design phases the payoffs of probabilistic approaches can be significant. Furthermore, for simple control systems with high uncertainties low order dynamic models can be utilized and such stochastic controllers can then be implemented without any difficulty.

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SECTION 1.1
INTRODUCTION

Very large-order finite element approximations are utilized to generate linear finite dimensional models for LFSS. There are various approaches to reducing the large order models for implementation and analysis purposes [1.1]. Moreover, random disturbances and uncertainties in the model are normally treated by including an additive white Gaussian noise vector. However, under conditions of high performance requirements and large degrees of uncertainty, simple, additive, white Gaussian noise will not be adequate and can even be the source of instability in a system's behavior. Multiplicative and additive noise in the system linear dynamic model, on the other hand, will be realistic and closely represent the real structure on an average basis [1.2].

A survey of literature is presented in the present chapter regarding various approaches to modeling and control of LFSS under high performance requirements and uncertainty [1.3]. Furthermore, each of the different methodologies are discussed briefly and their major contributions are highlighted. Both the continuous and the discrete time cases are presented and the references are evaluated in a chronological manner. In Section 1.2 Optimal Stochastic Control of Linear Systems with Multiplicative and Additive Noise is surveyed, and in Section 1.3 Modeling for Control Design of LFSS Under Uncertainty is reviewed. In Section 1.3.5 our approach to modeling and control of LFSS under uncertainty is introduced very briefly. In Section 1.3.6 conclusions are summarized; in Section 1.4 references of general importance are cited.

SECTION 1.2

OPTIMAL STOCHASTIC CONTROL OF LINEAR SYSTEMS WITH MULTIPLICATIVE AND ADDITIVE NOISE, A SURVEY

1.2.1 INTRODUCTION

System analysis and controller design of complex systems require a priori knowledge regarding overall performance, characteristics of the plant and the controller, and mutual interactions of the controller with the system at hand. However, the difficult issue often is the analytical description of the dynamic behavior of the plant. Moreover, the accuracy of the representation depends upon the assumptions made, the approximations used, the nonlinearities that are present and are identified, the unmodeled part of the system that is considered insignificant, the parameters involved, and the random disturbances that could affect the plant. Thus, modeling and control of modern complex systems, such as very large flexible space structures, entail an appreciable degree of uncertainty.

The sources of uncertainties are due to modeling approximations, noise and tolerances in components, random disturbances, parametric errors, configurational and random changes (that are often a function of time), and many other factors that cannot be accounted for with absolute accuracy. Statistical data regarding these uncertainties can play an important role in creating stochastic models for the control system under consideration. These models will be more realistic and close to the real plant on an average basis.

Various authors have addressed the problem of controlling linear/nonlinear systems with uncertainty in the model dynamics and control. Most of the work done in the past however, deal with system models with only additive noise to compensate for all uncertainties mentioned above; a large portion of the literature deals with linear models in the discrete-or continuous-time domains. For a good list of references on the subject see Kwakernaak and Sivan [1.1] and Mendel and Gieseking [1.2].

Optimal control theory is at a respectable level of maturity, thanks to mathematical developments such as the maximum principle, dynamic programming, stability theory, functional analysis, and filtering and estimation theory. However, stochastic optimal control theory for systems with multiplicative and additive noise needs more work for practical implementation and meaningful applications. The present article is concerned with research and theory in control of stochastic systems with multiplicative and additive noise that are represented by linear/bilinear models.

1.2.1.1 Historical Perspective

Methods for the optimal modeling and control of complex dynamic systems have been quite thoroughly investigated under deterministic conditions. However, when the plant to be controlled is subject to uncertainties (that are functions of the states and the controls), and are random in nature with known statistics, then we have a stochastic problem at hand [1.3]. Analysis and synthesis of the problem of optimal control design under multiplicative noise is relatively new. For, even though the general problem with stochastic variations is encountered in numerous references starting with the well known works of Wiener and Kalman, the optimal control of systems with state-dependent and control-dependent noise was not treated until the early part of the sixties.

According to the published literature, the first authors who solved a simple case of a control problem with random coefficients, were Drenick and Shaw in 1964 [1.4]. A similar problem with a little different setting was considered in a discrete-time function by Gunkel II and Franklin [1.5]. The stability of control systems was not addressed until 1965, by Krasovokiy [1.6]. Aoki [1.7] and others followed with various papers on the subject.

1.2.1.2 Status of Stochastic Control

The optimal control problem for infinite dimensional linear systems in Hilbert spaces is solved, both for deterministic and for stochastic systems with additive noise only [1.8]. Similarly, linear quadratic Gaussian theory is well established and is at a high level of maturity [1.1]. However, stochastic control and even deterministic control, for nonlinear systems is still a field open for research. The controllers that have to act under conditions of stochastic indeterminacy can only be functions of the observed data and thus of the "feedback" or "closed-loop" type. In situations of linear systems with only additive white Gaussian noise and quadratic performance indices the certainty equivalence principle is directly applicable. The certainty equivalence (or separation) principle states that the optimal stochastic feedback control is equivalent to the deterministic case with the state vector replaced by its estimate. When multiplicative noise is present, the above mentioned separability does not hold true. This aspect of the stochastic optimal control problem has been treated by various authors [1.7], [1.10].

The stochastic linear quadratic Gaussian (LQG) problem has a number of limitations. One major drawback is the fact that there is no systematic procedure for selecting appropriate weighting matrices, nor for choosing the covariances of the noise vectors of the measurement and the state equations [1.11].

All the limitations notwithstanding, linear quadratic Gaussian control is applied to a wide variety of systems ranging from large-scale integrated systems with multiple sensor-actuator loops to computer controlled systems with filters, adaptation, failure detection, reconfiguration and redundancy management. Moreover, the advent of very large flexible space antennas, and structures such as the space station, require new approaches to modeling under uncertainties due to the mere fact that it is virtually impossible to test these large structures under appropriate and realistic conditions. But, the need for more effective control and system integration, and more accurate modeling techniques dictate the use of stochastic modeling and control methodology as a viable approach for systems with complexities, uncertainty, and high performance requirements.

1.2.1.3 Application of Control Systems with Multiplicative and Additive Noise

The underlying objectives in control theory are the mathematical modeling and control design of complex dynamic systems and maintaining a set of variables within definite bounds. In operating physical systems, perturbations, which cannot be exactly predicted, subject the states of the system to strains and stresses and create the need to utilize stochastic models.

Various concepts of stochastic stability arise as a natural consequence of the study of the qualitative behavior of a system that is subject to random disturbances. Admissible controls for specific applications are determined, a priori, by establishing effective stability criteria. On the other hand, the criteria chosen often lead to quantitative performance indices on the basis of which admissible control strategies can be meaningfully compared. The latter in turn, leads to the well known problem of optimization, that is, determination of an optimal control function within an admissible class of controls [1.12].

The disturbances that we are concerned with herein are of three types: control-dependent noise, state-dependent noise and purely additive noise. Control-dependent noise arises from fluctuations in system structure, or of system energy content. The errors and disturbances during control tests can be modeled as multiplicative noise in the control matrix in a linear system. Additional causes of multiplicative noise that is control dependent is modeling of system parameters as idealized white Gaussian noise processes. Control-dependent noise tends to increase errors by diminishing the useful effect of the controller. Stabilization, especially by linear feedback, may not be possible in such cases, when the noise levels are high.

State-dependent noise, on the other hand, could be considered as internal dynamic disturbances which may be due to unmodeled dynamics, especially in systems with very high performance requirements. Additive noise is considered as an environmental disturbance which, together with the control, acts on the basic dynamics and affects the system. Large state-dependent noise has a destabilizing effect on the dynamics of the system and tends to increase the magnitude of the optimal gain. Stability may not be achieved even with a deterministic control input if the state-dependent noise is of large magnitude.

1.2.2 PROBLEM STATEMENT AND DISCUSSION

A relatively new accomplishment in modern control theory is the introduction of the concept of state space representation of a system. The optimal reconstruction of the state from observed data and the unification between the "state-space" theory and the classical transfer function theory have created opportunities for further research and understanding of complex practical issues. This has led to new approaches in systems control. In the present section we will state the general problem under consideration, the important relations between modeling and control, and the stochastic nature of modeling under uncertainty (specifically, the uncertainties that are state and control dependent and additive in nature). We will discuss both discrete-time and continuous time stochastic models. It should be noted that, while the minimal state space in deterministic control theory is almost unique, there are many solutions to the stochastic problem [1.13].

1.2.2.1 Mathematical Description

In this survey the optimal control problem of linear dynamical systems with state-dependent noise, control-dependent noise, and purely additive noise is considered. The most general representations in both continuous and discrete-time systems will be described and the necessary assumptions and restrictions will be stated.

The most general representation for a complex dynamic control system with high performance requirements will be a non-linear, infinite dimensional stochastic set of differential or difference equations [1.14]. However, for all practical purposes, approximations and simplifications are performed in order to obtain solutions, and to implement the system.

1.2.2.1.1 Continuous-time Case

Consider the following ordinary differential equations with multiplicative and additive noise, representing a linear control system:

$$\frac{d\,x\,(t)}{dt} = A(t)\,x\,(t) + B(t)U(t) + \sum_{i=1}^{n_1} F_i(t)\,x\,(t)\,\xi_i(t)$$

$$+ \sum_{j=1}^{n_2} G_j(t)U(t)\,\gamma_j(t) + E(t)\omega(t) \qquad (1.1)$$

where $x(t) \sim dim(n)$, $U(t) \sim dim(m)$ are the state and the control vectors respectively. ξ_i, $i=1, \ldots, n_1$, γ_j, $j=1, \ldots, n_2$ and $\omega(t) \sim dim(n)$ are zero mean white Gaussian independent random processes with given statistics (realistically these could be non-white, and non-Gaussian, however, for convenience quite often they are taken as Gaussian white processes). $A(t) \sim dim\ (n \times n)$, $B(t) \sim dim(n \times m)$, $F_i \sim dim(m \times n)$, $i=1, \ldots, n_1$, $G_j \sim dim(n \times m)$, $j=1, \ldots, n_2$, and $E(t) \sim dim(n \times n)$ are matrix functions all defined and bounded in the time interval of interest. $E(t)$ is of full rank.

The measurement system that provides noisy information is given as follows:

$$y(t) = C(t)\,x\,(t) + D(t)\nu(t) \qquad (1.2)$$

where: $y(t) \sim dim(p)$ is the measurement vector, and $\nu(t) \sim dim(p)$ is the zero-mean white Gaussian measurement noise vector with given statistics. $C(t) \sim dim(p \times n)$, and $D(t) \sim dim(p \times n)$ are matrix functions that are of full rank, bounded, and defined over the interval under consideration.

The performance functional that normally accompanies systems like (1.1) and (1.2) is given by the following quadratic measure:

$$J = E\ [x^T(t_f)\ H(t_f)\ x\ (t_f) + \int_{t_o}^{t_f} [x^T(t)\ Q(t)\ x\ (t)$$

$$+ U^T(t)\ R(t)\ U(t)]dt] \qquad (1.3)$$

where, H(t), Q(t) are positive semidefinite and R(t) is a positive definite matrix function defined and bounded over the time interval $[t_o, t_f]$. In Equation (1.3) $E[\cdot]$ is the statistical expectation operator. The optimal stochastic control problem is then to determine the admissible control input U(t) (that normally satisfies a given constraint, such as U(t)) such that J is minimized.

Under additive noise only the system above has an optimal control that feeds back the estimates, $\hat{x}(t)$ of the state x(t) and the "certainty equivalence" principle [1.10] holds true. However, under the present conditions it turns out that a set of non-linear matrix differential equations have to be solved and the separation principle mentioned above does not apply. Thus, the estimation and control problem have to be addressed simultaneously.

1.2.2.1.2 <u>Discrete-time Case</u>

Similar to the continuous case, the control system in the discrete-time domain is given by:

$$x(k+1) = \Phi(k)\ x\ (k) + \Gamma(k)U(k) + \sum_{i=1}^{r_1} \zeta_i(k)\ \Theta_i(k)\ x\ (k)$$

$$+ \sum_{j=1}^{r_2} \rho_j(k)\ \Psi_j(k)U(k) + \Lambda(k)\alpha(k) \qquad (1.4)$$

where $x \sim \dim(n)$ is the state vector, $U \sim \dim(m)$ is the control vector, ζ_i, $i=1, \ldots r_1$ and ρ_j, $j=1, \ldots r_2$, are zero-mean white Gaussian, independent, noise elements with given statistics, and $\alpha \sim \dim(n)$ is an additive, zero mean white Gaussian noise vector with given statistical data and independent from the other noise elements. $\phi \sim \dim(n \times n)$ and $\Gamma \sim \dim(n \times m)$ are the state-transition and the control matrices respectively of appropriate characteristics. Also, $\Theta_i \sim \dim(n \times n)$, $i=1, \ldots, r_1$ are coefficient matrices, and so is $\Lambda \sim \dim(n \times n)$, all of full rank.

The measurement system for Equation (1.4) is

$$y(k) = H(k)\, x(k) + E(k)\, \lambda(k) \qquad (1.5)$$

where $y(k) \sim \dim(p)$ is the measurement vector at time k, and $\lambda(k) \sim \dim(p)$ is the zero-mean white Gaussian noise vector with given statistics and independent from the rest of the noise vectors. H and E are coefficient matrices of full rank and appropriate dimensions.

The performance criterion for such a system is normally given by:

$$J_d = E\left[x^T(N) S\, x(N) + \sum_{i=0}^{N-1} \left[x^T(i) Q(i)\, x(i) + U^T(i) R(i) U(i)\right]\right] \qquad (1.6)$$

where N is the final time-step (in the infinite time situation $N \to \infty$), S and Q are matrices of appropriate dimensions and positive semi-definite, while R is a positive definite matrix of compatible dimensions. The discrete-time optimal stochastic control problem is to determine the control law that will minimize J_d using the information from Equation (1.5). Thus, as in the continuous case, there could be some constraint on U. Moreover, the separation principle being inapplicable, the filtering and control have to be solved simultaneously.

1.2.2.2 Relations Between Modeling and Control

In modeling and design of a control system the important factors that need careful consideration are (a) performance requirements, (b) uncertainties, (c) constraints, and (d) available measurement/information system. The level of performance dictates the degree of accuracy of the model. For, the higher the performance requirements the more effective a controller is needed, thus creating the need for accuracy. Uncertainties in parameters may be left as they are, and a "robust" control system may be designed that is essentially insensitive to parameter variations. Alternately, the levels of uncertainties can be reduced through extensive testing and verification, (whenever possible) or by means of real-time, on-line (or non-real-time) system identification. It is the former approach that is advocated and surveyed herein. The main reason for such an approach is twofold. First, system identification of complex control systems is often costly and practically not feasible. Secondly, a "robust" control system has some advantages that makes it more desirable as the "least of evil" [1.15]. Moreover, it is not always possible to generate an optimal (even a sub-optimal) closed loop system from an open-loop one, and it could be very involved. There does not seem to be a general approach to approximating non-linear, stochastic models with linear ones.

1.2.2.3 Stochastic Modeling with Multiplicative and Additive Noise

Various fields of application have motivated research in the analysis and control design of systems with multiplicative and additive noise. Thus, control systems that involve human operators [1.17-1.19], complex econometric systems with stochastically varying delays [1.19,1.20], mechanical systems with random vibrations, aerospace systems with high performance requirements (e.g., momentum exchange for regulating the angular precision of rotating spacecraft [1.18]) can all be cast into linear mathematical models with multiplicative noise. In addition, problems associated with reflections of transmitted signals from the ionosphere, as well as certain processes that involve random sampling errors can be formulated in the above mentioned fashion [1.5].

Further examples of systems with multiplicative noise models are: nuclear fission and heat transfer processes; migrations of people; migration of biological cells (as a consequence of the stochastic nature of cell divisions and separation), and noisy measurements on input and output variables. Furthermore, in pursuit-evasion game theory, the response trajectories of the pursuer and the evader may deviate from their nominal paths due to random parameter variations, thus resulting in a situation whereby state-dependent and control-dependent noise is realistically included in the system dynamic model [1.22]. Modeling of process disturbances with Gaussian white noise often results in multiplicative noise models as well [1.21].

The control and the stability characteristics of systems with the abovementioned formulations are rather different from deterministic systems, or systems with only additive noise elements [1.23,1.24, 1.25]. The formulation of stochastic models should be carried out with caution since optimal control laws that are derived from incorrectly specified stochastic disturbances may lead to instability [1.10,1.27,1.23].

1.2.3 VARIOUS APPROACHES TO STOCHASTIC MODELING AND CONTROL

1.2.3.1 Additive Gaussian Noise Model

It is well established that the optimal control law for linear stochastic systems with only additive, Gaussian noise, and a quadratic performance criterion is of the feedback type, and is accomplished by cascading the optimal state estimate with the deterministic optimal controller [1.1]. Thus, two Riccati equations are solved, the appropriate estimates and the optimal feedback gains are derived, which furnish the engineer with the optimal controller. We underline the fact that, by virtue of the separation principle (certainty equivalence), the two above-mentioned riccati equations are independent [1.8].

Under these circumstances, we have in Equation (1.1) $\xi_i = \gamma_j = 0$ and in Equation (1.4) $\zeta_i = \rho_j = 0$ for all i and j. Thus, if the new system is stabilizable and observable the optimal control always exists and is unique. The controller is a linear function of the state estimate and is independent of the intensity of the additive noise. The optimal problem is completely solved under the above conditions for both the discrete-time and the continuous-time situations, for full-state information or partial measurements, for simple regulation or tracking [1.1,1.8].

Extensive literature and theory exist on the topic of Linear Quadratic Gaussian (LQG) control systems [1.27]. It is the purpose of the present endeavor to review the state-of-the-art in stochastic control systems with multiplicative and additive noise. Hence, the reader interested in more details regarding LQG systems is referred to many of the existing books on the subject.

1.2.3.2 State-dependent and Control-dependent Noise

The monumental works of Feldbaum, Bellman, and Pontryagin dealt with a wide range of control problems: stochastic; deterministic; linear; and non-linear. They were among the first researchers to realize the statistical nature of the problem and the need for stochastic modeling of control systems under high performance requirements and uncertainties [1.28].

1.2.3.2.1 The Continuous-time Case

Consider the system given by Equations (1.1), (1.2), (1.3). The control analysis of this system has created worldwide interest. Florentin [1.29-1.32] specially in [1.31], Gersch and Kozin [1.33], Drenick and Shaw [1.4], Krasovokiy [1.6], and Krasovskii [1.34], seem to be the first researchers to analyze the abovementioned control system in the early sixties. Even though their considerations were basically in the scalar, or single-input-single-output, situation, they all realized the complex nature of the system and did not fail to point out that the random variations could result in non-existance of an optimal control input under multiplicative noise. During the latter part of the sixties, interest increased in control systems with multiplicative noise. Thus, Wonham [1.35], Tou [1.38], Gorman and Zaborsky [1.36], Kleinman [1.37] and McLane [1.18] considered multivariable systems and derived their optimal control characteristics. Kleinman, Gorman and Zaborsky considered the case with control-dependent noise, while Wonham and McLane treated state-dependent noise situations. The stability of stochastic systems was reviewed by Kozin [1.39] that brought into light several practical issues.

During the next decade the control problem with multiplicative noise was expanded further. It was McLane [1.18] who determined a linear feedback controller for systems with multiplicative and additive noise through the Hamiltonian approach. In his analysis, the measurement system was assumed perfect. He arrived at an optimal control law given by

$$U(t) = K\, x(t) \tag{1.7}$$

where

$$K(t) = -[R(t) + \Omega(P,t)]^{-1} B^T PSC(CSC^T)^{-1} \tag{1.8}$$

In Equation (1.8)

$$[\Omega]_{ij} = \mathrm{tr}\ [G_i^T(t)\ J_{xx} G_j(t)\ \Gamma(t)]$$

S(t) = the costate of the system

P(t) is the solution of the matrix Riccati equation, $\Gamma(t) = E\ [\gamma\gamma^T]$ and $J_{xx} = \partial^2 J/\partial x^2$.

Simultaneously Haussmann [1.40] studied the same problem using Lyapunov methods and derived conditions under which an optimal control law exists. The above researchers concluded that, in the use of control-dependent noise the control is cautious (small gains), while for state-dependent noise more active controls are required (large gains). Feedback stabilizability in the mean square sense was taken up by Willems and Willems [1.21] and necessary and sufficient conditions were presented.

During the latter part of the seventies several authors tackled the control problem with multiplicative noise [1.41]. Timofeev and Chernyavskii [1.42] considered a worst case situation in which the statistics of the random variables are not known but only an admissible set for their distributions is given. While Bismut [1.43] derived existance results for an optimal control in a random feedback form, using functional analysis. Moreover, he showed that a unique solution to the matrix riccati equation exists under assumptions of independence of the coefficients of the equation and the criteria from the noise parameters.

For the first time in the published literature, the problem of estimation of Equations (1.1), (1.2), (1.3) was studied by Bondaros and Konstantinov [1.44] through a Hamiltonian procedure. In this model, multiplicative and additive noise contaminated not only the state equation, but also the measurement equation. The analysis proved that the uncorrelated additive perturbations in the dynamics and observation equations increase the estimation error. A similar increase in the estimation error occurs if the additive and multiplicative noises of the measurement system are correlated. Milshtein [1.45,1.46] derived stabilizing controllers for the steady-state problem with both perfect and noisy measurements by reducing it to a constrained minimization problem. While Katayama considered the related problem of assymptotic stability properties of the Riccati equation with constant but unknown coefficients. Several other authors have studied the continuous-time problem of stochastic control Equations (1.1), (1.2), (1.3) [1.47,1.48] and most of the aspects treated. These studies are modifications to their predecessors' works [1.49-1.53]. Hyland [1.54] used the maximum enthoropy approach to control design and regulation for uncertain structural systems with uncertainties in frequencies.

1.2.3.3 The Discrete-time Case

There is definitely a richer literature for the discrete time problem Equations (1.4), (1.5). (1.6). The first article on the subject was published by Gunkel and Franklin [1.5] in which the effect of random sampling in sampled-data control systems were presented in the form of multiplicative noise. Tou [1.38] followed by advocating the concept of adaptive and learning control under large parameter fluctuations, and used statistical decision theory and dynamic programming. Aoki [1.7] pointed out that the certainty equivalence principle does not result in optimal control laws under multiplicative noise situations. Controllability of stochastic linear systems was taken up by Connors [1.55]. His analysis utilizes dynamic programming and derives necessary and sufficient conditions of controllability for systems with multiplicative noise and perfect measurements. Murphy [1.56] and Grammaticos and Horowitz [1.57] considered linear systems with unknown gains, while Ku and Athans [1.58] showed that the open-loop feedback optimal adaptive gains are functions of current and future uncertainty of the parameter estimates. Bar-Shalom and Sivan [1.59] studied linear systems with random parameters and derived optimal open-loop and open-loop feedback controllers under a quadratic criterion. Stability characteristics for stochastic nonlinear difference systems perturbed by random disturbances was treated for the first time by Konstantinov [1.60] and Mishra and Mahalanabis [1.61]. Kendrick [1.62], Shupp [1.63] and Aoki [1.7,1.20], presented applications of linear stochastic systems like Equations (1.4), (1.5), (1.6) to macroeconomic and economic systems. Katayama [64] treated the asymptotic properties of the matrix Riccati equation with random coefficients, while Athans, Ku and Gershwin [1.65] and Ku and Athans [1.66] studied the limitations and conditions under which the infinite horizon solution of the optimal stochastic control problem does not exist. Wittenmark presented a survey of stochastic self organizing, self-optimizing control methods whereby he also mentioned some aspects of multiplicative noise.

It was Zabczyk [1.67] who studied the general, infinite dimensional stochastic control problem of linear systems with multiplicative and additive noise in Hilbert spaces. His analysis contributes significantly to discrete-time systems' stochastic observability, controllability, existence and uniqueness of solutions, as well as the characteristics of the Riccati equation, both under finite and infinite time situations. He derived the optimal control law:

$$U(n) = - G(n)\ x\ (n) \tag{1.9}$$

where

$$G(n) = [R(n) + \Gamma^*(n)W(n+1)\Gamma(n) + \Delta(u)]^{-1} \cdot \Gamma^*(n)W(n+1)\Phi(n) \tag{1.10}$$

$$W(n) = Q(n) + \Phi^*(n)W(n+1)\ (n)$$

$$+ E[(W(n+1) \sum_{i=1}^{r1} \zeta_i(n)\theta_i(n), \sum_{i=1}^{n} \zeta_i(n)\theta_i(n))] \quad (1.11)$$

$$\cdot \Phi^*(n)W(n+1)\Gamma(u)\ [R(n) + \Gamma^*(n)W(n+1)\Gamma(n) + \Delta(n)]^{-1}$$

$$\Gamma^*(n)W(n+1) \cdot \Phi(n)$$

(.,.) is the inner product operator

(*) is the complex conjugate transposition operator

$[\cdot]^{-1}$ is the inversion operator, $R = \mathrm{cov}(\rho_j)$, $j = 1, \ldots$

$$\Delta(n) = E[(W(n+1) \sum_{j=1}^{r2} \rho_j(n)\psi_j(n), \sum_{j=1}^{r2} \rho_j(n)\psi_j(n))]$$

Zabczyk proved that both $\Delta(n)$ and the quantity with the expectation in Equation (1.11) are monotonic under some condition. Moreover, he showed that for the steady-state solution W of the riccati equation the cost is given by

$$J_d = (Wx, x) + N\mathrm{tr}(\Lambda^* W \Lambda V) \quad (1.12)$$

where $V = \mathrm{cov}\ (\alpha)$

Existence and uniqueness issues are also tackled in Zabczyk's article and, so are stabilizability and detectability conditions. Stochastic observability is introduced for the discrete-time stochastic system considered and novel results related to the finite dimensional applications are discussed. Joshi [1.68], Harris [1.69] also treated the same problem. The latter presented results on controllability for discrete stochastic systems with the random variables in the state and control matrices drawn from different distributions. Pakshin [1.70,1.71,1.72] analyzed the estimation and control synthesis of discrete-time linear systems by deriving a filter and a controller that are optimal in the class of linear systems. Furthermore, he drived a suboptimal solution for systems with non-quadratic criteria. Tugnait [1.73] presented results on uniform asymptotic stability of linear stochastic estimators with white multiplicative noise (as well as additive) contaminating the measurement system.

Panossian [1.23] and Panossian and Leondes [1.72,1.73] studied various aspects of multivariable linear stochastic, discrete-time systems that are partly deterministic, and partly stochastic with multiplicative and additive noise. Furthermore, they analyzed the estimation problem under partly exact and partly noisy measurements (the latter having multiplicative and additive components as well). Results on a reduced-order linear stochastic observer was also presented that will produce estimates that are optimal in the subclass of linear reduced-order stochastic systems. De Koning [1.24,1.76,1.77] on the other hand, reported results on linear discrete-time systems with stochastic parameters having models whereby the state and control matrices are sequences of random matrices with fixed statistics. These articles dealt with the behavior of the first and second moments of the random variables, and through the characteristics of these moments, De Koning addressed the issues of stability, detectability, stability in the mean and mean square sense, and optimal estimation. The subject of systems with random coefficients is still of current research interest. Moreover, design and control of uncertain linear systems was last considered by Petersen [1.78].

1.2.3.4 Bilinear Stochastic Systems

There has been an appreciable amount of literature on bilinear control systems during the past two and a half decades. However, most of the articles deal with deterministic cases. The stochastic problem of bilinear systems with random disturbances has been a topic of interest for many researchers [1.79]. Recent interest in the controllability, stability, and other aspects of the problem are also reported [1.80].

A wide range of problems may be approximated by bilinear stochastic systems [1.81]. Diffusion processes, specially in nuclear fission, inheat transfer, and in biological systems, may be modeled appropriately by bilinear stochastic systems [1.82]. An additional term in Equation (1) will result in a general form of stochastic bilinear systems. Thus, a term of the form N(t, U(t)) x (t) added to Equation (1.1) on the right-hand side results in such a system. The characteristics of bilinear stochastic systems are derived in several papers [1.83,1.84]. Most of the studies treat such systems as a first stage generalization of linear stochastic systems, especially under finite dimensional suboptimal filters. The problem of identification, on the other hand, needs further research [1.84]. State-dependent and control-dependent noise problems can be considered special cases of bilinear stochastic systems and most of the research is in this area (see previous sections). We will not treat this subject any further, since the main interest of the paper is in systems with multiplicative and additive noise. The analysis and synthesis of bilinear stochastic systems need further development and to this end many researchers continue to study various aspects of the problem [1.85,1.86].

1.2.4 LINEAR STOCHASTIC SYSTEMS

For the past two decades linear stochastic systems in the infinite dimensional spaces has been of interest to many researchers [1.8]. The theory of semigroups in functional analysis, specially that of linear operators developed recently, has proved very valuable and advantageous in solving very general classes of optimal control problems. Some of the disadvantages of semigroup theory, such as its applicability to only time invariant systems, has created the need to extensions to time dependent "evolution" equations [1.87]. However there seems to be a great deal of work still ahead in this area for this approach to lend itself to practical aspects of the problems in stochastic control theory.

1.2.4.1 Distributed Parameter Systems

In Equation (1.1) A(t) would represent an infinitesimal generator of a strongly continuous semigroup S(t) over an appropriate Hilbert space when $A(t) = A$ for all t and $I = \{t: 0 < t < T\}$. Moreover, B, F_i, G_j and E would all be linear bounded transformations mapping appropriately defined separable Hilbert spaces into the Hilbert space over which S(t) is defined. In a similar manner, the observation equation can be generalized to infinite dimensional Hilbert spaces. The quadratic cost functional in Equation (1.3) will also have to be placed in an appropriate infinite dimensional setting and then, the control problem is to find an admissible controller that will minimize Equation (1.3).

The closed-loop optimal stochastic infinite dimensional problem under partial and noisy measurements (to the author's knowledge) and with multiplicative and additive noise has not been solved to this day.

1.2.4.2 Approximations to Finite Dimensional Systems

The infinite dimensional representation of control systems is only an idealization of reality under the assumption that matter is a continuum and that internal and external forces and moments are distributed. However, under practical circumstances, when real control systems have to be implemented on real structures only finite dimensional models and controllers are meaningful (at least until the present). Hence, even though infinite dimensional models, for instance models composed of partial differential equations, can give deeper understanding and insight into subtle characteristics of systems' behavior. There are no physically implementable distributed controllers (in the practical sense of the word), nor are there infinitely distributed sensors for observation that can be practically useful.

There are various approaches of approximating infinite dimensional distributed parameter systems by finite dimensional ordinary differential (difference) equation systems. The widely used techniques known as Rayleigh-Ritz-Galerkin, finite element, finite difference, etc. are but a few of many others [1.88]. These numerical methods are translated into computer codes and are widely utilized in generating finite dimensional ordinary differential equation models for structural systems.

1.2.4.3 Nonlinearities in Linear Stochastic Systems.

Linear stochastic systems with multiplicative and additive noise can actually be considered a nonlinear system with respect to stochastic variations. Since, the state vector, the control vector, and the multiplicative noise matrices are expressed as products of each other, they create a nonlinear setting. It is well known that the optimal filter for such systems is an infinite dimensional non-linear filter [1.44,1.45]. Only suboptimal linear filters can be formulated for linear stochastic systems with multiplicative and additive noise, or optimal filters in a class of linear filters [1.48].

1.2.4.4 Computational and Realization Difficulties

The complexity of linear systems with multiplictive and additive noise speaks for itself. It is obvious that for large order finite dimensional systems there could be an unrealistically large amount of statistical information required. For situations of non-Gaussian distributions, the problem gets more complicated specially when filtering under partial and noisy observation.

The computational aspect of the problem is extremely complex, even if the distributions of the statistical parameters are well known and the order of the system is relatively small [1.23]. Thus, in order to formulate a control system model in the stochastic setting presented herein, it is necessary to make certain assumptions, simplifications and approximations. However, this is the case in any modeling situations, and it is not considered a severe limitation. The important fact in a control system remains to be, a valid approximation in the form of a mathematical model, whether it be stochastic or deterministic, linear or non-linear, distributed or lumped, be as close to the real plant as practically possible.

1.2.5 CONCLUSIONS

We have presented a brief review of the state of the art in stochastic linear-bilinear control sytems with multiplicative and additive noise. Both the discrete-time and continuous-time cases were exposed in a general setting and some of the important developments in the field were brought to light. It should be noted that, it is very difficult to iterate many of the significant theoretical achievements in this area and still remain within the limitations of publication guidelines. However, we have tried to include those which we thought should be discussed.

The advent of supersonic aircraft with very high performance requirements, and the futuristic space structures, such as the space station, seem to promise some realistic ground for the approach presented herein. Stochastic modeling and control is the realistic alternative to controlling systems with high performance requirements and uncertainty.

SECTION 1.3

MODELING FOR CONTROL DESIGN OF LARGE SPACE STRUCTURES UNDER UNCERTAINTY: A SURVEY

1.3.1 INTRODUCTION

The issue of mathematical modeling of large flexible space structures (LFSS) presently is a topic of extensive research. The theory of elasticity forms the core of modeling of the flexible body dynamics. Thus, stresses and strains in structures, created by various effects, result in deformations and displacements that can mathematically be approximated by partial differential equations with forcing functions and random disturbances. However, it is well known that engineering structures generally consist of discrete parts of finite length fastened together into a complete, integrated system [1.89,1.90,1.91]. The idealized representation of each of these parts is by infinite dimensional distributed parameter systems, given by partial differential equations (linear or nonlinear, depending on the specific structure and the performance requirements at hand). The derivation of a finite dimensional model hereon, is normally carried out through the use of approximations that reduce partial differential equation models to ordinary differential equation models via various techniques that essentially project the infinite dimensional space into a finite dimensional one. The most commonly used of these techniques are the finite element, the Rayleigh-Ritz-Galerkin, and the lumped parameter methods [1.92]. The rigid body dynamics, on the other hand, are described by ordinary differential equations. Thus, LFSS are coupled systems of elastically deformable and rigid bodies whose behavior is characterized by nonhomogeneous, hybrid equations with uncertain parameters and random disturbances [1.93].

There are some fundamental assumptions inherent in generating finite dimensional models for LFSS. Namely, the existance of: (1) an "accurate" finite dimensional model arbitrarily close to the ideal infinite system, (2) a maximum fundamental natural frequency of vibrations such that all the modes with higher fundamental frequencies can be neglected, (3) interaction of various modes, either stable or unstable, that can be modeled, (4) a finite control bandwidth vis-a-vis expected disturbances and desired performance specifications, and (5) a finite amount of structural damping.

The inherent reason behind the analytical representation of the dynamics of LFSS is to design a control system in order (1) to stabilize the vehicle with reference to an appropriate coordinate system, (2) to point the instrumentation with some a-priori constraint on accuracy and performance, and (3) to control shape variations. While only an inertial, six-degree-of-freedom, model is required in the rigid body dynamics situation, consideration of control of elastic modes (as well as the standard rigid body coordinates) must be made whenever the structural configurations are very large, or when stringent performance specifications dictate robust maneuverability requirements. The above mentioned unified approach to the active control of flexible body responses, in addition to ridig body dynamics, is often referred to as control/structure interaction [1.94].

The traditional approach to LFSS modeling is by the finite element (FE) method. There are various computer programs that can generate large-order FE models for complex control systems. Examples of these codes include NASTRAN and DISCOS. The FE technique will normally generate large dimensional models that have relatively good accuracy in the lower frequencies and their corresponding modes, and the uncertainities and errors increase drastically as higher frequencies are included. Furthermore, for every new parametric value, a new model has to be generated, which rules out any insight to the physical behavior of the system relative to parameter variations. Recently a great deal of interest was focused on distributed parameter modeling of LFSS [1.95]. This latter approach is definitely more concise in mathematical notation, provides some insight relative to physical behavior and design variations, and renders parametric studies possible. However, not withstanding all the above mentioned facts, uncertainties inherent to LFSS are so large that any approach to their mathematical modeling will still be inadequate for designing robust control systems without addressing the stochastic nature of the problem or performing on-line identification (which is very costly). Thus, theoretically, it is possible to develop a "best" deterministic model for LFSS. Then one can incorporate all uncertainties, that can statistically be identified, into this model for a realistic stochastic model that is closer to the real system, in an average sense [1.96].

There is a large amount of literature on modeling and control design of LFSS. Two quite comprehensive surveys related to dynamics and control of LFSS were published recently [1.97,1.98]. Nevertheless, most of the existing literature deal with methods of generating deterministic analytical models, and in the best case, discuss stochastic models with some additive white Gaussian noise vector in the linear dynamic model. This additive random vector, is supposed to account for all uncertainities due to modeling approximations, noise and tolerances in components, random effects, parametric errors, etc. Nevertheless, several authors [1.99-1.101] pointed out, in the early sixties, that there are various uncertainties that have to be accounted for in modeling of flexible structures. Hoshiya and Shah [1.101], in particular, considered the free vibration of a beam that has random material and dimensional parameters with given statistics, and they generated the general stochastic equations relative to the n^{th} natural frequency. Moreover, they performed sensitivity analyses between random input and output parameters of the stochastic system under consideration. Collins and Thomson [1.102] also investigated the statistical eigenvalue-eigenvector problem under random mass and stiffness perturbations. Several other authors continued this trend by addressing uncertainty in eigeuvalues and eigenvectors due to randomness in structural properties and inputs [1.103-1.108]. Hyland [1.109-1.111] was the first to address the stochastic closed-loop problem for LFSS. He presented the analytical model of structures with uncertainties in the frequencies and analyzed the optimal control problem under a maximum entropy setting. On the other hand, the author has presented another approach to stochastic modeling of LFSS [1.96] by incorporating statistical data into the best system dynamic model available. The frequency, damping, and mode shape parameters were considered to be stochastic processes with known statistics and their closed-loop stability characteristics, under various considerations were analyzed.

1.3.2 THE MODELING PROBLEM

Modeling of a control system is a direct function of the performance requirements, the size, and complexity of the system. However, accuracy requirements and the degree of detailed modeling are related to performance specifications and expected disturbances more than anything else [1.98]. The control system model may be generated through simple procedures if accuracy requirements permit leniency, or complicated FE or distributed parameter models may be necessary for high accuracy and stringent performance requirements. However, it is the balance between analysis and testing that renders the derivation of acceptable analytical models possible. Knowledge of structural characteristics can also improve the model.

There are many problems that face the designer/dynamicist which include data acquisition, excitation, hardware, and testing limitations and many other constraints. Currently there are two basic testing techniques. Namely: (1) multiexciter normal mode method, and (2) single excitation source frequency response matrix approach. Each of these approaches has its advantages and disadvantages. One underlying problem with all testing methods is relating the number of measurements, the number of identified mode shapes, and the order of the mathematical model of the system [1.100]. Moreover, determination of modal characteristics of structures is the core of experimental testing. Thus, natural frequencies, damping ratios, and modes are very important, physically meaningful, elements for applications in stability and control, prediction of response and loads, vibration, and modeling, among others [1.112]. In the abovementioned parameters, damping is the hardest to identify and model, especially in the case of LFSS, since these have inherently very low damping [1.113].

In the final analysis, a dynamical system model is a mathematical abstraction that represents the input-output relationship of a "state" vector (which in turn, represents some internal characteristics) with respect to an ordered set, time. Furthermore, a system model is called a finite dimensional realization, on a given interval and with known input-output characteristics, if it is completely reachable and completely observable. These conditions are in general very hard to meet and thus, the realization issue of dynamic control systems remains a nontrivial one [1.114].

1.3.2.1 Infinite Dimensional Distributed Parameter Method

There are many advocates to distributed parameter modeling and control design of LFSS [1.95]. Moreover, several authors assess the theoretical and practical advantages of partial differential equation representation of LFSS in terms of suitability for analysis, conciseness, and provision for physical understanding [1.115]. The usual procedure followed for modeling of distributed structures is the extended Hamilton principle, whereby expressions are derived for the Kinetic and potential energy and for the virtual work of the sytem and then the mathematical model is generated using the variational approach [1.116]. For any virtual displacement from the system's trajectory, the following is true

$$\delta \int_{t_1}^{t_2} (T-V)\, dt = \delta \int_{t_1}^{t_2} FU \cdot r\, dt = 0 \quad \forall\, t_1, t_2 \tag{1.13}$$

where T and V are the kinetic and the potential energies respectively and $FU \cdot r$ is the virtual work of the applied forces during the displacement. The modeling problem is now the computation of the differential terms in Equation (1.13) and their variations [1.117]. Furthermore, application of the variation principle and some manipulations yield the following:

$$L\,[x(s,t)] = [M(s)]\,\ddot{x}(t) + [D(s) + G(s)]\,\dot{x}\,(t) + [K[s) + H(s)]\,x\,(t) = F(s)\,u(t) \tag{1.14}$$

where x(t) is the spacial displacement vector, L is a linear transformation, [M], [D], [G], [S], [H] are matrices whose elements are scalars and functions of spacial variables. Also, the latter matrices are bounded operators with domains in appropriate spaces. In Equation (1.14), [M] is known as the mass or inertia matrix, [D] is called the damping matrix, and [K] is the stiffness matrix. [G] is often referred to as the gyroscopic or coriolis matrix, and [H] is referred to as the circulatory matrix. Appropriate transformations will transform Equation (1.14) into the state-space representation. Thus, by taking $y = (x,\dot{x})^T$ we have:

$$\dot{y} = Ay + Bu \tag{1.15}$$

where

$$A = \begin{bmatrix} 0 & I \\ -[M]^{-1}[K+H] & -[M]^{-1}[D+G] \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ [M]^{-1}F \end{bmatrix}$$

In the above equations, $(\cdot)^{-1}$ is the inversion operator, $(\cdot)^T$ is the transposition operator, t is time, and s is the spacial variable. For a stable system, [D+G] is a positive definite matrix operator. In most cases it is also self adjoining [1,117]. Moreover, [M] and [K+H] are real, positive, self adjoint operators in most applications.

In general, s will be a stochastic process, thus making [M], [D+G], [K+H] and [F] random operators. Hence, Equation (1.15) is a stochastic system with multiplicative and additive noise in the infinite dimensional space [1.119]. To the author's knowledge, there is no published literature relative to the abovementioned stochastic modeling and control problem in infinite dimensional spaces for LFSS.

1.3.2.2 Finite Dimensional Methods, Model Reduction

For implementation and controller design purposes any infinite dimensional system model will have to be reduced, somehow, to a finite dimensional system model. There are various approximation techniques to this end, all of which essentially project the infinite dimensional spaces onto finite dimensional ones and thus reduce partial differential equations to ordinary differential equations, [1.119] or large order models to smaller ones [1.120].

For structural response analysis, accurate expressions for the most significant (in some sense) normal modes are required. However, most real-life structures (especially the futuristic LFSS) have very complex geometries, attachments, and boundary conditions;

thus, making it almost impossible to derive exact expressions for the normal modes. Herein lies the need for approximation. This model reduction is accomplished by utilizing separation of variables and reducing Equation (1.15) to a set of ordinary differential equations. The latter have to at least satisfy the rigid boundary conditions [1,117]. Three basic approaches widely used in the industry are: Rayleigh-Ritz, Galerkin, and the FE methods. The first method performs the approximations from the variational statement of the equilibrium conditions and involves choosing of an appropriate sequence of shape functions that converge to a solution. The second method, on the other hand, requires minimization of the approximation error with the right sequence of admissible functions that converge to a solution. The third approach is more direct and general, in that it treats the system as an assemblage of discrete elements that have balanced displacements and internal forces at the nodal locations. Thus, a sequence of approximation functions are chosen that converge to a solution for each discrete element. Furthermore, the selected structural elements must be sufficiently simply in order to match the overall repertoire. Otherwise, submodels should be developed and boundary conditions carefully matched [1.98].

The capabilities and wide use of the above mentioned approaches notwithstanding, accurate modeling is still an art that is learned from experience and perfected by personal ingenuity. Moreover, even when accurate large dimensional models are developed, the issue of reducing the model down to a practically implementable order is of paramount importance in control design. There is a great deal of research interest in this particular area. Thus, several model order reduction techniques have been proposed by various authors, most of which deal with the problem as a mode selection process based on an appropriate error criterion [1.121,1.122].

LFSS control design and analysis entails development of a very large order finite dimensional model which is followed by a large order (50 or more) evaluation model. However, the evaluation model is usually reduced down further for control synthesis. Moreover, for practical on-board implementation purposes, a reduced-order (< 10 modes) controller model has to be generated with special consideration for spillover, i.e., the effect of sensor-actuator locations on the unmodeled or truncated modes of the structure [1.123-1.124]. Thus, Equation (1.15) is now a large-order, finite dimensional model and it can be represented by the following:

$$\begin{bmatrix} \dot{y}_C \\ \dot{y}_R \\ \dot{y}_S \end{bmatrix} = \begin{bmatrix} A_C & 0 & 0 \\ 0 & A_R & 0 \\ 0 & 0 & A_S \end{bmatrix} \begin{bmatrix} y_C \\ y_R \\ y_S \end{bmatrix} + \begin{bmatrix} B_C u \\ 0 \\ 0 \end{bmatrix} \tag{1.16}$$

where y_C is the controlled, y_R is the reduced, and Y_S is the suppressed state vector components (similarly for A_C, A_R, A_S, B_C). A part of the large dynamic model is considered absolutely insignificant, and hence, it is suppressed. Another part, which is considered for spillover effects and evaluation purposes represented by y_R, are eventually truncated and all that remains for the control design is y_C. Most of the approaches used in dynamics and control analysis is normally performed through model truncation, whereby modes that have fundamental natural frequencies above an a-priori chosen frequency are simply discarded. For LFSS, however, the frequency criterion for structural control is not sufficient in general and specialized selective removal of modes is more appropriate [1.125].

1.3.3 UNCERTAINTY MANAGEMENT IN MODELING AND CONTROL OF LARGE SPACE STRUCTURES

A very important consideration in controller design for LFSS aims at ensuring stability under modeling and parametric errors, and unmodeled or truncated modes. The simplest approach to the solution of the above mentioned robustness condition involves direct output feedback, which requires actuators and sensors to be collocated and placed appropriately. There are various other approaches which in the presence of uncertainties and realistic actuators and sensors, have their respective limitations. All these techniques have disadvantages that could most often lead to serious stability problems [1.126].

Uncertainty in modeling and control design of LFSS may arise either from randomness in the properties of the structure itself or from modeling approximations and process idealization [1.127]. Experimentation and testing is one way of reducing uncertainty in the analytical model, or in verifying and modifying it. However, LFSS are intended to be deployed in space under near-zero gravitational force and testing of such systems on earth is virtually impossible [1.128]. Moreover, the modal characteristics of LFSS are very dense and some of their eigenvalues are very low and nearly identical. To overcome the above mentioned difficulties, several authors have presented various nonconventional approaches to testing and data acquisition techniques, one of which is called multiple boundary condition test (MBCT) [1.129]. In this approach, a flexible beam is tested and analyzed with a variety of constraint conditions and constraint locations and the test results are used to modify parameters that are in error. All the recent developments in techniques of testing LFSS notwithstanding, some underlying requirements still remain and must be addressed. Namely, treatment of nonlinearities and randomness, design growth and complexity, coupling, and transformation of test results from scaled-down microstructures to derive characteristics of LFSS [1.98].

1.3.3.1 Uncertainties: Occurrence and Management

Modeling and control design of LFSS entail three basic types of randomness. Namely, uncertainties in the dynamic model, uncertainties in the control system, and random disturbances which have a diverse effect on the performance of the system. The first of the above mentioned stochastic phenomena is due to modeling nonlinear effects by approximate linear functions, model parameter errors, configuration growth and change, as well as internal and external disturbances. Uncertainties in the control system are due to errors in positioning and actuating of controllers, as well as to internal and external control dependent noise. The purely stochastic phenomena of the random disturbances are very hard to account for, since information relative to their statistical characteristics is often limited [1.127].

In situations of very high performance requirements for large and complex control systems under uncertainty, the problem of initial data for modeling purposes is a nontrivial and serious one [1.130]. Moreover, modeling of LFSS with appropriate consideration of all important uncertainties comprise a stochastic problem of high complexity. Two issues are of paramount importance. Firstly, the objective of the control system should be identified. Secondly, the initial data with consistent probability distributions, should be specified. Moreover, an appropriate measurement system should be selected based on the control performance and objective. Even with the best modeling and model reduction, however, it is conceivable that better and more robust control performance can be achieved when uncertainties are modeled through stochastic multiplicative and additive noise elements. For optimal control strategies derived under a wide range of parameter variations and random disturbances will result in robust control systems, under controllability and observability assumptions [1.131].

For uncertainty management purposes, consider Equation (1.15) in a finite dimensional setting. Under appropriate conditions of stability, symmetry, and positivity, Equation (1.15) can be transofrmed such that

$$A = \begin{bmatrix} [0] & I \\ -[\omega^2] & -[2\zeta\omega] \end{bmatrix} \text{ and } B = \begin{bmatrix} & 0 \\ [\Phi^T] & [M]^{-1}F \end{bmatrix}$$

where is the frequency, is the damping ratio and is the modal matrix of the system and $[\omega^2]$ and $[2\zeta\omega]$ are diagonal matrices with entries shown in the brackets. Formulation of the mathematical model in such a "modal" setting reduces the problem of uncertainties to three different sets. Namely, uncertainties in the frequencies, uncertainties in the damping ratios, and uncertainties in the mode shapes. Moreover, if the Vandermonde matrix [1.131-1.133] is utilized, then the uncertainties of the models can be deduced from uncertainties of the frequencies and damping ratios. A similar modeling approach with only uncertainties in the frequencies was presented by Hyland [1.110]. However, his approach to control design is based on minimum entropy, while Reference [1.96] treats the optimal control problem via dynamic programming.

There are other approaches to treatment of uncertainties in modal parameters of structural systems that were mentioned earlier. The basis for all these, however, is quantifying and reducing uncertainty via testing and experience [1.108]. Structural uncertainty is taken as the difference between prediction and measurement and statistical correlation analysis is utilized to generate the "true" values of the modal characteristics. In essence, the final model is still deterministic. However, the model suggested in Reference [1.96] and Reference [1.110] will take the best available model, incorporate statistical data into the uncertain elements and thus generate a stochastic model that is closer to the real structural system, on an average basis.

1.3.3.2 Control Design and Uncertainties

Development of control design under stochastic indeterminacy is a relatively new approach to the control problem dictated by high performance and stringent accuracy requirements of modern-day spacecraft. The controller under such circumstances influences the system dynamics in two ways: (1) through the dynamics of the control system and (2) through its dependence on the conditional distributions and their respective moments [1.134]. This "duality" is a very important consideration in any stochastic control problem [1.135]. Furthermore, the analysis of a control system and the synthesis of its corresponding regulating member can be regarded as stochastic problems. For, the fundamental disturbances and stochastic uncertainty of control systems often cannot be neutralized by simple regulation; their direct measurement is not, in general, possible. Indirect determination is possible, however. Thus, measuring inputs and outputs and analyzing their characteristics furnishes valuable information. The lack of complete information on the stochastic uncertainties leads to a-posteriori probability distributions reflects of their parameters. Although the latter do not provide exact values of the parameters, it is more accurate than a-priori distributions. A-poseteriori probability distributions reflect the real characteristics of the uncertainties. In general, the controllers whose regulation procedures encompass investigating and directing simultaneously, through their feedback processes, are referred to as "dual" controllers [1.135]. This duality is a significant feature of stochastic control systems.

The stochastic problem described above can be represented via several different approaches. The most common of these approaches is the Bayesian method, whereby a-priori distributions of all random variables are available. Another approach is the minimax method. There are several other non-Bayesian approaches as well (the interested reader can consult Reference [1.135]).

1.3.4 CHARACTERISTIC FEATURES OF FUTURE LFSS

The most significant features of LFSS are related to their vibrational modes. The vibrational modes of such structures are numerous, very densely populated, and have very low frequencies, often coinciding with the on-board controller bandwidth. Moreover, the difficulty associated with the uncertainty involved in predicting these characteristics renders its analysis and design a difficult problem [1.136].

1.3.4.1 Characteristics Different from Regular Spacecraft

The special features relative to control design of LFSS are many, a few important ones of which will be mentioned below. The first and foremost of these is that "theoretically", there exist an infinite number of elastic modes (in addition to the rigid body modes) that have low and uncertain natural damping. The controller bandwidth and usually a significant number of the system modes have an overlapping region. This latter feature is the underlining characteristic for the accurate formulation of the structural, vibrational, and attitude control problem for LFSS.

The interaction and coupling that exist between the flexible modes of LFSS and the controller of both the attitude and the shape control systems contribute to the complexity of such inherently complex problems [1.137]. Consequently, a well posed control problem for LFSS entails a precise formulation of the performance criteria, careful modeling of couplings, inclusion of disturbances and statistical data in the dynamic model, and selection of a practical and appropriate measurement system. Another important complication in generating reliable dynamic models for LFSS is the fact that not only the higher frequencies of the flexible modes are in error but the lowest ones, being very close to each other, are also very difficult to predict with high accuracy [1.138].

The enormous size and number of geometrical attachments as well as the new materials that are presently being designed and tested for use in LFSS, create a large scale problem of complexity. Moreover, the need for such structures to be transported by the space shuttle and deployed in space is even a more formidable source of uncertainty and complexity. Thus, consideration of all the above mentioned salient features of LFSS renders its modeling, analysis, and control design a formidable task.

1.3.4.2 Model Verification and Validation Problems and the Need for Advancement in Technology

There are numerous approaches to validation and verification of dynamic models of spacecraft, all of which deal with experimentation and/or on-board identification techniques [1.139]. However, it is widely believed that current techniques that are used to treat dynamic problems in LFSS are inadequate and that technological innovations and advancement are necessary in the areas of efficient modeling, nonlinear analysis, model verification and validation, as well as other areas [1.94,1.98,1.138].

Normally, ground tests are structured and performed to provide correlation with the analysis and thus lead to necessary modifications and refinement of the analytical model. Nevertheless, in-orbit tests are crucial in the case of LFSS because it could reveal unexpected nonlinearities, couplings, and other interactions, as well as provide some means of validating ground test results and verifying analytical models. In-orbit tests are, however, costly and harder to design and implement. Scaled testing is another approach to validation and verification. However, the technology involved in scaled testing is not thoroughly adequate for LFSS for the same reasons mentioned above. Furthermore, technology advancement is needed in adapting scaling experimentation techniques to LFSS in order to address all the different features and uncertainties involved [1.112].

System identification for LFSS in-orbit is a difficult task as yet unresolved. Because of modeling uncertainties and high performance requirements of LFSS control systems, it is necessary to design identification systems that start performing as soon as the structure is deployed. Since fine modifications of the control laws might be required before the LFSS goes into orbit. Moreover, identification routines will be required that can handle any system and parameter variations while in orbit. However, systems identification, and even parameter identification, are difficult to achieve in orbit because of insrumentation constraints, cost, and various other aspects. Thus, alternate approaches should be developed to tackle all the problems involved in LFSS modeling for control [1.97].

Modeling for control design of dynamic systems under high uncertainty entails several important issues that are absent where complete information and certainty prevail. Moreover, modeling for control design of such complex systems as the futuristic LFSS encompass application of diverse technological fields, including control, modeling, identification, testing, and many others. Thus, LFSS control design and analysis present complex problems that are of paramount importance for future space activities and mission requirements.

The main objective of the present article is to review the various issues relative to modeling for control of LFSS under high degrees of uncertainty, modeling/parameteric errors, and random effects. Various existing approaches to uncertainty in modeling of LFSS have been briefly introduced and two alternate recent developments in this regard were discussed. The need for further advancement in the technologies were highlighted. The advantages of stochastic modeling, in regard to the robustness of the controller, was briefly discussed. Moreover, the realistic representation of complex systems under uncertainties with stochastic models with multiplicative and additive noise was presented and reviewed.

The modeling issue of LFSS still remains a very important one and a unified approach is required that can generate a realistic model, that is close to the real system under consideration, in some predetermined sense. Moreover, uncertainties in LFSS modeling and control design being so high, it is essential to treat the problem in a stochastic setting for accuracy and robustness.

1.3.5 A NOVEL APPROACH TO STOCHASTIC MODELING AND CONTROL OF LFSS

System and parameter uncertainty and random effects render it necessary to address the problem of modeling and control of LFSS in a stochastic setting. Even with the best of the existing modeling and model validation and verification techniques, structural engineers and dynamicists are faced with the difficult task of predicting the dynamics of complex LFSS with the desirable degree of accuracy. Because no deterministic model will be able to stand the test of uncertainty and randomness due to the various factors mentioned in the previous pages. Moreover, it is widely established that linear stochastic models with multiplicative and additive noise will best represent complex control system dynamics with high performance requirements and large degree of uncertainty [1.140].

The approach that will be taken in the present research program is: (a) take the best available analytical model of a LFSS control and dynamic system; (b) transform this model into modal coordinates and evalute the uncertainties in the frequencies, the damping ratio, and the mode shape vector elements; (c) reduce the order of the model to a degree suitable for analysis and control implementation; and (d) incorporate the statistical variations of the uncertainties within the model thus creating a stochastic model that is closer to the real structure on a statistical basis. In a similar manner, develop a measurement system with all uncertainties included in the measurement model [1.132].

The next phase consists of developing the optimal control algorithm for such systems and analyzing the performance and stability conditions [1.14]. State estimation and filtering will also be addressed and appropriately trated. Furthermore, generation of statistical characteristics for the elements of the natural modes from those of the frequencies and the damping ratios will be studied and realistic simplifications for the practical application of the present approach will be hinted upon. The next phase of the program is the analytical formulation of the problem which entails detailed development of the mathematical model.

1.3.6 CONCLUSIONS

A multitude of approaches exist presently in modeling and control design of flexible spacecraft. The advent of the space program, however, has created challenging engineering problems involving design, deployment, and control of very large flexible space structures, with characteristics that encompass a large degree of uncertainty and randomness. Thus, many diverse technologies have to be integrated in order to develop viable space systems with adequate reliability, maintainability, and performance. Moreover, new modeling and control techniques (as well as techniques in other disciplines) will be required to answer the numerous questions that such systems have raised.

Linear multiplicative and additive noise stochastic models have the potential of being a viable approach that will address many of the issues treated in the present report. Having a model that is closer to the real structure on a statistical basis is better than having a deterministic model that can become unstable due to random effects and errors that are not included in the system model.

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CHAPTER 2.

ANALYTICAL FORMULATION OF THE CONTROL PROBLEM FOR LARGE SPACE STRUCTURES

2.1 INTRODUCTION

The mathematical formulation of a dynamical system whether it be physical, biological, economic, or space, entails the approximate analytical representation of input-output phenomena relative to its internal characteristics. Recent developments in system theory present unprecedented capabilities vis-a-vis the solution of complex dynamic problems.

There are four important notions that integrate modeling and control design analysis and synthesis; namely, state, control, optimization, and realization. The state expresses the value of some internal characteristic of a system at a specific moment which determines the current value of the output function and which influences its future values. The state is the most important notion in systems theory in that it contains sufficient information about the past history of the system behavior to characterize the system. Furthermore, the information provided by the state at a given time t_1 together with the affect of a given input applied during a given interval of time (t_1,t_2), $t_2>t_1$ is necessary and sufficient to determine the state at time t_2[2.1]. The control is the force, or input that regulates, directs, commands, or just influences and changes the state of a system. Optimization, on the other hand, is the mathematical process whereby the best or the most favorable outcome or performance is sought for based on some a-priori constraints and performance criteria.

Realization theory is a relatively new development. Solution of the abstract realization problem involves the mathematical construction of the internal structure of a dynamic system via input-output relations. For a complete survey of realization theory see Krylov [2.2].

Dynamical systems are, in general, nonlinear and stochastic in nature. Nonlinear in that their outputs are not linearly related to their inputs and stochastic in that their initial conditions and/or forcing functions are not well known and random disturbances and noise contaminate the system outputs and degrade performance. Real systems should be modeled by stochastic differential equations as a generalized representation of the system characteristics and, in the limiting situation, when randomness can be neglected the deterministic form is derived [2.3]. Moreover, performance and optimality requirements dictate the need for and the degree of detailed and precise formulation of a control system. The laws of physics and science in general provide an adequate tool for approximate modeling of real systems. However, under certain conditions, when parameters cannot be determined accurately or when random phenomena create an unpredictable environment and cause performance degradation, linear stochastic (and even nonlinear stochastic) models might be necessary to satisfy the requirements [2.4].

When it pertains to system analysis and control design of large flexible space structures (LFSS), numerous difficulties must be overcome in order to have a viable and realistic control system that satisfies stringent performance requirements. The control system model may be a simple linear deterministic model, if accuracy requirements permit leniency, or it may be a complicated nonlinear, stochastic model, represented by a set of ordinary differential and partial differential equations, when accuracy requirements are high [2.5]. However, a balance between experimental testing and analysis renders the generation of adequate analytical models for control and analysis of LFSS possible. Direct methods of testing and experimentation under almost zero gravitational environment is still in the elementary stages and approaches that can be utilized for model validation and verification as well as model test procedures do not seem to exist presently.

2.2 VARIOUS WAYS OF GENERATING FINITE DIMENSIONAL MODELS

Every scientific or engineering problem encompasses a chain of actions that is directed towards the generation of an admissible (according to some a-priori criteria) solution. Some of the underlying elements of this problem solving methodology include: (1) problem formulation, (2) analytical modeling, (3) model reduction, (4) analysis and synthesis, (5) simulation, (6) testing (whenever possible), (6) modifications. The first three tasks comprise the most difficult and the most vital part of the problem solution. The solution or the outcome will be as good and as close to reality as the modeler has made it.

There are essentially three types of dynamic problems. Namely, (1) analysis, (2) synthesis, (3) control and instrumentation. The first of these seeks for outputs for a given system model and given inputs. The second utilizes input and output information to generate a good system model, while the third determines the realistic inputs for a known system model and known, desired outputs. The methodology for the formulation of analytical models for systems entails development of mathematical equations of three different types: (1) distributed parameter, (2) lumped parameter, (3) discrete. All of these types of models are based on fundamental assumptions such as (1) casuality (whereby it is assumed that the inputs and the outputs are related through the system), (2) separability (whereby a given system is assumed isolated from its surroundings), among others. Furthermore, the derivations of these models are carried out by utilizing certain laws and approximations and they are, in general, functions of some parameters. The various approaches to generating finite-dimensional system models will be presented in the following pages.

2.2.1 Modeling of Large Flexible Space Structures (LFSS)

Any analytical model is the idealization of reality based on assumptions, siplifications, approximations, and laws or relations. In realistic applications, engineers are concerned with modeling real dynamical systems with bonafide inputs and outputs.

It is widely accepted that LFSS are basically distributed parameter systems that can be described analytically by partial differential equations [2.6]. Moreover, the control system for LFSS should be designed so that both attitude and shape regulation can be handled effectively. Practical and implementation considerations create the need for reduction of infinite-dimensional partial differential equation models to ordinary differential equations via discretization. Furthermore, uncertainties due to modeling approximations and other random effects dictate the use of stochastic modeling of LFSS. In any event, a generalized model of a LFSS consists of coupled systems of partial differential and ordinary differential equations that express the elastically deformable and the rigid body dynamic characteristics of the system respectively. Subsequently, the main issue remains to be the reduction of the above-mentioned complex set of equations into a finite and implementable set, or at least into a set that is appropriate for analysis and synthesis.

Most real structures consist of complex combinations of geometrical and material characteristics. The development of a mathematical model of such systems for dynamic analysis or for control design entails idealization of inertia, damping, stiffness and other properties by discrete or continuous elements. Normally, the physical system is approximated by an assemblage of discrete elements, such as masses, springs, and dashpots, or continuous elements [2.7]. Application of laws of physics and mechanics to the system and its elements yields the above-mentioned set of equations relating the inputs (excitations, etc.) to the outputs (responses) as follows:

$$L(P,d,s,t)\ x(s,t) = F(s,t) \tag{2.1}$$

where $L(\cdot)$ is an appropriate mathematical operator, p is a vector of system parameters, d is a vector of design variables, s is the spacial parameter, and t is time, x(s,t) is the response vector, and F(s,t) is the input or excitation vector.

A large class of structural/mechanical systems can be described by the following partial differential equations:

$$\rho_i(s)\ \frac{\partial^2\ x_i(s,t)}{t^2} + C\ (s)\ \frac{\partial x_i(s,t)}{t} + \sum_{j=1}^{M} L_{ij}(s)\ x_j(s,t)$$

$$= F_i(s,t),\ i=1,\ 2,\ \ldots,\ m \tag{2.2}$$

where:

ρ_i are the inertia coefficients, c_i represents the damping coefficients, and L_{ij} are a set of self-adjoint, linear, partial differential operators, called structural operators. In addition to (2.2) the displacements $x_i(s,t)$, i=1, 2, ..., M must satisfy the initial conditions, $x_i(s,0) = x_{io}(s)$ and $\dot{x}_i(s,0) = x_{io}(s)$ and homogeneous or nonhomogeneous boundary conditions. In solid mechanics there could be rigid or forced boundary conditions involving constraints on displacements and moments (among others) or natural boundary conditions involving kinetic elements such as forces and moments or a combination thereof (see Nigam [2.7] for details). The idealized expressions for $L_{ij}(s)$ in (2.2) will be given by:

(1) $\frac{\partial^2}{\partial s^2}\left[EI(s)\ \frac{\partial^2}{\partial s^2}\right]$

for the transverse vibration of an Euler beam, where EI(s) is the flexural stiffness.

$$(2)\quad D\left[\frac{\partial^4}{\partial s_1^4} + 2\frac{\partial^4}{\partial s_1^2\,\partial s_2^2} + \frac{\partial^4}{\partial s_2^2}\right]$$

for the transverse vibration of a thin plate, when

$$D = \frac{Eh^3}{(2(1 - \nu^2)}$$ is the flexural rigidity.

The vibrational behavior of LFSS can be described as free and forced. The free vibrations occur at the natural frequencies of the structures and, in general, the motions consist of several simultaneous oscillations at the various natural frequencies of the system. However, under certain conditions, all the system coordinates undergo harmonic motion corresponding to one of the natural frequencies of the LFSS. The condition whereby such motion occurs at every part of the LFSS is called the principal mode of vibration or the normal-mode vibration. The number of such normal-mode vibrations correspond to the degrees-of-freedom (DOF) of the system. Thus, a LFSS represented by n-DOF will have n natural frequencies characterized by their normal modes of vibration. In general, any motion of LFSS can be represented by the superposition of normal-mode vibrations under the assumption of linearity.

2.2.2 Approximation Methods

As a rule, an analytical model of a control system should be sufficiently simple with the provision that the results based on the model lead to realistic conclusions regarding the behavior of the system at hand. By and large the most common models are linear due to their analytical simplicity. However, nonlinearities arise usually through materiel properties, in particular damping, large deformation, and nonlinear couplings between terms.

Determination of the exact response of a structure to excitation via the influence function or the normal mode methodology requires accurate expressions for the complete set of influence functions or normal modes, respectively. Practically it is virtually impossible to get exact expressions for the usual modes or the influence functions. Thus, approximations to the response characteristics are generated via the separation of variables principle by expanding the displacement function in Equation (2.2) as a finite summation of a complete sequence of admissible functions and associated generalized coordinates. The resulting approximations normally lead to a set of coupled, linear second-order differential equations.

2.2.2.1 Rayleigh-Ritz Method. The equilbrium conditions from the variational formulation of a structural system are utilized in the Rayleigh-Ritz approach to derive the finite dimensional approximation of the dynamics. The basic feature of this method is the choice of a sequence of basic functions that converge to a solution.

Thus, let $x_i(s,t)$, i=1, 2, ..., n, from Equation (2.2) denote the displacements of the self-adjoint system and

$$x_i(s,t) = \sum_{j=1}^{M} U_{ji}(s)\, q_j(t) \tag{2.3}$$

where $U_{ji}(s)$ are admissible functions. By expressing the displacements in terms of generalized coordinates $q_j(t)$, relations for the Kinetic (T) and potential (K) energies, as well as for the dissipation function (R) can be derived as follows:

$$T = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} \dot{q}_i \dot{q}_j \tag{2.4}$$

$$V = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \dot{q}_i \dot{q}_j \tag{2.5}$$

$$K = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} q_i q_j \tag{2.6}$$

where m_{ij}, c_{ij}, and k_{ij} represent the mass, the damping, and the stiffness characteristics of the structure at hand and have appropriate integral representations [2.7].

The Euler-Lagrange equations of motion of the system can be written as:

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial T}{\partial q_i} + \frac{\partial K}{\partial \dot{q}_i} + \frac{\partial V}{\partial q_i} = Q_i(t), \; i=1, \ldots, n \tag{2.7}$$

where $Q_i(t)$ are generalized forces that are functions of F in Equation (2.2). Substitution of Equations (2.4) - (2.6) into Equation (2.7) yields:

$$M \ddot{q}(t) + C \dot{q}(t) + K q(t) = Q(t) \tag{2.8}$$

which is the general finite dimensional expression of a structural system with M = the mass, C = damping, K = stiffness matrices, respectively, and Q(t) is the generalized forcing vector.

2.2.2.2 <u>Galerkin Method</u>. The similarity between the Galerkin and the Rayleigh-Ritz methods is that both are based on the choice of a sequence of admissible approximation function that converg to a solution under the separation of variables assumption. However, the Galerkin method chooses the generalized coordinate functions by minimizing and the L_2-norm of the error between L(p,d,s,t) of Equation (2.1) and Γ(s,t) where x(s,t) is approximated by the sequence of admissible functions.

The result of convergence is basesd on the assumption that the admissible functions also satisfy the natural boundary conditions. It is well known that under self-adjoint differential equations and boundary conditions, and when the functional in the variational problem is positive definite, then both the Raleigh-Ritz and Galerkin methods give identical results.

2.2.2.3 Finite-Element Method. A more direct and general approach to discretization of continuous systems than the Rayleigh-Ritz and the Galerkin methods is the Finite-Element method. In this approach the system is treated as an assemblage of discrete elements (normally triangles or rectangles). Each member of which is a bona-fide continous structure. The displacements and the internal forces of each of the above-mentioned discrete members are required to balance at the nodes in order to ensure unity and continuity of the assemblage of elements [2.8]. Then, within each piece trial functions are given in a simple form, like polynomials of (usually) at most fifth degree. Boundary conditions are then imposed locally, along the edges of the above-mentioned triangles or rectangles. The accuracy of the approximations may be increased by refining the subdivisions. The fundamental problem in the finite element procedure is to discover how closely piecewise polynimials can approximate an unknown solution. Mathematically, the main task is to identify the error as accurately as possible and to determine the rate of convergence relative to the number of pieces or the degree of the approximation polynomial within each piece.

The finite element approach can be summarized in the following manner: suppose the problem is posed in a variational setting of finding a function U that minimizes a certain expression of the potential energy. The minimization leads to an Euler equation for U that, normally, has no closed form solution. The Rayleigh-Ritz-Galerkin idea is then utilized to choose a finite number of shape functions Q_1, Q_2, ..., Q_n, and to determine the minimizing linear combination with the weight q_j, via a system of N discrete algebraic equations. The minimization process seaks out the combination that is closest to the solution U [2.9].

2.2.2.4 The Finite Difference Method. By dropping the variables and parameters in the parenthesis of Equation (2.1), we get L x = F. Replacing the derivatives with their respective difference quotients results in a finite linear system given by:

$$L^h x^h = F^h \tag{2.8}$$

Equation (2.8) is the discrete operation form of the original equation.

In finite difference approximation processes, the derivatives are normally replaced with their centered difference quotients. Thus

$$x'(s) \rightarrow \Delta^h x(s) = \frac{x(s+h/2) - x(s-h/2)}{h} \tag{2.9}$$

Performing all the necessary operations results in Equation (2.8).

The basic requirements for the analysis of Equation (2.8) are: (1) the computation of the local truncation or discretization error via a Taylor series expansion, (2) the determination of global stability. The latter is accomplished by showing the continuous dependence of x^h on F^h as the grid size h approaches zero. These steps establish the rate of convergence of x^h to x as h reduces to zero [2.9].

2.3 LINEAR CONTINUOUS STOCHASTIC MODEL

The equations of motion of a structure can be formulated in a number of different coordinate systems. Dynamic, static, or both forms of coupling could be present, depending on the choice of coordinates. The general behavior of the structure, however, is independent of the coordinate system.

In general, free vibrations take place at the natural frequencies of a structure. The motion under such conditions will consist of several superimposed simultaneous oscillations at various natural frequencies. It is, theoretically, possible to represent the vibrations of LFSS in a coordinate system that will eliminate dynamic and static couplings and will result in diagonalized equations of motion. Such a special kind of coordinate system is called principal coordinates. Moreover, it is possible, in theory, for a system to oscillate at a single frequency under appropriate initial conditions [2.10]. For the general case, however, the initial conditions cannot match a single mode and a linear combination of modes is required.

2.3.1 State Matrix

The analysis and design of LFSS entails construction of an idealized mathematical model that accounts for major interactions between the system and the environment, excitations, random inputs, constraints, and parameter, physical, and operational requirements. Thus, inertia, damping, and stiffness properties are, normally, idealized via discrete or continous elements. Application of the laws of mechanics to the system and its elements then yields a system of differential equations relating the inputs or excitations to the outputs or the responses as in Equation (2.1).

Normally, the objective of experimental analysis of structural systems is to obtain the frequency response function by dividing the fourier transform of the response by the fourier transform of the excitation at discrete frequencies. This process assumes linearity of the system. However, most structures have only limited ranges of excitation levels and frequency content in which the system is linear. It is important to underline the fact that the definition of mode shapes and frequencies apply only when excitation and response of a system remain in the linear regions and cannot be used to predict the system's characteristics or response outside that range. A few of the salient features of nonlinear systems include (1) the response of a harmoniclly excited nonlinear system at a given frequency can be at other frequencies, (2) superposition does not apply for nonlinear systems, and (3) the principle of homogeniety does not apply, i.e., the response of two simultaneously applied inputs is not equal to the sum of the responses of each input applied separately [2.11].

Notwithstanding the fact that many systems are nonlinear and inhomogenous in nature, the most common models are linear due to their analyticl simplicity, as well as the realistic results that are obtained when using the linear models to represent (a larger class of) structural problems. However, while it is common practice to consider deterministic linear models for most structures, various errors, such as those due to modeling, parameters, modeling uncertainities, and linearization of nonlinearities create the need for the stochastic formulation of the problem. Better performance can be anticipated when the above-mentioned errors and uncertainties are modeled appropriately via a good stochastic model that is close to the realistic structure, on the average. Moreover, some of the uncertainties will be state- and control-dependent, especially under very high performance requirements, and a linear model with multiplicative and additive noise will ensue.

Consider Equation (2.8) expressed in state-space form given by:

$$\dot{x} = A\,x + Bu \tag{2.10}$$

where:

$$A = \begin{bmatrix} [0] & I \\ -[\omega^2] & -[2\zeta\omega] \end{bmatrix}, \quad B = \begin{bmatrix} [0] \\ [U^T]Q \end{bmatrix}$$

$[\omega^2]$ is a diagonal matrix with the frequencies ω^2 as entries, and $[2\zeta\omega]$ is a diagonal matrix with entries as twice the damping ratios ζ times the frequencies ω. Also the matrix [U] is the modal matrix, discussed earlier, composed of the mode shape vectors.

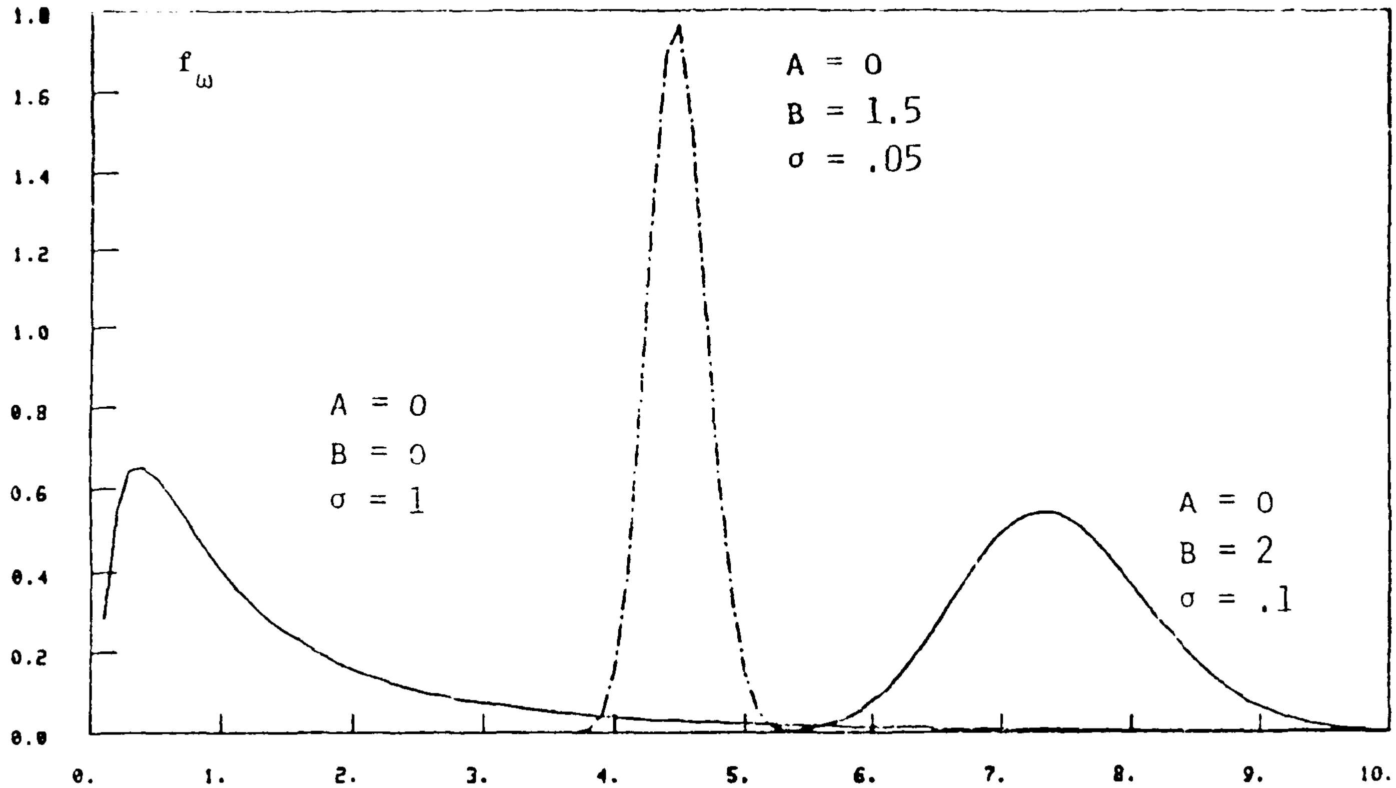
We note here that, under specific assumptions, $U^T[M]U = I$, $U^T[C]U = [2\zeta\omega]$ and $U^T[K]U = [\omega^2]$, where [M] [C] and [K] are as in Equation (2.8) and that setting $x = [q\ \dot{q}]^T$ yields Equation (2.10).

The matrix A is called the state matrix and is a function of the frequencies and the damping ratios of the structure under consideration. The uncertainties of A are mainly due to the dynamic modeling and random effects that can have a diverse effect on the performance of the system. Thus, it is very reasonable to look for data regarding analytical versus measured values of both frequencies and damping ratios of flexible structures. For it is intuitively apparent that if there is "sufficient" statistical data of uncertainties of the above-mentioned parameters, then it is possible to develop a realistic model of real LFSS. Hence, one way of generating a stochastic state matrix for a LFSS is to incorporate all statistical data on uncertainties of the above-mentioned elements within the "best" available state matrix of the system.

In the control design and analysis of LFSS it is of particular interest, and physically meaningful, to have the fundamental frequencies and fundamental modes at the engineers disposal. It is natural then to classify structural uncertainties according to the type of analysis to be performed and the specific kind of structures that are involved. It is very important, for instance, to separate analyses that are performed for structural integrity from that done for jitter or vibration or shape control. Similar characteristics probably could separate structures of different categories.

From the analysis of very limited data of LFSS, specifically satellites with flexible appendages, it was concluded that perhaps a "good" representation of the probability density function for the frequencies of LFSS is that of log normal distribution shown in Figure 2.1. In a similar manner the probability distribution that seems to fit the damping ratio data available from real measurements and analytical simulations is the Beta distribution given in Figure 2.2.

It is widely accepted that by analytical simulations of flexible structures it is possible to determine the lower frequencies with a high accuracy via large finite element models. However, in general, finite element models are too stiff ([illegible] large). Furthermore, the mass of space vehicles increase in the course of the design process. Also, modeling of stiffness characteristic is very difficult. Experimentation, whenever possible, will improve knowledge. When it is not possible to test a system, online identification is necessary for adjustment of parameters unless the stochastic formulation of the problem proves more practical. Also, in analysis of



LOG NORMA PROBABILITY DENSITY FUNCTION

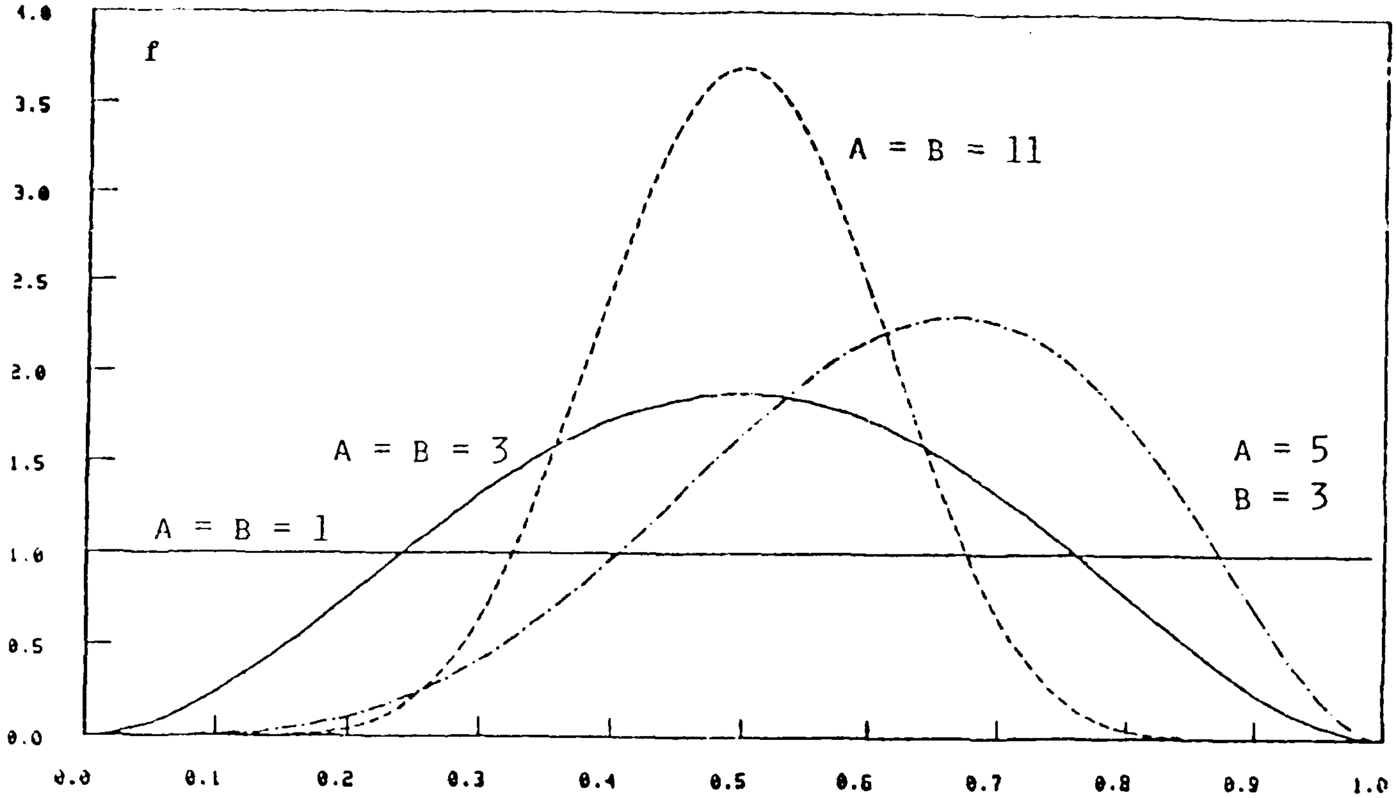
$$f_{\omega}(\omega) = \text{Exp}\ [-\left(\text{Ln}(\omega - A - B)^2/2\sigma^2\right]$$

$$\omega > A,\quad -\infty < B < \infty \qquad \sigma > 0$$

$$E[\omega] = A + \text{Exp}(B + \sigma^2/2)$$

$$E\left[(\omega - E[\omega])^2\right] = 2^{2B+\sigma^2}(\text{Exp}[\sigma^2] - 1)$$

Figure 2.1. Log Normal Probability Density Function



BETA PROBABILITY DENSITY FUNCTION

$$f_{\zeta}(\zeta) = C(A,B) \quad \zeta^{A-1} \; (1-\zeta)^{B-1}$$

$$0<\zeta<1, \; A>0, \; B>0$$

$$C(A,B) = \frac{(A+B-1)!}{(A-1)!(B-1)!}$$

$$E[\zeta] = \frac{A}{A+B}$$

$$E\left[(\zeta - E[\zeta])^2\right] = \frac{AB}{(A+B)^2(A+B+1)}$$

Figure 2.2. Beta Probability Density Function

LFSS, damping is usually known to be within a 100 percent error range. Even by experimentation, it is very hard to find exact values of damping. For there does not seem to exist good analytical models of damping that can represent all kinds of flexible structural damping. All the above-mentioned difficulties and facts contribute to the complexity of the modeling of LFSS and thus create the need for the stochastic formulation of the problem.

2.3.2 The Control Matrix

Design of LFSS is a challenging process both for structural and for control engineers because each must, in some sense, obtain an optimal design. Both the academia and the industry are aware of the high complexity and the interdisciplinary character of LFSS design and control and steps are being taken to remedy the lack of coordination and cooperation between the control engineer, design engineer, and the dynamicist [2.12]. There are various Government-supported programs that reflect the above-mentioned awareness and that are intended to produce more cost-effective, practical, and "totally optimal" designs of LFSS [2.13].

The development of a control matrix is highly dependent on the design constraints, physical constraints, and instrumentation constraints (among others) and it is conceivable that control/structure integration will help in developing stable and more robust control systems. The above-mentioned constraints and other difficulties raise the problem of uncertainty in the control system. This uncertainty is a direct function of performance requirements, random effects that are related to instrumentation (actuator/sensor) location, and to various other influences that have to be dealt with appropriately for a stable control system that works.

Furthermore, the control matrix B in Equation (2.10) is related to the modal matrix U, the mass matrix [M], as well as the actuator locations via Q. The various uncertainties of its elements are then functions of the modal paramerers. The mass elements, misallignment and tolerances of the actuators, and random effects related to accuracy of the instruments, inter alia. Suppose $\{\lambda_i, i=1, \ldots, n\}$ represents the set of eigenvalues of the system in Equation (2.8). Then the Vandermonde matrix given by:

$$U = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & & n \\ \lambda_1^2 & \lambda_2^2 & & \lambda_n^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & & \lambda_n^{n-1} \end{bmatrix}$$

will satisfy the characteristics of a "modal" matrix. For, in Equation (2.10)

$$U^{-1} A U = \Lambda \qquad (2.11)$$

where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$

Moreover, if a particular solution of Equation (2.10) is

$$x_p(t) = T(t)q(t)$$

where T(t) is the state transition matrix, in the autonomous case it is given by T(t) = exp(At) and q(t) is a vector valued function of time then it can be shown that

$$x(t) = T(t)\ x(0) + \int_0^t T(t-\tau)\ Bu(\tau)d\tau \qquad (2.12)$$

Also, we should note that $T(t) = U \exp(\Delta t) U^{-1}$, (see [2.14] for details).

The advantage of utilizing the modal matrix given by the Vandermonde matrix (it can be shown that any modal matrix can be transformed to the Vandermonde matrix via a nonsingular transformation) is that the uncertainties of the modal elements are related to the uncertainties of higher statistical measures of the frequencies. Theoretically, the statistics of the modal matrix can be generated by using the statistics of the frequencies. Hence, the overall problem of uncertainty for LFSS is then reduced to the uncertainties of the freuencies and the uncertainties of the damping ratios. It should be pointed out that the derivation of the probability density functions of each of the frequencies and each of the damping ratios, and then the generation of the statistics of the modal elements is by no means a trival one.

In case of repeated eigen values, the Vandermonde matrix should be modified to include the generalized eigenvectors. Moreover, if m of the n eigen values of A are distinct and one of them is repeated (n-m) times (or any other combinations thereof), say λ_i, the ith eigen value, then the corresponding generalized eigen vectors will be given by

$$e_i = (1, \lambda_i, \lambda_i^2 \ldots, \lambda_i^{n-1}),$$

$$e_2 = (0, 2, 2\lambda_i, 3\lambda_i^2, \ldots, (n-1)\ \lambda_i^{n-1}),$$

$$e_3 = (0, 0, 1, (\tfrac{3}{2})\ \lambda_i, \ldots, (n-1/2)\ \lambda_i^{n-3}), \ldots,$$

$$e_{n-m} = (0, 0, \ldots, 0, \binom{n-1}{n-m} \lambda_i^m)$$

where

$$\binom{n-1}{i} = (n-1)(n-2)\ \ldots\ (n-i)/(1 \times 2 \times 3 \times \ldots\ i)$$

See [2.15] for details.

2.3.3 Additive Noise Vector

Additive noise is incorporated in a linear system usually to compensate for environmental and other disturbances which act on the basic dynamics of the system (in parallel with the contoller) and influences its behavior. Normally, the additive disturbances are taken as white Gaussian noise with zero mean value and some constant covariance matrix. In general, however, the additive noise is modeled as the output of a linear system driven by white noise [2.17].

Thus, the linear model of Equation (2.10) with the additive noise vector will now take the form given by:

$$\dot{x}(t) = A\,x(t) + B\,u(t) + \xi(t) \tag{2.13}$$

where (t) is given by

$$\dot{\xi}(t) = A\,\xi(t) + \upsilon(t) \tag{2.14}$$

where $\upsilon(t)$ is a zero mean white Gaussian noise vector with given covariance.

It can further be assumed that the initial state is a stochastic variable and so is $\xi(t_o)$. Moreover, Equation (2.14) is augmented with Equation (2.13) to give the general dynamic equation with white Gaussian noise. However, we will assume (for convencience) at the outset that the additive noise is a Gaussian white noise vector with given statistics.

For completeness, we present a formal definition of white noise processes. Consider Equation (2.13) and suppose that (t) is a white noise process. Then its covariance

$$\mathrm{Cov}\,(\xi(t) = E[(\xi(t)] - E[\xi(t)])(\xi(\tau) - E[\xi(\tau)])^T] = R(t)\,\delta(t-\tau)$$

where R(t) is a positive semidefinite symmetric convariance matrix, $\delta(t-\tau)$ is the Dirac delta function, and E[·] is the statistical expectation operator, is delta-correlated. Moreover, if the correlation matrix for $\xi(t)$ is

$$R_x(\tau) = E[x(t+\tau)\ x^T(t)]$$

Then its power spectral density function is defined as the Fourier transform of its correlation by:

$$R_x(\omega) = \int_{-\infty}^{+\infty} e^{j\omega t}\ R_x(\tau)\ d\tau = \text{constant for white noise process.}$$

If in addition $\xi(t)$ is a zero mean Gaussian, random variable with covariance R, then its probability density function is given by

$$f(\xi) = \frac{1}{[(2\pi)^n\ \det R]^{1/2}} \exp\ [-1/2\ \xi^T R^{-1} \xi]$$

2.4 LINEAR DISCRETE-TIME STOCHASTIC MODEL

The basic property of dynamic systems is that their behavior at a given instant of time depends not only on the variables acting on them at the same instant, but also on the variables that have acted in the past. Often causality is the name describing such systems whereby the output at time t does not depend on the input applied after time t, but only on the input at and before time t. Discrete-time systems have the general form that makes the above-mentioned causality quite apparent. Moveover, the advent of digital computers has made discrete-time systems formulation even more attractive and practically advantageous.

2.4.1 The Discritization Process

The continuous time system presented in the previous section is defined for all t in $(-\infty,\infty)$ for the time-varying case or in $(0,\infty)$ for the time-invariant situation. The inputs and outputs of discrete-time systems, to be presented herein, are defined only at discrete instants of time. If we assume that in the continuous-time system of Equation (2.10) the responses are of interest and can be measured only at certain instants of time, then it can be represented by a discrete-time difference equation. Thus, consider Equation (2.12) representing the solution of Equation (2.10) and suppose the inputs u(t) are piecewise constant; that is, the input u changes values only at discrete instants of time. Inputs of this kind are dealt with in sampled data systems or in systems where digital computers are utilized to generate the inputs.

Let $u(t) = u(k)$ for $kT \leq t \leq (k+1)T$; k=0, 1, 2, ... where T is a positive constant, often referred to as the sampling period. The discrete-times 0, T, 2T, ..., are called sampling instants. If only the behavior of Equation (2.12) at the sampling instants is of interest, a discrete-time dynamical equation can be formulated giving the responses or the states $x(k) \triangleq x(kT)$ at k=0, 1, 2, ... Thus Equation (2.12) is now written as

$$\begin{aligned} x(k+1) &= e^{A(k+1)T} x_o + \int_o^{(k+1)T} e^{A((k+1)T-\tau)} Bu(\tau)d \\ &= e^{AT} [e^{AkT}x_o + \int_o^{kT} e^{A(kT-\tau)} Bu(\tau)d\tau \\ &\quad + \int_{kT}^{(k+1)T} e^{A(kT+T-\tau)} Bu(\tau)d\tau \end{aligned} \qquad (2.15)$$

But, the term within the brackets in Equation (2.15) is x(k) and $u(\tau)$ is assumed constant in the interval (kT,(k+1)T) and equals to u(k). Hence, a change of variables by $\tau = (k+1)T-\tau$ we have

$$x(k+1) = e^{AT} x(k) + (\int_0^T e^{A\tau}d\tau) Bu(k) \qquad (2.16)$$

Putting Equation (2.16) into a more general form, we get

$$x(k+1) = \phi x(k) + \psi u(k) \tag{2.17}$$

where:

$$\phi = e^{AT}$$

and

$$\psi = \left(\int_0^T e^{A\tau} d\tau\right) B$$

2.4.2 State Transition Matrix

To perform the discretization the following evaluations should be carried out:

$$\phi = \sum_{i=0}^{\infty} \frac{A^i t^i}{i!} = e^{At} \tag{2.18}$$

and

$$\psi = A^{-1} (e^{At} - I) B \tag{2.19}$$

where ϕ is called the state transition matrix and it satisfies the following properties:

$$\phi(-t) = e^{-At} = \phi^{-1}(t)$$

$$\phi(t+t_o) = \phi(t)\,\phi(t_o)$$

$$\phi(o) = I$$

A more general form of Equation (2.17) can be derived if we express the right hand side with respect to the initial state x(o). Thus

$$x(k) = \phi^k x(o) + \sum_{j=0}^{k-1} \phi^i Bu(k-j-1) \tag{2.20}$$

Φ^k is often referred to as the discrete-state transition matrix. There are various ways of computing ϕ and ψ via Equations (2.18) and (2.19) (see [2.16] for details).

For the system of Equation (2.8), where

$$A = \begin{bmatrix} [0] & I \\ -[\omega^2] & -[2\xi\omega] \end{bmatrix},$$

can be written as:

$$\phi = \exp\left(\begin{bmatrix} [0] & I \\ -[\omega^2] & -[2\xi\omega] \end{bmatrix}\right)t ,$$

$$\psi = \begin{bmatrix} -[\omega^2]^{-1}\,[2\xi\omega] & -[\omega^2]^{-1} \\ I & [0] \end{bmatrix} \left[\exp\left(\begin{bmatrix} [0] & I \\ -[\omega^2] & -[2\xi\omega] \end{bmatrix}\right) - I\right] B \qquad (2.21)$$

The numerical computations of ϕ and ψ could prove costly and tedious in certain cases. However, for symmetrical systems, like LFSS mass and stiffness matrices, the computational burden will be somewhat less. Moreover, approximate finite series expansions will often lead to good results, depending on the particular magnitudes of the elements of the matrices involved.

2.4.3 Additive Noise for Discrete Linear Stochastic Systems

Realization of realistic models for disturbances is a nontrivial task, to say the least. An underlying characteristic of random disturbances that occur in practical applications is that their future behavior is completely unpredictable. To develop a mathematical model for such a variable is very difficult. One reason is that an analytical function cannot be used. Since, by means of the analytical continuation property, it is possible to determine the values at every interval where the function is defined if its values are known at an infinitesimal interval. A statistical approach is feasible if one is careful enough [2.18]. Thus, modeling disturbances as a sequence of random variables is usually the right approach. In the continuous time case, the white noise process is considerably more involved, while for discrete time random processes, it reduces down to a sequence of independent, equally distributed, random variables.

Consider the discrete time linear model of Equation (2.17) and suppose additive white Gaussian noise is to be included in the model. Then, define the spectral density function by

$$\phi(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} R(k)\ e^{-ik\omega} \tag{2.22}$$

where

$$\begin{aligned} R(k,p) &= \text{covariance } [x(k),\ y(p)] \\ &= E[(x(k)-E[x(k)])(x(p) - E[x(p)])^T] \\ &= \int_{-\pi}^{\pi} e^{jn\pi}\ \phi(\omega)\, d\omega \end{aligned} \tag{2.23}$$

where $j = \sqrt{-1}$, $p = k = -\infty, \ldots, -1, 0, 1, \ldots$ and $E[\cdot]$ is the statistical expectation operator.

Now, for a weeakly stationary stochastic process, whereby the first and second moments of the distributions of $x(t_i)$ and $x(t_i+\tau)$, $i = 1, \ldots, k$ and $t, t_i \in T$, are the same, then the power spectrum of such a random variable will be constant [2.4].

There are two ways of generating the additive noise vector for the system in Equation (2.17). Namely, either by integrating step-by-step the Gaussian white noise $\xi(t)$ in Equation (2.13) or by deriving a comparable discrete noise vector with the characteristics to those of $\xi(t)$ at the sampling instants. In both situations suppose $\alpha(k)$ is the zero-mean white Gaussian noise vector of the additive noise for Equation (2.20). Then the general form of a linear distrete-time system will be:

$$x(k+1) = \phi x(k) + \psi u(k) + \alpha(k) \tag{2.24}$$

2.5 MODEL ORDER REDUCTION TECHNIQUES

2.5.1 Introduction and Perspective

One of the central issues in the active control of large flexible space structures (LFSS) is the derivation of a "correct" mathematical model for both the controlled and the uncontrolled dynamical systems. Theoretically, there are infinitely many elastic modes or degrees of freedom (DOF) in the distributed parameter (DP) models of LFSS, usually with very low natural damping. Moreover, the flexible modes contribute to the actual deformation of the structure. The abovementioned facts render the accurate modeling of LFSS a very complicated, nontrivial problem [2.19].

The truely infinite dimensional character of LFSS models has to be approximated by some "hign fidelity" finite (but usually very large) dimensional model. The normal approach taken by engineers to achieve this end is via modal models with a large number of modes that provide a reasonable representation of the spacecraft dynamic characteristics. However, a difficult problem still remains to be the development of a model of low enough dimensional order that it can be utilized by the onboard controller, yet high enough dimensional order that it preserves the dynamic characteristics of the real system represented and controlled [2.20]. The motivations for such a reduction are either to reduce computations for analysis and practical control design or to simplify the control system structure [2.21].

The dynamic analysis and design of structures by the finite element or other discretization procedure often lead to eigenvalue/ eigenvector problems of very large magnitudes; the solutions of which are very costly and plagued with numerous computational and numerical problems. Indeed, the structural dynamic matrix of the flexible modes, being a fully populated one, creates a drastically different situation from that of the static case [2.22]. Among the better known spacial discretization methods (other than the FE method) are the Rayleigh-Ritz/Galerkin and the finite difference methods. However, discretization procedures will not be discussed herein and it will be assumed that a large finite dimensional model has been generated some how. Furthermore, for implementation and other practical considerations, the model needs to be further reduced.

There are various approaches to model order reduction, some are optimal or pseudo-optimal, others are ad-hoc methods based on practical considerations and engineering experience and judgement. These techniques are known by different names such as "reduction", "condensation", "econmization", "aggregation", "optimal projection", and other combinations thereof. It has often been pointed out that such techniques generally constitute application of Rayleigh-Ritz/ Galerkin optimization, and matrix transformation methods to the eigenvalue/eigenvector formulation for structural dynamic problems [2.23].

The reduction or the condensation methods are based on transformations of the coordinates in the equations of motion that essentially, maintain the invariance of the quadratic forms of the potential and kinetic energies. An important feature of these methods is that the reduced order model often looses the basic characteristics of the original system. Considerable progress was made by Likins, Ohkami, and Wong [2.24] in this respect. However, they failed to develop a general enough criterion that could reduce the system model in an optimal sense without significantly affecting the eigenvalues of the original model.

The optimization or the mathematical reduction procedure, on the other hand, are based on the reduction of the eigenvalue problem to a smaller size based on some optimality criterion (usually quadratic) [2.25]. In these techniques the reductions are carried out without actually resorting to approximations, i.e., without truncating any coordinates or states of the original system.

Model order reduction for stochastic linear systems can be credited to the important works of Adamjan, Arov, and Krein [2.26 - 2.27] based on Hankel-norm procedures. The early work in this field could be credited to Desai and Pal [2.28], who formulated a "balanced reduction" approach for Markovian systems of stochastic processes [2.29]. However, this latter method to model reduction has yet to be extended to multivariable systems in order to be considered a viable method for structural dynamic problems.

"Balanced" model reduction of linear time-invariant dynamical systems is essentially based on the controllability and observability relations of the states of the system. Subsystem models are obtained by deleting those states that contribute the least to the controllability and observability (or the impulse response) of the original system [2.30] and thus cannot be expected to have any optimal character.

A more recent approach to model reduction is proposed in Skelton's work [2.31], where each state of the system model is assigned a "cost" relative to a given basis, via a quadratic criterion, and the states with the least cost are deleted in a systematic manner. The resulting reduced model of this method is a function of the state-space basis and thus there is no guarantee for optimality for all choices.

The latest development in model order reduction techniques is the work by Hyland and Bernstein [2.32]. Herein, first order necessary conditions for reduced order modeling of linear time-invariant systems are derived via a pair of modified Lyapunov equations coupled by a nonorthogonal projection. This aproach reveals the possibility of multiple exrema for some of the abovementioned methods.

In the following pages the various techniques in model order reduction will be presented and appropriately discussed. The advantages and disadvantages of each will be commented upon and further requirements and trends will be briefly touched. The development will include all the past and present model reduction procedures with their characteristic similarities and differences in a rather historical perspective. The motivation behind each of these techniques and their applications will be highlighted.

2.5.2 Problem Statement and Scope

The usual procedure for the development of a model for a distributed structure is by the use of the extended Hamilton principle, whereby expressions for the kinetic and the potential energies and for the virtual work of the system are derived. Subsequently, a mathematical model is developed via the variational or other approaches [2.33]. The most general finite dimensional model can be expressed as follows:

$$M\,\ddot{x}(t) + C\dot{x}(t) + K\,x(t) = F\,u(t) \tag{2.24}$$

where M, C, and K are termed the mass, the damping, and the stiffness matrices respectively and they could be functions of spacial variables. Appropriate transformations will change Equation (2.24) into the state-space representation given by

$$\dot{y} = Ay + Bu \tag{2.25}$$

where

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ M^{-1}F \end{bmatrix}$$

If the original model (2.24) has n DOF then Equation (2.25) will be a 2nth-order system. Thus A-(2n x 2n), B-(2n x 2m) given that u(t), the input or excitation vector is m-dimensional.

Real life systems that have motivated model order reduction range from chemical to electrical systems, from the analysis and control design of LFSS to turbines and power plants [2.34] There is a vast amount of literature regarding methods for simplifying or reducing models like that of Equation (2.25). However, for practical reasons and space limitations, we will concentrate only on those model reduction techniques that are relevant to flexible space structures and hence, to methods that are applicable to coupled multivariable systems. This constraint will reduce the scope of this endeavor, will exclude some of the procedures that are in existance, and will hopefully create uniformity and "completeness" in the pages that follow.

It is unavoidable, in any survey to leave out some references that should have been cited for their pertinent accounts, and the present one will not be an exception. Moreover, with the various limitations and the enormous amount of literature on model reduction scattered in numerous journals and conference proceedings, it is virtually impossible to be all inclusive. Nevertheless, we will try to be as thorough as possible and as concise as possible to do justice to the subject itself and to the researchers in model reduction techniques.

2.5.3 Condensation Methods

During the 1950's the classical approach to generating a reduced static model for a structural system was to use discrete masses associated with certain selected deflections [2.35]. In the early 60's, Guyan [2.36], Irons [2.35], and Davison [2.37] introduced ad-hoc methods of model reduction that is commonly known by various names, such as "Guyan reduction", "egenvalue economization", "mass condensation", or simply "reduction". These methods basically reduce the order of the original large dimensional linear model by discarding some of the modal deflections called "slave" DOF and retaining the remaining ones, called "master" DOF. The choice relative to the master and slave DOF is made, based on engineering judgement, in such a way that the lower frequencies in the eigen-spectrum of the structure are preserved as much as possible [2.38]. The resulting reduced order model has poles and zeros that are "closest" to the origin in the complex plane [2.37].

In the latter part of the 60's it was Aoki [2.39] who first introduced the idea of reducing dynamic models by "aggregation". This method reduces the dimension of a system model S_1, that is derived according to a given coordinate basis, to a lower dimensional model S_2 that uses fewer coordinates, via appropriate transformations. The aggregation technique can also be considered as a form of condensation. Since, once again, the criterion for the reduction is not based on any optimality concept, but rather on the preservation of the system characteristics in a simplified model.

Notwithstanding the fact that the component mode synthesis method, developed by Hurty [2.40], is also considered by many as a means of model simplification, it will be excluded from this study. Because it essentially divides a system into subsystems to be analyzed separately for simplification of analysis. Moreover, the overall system model generated by the component mode synthesis approach will still have to be simplified or reduced.

There are several other ad-hoc methods that can be considered condensation techniques. However, they are all similar to each other in certain respects. The main ideas of each "key" condensation technique will be presented in this section and their advantages and disadvantages will be pointed out. Modifications and improvements to these methods will be cited and commented upon.

2.5.3.1 Guyan Reduction

In structural analysis it is often necessary to reduce the very large mass and stiffness matrices, generated by finite element or other discretization procedures, for purposes of computational ease and simplification of the analysis. The underlying problem can be expressed as a general algebraic eigenvalue problem.

$$K\ x = \lambda M\ x$$

where K, M are as before and are the eigenvalues of the system. In what is commonly called "Guyan Reduction" the basis for the above order reduction is the elimination of coordinates at which there are no forces applied [2.36].

Thus the above mentioned eigenvalue problem can be expressed as follows:

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{11} & K_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.26}$$

where F_1, F_2 are the components of forces and F_2 is assumed to be zero. x_1, x_2 are the corresponding component coordinates, M_{ii} and K_{ii}, i=1,2, are the mass and stiffness components. This amounts to a coordinate transformation of the forces: $x = Tx_1$. After more manipulations it is found that the stiffness matrix corresponding to the forces F_1 is given by:

$$K_1 = K_{11} - K_{12}\ K_{22}^{-1}\ K_{21} \tag{2.27}$$

and the mass matrix corresponding to F_1 is:

$$M_1 = M_{11} - M_{12} K_{22}^{-1} K_{21} - K_{12} K_{22}^{-1} M_{21}$$

$$+ K_{12} K_{22}^{-1} M_{22} K_{22}^{-1} K_{21} \qquad (2.28)$$

As is mentioned in Reference [18], this transformation creates some discrepancy in the eigenvalue-eigenvector problem. For, from Equation (2.26)

$$x_2 = [I - \lambda K_{22}^{-1} M_{22}]^{-1} K_{22}^{-1} [\lambda M_{21} - K_{21}] x_1$$

and in expanded form

$$x_2 = [I + \lambda K_{22}^{-1} M_{22} + \frac{\lambda^2}{2!} (K_{22}^{-1} M_{22})^2$$

$$+ \ldots] K_{22}^{-1} (\lambda M_{21} - K_{21}) x_1 \qquad (2.29)$$

Thus, in the reduced matrices K_1 and M_1, the higher order terms are eliminated resulting in changes in eigenvalues and eigenvectors of the original system. Error bounds for the determination of eigenvalues for such reduction procedures are given in [2.41 - 2.42]. Furthermore, it has been established that eigenvalues are always increased due to the reduction of a model order [2.43].

Alternate approaches to model truncation or reduction are proposed by several authors that are variations of the Guyan reduction. Irons [2.35] presents an essentially similar approach working out in a rather step-by-step manner of eliminating "slave" modal deflections and keeping "master" nodal deflections. Davison [2.37] discusses his approach whereby only dominant time constants of a system transfer matrix are retained. Fried [2.44] related the reduction process to the power method where higher modes are suppressed. Kidder [2.45] and Geradin [2.41] derived the necessary approxiamtions for a more "accurate" reduced order system model. Flax [2.46], on the other hand, pointed out that the series in Equation (2.29) does not converge except for those eigenvalues of the retained terms that are less than those of the truncated terms. Wilson [2.47] showed the relationship between condensation and Gaussian elimination. While, Henshell and Ong [2.38] introduced an "automatic" reduction method that finds the master DOF, whereby all the natural frequencies in the lower portion of the eigen-spectrum of the original system are preserved. Various authors [2.48 - 2.49] have presented modifications that are meant to obtain more accurate reduced models.

The issue of truncation of modal coordinates has attracted the attention of many researchers, in particular, the various truncation criteria involved in model simplification and their interrelationships [2.50]. The two main criteria relative to eigenvalues are (1) those normal mode coordinates that have substancially larger frequencies than that of the highest significant harmonic in the forcing function can be truncated, (2) appendage modal coordiantes in hybrid coordinate systems can be reduced if the eigenvalues of the original and the reduced system approximate "adequately". Similarly, criteria related to eigenvectors and other controllability and observability issues are formulated in Reference [2.50].

Another approach to dynamic condensation is presented in Leung [2.51] and in Anderson, Irons, and Zienkiewiez [2.52] that is rather similar to the basic approach taken in Guyan [2.36] with some modifications. Downs [2.53] attempts to address several issues of accuracy and detection of master DOF. Namely, errors in the Guyan reduction process, bounds on the frequencies that can accurately be described by a reduced system, and a systematic procedure for selecting the master DOE in an improved manner. There are numerous criticisms directed towara the Guyan reduction or condensation procedures. The underlying problem is the influence of the reduction on the eigenvalues and the eigenvectors. The procedure presented in [2.52] reduces the model via elimination of those displacements that contribute the least amount to the kinetic energy of the system. However, the choice of the retained DOF is still guided by engineering skill and good judgement. Moreover, the convergence of the approximations in Equation (2.29) are functions of the choice of elements retained and the process of back transformation could lead to erroneous results [2.46]. Also, there seems to be some confusion in the number of retained DOF required for an accurate lower mode of vibration. Even if the number of retained DOF is fixed, an optimal procedure for selection of master DOF does not exist as yet. Downs [2.53] suggests that displacements, slopes, and twists should be the order of selection of the retained DOF. However, complex structures have to be treated under consideration of resonance and other aspects of stability.

Anderson and Hallauer [2.54] addressed the accuracy problem in their method of model reduction by representing the eigenvalue problem (2.26) in such a way that the eigensolutions (the modal matrix) of the original system is required. The inverse of the modal matrix with the diagonal eigenvalue matrix are utilized to generate a reduced order modal matrix that preserves the modes of the original system. However, the computational effort required seems to be much higher in this approach since an algebraic Riccati iteration has to be carried out. To address the proper selection of the retained modes Shah and Raymund [2.55] proposed a step-by-step algorithm based on bandwidth considerations while Thomas [2.56] examines the issue of errors in algebraic calculations of frequencies during the condensation process.

In many situations in structural analysis, the main reason for the derivation of the "exact" eigenvalues and eigenvectors is their use for the reduction of the order of the system model. Wilson, Yuan, and Dickens [2.57], and Arnold et.al. [2.58] present an approach whereby an "accurate dynamic analysis" is obtained at a reduced computational expense. Their method uses orthogonal ritz vectors, which are not the eigenvectors of the system, and thus derives a transformation that reduces the model. Paz [2.59] on the other hand, presents his "dynamic condensation" approach that starts by assigning to the eigenvalues an approximate or zero value for the first eigenvalue. It then proceeds by elementary operations to eliminate the undesired displacements. The resulting reduced eigenproblem is solved to derive a "virtually" exact eigenalue and the corresponding eigenvector of the first mode, as well as an approxiamte value for the second eigenvalue. Continuing in this fashion, one obtains a reduced order model with eigenvalues of the retained nodes very close to those of the original system. Sotiropoulos [2.60] points out methods that could lead to estimates of the error between the eigenvalues of the original and the reduced system. Moreover, he presents his own approach to dynamic condensation and calculations of the errors in the natural frequencies of structural systems in [2.62 - 2.66]. Other authors [2.63 - 2.66] have also addressed various aspects of dynamic condensation. One of the most important features of dynamic condensation methods is the fact that it does not require the series expansion inherent in static condensation or Guyan reduction procedures. Moreover, it remains to critically compare the dynamic condensation method with Guyan reduction and some of its better modifications in order to assess the value and the advantages of the former.

The problem of choosing a "good' reduced order system is still an art. A systematic procedure for generating the ideal reduced-order model for a given system remains to be developed and tested.

2.5.3.2 Aggregation Methods

Conceivably, valid analysis and design of large-scale systems can be carried out utilizing only aggregate models. There are situations when it is desirable and necessary to reduce the order of a given model like that of Equation (2.25) into the following form:

$$\dot{Z}(t) = F\,x\,(t) + Gu(t) \tag{2.30}$$

The state of this reduced model can be computed directly from the above equation and thus simulations can be performed with less computational effort. The aggregation method first introduced in control theory by Aoki [2.39], is such a technique. However, if $Z(t)$ only approximates the "real" states of the original model $x(t)$, then the validity of any analysis via (2.30) depends on the degree of accuracy of the approximations in the reduction process, which in turn is a function of the original system, the inputs, the choice of aggregation technique and the engineer's insight [2.21]. In the aggregation method, for dynamic exactness purposes, it is required that

$$Z(t) = C\,x(t) \tag{2.31}$$

(In the above equations, F, G, and C are constant matrices with appropriate dimensions). The above-mentioned exactness can be achieved if and only if the following matrix equations are satisfied:

$$FC = CA \tag{2.32}$$

$$G = CB \tag{2.33}$$

Equations (2.32) and 2.33) indicate that dynamic exactness exists only if $Z(t)$ is a linear combination of given modes of $x(t)$. In such a situation the eigenvalues of F are identical with the eigenvalues of A that correspond to the retained modes of $x(t)$. Looking at the aggregation method in this prespctive, it is apparent that, like the Guyan reduction method, the dominant modes of the original system model are retained. This characteristic definitely restricts the class of matrices C, that can serve as an aggregation matrix. Moreover, aggregation in linear systems is a form of minimal realization, just like "merging" in telecommunications [2.67].

Various factors should be considered in the process of determining the "best" aggregate model. Thus, the expected disturbances , the inputs, the structure of the feedback loops in the case of control systems, the bandwidth of the controller among others, should all be evaluated relative to system stability in the light of the stability of the reduced model. In linear quadratic control problems researchers have derived bounds on the performance index via the solution of the Riccati equation [2.68 - 2.70].

A different approach to aggregation is proposed by Tse, Medanic, and Perkins [2.71]. The new approach is called "generalized Hessenberg" method, whereby a sequence of aggregate models are utilized, each being a low order approximation to the given model, to obtain the "best" reduced order model under conditions of observability of modes and of coupling between components. In [2.72] a refinement to aggregation is presented whereby conditions are established that are necessary for aggregation of large scale systems, under incomplete controllability and observability. An error criterion is obtained for a quadratic performance functional and the subject of optimality of the aggregated model is discussed.

For time-invariant linear systems like that in Equation (2.25), the determination of the aggregate model depends on the choice of the eigenvalues of A and the choices of the aggregate matrix C. Commault [2.73] discusses an optimal choice of modes for the aggregate model in a combinatorial approach for a restricted class of aggregated systems. While, Ying-Ping [2.74] discusses methods for checking of the matrix C and for selecting its elements.

2.5.3.3 Perturbation Methods

Perturbation methods are often viewed as approximate aggregation methods [2.21]. There are two basic approaches to the perturbation technique, nonsingular and singular. The system model in equation (2.25) is expanded as follows:

$$\begin{bmatrix} \dot{x}_1(t) \\ \epsilon_1 \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & \epsilon_2 A_{12} \\ {}_2A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

where $\epsilon_1 = 1$ for conditions of weak coupling and ϵ_2 is a small positive parameter, while ϵ_1 is small and $\epsilon_2 = 1$ for strong coupling. In the first case two separate approximations are obtained, while in the second case, A_{22} is a stable matrix and the system order is reduced via zeroth order approximation.

In the first case of weak coupling, the Riccati equation of the corresponding optimal control problem can be expanded in terms of ϵ_2 and serves as a means of reducing the system model [2.75]. One disadvantage of this approach is that, in obtaining Zeroth or higher order terms there is no recursive procedure, and hence it is difficult to implement. Other drawbacks are discussed in Reference [2.21].

In the situation of strong coupling the eigenvalues of the first system (with x_1 as its state vector) is approximated by the eigenvalues of $(A_{11} - A_{12}A_{22}^{-1} A_{21})$, and the remaining eigenvalues are approximated (and their corresponding modes truncated) by those of A_{22}/ϵ_1 [2.76]. The need for systematic procedures in modeling and control design of large-scale systems such as LFSS is apparent in many applications. Moreover, the separation of modes into categories, such as fast and slow, important and noncritical, is a nontrivial task, very often, and it requires strong engineering judgement, considerable physical insight, and experience. An approach similar to the perturbation technique is presented in [2.71] whereby a reduced order model is obtained that preserves the input-output characteristics of the system behavior as shown by a set of measurements (assumed available). This method is especially useful in multiple decision processes. A collection of reduced-order models, each being a distinct low-order representation of the original model, are derived and a choice of the "best" reduced model with appropriate parameters is made under given criteria. This study uses a combination of aggregation and perturbation theories and is discussed in both subsections.

Perturbation methods for model reduction purposes are not very much applied in structural analysis and control design. However, it is presently recognized that singular perturbations exist in most control systems with reduced order models which disregard high frequency "parasites" [2.77]. Both continuous-time and discrete-time sysems can be treated by this approach. Several surveys and monographs present the various features and applications of singular perturbation methods [2.78 - 2.86]. Linear systems and time scale characteristics are treated in [2.87 - 2.93], state and output feedback are discussed in [2.94 - 2.96]. Optimization, dynamic programming, and other fields are also applied to perturbation theory [2.97 2.107].

2.5.3.4 Cost Decomposition Method

The requirement to determine the relative importance of modal vectors in a structural analysis and control design setting is apparent in various applications [2.108]. A systematic approach to establish the contribution of "cost" of each component mode of a large order system, based on a quadratic cost or performance index, and accordingly truncate those that have a minimal contribution is presented by Skelton [2.31]. This method can be considered a condensation method that addresses one of the important issues in model reduction. Namely, the "best" choice of the retained modes. Several choices of error criteria are presented on the basis of output, component, disturbance, and cost decoupling transformations.

The second stage of the structural analysis and control design of LFSS, or any large-scale system, according to this approach consists of the computation of the cost of each closed-loop eigenvector of an optimal system. Similar to establishing the most significant modes in the open-loop dynamic model of a LFSS, Skelton and Hughes [2.109] present a procedure for determining the most significant closed-loop modes. The resulting control system is endowed with the property of having the same eigenvalues and eigenvectors as the optimal system. The remaining modes will have shifted eigenvalues that can be predicted by modal controllability calculations. Furthermore, Yousuff and Skelton [2.110] give a controller reduction shceme that utilizes component cost analysis to reduce the order of the controller. Controller error measures are also defined and their mathematical expressions are derived with indications of their upperbounds and their relations to the observability characteristics of the states. The presentation includes the development of a cost decoupled control coordinate system similar in nature to the generalized Hessenberg representation in [2.71]. The resulting reduced states have the least control component cost, the least sensitivity to the control weighing matrix in the quadratic performance index, and the least observability in the controller.

The specific application of modal cost analysis to space structures is presented in References [2.111 - 2.113]. The reduced order model will have s modes of the n-dimensional original model (s < n) that has the largest contribution to the quadratic performance functional during closed-loop implementation. The approach is an iterative one. However, all the component modes have to be evaluated and compared for a valid reduction. There does not seem to exist an optimal procedure for this approach. Moreover, residual interactions of the truncated modes could have some impact on the performance of the control system with the reduced order model and controller. But, this is common to all reduction methods.

Several papers exist [2.114 - 2.117] that treat the model reduction technique by cost decomposition and address the issue of control system performance versus complexity in deriving a lower-order model. Moreover, model validation [2.118] is an important issue that recognizes the connection between model reduction and control design. The procedure with wich the reduced model is evaluated is by comparing the closed-loop suboptimal controller performance with that of the original system.

2.5.3.5 Balanced State-Space Representation

The model reduction problem relative to minimal realization theory was first introduced in control design by Moore [2.30]. His main idea was to be able to drive approximations to a given control system model that has a lower-order and that has virtually the same impulse response matrix. Thus, the states which are highly excited by the inputs to the system and which contribute most to the outputs are retained in this method. A normalization or balancing procedure is presented whereby coordinates are derived that are equally controllable and observable via transformations that equalize the controllability and the observability "Grammians".

The question of obtaining a "sufficiently accurate" reduced order model is an important issue in systems analysis and control design. The introduction of balancing in the coordinate system does not furnish an answer to the above-mentioned question. Moreover, the balanced coordinate procedure requires that the system be asympetically stable such that the observability and controllability Grammians be finite. The above mentioned Grammians can be defined as follows:

$$W_C(t_O,t) = \int_{t_O}^{t} e^{A(t-\tau)}\, B\, B^T e^{A^T(t-\tau)} d \qquad (2.35)$$

for the controllability Grammian, and

$$W_O(t_O,t) = \int_{t_O}^{t} e^{A^T(\tau-t_O)} C^T C\, e^{A(\tau-t_O)} d\tau$$

for the observability Grammian.

In the above equations, C is the matrix of the observation system of Equation (2.25) given by Z = Cy, and t_O is the initial time. These are defined for continuous deterministic systems. The procedure used by Moore [2.30] to generate a transformation T that simultaneously diagonalizes W_O and W_C and sets them equal is the singular value decomposition. Thus, after equalization the new coordinates (that are refered to as "balanced coordinates") are reduced by deleting the diagonal elements in $W_O = W_C$ with smallest magnitudes and thus keeping the dominant components. It is shown in [2.119] that the reduced order models thus obtained are stable only if the original model is stable.

Several other researchers have addressed various aspects of balanced coordinate model reduction techniques [2.120 - 2.122]. In Reference [2.121] it is shown that unstable systems could also be reduced and model reduction is extended to uniform realization by giving existance proofs of balanced realization for the class of analytic systems. Error bounds for reduced order models is obtained from internally balanced realizations in [2.123]. In [2.124 - 2.126] comparisons to other methods and stochatic systems are considered. All the various treatments based on balanced realizations involve systems-theoretic arguments rather than optimality criteria. Moreover, Kabamba [2.127] has pointed out that in the L^2 space the balancing method is not optimal, as can be expected.

2.5.3.6 Optimal Projection Methods

Optimal selection of reduced model coordinates was first treated by Wilson [2.128 -2.129]. In this approach the steady-state output error, that is quadratically weighted, is minimized when both the original system and the truncated model are effected by white-noise inputs. First order necessary conditions (similar to those in [2.39]) involving two Lyapunov equations, each of order equal to the sum of the orders of the original model and the reduced model, are obtained via parameter optimization. The basic procedure generates optimal reduced order matrices in an iterative manner, however, the guarantee of convergence of the iteration has not been established.

Another optimality criterion for model reduction is presented by Kabamba [2.25] where the original generalized coordinate vector is projected on a smaller subspace and the error norm for chosen scalar product is minimized. He refers to this method as "modal condensation". The essence of modal condensation is reducing an n-dimensional vector x to an m-dimensional one Z by a matrix transformation D and

$$Z = Dx \tag{2.37}$$

Where DD = D and D*A = AD for A = nxn, hermetian positive definite matrix, D self adjoint and idempotent. He has shown that truncation in this manner is a form of orthogonal projection.

In the case of structural systems, it is shown that the aim of truncation is the minimization of the kinetic and the potential energies of the residual modes, once the order of the reduced model is fixed. Moreover, truncation leads to an underestimation of potential and kinetic energies. Once again, the choice of the order of the reduced model is not addressed in this research article. However, the method lends itself to reduced computational effort and it can lead to indications of decoupling properties in a system.

Several papers appeared recently on optimal projection equations for model reduction [2.130 - 2.137]. The main result that appeared in [2.32] concerns the derivation of necessary conditions (in the form of two modified Lyapurov equations and rank conditions with non-negative definite solutions) for an optimal reduced-order model to exist. Multiple solutions to these optimal projection equations may exist corresponding to various local extrema. The balanced realization method is analyzed and compared with the optimal projection approach. Moreover an algorithm is presented that computes the local extrema by choosing eigenprojections. Also, component-cost decomposition is utilized to direct the algorithm to a global optimum.

Designing low-order controllers for flexible structures is taken up in [2.138]. The optimal projection equations for fixed-order dynamic compensation in the presence of state-, control-, and measurement-dependent noise are derived as a generalization of linear quadratic Gaussian theory.

The main idea behind the optimal projection approach is to directly characterize the optimal reduced-order controller with a quadratic performance functional for a given large-order model. Specifically, the optimal projection equations comprise a system of four matrix Riccati and Lyapunov equations coupled with an oblique projection that determines the optimal control gains.

The question of where the projection should take place is still unclear in this approach and engineering insight and practical experience apparently has it place. Computationally, solving Riccati and Lyapunov equations is no minor task. However, for low-order systems it is probably practical.

2.5.3.7 Other Model Reduction Techniques

Literature pertaining to model order reduction methods is enormous and scattered in various journals, conference proceedings, books, and reports. The survey article [2.139], by Genesio and Milanese, gives a long list of references related to model reduction techniques. Fromthe date of its publication,howver, there has been numerous papers written on various aspects of the subject. Most of the techniques that exist are ad-hoc methods. We will refer to some of them in this subsection and point out their significance as much as possible.

The "transfer matrix" method was developed in structural dynamic and was first used for model reduction by Dokainish [2.140]. In his approach, the stiffness and inertia matrices of individual elements are first derived (similar to the derivation of displacements in the finite element analysis). Then the matrices of only the retained part of the system are assembled and the natural frequencies are computed. The reduced modes are eliminated via algebraic transformation. Several papers [2.141 - 2.146] followed this presentation that either formulated modifications to the approach of Dokainish or analyzed the characteristics relative to its validity and applicability.

The polynomial approximation methods, on the other hand, that use low-order transfer functions to replace the original transfer function of the given system, calculate the coefficients of the reduced transfer function such that the output of the reduced model compares favorably with the output of the original system [2.147 - 2.152]. Methods that are often refered to as "moment mathcing", "continued fraction", "padè approximation", and "Error minimization" techniques are also ways of generating reduced-order models, specially for single input-single output situations [2.153 - 2.173] based on the same criteria.

Recent developments [2.26 - 2.27] related to Hankel norms has led to model reduction techniques of stochastic systems [2.29, 2.174 - 2.184]. Most of these approaches, however, deal with single-input single-output systems and utilize phase Hankel-norm techniques for reducing the order of a system.

2.6 CONCLUSIONS

Linear stochastic models that have state- and control-dependent noise in addition to additive noise comprise a realistic representation of the dynamics of LFSS. Uncertainties in various parameters can be represented as frequency, damping ratio, and modal paramter uncertainties, resulting in a mathematical formulation that requires the statistics of the frequency and the damping ratio uncertainties. The mechanism for referring the statistical measures of the modal parameters from the frequency and the damping ratio statistics is rather complex. However, considering the complexity of LFSS and utilizing appropriate model reduction techniques to derive a relatively small order model, the statistical characteristics might not be too difficult to generate.

The practical implications of such an approach should be treated separately on a realistic structure, and the advantages and disadvantages evaluated thereof. The intuitive and theoretical appeal of such an approach is definitely justifiable. Moreover, many realistic simplifications and assumptions could lead to practical results that could be valuable in LFSS analysis and control design.

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CHAPTER 3.

DETERMINATION OF UNCERTAINTIES AND PERFORMANCE INDEX

3.1 INTRODUCTION

There are many real situations where accurate mathematical modeling is a prerequisite for performance or response prediction, system analysis, and control. The type and the degree of accuracy of a model depends upon the application for which the above mentioned tasks are to be performed. For instance, in aerospace applications, where inherently high performance requirements exist, normally very precise mathematical models are needed. Models for industrial, economic, and other processes, on the other hand can be very approximate. Thus, the performance requirments and the overall objective dictate the need for accuracy of an analytical model [3.1, 3.2].

The mathematical model of a system can either be obtained via physical reasoning and application of fundamental conservation "laws" or by analyzing experimental data from system tests. In the former case modeling errors and approximations are unavoidable and in the latter situation the ability of the engineer to obtain an accurate representation of the system at hand is limited by the presence of random fluctuations such as unmeasured disturbances and measurement errors [3.3]. Furthermore, systems are in general dynamical and inherently nonlinear in nature. Thus, the response or the output is not linearly related to the input or excitations. The more serious aspect of the modeling problem is the stochasticity or randomness, that is always present and that should be incorporated within the model in a realistic manner. Stochastic differential equations could be utilized to represent such systems that have stochastic processes as coefficients of the differential operator, or the initial conditions could be random processes, and even the forcing functions or the input excitations could be stochastic processes and thus result in a stochastic system [3.4].

Deterministic system and control theories do not provide sufficient tools to perform analysis and control design of complex systems. The inherent reasons for the above assessments are varied. First of all, mathematical models are imperfect and only represent those characteristics of a system that are of direct interest to the engineer. For instance, complete vibration control of a large flexible space structure (LFSS) will require an infinite number of modes. However, "sufficient" control is accomplished most often by taking a small number of dominant or critical modes to form a finite dimensional model. This way of reducing the size of a real structure and the approximate manner of generating analytical models result in variations from reality that are random in nature. Second of all, unpredictable disturbances which do not have deterministic representations often influence the behavior of systems. Moreover, even if appropriate sensors are provided to gather test data for model perfection, the inherent shortcomings of the sensors cause random fluctuations in the supplied information, thus creating the need for stochastic representation of systems [3.5].

The level of uncertainty or the degree of randomness varies as a function of performance requirements, information at hand, and system complexity. However, since uncertainty is intrinsic to any problem, the probabilistic approach provides a useful framework to treat randomness quantitatively [3.6]. It essentially involves identification of sources of uncertainty, formulation of stochastic models, and incorporation of all the information and the lack of it in the stochastic model. The fundamental characteristic of the abovementioned approach is that each source of uncertainty is accounted for in a systematic manner and the relative sensitivity of the system to random variatons is assessed.

3.2 DEFINITION OF BASIC STATISTICAL APPROACH

The purpose of this section is to provide a concise account of the results from probability theory which will be utilized in the development of the ensuing work. The material covered will provide completeness, will motivate the usefulness of probability and statistics as a powerful mathematical tool, and will serve as a means of understanding the development and results in the research work that will be presented in the following pages.

Probability and statistics are of course related fields. In probability theory a model is assumed that can furnish quantitative information regarding the possibility of occurrence of various events. Whereas in statistics, a model is developed from some given observation data. Uses of probability and statistics are numerous [3.7]. Some examples include sampling, redundancy, confidence intervals, reliability, fracture mechanics, system identification, estimation, optimal control of stochastic systems, calibration, etc.

3.2.1 Probability Theory and Random Variables

Problem solving in stochastic systems entails random physical phenomena (experiments) with unpredictable outcomes. The totality of outcomes is often termed the sample space, and a subset of it an event, A. A probability measure P(·) is assigned to each event A such that $P(A) \geq 0$, $P(\Omega) = 1$, and $P(\cup A_i) = \Sigma P(A_i)$ for A_i independent for all i. The collection of all events must form a sigma field [3.8]. That is, A and its complement must both be bonafide events. Also, given a countable sequence of events $\{A_i\}$, then their union must also be an event, so must . The probability of two events is called a joint probability and the probabilitiy of event A occurring given that event B has already occurred is termed conditional probability. Thus $p(A/B) = p(AB)/p(B)$ if $p(B) \neq 0$. Moreover, if A_i, i = 1, n are independent, then the probability of their intersetion equals the product of the probability of all events. Bayes' rule is a very important result in probability theory that relates conditional probabilities, such that

$$p(A/B) = \frac{p(B/A)\ p(A)}{p(B)} \text{ if } p(B) \neq 0 \quad (3.1)$$

3.2.1.1 Random Variables

Normally the outcome of experiments are represented by quantities that are random variables. Thus, a random variable is a real-valued function $X(\cdot)$ defined on Ω for an outcome $\omega \epsilon \Omega$. The set given by $\{\omega : X(\omega) < x\}$ is a class of sets of Ω that is closed under all countable set operations. The probability of the latter set is called the distribution function and is denoted by $F_X(x)$. If $F_X(x)$ is absolutely continuous, then it can be expressed with respect to a probability density function $p_X(x)$ by:

$$F_X(x) = \int_{-\infty}^{x_1} \ldots \int_{-\infty}^{x_n} p_X(y)\ dy_i \ldots dy_n \quad (3.2)$$

where x_i and y_i are the ith components of x and y, respectively. The above definitions indicate that for some $\omega \epsilon \Omega$, $X(\omega) = x$. The distribution function completely describes the properties of a random variable. If there is a mass function $m_X(\cdot)$ such that

$$F_X(x) = \sum_{\substack{\xi \leq x \\ m_X(\xi) > 0}} m_X(\xi) \quad (3.3)$$

then, X is called a discrete random variable and it can assume only a countable number of values.

If $F_X(x)$ is differentiable then

$$p_X(x) = \frac{dF_X(x)}{dx} \quad (3.4)$$

the density function of a discrete random variable consists of a sum of delta functions [3.9].

The <u>mean value</u> or <u>expectation</u> (or first moment) of a continuous random variable X usually written as E[X], is the scalar defined by

$$E[X] = \int_{-\infty}^{\infty} x\ P_X(x)\ dx \tag{3.5}$$

The above value exists if $E[|X|] < \infty$. Moreover, a fixed deterministic real function $f(\cdot)$ of a random variable X is itself a random variable. Thus, $Y = f(X)$ and its expectation is given by:

$$E[Y] = \int f(x) p_X(x)) dx \tag{3.6}$$

The expectation operation is linear. The kth moment of a random variable is $m_k = E[X^k]$ and its kth <u>central moment</u> is $\mu_k = E[(X - E[X])^k]$. The joint moments of two random variables X and Y are given by the set of numbers $E[X^kY]$ and E[XY] is called the <u>correlation</u> of X and Y. Their joint central moment is

$$E[(X - E[X])\ (Y - E[Y])] \tag{3.7}$$

and is the <u>covariance</u> of X and Y. Furthermore, if E[XY] = E[X] E[Y] then X and Y are <u>uncorrelated</u>. Independent random variables are always uncorrelated. A random variable may also be characterized by its characteristic function given by:

$$\phi_X(s) = E[\exp(jsX)],\ j = \sqrt{-1} \tag{3.8}$$

Theoretically, one can obtain all the moments of the random variable X through differentiation of $_X$, if the latter is analytic. Thus,

$$m_k = j^k \frac{d^k}{ds^k} \phi_X(s) \ |_{s=0} \tag{3.9}$$

Given that an event A has occurred, the conditional moment of X is defined as

$$E[X|A] = \int_{-\infty}^{\infty} x \, p_{X|A}(x|A)dx \quad (3.10)$$

Thus, $E[X|Y]$ is a function of the random variable Y, moreover, $E[E[X|Y]] = E[X]$.

A very important theorem in probability theory is the central limit theorem. It states that given X_i, independent, then, under general conditions

$$Y_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

is approximately Gaussian with mean value $\frac{1}{n} \sum_i \mu_i$ and variance of $\frac{1}{n} \sum_i \sigma_i^2$. Gaussian distribution (or normal distribution) is that which has a probability density function given by:

$$p_X(x) = [(2\pi)^{\frac{n}{2}} |V|^{\frac{1}{2}}]^{-1} \exp[-\frac{1}{2}(x - E[X])^T V^{-1}(x - E[X])] \quad (3.11)$$

where $|V|$ = determinant of V, V is the covariance matrix of X, and $[\ ]^{-1}$ is the inversion operator. The symbol often used for Gaussian or normal distributions is

$$X \sim N(E[X], V)$$

3.2.2 Stochastic Processes

A random variable defines a mapping from the sample space into the real line, while a stochastic process is a mapping from the sample space into the function space of $x(t, \omega)$ for each in the sample space. Thus, for different observations $\omega \epsilon \Omega$, different realizations of the stochastic process are derived, that are functions of time. Hence, a stochastic process is a function of two parameters, the time parameter $t \epsilon T$, and the probabilty parameter ω.

In general, a vector stochastic process is represented by $\{x_t(\omega), t \epsilon T, \omega \epsilon \Omega\}$, or in short notation $\{x_t, t \epsilon T\}$, and is actually a family of random vectors indexed by the parameter set T. For a stochastic process an experiment results in a <u>function</u> mapping an underlying time set (all integers, reals, or subsets of these) into the reals.

A stochastic process $\{x_t, t \epsilon T\}$ is <u>strictly stationary</u> if for any real Δt, the joint probability density function is invariant at $t + \Delta t$. Thus

$$p(x_{t_1}, \ldots, x_{t_n}) = p(x_{t_1}+\Delta t, \ldots, x_{t_n}+\Delta t) \tag{3.12}$$

for all finite sets $\{\tau_i\} \epsilon T$ and $(t_i + \Delta t_i) \epsilon T$. A special case of (3.12) is:

$$p(x_t) = p(x_t + \Delta t) \text{ for all } \Delta t,$$

which means that $p(x_t) = p(x,t) = p(x)$ and that the mean value of the process is a constant. A real stochastic process $\{x_t, t \epsilon T\}$ is <u>wide sense stationary</u>, or <u>covariance stationary</u> if it is strictly stationary of the first order, has a finite second moment and $E[(x_t - E[x_t])(x_s - E[x_t])(x_s - E[x_s])]$ depends only on $(t - s)$ and not on either t or s [3.10]. A random sequence $\{x_n\}$ is said to converge to x with probability 1 (wpl) if

$$\lim_{n \to \infty} x_n(\omega) = x(\omega) \tag{3.13}$$

for all ω except for a set A such that $p(A) = 0$. The convergence is termed "in the mean square sense" if $E[|x_n|^2] < \infty$ for all n, $E[|x|^2] < \infty$ and $\lim_{n \to \infty} E[|x - x_n|^2] = 0$. Then

$$\text{l.i.m } x_n = x \qquad (3.14)$$

In general, convergence wpl neither implies nor is implied by mean square convergence or by l.i.m. A necessary and sufficient condition for mean square convergence is the cauchy criterion:

$$\lim_{n,\, m \to \infty} E[|\, x_n - x_m \,|^2] = 0 \qquad (3.15)$$

x_t is mean square continuous at $t \epsilon T$ iff $E[x_t x_\tau]$ exists and is continuous at (t, τ).

Processes for which only the first and the second moments of their joint distribution functions are finite are termed second-order stochastic processes. The above definitions extend to discrete stochastic processes with minor modifications. Thus, a discrete-time stochastic process is said to be an independent sequence if for any set $(t_1, \ldots, t_n) \epsilon T$, the corresponding random variables $X_{t_1}, \ldots, X_{t_n}$ are independent. Moreover, the joint distribution function F can be factored:

$$F\,(X_{t_1}, \ldots, X_{t_n}) = F_1(X_{t_1})\, F_2(X_{t_2}) \ldots F_N(X_{t_N}) \qquad (3.16)$$

where $F_i(X_i)$ are the marginal distribution functions of X_{t_i}, $i = 1, \ldots, N$. In addition, if $F_1, \ldots, F_N$ are identical functions, then the sequence is termed an independent and identically distributed sequence.

A discrete-time stochastic process is called white noise if its covariance matrix can be expressed in the form $\Sigma(t)\, \delta_{st}$ where δ_{st} is the Kronecker delta function and $\Sigma(t)$ is a non-negative definite matrix.

The spectral density matrix is the Fourier transform of the covariance function and in discrete form it is given by:

$$\phi(e^{j\omega}) = \sum_{-\infty}^{\infty} R(\tau)\, e^{-j\omega\tau} \tag{3.17}$$

where $R(t, \tau) = E[(X_t - E[X_t])\,(X_\tau - E[X_\tau])^T]$

and

$$R(\tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(e^{j\omega})\, e^{j\omega\tau}\, d\omega \tag{3.18}$$

where $R(\tau) = R(t, t - \tau)$ and $\phi(e^{j\omega})$, $\omega \epsilon [-\pi, \pi]$ is the spectral density matrix. An important result of the above is the power spectral density for a wide sense stationary white noise process with covariance of δ_{st} is

$$\phi_n(z) = \Sigma \tag{3.19}$$

where $z = e^{j\omega}$.

Markov processes: Heuristically, a Markov process is a stochastic process if, given the present information, the past does not influence the future, i.e., if $k_1 > k_2 > \ldots > k_n$, then

$$P_{X_{k_1} \mid X_{k_2}, \ldots, X_{k_n}} (X_{k_1} \mid X_{k_2}, \ldots, X_{k_n})$$

$$= P_{X_{k_1} \mid X_{k_2}} (X_{k_1} \mid X_{k_2}) \tag{3.20}$$

This type of a process is often called a first order Markov process. In a similar manner a second, third, and higher order Markov processes are defined.

Ergodic processes: In ergodic stochastic processes, time averages can be replaced by averages over the set of experiment outcomes or expectations. One definition of an ergodic process is: $\{X_k\}$ is ergodic if for any suitable function $f(\cdot)$,

$$E[f(\{X_k\})] = \lim_{k \to \infty} \frac{1}{2k+1} \sum_{-k}^{k} f(\{X_i\}) \qquad (3.21)$$

exists almost surely or wpl. In case $\{X_k\}$ is Gaussian with covariance sequence R_k,

$$\sum_{k=-\infty}^{\infty} |R_k| < \infty \text{ is sufficient for ergodicity.}$$

3.3 IDENTIFICATION OF SOURCES OF UNCERTAINTIES IN THE ANALYTICAL MODELS OF LFSS

The advent of LFSS has created the need for vibration and shape control of structural deformations within close tolerances. For instance, the satisfactory operation of a 100m antenna can impose constraints on surface distortions down to a few millimeters. Stringent operational requirements such as these further complicate the difficult modeling problem. Moreover, a particularly important factor emphasizing the necessity for advanced analysis methods is that the size of the abovementioned LFSS will make ground testing prohibitive [3.11]. The effective control of deformations within very small tolerances underlines the careful consideration of random loads such as pretensioning, orbital positioning thrusts, gravity gradients, atmospheric drag, and thermal forces, among others. Furthermore, the effect of uncertainties, that exist within the best analytical model of a LFSS, on the accurate response prediction or the control performance could be tremendous

The very high stabilization bandwidth requirements of precision pointing and shape control systems coupled with the higher degree of flexibility of the emerging LFSS, has driven the minimum frequency of the elastic modes of the structures well into the control bandwidth. Under such conditions, the control sytem design engineer has to recognize and compensate for multiple-elastic-mode behavior, as well as stochastic vibrations that could play a prominant role in the stability of the system.

3.3.1 Uncertainties in LFSS Contributing to Frequency and Modal Element Variations

There is little fundamental published research literature on uncertainties in frequencies and, particularly in modal elements of LFSS. This, notwithstanding the fact that researchers realize the large degree of uncertainty, specially in the higher frequencies and their corresponding modes. Modern computer-based methods, like the finite element modeling technique, render it possible to make highly precise calculations of the mass and stiffness properties of very complex structures. However, this requires large financial and manual resources. Moreover, the higher frequencies and their corresponding modes are still highly uncertain. Furthermore theoretical procedures (comparable to those used for frequencies and modes) relative to modeling of damping is almost non-existent. Generating numerical values for damping ratios in structures is based on "intelligent guesswork," as a rule.

Major efforts have been made in the field of risk analysis for various structures for the purpose of evaluating safety and stability margins. Especially important is the analytical work peformed for offshore platforms, which are being utilized for oil extraction from fields of deposits along the outer continental shelf [3.12]. One very important consideration in determining the degree of uncertainty in the analytical models of LFSS is its direct relationship to the accuracy requirements, which in turn, is a function of the performance constraints. The abovementioned uncertainties may be due to randomness in the structural characteristics or to approximations and idealizations or both. In any event, the modal characteristics may be treated in a probabilistic sense for performance or response prediction purposes. Furthermore, the randomness in complex structures may be attributed to many independent random variables. When their respective variables are small in magnitude relative to their mean values, perturbation theory is usually used for analysis [3.13 - 3.16]. The modal parameters may be treated as random variables which are linear combinations of those associated with specific structural properties. The validity of a linear model is dependent upon the magnitude of the deviation with respect to the mean of the random variables. In general, as the magnitudes of the deviations get large, non-linear effects start to cause problems and linearity is not valid any longer.

In structural analysis the solution of an eigen value problem is an essential step for the overall structural control. For instance, in calculating static linearly elastic buckling loads and undamped natural frequencies of vibration, the normal approach consists of solving the following:

$$Kq = \lambda Mq \qquad (3.22)$$

where K and M are n x n square symmetric matrices and their elements are in general functions of random variables. Hence, the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and the elements of the corresponding eigenvectors $q_1, q_2, \ldots, q_n$ are also random variables [14].

The nonlinear least-squares technique is used in estimating the parameters in a mathematical model of a structure by utilizing modal test data [3.17]. The first step consists of linearizing the equations from which analytical modal data are calculated and curve fitted to the test data. This results in revised estimates for the underlying mass and stiffness parameters. The latter revised estimates are then used to recalculate the modal data. This recursion continues until convergence occurs, or the correlation between analytical and test results is acceptable [3.18]. However, even when simple structures such as cantilevered beams are concerned, there are discrepancies between test results and analytical calculations of the frequencies and the modal parameters, specially for higher modes and frequencies. Furthermore, the case of LFSS is even more difficult to handle due to the mere fact that only minimal and partial testing is possible. Thus, comparisons of test data with analytical calcualtions is virtually impossible and essentially, the model verification and validation task has no practical meaning.

In the process of space discretization for a finite dimensional model of LFSS, the eigenvalue problem can be written with respect to the order of the system as follows:

$$K^{(n)} q_i^{(n)} = \lambda_i^{(n)} M^{(n)} q_i^{(n)} \quad i = 1, 2, \ldots, n \tag{3.23}$$

The superscript (n) indicates the n-degrees-of-freedom (DOF) model. Now, when a model of the same LFSS is considered with (n + 1) DOF, then

$$K^{(n+1)} q_i^{(n+1)} = \lambda_i^{(n+1)} M^{(n+1)} q_i^{(n+1)} \lambda_i \qquad i = 1, \ldots, n+1 \tag{3.24}$$

It can be shown [3.19] that the eigenvalues $\lambda_1^{(n)}, \lambda_2^{(n)}, \ldots, \lambda_n^{(n)}$ are only approximations to the lowest eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$ of the actual structure. Also, the computed eigenvalues approach the real eigenvalues of the LFSS from above as n tends to infinity. Thus the cruder the approximation the more the uncertainty in the values of the parameters. The higher the order n, the more accurate are the lower values of the parameters. However, in LFSS the lower eigenvalues are so closely spaced that even a slight degree of uncertainty could be very critical under high performance requirements.

3.3.2 Uncertainties in Damping Ratios of LFSS

The integration of control and structures technologies is presently a key issue in the analysis, synthesis, and instrumentation of future spacecraft. A very important factor in structural control is energy dissipation through damping. However, the latter parameter is very poorly understood and in practical applications it is modeled on the basis of "engineering" experience and "good" judgment. It has been established that damping is essential for a structural system to be stable and that the response near resonance peaks increases without bound in the absence of damping [3.20]. Moreover, various approaches exist that are directed towards augmenting natural damping by artificial means: passively, through the addition of special materials on the surface of a structure or actively, by means of feedback [3.21 - 3.23].

If one is to evaluate design stresses and responses, the uncertainties in damping coefficients would be exactly equal to the uncertainties in such stresses and responses. Thus, what is really missing is the basic conceptual understanding of damping and the numerical values of the relevant coefficients, for which only empirical rules exist. There are various models of damping ranging from the very simple viscous model to the more sophisticated hysteretic damping concept, and even to multiparameter approaches [3.24]. One underlying difficulty is the fact that if the damping properties of each individual component in LFSS are known apriori, this by no means implies that, by any discretization technique, the complete damping matrix for the structure may be constructed. Moreover, nonproportional damping, i.e., damping that does not exhibit proportional increase in magnitude as the frequencies increase, results in coupled dynamics and nondiagonal damping matrices [3.25].

The overall damping problem in structures can be summarized as the exchange between the kinetic and potential energies within the structure. During this interaction damping causes the removal of energy from the vibrating structure by radiation and dissipation. The radiated part is associated with the aerodynamic damping and the dissipated portion with structural damping. Most often, however, these two entities cannot be separated. The main part of damping that concerns LFSS is the structural damping, since these structures will be deployed in the absence of air, in space.

3.3.2.1 Viscous Damping Model

In the viscous damping model the damping force F_d is porportional to the velocity of motion, $\dot{y}$. Thus

$$F_d = -c\dot{y} \tag{3.25}$$

where c is the viscous damping factor. The energy dissipated during a single cycle of a harmonic motion with frequency ω is given by:

$$D = \pi c a^2 \omega \tag{3.26}$$

where a is the amplitude of motion.

The common approach is to express the damping factor as a fraction of the critical damping, which is the damping for which motion is just not harmonic any longer. The critical value is:

$$C_{cr} = 2m\omega_n \tag{3.27}$$

where m and ω_n are the mass and the natural frequency of the mass-spring system. The viscous damping ratio is then,

$$\beta = c/C_{cr} \tag{3.28}$$

One important advantage of the viscous damping model is its applicability in both transient and steady-state response analysis [3.24].

Moreover, when a structure is built from basic elements with known damping characteristics a proper damping matrix can be determined which leads to the general linear equation of motion as follows:

$$M\ddot{y} + C\dot{y} + Ky = bU(t) \qquad (3.29)$$

where y is the vector representing displacements at discrete points of the structure, M is the mass matrix, the damping matrix is C, whose elements are derived by various experimental or ad-hoc methods, and K, the stiffness matrix, is a function of material characteristics.

3.3.2.2 <u>Hysteretic Damping Model</u>

There are many situations in various structures when, unlike the case of viscous damping, the loss factor, i.e., the ratio between the frequency of vibration and the natural frequency multiplied by twice the viscous damping ratio β, is independent of the frequency, at least in some frequency ranges. Hysteretic damping is meant to include such situations. In the hysteretic damping model the damping force F_d is proportional to the displacement y and is 90 degrees out of phase with it.

$$F_d = -jhy \qquad (3.30)$$

where $j = \sqrt{-1}$ and h is the hysteretic damping factor. The maximum potential energy stored is $V = 1/2Ka^2$ and the loss factor now becomes

$$\eta = \frac{h}{k} \qquad (3.31)$$

where k is the spring stiffness, and a is the amplitude of motion. Material hysteresis due to plasticity and other nonlinear effects are not included herein. It has been shown that hysteretic damping models can be used only for a steady-state response analysis but not for transient response [3.26].

Once more, when a structure is built with elements of known damping characteristics the equations of motion of the system become:

$$M\ddot{y} + (K + iH)y = bU(t) \tag{3.32}$$

Here, H is the damping matrix.

3.3.2.3 More Advanced Damping Models

To describe damping characteristics with the hysteretic damping model the stiffness matrix and the damping factor need only be known, while with the viscous damping model in addition to these the resonance frequency is also required. To extend the results obtained by viscous and hysteretic damping, researchers have developed three parameter models (the so called Kelvin-Voigt model [3.24]) whereby not only the instantaneous applied force but also the pass loading is considered.

$$F_d = \int_{-\infty}^{t} \psi(t,\tau)\ y(\tau)\ d\tau \tag{3.33}$$

where ψ is a memory function. This model is sometimes considered as a viscous damping model with frequency dependent damping and stiffness parameter [3.27].

In the modal coordinate formulation of the analytical models the equation of motion is given by:

$$M\ \ddot{q} + [\omega^2]q = F(t) \tag{3.34}$$

where $M_{ii} = \phi_i^T m \phi_i$, $F_i = \phi_i^T\ bU(t)$ are the ith elements and ϕ is the modal matrix. The corresponding damping matrix elements will be

$$\overline{C}_{ii} = \phi_i^T c \phi_j \text{ and } \overline{H}_{ij} = \phi_i^T H \phi_j$$

which are nondiagonal, in general, as a sign of coupling between modes. The coupling is minimal under certain conditions related to damping ratios [3.28].

3.3.2.4 Uncertainty in Damping

Throughout the extensive literature of active control of LFSS, the role of structural damping is often omitted or relegated to the inclusion of small first derivative terms in the equations of motion. However, it is well established that satisfactory performance and even stability of the system rely essentially on the presence of such damping. For LFSS will not be surrounded by air and thus lack a very important "sink" for energy dissipation. Their lowest natural frequencies may be up to 1000 times less than those for typical aircraft vibrations. In the case of plate- or shell-like structures their spectra will have a very rich spectrum of modes [3.29]. These and other considerations raise the need for better damping prediction and modeling, and more accurate information about damping.

One of the most important and basic relations in dynamic modeling of structures is that the measured modes satisfy the theoretical requirement of weighted orthogonality with respect to the mass and the stiffness matrices. Satisfaction of such a requirement implies no or proportional damping (C = aM + bK) and a symmetrical stiffness matrix [3.30]. In the above situation the damped modes are identical with the normal modes. For simple structures the measured modes (complex modes) are very close to the normal modes. However, for more complex structures, this is not the case. Complexity of modes could be due to errors in measurements, identification, modeling, or merely as a result of nonproportional damping [3.31]. For structures with nonproportional damping it is extremely difficult to measure normal modes. Moreover, for LFSS where only parts of the whole system are amenable to some sort of testing it is virtually impossible to have any reasonable certainty of modal or damping parameters.

The analytical modeling under very limited testing for validation and verification creates an environment of high uncertainty. In order to achieve stability and control and appropriate compensation for multiple-elastic-mode behavior in LFSS a large order finite element model is usually used. However, due to the stringent limitations on the number and location of implementable controllers and sensors, as well as the limited on-board computational capability, a lumped parameter (usually linear) controller design is possible for only very low order models. Hence, reduced order models are necessary for the purpose of control system design, resulting in control and observation "spillover", that involve the residual vibration modes ignored in the reduced order model, or not represented accurately due to the high degree of uncertainty in the modeling problem [3.21, 3.32].

Vibrations involving single fixed frequency excitations with low stresses and relatively low performance requirements are virtually nonexistant in modern aerospace systems. On the contrary, excitations are invariably broadband random or arise from harmonic loads that change in frequency. Thus, stresses are high, resonance is almost unavoidable, and performance requirements are extremely exacting. Under such circumstances damping plays a very vital role, whether it be active or passive [3.33].

It is common practice to introduce damping in the modal model of spacecraft, whereby each mode has its own viscous or hysteretic damping value. Moreover, these modal damping values are normally "estimated" from previous experience on similar structures or are measured (when possible) a posteriori by means of ground vibration testing. The common method of generating modal damping valves from tests is through the loss factor given by

$$\beta_i = \frac{\sum_j \frac{\eta_j}{2} V_{j,max}^{(i)}}{\sum_j V_{j,max}^{(i)}} \tag{3.35}$$

where η_j is the loss factor of the jth element of the structure and $V_{j,\,max}^{(i)}$ is its maximum potential energy in the ith mode [34]. It is theoretically possible to utilize damping values from tests performed on substructures in order to infer the damping characteristics for the whole system. However, the complexities involved in LFSS and the material, model and other uncertainties inherent to such systems render such approaches very difficult and of limited use [3.35].

3.3.2.5 Propagation of Statistics in Simple Linear Models

Consider a simple relationship between two vectors x and y given by:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{matrix} = \\ = \\ = \end{matrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{27} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{3.36}$$

or x = Ay

If the components of y are random, then x_1, x_2, and x_3 are also random. In order to compute the variances of x_1, x_2, and x_3 it is necessary to perform matrix manipulations with the matrix A and the covariance matrix of the vector y. The covariance of y is given by:

$$\mathrm{cov}\ (y) = \Sigma_y = \begin{bmatrix} \sigma^2_{y_1} & \sigma_{y_1 y_2} & \sigma_{y_1 y_3} \\ \sigma_{y_2 y_1} & \sigma^2_{y_2} & \sigma_{y_2 y_3} \\ \sigma_{y_3 y_2} & \sigma_{y_3 y_2} & \sigma^2_{y_3} \end{bmatrix} \qquad (3.37)$$

The diagonal elements of the covariance of y, Σ_y, are the squares of the standard deviations (the variances) of each of the components of y. The off-diagonal elements of the covariance matrix show how two elements of y are statistically correlated. Under the assumption of statistical independence of the components of y, Σ_y becomes a diagonal matrix. However, statistical independence of the components of y does not imply, in general, the statistical independence of the elements of x.

As a function of A and Σ_y, the covariance matrix for the components of x can be expressed as

$$\Sigma_x = A\ \Sigma_y\ A^T \qquad (3.38)$$

The covariance matrix of x, Σ_x, now has the variances of its elements along the diagonal of Σ_x and the covariances of its elements of the diagonal of Σ_x.

This presentation is intended to clarify the linear relationship of statistical interdependence of vectors in linear models of LFSS.

3.3.3 Uncertainties in LFSS Models as State and Control Dependent Noise

Consider the modal dynamic model of LFSS in the state-space representation

$$\dot{x} = Ax + Bu \qquad (3.39)$$

where

$$x = \begin{bmatrix} \dot{q} \\ q \end{bmatrix} \quad \text{and } A = \begin{bmatrix} [-2\xi_i \omega_i] \\ I \end{bmatrix} \begin{bmatrix} [-\omega_i^2] \\ 0 \end{bmatrix} \quad B = \begin{bmatrix} \phi^T b \\ 0 \end{bmatrix}$$

$y = \phi q$, ω_i^2 and ϕ^i, $i = 1, 2, \ldots, n$ are the eigenvalues and their corresponding normalized eigenvectors.

Also, $[-2\xi_i\omega_i] = [\text{diag}\ [-2\xi_i\omega_i]]$ $i = 1, 2, \ldots, n$.

The above formulation permits the generalization and the concise statement of the uncertainty in the model. Thus, as discussed in the previous section the uncertainties in the frequencies and the damping ratios create a state-dependent noise via the matrix A. The control-dependent noise, on the other and, will be effected through the modal parameters ϕ. Hence, the overall problem of uncertainty in linear dynamic models for LFSS results in control-dependent and state-dependent noise.

Any additional errors caused by modeling uncertainties such as modeling of nonlinearities with linear approximations, or biases, could be incorporated within the model of Equation (3.39) as an additive noise vector. The resulting stochastic system model can be represented by:

$$\dot{x} = Ax + Bu + \xi \tag{3.40}$$

where ξ is the additive noise vector with known statistics.

The statistical information necessary to completely define all the uncertainties presented in the past pages is not a trivial task, by any means. If, as was conjectured earlier, the distributions of the random variables involved are non-Gaussian, then all their statistical moments will be required for complete information. All the essential assumptions and constraints that are practically meaningful should be utilized to accomplish this requirement for a tractable solution of the stochastic optimal control problem for LFSS.

Most modeling and control design approaches to LFSS thus far assume the existence of a very large order design model that is based on complete deterministic information on the values of all structural parameters and geometry. These models are then reduced and simplified using model order reduction techniques.

The next step consists of computing an "optimal" control design via linear quadratic Gaussian (LQG) formulations. For stability and robustness purposes, some nominal LQG-based design is developed by some researchers in order to recover essential system characteristics via lengthy and complex iterations. Alternately, a simple control design that accounts for spillover and other errors and anomalies, is also derived that has inherently lower overall performance than the previous approach [3.36].

The above approaches are not at fault in terms of their logical procedures under fully deterministic conditions. Rather, the uncertainties that contaminate the issue render the presumption of completely accurate, deterministic models incorrect. Moreover, there is normally very insufficient data relative to deviations of parameters from their nominal values. Thus, to be realistic, one has to design control systems based upon stochastic models which incorporate the state-dependent, the control-dependent and additive random uncertainties within the best deterministic analytical model.

A very essential and timely task will be a formal analysis of existing analytical and test data of LFSS for generation of statistical information about their main parameters, namely the frequencies, the damping ratios and the modal parameters.

Presently, there is quite a number of flexible spacecraft, both military and civilian, that have available test and analytical data regarding their static and dynamic behavior. It is, of course, a difficult undertaking to accomplish such a goal. However, the importance of statistical data on the uncertainties of LFSS models with the information it can supply and the benefits it can produce could be immeasurable. One of two approaches could be taken: 1) either the statistics of the underlying elements (frequencies, damping ratios, and modal parameters) could be derived; or 2) the statistical measures and distributions of the uncertainties of these could be generated under realistic assumptions and restrictions. This is an essential effort that has to be studied fully. The prospects for success should be evaluated and an appropriate plan of action should be formulated.

3.4 PERFORMANCE INDEX

Typical performance criteria for classical control systems encompass two basic response characteristics. Namely: a) system response to a step or ramp input that is characterized by rise time, settling time, peak overshoot, and steady-state accuracy, b) frequency response of the system that is characterized by gain and phase margins, peak amplitude, and bandwidth. Classical techniques have definitely proven successful in many applications of control systems with single-input single-output features, time invariant, and (normally) zero initial conditions [3.37]. However, multiple-input multiple-output control systems with complex characteristics cannot effectively be handled through classical means and thus, state-space representations and modern control techniques are necesary to accomplish required performance criteria.

3.4.1 The Basis for a Performance Criterion

A differential equation of the general form

$$\dot{x}(t) = f(x(t),\ u(t),\ \xi(t),\ t) \tag{3.41}$$

where x(t) is an nth order state vector, u(t) is an mth order control or input vector, $\xi(t)$ is an nth order random vector with known statistics, f is an nth order vector stochastic function with some desirable properties and t is time, can represent a complex dynamical system with its characteristic behavior embedded within the equation, relative to time. This will represent the important quantitative characteristics of the system under consideration. However, as was mentioned in the previous sections, there are approximations, simplifying assumptions, and constraints inherent to the system in Equation (3.41). The engineer or the scientist who has the task of designing and analysis for the system, has to carefully consider all these conditions. In order to evaluate the performance of the system one can compare the actual states x(t) (from measured data) with some desired state vector z(t). Alternately, it is possible to evaluate system performance in an absolute sense, with some given desired characteristics [3.38]. Quite often the rate of deviation is measured and averaged over the time interval of concern. Thus the performance index may be given by an integral of the following form:

$$J = \int_0^T Q(x - z)\, dt \tag{3.42}$$

where Q is a scalar function of the vector (x - z). The most common performance measure of this type is given by

$$J_d = \sum_{i=1}^{n} (x_i - z_i)^2 \qquad (3.43)$$

which is based on mean square deviation and is for sampled data systems. While for continuous time systems the time average is utilized.

$$J = \int_0^T (x - z, x - z)\, dt \qquad (3.44)$$

where (.,.) is an appropriate inner product. These criteria can be a representation of power loss or energy dissipation. Other measures could also serve as a means of evaluating performance of a system. For instance, if one wishes to devote the least amount of time for the performance to reach a desired state, then a time optimal problem will be defined whereby the average time is minimized.

3.4.2 Various Forms of Performance Criteria

The most general performance functional for problem (3.41) can be expressed as:

$$J = h(x(T), T) + \int_0^T g(x(t), u(t), t)\, dt \qquad (3.45)$$

where h and g are functions of appropriate properties. However, typical control problems are based on simplified functions replacing h and g, usually quadratic, to provide physical motivation for the selection of a performance criterion. For the discrete time situation the functional takes the following form:

$$J_d = h(x(N), N) + \sum_{i=0}^{N-1} g_d^i (x(i), u(i), i) \qquad (3.46)$$

The optimal control problem is now to find the sequence of control inputs u(i) such that J_d is minimized.

In the case of stochastic systems the states and parameters are stochastic processes or random variables and thus the statistical expectation of J and J_d will be the appropriate cost functionals and the optimal controller will only minimize the performance index on an average basis.

3.4.2.1 Minimum Time Problems

The problem of transferring a system from an arbitrary initial state, known or random, $x(0) = x_o$ to a predetermined target set S in the least amount of time is referred to as the time optimal problem. The performance measure is thus

$$\begin{aligned} J &= T - 0 \\ &= \int_o^T dt \end{aligned} \tag{3.47}$$

where T stands for the instant when x(t) and the target set S intersect.

3.4.2.2 Terminal Control Problems

There are cases when the final state of a system is required to have a certain value r(T). An appropriate cost functional will then be

$$J = \sum_{i=1}^{n} (x_i(T) - r_i(T)]^2 \tag{3.48}$$

This performance index penalizes both positive and negative deviations from the desired value. In matrix notation this can be represented by:

$$J = || x(T) - r(T) ||_H^2 \quad (3.49)$$

where $||\cdot||$ is an appropriate norm and where H is a weighting matrix of appropriate dimension. H is at the disgression of the designer and there does not seem to exist any rigorous means of choosing its elements. H is invariably chosen to be positive definite and often diagonal. The relative importance of the various states will be reflected in the magnitudes of the elements of the H matrix. Thus, if H is chosen to be the identity matrix then the implication follows that all states are equally important.

3.4.2.3 Minimum Control Effort

The general problem of being able to transfer a system from an arbitrary initial state $x(0) = x_o$ to a specified target set S, under the constraint of the least amount of control effort is termed minimum control effort problem. The particular application at hand will determine the clear implication of the minimum control effort. Hence, there could be various forms of the performance measure depending on the requiremnts and constraints. One general form of cost functional for such a case is

$$J = \int_o^T || u(t) ||_R^2 \, dt \quad (3.50)$$

where, as previously, R is a weighting matrix to be chosen by the designer and that has to have appropriate dimensions and characteristics that reflect the desired performance requirements. In the discrete-time case, the integration is replaced by summation. When the problem is placed in a stochastic setting then the expectation operator acts on the integral or the summation for minimization on an average basis.

3.4.2.4 Tracking and Regulator Problems

Most often, the need arises for maintaining a system state x(t) as close as possible to a predetermined state r(t) on [o, T]. In such situations, a quadratic performance index with an appropriate positive semi-definite weighting matrix Q, ensures the minimal deviation from a desired trajectoroy. Of course, once again for the most general linear case, the most common performance measure will be a quadratic cost functional given by:

$$J = \int_{o}^{T} || x(t) - r(t) ||^2_{Q(t)} dt \tag{3.51}$$

and in discrete-time situations we will have a summation of the form:

$$J_d = \sum_{i-1}^{n-1} (x_i(i) - r_i(i))^2 \tag{3.52}$$

When stochastic systems are involved, then

$$J = E \left[\int_{o}^{T} || X(t) - r(t) ||^2_{Q(t)} dt \right] \tag{3.53}$$

or

$$J_d = E \left[\sum_{i=1}^{n} (x_i(i) - r_i(i))^2 \right] \tag{3.54}$$

We should note here that a regulator problem is a special case of a tracking problem which results if the value of the desired state is the zero vector for all t in [o, T].

3.4.3 Selection of a Performance Measure

The underlying purpose in selecting a performance criterion is to attempt to define a mathematical expression which when minimized ensures desired system performance according to requirements and constraints. Thus, choosing a cost functional is a translation of a given system's physical requirements, based on some constraints and assumptions, into analytical terminology. The more general performance indices normally chosen in LQG problems consist of

$$J = E\,[x^T(T)\; Q(T)\; x(T) + \int_0^T [x^T(t)\; Q(T)\; x(T) + u^T(t)\; R(t)\; u(t)]\, dt] \tag{3.55}$$

for the continuous time case. And for the discrete time situation the form becomes:

$$J = E\,[x^T(N)\; Q(N)\; x(N) + \sum_{i=1}^{N-1} [x^T(i)\; Q(i)\; x(i) + u^T(i)\; R(i)\; u(i)]\} \tag{3.56}$$

where E [·] stands for the statistical expectation operator, Q is a real positive semidefinite, symmetric matrix, and R(t) is a real positive definite symmetric matrix, all of appropriate dimensions. Both Q and R are at the disgression of the design engineer to be selected with good judgment in order to weigh the appropriate state and control components in a physically meaningful manner. The normal procedure is to iterate through several choices of Q and R and evaluate the results according to performance requirements and realistic system behavior constraints. There are some ad-hoc rules suggested by various authors in selecting the "best" performance index [3.39]. Moreover, practical situations render more physical insight to the selection process [3.40].

In the control of LFSS, the development that will follow consists of the linear space representation of the dynamic system. A generalized linear stochastic model will be assumed in which multiplicative and additive noise contaminate the system dynamics. Then the statistics of all the random processes (assumed known apriori) will be utilized for the development of the optimal stochastic control input.

Moreover the expected value of a quadratic performance index of the type shown in Equation (3.56) will be used for minimization purposes that will result in an optimal sequence of control inputs.

3.4.3.1 Various Possible Choices of Performance Index for LFSS Problems

Depending upon the particular control problem at hand, there are many possible choices of performance functionals for LFSS. Rigid body rotations and translations would normally be undesirable and thus penalized in the cost function. Moreover, the control effort should be bounded in order to conserve fuel and not to create any excessive deflections that can cause resonances or disturbances. Another important factor of motion that is also considered unwanted is vibration or displacement of the various parts of the LFSS. Thus, vibration and shape control is one of the key issues in LFSS control. Therefore, when dealing with LFSS control problems (as in other cases) it is essential to analyze the problem thoroughly and determine which of the above, namely, rigid body motion, vibration, and shape, should be minimized or controlled and accordingly include an appropriate penalty function within the performance index. Normally, total strain energy, total kinetic energy, and total displacement at discrete points on the LFSS are considered viable elements of performance functionals. Thus,

$$J = \lim_{T \to \infty} E \left\{ \frac{1}{T} \int_0^T \left[x^T(t)\, Q(t)\, x(t) + u^T(t)\, R(t)\, u(t) \right] dt \right\} \tag{3.57}$$

is the general form of the performance index in steady-state situations, and averaged overtime. In the above equation x^TQx could consist of total energy, plus total displacement energy [3.41]. The case of displacements at discrete points on the structure could be represented by a summation of the squares of the displacments at the chosen discrete locations.

3.4.3.2 The Discounted Cost Problem [3.42]

The space where the optimal control for the infinite horizon problem is well defined and trackable can be extended into a larger space by weighting the cost functional with a discount factor α^k, where $0 < \alpha < 1$ [3.43]. Such discount factors are used extensively in economics in order to emphasize the short-term worth of utility functions as compared to long-term worth [3.44].

In applied dynamic systems control the discount factor has been used for infinite-time control problems [3.45]. In the infinite horizon problems of discrete-time systems, since the cost is infinite, it is usually normalized by the planning horizons N (just like in the continuous time case it is normalized by the magnitude of time interval T). Thus, discounted cost functions for discrete-time systems is expressed by:

$$J_{av} = \lim_{N \to \infty} \frac{1}{N} E \left\{ \sum_{k=1}^{N} [x^T(k)\ Q(k)\ x(k) + u^T(k)\ R(k)\ u(k)] \right\} \tag{3.58}$$

It can be shown that J_{av} can be closely approximated by

$$J_d = E \left\{ \sum_{k=0}^{\infty} \alpha^k [x^T(k)\ Q(k)\ x(k) + u^T(k)\ R(k)\ u(k)] \right\} \tag{3.59}$$

where 0 < α < 1 [3.42]. The use of the discount factor guarantees the finiteness of the performance index for any stabilizing controller. Furthermore, in the above situation the initial performance is emphasized, and if detectability and stabilizability conditions [3.42] are not met, the problem might have a nonunique solution. In all the above, Q and K are assumed to be positive definite symmetric matrices with appropriate dimensions. Also, note that the case of α = 1 is simply the undiscounted cost problem. Moreover, the normal approach to the solution of the discrete-time optimal control problem is by dynamic programming.

3.5 CONCLUSIONS

Generalized linear stochastic modal models of LFSS with state- and control-dependent and purely additive noise are realistic representations of real structures. The main reasons for uncertainties due to modeling techniques and approximations, as well as to high performance requirements are discussed and related to basic characteristics of geometry, material properties, inaccurate data, tolerances and various other things. The performance functions appropriate for LFSS control are highlighted and the best possible cost index is suggested and justified.

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CHAPTER 4.

DERIVATION OF CONTROL ALGORITHM AND STABILITY CHARACTERISTICS

SECTION 4.1

INTRODUCTION

Desired dynamic behavior of a control system is contingent upon (1) accurate configuration and modeling, (2) adequate representation of damping effects, nonlinearities, and random effects, (3) advanced algorithms, (4) hardware/software implementation and approaches, and many other factors. Control of large flexible space structures (LFSS) requires a multi-input, multi-output distributed approach. Complicating the design and analysis of LFSS are many features that are characteristic of such systems. Some of these features are: (1) LFSS cannot be treated as rigid bodies. (2) In addition to pointing and shape control, LFSS require several orders of magnitude of vibration suppression (beyond what can be achieved by natural damping or by using viscoelastic materials). (3) LFSS have extensive structural/control, interaction since controller bandwidth and structural modes often coincide. (4) There are appreciable uncertainties in LFSS modeling, controller actuation and location, sensor outputs and location, geometry, damping and other parameters, among others [4.1].

An accurate representation of LFSS by linear dynamic models requires large order systems of equations. The closed-loop gains of a low, or reduced order model plays a very important role in the control of LFSS. For, a high gain system means a large control bandwidth requirement. And a large bandwidth implies that more flexible modes should be considered. On the other hand, the controller bandwidth is determined by establishing the desired performance and the expected or existing disturbances. Thus, a dilemma exists in modeling and control design of LFSS, that is solved by compromising. First the expected disturbances are determined and then the control bandwidth that can adequately reduce such disturbances is identified appropriately [4.2]. Once the model has been established, there remains to consider the available measurements.

In this part of the report, our assumption will be that all the states are available through perfect measurements. The main objective will be to find an optimal feedback control law that uses all states for feedback. The system being stochastic, as considered earlier, a stochastic optimal control law will be developed under multiplicative and additive noise. The frequencies, damping ratios and the modal parameters of the LFSS modal model are considered random in nature, thus creating a linear system with multiplicative and additive noise.

The stochastic optimal control problem of linear multidimensional systems with purely random parameters will be formulated and solved herein. Both the discrete time and the continuous time cases will be treated. Dynamic programming will be utilized for the discrete case and the Hamiltonian approach will be taken for the continuous time case. Stability in a stochastic setting will be defined and appropriately discussed. Mean- and mean-square stability conditions will be determined for the infinite horizon situation.

SECTION 4.2
OPTIMAL LINEAR FEEDBACK CONTROL UNDER FULL STATE INFORMATION

For stochastic optimal control problems it is vital to furnish the necessary information for control. Hence, presently we will assume that the states of the sytem, denoted by $x(t)$ are available at every instant from exact measurements. We will also assume that the admissible controls are real-valued and of state-feedback type: $u(k) = f(x(k), k)$ for the discrete-time case, and $u(t) = g(x(t), t)$ for the continuous time situation.

4.2.1 PROBLEM STATEMENT AND SOLUTION: THE DISCRETE-TIME CASE

Consider a dynamical sytem represented by the following linear discrete-time vector difference equation:

$$x(k+1) = \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A(k) & 0_{n_1 x n_2} \\ 0_{n_2 x n_1} & \theta(k) \end{bmatrix} x(k) + \begin{bmatrix} B(k) \\ \Gamma(k) \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ \xi_o(k) \end{bmatrix} \qquad (4.2.1)$$

for $k = 0, 1, \ldots, N$.

For brevity, we can write Equation (4.2.1) as:

$$x(k+1) = \Phi(k)\, x(k) + \psi(k)u(k) + \xi(k) \qquad (4.2.2)$$

where:

1. x is an n-dimensional state vector composed of $x_1(k+1)$, an n_1-dimensional vector and $x_2(k+1)$, and n_2-dimensional random vector.

2. u(k) is the m-dimensional control vector.

3. $\theta(k)$ and $\Gamma(k)$ are $n_2 \times n_2$ - and $n_2 \times$ m-dimensional matrices of randomly varying parameters, which are assumed to be Gaussian and white (uncorrelated in time), with known means of:

$$E[\theta(k)] = \bar{\theta}(k) \text{ and } E[\Gamma(k)] = \bar{\Gamma}(k) \quad (4.2.3)$$

and covariances of:

$$E[(\theta(k) - \bar{\theta}(k))(\theta(j) - \bar{\theta}(j))^T] = \Sigma^{\theta\theta}(k)\delta_{kj} \quad (4.2.4)$$

$$E[(\Gamma(k) - \bar{\Gamma}(k))(\Gamma(j) - \bar{\Gamma}(j))^T] = \Sigma^{\Gamma\Gamma}(k)\delta_{kj} \quad (4.2.5)$$

and cross covariance matrix given by:

$$E[(\theta(k) - \bar{\theta}(k))(\Gamma(j) - \bar{\Gamma}(j))^T] = \Sigma^{\theta\Gamma}(k)\delta_{kj} \quad (4.2.6)$$

4. A(k) and B(k) are $n_1 x n_1$ and $n_1 \times$ m-dimensional deterministic matrices, respectively.

5. $\xi_0(k)$ is an n_2-dimensional zero mean white Gaussian noise vector which is assumed to be independent of all the other random variables of the control system and

$$E[(\xi_o(k)\ \xi_o^T(j)] = \Sigma^{\xi\xi}(k)\delta_{kj} \qquad (4.2.7)$$

6. $0_{n_1xn_2}$ is the n_1xn_2 null matrix.

7. $\Phi(k) = \begin{bmatrix} A(k) & 0_{n_1xn_2} \\ 0_{n_2xn_1} & \theta(k) \end{bmatrix}$ and $\psi(k) = \begin{bmatrix} B(k) \\ \Gamma(k) \end{bmatrix}$

8. x(o) is deterministic (for convenience).

We note that δ_{kj} is the Kronecker delta operator.

The scalar index of performance for the above control system is defined by the following quadratic cost functional:

$$J(u) = E\left[\frac{1}{2}\ x^T(N)F\ x(N) + \frac{1}{2}\sum_{k=0}^{N-1}(x^T(k)Q(k)x(k) + u^T(k)\ R(k)u(k))\right] \qquad (4.2.8)$$

where F and Q are positive semi-definite and R positive definite matrices of appropriate dimensions.

The stochastic optimal control problem now is to determine a control sequence $\{u_i\}_{i=o}^{k-1}$ such that the functional J(u) in Equation (4.2.8) is minimized.

Under the circumstance of a general linear stochstic system of the form of Equation (4.2.1) where $\xi(k)$ is a vector of correlated noise, it is possible [4.3] to model the additive colored noise by a system of first order difference equations driven by Gaussian white noise. Thus, if

$$\xi(k+1) = \theta(k)\xi(k) + \xi_o(k) \qquad (4.2.9)$$

then Equation (4.2.1) augmented with Equation (4.2.9) would take the shape given in Equation (4.2.2). Hence, one reason for the particular diagonal matrix type form that appears in Equation (4.2.2) is due to the abovementioned generalizations. See Figure 4.1 for block diagram of the system.

4.2.1.1 Optimal Control of Linear Stochastic Discrete-time Systems

We make the assumption that all the state variables can be measured exactly. That is, the information set consists of the following:

$$I(k) = \{x(0),\ x(1),\ \ldots,\ x(k),\ u(0),\ u(1),\ \ldots,\ u(k-1)\} \tag{4.2.10}$$

Theorem 4.1:

The optimal control for the system given by Equation (4.2.2) minimizing J(u) in Equation (4.2.8) is [4.4].

$$u^*(k) = -G^*(k)x^*(k) \tag{4.2.11}$$

where

$$G^*(k) = \left[R(k) + E\left\{ \begin{bmatrix} B(k) \\ \Gamma(k) \end{bmatrix}^T K(k+1) \begin{bmatrix} B(k) \\ \Gamma(k) \end{bmatrix} \right\} \right]^{-1} E\left\{ \begin{bmatrix} B(k) \\ \Gamma(k) \end{bmatrix}^T K(k+1) \begin{bmatrix} A(k) & 0 \\ 0 & \theta(k) \end{bmatrix} \right\} \tag{4.2.12}$$

and where K(k) is given by the following Riccati-like matrix difference equation:

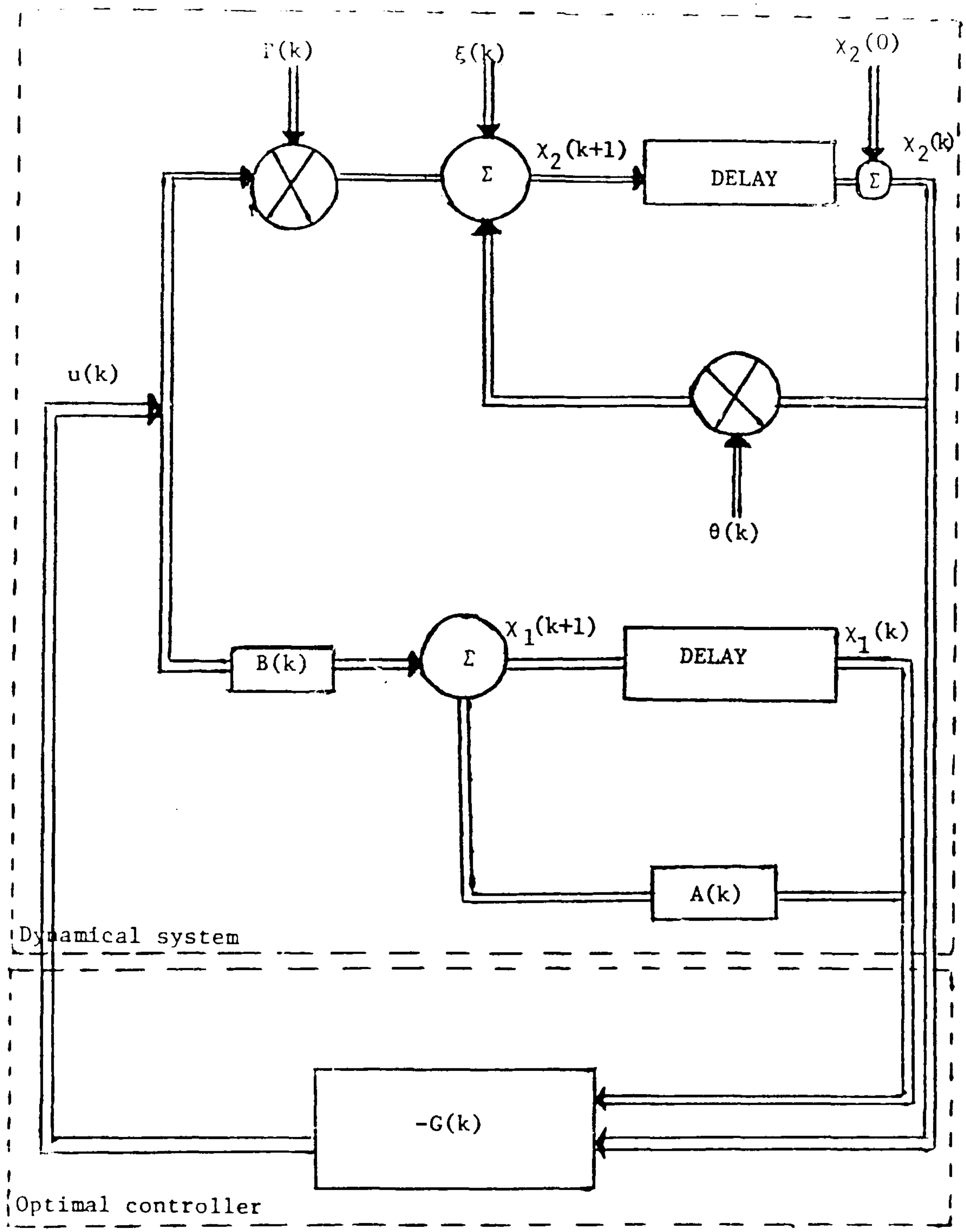

Figure 4.1. Optimal Controller for System Equation (4.2.1)

$$K(k) = Q(k) + E\left\{\begin{bmatrix} A(k) & 0 \\ 0 & \theta(k) \end{bmatrix}^T K(k+1) \begin{bmatrix} A(k) & 0 \\ 0 & \theta(k) \end{bmatrix}\right\} - E\left\{\begin{bmatrix} A(k) & 0 \\ 0 & \theta(k) \end{bmatrix}^T\right.$$

$$\left. K(k+1) \begin{bmatrix} B(k) \\ \Gamma(k) \end{bmatrix}\right\} \left[R(k) + E\left\{\begin{bmatrix} B(k) \\ \Gamma(k) \end{bmatrix}^T K(k+1) \begin{bmatrix} B(k) \\ \Gamma(k) \end{bmatrix}\right\}\right]^{-1}$$

$$E\left\{\begin{bmatrix} B(k) \\ \Gamma(k) \end{bmatrix}^T K(k+1) \begin{bmatrix} A(k) & 0 \\ 0 & \theta(k) \end{bmatrix}\right\} \quad (4.2.13)$$

with

$$K(N) = F \quad (4.2.14)$$

For a proof, see Appendix 4.1. The optimal state trajectory can now be derived from the solution of the following difference equation:

$$x(k+1) = (\Phi(k) - \psi(k)\ G^*(k))\ x(k);\ x(o) = x_o \quad (4.2.15)$$

Clearly, the optimal control vector given by equation (4.2.11) is a random vector since x(k) is itself a random variable. Due to the uncertainties within the system and the control, we note that the state and the control vectors are both weighted by the covariances and the means of the random parameters and that the Riccati-like Equation (4.2.13) cannot be reduced to coupled linear equations.

The extremal control of a stochastic multivariable discrete time dynamical system is not necessarily the unique optimal control, we must establish that the second order partial derivative of the Hamiltonian function of our problem with respect to u is positive definite [4.5]. The Hamiltonian function is given by:

$$H_d = \frac{1}{2} x^T(k)\ K(k)x(k) + \sum_{i=k}^{N-1} Tr\ (K(i + 1)\,\Sigma^{\xi\xi}) + P^T(\Phi(k)x(k)$$

$$+ \psi(k)u(k) + \xi(k)) \qquad (4.2.16)$$

Hence, $\frac{\partial^2 Hd}{\partial u^2}$ is given by:

$$\frac{\partial^2 H_d}{\partial^2 u} = R(k) + E[\psi(k)^T K(k + 1)\psi(k)] > 0 \qquad (4.2.17)$$

The solution K(k) to the Riccati-like Equation (4.2.13) is non-negative definite and unique for $N < \infty$. For a proof see Reference [4.6].

In what follows we will need to establish certain rules regarding the expected value of products of matrices that apear in Equations (4.2.12) and (4.2.13). Hence, for the proof of Theorem 4.1, we note the following facts.

In [4.7] p. 262, it is established for a random vector x with mean m and covariance R, that

$$E[x^T S x] = m^T\ Sm + tr\ SR \qquad (4.2.18)$$

where S is a deterministic matrix and tr represents the trace operator.

In the sequel, we will frequently have to calculate quanitities of the following form:

$$E[ABA^T] \qquad (4.2.19)$$

where A and B are random matrices with stochastic Gaussian elements.

To this end, we propose the following lemma:

Lemma 4.1

Let A and B be stochastic matrices with random Gaussian white elements and given statistics of:

$$E[A] = \bar{A}, \; E[B] = \bar{B} \quad (4.2.20)$$

$$E[(A-\bar{A})(A-\bar{A})^T] = \Sigma^{AA}, \; E[(B-\bar{B})(B-\bar{B})^T] = \Sigma^{BB} \quad (4.2.21)$$

$$E[(A-\bar{A})(B-\bar{B})^T] = \Sigma^{AB} \quad (4.2.22)$$

Then,

$$E[ABA^T] = \bar{A}\bar{B}\bar{A}^T = Tr(\Sigma^{AA}\bar{B}) \quad (4.2.23)$$

where Tr represents the matrix trace operator to be defined in Appendix 4.1 and A^T is the transpose of A. For a proof of Lemma 4.1, see Appendix 4.1.

4.2.1.2 Asymptotic Behavior of the Solution of the Riccati-like Equation

The existence of an optimal control law is closely related to the behavior of the Riccati-like difference Equation (4.2.13) associated with the problem at hand. Hence, the latter equation is studied carefully.

Throughout this section it will be assumed that the stochastic linear system given in Equation (4.2.2) has wider sense stationary statistics and the state and control weighting matrices Q(k) and R(k) are constant. Then, the Riccati-like difference Equation (4.2.13) will take the following form:

$$K(k) = Q + \bar{\Phi}^T K(k+1) \bar{\Phi} + Tr (\Sigma^{\Phi\Phi} K(k+1)) - [\bar{\Phi}^T K(k+1) \bar{\Psi}$$

$$+ Tr(\Sigma^{\Phi\Psi} K(k+1)] [(R + \bar{\Psi}^T K(k+1) \bar{\Psi} + Tr (\Sigma^{\Psi\Psi} K(k+1))]^{-1} \quad (4.2.24)$$

$$[\bar{\Psi}^T K(k+1) \bar{\Phi} + Tr (\Sigma^{\Phi\Phi} K(k+1))]$$

with

$$K(N) = 0 \quad (4.2.25)$$

Here, the ij^{th} element of the covariance matrix $\Sigma^{\Phi\Phi}$ is given by $\Sigma^{\Phi i \Phi j}$ where Φ_i and Φ_j are the i^{th} and j^{th} rows of Φ respectively.

It is worth noting that Equation (4.2.24) does not always have a steady-state solution "backward in time." Because, unlike the constant parameter situation, here we have constant but unknown parameters. The solution for the infinite-time of the scalar form of Equation (4.2.24) has been treated previously [4.8]. Furthermore, unlike the Riccati equation, Equation (4.2.24) cannot be related to a coupled set of linear equations. Hence, only under some conditions the existence of a steady state solution could be investigated. Reference [4.8], studied "The Uncertainty Threshold Principle", the existence of a steady-state solution for the one-dimensional scalar case of the abovementioned problem. In a similar effort, we develop an argument to derive the existence of a solution of Equation (4.2.24) under preassigned constraints.

It is not hard to show that there exist matrices R_1, T_2 and T_3 such that the following equality holds [4.9 - 4.11].

$$\bar{\Phi}^T K(k+1) \bar{\Phi} + Tr (\Sigma^{\phi\phi} K(k+1)) = T_1 \bar{\Phi}^T K(k+1) \bar{\Phi} T_2 \quad (4.2.26)$$

Assuming the above is true, we can rewrite Equation (4.2.24) in the following manner:

$$K(k) = Q + T_1\bar{\Phi}^T K(k+1)\bar{\Phi}\ T_2 - T_2^T\ \bar{\Phi}^T K(k+1)\bar{\Psi}\ [R + T_3\bar{\Psi}^T K(k+1)\bar{\Psi}]^{-1}$$

$$\cdot\ \bar{\Psi}^T K(k+1)\ \bar{\Phi}\, T_2 \qquad (4.2.27)$$

By adding and subtracting $T_2\ ^{-T}K(k+1)\bar{d}\ T_2$ to Equation (4.2.27), we get

$$K(k) = Q + (T_1 - T_2^T)\ \bar{\Phi}^T K(k+1)\ \bar{\Phi}\ T_2 + T_2^T\ \bar{\Phi}^T\ \{K(k+1)$$

$$- K(k+1)\ \bar{\Psi}\ [R + T_3\ \bar{\Psi}^T K(k+1)\ \bar{\Psi}]^{-1}\ \bar{\Psi}^T K(k+1)\}\ \bar{\Phi}\ T_2 \qquad (4.2.28)$$

Now, consider the quantity inside the curly brackets and define

$$M(k+1) = K(k+1) - K(k+1)\ \bar{\Psi}[R + T_3\ \bar{\Psi}^T K(k+1)\ \bar{\Psi}]^{-1}\ \bar{\Psi}^T K(k+1) \qquad (4.2.29)$$

In [4.12] it is given that for matrices A, B and C of appropriate dimensions it is true that

$$(A + BCB^T)^{-1} = A^{-1} - A^{-1}\ B(C^{-1} + B^T\ A^{-1}B)^{-1}\ B^T\ A^{-1}$$

Hence, in Equation (4.2.29) M(k+1) can be rewritten as:

$$M(k+1) = [K^{-1}(k+1) + \bar{\ }^{r}\ R + T_2(\Sigma^{\psi\psi} K(k+1)^{-1}\ \bar{\Psi}]^{-1}$$

It is now clear why M(k+1) need be positive-definite. Note, also, that for $||K(k+1)||$ very large, we can approximate Equation (4.2.28) by

$$K(k) \approx Q + (T_1 - T_2^T)\ \bar{\Phi}^T\ K(k+1)\ \bar{\Phi}\ T_2$$

which clearly is a Lyapunov-type equation and can be treated as such.

Matrices of the form of Equation (4.2.29) arise in the matrix Riccati equation of standard linear quadratic Gaussian problems, where the weighting matrices T_1, T_2, and T_3 are not necessarily unique (see [4.13]).

If we assume that the systems of $(\bar{\Phi}, \bar{\Phi})$ and $(\bar{\Phi}, Q^2)$ are controllable and observable respectively, then it is established [4.13], [4.8] that

$$M(k+1) = M^T(k+1) > 0 \text{ (is positive definite)} \qquad (4.2.30)$$

and that there exists a bound L such that

$$L \geq M(k+1) \text{ for all } k \qquad (4.2.31)$$

Hence, it follows that

$$T_2^T \bar{\Phi}^T M(k+1) \bar{\Phi}^T T_2 > 0 \qquad (4.2.32)$$

Thus,

$$K(k+1) \geq (T_1 - T_2) \bar{\Phi}^T K(k+1) \bar{\Phi} + Q \qquad (4.2.33)$$

Obviously, if any eigen-value of $(T_1 - T_2) \bar{\Phi}^T$ is greater than unity (i.e., if the spectral radius is greater than one) then K(k) grows without bound "backward in time." In other words, $\lim_{k \to \infty} K(k)$ does not exist. So, the optimal cost grows exponentially in a manner given below:

$$J^* (N) \geq D \exp (\max_i | \lambda_i | N) \qquad (4.2.34)$$

Where D is a constant matrix and $\max_i |\lambda_i|$ denotes the magnitude of the maximum eigenvalue of $(T_1 - T_2) \Phi^T$. In such a case, of course, only short term controls can be implemented.

From Equations (4.2.29) and (4.2.31) we have

$$K(k) \leq (T_1 - T_2)\ \bar{\Phi}^T\ K(k+1)\ \Phi^T + Q + T_2^T\ \bar{\Phi}^T\ L\ \bar{\Phi}^T\ T_2 \qquad (4.2.35)$$

Consequently, if $|\lambda_i| < 1$ for all i, the right-hand-side of inequality (4.2.35) will be a constant bounded solution matrix and so will K(k). Thus, the limiting solution lim K(k) is well defined. For more details of the above argument, see [4.14], [4.15] where it is required that Ψ be nxn nonsingular.

For $\Sigma^{\Phi\Phi} = \Sigma^{\Psi\Psi} = \Sigma^{\Phi\Psi} = 0$ the infinite time problem has a solution independent of λ_i [4.16]. While, when the above covariances get larger no stability can be expected in K(k). Hence, if we define

$$\beta = \max_{i\ j} |(T_1 - T_2)_{ij}| \qquad (4.2.36)$$

as the maximum value of the elements of the matrix $(T_1 - T_2)$ then, if

$\max_i |\lambda_i| < \frac{1}{\beta}$ the solution of the steady-state will exist and $\frac{1}{\beta}$ will give the radius of a shrinking disc which will contain all open-loop eigenvalues of Φ that make the problem solvable.

4.2.1.3 Stabilizability of Linear Stochastic Discrete-Time Systems

Digital control systems are very widely used nowadays due to the prominent presence of the digital computer. Also digital control systems under random sampling rates result in systems with stochastic parameters. It is a well known fact [4.17] that the infinite interval linear quadratic problem for a deterministic system has a solution if the system is stabilizable. If the problem is also observable then the solution is unique and the control system is asymptotically stable. In this subsection the conditions on the statistical measures of a discrete-time linear stochastic system for which the infinite interval problem has a solution will be discussed. The development is based on Reference [4.18].

Consider the following discrete-time control system:

$$x_{i+1} = 2\phi_i \, x_i + \Gamma_i \, u_i + \xi_i \tag{4.2.37}$$

where $x_i \in R^n$ is the state, $u_i \in R^m$ the control, Φ_i is an nxn and Γ_i is an nxm system and control matrices respectively. Here $\{\Phi_i\}$ and $\{\Gamma_i\}$ are sequences of independent random matrices with constant statistics. Also, at i = 0, x_o is deterministic and u_i is a deterministic function of x_j, $j \leq i$. Then, Φ_i and Γ_i are both independent of x_j for all $j \leq i$. Suppose the control law is of the feedback type given as follows:

$$u_i = -L \, x_i \tag{4.2.38}$$

where L is an mxn matrix. Also let

$$\psi_i = \Phi_i - \Gamma_i L \tag{4.2.39}$$

Then the closed loop system can be written as:

$$x_{i+1} = \psi_i \, x_i + \xi_i \tag{4.2.40}$$

Definitions: 1. Equation (4.2.40) is mean stable if $E[x_i] = \overline{x}_i \rightarrow 0$, and mean square stable if $E[||x_i||^2] \rightarrow 0 \; \forall x_o$.

2. Equation (4.2.38) is mean stabilizable if an L can be found such that Equation (4.2.40) is mean square stable.

In the above $E[\cdot]$ is the statistical expectation oprator and $||.||$ is the usual Euclidean norm.

Theorem 4.2

The system given by Equation (4.2.40) is mean stable if and only if $E[\psi_i] = \bar{\psi}_i$ is a stable matrix and it is mean square stable if and only if $E[\psi^T X \psi]$ is stable for every real symmetric nxn matrix X. Moreover, mean square stability implies mean stability.

Theorem 4.3

Equation (4.2.38) is mean stabilizable if there is an L such that $E[\psi]$ is a stable matrix and mean square stabilizable if there is an L such that $E[\psi^T X\Phi]$ is stable for every real symmetrix nxn matrix X.

For the optimal control in a finite interval, consider the system of Equation (4.2.38) with a performance functional given by the following:

$$J_N = E[\sum_{i=o}^{N-1} (x_i^T Q_i x_i + u_i^T R_i u_i) + x_N^T H x_N] \qquad (4.2.41)$$

where: Q_i - nxn, R_i - mxm and, H - nxn are weighting matrices such that Q and H are positive semidefinite and R is positive definite.

Theorem 4.4: for a feedback control of the form of Equation (4.2.38). The optimal cost given by:

$$J_N = x_o^T E[\sum_{i=0}^{N-1} (\psi^i)^T (\bar{Q} + L^T\bar{R}L) \psi^i + (\psi^N) \overset{T}{H} {}^N] x_o \qquad (4.3.42)$$

And the optimal control law is as given in Equations (4.2.11 - 4.2.13).

For the infinite interval case it can be shown [4.18] that if system (4.2.37) is mean square stabilizable then the infinite interval optimal control probelm has a solution. Moreover the mean square stabilizability is also possible if the Riccati Equation (4.2.13) converges in the steady state, resulting in a solution. Thus, under certain conditions [4.18], mean square stabilizability is necessary and sufficient for the existence of the infinite interval optimal control problem under full state information.

4.2.2 PROBLEM STATEMENT AND SOLUTION - THE CONTINUOUS-TIME CASE

Optimal control of stochastic continuous systems have been studied by several researchers [4.19 - 4.27]. In this section the general form of the optimal control and the performance functional are assumed known. Lyapunov functions are utilized to guarantee optimal stabilization of the control system, in a probabilistic sense. Thus, the states are assumed to be known at every instant and the cost functional associated with the given system is of the quadratic type.

4.2.2.1 Problem Statement in a General Setting

Consider a general system modeled by the following Ito type stochastic differential equation:

$$dx(t) = f(x(t), t)\, dt + G(x(t), t)\, d\beta(t) \tag{4.2.43}$$

where $x(t, \cdot)$ R^n is the sate vector stochastic process, $\beta(t)$ is a p-dimensional Wiener process, $f(x(t), t)$ is an n-vector function describing the system dynamics and $G(x(t), t)$ is an n x s matrix valued function, both continuous in t. Also

$$x(t_o) = x_o$$
$$E\,[d\beta(t)\, df^T(t)] = Q(t)dt$$

Here, Equation (4.2.43) is to be taken in the following sense:

$$x(t) - x(t_o) = \int_{t_o}^{t} f(x(\tau), \tau)\, d\tau + \int_{t_o}^{t} G(x(\tau, \tau)\, d\beta(\tau) \tag{4.2.44}$$

where the first integral is an ordinary integral for a given sample function of the process, while the second integral is an Ito stochastic integral. The assumption that f and G are admissible functions (according to some rule) yields solution processes of the Ito type [4.28 - 4.29]. Both f and G are defined and assumed measurable on $[t_o, \infty] \times R^n$ and the following conditions are true:

For every $t \geq t_o$ and, u, v R^n, there is a k>0 such that

1. $||f(u, t) - f(v, t)|| + ||G(u, t) - G(v, t)|| \leq k\,||u - v||$
2. $||f(u, t)||^2 + ||G(u, t)||^2 \leq k^2(1 + ||u||^2)$.

The first is a Lipschitz condition, while the second one restricts the growth of f and G. It is also assumed that f(0, t) = 0 G (0, t) = [0] for $t \geq t_o$. Under the above conditions [30] there exists an n dimensional diffusion process $x(t, x_o)$ that is the unique solution of Equation (4.2.44).

The following stability conditions [25] will be used in the sequal for the equilibrium solution $x(t, x_o)$ of Equation (4.2.44):

<u>Def. 4.2.1</u>: $x(t, x_o)$ is stochastically stable (ss) if $\forall\ \Sigma > 0$

$$\lim_{x_o \to 0} p\left(\sup_{t_o \leq t < \infty} ||x(t, x_o)|| \geq \epsilon \right) = 0$$

<u>Def. 4.2.2</u>: $x(t, x_o)$ is stochastically asymptotically stable (sas) if $\lim_{x_o \to 0} p\left(\lim_{t \to \infty} x(t, x_o) = 0 \right) = 1$

<u>Def. 4.2.3</u>: $x(t, x_o)$ is stochastically asymptotically stable in the large (sasil) if $p\left(\lim_{t \to \infty} x(t, x_o) = 0 \right) = 1$, for every $x_o \epsilon R^n$.

An optimal controller will now make the above stochastic system stable according to one or all of the above definintions and will minimize a certain performance functional to be defined later. To this end consider a more general (than Equation 4.2.44) system given by

$$dx(t) = A(x(t), t)\, dt + B(x(t), u(t), t)\, dt + C(x(t), t)\, d\xi(t) + D(u(t), t)\, d\gamma(t) + E(t)\, d\upsilon(t) \qquad (4.2.45)$$

where ξ, γ and υ are independent Wiener processes A, B, C, D, and E are of appropriate dimensions and satisfy conditions 1 and 2 above. Moreover, at $x_o = 0$ $A(0, t) = B(0, 0, t) = C(0, t) = D(0, t) = o$ for all $t \geq t_o$ where $u(x_o, t)$ is an m-dimensional condition input such that $u(0, t) = 0$ for all $t \geq t_o$.

Under complete controllability conditions the admissible set of controllers will be those that guarantee the continuity of B and D and their first and second derivatives with respect to the equilibrium solution $x(t, x_o)$. Define a performance functional that is to be minimized by the optimal controller as follows:

$$J = \lim_{t \to \infty} \frac{1}{t} E \left[\int_{t_o}^{t} g(x(\tau, x_o), u(\tau), \tau)\, d\tau \right. \qquad (4.2.46)$$

where, as before $E[\cdot]$ is the statistical expectation, g is a non-negative continuous function of $x(t, x_o)$ and $u(t)$, and $t \in [t_o, \infty]$.

The Jacobi-Bellman equation associated with Equation (4.2.45) for a Liapunov function $V(x(t), t) : R^n \times R^+ \to R^+$ is given by [30].

$$LV = \frac{\partial V}{\partial t} + (A + B)^T \frac{\partial V}{\partial x} + \frac{1}{2} \operatorname{tr} (CC^T + DD^T + EE^T) \frac{\partial^2 V}{\partial x^2}$$

$$= M(x(t), t) + B^T \frac{\partial V}{\partial x} + \frac{1}{2} \operatorname{tr} \left(DD^T \frac{\partial^2 V}{\partial x^2}\right) \qquad (4.2.47)$$

Seeking the minimum for

$$H = M(x(t), t) + \sum_{i=1}^{n} B_i \frac{\partial V}{\partial x_i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} [DD^T]_{ii} \frac{\partial^2 V}{\partial x_i \partial x_j}$$
$$+ g\,(x(t, x_o), u(t), t) \tag{4.2.48}$$

via dynamic programming, we have

$$\frac{\partial H}{\partial u} = \sum_{i=1}^{n} \frac{\partial B_i}{\partial u} \frac{\partial V}{\partial x_i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial}{\partial u} [DD^T]_{ij} \frac{\partial^2 V}{\partial x_i \partial x_j}$$
$$+ \frac{\partial g(x(t, x_o), u(t), t)}{\partial u} = 0 \tag{4.2.49}$$

From Equation (4.2.49) the optimal control law is derived usually by numerical means. Equation (4.2.46) can be expressed with respect to the Liapunov function for the case of finite time interval as follows:

$$J = - E \left[\int_{t_o}^{t_f} LV\,(x(t), t)\, dt \right] \tag{4.2.50}$$

and

$$E\,[V(x(t), t)] = E\,[V(x_o, t_o)] + E \left[\int_{t_o}^{t} LV\,(x(\tau), \tau)\, d\tau \right] \tag{4.2.51}$$

Since V is positive-definite and LV is negative definite the integral in Equation (4.2.51) converges as $t \to \infty$. The first term on the right-hand side of Equation (4.2.51) is finite and $V(x(t),t)$ is a positive supermartingale, thus making J finite.

The following conclusion can be drawn from the above development:

If

(i) $LV(x(t), t) \leq 0$ for $||x|| > \epsilon$, $\epsilon > 0$

(ii) $LV(x(t), t) < 0$ and $V(x(t), t) \leq v(x)$ where $v(x) > 0$ and $||x(t)|| < \epsilon$, $\epsilon > 0$

(iii) $V(x(t), t)$ is radially unbounded, (ii) is satisfied for all x R^n and $V(0, t) \equiv 0$ for all $t \geq t_o$

Then if (i) is satisfied the system in Equation (4.2.45) is ss, if (ii) is satisfied it is sas and if (iii) is satisfied then it is sasil.

To solve the above Jacobi-Bellman equation or for the minimizing optimal control law some simplifying assumptions are necessary.

To this end let us consider a special case of Equations (4.2.45-46) and

$$dx(t) = A(x(t), t)\,dt + P(x(t), t)\,u(t)\,dt + \sum_{i=1}^{n} Q_i(t)\,x(t)\,d\xi_i(t) + \sum_{j=1}^{m} N_i(t)\,u(t)\,d\gamma_i(t) + E(t)\,\upsilon(t) \tag{4.2.52}$$

with

$$J = E\left[\int_{t_o}^{\infty} (\varphi(x(t), x_o, t) + u^T R u)\,dt\right] \tag{4.2.53}$$

Then LV is given by

$$\begin{aligned} LV &= N(x(t), t) + P^T(x(t), t)\,u(t)\,\frac{\partial V}{\partial x} + \frac{1}{2}\,\mathrm{tr}\left(u^T N^T \frac{\partial^2 V}{\partial x^2} N u\right) \\ &\quad + \frac{1}{2}\,\mathrm{tr}\left(x^T(t)\,Q^T \frac{{}^2 V}{x^2}\,Q\,x(t)\right) \\ &= LV + \varphi(x(t), t) + u^T R u \end{aligned} \tag{4.2.54}$$

Fr·· he above we can obtain the optimal control law.

4.2.2.2 Linear Stochastic Control Systems

Consider the case of regular stochastic linear quadratic systems given in the following form.

$$dx(t) = A(t)\ x(t)\ dt + B(t)\ u(t)\ dt\ \sum_{i=1}^{m} C_i(t)\ x(t)\ d\xi_i(t)$$

$$+ \sum_{j=1}^{q} D_j(t)\ u(t)\ d\gamma_j(t) + E(t)\ d\nu(t) \qquad (4.2.55)$$

with a cost functional given by:

$$J = E\left[\int_{t_o}^{\infty} (x^T(t)\ Q(t)\ x(t) + u^T(t)\ R(t)\ u(t))\ dt\right] \qquad (4.2.56)$$

where ξ, γ, and ν are independent Wiener processes with elements ξ_i and γ_j, and Q is a positive semidefinite and R a positive definite symmetric matrices for all t respectively.

The Liapunov function for the above system is given by

$$V\ (x(t),\ t) = x^T(t)\ F(t)\ x(t) + P(t) \qquad (4.2.57)$$

with $F(t) > 0$ and $P(t) \geq 0$ for all $t > t_o$. Application of the Jacobi-Bellman equation with the Hamiltonian conditions yields the following optimal control law:

$$u(t) = -\left(\sum_{i=1}^{q} D_i^T\ K\ D_i + R\right)^{-1} B^T K x \qquad (4.2.58)$$

with

$$x^T\ (\dot{K} + A^T K + KA + \sum_{i=1}^{m} C_i^T\ K\ C_i - KB\ \left(\sum_{j=1}^{q} D_j^T\ KD_j + R\right)^{-1} B^T K)\ x$$

$$+ \mathrm{tr}\ (E \Sigma K) + \dot{P} = 0 \qquad (4.2.59)$$

Since the above equation must be true for all x(t), then

$$\dot{K} = -Q - A^T K - KA - \sum_{i=1}^{m} C_i^T K C_i + KB \left(\sum_{j=1}^{q} D_j^T K D_j + R \right)^{-1} B^T K \tag{4.2.60}$$

and

$$\dot{p} = - \operatorname{tr} (EE^T K) \tag{4.2.61}$$

It can be shown [4.23] that if E(t) is a square integrable function and if (4.2.60) has a solution then (4.2.58) is the optimal control law for the system of (4.2.55) - (4.2.56).

4.2.2.3 Stabilizability of Stochastic Systems

In this subsection the probelm of stabilization of stochastic linear continuous time systems with multiplicative and additive noise will be studied. The control inputs that ensure asymptotic stability of the equilibrium solution $x(t, x_o)$ of equation (4.2.45) minimizing the performance functional of Equation (4.2.39) can be determined by using a modification of the Liapunov stability theorem on asymptotic stability and certain considerations of Bellman's dynamic programming method [4.26]. The problem then reduces to the determination of the appropriate Liapunov function V(x, t) and the optimal control law. The first can be determined by solving a partial differential equation with an additional inequality constraint (which is quite a complex problem to solve).

Consider Equation (4.2.61). Under a fixed final time t_f and with $p(t_f) = 0$ we have

$$p(t) = \int_t^{t_f} \operatorname{tr} (EE^T K)\, dt \tag{4.2.62}$$

If the $\lim_{t \to \infty} P(t)$ exists and Equation 4.2.60 has a solution K(t), then tr (K(t)) < a, for a > 0. Moreover,

$$P(t) \leq \int_{t_o}^{\infty} \text{tr } (GG^T D) \, d\tau < a \int_{t}^{\infty} \text{tr } (GG^T) \, d\tau$$

Hence if G is a square integrable function matrix for $t \geq t_o$ and the above conditions hold then the optimal control law of Equation (4.2.58) is stabilizing in the sense of stochastic asymptotic stability in the large [4.25].

Consider a simplified form of Equation (4.2.60) given by:

$$\dot{K} = Q - A^T K - KA - \sum_{i=1}^{m} C_i^T K C_i + KBR^{-1}B^T K \qquad (4.2.63)$$

The state equation corresponding to (4.2.63) is

$$\frac{dx(t)}{dt} = A(t)\; x(t) + B(t)\; u(t) + \sum_{i=1}^{m} C_i(t)\; x(t)\; \xi_i(t) \qquad (4.2.64)$$

Since it is assumed that $D_j = 0$ for all j and E = 0. It can be shown that the equation for the covariance of x(t) is given by:

$$P(t) = \Phi(t, t_o)\; P_o\; \Phi^T(t, t_o) + \int_{t_o}^{t} \Phi(t,\tau) \left[\sum_{j=1}^{m} C_j^T(\tau)\; P(\tau) C_j(\tau)\right] \cdot \Phi^T(t, \tau)\; d\tau \qquad (4.2.65)$$

Where (t, t_o) is the state transition matrix. Moreover, under the assumption of uniform controllability of $(A(t), B(t))$, $A(t) - B(t)R^{-1}(t) B(t) K(t)$ can be made stable [31] with the desired degree of stability. Thus if there is a k_1, k_2 such that $x^T_{(t)}Q(t) x(t) \geq k_1 ||x||^2$ and

$$\sum_{i=1}^{m} ||C_i||^2 \leq k \text{ for } k_1, k_2 > 0$$

Then

$$||P|| \leq ||\Phi||^2 ||P_o|| + m\, k_2 \int_{t_o}^{t} ||\Phi||^2 ||P|| \, d\tau$$

Also, $||\Phi(t, \tau)||^2 \leq k_3 e^{-k_4(t-\tau)}$ for some $k_3, k_4 > 0$.

Hence

$$||P|| e^{k_4 t} \leq k_3 ||P_o|| + \int_{t_o}^{t} m\, k_2 k_3 e^{k_4 \tau} ||P|| \, d\tau$$

and by Gronwall's inequality we have

$$||P|| e^{k_4 t} \leq k_3 ||P_o|| \exp(-m\, k_2 k_3 t) \tag{4.2.66}$$

which implies that as $t \to \infty$, $||P|| \to 0$. The above result leads to the stability of $P(t)$, the state covariance matrix with the implication that for every positive symmetric matrix $Q(t)$, for all $t \geq t_o$. The Equation (4.2.63) has a unique solution. Thus Equation (4.2.64) is stochastically stable in the large and under time-invariant situations, the stability is asymptotic [4.22].

Following Reference [4.31] we assume the following control law for the system of Equation (4.2.55) with $E = 0$:

$$u(t) = -q(t)\, B^T(t)\, S(t, t + \Delta t)^{-1}\, x(t) \tag{4.2.67}$$

where $q(t) \geq 1/2$ is a scalar piecewise continuous time function and $S(t, t + \Delta t)$ is defined by:

$$S(t, t + \Delta t) = \int_t^{t+\Delta t} e^{\alpha(t-\tau)} \Phi(t, \tau) B(\tau) B^T(\tau) \Phi^T(t, \tau) d\tau$$

with $\alpha > 0$. Let $V = x^T S(t, t + \Delta t)^{-1} x$ and

$$LV = - x^T S(t, t + \Delta t)^{-1} [\dot{S}(t, t + \Delta t) - S(t, t + \Delta t) A^T(t)$$
$$- A(t) S(t, t + \Delta t) + 2q(t) B(t) B^T(t) - S(t, t + \Delta t) \sum_{i=1}^{m} C_i^T(t)$$
$$\cdot S(t, t + \Delta t)^{-1} C_i(t) S(t, t + \Delta t)$$
$$- q^2(t) B(t) \sum_{j=1}^{q} D_j^T(t) S(t, t + \Delta t)^{-1}$$
$$\cdot D_j(t) B^T(t)] S(t, t + \Delta t)^{-1} x$$
$$= x^T W^{-1} [e^{-\alpha \Delta t} \Phi(t, t + \Delta t) BB^T \Phi^T (t, t + \Delta t) + (2q(t) -1) BB^T$$
$$+ \alpha S - S \sum_{i=1}^{m} C_i^T S^{-1} C_i S - q^2 B \sum_{j=1}^{q} D_j^T S^{-1} D_j B^T] S^{-1} x$$

If we assume that for positive constants C_1 and C_2, $0 < C_1 I \leq S(t, t + \Delta t) \leq C_2 I$ then

$$LV \leq - \alpha x^T S^{-1} x - tr (S^{-1} [(2q - 1) BB^T - S \sum_{i=1}^{m} C_i^T S^{-1} C_i S$$
$$- q^2 B \sum_{j=1}^{q} D_j^T S^{-1} D_j B^T] xx^T$$
$$\leq \frac{\alpha}{C_2} ||x||^2 + tr (\sum_{i=1}^{m} C_i^T S^{-1} C_i + q^2 S^{-1} B \sum_{j=1}^{q} D_j^T S^{-1} D_j B^T S^{-1}] xx^T$$
$$\leq - \frac{\alpha}{C_2} ||x||^2 + [\sum_{i=1}^{m} ||C_i||^2 \frac{n}{C_1} + q^2 \frac{n^3}{C_1^3} ||B||^2 \sum_{j=1}^{q} ||D_j||^2] ||x||^2$$

Moreover, if

$$\frac{n}{C_1} \sum_{i=1}^{n} ||C_i||^2 \frac{n^3 q^2}{C_1^3} ||B||^2 \sum_{j=1}^{q} ||D_j||^2 < \frac{\alpha}{C_2}$$

for all $t \geq t_o$, then

$$\frac{dx(t)}{dt} = A(t)\ x(t) + B(t)\ u(t) + \sum_{i=1}^{m} C_i(t)\ x(t)\ \xi_i(t) \qquad (4.2.68)$$

$$+ \sum_{j=1}^{q} D_j(t)\ u(t)\ \gamma_j(t)$$

is stochastically asymptotically stable in the large.

4.3 CONCLUSIONS

The optimal stochastic control law of linear dynamical systems with multiplicative and additive noise reflects the influence of uncertainties on the dynamics of the control system via the presence of the covariances of the random elements in the system matrices. Thus, the Ricatti-like equations with constant but unknown coefficients have unique solutions under restrictive conditions. Moreover, under very high uncertainty no solution of the optimal control problem can be established. This makes a great deal of intuitive sense, since if a control system is highly unknown then we cannot expect to be able to stabilize any perturbations.

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APPENDIX 4.1

For a complete proof of Theorem 4.1 we will establish the validity of Lemma 4.1.

Before we begin the proof of Lemma 4.1 we will represent the matrix A by the product of its columns. Thus, let $A_1, A_2, \ldots, A_n$ be the column vectors of A^T. Then, we can express A in the following form:

$$A = \begin{bmatrix} A_1^T \\ A_2^T \\ \vdots \\ A_n^T \end{bmatrix} \quad \text{and } A^T = [A_1 \ A_2 \ldots A_n]$$

<u>Proof of Lemma 2.1</u>

Let $A_1^T = [A_{11} \ A_{12} \ldots A_{1n}]$, $A_2^T = [A_{21} \ A_{22} \ldots A_{2n}]$ ... $A_n^T = [A_{n1} \ A_{n2} \ldots A_{nn}]$

Now, let us define a matrix given by (B - E[B/A]) where E[B/A] is the conditional probabilistic expectation of B conditioned on the known values of the elements of A. It is easy to see that B - E[B/A] is independent of A. Hence, by appropriate use of Equation (4.2.18) we can start the proof of Lemma 4.1.

$$
\begin{aligned}
E[ABA^T] &= E[A(B - E[B/A] + E[B/A])A^T] \\
&= E_{(B - E[B/A])} \quad E_A \quad A(B - E(B/A])A^T + AE[B/A]A^T/(B - E[B/A]) \\
&= E_{(B - E[B/A])} \quad (\bar{A}(B - E[B/A])\bar{A}^T + AE[B/A]\bar{A}^T \qquad (4.1) \\
&\quad + Tr(\Sigma^{AA}(B - E[B/A]) + Tr(\Sigma^{AA} E[B/A]) \\
&= \bar{A}\bar{B}\bar{A}^T + Tr(\Sigma^{AA}\bar{B}^T)
\end{aligned}
$$

$$
Tr(\Sigma^{AA}B) = \begin{bmatrix} tr(\Sigma^{A_1A_1}\bar{B}) & tr(\Sigma^{A_1A_2}\bar{B}^T) & \dots & tr(\Sigma^{A_1A_n}\bar{B}^T) \\ tr(\Sigma^{A_2A_1}\bar{B}^T) & tr(\Sigma^{A_2A_2}\bar{B}) & \dots & tr(\Sigma^{A_2A_n}\bar{B}^T) \\ \vdots & & & \\ tr(\Sigma^{A_nA_1}\bar{B}^T) & tr(\Sigma^{A_nA_2}\bar{B}^T) & \dots & tr(\Sigma^{A_nA_n}\bar{B}) \end{bmatrix} \qquad (4.2)
$$

To see how Equation (4.2) is generated we note:

$$
ABA^T = \begin{bmatrix} A_1^T \\ A_2^T \\ \vdots \\ A_n^T \end{bmatrix} B[A_1 \; A_2 \dots A_n] = \begin{bmatrix} A_1^T B A_1 & A_1^T B A_2 & \dots & A_1^T B A_n \\ A_2^T B A_1 & A_2^T B A_2 & \dots & A_2^T B A_n \\ \vdots & & & \\ A_n^T B A_1 & A_n^T B A_2 & \dots & A_n^T B A_n \end{bmatrix} \qquad (4.3)
$$

We will also require the following relation [8]

$$
E[X^T(k+1) \; FX(k+1)] = \bar{X}^T(k+1) \; F\bar{X}(k+1) + tr(F\Sigma^X(k+1)] \qquad (4.4)
$$

where $\bar{X}(k+1) = E[X(k+1)]$ and the ijth element of the matrix $\Sigma^X(k+1)$ is given, using Equation 4.2.2, by:

$$
\Sigma_{ij}^X(k+1) = X^T(k)\Sigma^{\Phi i\Phi j} X(k) + 2X^T(k)\,\Sigma^{\Phi i\omega j}u(k) + u^T(k)\Sigma^{\omega i\omega j} u(k) + \Sigma^{\xi\xi} \qquad (4.5)
$$

where $\Sigma^{\Phi i \psi j}$ is the covariance matrix of the i^{th} row of Φ with the j^{th} row of ψ.

We now, prove theorem 4.1 by dynamic programming [4.2A] [4.3A].

Proof of Theorem 4.1

Let the final cost-to-go be given by

$$V(x(N), N) = \frac{1}{2} x^T (N) Fx(N) \qquad (4.6)$$

By applying the principle of optimality, the first stage optimization cost is:

$$V(x(N-1), N-1) = \min_{U(N-1)} \underset{\substack{\theta(N-1)\\ \Gamma(N-1)\\ \xi(N-1)}}{E} [\frac{1}{2} x^T(N-1)Q(N-1)x(N-1) + \frac{1}{2} u^T(N-1)R(N-1)u(N-1) + V(x(N), N)/I(N-1)] \qquad (4.7)$$

Equation (4.7) can be rewritten in the following form:

$$V(X(N-1), N-1) = \min_{u(n-1)} E[\frac{1}{2} X^T(N-1)Q(N-1)X(N-1)U(N-1) + \frac{1}{2} u^T(N-1)R(N-1)u(N-1) + \frac{1}{2} x^T(N) F x(N)/I(N-1)]$$

$$
\underset{\bar{\bar{u}}(n-1)}{\mathrm{Min}} E[\frac{1}{2} x^T(N-1)Q(N-1)x(N-1)u(N-1)
$$

$$
+ \frac{1}{2} u^T(N-1)R(N-1)u(N-1)
$$

$$
+ \left(\begin{bmatrix} A(N-1) & 0 \\ 0 & (N-1) \end{bmatrix} x(N-1) + \begin{bmatrix} B(N-1) \\ \Gamma(N-1) \end{bmatrix} u(N-1)\right.
$$

$$
\left. + \begin{bmatrix} 0 \\ \xi_o(N-1) \end{bmatrix}\right)^T F \left(\begin{bmatrix} A(N-1) & 0 \\ 0 & \theta(N-1) \end{bmatrix} X(N-1)\right.
$$

$$
\left. + \begin{bmatrix} B(N-1) \\ \Gamma(N-1) \end{bmatrix} u(n-1) + \begin{bmatrix} 0 \\ \xi(N-1) \end{bmatrix}\right) \Big/ I(N-1)]
$$

$$
\underset{\bar{\bar{u}}(n-1)}{\mathrm{Min}} \frac{1}{2} \left(\left[E \; [x^T(N-1)Q(N-1)x(N-1) + u^T(N-1)R(N-1)u(N-1)]\right.\right.
$$

$$
+ E\left[x^T(N-1)\begin{bmatrix} A(N-1) & 0 \\ 0 & \theta(N-1) \end{bmatrix}^T F \begin{bmatrix} A(N-1) & \\ 0 & \theta(N-1) \end{bmatrix}\right.
$$

$$
\left. \cdot\, x(N-1)\right] + E\left[x^T(N-1)\begin{bmatrix} A(N-1) & 0 \\ 0 & \theta(N-1) \end{bmatrix}^T F \begin{bmatrix} B(N-1) \\ \Gamma(N-1) \end{bmatrix} u(N-1)\right]
$$

$$
+ E\left[x^T(N-1) \begin{bmatrix} A(N-1) & \\ 0 & \theta(N-1) \end{bmatrix}^T F \begin{bmatrix} 0 \\ \xi_o(N-1) \end{bmatrix}\right]
$$

$$
+ E\left[u^T(N-1) \begin{bmatrix} B(N-1) \\ \Gamma(N-1) \end{bmatrix}^T F \begin{bmatrix} A(N-1) & 0 \\ 0 & \theta(N-1) \end{bmatrix} x(N-1)\right]
$$

$$
+ E\left[u^T(N-1) \begin{bmatrix} B(N-1) \\ \Gamma(N-1) \end{bmatrix}^T F \begin{bmatrix} B(N-1) \\ \Gamma(N-1) \end{bmatrix} u(N-1)\right]
$$

$$
+ E\left[u^T(N-1) \begin{bmatrix} B(N-1) \\ \Gamma(N-1) \end{bmatrix}^T F \begin{bmatrix} 0 \\ \xi_o(N-1) \end{bmatrix}\right] + E\left[\begin{bmatrix} 0 \\ \xi(N-1) \end{bmatrix}^T\right.
$$

$$
\left. \cdot \left[F \begin{bmatrix} A(N-1) & 0 \\ 0 & \theta(N-1) \end{bmatrix} x(N-1)\right]\right.
$$

$$
+ E\left[\begin{bmatrix} 0 \\ \xi(N-1) \end{bmatrix}^T F \begin{bmatrix} B(N-1) \\ \Gamma(N-1) \end{bmatrix} u(N-1)\right] + E\left[\begin{bmatrix} 0 \\ \xi(N-1) \end{bmatrix}^T F\right.
$$

$$
\left. \begin{bmatrix} 0 \\ \xi(N-1) \end{bmatrix}\right] \tag{4.8}
$$

To minimize Equation (4.8) over u(N-1) we set the derivative with respect to u(N-1) to zero and obtain:

$$u^*(N-1) = -\left[R(N-1) + \begin{bmatrix} B(N-1) \\ \Gamma(N-1) \end{bmatrix}^T F \begin{bmatrix} B(N-1) \\ \Gamma(N-1) \end{bmatrix} + Tr\left(\begin{bmatrix} 0 & 0 \\ 0 & \xi^{\Gamma\Gamma} \end{bmatrix} F\right)\right]^{-1} \begin{bmatrix} B(N-1) \\ \Gamma(N-1) \end{bmatrix}^T$$

$$\cdot F \begin{bmatrix} A(N-1) & 0 \\ 0 & \theta(N-1) \end{bmatrix} + Tr\left(\begin{bmatrix} 0 & 0 \\ 0 & \xi^{\theta\Gamma} \end{bmatrix} F\right) \Bigg] \; x(N-1) \qquad (4.9)$$

Substituting Equation (4.9) back into Equation (4.8) yields

$$V(x(N-1),\ N-1) = \frac{1}{2} x^T(N-1)K(N-1)x(N-1) + tr\left(\begin{bmatrix} 0 & 0 \\ 0 & \Sigma^{\xi\xi} \end{bmatrix} F\right) \qquad (4.10)$$

where

$$K(N-1) = Q(N-1) + \begin{bmatrix} A(N-1) & 0 \\ 0 & \bar{\theta}\ (N-1) \end{bmatrix}^T F \begin{bmatrix} A(N-1) & 0 \\ 0 & \bar{\theta}\ (N-1) \end{bmatrix} + Tr\left(\begin{bmatrix} 0 & 0 \\ 0 & \Sigma^{\theta\theta} \end{bmatrix} F\right)$$

$$- \left(\begin{bmatrix} A(N-1) & 0 \\ 0 & \bar{\theta}(N-1) \end{bmatrix}^T F \begin{bmatrix} B(N-1) \\ \bar{\Gamma}(N-1) \end{bmatrix} + Tr\left(\begin{bmatrix} 0 & 0 \\ 0 & \Sigma^{\theta\Gamma} \end{bmatrix} F\right)\right)$$

$$\left[R(N-1) + \begin{bmatrix} B(N-1) \\ \bar{\Gamma}(N-1) \end{bmatrix}^T F \begin{bmatrix} B(N-1) \\ \bar{\Gamma}(N-1) \end{bmatrix} + Tr\left(\begin{bmatrix} 0 & 0 \\ 0 & \Sigma^{\Gamma\Gamma} \end{bmatrix} F\right)\right]^{-1}$$

$$\left(\begin{bmatrix} B(N-1) \\ \bar{\Gamma}(N-1) \end{bmatrix}^T F \begin{bmatrix} A(N-1) & 0 \\ 0 & \bar{\theta}(N-1) \end{bmatrix} + Tr\left(\begin{bmatrix} 0 & 0 \\ 0 & \Sigma^{\theta\Gamma} \end{bmatrix} F\right)\right) \qquad (4.11)$$

Repeating the above procedure for time N-2 we get:

$$V(K-2,\ N-2) = \underset{\bar{u}(n-2)}{Min} E\ \frac{1}{2} x^T(N-2)Q(N-2)x(N-2) + u(N-2)R(N-2)u(N-2)U(N-2)$$

$$+ V(x(N-1),\ N-1) \qquad (4.12)$$

We note that the structure of the optimal policy rule derived in Equation (4.9) is modified to incorporate the difference between the cost expressions at times (N-1) and (N-2) respectively [4.4A]. Thus, after the usual repetition of the above steps we receive:

$$U(N-2) = -\left[R(N-2) + \begin{bmatrix} B(N-2) \\ \bar{\Gamma}(N-2) \end{bmatrix}^T K(N-1) \begin{bmatrix} B(N-2) \\ \bar{\Gamma}(N-2) \end{bmatrix} + \mathrm{Tr}\left(\begin{bmatrix} 0 & 0 \\ 0 & \Sigma^{\Gamma\Gamma} \end{bmatrix} K(N-1)\right)\right]^{-1} \left(\begin{bmatrix} B(N-2) \\ \bar{\Gamma}\ (N-2) \end{bmatrix}^T K(N-1) \begin{bmatrix} A(N-2) & 0 \\ 0 & \bar{\theta}(N-2) \end{bmatrix} + \mathrm{Tr}\left(\begin{bmatrix} 0 & 0 \\ 0 & \Sigma^{\theta\Gamma} \end{bmatrix} K(N-1)\right)\right) x(N-2) \tag{4.13}$$

The functional Equation 4.12) is now rewritten as

$$V(x(N-2),\ N-2) = \frac{1}{2}\, x^T(N-2K(N-2)x(N-2) + \mathrm{Tr}\left(\begin{bmatrix} 0 & 0 \\ 0 & \Sigma^{\xi o \xi o}_{(N-1)} \end{bmatrix}\right) \tag{4.14}$$

where

$$K(N-2) = Q(N-2) + \begin{bmatrix} A(N-2) & 0 \\ 0 & \bar{\theta}(N-2) \end{bmatrix}^T K(N-1) \begin{bmatrix} A(N-2) & 0 \\ 0 & \bar{\theta}(N-2) \end{bmatrix} + \mathrm{Tr}\left(\begin{bmatrix} 0 & 0 \\ 0 & \Sigma^{\theta\theta}_{(N-1)} \end{bmatrix} K(N-1)\right) - \left(\begin{bmatrix} A(N-2) & 0 \\ 0 & \bar{\theta}(N-2) \end{bmatrix}^T K(N-1) \begin{bmatrix} B(N-2) \\ \bar{\Gamma}(N-2) \end{bmatrix} + \left[\mathrm{Tr} \begin{matrix} 0 & 0 \\ 0 & \Sigma^{\theta\Gamma}_{(N-1)} \end{matrix}\right] \cdot K(N-1)\right)\right) \left[R(N-2) + \begin{bmatrix} B(N-2) \\ \bar{\Gamma}(N-2) \end{bmatrix}^T K(N-1) \begin{bmatrix} B(N-2) \\ \bar{\Gamma}(N-2) \end{bmatrix}\right.$$

$$+ \operatorname{Tr}\left(\begin{bmatrix} 0 & 0 \\ 0 & \Sigma^{\Gamma\Gamma}_{(N-1)} \end{bmatrix} K(N-1)\right)\Bigg]^{-1} \left(\begin{bmatrix} B(N-2) \\ \bar{\Gamma}(N-2) \end{bmatrix}^{T} K(N-1)\right.$$

$$\left.\begin{bmatrix} A(N-2) & 0 \\ 0 & \bar{\theta}(N-2) \end{bmatrix} + \operatorname{Tr}\left(\begin{bmatrix} 0 & 0 \\ 0 & \Sigma^{\theta\Gamma}_{(N-1)} \end{bmatrix} K(N-1)\right)\right) \tag{4.15}$$

Since the structural form of Equation (4.15) is exactly the same as that of Equation (4.10), by induction, with the usual steps of dynamic programming we get the following recursively general equations for the control law at any time k.

$$u^{*}(k) = - G^{*}(k)\; x^{*}(k) \tag{4.16}$$

where

$$G^{*}(k) = [R(k) + E[\psi^{T}(k)\; K(k+1)\; \psi(k)]]^{-1}\; E[\psi^{T}(k)\; K(k+1)\Phi(k)] \tag{4.17}$$

and K(k) is given by the matrix Riccati-like difference equation that follows:

$$K(k) = Q(k) + E[\Phi^{T}(k)\; K(k+1)\Phi(k)] - E[\Phi^{T}(k)\; K(k+1)\psi(k)]\; [R(k)$$
$$+ E[\psi^{T}(k)\; K(k+1)\psi(k)]]^{-1}\; E[\psi^{T}(k)\; K(k+1)\Phi(k)] \tag{4.18}$$

with

$$K(N) = F \tag{4.19}$$

The recursive functional equation for the optimal cost can now be written as [4.5A].

$$V(x(N), N) = x^{T}(N) F x(N)$$
$$V(x(k), k) = \frac{1}{2}\, x^{T}(k) K(k) x(k) + \sum_{i=k}^{N-1} \operatorname{tr}\left(K(i+1)\begin{bmatrix} 0 & 0 \\ 0 & \Sigma^{\xi o \xi o} \end{bmatrix}\right) = J^{*}(u) \tag{4.20}$$

Remark 4.1

The quantities $Tr(\Sigma^{\theta\psi}(N-1)\ K(k+1))$ and $Tr(\Sigma\ \ (N-1)$ $K(k+1))$ are monotonic and both satisfy the conditions that follow.

If there exists a K_k sequence such that $\lim_{k\to\infty} K_k \to K$ then $\lim_{n\to\infty} Tr(\Sigma^{\Phi\psi} K_n(k+1)) \to TR(\Sigma^{\psi\psi} K(k+1))$.

Also, $\lim_{n\to\infty} TR\ (\Sigma^{\psi\psi} K_n(k+1)) \to Tr(\Sigma^{\psi\psi} K(k+1))$.

Now, if we define

$$F_G(k) = Q^TQ + G^TRG + Tr(\Sigma^{\Phi\Phi}K(k+1) + G^T Tr(\Sigma^{\psi\psi}K(k+1))\ G + (\Phi-\psi G)^T K(\Phi-\psi G) \tag{4.21}$$

where G is given by Equation (4.17). Then it is proved that [4.5A]

$$J^*(k) = x^T(k)K(k+1)x(k) + \sum_{k=o}^{N-1} Tr(\Sigma^{\xi\xi}K(k+1)) \tag{4.22}$$

In particular, if G is fixed for $k = 0,1,\ldots,N-1$ and $F_G(k) = K$, then

$$J^* = x^TKx + N\ Tr(\Sigma^{\xi\xi}K) \tag{4.23}$$

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CHAPTER 5

FORMULATION OF MEASUREMENT SYSTEM AND DETERMINATION OF CONTROLLABILITY, OBSERVABILITY, AND DETECTABILITY

SECTION 5.1
INTRODUCTION

Data and information gathering has been of great concern for engineers and scientists for the past several decades and continue to be so for the present. Moreover, the futuristic complex control systems that involve multidimensional, large-scale dynamic couplings present the need for high accuracy observation systems for reliable information and decision making.

An observation system designed for data or information gathering can serve one of two main purposes. Namely, estimation of system states and parameters, and hypotheses testing or decision making [5.1]. Decision theory is applicable to a wide range of problems ranging from radar target detection and identification, to stock exchange and economic decisions. The decisions are of the "yes" or "no" types, referred to as H_0 and H_1 hypotheses in communication theory. Each of the hypotheses in a given situation results from an observation source which generates one of the hypotheses as an output. Here, the hypotheses are observed only indirectly and a probabilistic "transition mechanism" sorts the hypotheses out from the observations. This mechanism assumes a priori knowledge about which hypothesis is true and generates points in the observation space based on some rule of probability. Then, utilizing the observation points or elements, a decision is made on the basis of some decisiion rule as to which hypothesis to accept, given the probability measures of the various hypotheses.

On the other hand, estimating the states and parameters of a system from observations that are noisy, or are virtually determiistic, or are partly noisy and partly exact, is a nontrivial problem. The main purpose of state estimation is reconstruction of the missing states, while for parameter estimation, the underlying idea is to formulate an appropriate model that fits the given or measured characteristics of the system at hand.

During the initial phase of the design of a feedback control system, it is convenient to assume that the entire state vector is available via measurements. However, in most complex control systems, constraints on sensors and sensor locations render only partial state measurements. In the case of linear optimal control problems with quadratic performance functionals, it is well known that [5.2] the optimal control law is of the feedback type and it utilizes all of the states as feedback. When only some of the states are available from measurements, full state feedback cannot be implemented in the usual sense of the word. Thus, either a different approach, accounting for the partial measurements, must be developed or approximations of the missing states will have to be utilized for feedback. Invariably, this latter approach is followed and various techniques exist for estimating approximate values of the missing states from measurements of the available states.

The latter method results in the decomposition of the control design problem into two main tasks [5.3]. During the initial task, the control law is designed under the assumption of full state availability. Here, the control law derivation can be carried out via optimization or some other technique. In all cases, the resulting control law is normally without dynamics. The second step is the control system design with an approximation to the state vector. That part of the control system design which produces the approximation to the missing states is called an observer [5.4] in the deterministic setting and a Kalman filter in the case of additive white Gaussian noise contaminating the system and the measurement dynamics [5.5].

The optimal state estimation problem has recently attracted considerable attention, especially relating to complex dynamical systems with appreciable process noise. For linear systems with additive white Gaussian noise, the optimal unbiased minimum viariance estimator was first derived by Kalman and Bucy [5.6] and it has been the subject of extensive research and applications.

In many practical applications, the system dynamics and the measurement system are inherently nonlinear and the truly optimal nonlinear filter entails infinite dimensional systems for their solution [5.7]. Thus, methods for approximating the a-posteriori probability density functions based on perturbations and Taylor's series expansions have been developed [5.8]. In nonlinear filtering applications, first-order extended Kalman filers yield valid results under bounded (small) second-order perturbation terms.

The overall problem involving an appropriate measurement system for the purose of data and information gathering is very complex. The procedure entails test planning, instrumentation specification, choice of an appropriate (linear or nonlinear) mathematical model, a particular numerical technique suitable for the task at hand, and a special statistical approach for interpretation of results. However, most of the time, linear quadratic Gaussian models are found to be satisfactory and the usual Kalman-Bucy filtering algorithms give sufficiently accurate results for state vector estimates. However, in some situations, some of the measurement output can be considered practically noise-free. This type of system often leads to singular measurement covariance matrices and such ill-conditioned matrices create numerical problems [5.9]. If the output variables are noise free and the system is linear, then a Luenberger observer can be utilized for the state estimates [5.3]. When all the output variables are corrupted by additive white Gaussian noise, then a Kalman filter is used [5.4] to reconstruct the states.

There are many situations in which the measurement system can be considered linear. Furthermore, part of the measurement output can be taken as deterministic and the remaining part as stochastic. In some cases, the measurement noise influences the output dynamics in a state- and control-dependent fashion, in addition to the regular additive noise that contaminates the output [5.10-5.11]. Moreover, measurement systems can be treated in discrete-time or continuous-time settings, depending on the specific application, instrumentation, and implementation. In particular, measurement systems with multiplicative noise arise in applications of signal processing (in phenomena related to reflection of signals from the ionosphere), sampling (random data and measurement sampling), and gating or amplitude modulation.

Many researchers have presented various situations involving linear measurement systems with multiplicative noise. De Koning [5.11] derived the optimal linear state estimation algorithm for linear discrete-time systems with multiplicative measurement noise, while Tugnait [5.12] derived conditions regarding the uniform asymptotic stability of the optimal linear continuous filter with an observation system that is contaminated by white multiplicative noise. In References [5.13-5.14], optimal linear one-stage predictors are generated for continuous and discrete systems, respectively, where multiplicative noise contaminates the measurement outputs.

In this chapter, a general measurement system, both in the discrete-time and continuous-time domains, is presented whereby the measurement output is a linear function of the states and the state measurement matrix includes random Gaussian elements. Some extensions to the theory of stochastic controllability and observability have been achieved in the area of linear systems with multiplicative and additive noise contaminating not only the state equations, but also the observation system.

SECTION 5.2
GENERAL MEASUREMENT SYSTEM

The most general measurement system is a nonlinear stochastic system where the measurement output is a function of the states, the controls, and time. Moreover, some additive noise may be involved that will compensate for the modeling errors and other random effects. Thus, a measurement system of the It_o [5.15] type can be written as follows:

$$dy(t) = f(x(t), u(t), t)dt + g(x(t), u(t), t)dw(t) \tag{5.1}$$

where the observed process $\{y(t), t \geq t_o\}$ is an m-vector stochastic process, f and g are mxn, random, continuous matrix time-function processes and $\{w(t), t \geq t_o\}$ is an m-vector Brownian motion process with $E[w(t)w^T(t)] = R(t)dt$, $R(t)>0$. Moreover, $x(t_o)$, and w(t) are assumed independent of each other and of the noise dynamics of the state vector.

In a similar manner, the discrete-time nonlinear stochastic measurement system may be written as follows:

$$y_k = \phi(x_k, u_k, k) + \Gamma(x_k, u_k, k)\nu_k \tag{5.2}$$

where y_k is the m-dimensional observation vector, Φ and Γ are nxn and mxn random, bounded matrix functions respectively, and $\{\nu_k, k = 1,2,...\}$ is an m-dimensional white Gaussian noise sequence, $\nu_k \sim N(0,R_k)$, $R_k>0$. Similarly, x_o is Gaussian with $x_o \sim N(\hat{x}_o, P_o)$. x_o and ν_k are independent. If there is no white Gaussian noise assumption, then Equation (5.1) can be written in a more general setting as follows:

$$z(t) = \frac{dy(t)}{dt} = f(x(t),u(t),T) + g(x(t),u(t),t)\ \gamma(t) \qquad (5.3)$$

where

$$\gamma(t) = \frac{dw(t)}{dt}$$

A similar generalization is true for Equation (5.2).

Since the well-known monogram of K. Itô on Itô stochastic differential and integral equations, the rigorous formulation of complex dynamic systems with a high degree of uncertainty and randomness has taken a new outlook. The determination of the Kolmogorov [5.16] diffusion process has now been facilitated through Ito integral equations. Thus, when the linear dynamical system associated with the linear version of Equation (5.1) is given by:

$$dx(t) = F(t)\ x(t)dt + G(t)d\ \beta(t) \qquad (5.4)$$

then Kolmogorov's forward equation becomes

$$dp(x,t/Y_t) = L(p)dt \quad t_k \le t < t_{k+1} \qquad (5.5)$$

where

$$L(\cdot) = -\sum_{i=1}^{n} \frac{\partial(\cdot, f_i)}{\partial x_i} + 1/2 \sum_{i,j=1}^{n} \frac{\partial^2(\cdot, (GQT)_{ij})}{\partial x_i\ \partial x_j} \qquad (5.6)$$

becomes

$$\frac{\partial p}{\partial t} = -ptr(F) - P_x{}^T F\ x + 1/2\ tr(GQG^T P_{xx}) \qquad (5.7)$$

yielding the conditional probability density function [5.17].

SECTION 5.3
LINEAR NOISY MEASUREMENTS

<u>The continuous Time Case</u>

The simplest measurement system is given by the linear time-invariant equation

$$y(t) = C\,x\,(t) \tag{5.8}$$

where $y(t)$ is an m-dimensional measurement output vector and C is the constant mxn measurement matrix and $x(t)$ is the n-dimensional state-vector that is the solution to the following time-invariant dynamic equation:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{5.9}$$

where A is an nxn and B an nxm constant matrices respectively. The dimension of the observation vector $y(t)$ is in general lower than that of the state vector. The main reason being that in practice it seldom occurs when the whole state vector is thoroughly (directly or indirectly) measurable. Thus, the realistic approach is to have $m<n$, whereby a set of linear combinations of the state vector components, that is lower in dimension than the state vector, is available from the measurement output that is reflected via Equation (5.8).

For more realistic situations that involve systems of relatively high complexity, the appropriate measurement system that is usually considered and widely practiced is represented by an equation similar to Equation (5.8) with an additive noise element representing the inaccuracy or the error in the measurement sensors. Thus, a "stochastic" linear measurement system with only additive noise is expressed as follows:

$$y(t) = C\,x(t) + \gamma(t) \tag{5.10}$$

with $\gamma(t)$ an m-dimensional random noise vector that is usually taken to be a zero-mean, white Gaussian noise with a covariance matrix given by:

$$\mathrm{cov}(\gamma(t)) = E[\gamma(t)\,\gamma^T(\tau)] = R(t)\,\delta(t-\tau)$$

where R(t) is an mxm positive definite matrix and E[·] is the statistical expectation operator. Furthermore, the linear dynamic system equation that expresses the diffeential behavior of the state vector is given by:

$$\dot{x}(t) = Ax(t) + Bu(t) + E\xi(t) \tag{5.11}$$

with E an nxn constant matrix and $\xi(t)$ an n-dimensional (usually) zero-mean, white Gaussian noise vector with covariance of

$$E[\xi(t)\,\xi^T(\tau)] = Q(t)\,\delta(t-\tau)$$

Also, it is usually assumed that the initial state vector $x(t_o)$ is a random vector with zero-mean and covariance of

$$E[x(o)\;x^T(o)] = P(o) = \text{a known, nxn, covariance matrix.}$$

The most general linear measurement system that is closer to reality than the previous ones is the stochastic model given by:

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} C(t) \\ \theta(t) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \nu(t) \end{bmatrix} \tag{5.12}$$

where y_1, y_2 are the exact and noisy parts of the measurement output vector y(t) and y_1 is p-dimensional and y_2 is (m-p)-dimensional, C is a pxn- and θ is a (n-p) x n-dimensional matrices with C deterministic and θ a random matrix with zero-mean, white Gaussian noise elements of known covariance of $E[\theta(t)\theta^T(\tau)]\Sigma = \delta^{\theta\theta}(t-\tau)$. Also, $\nu(t)$ is a p-dimensional zero-mean white Gaussian noise vector with covariance of $E[\nu(t)\nu^T(\tau)] = V(t)\,\delta(t-\tau)$. Thus, a part of the measurements are assumed to be perfect (the noise level is minimal) and the rest of the measurement output is considered noisy, with state-dependent noise.

The corresponding state equation for Equation (5.12) will have the following form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A(t) & 0 \\ 0 & \Phi(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B(t) \\ \Gamma(t) \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ \gamma(t) \end{bmatrix} \quad (5.13)$$

where, similar to Equation (5.12), $\Phi(t)$ is an $(n_2 x n_2)$ random matrix with zero-mean white Gaussian noise elements and known covariance of $\Sigma^{\Phi\Phi}\delta(t-\tau)$, and that of Γ given by $\Sigma^{\Phi\Phi}\delta(t-\tau)$. Here δ is the delta function and Γ is of appropriate dimension [5.10].

The Discrete-Time Case

Similar to the continuous-time case, the discrete-time observation system is given in the simplest form by

$$y(k) = H\,x(k) \quad (5.14)$$

where x is the state vector, y the output vector (with dimension usually less than that of x) and H is a constant matrix with appropriate dimensions. When the output is also a function of the inputs, then (5.14) is written as

$$y(k) = H\,x(k) + Eu(k) \tag{5.15}$$

where E is a constant matrix of appropriate dimensions and u is the control input vecotr with dimension less than (normally) that of the state vector and different from the dimension of the output vector y. The dynamics equation in the state-space form for the discrete-time case is given by:

$$x(k+1) = F(k)\,x\,(k) + G(k)\,u(k) \tag{5.16}$$

where F and G are coussant matrices of appropriate dimensions.

The interconnection between the discrete-time and the continuous-time systems is given as follows:

$$x(t_{i+1}) = \Phi(t_{i+1},t_i)x(t_i) + \left[\int_{t_i}^{t_{i+1}} \Phi(t_{i+1},\tau)B(\tau)d\tau\right]u(t_i) \tag{5.17}$$

where $\Phi(t,t_o)$ is the state-transition matrix of the system in Equation (5.9). Also, $u(t_i) = u(t)$, $t_i \le t < t_{i+1}$ $i = 0,1....$ To transform Equation (5.9) into Equation (5.15), let

$$F = \Phi(t_{i+1},\, t_i)$$

$$G = \int_{t_i}^{t_{i+1}} \Phi(t_{i+1},\, s)\ Bds$$

$$H = C\ (t_i',\, t_i)$$

$$E = C\int_{t_i}^{t_i'} \Phi(t_i',\, s)\ Bds + D(t_i')$$

when the sampling periods are equally spaced, $t_{i+1} - t_i = \Delta$, and $t_i' - t_i = \Delta'$, then $F = e^{A}\Delta$, $G = (\int_o^{\Delta} e^{As}ds)B$, $H = Ce^{A\Delta'}$ and $E = C(\int_o^{\Delta W'} e^{As}ds)B+D$, see [5.18] for details.

The most general linear discrete-time measurement system can be written as follows [5.19]:

$$y(k) = \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} H(k) \\ \Omega(k) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ \nu(k) \end{bmatrix} \qquad (5.18)$$

where y_1 is the l-dimensional vector of exact measurements, y_2 is the q-dimensional vector of noisy measurements, H is the lxn deterministic matrix and $\Omega(k)$ the qxn stochastic matrix of Gaussian elements with given statistics of:

$$E[\Omega(k)] = \bar{\Omega}(k)$$

and

$$E[(\Omega(k) - \bar{\Omega}(k))(\Omega(i) - \bar{\Omega}(i))^T] = \Sigma^{\Omega\Omega}(k)\delta_{ki}$$

Also, $\nu(k)$ is the q-dimensional vector of Gaussian (usually white) noise with statistics of

$$E[\nu(k)] = 0$$

and

$$E[\nu(k)\nu^T(i)] = \Sigma^{\nu\nu}(k)\delta_{ki}$$

and ν is assumed statistically independent of all the other random variables.

SECTION 5.4
STOCHASTIC CONTROLLABILITY

The question of determining stochastic controllability in linear time-dependent systems with multiplicative and additive noise has been the subject of extensive research during the past decades. The fundamental issue in the determination of controllability, with even the best definition, is the derivation of sufficiently simply and easy-to-use algebraic conditions that can be applied as a test for stochastic controllability.

Work in this field commenced primarily by Kalman [5.20] and later was extended by Connors [5.21]. Stochastic continuous-time systems have been considered by several authors [5.22-5.23]. It was Harris [5.24] who extended the work on discrete-time controllability under multiplicative noise. We will present herein a brief description of controllability of stochastic linear dynamical systems under multiplicative and additive noise.

In general, it is said that a system is stochastically controllable if "the initial state of the stochastic system under consideration can be transferred, in some stochastic sense, to any desired state within a finite time by some control action." We base our arguments on this definition.

5.4.1 Continuous-Time Case

For a linear deterministic time-dependent control system similar to Equation (5.9) with A and B both functions of time, if there exist t_0 and t_1, with $0 < t_0 < t_1 < \infty$ such that the following "controllability Gramian" is nonsingular:

$$W(t_0,t) = \int_{t_0}^{t} \Phi(t,\tau)B(\tau)B^T(\tau)\Phi^T(t,\tau)d\tau \qquad (5.19)$$

(where $\Phi(t,t_o)$ is the transition matrix of the system), then the system is said to be completely controllable [5.25]. In the case that A and B are constants, the corresponding controllability condition is that the matrix $[B, AB, A^2B, \ldots A^{n-1}B]$ is of full rank (n).

In the case of linear quadratic Gaussian (LQG) systems of the form of Equation (5.11) with A and B functions of time, the controllability condition is more involved [5.26]. The basic controllability criterion is the existance of a piecewise continuous control function u(•) such that the solution of the system differential equation with initial state of $x(t_o) = x_o$ can be transferred to any final state $x(t_1) = x_1$ in some finite time t_1 via the application of a controller u(t) for t in $[t_o, t_1]$.

It is easy to see that if a system such as

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{5.20}$$

is completely controllable, then one controller that can transfer the initial state to any final state in a finite time is given by:

$$u(t) = -B^T(t)\Phi^T(t_o,t)[W(t_o,t_1)]^{-1}[x_o - \Phi(t_o,t_1)x_1] \tag{5.21}$$

If W is singular and of rank $k < n$, then (n-k) of the states will be controllable.

For systems of the form of Equation (5.11), stochastic controllability is defined as the condition whereby positive numbers a and b in $[o,^{\infty}]$ as well as a time interval $[t_o,t_1]$ can be found, such that for all t inside the interval $[t_o,t_1]$ the following hold:

$$a\,I - \int_{t_o}^{t_1} \Phi(t,s)\, E\, Q(s)\, E^T\Phi^T(t,s)ds \tag{5.22}$$

is negative-semidefinite and simultaneously

$$\int_{t_o}^{t_1} \Phi(t,s)\, E\, Q(s) E^T \Phi^T (t,s)^T ds^T - bI \tag{5.23}$$

is positive-semidefinite, where I is the identity matrix. The reason for the above conditions is due to the input noise, since the noise effects all the states. Thus, for stochastic controllability, the integral in Equations (5.22, 5.23) must be bounded above and below and it should be positive-definite for the time interval under consideration.

There are two main definitions to stochastic controllability for systems with multiplicative and additive noise, such as Equation (5.13). One of the definitions is referred to as ε-controllability and the other controllability with probability one [5.23]. These two definitions with their conditions will be briefly presented.

Definition 5.1 - An initial state x_o for a system such as Equation (5.13) is stochastically ε-controllable in probability ρ, in the interval $[t_o,t_1]$, if there exists a controller $u(t,\cdot)$ such that

$$pr[\,||x(t_1)||^2 \geq \varepsilon \,|\, x(t_o) = x_o] \leq 1-\rho \tag{5.24}$$

where the norm is the usual Euclidean norm and $0<\rho<1$. If, in addition, the above definition is true for all $x_o \varepsilon R^n$, then it is called completely ε-controllable.

The approach taken in showing ε-controllability is by finding a Lyapunov-like function $V(x,t)$ that has bounded first and second derivatives with respect to every component of x, and a first derivative with respect to $t = t_1$, in the interval of interest. Moreover, $V(x,t_1) \geq \frac{1}{\alpha}/\, x^T(T_1\, x\, (t_1)$, where $o<\alpha<<\epsilon$ and $V(x_o,t_o) \leq (1-\rho)\varepsilon/\alpha$. When these conditions (along with negative definiteness constraints [5.23] are satisfied, the system can be proven to be stochastically ϵ-controllable.

Definition 5.2 - The initial state x_o of the system (5.13) is stochastically controllable with probability one in the interval $[t_o, t_2]$ if there is a controller u such that

$$\Pr[\lim_{t \to t_1} ||x(t)|| = 0 \mid x(t_o) = x_o] = 1$$

This definition too can be extended to complete controllability. Sufficient conditions for such stochastic controllability are as follows: (5.1) V(x,t) exists and satisfies the boundedness of the aforementioned derivatives, (5.2) for all functions f(t) such that $\lim_{t \to t_1} f(t),t) \neq 0$ the following holds

$$\lim_{t \to t_1} V(f(t),t) \to \infty$$

and an additional constraint related to the Jacobi-Bellman Equation [5.23].

It is a nontrivial task to check for stochastic controllability with probability one or even ϵ-controllability. The appropriate Lyapunov-type function is often very difficult to find. However, stochastic controllability checks are sometimes essential in order to avoid extensive analysis.

5.4.2 Discrete-Time Case

In the constinuous-time situation, controllability can be defined as the propety that any initial state can be transferred to the zero state in finite time. However, such a definition will make a discrete-time system that is always zero (: x (k+1) = 0) controllable. To avoid such a confusing situation, we take the definition of transferring the initial state to any final state in a finite number of steps. Such a behavior can be clarified further by the following development. Consider a time-dependent discrete system given by

$$x(k+1) = A(k)\, x(k) + B(k)u(k) \tag{5.25}$$

The solution to this system is given by:

$$x(k) = \Phi(k,i)\, x(i) + \sum_{j=i}^{k-1} \Phi(k,j+1)B(j)u(j) \tag{5.26}$$

for $k \geq i+1$. Here $\Phi(k,i)$ is given by

$$\Phi(k,i) = \begin{cases} A(k-1)A(k-2) \ldots A(i) & \text{for } k \geq i+1 \\ I \quad \text{for } k=i \end{cases}$$

In the case when $A(k) = A$ for all k then

$$\Phi(k,i) = A^{k+i} \tag{5.27}$$

The controllability of system (5.25) can be expressed as follows:

Equation (5.25) is completely controllable if and only if the following (controllability Gramian) symmetric, nonnegative-definite matrix

$$W(i,k) = \sum_{j=i}^{k-1} \Phi(k,j+1)B(j)B^T(j)\Phi^T(k,j+1) \tag{5.28}$$

is nonsingular. Uniform controllability can also be singularly defined [5.18]. For the time invariant case like that of (5.16), complete controllability holds if and only if the following controllability matrix $[G, FG, F^2G, \ldots, F^{n-1}G]$ has full rank (n).

When a stochastic linear system with only additive noise is considered, such as

$$x(k+1) = F(k)x(k) + G(k)u(k) + E(k)\omega(k) \quad (5.29)$$

with $\omega(k)$ a zero-mean white Gaussian noise sequence and with known covariance of:

$$E[\omega(k)\omega^T(j)] = Q_d(k)\delta_{kj}$$

then system (5.29) is stochastically controllable if there exists a and b in the interval $(1,\infty)$, and a positive integer N such that, for all $k \geq N$

$$aI \leq \sum_{j=k-n+1}^{k} \Phi(k,j)E(j)Q_d(j)E^T(j)\Phi^T(k,j) \leq bI \quad (5.30)$$

The condition implies complete controllability with respect to the sequence of the noise $\omega(k)$ [5.26].

The more general situation is the one with multiplicative and additive noise (the discrete-time version of Equation (5.13)) and the controllability of such systems is very complex. Similar to the continuous-time situation, the Lyapunov function approach is the most convenient and popular: a few definitions regarding discrete-time controllability are in order.

Definition 5.3 - The initial state x_o of a discrete-time stochastic linear system like Equation (5.13) is stochastically ε-controllable with respect to a final state x_f, with probability ρ, in the Euclidean norm sense, in the time interval $[o,N]$ (for a fixed N) if there is a control law $u_{[o,N]} = \{u(x_n,n),\ n = o,1,\ldots,N-1\}$ such that prob $\{|| x_N - x_j || \geq \varepsilon \mid x_o\} \leq \rho \quad o < \rho < 1$ and $u \varepsilon UCE^{(m)}$. If the last two conditions hold true for every initial state x_o, then the system is called stochastically completely controllable.

The following is proved in [5.24]. An initial state x_o is stochastically ε-controllable with respect to a known final state x_f, if all of the following statements hold true:

a. A scalar non-negative definite function $V(x,k)$ can be defined in $[o,N]$, such that for $\alpha > o$ and $\alpha >> \varepsilon$,

$$V(x_n - x_f, N) \geq [\frac{1}{a_\sigma 2}(x_N - x_f)^T (x_N - x_f)]^{1/2}$$

b. $V(x_o - x_f, 0) \leq \rho \, \varepsilon / \alpha$

c. A control law u exists such that

$$E[V(x_{n+1} - x_f, n+1) - V(x_n - x_f, n) \mid x_n] \leq 0$$

with probability 1 on the set $\{x_n \mid V(x_n - x_f, k) < \lambda$, for $0 \leq k \leq N\}$

There are other approaches to stochastic controllability that have some advantages and some disadvantages over the ones presented herein. One such approach is known as the Chebyshev approach, and it can be represnted in a simple algebraic bound, thus making it relatively easy to use. However, both the Lyapunov and the latter Chebyshev approaches are only sufficient conditions for stochastic controllability when the bound is less than unity. Otherwise, simulation-type approaches might be more useful [5.23].

SECTION 5.5
STOCHASTIC OBSERVABILITY

The usual definition to observability is that if a system is observed up to any given time t_1, there always exists a time $t_o<t_1$ when the state of the system can be uniquely determined. Observability is concerned with the effect of the states of a system on the outputs that are available from measurements. An observable system is one in which the output is effected in some way by the variations of any state variable. Also, the effect of any state variable of a system on the output of that system must be unique to be separable from the effects of other state variables.

Observability for stochastic linear systems with multiplicative and additive noise has been addressed by a few authors [5.10,5.11,5.19]. In [5.11], mean- and mean-square-observability are considered for a linear stochastic system with a sequence of random matrices. Zabczyk [5.28], on the other hand, considers a Hilbert space setting of discrete-time linear stochastic systems with multiplicative noise. We will present some of the important characteristics of stochastic observability, various definitions and conditions involved, and their applicability both in continuous- and discrete-time domains.

5.5.1 Continuous-Time Observability

Consider system (5.9) with the measurements given by Equation (5.8). Then, it is well known [5.18] that this system is completely observable if and only if the following observability matrix

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

is of full rank (n). Consider a time-dependent system:

$$\dot{x}(t) = A(t)x(t) + B(t)\,u(t) \tag{5.31}$$

with

$$y(t) = C(t)\,x\,(t) \tag{5.32}$$

Then, there exists a time t_o for all t_1 such that $-\infty < t_o < t_1$, system (5.31,5.32) is completely observable if and only if the following observability Gramian:

$$\int_{t_o}^{t_1} \Phi^T(s,t)C^T(s)C(s)\,\Phi\,(s,t)ds \tag{5.33}$$

is nonsingular. This definition can be extended for uniformly complete-observability. Moreover, the "duality" between observability and controllability is discussed in [5.18].

For stochastic systems with only additive noise such as in system (5.10) with (5.11), the stochastic controllability is defined as follows: System (5.10) and (5.11) is stochastically observable if one can find finite scalars $\beta > \alpha > 0$ and times t_o and t_1 such that $t_1 > t_o > 0$ and for all $t \geq t_1$

$$\alpha I < \int_{t_o}^{t_1} \Phi((s,t)H^T(s)R^{-1}(s)H(s)\Phi(s,t)ds < \beta I \tag{5.34}$$

where the integral is similar to that of (5.22) - (5.23). Obviously, stochastic controllability is violated whenthe covariance matrix R is singular over the time interval of concern.

For a stochastic system with multiplicative and additive noise such as given in Equation (5.13) with the measurement system of equation (5.12), stochastic observability can be defined in several different ways. One approach would be to consider an appropriate filter that will give an estimate of the state x(t). The usual Kalman-Bucy filter would give a suboptimal estimate of the form

$$\dot{\hat{x}}(t) = \begin{bmatrix} A(t) & 0 \\ 0 & \Phi(t) \end{bmatrix} \hat{x}(t) + \begin{bmatrix} B(t) \\ \Gamma(t) \end{bmatrix} u(t) + L(t)\left[\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} - \begin{bmatrix} C(t) \\ \Theta(t) \end{bmatrix} \hat{x}(t)\right] \tag{5.35}$$

Then, the error vector $\tilde{x}(t) = x(t) - \hat{x}(t)$ satisfies:

$$\dot{\tilde{x}}(t) = \left[\begin{bmatrix} A(t) & 0 \\ 0 & \Phi(t) \end{bmatrix} - L \begin{bmatrix} C(t) \\ \Theta(t) \end{bmatrix}\right] \tilde{x}(t) + \rho(t) \tag{5.36}$$

where

$$\rho(t) = L \begin{bmatrix} 0 \\ \nu(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \gamma \end{bmatrix}$$

Usually a feedback control is chosen by

$$u(t) = -K\hat{x}(t) \tag{5.37}$$

thus determining a stochastic feedback regulator. The dynamics of the closed-loop system is now given by:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\tilde{x}}(t) \end{bmatrix} = \begin{bmatrix} \left(\begin{bmatrix} A & 0 \\ 0 & \Phi \end{bmatrix} - \begin{bmatrix} B \\ \Gamma \end{bmatrix} K\right) & - \begin{bmatrix} B \\ \Gamma \end{bmatrix} K \\ 0 & \left(\begin{bmatrix} A & 0 \\ 0 & \Phi \end{bmatrix} - L \begin{bmatrix} C \\ \Phi \end{bmatrix}\right) \end{bmatrix} \begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} 0 \\ \gamma \end{bmatrix} \\ \rho \end{bmatrix}$$

The, stochastic observability in the mean sense can be related to stochastic stability in the mean-square sense. Thus, writing system (5.38) in simplified notation:

$$\dot{X} = MX + \Psi \tag{5.39}$$

where

$$M = \begin{bmatrix} \left(\begin{bmatrix} A & 0 \\ 0 & \Phi \end{bmatrix} - \begin{bmatrix} B \\ \Gamma \end{bmatrix} K\right) & - \begin{bmatrix} B \\ \Gamma \end{bmatrix} K \\ 0 & \left(\begin{bmatrix} A & 0 \\ 0 & \Phi \end{bmatrix} - L \begin{bmatrix} C \\ \theta \end{bmatrix}\right) \end{bmatrix} \quad \text{and} \quad \psi = \begin{bmatrix} \begin{bmatrix} 0 \\ \gamma \end{bmatrix} \\ \rho \end{bmatrix}$$

One can state that system (5.39) is exponentially stable in the mean-square sense if and only if there exist positive definite matrices P and N such that

$$M^*P + PM + \Psi = -N \tag{5.40}$$

Then system (5.12), (5.13) is stochastically completely observable in the mean-sense if and only if there exists a matrix L such that

$$E\left(\begin{bmatrix} A & 0 \\ 0 & \Phi \end{bmatrix} - L \begin{bmatrix} C \\ \theta \end{bmatrix}\right)$$

is a stable matrix [5.29].

Moreover, following Zabczyk [5.28], we can also formulate a stochastic observability criterion given by the following:

The above-mentioned system (5.12), (5.13) is stochastically completely observable if and only if for every non-zero initial state the following inequality will be true with positive probability:

$$\sup_t \left\| \begin{bmatrix} C \\ \theta \end{bmatrix} x(t) \right\| > 0$$

for all t and with an appropriate choice of norm in an appropriate Hilbert space.

5.5.2 Discrete-Time Observability

For a discrete-time, time-invariant linear deterministic system of the form (5.14), (5.16), complete observability refers to the condition of the following matrix

$$\begin{bmatrix} H \\ HF \\ HF^2 \\ \vdots \\ HF^{n-1} \end{bmatrix} \qquad (5.41)$$

being of full rank (n) as a necessary and sufficient condition [5.18]. Moreover, for time-dependent discrete-time deterministic systems of the form (5.14), (5.16), complete observability is fulfilled if and only if for every k there exists an $i \leq k-1$ such that the symmetric non-negative definite matrix given by the following summation

$$\sum_{j=i+1}^{k} \Phi^T(j,i+1)H^T(j)H(j)\Phi(j,i+1) \qquad (5.42)$$

is nonsingular. In the above summation Φ is the transition matrix of the system (5.16). These definitions can be easily extended to include uniform complete observability.

Consider the following linear discrete-time system that has additive white Gaussian noise

$$x(k+1) = F(k)x(k) + G(k)u(k) + \alpha(k) \qquad (5.43)$$

with an appropriate measurement system given by

$$y(k) = H(k)x(k) + w(k) \qquad (5.44)$$

where

$$E[\alpha(k)\alpha^T(j)] = Q(k)\delta_{kj}$$

and

$$E[w(k)w^T(j)] = R(k)\delta_{kj}$$

both and w are zero-mean white Gaussian independent noise vectors and is the Kronecker delta function. Then observability of the above system is satisfied if there exist finite numbers a and b such that $o<\alpha<b$ and a positive integer N such that, for all $i \geq N$

$$aI \leq \sum_{j=i-N-1}^{i} \Phi^T (j,i) H^T (j) R^{-1} (j) H(j) \Phi (j,i) \leq bI$$

This condition implies complete observability with respect to the output of the measurements from the system model. Hence, it reflects the effect of change of any state in the output. The summation between the inequalities is often referred to as the information matrix.

In order to analyze stochastic under multiplicative and additive noise, let's assume that in system [5.43 and 5.44] $F(k) = F_i(k)$, $G(K) = G_i(k)$ and $H(k) = H_i(k)$ are all sequences of matrices that are random and time-invariant for each k_i and each have constant statistics. Moreover, assume that the initial state $x(o) = x_o$ is deterministic and that u(k) is a deterministic function of x(j) for all $j \leq k$. We will now define mean observability and mean-square observability based on the work in [5.27].

We say that the system [5.43], [5.44] with the random characteristics defined above are mean observable if there exists a time-step N such that $E[y(k)] = \bar{y}(k)$ for all k = o, 1,..., N-1 then x(o) =0. Moreover it is mean square observalbe if one can find an N such that $E[||y(k)||^2] = o$ for k=o,1,...,N-1 than $x_o = o$. Obviously, this shows that mean observability implies that x_o can be reconstructed from $\tilde{y}(k)$ fa k=o,....,N-1, for some N. Similarly, for mean-square observability. Also, we should note that for such a system-observability there is a mean value observability matrix similar to the deterministic case given in the previous pages.

A more general case was considered by Zabczyk [5.28] in the following formulation: Consider a system of the form:

$$x(k+1) = Fx(k) + E(x(k), \varepsilon(k)) \qquad (5.46)$$

with an observation system given by:

$$y(k) = Hx(k) \qquad (5.47)$$

The above system is stochastically completely observable if and only if for every non-zero initial state, the following inequality will hold with positive probability:

$$\sup_k |Hx(k)| > o \qquad (5.48)$$

where x(k) is now the solution of equation [5.46]. It can be shown that the above system is stochastically observable if and only if, for all x in an appropriate Hilbert space H, $x \neq o$

$$\mathrm{Sup}[(L_o^k(o)x,x); K=o,1,\ldots] > o \qquad (5.49)$$

and if and only if

$$\mathrm{Sup}[(M^k(o)x,x); k=o,1,\ldots] > o \qquad (5.50)$$

where:

for any W in a linear bounded space

$$L_W(K) = H^*H + W^*RW + \Lambda_1(K) + (F-GW)^*K(F-GW) \qquad (5.51)$$

where R is a positive semidefinite linear bounded operator in an appropriate Hilbert space and

$$\Lambda_1(K) = \sum_{k=1}^{\infty} \lambda k F_k K F_k \qquad (5.52)$$

where $F_k(\cdot) = F(\cdot, e_k)$ $k=1,2,\ldots$ and λ_k, e_k, are the eigenvalues and eigenvectors of the covariance operator of the state-dependent noise vectors ξ. Moreover, $(\cdot\ ,\cdot)$ is an inner product in an appropriate Hilbertspace [5.28]. Also,

$$M(K) = H^*H + \Lambda_1(K) \tag{5.53}$$

and (*) indicates complex conjugate transposition. Stochastic stability, stabilizability, and detectability can also be related to observability. [5.28].

5.6 CONCLUSIONS

A general theory of measurement systems and applications is presented in the previous pages. Linear noisy measurement systems are treated under various situations: Deterministic, complete measurements, simple incomplete measurements with additive zero-mean white Gaussian noise, and systems with incomplete measurements contamnated by multiplicative and additive zero-mean white Gaussian noise elements. Moreover, stochastic and deterministic controllability and observability issues are treated in appreciable detail and extensions to the existing theory of stochastic controllability and observability of linear dynamic systems.

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CHAPTER 6

DERIVATION OF STATE ESTIMATION EQUATIONS UNDER REALISTIC MEASUREMENT ASSUMPTIONS

6.1 INTRODUCTION

In recent years, interest in control systems with multiplicative and additive noise has increased considerably. One of the reasons for this is the increased complexity of systems and the high uncertainty involved in their modeling. Thus, modeling of modern complex control systems has become a nontrivial task, especially due to the very high performance requirements and the virtually unknown new instrumentation and materials being utilized. Moreover, model accuracy requirements and the extent of detailed modeling are a function of performance specifications and the expected disturbances that could effect the system [6.1].

Typically, in most complex systems, the entire state vector is not available from measurements, but only a linear combination of a set of the state variables is given. Under such circumstances, either a new procedure should be developed whereby the missing states are accounted for, or an appropriate approximation of the complete state vector must be derived for the optimal control law. In virtually all complex situations, approximations to the state vector generated from the available partial measurement information, are developed and utilized. Thus, observers are derived that will reconstruct the missing state information via appropriate estimation and filtering techniques. A great deal of theory and many algorithms exist for the deterministic [6.2] and linear quadratic Gaussian (LQG) [6.3] systems estimation and filtering methods. In contrast to the above-mentioned case of additive white Gaussian noise for which even nonlinear filters or extended Kalman filtering algorithms exist, there are a very small number of papers that address the issue of state estimation of linear (let alone nonlinear) stochastic systems with multiplicative and additive measurement noise.

The limited amount of research in the state estimation and control of systems with multiplicative noise can be related to the following fact. The extension of deterministic control system and state estimation theory to stochastic systems often poses serious complications due to the consequences of Ito calculus. Thus, the number of equations as compared with the deterministic case increases mainly because of the appearance of second powers of system matrices in linear stochastic systems with multiplicative and additive noise [6.4]. The problem of optimal estimation under multiplicative and additive noise is essentially a nonlinear filtering problem.

The problem of optimal state estimation of linear stochastic systems with multiplicative noise from noisy measurements has been the topic of considerable research in recent years. Optimal linear recursive estimation schemes have been especially useful in the LQG system setting particularly because of their computational features and structural simplicity. Many different optimality criteria such as minimum-mean-square error, maximum lilelihood, least-square errors, etc., have been in common use in a wide variety of fields. One fundamental assumption in all observer and LQG filtering techniques is that the system dynamic model is accurately known. In practice, this is far from reality and often uncertain parameters, modeling errors, and other factors are involved. Thus, performance levels are often reduced due to a-priori chosen nominal model-parameter values when these parameters experience deviations.

Robustness is significantly more important nowadays than in the past due to the complex nature of control systems, whether these are under development or already in application. Thus, for least performance degradation due to parameter variations, it is highly desirable that one obtain robust state estimation algorithms and methods that provide a high degree of performance. This high performance even under worst possible system or prarmeter variations over the range of parametric undertainty [6.6.].

The reduced order state-estimation problem is another topic of extensive research interest. The need for this type of an optimal reduced order approach to state estimation is from practical constraints on computational capabilities, complexity, instrumentation and reliability, among others. Moreover, often only a small number of estimates would be quite adequate to perform the task at hand. However, reduced order estimators designed by means of either model reduction followed by "full-order" state estimation or full-order estimation followed by estimator order reduction is not optimal for the given order of the system [6.6]. It has recently been shown [6.11] that solutions to the steady-state reduced-order state-estimation problem is characterized by a system of modified Riccati and Lyapunov equations that are coupled by an "oblique" projection.

The case when measurement noise is singular (i.e., colored noise), the optimal state-estimation does not exist since the filter gains of linear quadratic (LQ) estimation algorithms are given in terms of the inverse of the noise intensity matrix [6.6]. A good reference on research related to stochastic observer theory is the book by O'Reilly [6.22]. The usual approach (by most researchers) to overcome noise singularily is to introduce new measurements via differentiation of existing noise-free meassurements. Haddad and Bernstein [6.23] complemented the above technique by simultaneously designing an optimal reduced-order dynamic estimator for the noisy measurements and an observer for the noise-free measurements. The results obtained include modified Riccati-Lyapunov equations coupled with two "oblique" projections which essentially generalizes the classical steady-state Kalman-Bucy filtering theory.

Further extension of the classical Kalman-Bucy state estimation of the case of state-, control-, and measurement-dependent noise, is a relatively new development [6.24]. The fundamental motivation of such a modeling approach is to help desensitize the controller and the state estimator which will result in robust filtering algorithms that have virtually invariant performance relative to actual parameter or system variations [6.25-6.30]. Specially in discrete-time systems with stochastic parameters (that often arise in sampling, gating or amplitude modulation) mean square system stability is required for the existence, uniqueness, and stability of time invariant estimators [31-32].

The problem of estimation of signals that are contaminated by multiplicative noise under restrictive assumptions of semi-infinite observation interval and stationarity has been addressed by Loo [6.33]. Nahi [6.34], on the other hand, has studied uncertainties in the observation via hypothesis testing. Rajasekaran, Satyanarayana and Srinath [6.35] presented estimation algorithms that are optimal in the class of linear minimum mean-square error estimators in the form of prediction, filtering and smoothing. Both discrete-time and continuous-time situations are considered for nonstationary stochastic systems. Tugnait [6.36] presents conditions for uniform asymptotic stability in the large of the optimal minimum mean-square error linear filter for linear systems with multiplicative observation noise. The implications for such stability conditions are that the computations of the filter gains and error covariances are stable. In the above, only scalar multiplicative noise was considered.

Accelerated interest has been shown in recent years regarding the optimal control, stabilization, state estimation, and identification of linear stochastic systems with multiplicative and additive noise [6.37-6.49]. State-estimation has been of special interest due to the practical implications relative to realistic complex systems. According to the published literature, the first authors who considered state estimation of linear systems with multiplicative noise in their measurement system were Bondaros and Konstantinov [6.50]. Simultaneously Milshtein [6.51] considered the optimal controller of linear systems with multiplicative and additive noise, but with partial deterministic observation dynamics. Recent advances showing performance and stability robustness of such has made state estimation of stochastic linear systems practically attractive.

6.2 THE STATE ESTIMATION PROBLEM: THE CONTINUOUS-TIME OBSERVER

Consider a linear dynamic system represented by the ing stochastic differential equation:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A(t) & D \\ O & \Phi(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B(t) \\ \Gamma(t) \end{bmatrix} u(t) + \begin{bmatrix} O \\ \gamma(t) \end{bmatrix} \quad (6.1)$$

where: $n_1 \geq o$, $n_2 \geq o$, $n_1+n_2 = n>o$,

$x_1 \varepsilon R^{n_1}$, $x_2 \varepsilon R^{n_2}$, $A\ (n_1 x n_1)$, $\Phi \sim n_2 x n_2$, $B\ n_1 xm$, $\Gamma \sim n_2 xm$, $u \varepsilon R^m$, and

$\gamma \varepsilon R^{n_2}$ are the state vectors, the state matrices, the control matrices, the control input vector, and the additive zero-mean white Gaussian noise vector, respectively. In the above, Φ and Γ are matrices with random zero-mean, white Gaussian noise (for convenience) elements and covariances of $\Sigma^{\Phi\Phi}\delta(t-\tau)$ and $\Sigma^{\Gamma\Gamma}\delta(t-\tau)$ here δ is the usual delta function). The covariance of γ is $\Sigma^{\gamma\gamma}\delta(t-\tau)$.

The most general linear measurement system for Equation (6.1) is given by:

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} C(t) \\ \Theta(t) \end{bmatrix} x(t) + \begin{bmatrix} O \\ \nu(t) \end{bmatrix} \quad (6.2)$$

where: $y_1 \varepsilon R^{p_1}$, $y_2 \varepsilon R^{p_2}$, $p_1+p_2=p$, $C\ p_1 xn$, $\Theta\ p_2 xn$, and $\nu \varepsilon R^{p_2}$, are the measurement vectors, the measurement matrices and the additive measurement noise vector. In the above Θ is a matrix of zero-mean white Gaussian noise elements with covariance of $\Sigma^{\Theta\Theta}\delta(t-\tau)$ and ν is a zero-mean white Gaussian noise vector with covariance of $\Sigma^{\nu\nu}\delta(t-\tau)$ and independent of all the other random elements.

The above system is a worst case situation for a linear system since it has state-, control-, and measurement-dependent noise. Moreover, it is assumed that in realistic situations, a part of any complex system can be considered to be adequately modeled and thus, there is no need for any stochastic representation. Similarly, a part of the measurements can be considered accurate enough not to warrant any probabilistic representation. Under these conditions, the state estimation problem is to find an "optimal" filtering algorithm that will furnish the required estimates of the states for an acceptable control system performance.

The deterministic part of Equation (6.1) will be neglected and so will that of Equation (6.2). For, there are numerous observer algorithms that handle such cases [6.2]. We will concentrate on $x_2(t)$ with measurements of $y_2(t)$ and call them $x(t)$ and $y(t)$ respectively for convenience.

6.2.1 The Stochastic Observer

The approach we will take is to find an unbiased optimal linear state estimator $x(t)$ of the state $x(t)$ given by a stochastic differential equation of the following general form:

$$\dot{\hat{x}}(t) = F_1\hat{x}(t) + F_2y(t) + F_3 \tag{6.3}$$

where F_1, F_2, F_3 are unknown matrix coefficients that will be determined.

Let us define the error vector by $\tilde{x}(t) = x(t) - \hat{x}(t)$. Then, if we assume that the optimal feedback control is $u(t) = -G\hat{x}(t)$, we have:

$$\dot{\tilde{x}}(t)=F_1\tilde{x}(t)+(\Phi+F_1-F_2\theta)x(t) - \Gamma G\hat{x}(t)+(\gamma(t)-F_2\nu(t)-F_3) \tag{6.4}$$

The equation of the mean error is given by:

$$E[\dot{\tilde{x}}(t)] = \frac{dE[\tilde{x}]}{dt} = F_1(E[\tilde{x}] + E[x]) - F_3 \tag{6.5}$$

with $E[\tilde{x}(o)] = o$, $E[\dot{\tilde{x}}(o)] = o$, from unbiasedness we have

$E[\tilde{x}] = o$ and $E[\dot{\tilde{x}}] = o$ for all t, where $\hat{x}(o) = E[x(o)]$

Hence,

$$F_1E[x]=F_1 \bar{x} = F_3 \tag{6.6}$$

and

$$\dot{\tilde{x}}(t)=F_1(\tilde{x}(t)-E[x]) + (\Phi+F_1-F_2\theta)x(t) - \Gamma G\hat{x}(t) + \gamma(t)-F_2\nu(t)$$

Define the error covariance matrix P by $P=E[\tilde{x}\tilde{x}^T]$ then the differential equation for P is:

$$\dot{P}=F_1P+PF_1{}^T + Tr(\Sigma^{\Phi\Phi}\Sigma_{xx}) + F_1\Sigma_{xx}F_1{}^T+F_2Tr(\Sigma^{\Phi\Phi}\Sigma_{xx})F_2{}^T + GTr(\Sigma^{\Gamma\Gamma}\Sigma_{\tilde{x}\tilde{x}})G^T+$$

$$+ \Sigma^{\gamma\gamma}+F_2\,\Sigma^{\nu\nu}F_2{}^T - F_1\bar{x}\bar{x}^TF_1 \tag{6.7}$$

and its initial condition is $P(o) = E[x(o)x^T(o)]$.

Taking the Hamiltonian (H) approach with the minimization criterion being the error covariance, we have the following external conditions:

$$\frac{\partial H}{\partial F_1} = \frac{\partial H}{\partial F_2} = 0 \tag{6.8}$$

$$\frac{dS}{dt} = \frac{\partial H}{\partial P} \qquad (6.9)$$

where H is given by:

$$H=F_2^T P+S^T(F_1P+PF_1^T+T_r(\Sigma^{\Phi\Phi}\Sigma_{xx})+F_1\Sigma_{xx}F_1^T+F_2T_r(\Sigma^{\Phi\Phi}\Sigma_{xx})F_2^T$$

These, together with the state equation and the performance functional, form the new deterministic control problem. From the above, with some manipulations, the following observer equation is derived [49].

$$\dot{\hat{x}} = (\Phi-P\theta-2S^TP\Sigma_{xx}^{-1})\hat{x} - P[Tr(\Sigma^{\theta\theta}\Sigma_{xx})]^{-1}y-2S^TP\Sigma_{xx}^{-1}\bar{x} \qquad (6.11)$$

with $\hat{x}(o) = E[x(o)] = \bar{x}_o$, and $\bar{x} = E[x]$

Equation (6.11) is the observer equation for the optimal linear stochastic filter that is unbiased and that is optimal only in the class of linear filters.

6.3 THE STATE ESTIMATION PROBLEM: THE DISCRETE-TIME OBSERVER

There are several advantages to discrete-time observers. The most imporetant one is the fact that most data processing systems are digital and it makes a great deal of sense to have algorithms that are discrete-time. In addition, most measurement systems give information at discrete-time points and their realistic representation is by difference rather than differential equations. Moreover, optimal recursive linear estimation routines have beenparticularly attractive due to their computational advantages performed via digital processing, as well as because of their simple structure. Various criteria of optimality such as minimum mean-square error (MMSE), maximum likelihood (ML), and least square error (LSE) have been estensively utilized upon the problem of interest at hand [6.52].

Various researchers have developed recursive estimation algorithms for discrete-time systems with state- and measurement- dependent noise [6.1, 6.9, 6.25, 6.30, 6.32, 6.35]. Presented herein is a summary of the major algorithms with a brief discussion of their advantages and disadvantages.

6.3.1 The Optimal MMSE Linear Recursive Estimator [6.35]

Nahi [6.34] has developed an algorithm for uncertain observation systems (with scalar multiplicative noise) in an hypothesis testing approach that can be considered a special case of what is presented herein. The problem considered in this subsection involves a discrete-time system that is a special case of the system presented in Equations (6.1) - (6.2). Thus, the stochastic measurement equation at the k-th instant is given by:

$$y(k) = \mu(k)\ H(k)x(k) + \nu\ (k) \tag{6.12}$$

where, as previously defined $y \varepsilon R^m$ is the measurement vector, $x \varepsilon R^n$ is the state vector, $\mu \varepsilon\ R$ is a scalar white noise process with mean value m(k) and covariance of N(k), $H \sim (mxn)$ and $\nu \varepsilon R^m$ is a zero-mean white Gaussian noise vector with covariance of $\Sigma^{\nu\nu}$. The state equation for the above system is given as follows:

$$x(k+1) = A(k+1,k)x(k) + B\ (k+1,k)\ \omega\ (k) \tag{6.13}$$

where A nxn, B nxr are constant matrices and ω (k) is a zero-mean white Gaussian noise vector with covariance of $\Sigma^{\omega\omega}$. The initial state x(o) is assumed to be random and independent of the other noise elements. The problem is to find an optimal estimate $x(k/\ell)$ of the states x(k) which is a linear combination of the measurements up to the ℓth instant such that the expected value of the estimation error squared $E([x-\hat{x})^T(x-\hat{x}]$ is a minimum. The problem we will address here is that of filtering, when $\ell = k$. It is known that the necessary and sufficient condition to be satisfied for the optimal filter is:

$$E[x(k)-\hat{x}(k/k)y^T(j)] = 0 \text{ for } j = 0, 1, \ldots, k \quad (6.14)$$

The innovations process defined by $I(j)=y(j)-m(j)Hx(j/j-1)$ can replace the observation vector in Equation (6.14). Thus, the necessary and sufficient condition is now:

$$E[x(k/k)I^T(j)]=o, \; j=o, 1, \ldots, k \quad (6.15)$$

It is shown elsewhere [6.35] tha the innovations process is a zero-mean white sequence with covariance

$$\Sigma^I(k)=m^2(k)H(k)P(k)H^T(k)+N(k)H(k)E[(x(k)x^T(k)]H^T(k)+\Sigma^{\nu\nu} \quad (6.16)$$

and P = the error covariance matrix as defined earlier. Continuing the derivations to satisfy the necessary and sufficient condition for optimal estimate, the filtering equation is given as follows:

$$\hat{x}(k/k)=\hat{x}(k/k-1)+m(k)P(k)H^T(k)(\Sigma^I)^{-1}[y(k)-m(k)H(k)\hat{x}(k/k-1)] \quad (6.17)$$

and the error covariance equation is given by:

$$P(k)=A(k,k-1)P(k-1)A^T(k,k-1)-m^2(k)P(k)H^T(k)(\Sigma^I)^{-1}H(k)P(k)$$
$$+B(k,k-1)\Sigma^{\omega\omega}(k)B^T(k,k-1) \quad (6.18)$$

with initial condition of $P(o)=E[x(o)x^T(o)]$. Similar results can easily be derived for the continuous-time case.

6.3.2 Optimal Estimation of Linear Discrete-Time Systems with Continuous-Valued Stochastic Parameters [6.32]

Consider a system of the following sequential form:

$$x(k+1)=\emptyset(k)x(k)+\omega(k) \quad (6.19)$$

with a measurement system given by:

$$y(k) = H(k)\ x(k) + \nu(k) \quad (6.20)$$

where, as before, $x\varepsilon R^n$ is the state, $y\varepsilon R^\ell$ is the measurement, $\omega\varepsilon R^n$ is the zero-mean system noise, $\nu\epsilon R^\ell$ is the zero-mean measurement noise, and $\emptyset$ and H are random vector α_1 as follows:

$$\alpha_i= [St(\Phi^T(k))St(H^T(k)\omega^T(k))\nu^T(k)] \quad (6.21)$$

where St is the stacking operator that transforms a matrix into a vector by $St(\emptyset^T(k)) = [\emptyset^1\ \emptyset^2 ... \emptyset^n]^T$ where $\emptyset^i$ is the i^{th} row of $\emptyset^T$. Now, assume that α_i is a sequence of independent random variables with contstant and known statistics and independent of $x(o)=x_o$. Moreover, assume that the elements of $\emptyset$ and H are independent of those of ω and ν. The following statistics are assumed known:

$$E[x_o]=\overline{x}\ , E[(x_o-\overline{x}_o)(x_o-x_o)^T]=P_o,\ E[\omega(k)\omega^T(k)]=W,\ E[\nu(k)\nu^T(k)]=\nu$$

Suppose $\emptyset$ and H can be expressed as follows:

$\emptyset(k) = \overline{\emptyset}+\widetilde{\emptyset}(k)$, $H(k) = \overline{H} + \widetilde{H}(k)$. Using these expression in Equations (6.19) - (6.20) we get:

$$x(k+1)=\overline{\emptyset}x(k)+ \omega'(k) \text{ and } y(k)=\overline{H}x(k)+ \nu'(k) \quad (6.22)$$

where: $\omega^1 = \omega + \tilde{\phi} x$ and $\nu^1 = \nu + \tilde{H}x$

Now the statistics are given by:

$$E[\omega^1(\omega^1)^T] = W + E[\tilde{0}\ x(k)\ x^T(k)\ \tilde{0}^T] = W + E[\tilde{0}\ X(k)\ \tilde{0}^T]$$

$$E[\nu^1(\nu^1)^T] = V + E[\tilde{H}\ x(k)\ x^T(k)\ \tilde{H}^T] = V + E[\tilde{H}\ X(k)\ \tilde{H}^T]$$

$$X(k+1) = E[0\ X(k)\ o^T] + W \qquad (6.23)$$

with $\bar{\underline{X}}(o) = \bar{x}_o \bar{x}_o^T + P_o$

The minimum variance linear estimate $\hat{x}(k+1/k)$ of $x(k+1)$ given the known measurement information $y(o),\ldots,y(k)$ is given by:

$$\hat{x}(k+1/k) = \bar{\Phi}\hat{x}(k/k-1) + K(k)[y(k) - \bar{H}\hat{x}(k/k-1)],\ \hat{x}(0/-1) = \bar{x}_o \qquad (6.24)$$

$$K(k) = \bar{\Phi}P(k/k-1)\ \bar{H}^T[\bar{H}\ P(k/k-1)\bar{H}^T + W + E[\tilde{\phi}X\tilde{\phi}^T]]^+ \qquad (6.25)$$

$$P(k+1/k) = \bar{\Phi}P(k+1/k-1)\bar{\Phi}^T - \bar{\Phi}P(k/k-1)\bar{H}^T[\bar{H}\ P(k/k-1)\bar{H}^T +$$
$$V + E[\tilde{H}\ X\ \tilde{H}^T]]\ \ \bar{H}\ P(k/k-1)\ \bar{\Phi}^T + W + E[\tilde{\phi}X\tilde{\phi}^T] \qquad (6.26)$$

with $P(0/-1) = P_o$ and $[\ ]^+$ denoting pseudo-inverse.

DeKoning [32] has shown that if the spectral radius $\rho(\phi \otimes \phi)<1$. Here, $\otimes$ denotes the Kronecker product $\phi \otimes \phi = [\phi_{ij}\phi]$, where ϕ_{ij} is the $_{ij}$th element of ϕ. Moreover, the steady-state lim $X(k+1) = X$ as $k\to\infty$ of Equation (6.23) has a unique solution when $\rho(\phi X \phi)<1$ holds.

Thus, mean square stability of the above system is sufficient and almost necessary condition for the existence, uniqueness, and stability of the time invariant estimator.

6.3.3 The Linear Stochastic Observer

Consider the discrete-time form of Equations (6.1) - (6.2) with a random Gaussian initial state vector x(o), that is, independent of all the other random elements and given statistics of $E[x(o)] = \bar{x}_o$ and $E[(x(o) - \bar{x}_o)(x(o) - \bar{x}_o)^T] = X_o$.

$$x(k+1) = \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A(k) & 0 \\ 0 & \theta(k) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} B(k) \\ \Gamma(k) \end{bmatrix} U(k) + \begin{bmatrix} 0 \\ \gamma(k) \end{bmatrix} \tag{6.28}$$

Similarly, the corresponding measurement system is given by:

$$y(k) = \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} c(k) \\ \Omega(k) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ \nu(k) \end{bmatrix} \tag{6.29}$$

where x_i, y_i, i=1, 2, A, B, θ, Γ, u, γ, C, Ω and ν are defined in the same manner as previously. The optimal control problem is to derive a control sequence $\{(k), k=0, 1,...,N\}$ that minimizes the following performance functional:

$$J = E\left\{x^T(N)\, S_N\, x(N) + \sum_{k=0}^{N-1} (x^T(k)\, Q(k)\, x(k) + u^T(k) R(k)\, u(k))\right\} \tag{6.30}$$

where S_N and Q(k) are symmetric non-negative definite nxn matrices and R(k) is a symmetric positive definite mxm matrix.

The solution to the above stochastic optimal control problem with multiplicative and additive random parameters shows the effects of uncertainties in the performance of the control system. Thus, the control gain will be a function of the unconditional means and covariances of the uncertainties creating nonlinear effects. For simplicity and practical considerations, only a suboptimal stochastic linear feedback control law will be considered [5.53]. The following control law:

$$u(k) = -L(k)\hat{x}(k) \qquad (6.31)$$

is assumed, where $\hat{x}(k)$ is the estimate of $x(k)$ and is found by means of a Kalman-Bucy type filtering algorithm. This algorithm is of lower dimension than that of the Kalman-Bucy filter, since the optimal mean-square unbiased estimate requires roughly n^3 multiplications for the above system, while it only requires $(n-p_1)^3$ multiplications for the observer we will present.

6.3.3.1 Reformation of the Problem

Choose a vector $Z(k)$ such that when augmented with the deterministic vector $y_1(k)$, it forms an nth order vector. Thus,

$$\begin{bmatrix} Z(k) \\ y_1(k) \end{bmatrix} = M(k)\ x(k) \qquad (6.32)$$

where:

$$M(k) = \begin{bmatrix} M_1(k) \\ C(k) \end{bmatrix} \qquad (6.33)$$

and $\det M(k) \neq o$, with the dimensions of $M_1(k)$ given by $(n-p_1) \times n$. Hence, Equation (6.28) can be rewritten as:

$$x(k+1) = M^{-1}(k+1) \begin{bmatrix} Z(k+1) \\ y_1(k+1) \end{bmatrix}$$

$$= \begin{bmatrix} A(k) & O \\ O & \theta(k) \end{bmatrix} \begin{bmatrix} M_1(k) \\ C(k) \end{bmatrix}^{-1} \begin{bmatrix} Z(k) \\ y_1(k) \end{bmatrix} + \begin{bmatrix} B(k) \\ \Gamma(k) \end{bmatrix} u(k) + \begin{bmatrix} O \\ \gamma(k) \end{bmatrix} \quad (6.34)$$

and

$$\begin{bmatrix} Z(k+1) \\ y_1(k+1) \end{bmatrix} = M(k+1) \begin{bmatrix} A(k) & O \\ O & \theta(k) \end{bmatrix} \begin{bmatrix} M_1(k) \\ C(k) \end{bmatrix}^{-1} \begin{bmatrix} Z(k) \\ y_1(k) \end{bmatrix} + M(k+1) \begin{bmatrix} B(k) \\ \Gamma(k) \end{bmatrix} u(k)$$

$$+ M(k+1) \begin{bmatrix} O \\ \gamma(k) \end{bmatrix} \quad (6.35)$$

Rewriting Equation (6.35) yields:

$$Z(k+1) = A_{11}(k,k+1)\ x(k) + A_{12}(k,k+1) y_1(k) + B_1(k,k+1) u(k) + \gamma_1(k) \quad (6.36)$$

$$y_1(k+1) = A_{21}(k,k+1)\ x(k) + A_{22}(k,k+1)\ y_1(k) + B_2(k,k+1) u(k) + \gamma_2(k) \quad (6.37)$$

where: $$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} M_1(k+1) \\ C(k+1) \end{bmatrix} \begin{bmatrix} A(k) & O \\ O & \theta(k) \end{bmatrix} \begin{bmatrix} M_1(k)^{-1} \\ C(k) \end{bmatrix}$$

$$\begin{bmatrix} B_1(k,k+1) \\ B_2(k,k+1) \end{bmatrix} = M(k+1) \begin{bmatrix} B(k) \\ \Gamma(k) \end{bmatrix} \text{ and } \begin{bmatrix} \gamma_1(k) \\ \gamma_2(k) \end{bmatrix} = M(k+1) \begin{bmatrix} O \\ \gamma(k) \end{bmatrix}$$

where: $A_{11} \sim (n-p_1)$, $A_{12} \sim (n-p_1) \times p_1$, $A_{21} \sim p_1 \times (n-p_1)$, $A_{22} \sim p \times p_1$, $B_1 \sim (n-p_1) \times m$, $B_2 \sim p_1 \times m$, and $\gamma_1 \sim n-p_1$, and $\gamma_2 \sim p_1$ dimensional matrices and vectors.

Consider Equation (6.37) rewrite it as follows:

$$Z_2(k) = y_1(k+1) - A_{22}(k,k+1)\ y_1(k) - B_2(k,k+1)\ u(k)$$
$$= A_{21}(k,k+1)\ Z(k) + \gamma_2(k) \qquad (6.38)$$

A new system of equations representing a control systme can now be defined by Equation (6.36) as the state equation with a measurement system given by Equation (6.29) and an additional equation to be derived later: it is possible to choose M(k) such that $y_1(k+1)$ and $y_1(k+1)$ and $y_1(k)$ are deterministic. Then $Z_2(k)$ can be calculated from the exact measurements of $y_1(k+1)$ and $y_1(k)$.

Equation (6.29) can be rewritten as:

$$y_2(k) = \Omega k)\ x(k) + \nu(k)$$
$$= \Omega(k)\ M^{-1}(k) \begin{bmatrix} Z(k) \\ y_1(k) \end{bmatrix} + \nu(k)$$
$$= H_1(k)\ Z(k) + H_2(k)\ y_1(k) + \nu(k) \qquad (6.39)$$

which yields: $$Z_3(k) = y_2(k) - H_2(k)\ y_1(k)$$
$$= H_1(k)\ Z(k) + \nu(k) \qquad (6.40)$$

This implies that, since y_2 and y_1 give the value of Z_3 on the left hand side of Equation (6.40), we have a new measurement equation. Thus, the estimation problem reduces to estimating Z(k) in Equation (6.36) via the measurements of (6.38) and (6.40).

For convenience, introduce an augmented set of measurements by:

$$Z_a(k) = \begin{bmatrix} Z_2(k) \\ Z_3(k) \end{bmatrix} = H_1(k,k+1)\ Z(k) + \gamma_a(k) \qquad (6.41)$$

where: $$H_a(k,k+1) = \begin{bmatrix} A_{21}(k,k+1) \\ H_1\ (k,k+1) \end{bmatrix} \text{ and } \gamma_a(k) = \begin{bmatrix} \gamma_2(k) \\ \nu(k) \end{bmatrix}$$

6.3.3.2 Generation of the Observer Equation

Consider the new state estimation system given by:

$$Z(k+1) = A_{11}(k,k+1)\ Z(k) + A_{12}(k,k+1)\ y_1(k) + B_1(k,k+1)\ u(k) + \gamma_1(k) \quad (6.42)$$

$$Z_1(k) = H_1(k,k+1)\ Z(k) + \gamma_a(k) \quad (6.43)$$

The equation of the observer can now be rewritten as [30].

$$\hat{Z}(k+1) = K_1(k)\ \hat{Z}(k) + K_2(k)\ Z_a(k) + K_3(k)\ [B_1(k,k+1)\ u(k) + A_{12}(k,k+1)\ y_1(k)] \quad (6.44)$$

where K_1, K_2, K_3 are unknown observer matrix coefficients tha need to be determined for optimality. Define the estimation error by by $\tilde{x}(k) = \hat{Z}(k) - Z(k)$. Then,

$$\tilde{x}(k+1) = [A_{11}(k,k+1) - K_1(k) - K_2(k)\ H_a(k,k+1)]\ Z(k) + K_1(k)\ \tilde{x}(k) - K_2(k)\ \gamma_a(k) + [I - K_3(k)][B_1(k,k+1)\ u(k) + A_{12}(k,k+1)\ y_1(k)] + \gamma_1(k) \quad (6.45)$$

where I is the $(n-p_1)$ x $(n-p_1)$ dimensioned identity matrix. If we assume that the estimator is a minimum mean-square error, unbiased estimator, then,

$$E[\tilde{x}(k+1)] = E[A_{11}(k,k+1)\ Z(k)] - K_1(k)\ E(Z(k)) - K_2(k)\ E[H_a(k,k+1)\ Z(k)] + K_1(k)\ E[\tilde{x}(k)] - K_2(k)\ E[\gamma_a(k)] + [I-K_3(k)][E[B_1(k,k+1)\ u(k) + E[A_{12}(k,k+1)]\ y_1(k)] + E[\gamma_1(k)] = 0 \quad (6.46)$$

Ω and θ are assumed to have uncorrelated elements with those of x(o) and, similarly, A_{11} and H_a have uncorrelated elements with A(k). These assumptions and facts with the unbiasedness property yield:

$$K_3(k) = I \text{ and } K_1(k) = \overline{A}_{11} - K_2(k)\,\overline{H}_a \qquad (6.47)$$

where: $\overline{A}_{11} = E[A_{11}(k,k+1)]$ and $\overline{H}_a = E[H_a(k,k+1)]$.

Now Equation (6.44) becomes:

$$\hat{Z}(k+1) = [\overline{A}_{11} - K_2\overline{H}_a]\,\hat{Z}(k) + k_2(k)\,Z_a(k) + B_1(k,k+1)\,u(k) + A_{12}(k,k+1)\,y_1(k) \qquad (6.48)$$

with initial condition of:

$$\hat{Z}(o) = M_1(o)\,\overline{X}_o = M_1(o)\,E[x(o)]$$

Equation (6.45) can now be rewritten as:

$$\tilde{x}(k+1) = A_{11}(k,k+1)\,Z(k) - [\overline{A}_{11} - K_2\overline{H}_a]\,\hat{Z}(k) + K_2(H_a(k,k+1)\,z(k) - \gamma_a(k)] + \gamma_1(k) \qquad (6.49)$$

The error covariance matrix is given by:

$$P(k+1) = E[\tilde{x}(k+1)\,\tilde{x}^T(k+1)]$$

$$= E[A_{12}\,\tilde{x}\,\tilde{x}^T A^T{}_{11} - A_{11}\,\tilde{x}\,\hat{z}^T(\overline{A}_{11}-A_{11}-K_2\overline{H}_a)^T - A_{11}\,\tilde{x}\,z^T H_a^T\,K_2^T$$

$$- A_{11}\,\tilde{x}\,\gamma_a{}^T K_2{}^T + A_{11}\,\tilde{x}\,\gamma 1^T - (\overline{A}_{11}-A_{11}-K_2\overline{H}_a\hat{Z})\,\tilde{x}^T A_{11}{}^T$$

$$+ (\overline{A}_{11}-A_{11}-K_2\overline{H}_a)\,\hat{z}\,\hat{z}^T\,(\overline{A}_{11}-A_{11}-K_2\overline{H}_a)^T$$

$$+ (\bar{A}_{11}-A_{11}-K_2\bar{H}_a)\ \hat{Z}\ \hat{Z}^T\ H_a^T\ K_2^T + (\bar{A}_{11}-A_{11}-K_2\bar{H}_a)\ \hat{Z}\gamma_a^T\ K_2^T$$

$$- (\bar{A}_{11}-A_{11}-K_2\bar{H}_a)\ \hat{Z}\gamma_1^T - K_2\gamma_a\ \tilde{x}^T A_{11}^T + K_2\gamma_a\hat{Z}^T(\bar{A}_{11}-A_{11}-\bar{H}_a)^T$$

$$+ K_2\gamma_a\ Z^T H_a^T K_2^T + K_2\gamma_a\gamma_a^T K_2^T\gamma - K_2\gamma_a\gamma_1^T + \gamma_1^T + \gamma_1\ \tilde{x}^T A_{11}^T$$

$$- \gamma_1\ \hat{Z}\ (\bar{A}_{11}-A_{11}-K_2\bar{H}_a)^T - \gamma_1 Z^T\ H_a^T\ K_2^T - \gamma_1\gamma_a^T K_2^T + \gamma_1\gamma_1^T] \tag{6.50}$$

The optimal mean-square error unbiased estimate requires the minimization of the tr(P(k+1) with respect to K_2. The initial condition for Equation (6.50) is:

$$P(o) = E[Z(o)-M_1(o)\overline{x}_o) \cdot (Z(o)-M_1(o)\overline{x}_o)^T].$$

Performing the calculations and equating the partial derivative of the trace of P(k+1) with respect to K_2 to zero yields the optimal value of K_2:

$$K_2 = \tfrac{1}{2}\{\bar{A}_{11}\ E[ZZ^T]\ \bar{A}_{11}^T + Tr(\Sigma A_{11}\ E[ZZ^T]) + 2\ \bar{A}_{11}E[\hat{Z}\hat{Z}^T]$$

$$\bar{H}_a^T + 2\ \Sigma\gamma_1\gamma_a\}\ [\bar{H}_a^T\ E[\hat{Z}\hat{Z}^T]^T\bar{H}_a + \Sigma\gamma_a\gamma_a]^{-1} \tag{6.51}$$

K_2 minimizes the dispersion of the error at the next step. The optimal estimator is now given by Equations (6.48), (6.50), and (6.51).

The optimal mean-square error unbiased estimate of x(k) is given by:

$$\hat{x}(k) = M^{-1}(k) \begin{bmatrix} \hat{Z}(k) \\ y_1(k) \end{bmatrix} \tag{6.52}$$

The need to know the value of $y_1(k+1)$ at time instant $(k+1)$ for estimating $\hat{Z}(k)$ is definitely undesirable. To eliminate this nonreal-time computational requirement, define a vector $Z^*_1(k) \sim (n-p_1)$ such that

$$\hat{Z}(k+1) = Z^*_1(k+1) + K_2 Z'(k+1), \quad k=0,1,\ldots,N \tag{6.53}$$

where: $Z'(k) \sim (p_1 + p_2)$ dimensional vector given by:

$$Z'(k) = [y_{11}(k), y_{12}(k), \ldots, y_{1p_1}(k), 0, \ldots, 0]^T \tag{6.54}$$

and where $y_{1i}(k)$ is the ith element of $y_1(k)$. Now, define $\tilde{Z}(k), k=0,1,\ldots,N$ by $\tilde{Z}(k) = [Z^T(k)\ A_{22}{}^T(k,k+1):0]^T$ where $\tilde{Z}$ is a $(p_1 + p_2) \sim$ dimensional vector augmented with zeros. Similarly, let $\tilde{u}(k), k=0,1,\ldots,N$, be an $(p_1 + p_2) \sim$ dimensional vector given by $[u^T(k)\ B_2{}^T(k,k+1):0]^T$. Then, Equation (6.38) can be rewritten as follows:

$$y_1(k) = Z'(k+1) = \tilde{Z}(k) - u'(k) \tag{6.55}$$

where: $y_1(k) = [Z_{21}(k), \ldots, Z_{2p_1}(k), 0, \ldots, 0]^T$, is a $(p_1 + p_2)$ dimensional vector. Using the above vectors, the equation of the observer (6.48) can be rewritten as:

$$\begin{aligned} Z^*_1(k+1) + K_2(k)\, Z'(k+1) = {} & [\bar{A}_{11} - K_2(k)\bar{H}_a] Z(k/k-1) \\ & + B_1(k,k+1)\, u(k) + A_{12}(k,k+1)\, y_1(k) \\ & + K_2(k)\, [\tilde{Z}_3(k) + Z'(k+1) - \tilde{Z}(k) - \tilde{u}(k)] \end{aligned} \tag{6.56}$$

where: $\tilde{Z}_3(k) = [0, \ldots, 0 Z_{31}(k), Z_{32}(k), \ldots, Z_{3p_1}(k)]^T$ is $(p_1 + p_2)$ dimensional. Some manipulations yield the final form of (6.56) as:

$$\begin{aligned} Z^*_1(k+1) = {} & [\bar{A}_{11} - K_2\bar{H}_a]\, \hat{Z}(k/k-1) + B_1(k,k+1)\, u(k) + A_{12}(k,k+1)\, y_1(k) \\ & + K_2(k)\, [\tilde{Z}_3(k) - \tilde{Z}(k) - \tilde{u}(k)] \end{aligned} \tag{6.57}$$

The real-time minimum mean-square error linear unbiased stochastic estimator. The optimal estimation problem is solved by the use of Equations (6.57), (6.51), and (6.52) with all the newly defined augmented vectors and the following initial condition:

$$\hat{Z}(o) = M_1(o)\ \bar{x}_o$$

$$Z_{3(o)} = y_{2(o)} - H_2(o)y_1(o)$$

The case with all noisy measurements, $p_1=0$, coincides with the regular Kelman-Bucy filter with the removal of the multiplicative noise elements from the system equations and the measurements.

6.4 CONCLUSIONS

Herein presented were optimal linear state estimators, in the minimum mean-square sense and unbiased, both in the continuous-time and discrete-time cases. The presence of multiplicative noise in the system and measurements renders the conditional probability density function of the state vector non-Gaussian, notwithstanding the fact tha the random processes representing the multiplicative noises were assumed Gaussian.

The results of previous derivations on minimum mean-square error stochastic observer theory was extended. The linear stochastic filter derived is similar in nature to the regular Kalman-Bucy filtering algorithm. However, the lower dimension of the observer developed in the previous pages reduces the computational effort (for large order measurement systems) tremendously. While the Kalman-Bucy filter for an nth dimensional system requires n multiplications, the stochastic reduced order observer only required $(n-p_1)$ multiplications. The stability considerations for such systems needs to be modified [32].

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CHAPTER 7
THE CLOSED-LOOP STOCHASTIC
STABILIZATION AND COMPENSATION

SECTION 7.1
INTRODUCTION

The complexity of many physical systems, such as large space structures (LSS), combined with their inherent nonlinearities, randomness, and uncertainties, precludes any direct deterministic analytical approach and creates the need for stochastic analysis and approximation techniques. Theoretically, there are various methods, such as the method of operator equations using Green's functions [7.1] that can solve nonlinear stochastic multi-dimensional differential equations. However, in practice, their use is limited.

The pioneering work on the modern theory of differential equations was led by Liapunov and Poincarre at the turn of this century. Interest in stochastic differential equations was heightened by Ito [7.2] with his rigorous formulations of differential equations that led to the solution of Kolmogorov's diffusion process.

The analytical representation of complex dynamic systems entail nonlinear stochastic partial differential equations that characterize the overall behavior of the system in an approximate manner. The difficulty of generating solutions to such systems has resulted in various approximation techniques that reduce these systems to linear ordinary differential/difference equations. Under certain realistic situations, it becomes unavoidable to utilize the probabilistic approach for the analysis and optimal solution of such systems.

It is often the case that stringent stability requirements are difficult to meet due to the availability of only partial measurements of the state variables of a system. Moreover, systems subjected to random external disturbances and with a high degree of internal uncertainty in their analytical models have established the stochastic approach to stability theory as a viable engineering tool. This is especially true when the randomness or the uncertainty is of the state- and/or control-dependent type. The stability and stabilizibility characteristics of such stochastic systems cannot be treated on the basis of the separation or the certainty equivalence prnciple. Furthermore, an optimal control may not exist for control systems with multiplicative and additive noise, and the optimal filter is nonlinear and infinite dimensional [7.3-7.7]. Only suboptimal filters in the class of linear filters can be derived in practice, thus leading to suboptimal stabilization of stochastic linear systems. Various authors have addressed the problem of controlling linear systems with uncertainty in the model dynamics and control. Most of the past work deals with system models with only additive noise [7.8, 7.9].

It is an unquestionable fact that LFSS with very high performance requirements require active feedback control systems due to their high flexibility, high density modes in the lower part of their spectrum, and the (virtually inevitable) control/structure intraction problems inherent to such systems. Typically, LFSS with their broadband uncertainties have high order analytical representations, with fully coupled dynamic equations mainly due to the need for multiple input-multiple output feedback control systems. Finite element (FE) models of LFSS have inaccuracies in the very low frequency modes as well as in the very high frequency modes. Even though experimentation and identification techniques might be sufficiently advanced in the near future to be used to iprove model accuracy, yet residual uncertainties and time-dependent changes, as well as other random effects must still be handled appropriately. Thus, both synthesis and analysis techniques are necessary for the optimal control design of LFSS, since the implementation of coupled, multiple feedback loops, with their complex interactions, renders regular methods such as LQG very unattractive.

Moreover, LQG controllers are of the same order as the structural model and this requires a high degree of computational effort [7.10]. Application of stochastic techniques with stochastic reduced-order observers presents an approach with decreased computational burden and robust performance. This avenue is presently being pursued by many researchers.

In the analysis and control design of LFSS, modeling errors must be accounted for in order to provide adequate robustness for the control system. Multiple-loop frequency and time-domain techniques exist that deal with this model uncertainty and robustness issues (at least to some extent) [7.11]. However, the controllers derived with frequency domain techniques are usually of higher order [7.12]. LQG methods do not address the model uncertainties appropriate and do not guarantee gain margins [7.13]. In certain special cases, the so-called "LQG/LTR" technique guarantees stability under control matrix perturbations of only a certain kind [7.14-7.18]. Furthermore, realistic parameter variations are treated via loop shaping techniques based on singular value norm bounds. This could often lead to unrealistic constraints.

Modal characteristics of LFSS such as frequencies, damping ratios, and modal displacements represent physically meaningful parameters, and any uncertainties relative to these have a strong effect on the stability and robustness of the control system. LQG theory can be utilized for controller synthesis in multiple input-multiple output environments. However, since the underlying features of LQG comprise synthesis of feedback gains of an optimal dynamic compensator via the solutions of two uncoupled Riccati equations, it is very difficult to address performance robustness or computational matters adequately under high performance requirements or significant modeling uncertainties. Recently, optimal reduced-order compenstors were derived via optimal projection equations [7.19-7.20]. In these presentations, the structure and the order of the control law are fixed and the performance functional is minimized with respect to the control gains of the system, thus leading to necessary conditions for otpimality. These necessary conditions are then transformed into a coupled set of four algebraic Ricaati and Lyapunov equations via an oblique projection. The coupling between these equations is a strong indication of the breakdown of the separation principle under multiplicative noise [7.21].

In the present chapter, a unified theory of stability stabilizability and control of stochastic linear control systems with multiplicative and additive noise will be treated. The fundamental characteristics of such systems will be discussed and the stabilizing optimal/suboptimal controller will be derived. Both perfect information and partial and noisy mesurements will be considered and directions of future research in the area of linear/nonlinear systems with multiplicative and additive noise will be briefly commented upon. It is emphasized that multiplicative noise is a realistic way of expressing the influence of high uncertainty on the system performance via their statistical covariance. This, in a way, provides a natural performance robustness vis-a-vis random parametric and plant variations.

SECTION 7.2
THE STOCHASTIC STABILIZATION PROBLEMS

Consider the following Ito stochastic differential equation representing the perturbed motion of a system via a Markov process:

$$dx(t) = f[x(t),t]dt + \Gamma[x(t),t]d\ \zeta(t) \qquad (7.1)$$

with $f(o,t) = 0$ and $(o,t) = 0$ and where $x\ R^n$ is the state vector, $f(.,.)$ is a real n-dimensional continuous function such that $||f[x(t),t] - f[y(t),t]|| \leq k||x(t) - y(t)||$ for all $t > t_o$ and for all $x,y\ R^n$. $[x(t),t]$ is a real $n \times m$ matrix function continuous for $t \geq t_o$ and it in turn satisfies a similar Lipschitz condition. $\zeta(t)$ is an n-dimensional Wiener process with independent elements, zero mean, and $E[\zeta_j^2(t)] = 2t$ for $j = 1, \ldots, m$. Here, $E[.]$ denotes statistical expectation.

Under the above conditions, there is a strong Markov process that has almost sure continuity with a Feller trasition function as the solution to Eq. 1. The associated Jacobi-Bellman equation for Equation (7.1) is [7.5].

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i[x(t),t]\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}[x(t),t]\frac{\partial^2}{\partial x_i \partial x_j} \qquad (7.2)$$

where $[a_{ij}] = \Gamma\Gamma^T$ and $(.)^T$ denotes transposition.

The stability and stabilizability of the equilibrium solution of the linearized form of Equation (7.1) by a partial set of the state variable measurements of $x_1, x_2, \ldots, x_p$, $p<n$, is the main concern of the present article.

Stabilization of systems of the above form by a partial set of imperfectly mesured (noisy) states can be achieved by the synthesis of additional control forces u(t). Thus, let g[x(t),u(t),t] be a real n-dimensional continuous function that is uniformly Lipschitzian with respect to $t \geq t_o \geq o$ and let the control functions u(t) be Markovian with u(o,t) = 0; then for every control function u U (an admissible set), there is a corresponding Markov process that is almost sure continuous and is the solution to [7.7]:

$$dx(t) = f[x(t),t]dt + g[x(t),u(t),t]d\nu(t) + \Gamma[x(t),t]\ d\zeta(t) \quad (7.3)$$

The Jacobi-Bellman equation for Equation (7.3) is given by:

$$L_g = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i[x(t),t] \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} g_i[x(t),u(t),t] \frac{\partial}{\partial x_i}$$

$$+ \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}[x(t),t] \frac{\partial^2}{\partial x_i \partial x_j} \quad (7.4)$$

where ν is the zero-mean Weiner process independent of ζ. Under appropriate conditions, Equation (7.3) can be written in the following simple form:

$$dx(t)/dt = f[x(t),t] + g[x(t),u(t),t]\ \Upsilon(t) + \Gamma[x(t),t]\ \xi(t) \quad (7.5)$$

where Υ and ξ are zero-mean, independent, white Gaussian noise processes with covariances of Σ_Υ and Σ_ξ, respectively. A partial set of measurments of the states are provided by a system of the following form:

$$y(t) = h[x(t),t] + w(t) \quad (7.6)$$

where $y \epsilon R^{\ell}$ is a measurement vector, h is a continuous function that is Lipschitzian with respect to t and is uniformly bounded with respect to x(t). $w(t) \epsilon R^{\ell}$ is a zero-mean white Gaussian noise vector with covariances Σ_w and is independent of all the other noise elements.

The optimal stabilization problem is now to find an admissible control function u(t) that will make Equation (7.5) with the mesurements in Equation (7.6) stochastically stable with respect to x(t) and stochastically asymptotically stable with respect to the partialset of measurements. However, for the sake of being practical, the route that will be followed herein consists of treating the linearized version of the above problem. The approximation error is compensated for by incorporating the uncertainty terms in the dynamic equation.

In most practical situations, equations of the quations (7.5) and (7.6) can be approximated by a linear ith multiplicative and additive noise that cn be made otically close to the original nonlinear system. Thus, a linear stochastic system results, given by:

$$\dot{x}(t) = Ax(t) + Bu(t) + C(\xi)x(t) + D(\gamma)u(t) + E\omega(t) \qquad (7.7)$$

With Equation (7.6) replaced by its linearized version:

$$y(t) = Fx(t) + G(\alpha)x(t) + H\beta(t) \qquad (7.8)$$

Where A,B and F,H are constant (for convenience) matrices of appropriate dimensions, $x \epsilon R^n$ and $u \epsilon R^m$ are the state and the control vectors, respectively; ξ, γ, ω, α, and β are zero-mean, independent Gaussian noise vectors with covariances of Σ_ξ, Σ_γ, Σ_ω, Σ_α, and Σ_β, respectively, C, D, and G are matrix functions of ξ, γ, and α (usually taken as linear functions of the random variables) and

(A,B) is assumed stabilizable. Moreover, the initial state is assumed deterministic and known.

The stabilization problem is now to find appropriate control forces u(t) such that system Equations (7.7) and (7.8) are probabilistically stable with respect to all the variables $x_1, \ldots, x_n$ and is asymptotically probabilistically stable with respect to the estimates $\hat{x}(t)$ of the states x(t), generated from appropriate linear estimator/filters.

In order to establish the asymptotic stability in the large of the unperturbed motion

$$x(t) \triangleq 0,$$

of the dynamic system Equations (7.7) and (7.8), we start by a Lypunov function

$$V(x,t) = x^T(t)Px(t) \tag{7.9}$$

where P is a positive definite symmetric matrix that will be determined later and V(o,t) = 0.

Definition 7.1

System Equations (7.7) and (7.8) are quadratically asymptotically stabilizable if there exists a feedback control u = Kx, an n x n positive definite symmetric matrix P, and a constant $\rho > 0$, such that, for the Lyapunov function (7.9) and any admissible uncertainties $C(\xi)$ and $D(\gamma)$, the following is satisfied:

$$\begin{aligned} L[x(t),t] \triangleq\ & x^T(t)\{[A + C(\xi)]^T P + P[A + C(\xi)]\}\hat{x}(t) \\ & + x^T(t)\{K^T[B + D(\gamma)]^T P \\ & + P[B + D(\gamma)]K\}\hat{x}(t) < -\rho(||x||^2 + ||x||^2) \end{aligned} \tag{7.10}$$

Remark 7.1

When Equation (7.10) is satisfied, it can be shown that for any admissible uncertainties C( ) and D( ), the corresponding closed-loop system has x(t) 0 as an asymptotically stable equilibrium point [7.22-.25].

Definition 7.2

System Equations (7.7) and (7.8) is:

1. Probabilistically stable [7.5] if for any $t_o>0$, $\epsilon>0$, a>0 there exists a b>0 such that $||x||<\delta$

$$P\left[\sup_{t>t_o} ||\hat{x}(t)||>\epsilon\right]<a$$

2. Asymptoticallyprobabilistically stable if it is probabilisticlly stable and

$$\lim_{x\to 0}\left[\lim_{t\to\infty} ||\hat{x}(t)|| = 0\right] = 1$$

3. Asymptoticlly stable in the large if 1. is satisfied and

$$P\left[\lim_{t\to\infty} ||\hat{x}(t)|| = 0\right] = 1$$

Theorem 7.1

If in the set $\{x: ||\hat{x}(t)|| < d,\ d > 0\} \times R^+$, there exists a V(x,t) as defined above and satisfying:

$$LV[x(t),t]\leq 0 \qquad (7.11)$$

for $x(t) \neq 0$, then $x(t)$ 0 of Equations (7.7) and (7.8) is probabilistically stable. If, in addition,

$$\psi_1(||\hat{x}||) \leq V(x(t),t] \leq \psi_2(||\hat{x}||) \qquad (7.12)$$

and

$$LV[x(t),t] \leq -\psi_3(||\hat{x}||), \qquad (7.13)$$

where $_i(s)$ are continuous positive, monotonically increasing functions for all s $1[o,\infty)$ and $\psi_i(o) = 0$, i=1, 2, 3, then $x(t) = 0$ is asymptotically probabilistically stable in the large.

It can be proven [7.5] that if $\psi_1(||x||) = \infty$ as $||x|| \rightarrow \infty$, then the trivial solution is asymptotically probabilistically stable in the large. The P matrix above is the solution of a Lyapunov-like equation given by:

$$\{A + C(\xi) + K^T[B + D(\gamma)]^T\} P + P \{A + C(\xi) + [B + D(\gamma)] K\} = -Q \qquad (7.14)$$

and

$$P = \int_0^\infty \exp\{A + C(\xi) + K^T[B + D\gamma)]^T t\} Q \exp\{A + C(\xi)$$

$$[B + D(\gamma)]K\} t \, dt \qquad (7.15)$$

This formulation is an extension to the case given in Reference [7.5]; however, the proof of the theorem is very similar to the one given therein -- thus, it is omitted.

SECTION 7.3

STOCHASTIC CONTROL UNDER PARTIAL INFORMATION

System Equations (7.7) and (7.8) can be optimally stabilizable by the optimal controller that minimizes the following performance functional:

$$J = E\{x^T(T)\ H_c\ x(T) + \int_0^T [x^T(t)\ Q_c\ x(t) + u^T(t)\ R\ u(t)]dt\} \tag{7.16}$$

where H_c and R are positive definite and Q_c is positive semidefinite matrices of appropriate dimensions. The necessary and sufficient conditions for optimal stabilizability will be given and discussed. It is established [7.26] that when EE^T is positive definite, then an optimal control exists given that the control- and state-dependent noise are "small". Moreover, Haussman [7.27) has shown that when the above-mentioned noises affect only stable modes (thus, independent of the magnitudes of the control- and state-dependent noise vectors), an optimal control exists under complete state information. It is not hard to show [7.28] that, under a feedback control $u = -Kx(t)$, an optimal (in the class of linear controllers) K exists, if (A,B) is stabilizable and subject to the steady-state condition of Equation (7.16) and (here I is the identity matrix)

$$\inf \left|\left| \int_0^\infty e^{t(A-BK)^T} [K^T\ \Gamma(I)\ K + \Delta(I)] e^{t(A-BK)} dt \right|\right| < 1 \tag{7.17}$$

where $\Delta(S) = Tr(E[C^T(\xi)SC(\xi)])$, $\Gamma(S) = Tr\{E[D^T(\gamma)SD(\gamma)]\}$ and Tr is the matrix trace operator that was defined in earlier chapers. Also, E[.] is the statistical expectation operator and S is any positive semidefinite matrix of compatible dimensions. Then, K is found to be given by:

$$K = [\Gamma(P) + R + B^T PB]^{-1} B^T PA \tag{7.18}$$

and P is the unique positive definite solution of the following Riccati-like matrix algebraic equation:

$$Q + A^T P + PA + \Delta(P) - pB\ [\Gamma(P) + R + B^T PB]^{-1} B^T P = 0 \tag{7.19}$$

The optimal cost under the above conditions is found to be given by:

$$J^* = tr(E^T PE\Sigma_{\omega}] \tag{7.20}$$

Under conditions of partial and noisy measurement information like Equation (7.8), the state estimate that is optimal in the class of linear unbiased observers given by

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + M(t)\ [y(t) - (F + \Sigma_{\alpha})H^{-1}\ \hat{x}] \tag{7.21}$$

that minimizes the weighted estimation error given by:

$$J_e = E[\tilde{x}^T(t)\ U\tilde{x}(t)], \tag{7.22}$$

where $\tilde{x} = x - \hat{x}$ and U is an nxn positive definite matrix.

Let us define $X = \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$ and $\zeta = \begin{bmatrix} \omega \\ \nu \end{bmatrix}$, then

$$\dot{X}(t) = (\bar{A} + \bar{C} + \bar{D})\ X + \bar{G}\zeta, \tag{7.23}$$

where

$$\bar{A} = \begin{bmatrix} A - BK & BK \\ 0 & A - MF \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} C & 0 \\ C & 0 \end{bmatrix}, \bar{D} = \begin{bmatrix} -DK & DK \\ -DK & DK \end{bmatrix}, \bar{E} = \begin{bmatrix} E & 0 \\ E & -MG \end{bmatrix}$$

and

$$u(t) = -K(t)\ \hat{x}(t) \tag{7.24}$$

Also, if we define

$$\bar{H} = \begin{bmatrix} H & 0 \\ 0 & U \end{bmatrix}, \bar{Q} = \begin{bmatrix} Q + K^T RK & - K^T RK \\ -K^T RK & K^T RK \end{bmatrix}$$

then, the cost functional for Equation (7.24) will be

$$J_X = E[X^T (T) HX(T) + \int_0^T X^T (t)\bar{Q}X(t)\ dt] \tag{7.25}$$

Moreover, let

$$W = E\ [XX^T] = \begin{bmatrix} W_{11} & W_{12} \\ W_{12}{}^T & W_{22} \end{bmatrix}$$

Then,

$$J_X = tr[\bar{H}W(T)\ ] + \int_0^T tr(\bar{Q}W)\ dt \tag{7.26}$$

and

$$\dot{W} = \bar{A}W + W\bar{A}^T + \bar{C}W\bar{C}^T + \bar{D}W\bar{D}^T + \bar{E}\bar{E}^T \tag{7.27}$$

The optimal control problem is now transformed to that of minimizing J_X subject to Equation (7.27). The Hamiltonian approach will yield:

$$H = tr(\bar{Q}W) + tr\,(\bar{A}W + W\bar{A}^T + \bar{C}W\bar{C}^T + \bar{D}W\bar{D}^T)S \tag{7.28}$$

where S is the nxn symmetric costate matrix

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}$$

given by the solution of

$$S = -(\bar{Q}+\bar{A}^T S+S\bar{A}+\bar{C}^T S\bar{C}+\bar{D}^T S\bar{D}) \tag{7.29}$$

with

$$S(T) = \bar{H}$$

The necessary condition for optimality yields

$$M = W_{22}\, S^T + S_{22}^{-1}\, S_{12}^T\, W_{12} \tag{7.30}$$

and

$$\frac{\partial H}{\partial K} = \ 0 \text{ yields}$$

$$DK(W_{11} - W_{12}^T - W_{12} + W_{22}) - B^T(S_{11}W_{11} - S_{11}W_{12} + S_{12}W_{12}^T - S_{12}W_{22})$$
$$+ D^T(W_{11} + W_{12}^T + W_{12} + W_{22})K(W_{11} - W_{12} - W_{12}^T + W_{22}) = 0 \tag{7.31}$$

Some simplifying assumptions will lead to easily computable solutions of this system [7.21].

7.3.1 Statement of the Discrete-Time Compensation Problem

The following discrete-time linear stochastic system with purely random (white) parameters is considered:

$$x(k+1) = \Phi(k)x(k) + \Psi(k)u(k) + \xi(k) \tag{7.32}$$

where x(0) is a Gaussian random vector wiht $E[x(0)] = \bar{x}_o$ and $cov[x(0)] = X_o$. The measurements are given by:

$$y(k) = \begin{bmatrix} y_1(k) \\ \hline y_2(k) \end{bmatrix} = \begin{bmatrix} c(k) \\ \hline \Omega(k) \end{bmatrix} \begin{bmatrix} x_1(k) \\ \hline x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ \nu(k) \end{bmatrix} \tag{7.33}$$

where x, u, y_1, y_2, y are n-, m-, ℓ-, q-, and (ℓ+q)-dimensional state input and output vectors, respectively. $\Phi(k)$, $\Psi(k)$, C(k), $\Omega(k)$ are nxn, nxm, ℓxn and qxn matrices and Φ and Ψ are as described earlier. While C(k) and $\Omega(k)$ are deterministic and stochastic matrices, respectively, with $E[\Omega(k)] = \bar{\Omega}(k)$ and $cov[\Omega(k)] = \Sigma^{\Omega\Omega}$. Also, $\xi(k)$ and $\nu(k)$ are n- and c-dimensional mutally uncorrelated, zero-mean Gaussian white noise processes with given covaraínces of $cov[\xi(k)] = \Sigma^{\xi\epsilon}(k)$ and $cov[\gamma(k)] = \Sigma^{\nu\nu}$. Assume that $\Phi(k)$ and $\Psi(k)$ have some correlation with each other expressed by cov $[\Phi(k), \Psi(k)] = \Sigma^{\Phi\Phi}$. While $\Omega(k)$ is uncorrelated with $\nu(k)$ and $\Phi(k)$, $\Psi(k)$ are uncorrelated with $\xi(k)$ at all times.

The optimal control problem is now to generate a closed-loop control law based on past and current measurements and past controls that will minimize the following quadratic cost functional.

$$J = E\left[x^T(N)Fx(N) + \sum_{k=0}^{N-1} (x^T(k)Q(k)x(k) + u^T(k)R(k)u(k))\right] \tag{7.34}$$

where Q(k) is a symmetric, non-negative definite matrix and R(k) is a symmetric positive definite matrix of approximate dimensions.

The information set available at each instant of time is given by:

$$I(k) = \{(0), y(1), \ldots, y(k);\ u(0), u(1), \ldots, u(k-1)\} \tag{7.35}$$

As was previously done, it will be assumed that the admissible controls are measurable functions of the current and previous measurements. Furthermore, the control laws sought are of the feedback type, namely:

$$u(k) = f(\hat{x}(k), k) \tag{7.36}$$

where $\hat{x}(k)$ is the unbiased state estimate generated by the linear algorithm developed previously, $u \epsilon U$ (where U is the set of admissible controls) has full memory and completely nested information structure.

The cost functional can be expressed as:

$$J = E[L(u(k), \alpha(k)\ x(k)) + L(x(k+1))] \tag{7.37}$$

where $\alpha(k)$ are random vectors and L(k) is some convex function of its variables.

By the principle of optimality, we can get the minimum value of the cost as:

$$J^*(I(k)) = \underset{u(k)}{\text{Min}}\ E[L(u(k), \alpha(k), x(k)) + J^*(I(k+1)) | I(k)] \tag{7.38}$$

With the noisy observations given in Equation (7.33), we have to know the conditional probability function $P(x(k)|I(k))$. The perfect memory of the controls renders it possible to recursively compute $P(x(k+1)|I(k+1))$ from $P(x(k)|I(k))$ by some filtering algorithm. In the case when the filtering algorithm is independent of the control laws at all past time-indices, then the Separation Theorem can be applied for dynamic optimization.

7.3.2 Optimal Solution of the Stochastic Linear Problem with Incomplete State Information

Let us define the observed outputs by:

$$Y_k = [y^T(0), y^T(1), \ldots, y^T(k)]^T \tag{7.39}$$

Assume that the admissible controls u(k) are functions of the outputs Y_{k-1}, where clealy, Y_k is a vector in the (q+1) dimensional space and the dimension of Y_k increases with k.

Consider the stochastic problem defined by Equations (7.32), (7.33), and (7.34) with the appropriate statistics given as before. Then, the following identity can be verified easily [7.29].

$$x^T(N)Fx(N) = x^T(0)K(0)x(0) + \sum_{k=0}^{N-1} [x^T(k+1)K(k+1)x(k+1)$$

$$- x^T((k)K(k)x(k)] \tag{7.40}$$

$$K(N) = F \tag{7.41}$$

Take each term in the summation separately and expand it by using Equation (7.32) to get:

$$x^T(k+1)K(k+1)x(k+1) = [\Phi(k)x(k) + \Psi(k)\ u(k) + \xi(k)]^T K(k+1) \cdot$$

$$[\Phi(k)x(k) + \Psi(k)u(k) + \xi(k)] \tag{7.41}$$

and:

$$x^T(k)K(k)x(k) = x^T(k)\ [Q(k) + \bar{\Phi}^T(k)K(k+1)\bar{\Phi}(k) + Tr(\Sigma^{\Phi\Phi}K(k+1))$$

$$-\bar{\Phi}^T(k)K(k+1)\bar{\Psi}(k) + Tr(\Sigma^{\Phi\Phi}K(k+1)][R(k) + \bar{\Psi}^T(k)K(k+1)\bar{\Psi}(k)$$

$$+ Tr(\Sigma^{\Psi\Psi} K(k+1))]^{-1} \cdot [\bar{\Psi}^T(k) K(k+1) \bar{\Psi}(k) + Tr[\Sigma^{\Phi\Psi} K(k+1)) \quad x(k)$$

$$= x^T(k) \; [Q(k) \; + \bar{\Phi}^T(k) K(k+1) \bar{\Phi}(k) + Tr(\Sigma^{\Phi\Phi} K(k+1)) - G^T(k)$$

$$[R(k) + \bar{\Psi}^T(k) K(k+1) \bar{\Psi}(k) + Tr(\Sigma^{\Psi\Psi} K(k+1))] G(k)] x(k) \qquad (7.42)$$

where G(k) is the feedback gain derived earlier.

Now, substituting Equations (7.41) and (7.42) into (7.40), we get:

$$x^T(N) F x(N) = x^T(0) K(0) x(0) + \sum_{k=1}^{N-1} \{ [\Phi(k) x(k) + \bar{\Psi}(k) u(k)]^T K(k+1)$$

$$\cdot [\Phi(k) x(k) + \bar{\Psi}(k) u(k)] + [\Phi(k) x(k) + \Psi(k) u(k)]^T K(k+1) \xi(k) \}$$

$$+ \xi^T(k) K(k+1) [\bar{\Phi}(k) x(k) + \bar{\Psi}(k) u(k)] + \xi^T(k) K(k+1) \xi(k)$$

$$- \sum_{k=0}^{N-1} \{ x^T(k) [Q(k) + \bar{\Phi}^T(k) K(k+1) \bar{\Phi}(k) + Tr(\Sigma^{\Phi\Phi} K(k+1))) - G^T(k) [R(k)$$

$$+ \bar{\Psi}^T(k) K(k+1) \bar{\Psi}(k) + Tr(\Sigma^{\Psi\Psi} K(k+1))] \; x(k) + u^T R(k) u(k)$$

$$- u^T(k) K(k) u(k) \} \qquad (7.43)$$

By rearranging the terms in Equation (7.43), we end up with:

$$x^T(N) F x(N) + \sum_{k=0}^{N-1} x^T(k) Q(k) x(k) + u^T(k) R(k) R(k) = x^T(0) K(0) x(0)$$

$$+ \sum_{k=0}^{N-1} \{ [\Phi(k) x(k) + \Psi(k) u(k)]^T K(k+1) [\Phi(k) x(k) + \Psi(k) u(k)]$$

$$+ [\Phi(k) x(k) + \Psi(k) u(k)]^T K(k+1) \xi(k) + \xi^T(k) K(k+1)$$

$$[\Phi(k) x(k) + \Psi(k) u(k)] + \xi^T(k) K(k+1) \xi(k) \} \; - \sum_{k=0}^{N-1}$$

$$\{x^T(k)[\bar{\Phi}^T(k)K(k+1)\bar{\Phi}(k) + Tr(\Sigma^{\Phi\Phi}K(k+1) - G^T(k)[R(k) + \bar{\Psi}^T(k)$$

$$K(k+1)\bar{\Psi}(k) + Tr(\Sigma^{\Psi\Psi}K(k+1)]G(k)]x(k) + u^T(k)R(k)u(k)\} \quad (7.44)$$

The optimal cost functional is now given by [7.29]:

$$J = \underset{u(k)}{Min} \quad E[x^T(N)Fx(N) + \sum_{k=0}^{N-1} (x^T(k)Q(k)x(k) + u^T(k)k)k)u(k))]$$

$$= E\{\underset{u(k)}{Min} \quad [E[x^T(n)Fx(n) + \sum_{k=0}^{N-1} (x^T(k)u(k)x(k)$$

$$+ u^T(k)R(k)u(k))]Y_{k-1}]\} \quad (7.45)$$

Now, let us define the conditional mean of the state vector by:

$$\underline{x}(k) = E[x(k)/Y_{k-1}] \quad (7.46)$$

And the conditional covariance matrix of the state vector by:

$$\Sigma^{xx}(k) = E[(x(k) - \underline{x}(k))(x(k) - \underline{x}(k))^T) \quad (7.47)$$

The Equation (7.45) can now be written as:

$$J = E\left(\underset{u(k)}{Min} \quad E[x^T(0)K(0)x(0) + \sum_{k=0}^{N-1} \{(\Phi x+\Psi u)^T K(k+1)(\Phi x+\Psi u)\right.$$

$$+ (\Phi x+\Psi u)^T K(k+1)\xi + \xi^T K(k+1)(\Phi x+\Psi u) + \xi^T K(k+1)\xi\} - \sum_{k=0}^{N-1}$$

$$\{x^T[\bar{\Phi}^T K(k+1)\bar{\Phi}]x + Tr(\Sigma^{\Phi\Phi}K(k+1)) - G^T[R + \bar{\Psi}^T K(k+1)\bar{\Psi}$$

$$\left. + Tr(\Sigma^{\Psi\Psi}K(k+1))]G + u^T Ru\} | Y_{k-1}]\right)$$

$$= E[\min_{u(k)} [\underline{x}^T(0)K()\underline{x}(0) + Tr(\bar{x}_o K(0)) + \sum_{k=0}^{N-1} \{ \underline{x}^T \bar{\Phi}^T K(k+1) \bar{\Phi} \underline{x}$$

$$+ \underline{x}^T Tr\ (\Sigma^{\Phi\Phi} K(k+1))\ \underline{x} + \underline{x}^T \bar{\Phi}^T K(k+1) \bar{\Psi} u + u^T \bar{\Psi}^T K(k+1) \bar{\Phi} \underline{x}$$

$$+ 2\underline{x}^T Tr(\Sigma^{\Phi\Psi} K(k+1))u + u^T \bar{\Psi}^T K(k+1) \bar{\Psi} u + u^T Tr(\Sigma^{\xi\xi} K(k+1))\ u$$

$$+ Tr(\Sigma^{\xi\xi} K(k+1))\} - \sum_{K=0}^{N-1} \underline{x}^T [\bar{\Phi}^T K(k+1) \bar{\Phi}$$

$$+ Tr(\Sigma^{\Phi\Phi} K(k+1)) - G^T [R + \bar{\Psi}^T K(k+1) \bar{\Psi}$$

$$+ Tr(\Sigma^{\Psi\Psi} K(k+1))] \cdot G] \underline{x} - Tr(\Sigma^{xx} [\bar{\Phi}^T K(k+1) \bar{\Phi}$$

$$+ Tr(\Sigma^{\Phi\Phi} K(k+1)) - G^T [R + \bar{\Psi}^T K(k+1) \bar{\Psi}$$

$$+ Tr(\Sigma^{\Psi\Psi} K(k+1)) [G - u^T R u] |\ Y_{k-1})\} \quad (7.48)$$

Minimizing with respect to u(k) in the usual manner yields:

$$u(k) = -[R + \bar{\Psi}^T K(k+1) \bar{\Psi} + Tr(\Sigma^{\Psi\Psi} K(k+1))]^{-1}\ [\bar{\Psi}^T K(k+1) \bar{\Phi}$$

$$+ Tr(\Sigma^{\Phi\Psi} K(k+1))]\ \ x(k) = -G(k)\underline{x}(k) \quad (7.49)$$

The minimum value of the loss function then yields:

$$J = E(\underline{x}^T(0)K(0)\underline{x}(0) + Tr(\underline{x}_o K(0)) - \sum_{k=0}^{N-1} \{tr(\Sigma^{xx} G^T [R$$

$$+ \bar{\Psi}^T K(k+1) \bar{\Psi} + tr(\bar{\Sigma}^{CC} K(k+1))] G - tr(\Sigma^{\xi\xi} K(k+1)$$

$$+ \underline{x}^T \bar{\Phi}^T K(k+1) \bar{\Psi} G \underline{x} + \underline{x}^T G^T \bar{\Psi}^T K(k+1) \bar{\Phi} \underline{x}$$

$$+ 2\underline{x}^T Tr(\Sigma^{\Phi\Psi} K(k+1)) G \underline{x}\} |\ Y_{k-1}) \quad (7.50)$$

We summarize the above results in the following theorem.

Theorem 7.1

Consider the system given by Equations (7.32) with (7.33). Let the admissible controls u(k) be functions of Y_{k-1}. Assume that an optimal K has been found such that it is non-negative definite and $[R(k) + \bar{\Psi}^T K(k+1)\bar{\Psi} + Tr(\Sigma^{\psi\psi}K(k+1))]$ is positive definite for all k. Then there exists a unique admissible control given by Equation (7.49) which minimizes the expected cost functional. The minimal value of the cost is given by Equation (7.50).

Remark 7.1

It is possible to modify Theorem 7.1 to include the case when u(k) is a function of Y_k.

Remark 7.2

It is appropriate to note here that the conditional mean defined above is not computable in closed form because the truly optimal filter that will give the true estimate of x is infinite dimensional. So, we cascade a linear filter with a linear controller to get a fixed structure for our dynamic compensator.

7.3.3 Fixed Structure Linear Controller

The development that follows is a reformulation of the original stochastic control problem into a deterministic parameter optimazation problem using only first and second unconditional moments.

Consider the linear multivariable stochastic system given by Equations (7.32) - (7.34). The Optimal Stochastic Control for the above-mentioned system is to be generated at each instant of time by the following time-varying controller:

$$u(k) = -G(k)\hat{x}(k) \tag{7.51}$$

where $\hat{x}(k) \epsilon R^{\hat{\dot{n}}}$ (for $\hat{n} \epsilon R$ and $\hat{n}$) is the state-estimate of the true state vector $x(k)$. $\hat{x}(k)$ is generated by the linear unbiased estimator algorithm that was developed in previous chapters.

To achieve the transition from the stochastic to the deterministic control problem, we define a random vector consisting of the original state variable augmented with the estimation error:

$$\ell(k) = \begin{bmatrix} x(k) \\ x(k)-\hat{x}(k) \end{bmatrix} \tag{7.52}$$

Let us denote the second moment matrices of $\ell(k)$ by:

$$L(k) \triangleq E[\ell(k)\ell^T(k)] \triangleq \begin{bmatrix} L_{00}(k) & L_{01}(k) \\ L_{10}(k) & L_{11}(k) \end{bmatrix}$$

$$= E\begin{bmatrix} x(k)x^T(k) & x(k)[x(k)-\hat{x}(k)]^T \\ [x(k)-\hat{x}(k)]x^T(k) & [x(k)-\hat{x}(k)][x(k)-\hat{x}(k)]^T \end{bmatrix} \tag{7.53}$$

The original state equation considered was given by:

$$x(k+1) = \begin{bmatrix} A(k) & 0 \\ 0 & \theta(k) \end{bmatrix} x(k) + \begin{bmatrix} B(k) \\ \Gamma(k) \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ \xi(k) \end{bmatrix} \tag{7.54}$$

Now, this equation can be written (after some algebraic manipulation) as:

$$x(k) = \left(\begin{bmatrix} A(k-1) & 0 \\ \hline 0 & \theta(k-1) \end{bmatrix} - \begin{bmatrix} B(k-1) \\ \hline \Gamma(k-1) \end{bmatrix} G(k-1) \right) \hat{x}(k-1)$$

$$+ \begin{bmatrix} B(k-1) \\ \hline \Gamma(k-1) \end{bmatrix} G(k-1)\ (\hat{x}(k-1)-x(k-1) + \begin{bmatrix} 0 \\ \hline \xi(k-1) \end{bmatrix} \quad (7.55)$$

As previously shown, the state estimate x takes the following form:

$$\hat{x}(k) = M^{-1}(k) \begin{bmatrix} \hat{z}(k) \\ \hline y_1(k) \end{bmatrix} = M^{-1}(k) \begin{bmatrix} (\bar{A}_{11}-K_2\bar{H}_a)\hat{z}(k-1)+K_2H_a z(k-1) +K_2\xi(k-1)+B_1u(k-1)+A_{12}y_1(k-1) \\ \hline A_{21}z(k-1)+A_{22}y_1(k-1)+B_2u(k-1) +\xi_2(k-1) \end{bmatrix}$$

$$= M^{-1}(k) \left[\begin{pmatrix} A_{11}-K_2H_a & A_{12} \\ \hline 0 & A_{22} \end{pmatrix} M(k-1) - \begin{pmatrix} B \\ \hline \Gamma \end{pmatrix} G \right] \hat{x}(k-1)$$

$$+ M^{-1}(k) \begin{pmatrix} K_2H_a & 0 \\ \hline A_{21} & 0 \end{pmatrix} M(k-1)x(k-1) + M^{-1}(k) \begin{pmatrix} K_2\xi_a \\ \hline \xi_2 \end{pmatrix} \quad (7.56)$$

By appropriate use of Equations (7.53) - (7.56), the following component matrices of L(k) are obtained:

$$L_{00}(k) = E\left\{ \left[\begin{pmatrix} A(k-1) & 0 \\ \hline 0 & \theta(k-1) \end{pmatrix} - \begin{pmatrix} B(k-1) \\ \hline \Gamma(k-1) \end{pmatrix} G(k-1) \right] x(k-1) \right.$$

$$+ \begin{bmatrix} B(k-1) \\ \hline \Gamma(k-1) \end{bmatrix} G(k-1)\ (x(k-1)-x(k-1)) + \begin{bmatrix} 0 \\ \xi_0(k-1) \end{bmatrix} \left(\begin{pmatrix} A & 0 \\ \hline 0 & \theta \end{pmatrix} \right.$$

$$\left. \left. - \begin{bmatrix} B \\ \hline \Gamma \end{bmatrix} G \right) + \begin{bmatrix} B \\ \hline \Gamma \end{bmatrix} G(x-x) + \begin{bmatrix} 0 \\ \hline \xi_0 \end{bmatrix} \right] \right\}^T$$

$$= \bar{D}_1 L_{00}(k-1)\bar{D}_1^T + \bar{D}_1 L_{01}\bar{D}_2^T + \bar{D}_2 L_{11}\bar{D}_2^T + \Sigma^{\xi\xi}$$

$$+ \mathrm{Tr}(\Sigma^{D_1D_1} L_{00} + \Sigma^{D_1D_2}(L_{01}+L_{10}) + \Sigma^{D_2D_2} L_{11}) \quad (7.57)$$

where, for convenience, the time indices have been dropped and where:

$$D_1 = \begin{bmatrix} A & 0 \\ 0 & \theta \end{bmatrix} - \begin{bmatrix} B \\ \Gamma \end{bmatrix} G \qquad \bar{D}_1 = E[D_1]$$

$$D_2 = \begin{bmatrix} B \\ \Gamma \end{bmatrix} G(k-1) \text{ and } E\,[D_2] = \bar{D}2$$

$$\Sigma^{D_i D_j} = \text{cov}(D_i D_j) \text{ for } i = 1,2; j = 1,2$$

The equation for the estimation error is given by:

$$x(k)-\hat{x}(k) = \left(\left[\begin{array}{c|c} A & 0 \\ \hline 0 & \theta \end{array} \right] - M^{-1}(k) \left(\begin{array}{c|c} K_2 H_a & 0 \\ \hline A_{21} & 0 \end{array} \right) M(k-1) \right) x(k-1)$$

$$- \left[\begin{array}{c} B \\ \hline \Gamma \end{array} \right] G - M^{-1}(k) \left[\left(\begin{array}{c|c} \bar{A}_{11} - K_2 \bar{H}_a & A_{12} \\ \hline 0 & A_{22} \end{array} \right) M(k-1) \right.$$

$$\left. + \left[\begin{array}{c} B \\ \hline \Gamma \end{array} \right] G \right] \Big) \hat{x} + \left[\begin{array}{c} 0 \\ \hline \xi_o \end{array} \right] - M^{-1}(k) \begin{bmatrix} K_2 \xi a \\ 2 \end{bmatrix}$$

$$= \left(\left[\begin{array}{c|c} A & 0 \\ \hline 0 & \Gamma \end{array} \right] - M^{-1}(k) \left[\begin{pmatrix} K_2 H_a & 0 \\ A_{21} & 0 \end{pmatrix} M(k-1) - \left(\begin{array}{c|c} \bar{A}_{11} - K_2 \bar{H}_a & A_{12} \\ \hline 0 & A_{22} \end{array} \right) \right. \right.$$

$$\left. \left. M(k-1) + \left[\begin{array}{c} B \\ \hline \Gamma \end{array} \right] G \right] - \left[\begin{array}{c} B \\ \hline \Gamma \end{array} \right] G \right) x(k-1)$$

$$+ \left(\left[\begin{array}{c} B \\ \hline \Gamma \end{array} \right] G - M^{-1}(k) \left[\left(\begin{array}{c|c} \bar{A}_{11} - K_2 \bar{H}_a & A_{12} \\ \hline 0 & A_{22} \end{array} \right) \right. \right.$$

$$M(k-1) + \left[\begin{array}{c} B \\ --- \\ \Gamma \end{array}\right] G \Bigg]\Bigg) (x(k-1)-\hat{x}(k-1)) + \left[\begin{array}{c} 0 \\ --- \\ \xi_o \end{array}\right]$$

$$- M^{-1}(k) \quad \left[\begin{array}{c} K_2\xi_a \\ ---- \\ \xi_2 \end{array}\right]$$

$$= D_3x(k-1) + D_4(x(k-1)-\hat{x}(k-1)) + D_5 \tag{7.58}$$

Clearly, D_3, D_4 and D_5 are given by the expression for which they are substituted in Equation (7.58). Thus,

$$L_{01}(k) = E \left[\left(D_1x + D_2(x-\hat{x}) + \left[\begin{array}{c} 0 \\ -- \\ \xi_o \end{array}\right]\right) \left(D_3x + D_4(x-\hat{x}) + D_5\right)^T\right]$$

$$= \bar{D}_1L_{00}\bar{D}_3{}^T + \bar{D}_1L_{01}\bar{D}_4{}^T + \bar{D}_2L_{10}\bar{D}_3{}^T + \bar{D}_2L_{11}\bar{D}_4{}^T + E \left[\left[\begin{array}{c} 0 \\ -- \\ \xi o \end{array}\right] D_5{}^T\right]$$

$$+ Tr \left(L_{00}\Sigma^{D_1D_3}+L_{01}\Sigma^{D_1D_4}+L_{10}\Sigma^{D_2D_3}+L_{11} \Sigma^{D_2D_4}\right) \tag{7.59}$$

$$L_{10} = E \left[\left(D_3x + D_4(x-\hat{x}) + D_5\right)\left(D_1x + D_2(x-\hat{x}) + \left[\begin{array}{c} 0 \\ -- \\ \xi o \end{array}\right]\right)^T\right]$$

$$= \bar{D}_3L_{00}\bar{D}_1{}^T + \bar{D}_3L_{01}\bar{D}_2{}^T + \bar{D}_4L_{10}\bar{D}_1{}^T + \bar{D}_4L_{11}\bar{D}_2{}^T + E \left[D_5 \quad \left[\begin{array}{c} 0 \\ --- \\ \xi o \end{array}\right]\right]$$

$$+ Tr \left(L_{00} \Sigma^{D_3D_1} + L_{01}\Sigma^{D_3D_2} + L_{10} \Sigma^{D_4D_1} + L_{11} \Sigma^{D_4D_2}\right) \tag{7.60}$$

and

$$L_{11}(k) = E \left[\left(D_3x + D_4(x-\hat{x}) + D_5\right) \quad \left(D_3x + D_4(x-\hat{x}) + D_5\right)^T\right]$$

$$= \bar{D}_3L_{00}\bar{D}_3{}^T + \bar{D}_3L_{01}\bar{D}_4{}^T + \bar{D}_4L_{10}\bar{D}_3 + \bar{D}_4L_{11}\bar{D}_4{}^T + \Sigma^{D_5D_5}$$

$$+ Tr \left(L_{00} \Sigma^{D_3D_3} + L_{01} \Sigma^{D_3D_4} + L_{10} \Sigma^{D_4D_3} + L_{11} \Sigma^{D_4D_4}\right) \tag{7.61}$$

Note: In all of the above equations the time indices are dropped for convenience. Also, the actual value of κ_2 is given in a previous chapter on estimation.

We can now write the cost functional by:

$$J = tr(SL(N)) + \sum_{k=0}^{N-1} tr(\hat{Q}(k)L(k)) \qquad (7.62)$$

where:

$$S = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \hat{Q}(k) = \begin{bmatrix} Q + GRG & -G^TRG \\ -G^TRG & G^TRG \end{bmatrix}$$

Expanding Equation (7.62) in terms of its components we obtain:

$$J = tr(FL_{00}(n)) + \sum_{k=0}^{N-1} tr(QL_{00}(k)) + tr(G^TRG[L_{00}(k)-L_{01}(k) -L_{10}(k) + L_{11}(k)]) \qquad (7.63)$$

Let us formulate the Hamiltonian function:

$$\mathbf{H} = tr(\hat{Q}(k)L(k)) + tr([L(k+1)-L(k)]\ P^T(k+1)) \qquad (7.64)$$

Now, the canonical equations will be given by:

$$L^*(k+1)-L(k) = \left.\frac{\partial \mathbf{H}}{P(k)}\right|_* \text{ with } L^*(0) = L(0) \qquad (7.65)$$

and

$$P^*(k+1)-P^*(k) = -\left.\frac{\partial \mathbf{H}}{\partial L(k)}\right|_* \text{ with } P^*(N) = F \qquad (7.66)$$

where * stands for evaluation along the optimal trajectories.

We can expand the Hamiltonian function given in Equation (7.64) in the following manner:

$$H = tr(LQ_{00}(k) + tr(G^TRG[L_{00}-L_{01}-L_{10} + L_{11}])$$

$$+ tr\left(\begin{bmatrix} L_{00}(k+1)-L_{00}(k))P^T_{00}(k+1) + (L_{01}(k+1)-L_{01}(k))P^T_{10}(k+1) \\ L_{10}(k+1)-L_{10}(k))P^T_{00}(k+1) + (L_{11}(k+1)-L_{11}(k))P^T_{10}(k+1) \\ L_{00}(k+1)-L_{00}(k))P^T_{01}(k+1) + (L_{01}(k+1)-L_{01}(k))P^T_{11}(k+1) \\ L_{10}(k+1)-L_{10}(k))P^T_{01}(k+1) + (L_{11}(k+1)-L_{11}(k))P^T_{11}(k+1) \end{bmatrix}\right)$$

$$= tr(QL_{00}(k)) + tr(G^TRG(L_{00}(k)-L_{01}(k)-L_{10}(k) + L_{11}(k)])$$

$$+ tr\ ((L_{00}(k+1)-L_{00}(k))\ P^T_{00}(k+1) + L_{01}(k+1)-L_{01}(k))\ P^T_{10}(k+1)$$

$$+ (L_{10}(k+1)-L_{10}(k))\ P^T_{01}(k+1) + (L_{11}(k+1)-L_{11}(k))\ P^T_{11}(k+1)) \quad (7.67)$$

Substituting Equations (7.57) - (7.60) into Equation (7.67) one obtains:

$$H = tr(QL_{00}) + tr(G^TRG[L_{00}-L_{01}-L_{10} + L_{11}]) + tr\ \{(\bar{D}_1L_{00}\bar{D}_1{}^T+\bar{D}_1L_{01}\bar{D}_2{}^T$$

$$+ \bar{D}_2L_{10}\bar{D}_1{}^T + \bar{D}_2L_{11}\bar{D}_2{}^T + \Sigma^{\xi\xi} + Tr(\Sigma^{D_1D_1}L_{00} + \Sigma^{D_1D_2}(L_{01}+L_{10})$$

$$+ \Sigma^{D_2D_2}L_{11})-L_{00})P_{00}{}^T(k+1) + \bar{D}_1L_{00}\bar{D}_3{}^T + \bar{D}_1L_{01}\bar{D}_4{}^T + \bar{D}_2L_{10}\bar{D}_3{}^T$$

$$+ \bar{D}_2L_{11}\bar{D}_4{}^T + E\left[\begin{bmatrix}0\\ \xi_o\end{bmatrix} D_5{}^T\right] Tr(L_{00}\Sigma^{D_1D_3} + L_{01}\Sigma^{D_1D_4} + L_{10}\Sigma^{D_2D_3}$$

$$+ L_{11}\Sigma^{D_2D_4})-L_{01})P_{10}{}^T(k+1) + (\bar{D}_3L_{00}\bar{D}_1{}^T + \bar{D}_3L_{01}\bar{D}_2{}^T + \bar{D}_4L_{10}\bar{D}_1{}^T$$

$$+ \bar{D}_4L_{11}\bar{D}_2{}^T + E[D_5 \begin{bmatrix}0\\ \xi_o\end{bmatrix}]^T + Tr(L_{00}\Sigma^{D_3D_1} + L_{01}\Sigma^{D_3D_2} + L_{10}\Sigma^{D_4D_2})$$

$$- L_{10})P_{01}{}^T(k+1) + \bar{D}_3L_{00}\bar{D}_3{}^T + \bar{D}_3L_{01}\bar{D}_4{}^T + \bar{D}_4L_{10}\bar{D}_3{}^T + \bar{D}_4L_{11}\bar{D}_4{}^T + \Sigma^{D_5D_5}$$

$$+ Tr(L_{00}\Sigma^{D_3D_3} + L_{01}\Sigma^{D_3D_4} + L_{10}\Sigma^{D_4D_4}) -L_{11})\ P_{11}{}^T(k+1) \qquad (7.68)$$

The co-state equations can be derived from Equation (7.68) as follows:

$$\frac{\partial H}{\partial L_{00}(k)} = -P_{00}(k+1) + P_{00}(k)$$

$$= Q+G^TRG+\bar{D}_1{}^TP_{00}(k+1)\bar{D}_1{}^TP_{00}(k+1)\bar{D}_1{}^TP_{10}(k+1)$$

$$\bar{D}_3{}^TP_{10}(k+1) + \bar{D}_3{}^TP_{01}(k+1)\bar{D}_1-P_{01}(k+1)$$

$$+ \bar{D}_3{}^TP_{11}(k+1)\ \bar{D}_3-P_{11}(k+1) + Tr(\Sigma^{D_1D_1})P_{00}(k+1)$$

$$+ Tr(\Sigma^{D_1D_3})P_{10}(k+1) + Tr(\Sigma^{D_3D_1})P_{01}(k+1)$$

$$+ Tr\ (\Sigma^{D_3D_3}) \qquad (7.69)$$

$$\frac{\partial H}{\partial L_{01}(k)} = -P_{01}(k+1) + P_{00}(k)$$

$$= -G^T RG + \bar{D}_1{}^T P_{00}(k+1)\bar{D}_2 + \bar{D}_1{}^T P_{10}(k+1)\bar{D}_4{}^T P_{01}(k+1)\ \bar{D}_2$$

$$+ \bar{D}_3{}^T P_{11}(k+1)\bar{D}_4 - P_{10}(k+1) + Tr(\Sigma^{D_1D_2})P_{00}(k+1)$$

$$+ Tr(\Sigma^{D_1D_4})P_{10}(k+1) + Tr(\Sigma^{D_3D_4})P_{01}(k+1)$$

$$+ Tr(\Sigma^{D_3D_2})P_{01}(k+1) + Tr\ (\Sigma^{D_3D_4})P_{11}(k+1) \qquad (7.70)$$

$$\frac{\partial H}{\partial L_{10}(k)} = -P_{10}(k+1) + P_{10}(k)$$

$$= -G^T RG + \bar{D}_2{}^T P_{00}(k+1)\bar{D}_1 + Tr(\Sigma^{D_1D2})P_{00}(K+1)$$

$$+ \bar{D}_2{}^T P_{10}(k+1)\bar{D}_3 + TR(\Sigma^{D_2D_3})P_{10}(k+1)$$

$$+ \bar{D}_4{}^T P_{01}(k+1)\bar{D}_1 + Tr(\Sigma^{D_4D_1})P_{01}(k+1)$$

$$+ \bar{D}_4{}^T P_{11}(k+1)\bar{D}_3 + Tr(\Sigma^{D_4D_3})P_{11}(k+1) - P_{01}(k+1) \qquad (7.71)$$

$$\frac{\partial H}{\partial L_{11}(k)} = -P_{11}(k+1) + P_{11}(k)$$

$$= -G^T RG + \bar{D}_2{}^T P_{00}(k+1)\bar{D}_2 + Tr(\Sigma^{D_2D_2})P_{00}(k+1) + \bar{D}_2{}^T P_{10}(k+1)\bar{D}_4$$

$$+ Tr(\Sigma^{D_2D_4})P_{10}(k+1) + \bar{D}_4{}^T P_{01}(k+1)\bar{D}_2 + Tr(\Sigma^{D_4D_2})P_{01}(k+1)$$

$$+ \bar{D}_4{}^T\ P_{11}(k+1)\bar{D}_4 + Tr(\Sigma^{D_4D_4}P_{11}(k+1) - P_{11}(k+1) \qquad (7.72)$$

Since there is no constraint on the "control" matrix G, the optimal value of G can be computed as follows:

$$\frac{\partial \mathbf{H}}{\partial G} = 0 \qquad (7.73)$$

Obviously, to obtain the above optimal gain matrices, we have to deal with coupled, highly non-linear, very-hard-to-solve, two-point boundary value problems (TPBVP) which involve multi-dimensional matrix difference equations. Thus, no separation exists between control and filter equations and the only means of solving the above-mentioned TPBVP is successive approximations of numerical iteration techniques.

SECTION 7.4
CONCLUSIONS

The solution to the linear multivariable stochastic control system with a fixed structure feedback regulator was presented herein. The complex nature of the matrix equations generated, even if an analytical solution in a closed form were to be derived, would make computer simulations a non-trivial problem to say the least.

For the class of stochastic control problems considered in the previous pages, the optimum estimator is non-linear and requires computation of all the moments. Hence, adaptive sub-optimal controllers with given fixed structures were sought. The class of admissible controllers were thus fixed to be of linear feedback regulator type and subsequently, after the problem was reformulated in a deterministic framework, the free parameters of the compensator were optimized.

The final time-varying feedback controller wtih its implicit differential or difference equation requires severe off-line computations. Thus, to obtain the gains coupled, highly non-linear, two-point boundary value problems have to be solved. Even though different approaches (to the estimation problem) were considered, yet in all cases the high degree of complexity involved in the estimation problem from the matrix equations for the gains makes it impossible to obtain comparable analytical results, as compared to the perfect information situation discussed previously.

Presented herein was a unified theory of stochastic stability and stabilizability of linear systems with multiplicative and additive noise under partial information. The fundamental characteristics of such a system preclude the application of the certainty equivalence principle. Thus, a simultaneous solution of the estimation and stabilization/control problems was treated. The field of stochastic systems presents a challenging new research area, whereby a novel approach to robustness in the usual sense, that guarantees minimization of a given quadratic performance functional is possible.

Robust stability and performance is a crucial issue in complex control systems. Designing a controller that will sustain its effectiveness under parametric and other random variations is of paramount importance, especially in situations of complex nonlinear distributed parameter stochastic systems. There is a great deal of work in this branch of stochastic control and stabilization. Application of approximation theory and stochastic nonlinear control systems should be further reconciled and new methods of dealing with nonlinearities and randomness is needed for practical implementation.

SECTION 7.5

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CONCLUSIVE REMARKS

A unified theory of linear stochastic control systems with multiplicative and additive noise was presented in the previous pages. This theory, as applied to LFSS modeling and control design presents a new, probabilistic approach that has several advantages not the least of which is the inherent robustness it provides to the controller. Moreover, both stochastic continuous-time and discrete-time systems were considered for completeness. The stability characteristics and stabilizability of the closed-loop system was also treated.

There are extensive worldwide research activities pertaining to LFSS that are currently ongoing. An enormous amount of literature exists that covers various aspects of LFSS ranging from dynamics and nonlinearaties, to uncertainties and modeling techniques. Probabilistic approaches to structural dynamics and control is a relatively new phenomenon that is steadily expanding. The theory presented herein is an effort to present an approach for more realistic modeling of LFSS and the subsequent control design that is involved under the assumed uncertainty conditions. There are other such procedures that have been suggested by researchers in the field, (see the references). However, our approach has the advantage of relating the uncertainties to the model parameters; the statistics of which is possible to generate with some effort.

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